Unbiased confidence

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FocuStat, 22.05.2018

Overview

- Confidence distributions and focused confidence curves
- Unbiased estimates, tests and confidence sets
- Unbiased confidence curves,
- Unbiasedness and tail-symmetry
- Normal data
- Asymptotic unbiasedness
- Deviance based confidence curves
- Uniformly most powerful unbiased confidence curves

Confidence distributions and confidence curves

Definition

- $C(\theta; X)$ is cdf of a confidence distribution when $C(\theta; x)$ is a cdf in θ for all possible data x and $C(\theta_0; X) \sim U(0,1)$ when $X \sim f_{\theta_0}$.
- $cc(\theta; X)$ is a confidence curve when $0 \leq cc(\theta; x) \leq 1$ for all x and θ, and $cc(\theta_0; X) \sim U(0,1)$ when $X \sim f_{\theta_0}$. It is *focused* if it has finite level sets for low levels.

Cumulative confidence distributions are un-focused confidence curves.

Traditional unbiasedness

- $\hat{\theta} = C^{-1}(1/2)$ is median unbiased:
  $P_\theta(\hat{\theta} \leq \theta) = P_\theta(1/2 \leq C(\theta)) = 1/2$
- An $(1-\alpha)$-level confidence set $CS = \{\theta : cc(\theta) \leq 1-\alpha\}$ is unbiased when it excludes false parameter with at least probability α:
  $P_\theta(\theta_1 \in CS) \leq P_\theta(\theta \in CS) = 1-\alpha$ for false parameters θ_1
- An α-level test is unbiased when it rejects H_0 with at least probability α when the alternative is true.

Unbiased confidence curves

Definition

A confidence curve cc is unbiased when $cc(\theta_1, X) \overset{ST}{\geq} U(0,1)$ for false θ_1

- cc unbiased $\Rightarrow \{\theta : cc(\theta, X) \leq 1 - \alpha\}$ is an unbiased confidence set of level $1 - \alpha$
- C unbiased $\Rightarrow C(\theta_0)$ is an unbiased p-value for $H_0 : \theta \leq \theta_0$, reject H_0 when the p-value $\leq \alpha$ is an unbiased test.

The archetypal $C(\theta, X) = 1 - F(X; \theta)$ is unbiased.

Example: Exponential model

Example

Let X be exponentially distributed with rate parameter λ. The tail-symmetric confidence curve is

$$cc.ts(\lambda, X) = |1 - 2\exp(-\lambda X)|,$$

and the deviance based confidence curve

$$cc.d(\lambda, X) = K_1(Dev/1.1263).$$

K_1 is the cdf of the χ^2_1-distribution.

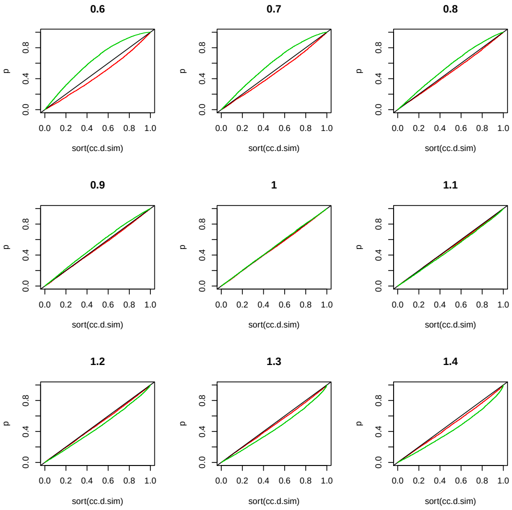


Figure: Cumulative distribution functions for $cc.d(\lambda_1)$ (panel title: λ_1) in the exponential model (red curve) and for $cc.ts(\lambda_1)$ (green curve). The black curve is the diagonal.

Normal data

For $X \sim N_p(\mu, \Sigma)$ the natural confidence curve for μ is

$$cc(\mu, X) = K_p((\mu - X)^t \Sigma^{-1} (\mu - X))$$

is unbiased, since The non-central χ^2, $(\mu_1 - X)^t \Sigma^{-1} (\mu_1 - X)$ is stochastically larger than χ^2_p. Here, K_p is the cumulative χ^2_p, Σ is assumed known, and $\mu_1 \neq \mu$.

Σ unknown:

$$cc(\mu) = H_p((\mu - X)^t \hat{\Sigma}^{-1} (\mu - X))$$

is unbiased, since the non-central Hotelling distribution is stochastically larger than the central one, H_p.

Large data

In smooth models, the deviance D_n is asymptotically Gaussian as the data size $n \to \infty$. The deviance based confidence curve

$$cc_n(\theta, X_n) = K_p(D_n)$$

is thus asymptotically unbiased.

Are deviance based confidence curves unbiased?

Lemma

The expected deviance is minimized at the true value.

Proof.

$$D(\theta_1) - D(\theta_0) = -2\log(f(X,\theta_1)/f(X,\hat{\theta})) + 2\log(f(X,\theta_0)/f(X,\hat{\theta}))$$

$$= 2\log(f(X,\theta_0)) - 2\log(f(X,\theta_1)),$$

thus

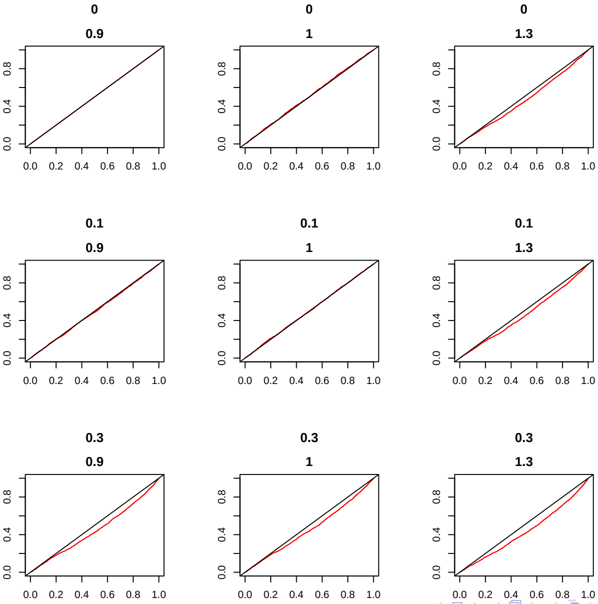
$$E_0 D(\theta_1) \geq E_0 D(\theta_0)$$

when $X \sim f(x,\theta_0)$ by the Kullback-Leibler divergence (Gibbs' inequality). □

Examples of unbiased deviance-based confidence curves

- Rate (or scale) of an exponential distribution
- scale, χ^2 $df = 3$
- Scale and location, Cauchy distribution
- Shape and scale, Weibull distribution
- Shape, Gamma distribution, scale=1
- shape and scale, Gamma distribution

Example: Cauchy, $(\mu, \sigma) = (0, 1)$, n=10.



Example: Gamma, shape, fixed scale=1, n=10.

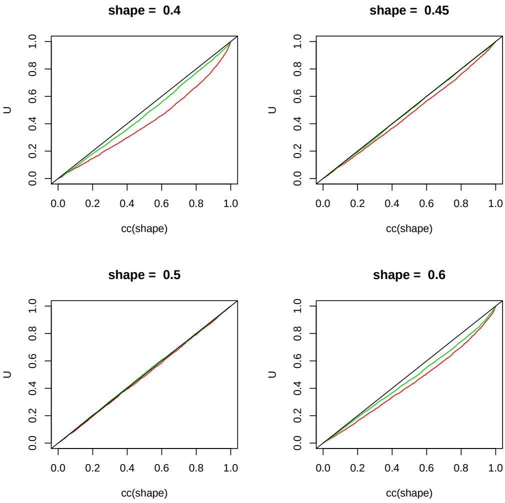


Figure: Cumulative distribution functions for $cc_d(\nu)$ (red curve) and for $cc_{ts}(\nu)$ (green curve). True shape = .5 The black curve is the diagonal.

Conjecture

Focused confidence curves based on deviance in parametric models are unbiased

Most powerful unbiased focused confidence curves

Lehmann (1959): A most powerful unbiased test of $H_0 : \psi = \psi_0$ versus $H_1 : \psi \neq \psi_0$ in an exponential family model must be conditional (when there are nuisance parameters), and have as acceptance region a (finite) interval of the sufficient statistic for ν.

$$\Downarrow$$

Focused confidence curves based on the (conditional) deviance are uniformly most powerful unbiased confidence curves in exponential family models (when unbiased).

Example: Gamma shape parameter, known scale

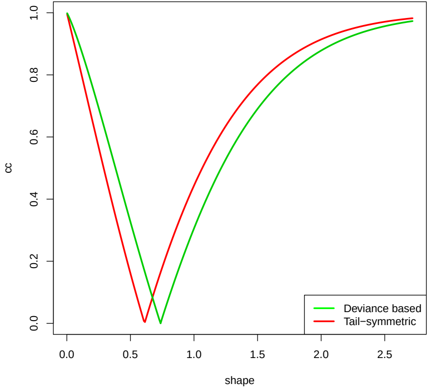


Figure: True shape = .5, $n = 10$. The tail-symmetric is $cc = |2C - 1|$.

Summing up

- The classical concept of unbiasedness for tests and confidence regions carries over to confidence curves: cc is unbiased whenever for $\psi_1 \neq \psi_0$

$$cc(\psi_1, X) \overset{ST}{\geq} cc(\psi_0, X) \sim U(0,1), \ X \sim f_{\psi_0}$$

- Confidence curves for a linear parameter in normal models are unbiased.
- The expected deviance is always at least as large at false values of the parameter than at true ones.
- Conjecture: Focused confidence curves based on the deviance D, $cc(\psi) = F(D(\psi))$, are unbiased.
- Tail-symmetric confidence curves $cc = |2C - 1|$ are typically (slightly) biased. Focused confidence curves based on the (conditional) deviance are UMPUcc in exponential family models.

To the young ones:

- Prove the conjecture!
- Are focused confidence curves based on profile deviances unbiased?