Potential and flux reconstructions for optimal a priori and a posteriori error estimates

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Given a scalar-valued discontinuous piecewise polynomial, we devise a potential reconstruction which is piecewise polynomial and (trace-)continuous, H^1-conforming. It is obtained via a solution of local homogeneous Dirichlet problems on patches of elements sharing a vertex, using the conforming finite element method. Similarly, given a vector-valued discontinuous piecewise polynomial, we devise a flux reconstruction which is piecewise polynomial and normal-trace- continuous, $H(\text{div})$-conforming. This is obtained via local homogeneous Neumann problems on patches of elements, using the mixed finite element method. We show that the concept of potential reconstruction leads to a simple proof of equivalence of local-best and global-best approximations in the H^1 seminorm, i.e., that the global best approximation of a given H^1 function in a H^1- conforming finite element space can be bounded above and below by the sum of the respective elementwise approximations without any continuity requirement along the interfaces. This yields a priori error estimates for conforming finite elements under minimal regularity. Similarly, the flux reconstruction leads to the equivalence of local-best and global-best approximations in the L^2 norm under a divergence constraint, thereby yielding optimal a priori error estimates for mixed finite elements. Moreover, a stable commuting local projector can be identified. For a posteriori error estimates, the role is simply switched: guaranteed, locally efficient, and (polynomial degree) robust estimates are obtained via the flux reconstruction for conforming methods, whereas the potential reconstruction plays a key role in mixed (as well as nonconforming and discontinuous Galerkin) methods.