Noncommutative Riemannian Geometry on finite groups and Hopf quivers

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Introduction

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- Gelfand-Naimark theorem
- Serre-Swan theorem
- Spectral triples by A. Connes

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Quantum groups approach to noncommutative geometry:

- allow one to generalize ‘classical’ ideas to ‘deformed’ versions: e.g. q-deformation $SU_q(2)$
- providing insights into a more general structure by using the Hopf algebra language to rephrase ideas and concepts
- quantum symmetry as a guide
- may hold the key to dealing with one of the major unsolved problems in physics: quantum gravity

Outline

I. Differential calculus and Quiver calculus

Noncommutative differential forms

Definition (1-forms)

Let A be an algebra. We say a pair (Ω^1, d) is **(generalised) first order differential calculus** over A, if

1) Ω^1 is a A-bimodule;
2) $\mathrm{d} : A \to \Omega^1$ linear map, called **derivative**, such that

$$\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b, \ \forall\, a, b \in A;$$

3) (dropped) $\Omega^1 = \mathrm{span}\{a\mathrm{d}b\}$.

When A is an Hopf algebra, one can require Ω^1 to be **left covariant** if 1) Ω^1 in addition is a left comodule with the coaction $\Delta_L : \Omega^1 \to H \otimes \Omega^1$ being a bimodule map and 2) the derivation is a left comodule map. A calculus Ω^1 is **bicovariant** if it is both left and right covariant.

Noncommutative differential forms

Definition (Higher forms)

We say $(\Omega(A) = \oplus_{n\geq 0}\Omega^n, \mathrm{d})$ with $\Omega^0 = A$ is a **(generalised) differential graded algebra (DGA)** over A if

1) Ω is a graded algebra, i.e., $\Omega^i \wedge \Omega^j \subseteq \Omega^{i+j}$ for $i, j \geq 0$;
2) $\mathrm{d} : \Omega^i \to \Omega^{i+1}$ is a degree 1 map such that $\mathrm{d}^2 = 0$ and graded Leibniz rule, i.e.,

$$\mathrm{d}(\xi \wedge \eta) = (\mathrm{d}\xi) \wedge \eta + (-1)^{|\xi|}\xi \wedge (\mathrm{d}\eta),\ \forall\, \xi, \eta \in \Omega;$$

3) (dropped) Ω is generated by A and Ω^1 as an algebra.

The advantage of Ω^1 being bicovariant is that one can construct DGAs via

- Woronowicz-Nichols algebra $\Omega_w(A) = A \ltimes B_-(\Lambda^1)$
- Quantum Shuffle algebra $\Omega_{\mathrm{sh}}(A) = A \ltimes \mathrm{Sh}_-(\Lambda^1)$ (generalised one), etc

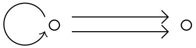
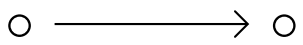
Quiver calculus on finite sets

Example (and Proposition)

Let $A = \Bbbk(X)$ be the algebra of functions on a finite set X.

$$\left\{\text{Differential calculi } \Omega^1 \text{ on } A\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{c}\text{Directed graphs } \bar{Q} = (X, E) \\ \text{without loops and multiple edges}\end{array}\right\}$$

$$\left\{\text{Generalised } \Omega^1 \text{ on } A\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{c}\text{Quivers } Q \text{ containing a digraph } \bar{Q} \\ \text{with } Q_0 = \bar{Q}_0 = X\end{array}\right\}$$

Q :  containing digraph $\bar{Q}$: 

Quiver allows for loops and multiple edges.

Hopf quivers

- The **path coalgebra** denoted by $\Bbbk Q^c$ is the $\Bbbk$-space spanned by the paths of Q with comultiplication and counit defined by $\Delta(x) = x \otimes x,\ \epsilon(x) = 1$ for each $x \in Q_0$, and

$$\Delta(p) = s(\alpha_1) \otimes p + \sum_{i=1}^{n-1} \alpha_1 \cdots \alpha_i \otimes \alpha_{i+1} \cdots \alpha_n + p \otimes t(\alpha_n), \quad \epsilon(p) = 0.$$

  for each non-trivial path $p = \alpha_1 \cdots \alpha_n$.

- A quiver Q is said to be a **Hopf quiver** if the corresponding path coalgebra $\Bbbk Q^c$ admits a length-graded Hopf algebra structure.
- For a Hopf quiver, Q_0 is necessarily a group and Q_1 is determined by **ramification datum** $R = \sum_{C \in \mathfrak{C}} R_C C$.
- Let G be a group and $S \subset G$ be a subset such that $e \notin S$. The **Cayley graph** associated to (G, S) is defined as the directed graph having one vertex at each $g \in G$ and directed edges $g \to h$ whenever $g^{-1}h \in S$.
- If a Hopf quiver has no loops and no multiple edges then it is a Cayley graph.

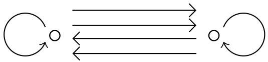
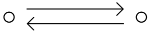
Hopf quiver calculus on finite groups

Example (and Theorem)

Let $A = \Bbbk(G)$ be the algebra of functions on a finite group G.

$$\{\text{Bicovariant } \Omega^1 \text{ on } A\} \xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{Cayley graphs } \bar{Q} \text{ w.r.t} \\ \text{a union of nontrivial conjugacy classes } \bar{C} \end{array} \right\}$$

$$\{\text{Generalised bicovariant } \Omega^1 \text{ on } A\}$$
$$\xleftrightarrow{1-1} \left\{ \begin{array}{c} \text{Hopf quivers } Q \text{ containing a Cayley digraph } \bar{Q} \\ \text{with } Q_0 = \bar{Q}_0 = X \end{array} \right\}$$

Q :  containing Cayley digraph $\bar{Q}$: 

Path algebra vs Path coalgebra

Corollary

Let $\Omega^1(\bar{Q}, Q)$ be a Hopf digraph-quiver calculus on $\Bbbk(G)$ of a finite group G. Then it extends to a DGA as a quotient of the path super-Hopf algebra $\Bbbk Q^a$ by the relation that the element $\sum_{x\in G, a,b\in\bar{C}} x \xrightarrow{(1)} xa \xrightarrow{(1)} xab$ is central.

Proof. $\Bbbk(G) \ltimes T_{-}\Lambda^1 \cong \Bbbk Q^a$. □

On group Hopf algebra $\Bbbk G$, where G is not necessary finite, we have

Theorem

Associated to a Hopf quiver containing loops, there is a bicovariant calculus on $\Bbbk G$. It extends to a DGA on the path super-Hopf algebra $\Bbbk Q^c$ with super-derivation given by $\mathrm{d} = [\theta, \}$, where θ is the sum of loops.

Proof. $\Omega_{\mathrm{sh}}(\Bbbk G) \cong \Bbbk Q^c$. □

II. Connections and quantum metrics

Connections

Let A be a unital algebra and (Ω^1, d) a (generalised) differential calculus over A.

- A **(left) connection** on a left A-module E is a linear map $\nabla : E \to \Omega^1 \otimes_A E$ such that

$$\nabla(a\omega) = \mathrm{d}a \otimes_A \omega + a\nabla\omega$$

  for all $\omega \in E$, $a \in A$.

- A connection ∇ is called a **(left) bimodule connection** if there exists a bimodule map $\sigma : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$ such that

$$\nabla(\omega a) = (\nabla\omega)a + \sigma(\omega \otimes_A \mathrm{d}a)$$

  for all $\omega \in E$, $a \in A$.

For a standard calculus, the map σ (if it exists) is fully determined by ∇.

Proposition (Connections given by quiver representations)

Let $A = \Bbbk(X)$ *and* $\Omega^1(\bar{Q}, Q)$ *digraph-quiver calculus*

1. *A connection* (E, ∇) *means a quiver representation i.e., a set of spaces* ${}_xE$ $(x \in X)$ *and maps* $L_\beta : {}_{s(\beta)}E \to {}_{t(\beta)}E$ $(\beta \in Q_1)$, *where we identify this information with*

$$E = \oplus_{x \in X}\, {}_xE, \quad \nabla v = \sum_{\alpha \in \bar{Q}_1} \alpha \otimes_A {}_{t(\alpha)}v + \sum_{\beta \in Q_1} \beta \otimes_A L_\beta({}_{s(\beta)}v).$$

   where ${}_xv$ *is the component of* v *in* ${}_xE$.

2. *A bimodule connection* (E, ∇, σ) *means a left connection and* $\sigma : E \otimes_A \Omega^1 \to \Omega^1 \otimes_A E$ *a bimodule map satisfying*

$$\sigma(v \otimes_A \alpha) = -\sum_{\beta \in Q_1} \beta \otimes_A L_\beta({}_{s(\beta)}v_{s(\alpha)})_{t(\alpha)}$$

   for all arrows α *in the digraph* $\bar{Q}$.

Metrics and Levi-Civita connections

A **metric** is an element $g \in \Omega^1 \otimes_A \Omega^1$ together with a bimodule map $(\ ,\ ) : \Omega^1 \otimes_A \Omega^1 \to A$ such that

$$g^{(1)} \otimes_A (g^{(2)}, \omega) = \omega, \quad (\omega, g^{(1)}) \otimes_A g^{(2)} = \omega, \quad \forall\, \omega \in \Omega^1,$$

where $g = g^{(1)} \otimes g^{(2)}$. One can require that g to be **central** in $\Omega^1 \otimes_A \Omega^1$, i.e. $ag = ga$ for any $a \in A$.

A bimodule connection (∇, σ) will be called **Levi-Civita** if

- torsion-free: $T_\nabla = 0$, where $T_\nabla := \wedge\nabla - \mathrm{d} : \Omega^1 \to \Omega^2$;
- torsion-compatible: $\mathrm{Im}(\mathrm{id} + \sigma) \subset \ker\wedge \implies T_\nabla$ is a bimodule map;
- metric-compatible:
  $\nabla g := \nabla g^{(1)} \otimes_A g^{(2)} + (\sigma \otimes \mathrm{id})(g^{(1)} \otimes_A \nabla g^{(2)}) = 0.$

Let $Q = (Q_0, Q_1)$ be a quiver and let $n(x,y) = \#\{x \to y\}$ in Q. We say Q is **symmetric** if $n(x,y) = n(y,x)$ for $\forall\, x, y \in Q_0$.

Proposition

The differential $\Omega^1 = kQ_1$ on $A = k(Q_0)$ admits a central metric if and only if the quiver is symmetric. The metric takes the form

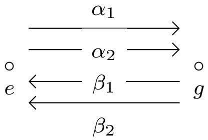
$$g = \sum_{x \to y \in E_Q} \sum_{i,j=1}^{n(x,y)} g_{x\to y}^{ij} x \xrightarrow{(i)} y \xrightarrow{(j)} x,$$

$$(y \xrightarrow{(j)} x, x' \xrightarrow{(k)} y') = (g_{x\to y})^{-1}{}_{jk} \delta_{x,x'} \delta_{y,y'} \delta_y,$$

where $g_{x\to y} = (g_{x\to y}^{ij})$ is an arbitrary $n(x,y) \times n(x,y)$ invertible matrices associated to index arrow $x \to y$.

Example

We have computed the Riemannian geometry of 4D (inner) generalised differential calculus of $A = k(\mathbb{Z}_2)$ associated to the following quiver:



There is a full 4-functional parameter moduli of quantum Levi-Civita connections for a given metric.

III. Quantum principal bundle

Quantum principal bundle

Definition

A **quantum principal bundle** (P, H, β) means

- P a right H-comodule algebra with coaction $\rho : P \to P \otimes H$;
- Let $A = P^{\mathrm{co}H}$. P together with a right-covariant standard differential $\Omega^1(P)$ such that

$$0 \to P\Omega^1(A)P \hookrightarrow \Omega^1(P) \to P \otimes \Lambda^1_H \to 0$$

  is a well-defined exact sequence, where $\Omega^1(H) = H \otimes \Lambda^1_H$ is bicovariant.

In the standard case, the exactness is interpreted as Hopf-Galois condition, which means the Galois map

$$\beta : P \otimes_A P \to P \otimes H, \; a \otimes b \mapsto a\rho(b)$$

is a linear isomorphism.

Here $A = P^{\,\mathrm{co}H}$ is the coordinate ring of the 'base' of the bundle, P the 'total space' and H the 'structure group'.

Example

Over $\mathbb{C}$, let $P = \mathbb{C}_q[SL_2]$ and $H = \mathbb{C}\mathbb{Z} = \mathbb{C}[g, g^{-1}]$ with Hopf algebra surjection

$$\pi : \mathbb{C}_q[SL_2] \to \mathbb{C}[g, g^{-1}], \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}$$

which induces a coaction $\beta = (\mathrm{id} \otimes \pi) \circ \Delta : \mathbb{C}_q[SL_2] \to \mathbb{C}_q[SL_2] \otimes \mathbb{C}\mathbb{Z}$. The 'quantum sphere' is defined as the coinvariant subalgebra

$$S^2_q = \mathbb{C}_q[SL_2]^{\,\mathrm{co}\,\mathbb{C}\mathbb{Z}}$$

Example

Consider a finite group G acting on a finite set X with $\mu : X \times G \to X$. Let $P = \Bbbk(X), H = \Bbbk(G)$ with $\mu^* : P \to P \otimes H$. Let $Y = X/G$ (G-orbits). Then $A = P^{\mathrm{co}H} = \Bbbk(Y) = \Bbbk(X/G)$ and $\beta(a \otimes_A b) = a\mu^*(b)$. One can check that β is an isomorphism if and only if the G-action is *free* (if g has a fix point,then $g = e$). This implies each orbit $\mathcal{O}_x$ has the same cardinality $|G|$.

Take $\Omega^1(P)$ as a digraph $Q = (Q_0 = X, Q_1)$ and $\Omega^1(H)$ as a Cayley graph of a conjugacy class $\bar{C}$ of G.

Proposition

In the above setting, G acting on a digraph on X gives a quantum principal bundle if and only if

1. *Each orbit $\mathcal{O}_x$ has cardinality $|G|$;*
2. *The graph within each orbit $\mathcal{O}_x$ has valency $|\bar{C}|$.*

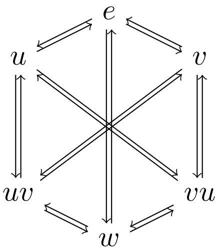
Corollary

Let $G \subseteq X$ be a nontrivial subgroup of a finite group X. Let $\bar{C}, \bar{C}_X$ define respectively bicovariant and right-covariant differentials. Then $X \to X/G$ gives a quantum principal bundle if and only if

$$\bar{C} = \bar{C}_X \cap G.$$

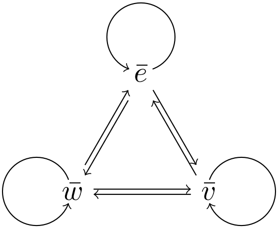
Example

We take $X = S_3$ with $\bar{C}_X = \{u, v, w\}$ (2-cycles) and take $G = \mathbb{Z}_2 = \langle u \rangle$ with $\bar{C} = \{u\}$. Let G act on digraph of X by right translation.



Then $X/G = \{\mathcal{O}_e, \mathcal{O}_v, \mathcal{O}_w\}$ consists of left cosets.

Example (cont.)



Thank you for your attention!