Synchronous games and game algebras

Let $\mathcal{G} = (X, X, A, A, \lambda)$ be a synchronous game, i.e.

$$\lambda(x, x, a, b) = 0 \text{ if } a \neq b,$$

(e.g. $\mathrm{Hom}(G, H)$) then winning strategies are characterised by traces on

$$\mathcal{A}(\lambda) = \mathcal{A}_{X,A}/J(\lambda) \text{ (the game algebra)}$$

with $J(\lambda) = \langle e_{x,a}e_{y,b} : \lambda(x, y, a, b) = 0\rangle$.

We have (Helton-Meyer-Paulsen-Satriano,Kim-Paulsen-Schafhauser):

- $\mathcal{G}$ has a perfect C_{qc} strategy iff there exists a non-zero tracial C^*-algebra $(\mathcal{A}, \tau)$ and a $*$-homomorphism $\pi : \mathcal{A}(\lambda) \to \mathcal{A}$. Then

$$p(a, b|x, y) = \tau(e_{x,a}e_{y,b}),$$

- C_{qa} strategy iff we can take $\mathcal{A} = \mathcal{R}^U$;
- C_q strategy iff we can take $\mathcal{A}$ finite-dimensional;
- C_{loc} strategy iff we can take $\mathcal{A}$ abelian.

Homomorphism and isomorphism games

For $t \in \{loc, q, qa, qc\}$ and graphs G and H, write $G \to_t H$ if there exists a perfect C_t strategy for the homomorphism game $\mathrm{Hom}(G, H)$. One can define quantum analogues of chromatic numbers and independence numbers $\chi(G)$ and $\alpha(G)$:

- The t-chromatic number of a graph G is
$$\chi_t(G) = \min\{d : G \to_t K_d\}$$
- The t-independence humber of G is
$$\alpha_t(G) = \max\{d : K_d \to_t \bar{G}\}.$$
- $\chi(G) = \chi_{loc}(G) \geq \chi_q(G) \geq \chi_{qa}(G) \geq \chi_{qc}(G)$,
  $\alpha(G) = \alpha_{loc}(G) \leq \alpha_q(G) \leq \alpha_{qa}(G) \leq \alpha_{qc}(G)$.

$\mathcal{A}(Hom(G,H))$, $\mathcal{A}(Iso(G,H))$

$Hom(G,H)$ games are rich enough to witness differences between C_q, C_{qa} and C_{qc}:
Dykema-Paulsen-Prakash (2019) using *Kruglyak-Rabanovich-Samoilenko* (2003):

$$\exists G : \chi_q(G) > \chi_{qa}(G)$$

Mancinska-Roberson-Varvitsiotis (2020) using MIP*=RE (2020):

$$\exists G : \alpha_{qc}(G) > \alpha_{qa}(G)$$

$Iso(G,H)$ games are examples of bisynchronous games where

$$\lambda(x,x,a,b) = 0 \text{ and } \lambda(x,y,a,a) = 0 \text{ if } x \neq y, a \neq b.$$

Perfect strategies are captured by quantum permutation group: $O(S_X^+)$ is generated by $p_{a,x} = p_{a,x}^* = p_{a,x}^2$ s.t. $U = (p_{a,x})_{a,x\in X}$ is a magic unitary, i.e. $\sum_a p_{a,x} = \sum_y p_{b,y} = 1$, x, $b \in X$.

If A_G and A_H are the adjacency matrices of G and H resp. then

$$\mathcal{A}(Iso(G,H)) = \langle p_{a,x} : U = (p_{a,x})_{a,x} \text{ magic unitary with } (A_G \otimes 1)U = U(A_H \otimes 1)\rangle.$$

Combining the previous results one gets:

For Iso(G,H) we have:

- $G \simeq_{qc} H$ iff there is a trace on $\mathcal{A}(Iso(G,H))$
- $G \simeq_{q} H$ iff there is a fin.-dim. repr. of $\mathcal{A}(Iso(G,H))$
- $G \simeq_{loc} H$ iff $G \simeq H$ iff there is a one-dim repr. of $\mathcal{A}(Iso(G,H))$.

$$G \simeq H \underset{[AMRSSV]}{\overset{\Rightarrow}{\not\Leftarrow}} G \simeq_q H \underset{[KPS]}{\overset{\Rightarrow}{\not\Leftarrow}} G \simeq_{qa} H \Rightarrow G \simeq_{qc} H$$

$$G \simeq_{qc} H \Leftrightarrow \underset{*\mathrm{Rep}\mathcal{A}(Iso(G,H))\neq\emptyset}{G \simeq_{C^*} H} \Leftrightarrow \underset{\mathcal{A}(Iso(G,H))\neq 0}{G \simeq_{alg} H} \quad \text{([BCEHPSW])}$$

Going quantum-to-quantum

Motivating question I: Suppose the game has quantum inputs/outputs. What kind of strategies can be used?

Motivating question II: Perhaps a suitable (simpler and genuinely) quantum game can disprove Tsirelson-Connes?

A classical input $(x, y) \rightsquigarrow$ the state $\epsilon_{x,x} \otimes \epsilon_{y,y} \in \mathcal{D}_X \otimes \mathcal{D}_Y$ (pos.-semidef. matrix of trace 1) here matrix units: $\epsilon_{x,y}$, $x, y \in X$ in M_X.

A correlation $p = \{(p(a, b|x, y)_{a,b} : x, y \in X\} \rightsquigarrow \mathcal{N}_p : \mathcal{D}_{X\times Y} \to \mathcal{D}_{A\times B}$,

$$\mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \sum_{a,b\in A} p(a, b|x, y)\epsilon_{a,a} \otimes \epsilon_{b,b}.$$

Note: p is no-signalling $\Leftrightarrow$

$\mathrm{Tr}_A\mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \mathrm{Tr}_A\mathcal{N}_p(\epsilon_{x',x'} \otimes \epsilon_{y,y})$ and $\mathrm{Tr}_B\mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y,y}) = \mathrm{Tr}_B\mathcal{N}_p(\epsilon_{x,x} \otimes \epsilon_{y',y'})$

Quantisation: (*Duan-Winter*) Quantum channels (completely positive trace preserving)

$\Gamma : M_{XY} \to M_{AB}$, satisfying no-signalling conditions.

No-signalling:

$\mathrm{Tr}_A\Gamma(\omega_X \otimes \omega_Y) = \mathrm{Tr}_A\Gamma(\omega'_X \otimes \omega_Y)$ and $\mathrm{Tr}_B\Gamma(\omega_X \otimes \omega_Y) = \mathrm{Tr}_B\Gamma(\omega_X \otimes \omega'_Y)$

Classes of quantum no-signalling correlations (Todorov-T. (2020))

A family of classical POVM's:

$\{(E_{x,a})_{a\in A} : x \in X\}$

$\rightsquigarrow$

$E = \sum_{x\in A}\sum_{a\in A} \epsilon_{x,x} \otimes \epsilon_{a,a} \otimes E_{x,a} \in M_{XA}(B(H))^+$

A family of quantum POVM's:

Stochastic operator matrix

$E = (E_{x,x',a,a'}) \in M_{XA}(B(H))^+$ such that $\mathrm{Tr}_A E = I \otimes I_X$.

Classes of quantum no-signalling (QNS) correlations

$\updownarrow$

Quantum channels $\Gamma : M_{XY} \to M_{AB}$ with specified Choi matrices $(\Gamma(\epsilon_{x,x'} \otimes \epsilon_{y,y'}))_{x,x',y,y'} \in M_{XYAB}$

Local QNS

Convex combinations of $\Phi \otimes \Psi$

Quantum QNS

$(\langle E_{x,x',a,a'} \otimes F_{y,y',b,b'}\xi, \xi\rangle)$, $\xi \in H_A \otimes H_B$

Quantum commuting QNS

$(\langle E_{x,x',a,a'} F_{y,y',b,b'}, \xi, \xi\rangle)$, $\xi \in H$

$$\mathcal{Q}_{\mathrm{loc}} \subset \mathcal{Q}_{\mathrm{q}} \subset \mathcal{Q}_{\mathrm{qa}} \subset \mathcal{Q}_{\mathrm{qc}} \subset \mathcal{Q}_{\mathrm{ns}}$$

Quantum non-local games

Classical games (X, Y, A, B, λ):

Rule function $\lambda : X \times Y \times A \times B \to \{0,1\} \rightsquigarrow (P_{(x,y)}, P_{\beta_{x,y}(\lambda)})$

$\beta_{x,y}(\lambda) = \{(a,b) : \lambda(x,y,a,b) = 1\}$,

P_α is a projection onto $\text{span}\{e_x \otimes e_y : (x,y) \in \alpha\}$

$\lambda \rightsquigarrow \varphi_\lambda : \text{Proj}(\mathcal{D}_{XY}) \to \text{Proj}(\mathcal{D}_{AB})$, $\vee$-preserving 0-preserving map.

Quantum games:

Replace (X, Y, A, B, λ) by $\text{Proj}(M_{XY}), \text{Proj}(M_{AB})$ and

$$\varphi : \text{Proj}(M_{XY}) \to \text{Proj}(M_{AB}),$$

$\vee$-preserving 0-preserving map.

Definition (Todorov-T, 2020)

A QNS correlation $\Gamma : M_{XY} \to M_{AB}$ is a perfect strategy for the quantum non-local game φ if $\langle \Gamma(P), \varphi(P)^\perp \rangle := \text{Tr}(\Gamma(P)\varphi(P)^\perp) = 0$, $\forall P \in \text{Proj}(M_{XY})$.

Questions and some references

- "Synchronous" and "bisynchronous" quantum games and associated algebras.

  Brannan-Harris-Todorov-T, 2021,2022, Bochniak-Kaspzak-Sołtan, 2021.
- Quantum graph homomorphisms and isomorphisms and associated algebras.

  Brannan-Chirvasitu-Eifler-Harris-Paulsen-Su-Wasilewski 2020,

  Brannan-Ganesan-Harris 2020, Ganesan 2022, Brannan-Harris-Todorov-T, 2021,2022
- Value of quantum non-local games.

  Cooney-Junge-Palazuelos-Pérez-García 2015, Crann-Levene-Todorov-T-Winter, in progress