Optimality conditions for approximate solutions of nonsmooth semi-infinite vector optimization problems

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Outline

- Semi-Infinite Programming (SIP): Optimization problems with an infinite number of constraints (providing that the decision space is finite-dimensional)
- Semi-Infinite Multi-objective Optimization Problems

$$\mathrm{Min}_{\mathbb{R}^m_+}\{f(x) \mid x \in C,\ g_t(x) \leq 0, t \in T\} \qquad \text{(SIMOP)}$$

⋆ $\mathrm{Min}_{\mathbb{R}^m_+}$ in problem (SIMOP) is understood with respect to the ordering cone $\mathbb{R}^m_+ := \{(y_1, \ldots, y_m) \mid y_i \geq 0, i = 1, \ldots, m\}$;

⋆ C: the *abstract set* of problem (SIMOP) is a nonempty closed (not necessarily convex) subset of $\mathbb{R}^n$;

⋆ $f : \mathbb{R}^n \to \mathbb{R}^m$ with every component $f_i, i = 1, \ldots, m$ being locally Lipschitz functions;

⋆ $g_t : \mathbb{R}^n \to \mathbb{R}, t \in T$, are locally Lipschitz with respect to x uniformly in t , and T is an index set (possibly infinite).

- Let F be the feasible set of problem (SIMOP), given by

$$F := \{x \in C \mid g_t(x) \leq 0, t \in T\} \tag{1.1}$$

- Observe that if $m = 1$, then the problem (SIMOP) is reduced to an SIP.

⋆ It is worth noting that problem SIP with linear and convex inequality constraints have been widely studies and applied, but with Lipschitzian data are not very much in the literature.

- If $m \geq 2$, for the case of (weakly) efficient solutions and (weakly) approximate efficient solutions to problem (SIMOP) with Lipschitzian data, necessary and sufficient optimality conditions were investigated by several works; e.g., [1,2].
  - ⋆ Note that, all results [1,2] were obtained in the sense of Clarke subdifferential.
  - **[Goal]** In this talk, we will report some results on optimality conditions for approximate solutions of problem (SIMOP), by invoking some advanced tools from generalized differentiation and variational analysis due to Mordukhovich [3].

[1] D. S. Kim and T. Q. Son, An approach to ϵ-duality theorems for nonconvex semi-infinite multiobjective optimization problems, Taiwanese J. Math. 22 (2018), 1261–1287.

[2] T. Shitkovskaya and D. S. Kim, ϵ-solutions in semi-infinite multiobjective optimization, RAIRO Oper. Res. 52 (2018), 1397–1410.

[3] B. S. Mordukhovich, Variational Analysis and Applications, Springer Monographs in Mathematics, XIX+622 pp., Springer, Cham, Switzerland, 2018.

2. Basic Tools

- Let $\mathbb{R}^n$ denote the Euclidean space equipped with the usual Euclidean norm $\|\cdot\|$.
- ⋆ The notation $\langle \cdot, \cdot \rangle$ signifies the inner product in $\mathbb{R}^n$.
- ⋆ The non-negative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$.
- The polar cone of a set $\Omega \subset \mathbb{R}^n$ is defined by

$$\Omega^\circ := \{y \in \mathbb{R}^n \mid \langle y, x \rangle \leq 0, \ \forall x \in \Omega\} \tag{2.1}$$

- Let h be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, +\infty]$. We say $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ is lower semicontinuous (l.s.c.) at $\bar{x} \in \mathbb{R}^n$ if $\liminf_{x \to \bar{x}} h(x) \geq h(\bar{x})$.
- Consider set-valued mapping (or multifunctions) $P : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, with values $P(x) \subset \mathbb{R}^m$ in the collection of all the subsets of $\mathbb{R}^m$.

⋆ The limiting construction

$$\operatorname{Limsup}_{x \to \bar{x}} P(x) := \{y \in \mathbb{R}^m \mid \exists x_k \to \bar{x}, y_k \to y \text{ with } y_k \in P(x_k), \forall k \in \mathbb{N}\} \quad (2.2)$$

is known as the Painlevé–Kuratowski upper/outer limit of P at $\bar{x}$, where $\mathbb{N} := \{1, 2, \ldots\}$.

Basic Notations

- Given a set $\Omega \subset \mathbb{R}^n$, associate with it,
  - the distance function

$$\mathrm{dist}(x;\Omega) := \inf_{z\in\Omega} \|x - z\|, \ x \in \mathbb{R}^n$$

  - the Euclidean projector of $x \in \mathbb{R}^n$ to Ω by

$$\Pi(x;\Omega) := \{w \in \Omega \mid \|x - w\| = \mathrm{dist}(x;\Omega)\}$$

Definition (3, Definition 1.1)

Let $\Omega \subset \mathbb{R}^n$ with $\bar{x} \in \Omega$. The (basic) normal cone to Ω at $\bar{x}$ is defined by

$$N_\Omega(\bar{x}) := \text{Limsup}_{x \to \bar{x}} \left[\text{cone}(x - \Pi(x;\Omega))\right]$$

via the outer limit (2.2).

Each $v \in N_\Omega(\bar{x})$ is called a basic or limiting normal to Ω at $\bar{x}$ and is represented as follows: there are sequences $x_k \to \bar{x}$, $w_k \in \Pi\left(x_k;\Omega\right)$, and $\alpha_k \geq 0$ such that $\alpha_k\left(x_k - w_k\right) \to v$ as $k \to \infty$.

[3] B. S. Mordukhovich, Variational Analysis and Applications, Springer Monographs in Mathematics, XIX+622 pp., Springer, Cham, Switzerland, 2018.

- For an extended real-valued function $h : \mathbb{R}^n \to \overline{\mathbb{R}}$ its epigraph is denoted by

$$\text{epi } h := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid h(x) \leq r\}.$$

- The limiting/Mordukhovich subdifferential of h at $\bar{x} \in \mathbb{R}^n$ with $|h(\bar{x})| < \infty$ is defined by

$$\partial h(\bar{x}) := \{y \in \mathbb{R}^n \mid (y, -1) \in N_{\text{epi}\, h}(\bar{x}, h(\bar{x}))\}$$

If $|h(\bar{x})| = \infty$, one puts $\partial h(\bar{x}) := \emptyset$.

Lemma (sum rule, [3, Corollary 2.21])

Let $h_i : \mathbb{R}^n \to \overline{\mathbb{R}}, i = 1, 2, \ldots, k, k \geq 2$, be lower semicontinuous around $\bar{x} \in \mathbb{R}^n$, and let all these functions except, possibly, one be Lipschitz[1] continuous around $\bar{x}$. Then one has

$$\partial (h_1 + h_2 + \ldots + h_k) (\bar{x}) \subset \partial h_1(\bar{x}) + \partial h_2(\bar{x}) + \ldots + \partial h_k(\bar{x}). \tag{2.3}$$

[1] A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be locally Lipschitz, if for any $x \in \mathbb{R}^n$ there exists a positive constant K and a neighborhood N of x such that

$$|\phi(y) - \phi(z)| \leqq K\|y - z\|, \quad \forall y, z \in N(x).$$

[3] B. S. Mordukhovich, Variational Analysis and Applications, Springer Monographs in Mathematics, XIX+622 pp., Springer, Cham, Switzerland, 2018.

Theorem (Ekeland Variational Principle)

Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a continuous function bounded from below. Let $\epsilon > 0$ and $x_0 \in \mathbb{R}^n$ be given such that

$$\inf_{x \in \mathbb{R}^n} \varphi(x) \leq \varphi(x_0) \leq \inf_{x \in \mathbb{R}^n} \varphi(x) + \epsilon.$$

Then for any $v > 0$ there is $\bar{x} \in \mathbb{R}^n$ satisfying

(i) $\varphi(\bar{x}) \leq \varphi(x_0)$;

(ii) $\|\bar{x} - x_0\| \leq v$;

(iii) $\varphi(\bar{x}) \leq \varphi(x) + \frac{\epsilon}{v}\|x - \bar{x}\|$ for all $x \in \mathbb{R}^n$.

$$\mathbb{R}^{|T|} := \left\{\lambda = (\lambda_t)_{t\in T} : \lambda_t = 0 \text{ for all } t \in T \text{ but finitely many } \lambda_t \neq 0\right\}.$$

With $\lambda \in \mathbb{R}^{|T|}$, its supporting set $T(\lambda) = \{t \in T : \lambda_t \neq 0\}$ is a finite subset of T.

- The nonnegative cone of $\mathbb{R}^{|T|}$ is denoted by:

$$\mathbb{R}^{|T|}_+ = \left\{\lambda = (\lambda_t)_{t\in T} \in \mathbb{R}^{|T|} : \lambda_t \geq 0, t \in T\right\}.$$

- For $g_t, t \in T$

$$\sum_{t\in T} \lambda_t g_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t g_t & \text{if} \quad T(\lambda) \neq \emptyset \\ 0 & \text{if} \quad T(\lambda) = \emptyset \end{cases}$$

Definition (solution concepts of the problem (SIMOP))

Let $\varepsilon := (\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}^m_+$. A point $\bar{x} \in F$ is said to be

(i) an ε-efficient solution to the problem (SIMOP) iff there is no $x \in F$ such that

$$f_i(x) + \epsilon_i \leqq f_i(\bar{x}), i = 1, \dots, m,$$

with at least one strict inequality;

(ii) a quasi ε-efficient solution to the problem (SIMOP) iff there is no $x \in F$ such that

$$f_i(x) + \epsilon_i \|x - \bar{x}\| \leqq f_i(\bar{x}), i = 1, \dots, m,$$

with at least one strict inequality;

(iii) a weakly quasi ε-efficient solution to the problem (SIMOP) iff there is no $x \in F$ such that

$$f_i(x) + \epsilon_i \|x - \bar{x}\| < f_i(\bar{x}), i = 1, \dots, m.$$

3.1. Fuzzy necessary optimality condition

Reformulation

- For fixed $\bar{x} \in C$ and $(\epsilon_1, \dots, \epsilon_m) \in \mathbb{R}^m_+ \setminus \{0\}$, we define a real-valued function ψ on C as follows:

$$\psi(x) := \sup_{i=1,\dots,m,t\in T} \{f_i(x) - f_i(\bar{x}) + \epsilon_i, g_t(x)\}, \quad x \in C. \tag{3.3}$$

- For simplicity, denote by $\widehat{T} := \{1, \dots, m\} \cup T$ satisfying $\{1, \dots, m\} \cap T = \varnothing$, and

$$\hat{g}_t(x) := \begin{cases} f_t(x) - f_t(\bar{x}) + \epsilon_t, & \text{if } t \in \{1, \dots, m\}; \\ g_t(x), & \text{if } t \in T. \end{cases} \tag{3.4}$$

- Rewrite (3.3) as

$$\psi(x) := \sup\left\{\hat{g}_t(x) : t \in \widehat{T}\right\}, \quad x \in C. \tag{3.5}$$

- Define the set of α-active indices at y by

$$\widehat{T}_\alpha(y) := \left\{t \in \widehat{T} : \hat{g}_t(y) \geq \psi(y) - \alpha\right\}, \quad \alpha \geq 0$$

with $\widehat{T}(y) := \widehat{T}_0(y)$ and clearly that $\widehat{T}_\alpha(y) \neq \varnothing$ for $\alpha > 0$.

Lemma (compare [3, Theorem 8.30 (ii)])

Given $\hat{g}_t$ as in (3.4) and consider the supremum function ψ as (3.3). Then there exist

1. $\tau_i \geq 0$, $i \in M(y) := \{i \in \{1, \dots, m\} : \psi(y) = f_i(y) - f_i(\bar{x}) + \epsilon_i\}$ and
2. $\lambda_t \geq 0$, $t \in T(y) := \{t \in T : \psi(y) = g_t(y)\}$ satisfying $\sum_{i \in M(y)} \tau_i + \sum_{t \in T(y)} \lambda_t = 1$

such that

$$\partial\psi(y) \subset \sum_{i \in M(y)} \tau_i \partial f_i(y) + \sum_{i \in T(y)} \lambda_t \partial g_t(y). \tag{3.6}$$

[3] B. S. Mordukhovich, Variational Analysis and Applications, Springer Monographs in Mathematics, XIX+622 pp., Springer, Cham, Switzerland, 2018.

Theorem (Jiao/K. — Fuzzy necessary optimality condition)

Let $\bar{x}$ be a weak ε-efficient solution to the problem (SIMOP). For any $v > 0$ small enough, there exist $x_v \in C$ and $\tau_i \geq 0, i \in M(x_v)$ and $\lambda_t \geq 0,\ t \in T(x_v)$ satisfying $\sum_{i\in M(x_v)} \tau_i + \sum_{t\in T(x_v)} \lambda_t = 1$, such that $\|x_v - \bar{x}\| \leq v$ and

$$0 \in \sum_{i\in M(x_v)} \tau_i \partial f_i(x_v) + \sum_{i\in T(x_v)} \lambda_t \partial g_t(x_v) + N_C(x_v) + \frac{\max_{i=1,\ldots,m}\{\epsilon_i\}}{v}\mathbb{B}, \quad (3.7)$$

where $M(x_v) := \{i \in \{1,\ldots,m\} : \psi(x_v) = f_i(x_v) - f_i(\bar{x}) + \epsilon_i\}$ and $T(x_v) := \{t \in T : \psi(x_v) = g_t(x_v)\}$.

Proof: by Ekeland Variational Principle!

3.2. Optimality conditions for (SIMOP)

A key finding [4]: the solution relationship between problem (SIMOP) and the following standard multi-objective optimization problem:

$$\mathrm{Min}_{\mathbb{R}^m_+}\{\varphi(x) \mid x \in F\}, \tag{MOP}$$

where

- $\varphi(x) = f(x) + \varepsilon\|x - \bar{x}\|$
- $\varepsilon\|x - \bar{x}\| := (\epsilon_1\|x - \bar{x}\|, \dots, \epsilon_m\|x - \bar{x}\|)$
- the feasible set F is same as (1.1).

[4] Liguo Jiao and **Do Sang Kim***, Weakly quasi ϵ-efficiency for semi-infinite multi-objective optimization problems with locally Lipschitzian data. *Applied Analysis and Optimization*, **4** (2020), no. 1, 65–78.

Proposition (Jiao/K. — solution relationship)

If $\bar{x} \in F$ is a weakly quasi ε-efficient solution to the problem (SIMOP), then it is a weakly efficient solution to the problem (MOP).

The proof is just by definition! This result is easy to understand, but also pretty powerful!

The set of active constraint multipliers at $\bar{x} \in C$:

$$A(\bar{x}) := \left\{ \lambda \in \mathbb{R}_+^{|T|} \mid \lambda_t g_t(\bar{x}) = 0 \text{ for all } t \in T \right\} \tag{3.1}$$

Definition (LCQ)

Let $\bar{x} \in F$. We say that the following limiting constraint qualification (LCQ) is satisfied at $\bar{x}$ iff

$$N_F(\bar{x}) \subseteq \bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \right] + N_C(\bar{x}).$$

Theorem (Jiao/K. — exact necessity)

Let (LCQ) be satisfied at $\bar{x} \in F$. If $\bar{x}$ is a weakly quasi ε-efficient solution to the problem (SIMOP), then there exist $\tau := (\tau_1, \ldots, \tau_m) \in \mathbb{R}^m_+$ with $\tau^T e = 1$ and $\lambda \in A(\bar{x})$ defined in (3.1) such that

$$0 \in \sum_{i=1}^{m} \tau_i \partial f_i(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N_C(\bar{x}) + \sum_{i=1}^{m} \tau_i \epsilon_i \mathbb{B} \qquad (3.2)$$

The proof is mainly based on

- extreme principle — variational counterpart of the separation theorem in nonconvex settings,
- variational analysis and generalized differentiation (like Fermat theorem, sum rule etc).

Example (the importance of (LCQ))

- Consider the problem (SIMOP) with $C = \mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f_1(x) = f_2(x) := x$, and let $g_t : \mathbb{R} \to \mathbb{R}$ be given by $g_t(x) := tx^4$ for $x \in \mathbb{R}$ and for $t \in T := [1, 2]$.
- Let $\varepsilon = (\epsilon_1, \epsilon_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ be given.
- Clearly, the feasible set $F = \{0\}$ and thus, $\bar{x} := 0$ is the unique efficient solution [thus a quasi ε-efficient solution] of this problem.
- Since $\partial g_t(\bar{x}) = 2t\bar{x} = 0$ at $\bar{x} = 0$ for all $t \in T$,

$$\bigcup_{\lambda \in A(\bar{x})} \left[\sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \right] + N_C(\bar{x}) = \{0\}.$$

- On the other hand, $N_F(\bar{x}) = \mathbb{R}$. Therefore, the (LCQ) fails to hold at $\bar{x}$, and the above theorem does not hold.

Definition (generalized convexity)

Let $f := (f_1, \ldots, f_m)$ and $g_T := (g_t)_{t\in T}$. We say that (f, g_T) is generalized convex on C at $\bar{x} \in C$ iff, for any $x \in C, \xi_i \in \partial f_i(\bar{x}), i = 1, \ldots, m$, and $\eta_t \in \partial g_t(\bar{x}), t \in T$, there exists $\omega \in N_C(\bar{x})^\circ$ satisfying

$$\begin{aligned} \langle \xi_i, \omega \rangle &\leq f_i(x) - f_i(\bar{x}), \quad i = 1, \ldots, m, \\ \langle \eta_t, \omega \rangle &\leq g_t(x) - g_t(\bar{x}), \quad t \in T, \end{aligned}$$

and

$$\langle b, \omega \rangle \leq \|x - \bar{x}\|, \quad \forall b \in \mathbb{B}.$$

Definition (strictly generalized convexity)

Let $f := (f_1, \ldots, f_m)$ and $g_T := (g_t)_{t \in T}$. We say that (f, g_T) is strictly generalized convex on C at $\bar{x} \in C$ iff, for any $x \in C, \xi_i \in \partial f_i(\bar{x}), i = 1, \ldots, m$, and $\eta_t \in \partial g_t(\bar{x}), t \in T$, there exists $\omega \in N_C(\bar{x})^\circ$ satisfying

$$\langle \xi_i, \omega \rangle < f_i(x) - f_i(\bar{x}), \quad i = 1, \ldots, m,$$
$$\langle \eta_t, \omega \rangle \leq g_t(x) - g_t(\bar{x}), \quad t \in T,$$

and

$$\langle b, \omega \rangle \leq \|x - \bar{x}\|, \quad \forall b \in \mathbb{B}.$$

Theorem (Jiao/K. — sufficiency)

Let $\bar{x} \in F$ satisfy (3.2).

(i) If (f, g_T) is generalized convex on C at $\bar{x}$, then $\bar{x}$ is a weakly quasi ε- efficient solution to problem (SIMOP).

(ii) If (f, g_T) is strictly generalized convex on C at $\bar{x}$, then $\bar{x}$ is a quasi ε-efficient solution to problem (SIMOP).

Example (the importance of generalized convexity)

- Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^3$, let $g_t : \mathbb{R} \to \mathbb{R}$ be given by $g_t(x) := tx^2, x \in \mathbb{R}, t \in T := [-2, -1]$, and let $C = \mathbb{R}$.
- Observe that the feasible set $F = \mathbb{R}$.
- Take $\bar{x} = 0 \in F$, clearly $\bar{x} = 0$ satisfies condition (3.2) in the above theorem.
- However, $\bar{x} = 0$ is not a quasi ϵ-solution to the problem (SIMOP) with $m = 1$.
- The reason is that the generalized convexity of (f, g_T) on C at $\bar{x}$ was not satisfied.

3.3. ε-Wolfe type duality

- For $y \in \mathbb{R}^n, \tau := (\tau_1, \ldots, \tau_m) \in \mathbb{R}^m_+$ with $\tau^T e = 1$ and $\lambda \in \mathbb{R}^{|T|}_+$, here $e := (1, \ldots, 1) \in \mathbb{R}^m$, put

$$\mathcal{L}(y, \tau, \lambda) := f(y) + \sum_{t \in T} \lambda_t g_t(y) e.$$

- Consider the Wolfe (in approximate form) dual problem of the problem (SIMOP) as follows:

$$\text{Max}_{\mathbb{R}^m_+} \left\{ \mathcal{L}(y, \tau, \lambda) \mid (y, \tau, \lambda) \in F_W \right\}, \qquad (\mathrm{D}_W)$$

where the feasible set F_W is given by

$$F_W := \{(y, \tau, \lambda) \in C \times (\mathbb{R}^m_+ \setminus \{0\}) \times \mathbb{R}^{|T|}_+ \mid 0 \in \sum_{i=1}^m \tau_i \partial f_i(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x})$$

$$+ N_C(\bar{x}) + \sum_{i=1}^m \tau_i \epsilon_i \mathbb{B}, \ \sum_{i=1}^m \tau_i = 1\}.$$

Definition (solution for dual problem)

Let $\mathcal{L} := (\mathcal{L}_1, \ldots, \mathcal{L}_m)$, and let $\varepsilon := (\epsilon_1, \ldots, \epsilon_m) \in \mathbb{R}^m_+ \backslash \{0\}$.

1. We say $(\bar{y}, \bar{\tau}, \bar{\lambda}) \in F_W$ is a quasi ε-efficient solution to problem (D_W) iff there is no $(y, \tau, \lambda) \in F_W$ such that
$$\mathcal{L}_i(y, \tau, \lambda) \geq \mathcal{L}_i(\bar{y}, \bar{\tau}, \bar{\lambda}) + \epsilon_i \|\bar{y} - y\|, i = 1, \ldots, m,$$
with at least one strict inequality.
2. We say $(\bar{y}, \bar{\tau}, \bar{\lambda}) \in F_W$ is a weakly quasi ε-efficient solution to problem (D_W) iff there is no $(y, \tau, \lambda) \in F_W$ such that
$$\mathcal{L}_i(y, \tau, \lambda) > \mathcal{L}_i(\bar{y}, \bar{\tau}, \bar{\lambda}) + \epsilon_i \|\bar{y} - y\|, i = 1, \ldots, m.$$

In what follows, we use the following notation for convenience.
$u \prec v \Leftrightarrow u - v \in -\operatorname{int}\mathbb{R}^m_+$, $u \not\prec v$ is the negation of $u \prec v$;
$u \preceq v \Leftrightarrow u - v \in -\mathbb{R}^m_+ \setminus \{0\}$, $u \not\preceq v$ is the negation of $u \preceq v$.

Theorem (ε-Weak Duality)

Let $x \in F$ and let $(y, \tau, \lambda) \in F_W$

(i) If (f, g_T) is generalized convex on C at y, then

$$f(x) \not\prec \mathcal{L}(y, \tau, \lambda) - \varepsilon\|x - y\|.$$

(ii) If (f, g_T) is strictly generalized convex on C at y, the

$$f(x) \not\preceq \mathcal{L}(y, \tau, \lambda) - \varepsilon\|x - y\|.$$

Theorem (ε-Strong Duality)

Let $\bar{x}$ be a weakly quasi ε-efficient solution to the primal problem (SIMOP) such that the (LCQ) is satisfied at this point. Then there exists $(\bar{\tau}, \bar{\lambda}) \in \mathbb{R}^m_+ \times \mathbb{R}^{|T|}_+$ such that $(\bar{x}, \bar{\tau}, \bar{\lambda}) \in F_W$ and $f(\bar{x}) = \mathcal{L}(\bar{x}, \bar{\tau}, \bar{\lambda})$. If in addition,

(i) (f, g_T) is **generalized convex** on C at any $y \in C$, then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a **weakly quasi ε-efficient solution** to problem (D_W);

(ii) (f, g_T) is strictly generalized convex on C at any $y \in C$, then $(\bar{x}, \bar{\tau}, \bar{\lambda})$ is a quasi ε-efficient solution to problem (D_W).

4. Conclusions

Conclusions:

- Working under the framework of vector optimization;
- Invoking locally Lipschitz data to this framework:
  - Basic Tools: variational analysis and generalized differentiation.
  - Main Tool: extreme principle — variational counterpart of the separation theorem in nonconvex settings.
  - Focus on: optimality conditions (fuzzy and exact forms) / duality for approximate solutions.

Main Reference:

⋆ Liguo Jiao and **Do Sang Kim***, Optimality conditions for approximate solutions of nonsmooth semi-infinite vector optimization problems. In Anurag Jayswal and Tadeusz Antczak (editors), Continuous Optimization and Variational Inequalities, pages 39–53. CRC Press Taylor & Francis Group, 2022. [**invited paper**]

⋆ Liguo Jiao and **Do Sang Kim***, Weakly quasi ϵ-efficiency for semi-infinite multi-objective optimization problems with locally Lipschitzian data. *Applied Analysis and Optimization*, **4** (2020), no. 1, 65–78. [**invited paper**]

Thank you for your attention!