# QUADRATIC ALGEBRAS AS QUANTUM LINEAR SPACES: MONOIDAL STRUCTURES, DUALITIES, AND ENRICHMENTS

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## SUMMARY

**In my Montreal lectures of 1988, I developed the approach to quantum group putting in the foreground non–commutative versions of their group rings rather than universal envelopping algebras.**

**In this approach, the classical category of vector spaces is replaced by the category of quadratic algebras.**

**In this talk, I make a survey of basic properties of these "quantum linear spaces", and then extend the relevant definitions and results to the category of operads whose components are quadratic algebras.**

## GROTHENDIECK–VERDIER CATEGORIES: DEFINITIONS AND EXAMPLES

**SOURCE: [BD] M. Boyarchenko, V. Drinfeld.** *A duality formalism in the spirit of Grothendieck and Verdier.* **Quantum Topology, 4 (2013), 447–489.**

• **DEFINITION. A Grothendieck–Verdier category is a monoidal category** $(\mathcal{M}, \otimes)$ **endowed with** *a duality functor* $D$ **and** *dualizing object* $K$**.**

**Duality functor** $D$ **is an antiequivalence** $D : \mathcal{M} \to \mathcal{M}^{op}$ **such that for each object** $\mathcal{M}$**, the functor** $X \mapsto \mathrm{Hom}(X \otimes Y, K)$ **is representable by the object** $DY$**.**

• **EXAMPLES. (i)** $\mathcal{M}$**:= Bounded derived category of constructible** $l$**–adic sheaves on a scheme of finite type over a field,** $D$ **:= the Verdier duality functor.**

**(ii)** $\mathcal{M}$ **:= the bounded derived category of** $l$**–adic sheaves on the quotient stack** $\mathrm{Ad}\, G) \setminus G$ **with monodical structure defined via convolution.**

**• BASIC CATEGORY IN THIS TALK: QUADRATIC ALGEBRAS.**
**SOURCE: Yu. Manin.** *Quantum groups and non–commutative geometry,* **CRM, Montréal, 1988.**

**(a)** *A quadratic algebra* **is an associative graded algebra** $A = \oplus_{i=0}^{\infty} A_i$**, where** $A_0 = k$ **is a fixed ground field,** $A_1$ **is a finite dimensional linear space generating** $A$ **over** $k$**, and the graded ideal of all homogeneous relations is generated by its quadratic part** $R(A) \subset A_1^{\otimes 2}$.

$$Shorthand: \quad A \leftrightarrow (A_1, R(A))$$

**(b)** *Category* **QA***:* **Objects := quadratic algebras; morphisms: = graded homomorphisms over** $k$**.**

**(c)** *Monoidal structure(s)***: there are in fact** *four* **natural symmetric monodical structures on** $QA$**: see [M88], p.19.**

**Here our starting point will be** *the black product:*

$$A \bullet B \longleftrightarrow \{A_1 \otimes_k B_1, S_{23}(R(A) \otimes R(B))\},$$

$$S_{23}(a_1 \otimes a_2 \otimes b_3 \otimes b_4) := a_1 \otimes b_3 \otimes a_2 \otimes b_4.$$

**(d)** *Duality functor* $QA \to QA^{op}$*:*

$$A \mapsto A^! \leftrightarrow \{A_1^*, R(A)^\perp\},$$

$$(f : A \to B) \mapsto f^! \ := \ \textbf{the lift of the dual linear map } f_1^* : B_1^* \to A_1^*.$$

• **THEOREM. (i)** $(QA, \bullet)$ **is a Grothendieck–Verdier category with the duality functor ! and dualizing object** $k[t]$**, that is quadratic algebra with one–dimensional generating space and no relations.**

**(ii) It is pivotal category ([BM88], Def. 6.1), but not** $r$**–category ([BM88], Def. 1.5), because its identity object** $k[\varepsilon]/(\varepsilon^2)$ **is not isomorphic to its dualizing object.**

• **WHITE PRODUCT IN** $QA$**. Although** $QA$ **is not an** $r$**–category, the construction of the second monoidal structure in** $QA$ **generally called** *white product* **works also for quadratic algebras.**

**Explicitly, put as in [M88]:**

$$A \circ B \ \longleftrightarrow \ \{A_1 \otimes B_1, S_{(23)}(R(A) \otimes B_1^{\otimes 2} + A_1^{\otimes 2} \otimes R(B))\}$$

**Then we have:**

• **THEOREM. ([M88], p. 25.) There is a functorial isomorphism in** $QA$**:**

$$\mathrm{Hom}(A \bullet B, C) \simeq \mathrm{Hom}(A, B^! \circ C).$$

**Thus,** $B^! \circ C$ **can be identified with the right internal** Hom **in the Grothendieck–Verdier monodical category** $(QA, \bullet)$ **in the sense of [BD13], (2.8):**

$$B^! \circ C \simeq \underline{\mathrm{Hom}}'(B, C)).$$

## QUANTUM COHOMOLOGY OPERADS AND QUADRATIC ALGEBRAS

• **OPERAD OF GENUS ZERO MODULI SPACES. The $n$–th component of this operad is the moduli space of stable curves of genus zero with $n+1$ marked points $\overline{M}_{0,n+1}$ for $n \geq 2$. For $n=1$, this component is just a point. Among $n+1$ marked points $(x_0, x_1, \dots, x_n)$ one is declared *initial one,* say, $x_0$.**

**The family of operadic composition maps, here morphisms of smooth algebraic varieties,**

$$\mu(k_1, \dots, k_j): \ \overline{M}_{0,j+1} \times \overline{M}_{0,k_1+1} \times \dots \times \overline{M}_{0,k_j+1} \to \overline{M}_{0,k_1+\dots k_j+1}$$

**represents the natural geometric operation which identifies the $0$–th marked point of the curve $C_l$ over $\overline{M}_{0,k_l+1}$ with the $l$–th marked point of the curve $C_{j+1}$ over $\overline{M}_{0,j+1}$.**

## ENRICHMENTS

**SOURCE: G. M. Kelly.** *Basic concepts of the enriched category theory.*
**Cambridge UP (1982).**

*Revised online version* **http//www.tac.mta.ca/tac/reprints/articles/10/tr10.pdf**

• **The general construction of** *enrichment of a category* $\mathcal{A}$ *by a category* $\mathcal{B}$ **starts with a replacement of all morphism sets** $Hom_{\mathcal{A}}(X, Y)$ **by objects of the category** $\mathcal{B}$**.**
**At the next step we must lift composition maps**

$$Hom_{\mathcal{A}}(Y, Z) \times Hom_{\mathcal{A}}(X, Y) \to Hom_{\mathcal{A}}(X, Z)$$

**to appropriate morphisms in** $\mathcal{B}$ **which requires also the introduction of a bifunctorial composition** $\otimes$ **between objects of** $\mathcal{B}$ **replacing set-theoretic direct prooduct** $\times$**.**

**It follows that $\mathcal{B}$ must be a monoidal category. Finally, all the usual categorical axioms must be lifted to a class of commutative diagrams in $\mathcal{B}$.**

**• An additional condition in the treatment of enrichment by monoidal categories is the idea of its *closedness*.**

**A monoidal category is called *closed* if each functor of right tensor multiplication by a fixed object $* \mapsto * \otimes Y$ has a right adjoint $* \mapsto [Y, *]$, that is:**

$$Hom_{\mathcal{V}_0}(X \otimes Y, Z) = Hom_{\mathcal{V}_0}(X, [Y, Z]).$$

**Kelly also introduces unit and counit functors**

$$d: \; X \mapsto [Y, X \otimes Y], \qquad e: [Y, Z] \otimes Y \mapsto [Y, Z] \otimes Z.$$

**THEOREM. The category of quadratic algebras QA admits the "self"−enrichment by the symmetric monoidal category $(\mathbf{QA}, \bullet)$ with unit $K[t]/(t^2)$, where the black product $\bullet$ is defined on objects by**

$$(A_1, R(A)) \bullet (B_1, R(B)) := (A_1 \otimes_K B_1, S_{(23)}(R(A) \otimes_K R(B))).$$

**PROOF. (i) We start with an explicit description of the lifts of sets $Hom_{\mathbf{QA}}(A, B)$. We denote such a lift by $\underline{Hom}_{\mathbf{QA}}(A, B)$ and define it as**

$$\underline{Hom}_{\mathbf{QA}}(A, B) := A^! \circ B$$

**where** *white product* $\circ$ **is defined on objects of QA by**

$$(A_1, R(A)) \circ (B_1, R(B)) := (A_1 \otimes_K B_1, S_{(23)}(R(A) \otimes_K B_1^{\otimes 2} + A_1^{\otimes 2} \otimes_K R(B))).$$

**(ii) Now we must define the enriched composition morphisms (Kelly's notation $M_{ABC}$)**

$$\underline{Hom}_{\mathbf{QA}}(B,C) \bullet \underline{Hom}_{\mathbf{QA}}(A,B) \to \underline{Hom}_{\mathbf{QA}}(A,C)$$

**that is**

$$(B^! \circ C) \bullet (A^! \circ B) \to A^! \circ C.$$

**We can use functorial identifications**

$$Hom_{\mathbf{QA}}(A \bullet B, C) = Hom_{\mathbf{QA}}(A, B^! \circ C)$$

**in which a morphism in QA induced by the linear map $f : A_1 \otimes B_1 \to C_1$ is identified with the morphism in QA induced by the linear map $g : A_1 \to B_1^* \otimes C_1$ as is standard in the category of vector spaces.**

**(III) The compatibility with quadratic relations is checked directly. In order to pass to the general multiplication morphisms, we must iterate these identifications.**

**Identity morphisms $id_A : A \to A$ in QA are lifted to the Kelly's identity elements $j_A : K[t]/(t^2) \to A^! \circ A$.**

**The composition law (Kelly's $M_{ABC}$) is our morphism $\mu = \mu_{ABC}$.**

**Finally, we must check the associativity and unit axioms for this enrichment.**

## OPERADS AND THEIR ENRICHMENTS

**SOURCE: [BM] D. Borisov, Yu. Manin.** *Generalized operads and their inner cohomomorphisms.*
**Birkhäuser Verlag, Progress in Math., vol. 265 (2007), 247–308.**

• **We will use here the version of definition of operads according to which an operad P over a symmetric monoidal category** $(\mathcal{A}, \otimes)$ **( "ground category") is a monoidal/tensor functor** $(\Gamma, \coprod) \to (\mathcal{A}, \otimes)$ **where** $\Gamma$ **is a category of finite (eventually labeled) graphs with disjoint union** $\coprod$ **and morphisms including** *graftings.*

• **In our context, graphs will be forests having one labeled** *root* **at each connected component, and a numbering (complete ordering) by** $\{1, \dots, n\}$ **of all leaves on each connected component. (In [BM], we say "flags" in place of more common "leaves"). Grafting will connect roots to leaves.**

• **Denote by $\mathcal{P}(n)$ the image of the tree with one root and $n$ leaves totally ordered by labels $\{1,\ldots,n\}$, $n\geq 1$. We will refer to the family of objects $\mathcal{P}(n)$, eventually endowed with right $S_n$-actions, as** *a collection*, **and refer to $\mathcal{P}(n)$ as** *$n$–ary component of $\mathcal{P}$*, **or else** *component of arity $n$.*

• **The data completely determining such an operad is a set of morphisms in the ground category**

$$\mathbf{P}(k)\otimes\mathbf{P}(m_1)\otimes\mathbf{P}(m_2)\otimes\cdots\otimes\mathbf{P}(m_k)\to\mathbf{P}(n),\ \ n=m_1+m_2+\cdots+m_k \qquad (*)$$

**indexed by unshuffles of $\{1,2,\ldots n\}$. They are called** *operadic compositions* **or** *multiplications.*

**The relevant notion of cooperad is obtained by inversion of arrows in (*).**

• **DEFINITION. Given a Kelly enrichment of the ground category $(\mathcal{A}, \otimes)$ by $(\mathcal{B}, \times)$, we will call the** *enriched operad* **the family of respective lifts of morphisms (*)**

$$I_{\mathcal{B}} \to \underline{Hom}_{\mathcal{A}}(\mathbf{P}(k) \otimes \mathbf{P}(m_1) \otimes \mathbf{P}(m_2) \otimes \cdots \otimes \mathbf{P}(m_k), \mathbf{P}(n)). \qquad (**)$$

**Consider now an operad $\mathcal{P}$ over the ground category $(\mathbf{QA}, \bullet)$.**

• **PROPOSITION. The enrichment of $\mathcal{P}$ in the Kelly enrichment of $(\mathbf{QA}, \bullet)$ by QA is given by a family of quadratic algebras**

$$(\mathbf{P}(k) \otimes \mathbf{P}(m_1) \otimes \mathbf{P}(m_2) \otimes \cdots \otimes \mathbf{P}(m_k))^! \circ \mathbf{P}(m_1 + m_2 + \cdots + m_k)$$

**endowed with a family of elements in the linear spaces**

$$(\mathbf{P}(k) \otimes \mathbf{P}(m_1) \otimes \mathbf{P}(m_2) \otimes \cdots \otimes \mathbf{P}(m_k))_1^* \otimes \mathbf{P}(m_1 + m_2 + \cdots + m_k)_1 \qquad (***)$$

**indexed by unshuffles and having vanishing squares.**

**PROOF. As was shown earlier, for any three quadratic algebras** $A, B, C$ **we have canonical identifications**

$$Hom_{\mathbf{QA}}(A \bullet B, C) = Hom_{\mathbf{QA}}(A, B^! \circ C).$$

**Putting here** $A = K[t]/(t^2)$ **which is the unit object in** $(\mathbf{QA}, \bullet)$**, we get**

$$Hom_{\mathbf{QA}}(B, C) = Hom_{\mathbf{QA}}(K[t]/(t^2), B^! \circ C)$$

$$= \{d \in B_1^* \otimes C_1 \,|\, S_{(23)}(d^{\otimes 2}) \in R(B)^{\perp} \otimes C_1^{\otimes 2} + (B_1^*)^{\otimes 2} \otimes R(C)\}.$$

**In order to pass from this general case to (***), it remains to choose**

$$B_1 = \mathbf{P}(k) \otimes \mathbf{P}(m_1) \otimes \mathbf{P}(m_2) \otimes \cdots \otimes \mathbf{P}(m_k))_1, \quad C_1 = \mathbf{P}(m_1 + m_2 + \cdots + m_k)_1.$$

**This completes the proof.**

**• REMARK. Family of elements (***) with vanishing squares satisfies also some additional identities that follow from the operadic axioms. Their explicit form can be obtained in several steps.**

**(i) Write the respective axiom as a class of commutative diagrams in QA.**

**(ii) Break each commutative diagram into a family of neighboring commutative triangles and replace it by a sequence of equalities of elements in the Kelly's enrichments.**

**Namely, a commutative square** $gf = eh$ **in QA where** $f : A \to B$**,** $g : B \to C$**,** $h : A \to D$**,** $e : D \to C$**, can be lifted to the equality of the respective elements defined with the help of Kelly's morphisms:**

$$M_{ABC} : \quad \underline{Hom}_{\mathbf{QA}}(B,C) \bullet \underline{Hom}_{\mathbf{QA}}(A,B) \to \underline{Hom}_{\mathbf{QA}}(A,C)$$

**that is**

$$(B^! \circ C) \bullet (A^! \circ B) \to A^! \circ C$$

**and similarly** $M_{ADE}$**.**

## GENUS ZERO MODULAR OPERAD

**Here I will describe the main motivating example of the shuffle operad in the category QA: the genus zero modular (co)operad (also called tree–level cyclic CohFT (co)operad) $P$.**

- **The component of arity $n$ for $n \geq 2$ of $P$ is the cohomology ring**

$$P(n) := H^*(\overline{M}_{0,n+1}, \mathbf{Q})$$

**where $\overline{M}_{0,n+1}$ is the moduli space (projective manifold) parametrising stable curves of genus zero with $n+1$ labelled points. Component of arity 1 is Q.**

- **Structure morphisms (cooperadic comultiplications)**

$$P(m_1 + m_2 + \cdots + m_k) \to P(k) \otimes P(m_1) \otimes P(m_2) \otimes \cdots \otimes P(m_k)$$

**are maps induced by the maps of moduli spaces defined point–wise by the glueing of the respective stable curves:**

$$\overline{M}_{0,k+1} \times \overline{M}_{0,m_1+1} \times \cdots \times \overline{M}_{0,m_k+1} \to \overline{M}_{0,m_1+\cdots+m_k+1}.$$

• **PROPOSITION. a) For every $n \geq 3$, $P(n)$ is a quadratic algebra with linear space of generators $P(n)_1 = H^2(\overline{M}_{0,n+1})$ of dimension**

$$2^n - \frac{n(n+1)}{2} - 1.$$

**b) Comultiplications are morphisms of quadratic algebras.**

**PROOF. For a) and further details, see Ch. III, sec. 3, in**

**Yu. Manin. Frobenius manifolds, quantum cohomology, and moduli spaces. AMS Colloquium Publications, Vol. 47 (1999), xiii+303 pp.**

**Part b) follows from the fact that any morphism of smooth projective manifolds $X \to Y$ induces a functorial homomorphism of Chow rings $f^* : A^*(Y) \to A^*(X)$. Indeed, $P(n) := H^*(\overline{M}_{0,n+1}, \mathbf{Q})$ are just Chow rings graded by algebraic codimension of respective cycles.**

**Algebras classified/encoded by $P$, will be directly described below.**

• **There is another interesting operad $G$ such that components of every arity in its dual cooperad are quadratic algebras as well. It encodes** *Gerstenhaber algebras.* **Each $G(n)$ can be represented as the homology ring of the Fulton–MacPherson compactification of the space of configurations of $n$ points in $\mathbf{R}^2$.**

**In the literature, one can find a few other operads such that components of their dual cooperads are quadratic algebras.**

• **Additional information about $P$ and $P$–algebras.**

**I will briefly recall here a description of $P$ as a functor on the category of trees/forests.**

**Start with the combinatorial definition of relevant graphs.**

**(i)** *A stable tree* $\tau$ **is a diagram of pairwise disjoint finite sets $(V_\tau, E_\tau, T_\tau)$ and boundary maps**

$$b_T : T_\tau \to V_\tau, \quad b_E : E_\tau \to \{unordered\ pairs\ of\ distinct\ vertices\}.$$

*A geometric realization of* $\tau$ **is the CW–complex whose 1–simplexes are (bijective to)** $E_\tau \cup T_\tau$ (*edges and tails*) **and** 0**–simplexes are (bijective to)** $V_\tau$ ( *vertices.*)
**The geometric realisation of** $\tau$ **must be connected and simply–connected, i. e. to be a tree.**
**Each vertex must belong to the boundary of either one tail, or one tail and** $\geq 2$ **edges, or else or** $\geq 3$ **edges (stability condition).**

**(ii) Stable trees are objects of a category, in which every morphism** $f : \tau \to \sigma$ **consists of three maps**

$$f_v : V_\tau \to V_\sigma, \quad f^t : T_\sigma \to T_\tau, \quad f^e : E_\sigma \to E_\tau.$$

**satisfying certain conditions that we omit.**

**(iii) Let now** $F$ **be a finite set of cardinality** $\geq 3$. **Below we will denote by** $\overline{M}_{0,F}$ **the moduli space of stable curves of arithmetic genus zero endowed with a collection of pairwise different smooth points labelled by** $F$**.**

**One can define a functor $\mathcal{M}$ from the category of stable trees above to the category of projective algebraic manifolds. On objects, it is defined by**

$$\mathcal{M} : \tau \mapsto \prod_{v \in V_\tau} \overline{M}_{0,F_\tau(v)}.$$

**Here $F_\tau$ denotes the set of flags of $\tau$ that is, (pairs $\{edge,\ one\ vertex\ of\ it\}$), and $F_\tau(v)$ denotes the set of all flags, containing the vertex $v$.**

**I omit the definition of $\mathcal{M}$ on morphisms.**

**Let $L$ be an object of the category $\mathbf{Lin}_K^s$ of finite–dimensional $K$–linear superspaces with a non–degenerate even scalar product. One can define the operad $\mathbf{OpEnd}\,L$ as the functor on stable trees defined on objects by**

$$\mathbf{OpEnd}\,L\,(\tau) := L^{\otimes F_\tau}.$$

**Again, the definition on morphisms is here omitted.**

• **DEFINITION. The structure of $\mathcal{M}$–algebra on $L$ is a morphism of functors $\mathbf{OpEnd}\,L \to H^*\mathcal{M}$ compatible with gluing.**

**Applying to this operad the general construction sketched above, we obtain the following concrete result:**

• **PROPOSITION. The enrichment of action of $P$ upon a quadratic algebra $Q$ is represented by the family of Kelly enrichments $P(n)^! \circ \underline{Hom}\,(Q^{\otimes n}, Q)$ endowed with a family of elements described above.**

**Unfortunately, in the vast supply of examples of $P$–algebras, furnished by quantum cohomology, I was unable to find nontrivial actions of $P$ upon** *quadratic algebras $A$* **rather than upon** *graded spaces* **obtained by forgetting multiplication in $A$.**

**Below I will give some more details about the operad $P$.**

**• Generally, an operad can be characterised by the category of algebras that it classifies.**

**The operad $P$ classifies algebras endowed with infinitely many multilinear operations satisfying infinitely many "multicommutativity" properties which I will briefly recall below.**

**Let $L$ be a linear (super)space with symmetric even non–degenerate scalar product $h$.**

**A morphism of $P$ to its endomorphism operad induces upon $L$ the structure that I will call here, following E. Getzler, hypercommutative (or $hyperCom$) algebra.**

• **DEFINITION. A structure of cyclic** $hyperCom$**–algebra on** $(L,g)$ **is a sequence of polylinear multiplications**

$$\circ_n : \ L^{\otimes n} \to L, \ \circ_n(\gamma_1 \otimes \cdots \otimes \gamma_n) =: (\gamma_1, \dots, \gamma_n), \ n \geq 2$$

**satisfying three axioms:**

**(i) Commutativity := S**$_n$**–symmetry;**

**(ii) Cyclicity:** $h((\gamma_1, \dots, \gamma_n), \gamma_{n+1})$ **is S**$_{n+1}$**–symmetric;**

**(iii) Associativity: for any** $m \geq 0$**,** $\alpha, \beta, \gamma, \delta_1, \dots, \delta_m$

$$\sum_{\{1,\dots,m\}=S_1 \amalg S_2} \pm((\alpha, \beta, \delta_i \,|\, i \in S_1), \gamma, \delta_j \,|\, j \in S_2) =$$

$$\sum_{\{1,\dots,m\}=S_1 \amalg S_2} \pm(\alpha, \delta_i \,|\, i \in S_1), \beta, \gamma, \delta_j \,|\, j \in S_2))$$

**with usual signs from superalgebra.**

**(iv) (Optional) identity Data and Axiom:** $e \in L_{even}$ **satisfying**

$$(e, \gamma_1, \dots, \gamma_n) = \gamma_1 \textbf{ for } n = 1;\ 0 \textbf{ for } n \geq 2.$$

• **FACT. This direct description of cyclic** $hyperCom$**–algebras produces the same family of algebras that was described above as** $\mathcal{M}$**–algebras.**

• **Here are some comments.**

**1) If** $\circ_n = 0$ **for** $n \geq 3$**, we get the structure of commutative algebra with invariant scalar product:** $g(\alpha\beta, \gamma) = g(\alpha, \beta\gamma)$.

**2) Associativity identities for** $m = 1$ **are:**

$$((\alpha, \beta), \gamma, \delta) + ((\alpha, \beta, \delta), \gamma) = ((\alpha, (\beta, \gamma, \delta)) + (\alpha, \delta, (\beta, \gamma))$$

**3) One of the earliest results of mathematical theory of quantum cohomology established that for any smooth projective manifold (or a compact symplectic manifold) $V$, the superspace**

$$(L, h) := (H^*(V),\ \textbf{Poincaré pairing})$$

**admits a canonical structure of cyclic** $hyperCom$**–algebra.**

**• ON THE SELF-REFLEXIVITY OF QUANTUM COHOMOLOGY.**

**The idea to introduce a higher level (enriched, or "quantised") operadic action of** $P$ **upon its own components** $\{P(n)\}$ **was motivated by the problem which seems as yet far away from its solution. In the language of classical algebraic geometry, this problem consists in calculation of Gromov–Witten invariants of genus zero of** $\overline{M}_{0,n},\ n \geq 6$**, corresponding to those effective curve classes** $\beta$ **which lie "to the wrong side" of the anticanonical hyperplane.**

**In order to solve this problem, it might be helpful to use very recent results and methods of V. Dotsenko showing that all cohomology algebras** $H^*(\overline{M}_{0,n}$ **are Koszul:**

**V. Dotsenko. Homotopy invariants for** $\overline{M}_{0,\bullet+1}$ **via Koszul duality. arXiv:1902.06318.**

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**THANK YOU FOR YOUR ATTENTION**