Explicit Rieffel Induction Module for quantum groups

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Work in progress

Motivations

G - complex semi-simple Lie group (Ex : $G = SL_n(\mathbb{C})$)
$G = KAN$ (Ex : $K = SU(n)$, $A = \{$diagonals with positives entries$\}$,
$N = \{$unipotent upper triagular matrices$\}$)
$B = TAN$ - Borel subgroup, $L = TA$ - Levi factor.

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there exists a $C^*(L)$-Hilbert module $\mathcal{E}(G/N)$ s.t. $\forall\, \mu \in \hat{T}$, $\lambda \in \hat{A}$:

$$\mathrm{Ind}_B^G \mu \otimes \lambda \otimes \mathbf{1} \cong \mathcal{E}(G/N) \otimes_{C^*(L)} \mathbb{C}_{\mu \otimes \lambda}$$

We can thus re-express Harish-Chandra results :

$$C_r^*(G) = \oplus_{\mu \in \hat{T}} \mathcal{K}(\mathcal{E}(G/N) \otimes_{C_r^*(L)} C_0(\hat{A})_\mu)^W$$

Convolution algebra

Let $(\mathcal{A}(\mathbb{G}), \phi_{\mathbb{G}})$ be an algebraic quantum group (or more generally, a bornological quantum group).
We define $\mathcal{D}(\mathbb{G})$ as the $*$-algebra s.t. $\mathcal{D}(\mathbb{G}) = \mathcal{A}(\mathbb{G})$ as a space and with product and involution

$$f * g = id \otimes \phi_{\mathbb{G}}((1 \otimes S^{-1}(g))\Delta(f)), \ \forall f, g \in \mathcal{D}(\mathbb{G}),$$
$$f^* = \overline{S(f)}\delta_{\mathbb{G}}.$$

In Sweedler notations it gives $f * g = \phi_{\mathbb{G}}(S^{-1}(g) f_{(2)}) f_{(1)}$.

Closed quantum subgroup

Let's consider another algebraic (or bornological) quantum group $(\mathcal{A}(\mathbb{B}), \phi_{\mathbb{B}})$. Suppose we have a (bounded) Hopf $*$-morphism

$$\pi : \mathcal{A}(\mathbb{G}) \twoheadrightarrow \mathcal{A}(\mathbb{B}).$$

We thus say that π identifies $\mathcal{A}(\mathbb{B})$ as a closed quantum subgroup of $\mathcal{A}(\mathbb{G})$.

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The map π can be seen as $\pi : \mathcal{D}(\mathbb{G}) \to \mathcal{D}(\mathbb{B})$, which is no longer a morphism but induces a right $\mathcal{D}(\mathbb{B})$-module structure on $\mathcal{D}(\mathbb{G})$:

$$f \cdot h = \phi_{\mathbb{B}}(S^{-1}(h)\pi(f_{(2)})\gamma)f_{(1)}, \; f \in \mathcal{D}(\mathbb{G}), \; h \in \mathcal{D}(\mathbb{B}).$$

Here $\gamma = \pi(\delta_{\mathbb{G}}^{-\frac{1}{2}})\delta_{\mathbb{B}}^{\frac{1}{2}}$ (we assume $\delta_{\mathbb{B}}^{\frac{1}{2}} \in \mathcal{M}(\mathcal{D}(\mathbb{B})), \delta_{\mathbb{G}}^{\frac{1}{2}} \in \mathcal{M}(\mathcal{D}(\mathbb{G}))$

Conditional expectation

The map

$$E : \mathcal{D}(\mathbb{G}) \to \mathcal{D}(\mathbb{B}), \ E(f) = \pi(f)\gamma$$

has the weak conditional expectation properties :

1. $E(f^*) = E(f)^*$,
2. $E(f \cdot h) = E(f) * h$.

Remark : In the case where $\mathbb{B}$ is also open in $\mathbb{G}$, this map is a conditional expectation, it has been treated by Kalantar, Kasprzak, Skalski, Sołtan [*Induction for locally compact quantum groups revisited*, 2017]

The induction module

The space $\mathcal{D}(\mathbb{G})$ can be seen as a right $\mathcal{D}(\mathbb{B})$-module. Then

$$\langle f, g\rangle_{C^*(\mathbb{B})} = E(f^* * g), \; f, g \in \mathcal{D}(\mathbb{G}),$$

defines a $\mathcal{D}(\mathbb{B})$-inner product. $\mathcal{D}(\mathbb{G})$ can be completed to a $C^*(\mathbb{B})$-Hilbert module $\mathcal{E}(\mathbb{G})$, equipped a left $*$-action of $C^*(\mathbb{G})$.

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$$\mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} K$$

which is a representation $C^*(\mathbb{G})$ that we call the induced representation of K.

Link with Vaes' approach to induction

[*A new approach to induction and imprimitivity results*, 2004], Sketch of Vaes approach to induction :

$$ {}_{vN(\mathbb{G})}\mathcal{I}_{vN(\mathbb{B})} = \{v \in B(L^2(\mathbb{B}), L^2(\mathbb{G})),\ vx = \hat{\pi}'(x)v\ \forall x \in vN(\mathbb{B})'\} $$

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$$ \Big\downarrow $$

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$$\Big\downarrow \rightsquigarrow$$

$$_{C^*(\mathbb{G})}\mathrm{Ind}\ K \leftsquigarrow {}_{vN(\mathbb{G})}\mathcal{I} \otimes_{vN(\mathbb{B})} {}^{L^\infty(\mathbb{G})'}\big(L^2(\mathbb{G}) \otimes K\big)_{vN(\mathbb{G})} \overset{\exists}{\cong} {}_{vN(\mathbb{G})}{}^{L^\infty(\mathbb{G})'}L^2(\mathbb{G}) \otimes \mathrm{Ind}\ K_{vN(\mathbb{G})}$$

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$$
\begin{array}{ccc}
{}_{C^*(\mathbb{B})}K & \rightsquigarrow & {}_{vN(\mathbb{B})}L^2(\mathbb{G}) \otimes K_{vN(\mathbb{G})} \\
\vdots & & \wr \\
\downarrow & & \downarrow \\
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\end{array}
$$

Idea of the proof

1. Build an injection $\mathcal{E}(\mathbb{G}) \to \mathcal{I}$ with dense image (w.r.t. the weak topology of $B(L^2(\mathbb{B}), L^2(\mathbb{G}))$)
2. Show that

$$\Psi : {}_{vN(\mathbb{G})}L^2(\mathbb{G}) \otimes \overset{L^\infty(\mathbb{G})'}{\mathcal{E}(\mathbb{G})} \otimes_{C^*(\mathbb{B})} K_{vN(\mathbb{G})} \to {}_{vN(\mathbb{G})}\overset{L^\infty(\mathbb{G})'}{\mathcal{I}} \otimes_{\mathbb{B}} (L^2(\mathbb{G}) \otimes K)_{vN(\mathbb{G})}$$

with

$$\Psi(\xi \otimes a \otimes v) = \Delta(\xi)(a \otimes 1) \otimes v$$

defines an equivalence of bicovariant correspondences

(Thus $L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} K \cong L^2(\mathbb{G}) \otimes \text{Ind } K$ and so $\mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} K \cong \text{Ind } K$)

Complex semi-simple quantum groups

K_q - compact semi-simple quantum group
$G_q = K_q \bowtie \hat{K}_q$ the Drinfeld double
$B_q = T \bowtie \hat{K}_q$ - Borel quantum subgroup
$L_q = T \times A_q$ (where A_q designate the weight lattice associated to K_q)

Ex : $K_q = SU_q(2)$, $G_q = SL_q(2, \mathbb{C})$, $T \approx \mathbb{T}$, $A_q \approx \mathbb{Z}$

Remark : L_q is not a quantum subgroup but we have

$$\mathcal{D}(B_q) \twoheadrightarrow \mathcal{D}(L_q)$$

Parabolic induction

B_q is a closed quantum subgroup of G_q :

$$\mathcal{A}(G_q) = \mathcal{A}(K_q) \otimes \mathcal{A}(\hat{K}_q) \twoheadrightarrow \mathcal{A}(B_q) = \mathcal{A}(T) \otimes \mathcal{A}(\hat{K}_q)$$

Let now $(\mu, \lambda) \in \mathbf{P} \times T$. The representation of G_q associated with (μ, λ) is originally defined as

$$\mathrm{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu,\lambda} = \left\{ \xi \in \mathcal{A}(G_q) \;\middle|\; (id \otimes \pi_{B_q})\Delta_{G_q}(\xi) = \xi \otimes e^{\mu} \otimes K_{\lambda+2\rho} \right\}$$

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Proposition

$$Ind_{B_q}^{G_q} \mathbb{C}_{\mu,\lambda} = \mathcal{E}(G_q) \otimes_{C^*(B_q)} \mathbb{C}_{\mu,\lambda}$$

The Parabolic Induction module

The parabolic induction module can be built as the $C^*(L_q)$-Hilbert module $\mathcal{E}(G_q) \otimes_{C^*(B_q)} C^*(L_q)$

But we can build the module of "functions on the quotient space G_q/N_q" :

$$\mathcal{A}(G_q/N_q) := \mathcal{A}(K_q) \otimes \mathcal{A}(A_q)$$

with the $C^*(L_q)$-inner product

$$\langle a \otimes f, b \otimes g \rangle_{C^*(L_q)} = \phi_{K_q}(\bar{a}b_{(1)})\pi(b_{(2)}) \otimes \phi_{A_q}(f^* g_{(1)})g_{(2)}K_{-2\rho}$$

The $C^*(L_q)$-Hilbert module $\mathcal{E}(G_q/N_q)$ obtained by completing $\mathcal{A}(G_q/N_q)$ is isometrically isomorphic to $\mathcal{E}(G_q) \otimes_{C^*(B_q)} C^*(L_q)$.