Existence Theorems for Optimal Solutions in Semi-algebraic Optimization

Gue Myung Lee

Department of Applied Mathematics,

Pukyong National University, Busan 48513, Korea:

email: gmlee@pknu.ac.kr

Jointly Worked with Dr. Jae Hyoung Lee and Prof. Tien Son Pham

.

Conference Dynamical systems and Semi-algebraic geometry: interaction with Optimization and Deep Learning

University of Dalat, Vietnam, July 17-21, 2023

1. Introduction

The existence of optimal solutions for optimization problems has been an essential research topic in optimization theory.

It is well-known that a linear function attains its infimum on a nonempty polyhedral set if it is bounded from below on the set.

In 1956, Frank and Wolfe proved that a quadratic function attains its infimum on a nonempty polyhedral set if it is bounded from below on the set.

In 1982, Andronov, Belousov and Shironin showed that this result is still true if the quadratic objective function is replaced by a cubic function.

In 2002, Belousov and Klatte established the existence of optimal solutions for convex polynomial optimization problems.

Very recently, for a polynomial optimization problem, Pham provided necessary and sufficient conditions for the existence of optimal solutions of the problem as well as the boundedness from below and coercivity of the objective function on the constraint set, where the results are presented in terms of the tangency variety of the polynomials defining the problem.

Since polynomials form a subclass of semi-algebraic functions, it is natural to extend these results for semi-algebraic optimization problems.

In this talk, we consider the problem of minimizing a lower semi-continuous semi-algebraic function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ on an unbounded closed semi-algebraic set $S \subset \mathbb{R}^n$.

Employing adequate tools of semi-algebraic geometry, we first establish some properties of the tangency variety of the restriction of f on S.

Then we derive verifiable necessary and sufficient conditions for the existence of optimal solutions of the problem as well as the boundedness from below and coercivity of the restriction of f on S.

We also present a computable formula for the optimal value of the problem.

2. Preliminaries

Throughout this talk, we shall consider the Euclidean vector space $\mathbb{R}^n$ endowed with its canonical scalar product $\langle \cdot, \cdot \rangle$ and we shall denote its associated norm $\| \cdot \|$.

The closed ball and the sphere centered at the origin $0 \in \mathbb{R}^n$ of radius $R > 0$ will be denoted by $\mathbb{B}_R$ and $\mathbb{S}_R$, respectively.

For a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we denote its *effective domain* and *epigraph* by, respectively,

$$
\begin{aligned}
\mathrm{dom} f &:= \{x \in \mathbb{R}^n \mid f(x) < +\infty\}, \\
\mathrm{epi} f &:= \{(x, \alpha) \in \mathbb{R}^n \times \mathbb{R} \mid f(x) \leq \alpha\}.
\end{aligned}
$$

The function f is said to be *lower semi-continuous* if for each $x \in \mathbb{R}^n$ the inequality $\liminf_{x' \to x} f(x') \geq f(x)$ holds. The *indicator function* of a set $S \subset \mathbb{R}^n$, denoted δ_S, is defined by

$$\delta_S(x) := \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{otherwise.} \end{cases}$$

2.1 Normals and subdifferentials Here we recall the notions of the normal cones to sets and the subdifferentials of real-valued functions used in this paper. ([Mordukhovich, Rockafellar]).

Definition 2.1. Consider a set $S \subset \mathbb{R}^n$ and a point $x \in S$.

(i) The *regular normal cone* (known also as the *prenormal* or *Fréchet normal cone*) $\widehat{N}(x;S)$ to S at x consists of all vectors $v \in \mathbb{R}^n$ satisfying

$$\langle v, x' - x \rangle \;\leq\; o(\|x' - x\|) \qquad \text{as } x' \to x \text{ with } x' \in S.$$

(ii) The *limiting normal cone* (known also as the *basic* or *Mordukhovich normal cone*) $N(x;S)$ to S at x consists of all vectors $v \in \mathbb{R}^n$ such that there are sequences $x_k \to x$ with $x_k \in S$ and $v_k \to v$ with $v_k \in \widehat{N}(x_k;S)$.

In particular, for all $t > 0$ and all $x \in \mathbb{S}_t$, we have $N(x; \mathbb{S}_t) = \{\mu x \mid \mu \in \mathbb{R}\}$.

Functional counterparts of normal cones are subdifferentials.

Definition 2.2. Consider a function $f\colon \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ and a point $x \in \mathrm{dom} f$. The *limiting* and *horizon subdifferentials* of f at x are defined respectively by

$$\begin{aligned} \partial f(x) &:= \Big\{ v \in \mathbb{R}^n \,\Big|\, (v,-1) \in N\Big((x,f(x)); \mathrm{epi} f\Big)\Big\}, \\ \partial^\infty f(x) &:= \Big\{ v \in \mathbb{R}^n \,\Big|\, (v,0) \in N\Big((x,f(x)); \mathrm{epi} f\Big)\Big\}. \end{aligned}$$

Lemma 2.3. For any set $S \subset \mathbb{R}^n$ and point $x \in S$, we have the representations

$$\partial \delta_S(x) = \partial^\infty \delta_S(x) = N(x;S).$$

Theorem 2.4 [Fermat rule] [Mordukhovich]. Consider a lower semi-continuous function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a closed subset S of $\mathbb{R}^n$. If $x \in \mathrm{dom} f \cap S$ is a local minimizer of f on S and the qualification condition

$$\partial^\infty f(x) \cap \Big(- N(x;S)\Big) \;=\; \{0\}$$

is valid, then $0 \in \partial f(x) + N(x;S)$.

2.2. Semi-algebraic geometry

Now, we recall some notions and results of semi-algebraic geometry, which can be found in [Bochnak et al and Ha et al.]

Definition 2.5. A subset S of $\mathbb{R}^n$ is called *semi-algebraic* if it is a finite union of sets of the form

$$\{x \in \mathbb{R}^n \mid f_i(x) = 0,\ i = 1, \ldots, p; f_i(x) > 0,\ i = p+1, \ldots, q\},$$

where all f_i are polynomials. In other words, S is a union of finitely many sets, each defined by finitely many polynomial equalities and inequalities.

A map $f\colon S \to \mathbb{R}^m$ is said to be *semi-algebraic* if its graph

$$\{(x, y) \in S \times \mathbb{R}^m \mid y = f(x)\}$$

is a semi-algebraic set.

A major fact concerning the class of semi-algebraic sets is its stability under linear projections.

Theorem 2.6 [Tarski–Seidenberg theorem]. The image of any semi-algebraic set $S \subset \mathbb{R}^n$ under a projection to any linear subspace of $\mathbb{R}^n$ is a semi-algebraic set.

Remark 2.7. As an immediate consequence of the Tarski–Seidenberg Theorem, we get semi-algebraicity of any set $\{x \in A : \exists y \in B, (x, y) \in C\}$, provided that A, B, and C are semi-algebraic sets in the corresponding spaces. Also, $\{x \in A : \forall y \in B, (x, y) \in C\}$ is a semi-algebraic set as its complement is the union of the complement of A and the set $\{x \in A : \exists y \in B, (x, y) \notin C\}$. Thus, if we have a finite collection of semi-algebraic sets, then any set obtained from them with the help of a finite chain of quantifiers is also semi-algebraic.

Definition 2.8. Let S, T and T' be semi-algebraic sets, $T' \subset T$, and let $f\colon S \to T$ be a continuous semi-algebraic map. A *semi-algebraic trivialization of* f *over* T', with fibre F, is a semi-algebraic homeomorphism $h\colon F \times T' \to f^{-1}(T')$, such that $f \circ h$ is the projection map $F \times T' \to T', (x,t) \mapsto t$.

We say that the semi-algebraic trivialization h is *compatible* with a subset S' of S if there is a subset F' of F such that $h(F' \times T') = S' \cap f^{-1}(T')$.

Theorem 2.9 [Hardt's semi-algebraic triviality]. Let S, T be two semi-algebraic sets, $f\colon S \to T$ a continuous semi-algebraic map, $\{S_i\}_{i=1,\dots,p}$ a finite family of semi-algebraic subsets of S.

Then there exists a finite partition of T into semi-algebraic sets $T = \cup_{j=1}^{q} T_j$ and, for each j with $f^{-1}(T_j) \neq \emptyset$, a semi-algebraic trivialization $h_j\colon F_j \times T_j \to f^{-1}(T_j)$ of f over T_j, compatible with S_i, for $i = 1, \dots, p$.

The following well-known lemmas will be of great importance for us.

Lemma 2.10 [monotonicity lemma]. Let $f\colon (a,b) \to \mathbb{R}$ be a semi-algebraic function.

Then there are finitely many points $a =: t_0 < t_1 < \cdots < t_p := b$ such that for each $i = 0, \ldots, p-1$, the restriction of f to the interval (t_i, t_{i+1}) is analytic, and either constant, or strictly increasing or strictly decreasing.

Lemma 2.11 [growth dichotomy lemma]. Let $f\colon (0,\epsilon) \to \mathbb{R}$ be a semi-algebraic function with $f(t) \neq 0$ for all $t \in (0,\epsilon)$.

Then there exist constants $a \neq 0$ and $\alpha \in \mathbb{Q}$ such that $f(t) = at^{\alpha} + o(t^{\alpha})$ as $t \to 0^{+}$.

Lemma 2.12 [curve selection lemma at infinity]. Let $S \subset \mathbb{R}^n$ be a semi-algebraic set, and let $f := (f_1, \ldots, f_m)\colon \mathbb{R}^n \to \mathbb{R}^m$ be a semi-algebraic map. Assume that there exists a sequence $\{x_k\}_{k \geq 1} \subset S$ such that $\lim_{k \to +\infty} \|x_k\| = \infty$ and $\lim_{k \to +\infty} f(x_k) = y \in (\overline{\mathbb{R}})^m$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

Then there exists an analytic semi-algebraic curve $\phi\colon (R, +\infty) \to \mathbb{R}^n$ such that $\phi(t) \in S$, for all $t > R$, $\lim_{t \to +\infty} \|\phi(t)\| = \infty$ and $\lim_{t \to +\infty} f(\phi(t)) = y$.

Lemma 2.13 [path connectedness]. (1) Every semi-algebraic set has a finite number of connected components and each such component is semi-algebraic.

(2) Every connected semi-algebraic set S is semi-algebraically path connected: for every two points x, y in S, there exists a continuous semi-algebraic curve ϕ: $[0, 1] \to \mathbb{R}^n$ lying in S such that $\phi(0) = x$ and $\phi(1) = y$.

Lemma 2.14 [piecewise continuity of semi-algebraic functions]. Given a semi-algebraic function $f: S \to \mathbb{R}$, where S is a semi-algebraic subset of $\mathbb{R}^n$,

there is a finite partition of S into path connected semi-algebraic sets $C_1, \ldots, C_p$, such that for each $i = 1, \ldots, p$, the restriction of f on C_i is continuous.

As a consequence of the curve selection lemma at infinity, we have the following fact.

Corollary 2.15 [Pham]. Let $S \subset \mathbb{R}^n$ be a semi-algebraic set.

Then S is unbounded if and only if there exists a real number $R > 0$ such that the set $S \cap \mathbb{S}_t$ is nonempty for all $t > R$.

We close this section with the following fact ([Ioffe, Pham]).

Lemma 2.16 [Chain Rule] [Ioffe]. Consider a lower semi-continuous and semi-algebraic function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and a semi-algebraic curve $\phi\colon [a,b] \to \mathrm{dom} f$.

Then for all but finitely many $t \in [a,b]$, the maps ϕ and $f \circ \phi$ are analytic at t and satisfy

$$\begin{aligned} v \in \partial f(\phi(t)) \;&\Longrightarrow\; \langle v, \dot{\phi}(t)\rangle \;=\; \frac{d}{dt}(f \circ \phi)(t), \\ v \in \partial^{\infty} f(\phi(t)) \;&\Longrightarrow\; \langle v, \dot{\phi}(t)\rangle \;=\; 0, \end{aligned}$$

where $\dot{\phi}(t) := \frac{d}{dt}\phi(t)$.

3. Tangencies

In order to formulate and prove the main results of the paper, we need some notation and auxiliary results.

Throughout the talk, let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous and semi-algebraic function and let S be a closed semi-algebraic subset of $\mathbb{R}^n$ such that the set $\mathrm{dom} f \cap S$ is nonempty and unbounded.

Definition 3.1. By the *set of critical points of f on S* we mean the set

$$\Sigma\,(f, S) := \{x \in \mathrm{dom} f \cap S \mid 0 \in \partial f(x) + N(x; S)\}\,.$$

By the Tarski–Seidenberg theorem (Theorem 2.6), $\Sigma(f, S)$ is a semi-algebraic set. Moreover, we have, by Lemmas 2.13 and 2.14, the following lemma.

Lemma 3.2. $f\left(\Sigma\left(f, S\right)\right)$ is a finite subset of $\mathbb{R}$.

By Lemmas 2.10 and 2.16, we have the following lemma,

Lemma 3.3. There exists a real number $R > 0$ such that for all $t > R$ and all $x \in S \cap \mathbb{S}_t$ we have

$$N(x; S) \cap \Big(- N(x; \mathbb{S}_t)\Big) = \{0\}.$$

By Lemma 3.3, we have the following lemma,

Lemma 3.4. There exists a real number $R > 0$ such that for all $t > R$ and all $x \in S \cap \mathbb{S}_t$ we have the inclusion

$$N(x; S \cap \mathbb{S}_t) \subset N(x; S) + N(x; \mathbb{S}_t).$$

Definition 3.5. We say that the *qualification condition* ((QC) for short) holds if

$$\partial^\infty f(x) \cap \Big(- N(x; S)\Big) = \{0\} \quad \text{for all} \quad x \in \text{dom} f \cap S.$$

We say that the *qualification condition at infinity* ($(\text{QC})_\infty$ for short) holds, if there exists $R > 0$ such that

$$\partial^\infty f(x) \cap \Big(-N(x; S)\Big) = \{0\} \quad \text{for all} \quad x \in (\text{dom} f \cap S) \setminus \mathbb{B}_R.$$

Note that if f is locally Lipschitz, then $\partial^\infty f(x) = \{0\}$ for all x, and so the conditions (QC) and $(\text{QC})_\infty$ hold.

Lemma 3.6. If $(\text{QC})_\infty$ holds, then there exists $R > 0$ such that for all $t > R$ and all $x \in \text{dom} f \cap S \cap \mathbb{S}_t$,

$$\partial^\infty f(x) \cap \Big(- N(x; S \cap \mathbb{S}_t)\Big) = \{0\}.$$

Definition 3.7. By the *tangency variety of f on S*, we mean the set $\Gamma(f, S) := \{x \in \mathrm{dom} f \cap S \mid$ there exists $\mu \in \mathbb{R}$ such that $\quad 0 \in \partial f(x) + N(x; S) + \mu x\}$.

Observe that $\Gamma(f, S)$ is a semi-algebraic set containing $\Sigma(f, S)$. Moreover, we have

Lemma 3.8. Assume that $(\mathrm{QC})_\infty$ holds. Then the tangency variety $\Gamma(f, S)$ is nonempty and unbounded.

By Lemma 2.14, there is a finite partition of $\Gamma(f,S)$ into semi-algebraic sets $C_i, i = 1,\ldots,\ell$ such that the restriction of f on C_i is continuous.

Applying Hardt's triviality theorem (Theorem 2.9) for the continuous semi-algebraic function

$$\rho\colon \Gamma(f,S) \to \mathbb{R}, \quad x \mapsto \|x\|,$$

we find a real number $R > 0$, semi-algebraic sets $F_i, i = 1,\ldots,\ell$ and a semi-algebraic homeomorphism

$$h\colon (\cup_{i=1}^{\ell} F_i) \times (R,+\infty) \to \Gamma(f,S) \setminus \mathbb{B}_R$$

such that $h(F_i \times (R,+\infty)) = C_i \setminus \mathbb{B}_R$ for $i = 1,\ldots,\ell$ and the following diagram commutes:

$$\begin{tikzcd}
\Gamma(f,S) \setminus \mathbb{B}_R \arrow[r, "\rho"] & (R,+\infty) \\
(\cup_{i=1}^{\ell} F_i) \times (R,+\infty) \arrow[u, "h"] \arrow[ur, "\pi"] &
\end{tikzcd}$$

where π is the projection on the second component of the product, i.e., $\pi(x,t) = t$. Since F_i is semi-algebraic, the number of its connected components, say, p_i, is finite.

So, $C_i \setminus \mathbb{B}_R$ has exactly p_i connected components, which are unbounded semi-algebraic sets. Therefore,

we may decompose the set $\Gamma(f, S) \setminus \mathbb{B}_R$ as a disjoint union of finitely many semi-algebraic sets $\Gamma_k, k = 1, \ldots, p := \sum_{i=1}^{\ell} p_i$ such that the following conditions hold:

(i) Γ_k is connected and unbounded;

(ii) for each $t > R$, the set $\Gamma_k \cap \mathbb{S}_t$ is nonempty and connected; and

(iii) the restriction of f on Γ_k is continuous.

Corresponding to each Γ_k, let

$$f_k\colon (R, +\infty) \to \mathbb{R}, \quad t \mapsto f_k(t),$$

be the function defined by $f_k(t) := f(x)$, where $x \in \Gamma_k \cap \mathbb{S}_t$. The definition is well-posed as shown in the following lemma.

Lemma 3.9. Assume that $(\mathrm{QC})_\infty$ holds. For all R large enough and all $k = 1, \ldots, p$, the following statements hold:

(i) The function f_k is well-defined and semi-algebraic;

(ii) The function f_k is either constant or strictly monotone;

(iii) The function f_k is constant if and only if $\Gamma_k \subset \Sigma(f, S)$.

For any $t > R$, the set $\mathrm{dom} f \cap S \cap \mathbb{S}_t$ is nonempty and bounded. Since f is lower semi-continuous and semi-algebraic, the function

$$\psi\colon (R, +\infty) \to \mathbb{R}, \quad t \mapsto \psi(t) := \min_{x \in S \cap \mathbb{S}_t} f(x),$$

is well-defined and semi-algebraic. With this definition, we have the following three lemmas;

Lemma 3.10. Assume that $(\mathrm{QC})_\infty$ holds. Then for R large enough, the following statements hold:

(i) Any two of the functions $\psi, f_1, \ldots, f_p$ either coincide or are distinct.

(ii) $\psi(t) = \min_{k=1,\ldots,p} f_k(t)$ for all $t > R$.

(iii) There is an index $k \in \{1, \ldots, p\}$ such that $\psi(t) = f_k(t)$ for all $t > R$.

Lemma 3.11. Assume that $(\mathrm{QC})_\infty$ holds. Then

$$\lim_{t\to+\infty}\psi(t) \;=\; \min_{k=1,\ldots,p}\lambda_k.$$

Lemma 3.12. We have

$$\lim_{t\to+\infty}\psi(t) \;\geq\; \inf_{x\in S} f(x)$$

with the equality if f does not attain its infimum on S.

4. Results

Let $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous and semi-algebraic function and let S be a closed semi-algebraic subset of $\mathbb{R}^n$ such that the set $\mathrm{dom} f \cap S$ is nonempty and unbounded.

Consider the constrained optimization problem:

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in S. \tag{P}$$

Following the approach in [Pham], we provide verifiable necessary and sufficient conditions for the existence of optimal solutions of the problem (P) as well as the boundedness from below and coercivity of the restriction of f on S. We also present a computable formula for the optimal value of the problem.

Keeping the notation as in the previous section, we can write $\Gamma(f, S) \setminus \mathbb{B}_R = \cup_{k=1}^{p} \Gamma_k$, where each Γ_k is an unbounded connected semi-algebraic set. Corresponding to each Γ_k, the semi-algebraic functions

$$f_k\colon (R, +\infty) \to \mathbb{R}, \quad t \mapsto f_k(t) := f|_{\Gamma_k \cap \mathbb{S}_t},$$

are well-defined, and so are the real numbers

$$\lambda_k := \lim_{t \to +\infty} f_k(t) \in \mathbb{R} \cup \{\pm\infty\}.$$

Also, recall that the semi-algebraic function $\psi\colon (R, +\infty) \to \mathbb{R}$ is defined by

$$\psi(t) := \min_{x \in S \cap \mathbb{S}_t} f(x).$$

Here and in the following, R is chosen large enough so that the conclusions of Lemmas 3.3, 3.4, 3.6, 3.9 and 3.10 hold.

4.1. Boundedness from below

In this subsection we present necessary and sufficient conditions for the boundedness from below of the objective function f on the constraint set S.

Theorem 4.1. Assume that $(\mathrm{QC})_\infty$ holds. Then f is bounded from below on S if and only if it holds that

$$\min_{k=1,\ldots,p} \lambda_k > -\infty.$$

In what follows we let

$$K := \{k \mid f_k \text{ is not constant}\}.$$

By the growth dichotomy lemma (Lemma 2.11), we can assume that each function $f_k, k \in K$, is developed into a fractional power series of the form

$$f_k(t) \;=\; a_k t^{\alpha_k} + \text{ lower order terms in } t \quad \text{as} \quad t \to +\infty,$$

where $a_k \in \mathbb{R} \setminus \{0\}$ and $\alpha_k \in \mathbb{Q}$.

Theorem 4.2. Assume that $(\mathrm{QC})_\infty$ holds. Then f is bounded from below on S if and only if for any $k \in K$,

$$\alpha_k > 0 \quad \Longrightarrow \quad a_k > 0.$$

The following result shows that to compute the optimal value of the problem (P) it suffices to know the *finite* set $f(\Sigma(f, S))$ and the values $\lambda_k, k = 1, \ldots, p$.

Theorem 4.3. Assume that (QC) holds. Then

$$\inf_{x \in S} f(x) \;=\; \min\left\{ \min_{x \in \Sigma(f,S)} f(x), \min_{k=1,\ldots,p} \lambda_k \right\}.$$

4.3. Existence of optimal solutions

In this subsection we provide necessary and sufficient conditions for the existence of optimal solutions to the problem (P). We start with the following result.

bf Theorem 4.4. Assume that (QC) holds. Then f attains its infimum on S if and only if it holds that

$$\Sigma(f, S) \neq \emptyset \quad \text{and} \quad \min_{x \in \Sigma(f,S)} f(x) \ \leq \ \min_{k \in K} \lambda_k.$$

Theorem 4.5. Assume that (QC) holds. Then the set of all optimal solutions of the problem (P) is nonempty compact if and only if it holds that

$$\Sigma(f,S) \neq \emptyset, \qquad \min_{x \in \Sigma(f,S)} f(x) \;\le\; \min_{k \in K} \lambda_k,$$

$$\text{and} \quad \min_{x \in \Sigma(f,S)} f(x) < \min_{k \notin K} \lambda_k.$$

4.4. Coercivity

The function f is *coercive on* the set S if for every sequence $x_k \in S$ such that $\|x_k\| \to +\infty$, we have $f(x_k) \to +\infty$. It is well known that if f is coercive on S, then f achieves its infimum on S.

A necessary and sufficient condition for the coercivity of f on S is as follows.

Theorem 4.6. Assume that $(\mathrm{QC})_\infty$ holds. Then the following statements are equivalent:

(i) The function f is coercive on S.

(ii) $\lambda_k = +\infty$ for all $k = 1, \ldots, p$.

5. Examples

In this section we give examples to illustrate our main results.

Example 5.1. Let $S := \mathbb{R}^2$ and $f(x,y) := x^2 + |y|$. A direct calculation shows that $N((x,y);\mathbb{R}^2) = \{(0,0)\}$, $\partial^\infty f(x,y) = \{(0,0)\}$ (as f is locally Lipschitz) and that

$$\partial f(x,y) = \begin{cases} \{(2x,\xi) \mid \xi \in [-1,1]\} & \text{if } y = 0, \\ \{(2x,1)\} & \text{if } y > 0, \\ \{(2x,-1)\} & \text{if } y < 0. \end{cases}$$

It follows that $\Sigma(f,\mathbb{R}^2) = \{(0,0)\}$ and

$$\Gamma(f,\mathbb{R}^2) = [\mathbb{R} \times \{0\}] \cup [\{0\} \times \mathbb{R} \setminus \{0\}] \cup \{(x,y) \mid x \in \mathbb{R},\ y = \pm\frac{1}{2}\}.$$

Hence, for $R > \frac{1}{2}$, the set $\Gamma(f, \mathbb{R}^2) \setminus \mathbb{B}_R$ has eight connected components:

$$\begin{aligned}
\Gamma_{\pm 1} &:= \{(\pm t, 0) \mid t > R\}, \\
\Gamma_{\pm 2} &:= \{(0, \pm t) \mid t > R\}, \\
\Gamma_{\pm 3} &:= \left\{\left(t, \frac{1}{2}\right) \mid t > \sqrt{R - \frac{1}{4}}\right\}, \\
\Gamma_{\pm 4} &:= \left\{\left(t, -\frac{1}{2}\right) \mid t > \sqrt{R^2 - \frac{1}{4}}\right\}.
\end{aligned}$$

Consequently, the restriction of f on these components are given by

$$\begin{aligned}
f|_{\Gamma_{\pm 1}} &= t^2, \quad f|_{\Gamma_{\pm 2}} = t, \\
f|_{\Gamma_{\pm 3}} &= f|_{\Gamma_{\pm 4}} = t^2 + \frac{1}{4}.
\end{aligned}$$

Thus

$$\lambda_{\pm 1} \;=\; \lambda_{\pm 2} = \; \lambda_{\pm 3} \;=\; \lambda_{\pm 4} \;=\; +\infty.$$

The results presented in the previous section show that the set of global minimizers of f on S is nonempty compact and that

$$\inf_{(x,y)\in\mathbb{R}^2} f(x,y) \;=\; \min_{(x,y)\in\Sigma(f,S)} f(x,y) \;=\; f(0,0) \;=\; 0.$$

Furthermore, in light of Theorem 4.6, f is coercive.

Example 5.2. Let $S := \mathbb{R}^2$ and $f(x, y) := x + y$. Then, by simple calculations, we have

$$\Gamma(f, \mathbb{R}^2) = \{(x, y) \mid x = y\}.$$

For $R > 0$, let $\Gamma_1 := \{(t, t) \mid t \geq R\}$ and let $\Gamma_2 := \{(-t, -t) \mid t \geq R\}$. Then we see that the restriction of f on these components are given by

$$f|_{\Gamma_1} = \sqrt{2t}, \quad f|_{\Gamma_2} - \sqrt{2t}.$$

So, we have

$$\lambda_1 = \lim_{t \to \infty} f|_{\Gamma_1} = +\infty, \quad \lambda_2 = \lim_{t \to \infty} f|_{\Gamma_2} = -\infty.$$

and thus, by Theorem 4.1, f is not bounded from below on S.

Example 5.3. Let $S := \mathbb{R}^2$ and $f(x,y) := (xy-1)^2+|y|$. We have $N((x,y);\mathbb{R}^2) = \{(0,0)\}$, $\partial^\infty f(x,y) = \{(0,0)\}$ (as f is locally Lipschitz) and

$$\partial f(x,y) = \begin{cases} \{(0,-2x+\xi) \mid \xi \in [-1,1]\} & \text{if } y = 0, \\ \{(2(xy-1)y, 2(xy-1)x+1)\}, & \text{if } y > 0, \\ \{(2(xy-1)y, 2(xy-1)x-1)\}, & \text{if } y < 0. \end{cases}$$

It follows that $\Sigma(f,\mathbb{R}^2) = [-\frac{1}{2}, \frac{1}{2}] \times \{0\}$ and

$$\Gamma(f,\mathbb{R}^2) = \Sigma(f,\mathbb{R}^2) \cup \{(x,y) \mid g_+(x,y) = 0, y > 0\} \cup \{(x,y) \mid g_-(x,y) = 0, y < 0\},$$

where $g_\pm(x,y) := -2\,x^3y + 2\,xy^3 \mp x + 2\,x^2 - 2\,y^2$. Then we can see that* for R large enough, the set $\Gamma(f,\mathbb{R}^2)\backslash\mathbb{B}_R$

*The computations are performed with the software Maple, using the command "puiseux" of the package "algcurves" for the rational Puiseux expansions.

has eight connected components:

$$\begin{array}{lll}
\Gamma_{\sigma,1}: & x := (-t^{-1} - \frac{1}{2}\,\sigma\,t^2 + O\left(t^4\right)), & y := (-t^{-1} - \frac{1}{4}\,\sigma\,t^2 + O\left(t^4\right)), \\
\Gamma_{\sigma,2}: & x := (\frac{1}{3}t^{-1} + \frac{3}{2}\,\sigma\,t^2 + O\left(t^4\right)), & y := (-\frac{1}{3}t^{-1} + \frac{3}{4}\,\sigma\,t^2 + O\left(t^4\right)), \\
\Gamma_{\sigma,3}: & x := (-2\,t + 4\,t^3 + O\left(t^4\right)), & y := (-\frac{1}{2}t^{-1} - t + 2\,t^3 + O\left(t^4\right)), \\
\Gamma_{\sigma,4}: & x := (t^{-1} + 2\,t - \sigma\,t^2 - 4\,t^3 + O\left(t^4\right)), & y := (t - \frac{1}{2}\,\sigma\,t^2 - 2\,t^3 + 3\,\sigma\,t^4),
\end{array}$$

where $\sigma = \pm 1$ and $t \to \mp 0$ for $k = 1, 2, 3$, and $t \to \pm 0$ for $k = 4$. Then substituting these expansions in f we get

$$\begin{aligned}
f|_{\Gamma_{\sigma,1}} &= (t^{-4} - 2\,t^{-2} + \frac{1}{2}\,\sigma\,t^{-1} + 1 + O\left(t\right)), \\
f|_{\Gamma_{\sigma,2}} &= (\frac{1}{81}t^{-4} + \frac{2}{9}t^{-2} - \frac{5}{18}\,\sigma\,t^{-1} + 1 + O\left(t\right)), \\
f|_{\Gamma_{\sigma,3}} &= (-\frac{1}{2}\,\sigma\,t^{-1} - \sigma\,t + 2\,\sigma\,t^3 + O\left(t^4\right)), \\
f|_{\Gamma_{\sigma,4}} &= (\sigma\,t - \frac{1}{4}\,\sigma^2 t^2 - 2\,\sigma\,t^3 + 2\,\sigma^2 t^4 + O\left(t^5\right)).
\end{aligned}$$

It follows that

$$\lambda_{\sigma,1} = \lambda_{\sigma,2} = \lambda_{\sigma,3} = +\infty, \quad \lambda_{\sigma,4} = 0.$$

In light of Theorem 4.1, f is bounded from below. Note that

$$f|_{\Sigma(f,\mathbb{R}^2)} \equiv 1 > 0 = \min_{k=1,\ldots,4} \lambda_{\sigma,k}.$$

Hence, by Theorem 4.4, f does not attain its infimum. In view of Theorem 4.3, we have

$$\inf_{(x,y)\in\mathbb{R}^2} f(x,y) = 0.$$

Furthermore, by Theorem 4.4, f is not coercive.

Example 5.4. Let $f(x,y) := \min\{x+y, 1\}$ and let $S := \mathbb{R}^2_+$. Then the function f is semi-algebraic and (QC) holds. Note that the function f is continuous and concave. Then it follows from [Ioffe, Proposition 7] that we have

$$\partial f(x,y) = \begin{cases} \{(0,0)\} & \text{if } x+y>1, \\ \{(0,0),(1,1)\}, & \text{if } x+y=1, \\ \{(1,1)\}, & \text{if } x+y<1. \end{cases}$$

Moreover, by a simple calculation, we see that

$$N((x,y);S) = \begin{cases} -\mathbb{R}^2_+ & \text{if } (x,y)=(0,0), \\ \{0\}\times(-\mathbb{R}_+), & \text{if } x>0,\ y=0, \\ -\mathbb{R}_+\times\{0\}, & \text{if } x=0,\ y>0, \\ \{(0,0)\}, & \text{if } x>0,\ y>0, \end{cases}$$

and so,

$$\begin{aligned} \Gamma(f,S) \;\; = \;\; & \{(x,y)\in S \mid x=y,\ x+y<1\} \cup \{(x,y)\in S \mid \ x+y\geq 1\} \\ & \cup\{(x,y)\in S \mid \ x=0,\ y>0\} \cup \{(x,y)\in S \mid \ x>0,\ y=0\} \cup \{(0,0)\}. \end{aligned}$$

Note that

$$\Sigma(f, S) = \{(x, y) \in S \mid \ x + y \geq 1\} \cup \{(0, 0)\}$$

Now, for $R > 1$, let $\Gamma_1 := \Gamma(f, S) \setminus \mathbb{B}_R$. Then we have $f|_{\Gamma_1} \equiv 1$, and so $\lambda_1 = \lim_{t\to\infty} f|_{\Gamma_1} = 1$. Thus,

$$\min_{(x,y)\in\Sigma(f,S)} f(x, y) = 0 < 1 = \lambda_1.$$

By Theorem 4.5, the set of all optimal solutions of f on S is nonempty compact, that is $\{(0, 0)\}$. Moreover, it follows from Theorem 4.6 that the function f is not coercive on S.

Example 5.5. Let $S := \mathbb{R}^2_+$. Consider the following function from $\mathbb{R}^2$ to $\mathbb{R}$:

$$f(x,y) = \begin{cases} 0, & \text{if } (x,y) \in A, \\ 1, & \text{if } (x,y) \notin A, \end{cases}$$

where $A := \{(x,y) \mid x \in \mathbb{R},\ \ y \geq 0\}$. Note that the function f is lower semi-continuous and semi-algebraic, but not local Lipschitz. Note also that Normal cone to S at x is same in Example 5.4. So, by a direct calculation, we see that

$$\partial f(x,y) = \begin{cases} \{0\} \times (-\mathbb{R}_+) & \text{if } x \geq 0,\ y = 0, \\ \{(0,0)\}, & \text{if } x \geq 0,\ y > 0. \end{cases}$$

and $\Gamma(f,S) = \mathbb{R}^2_+$. Let $\Gamma_1 := \Gamma(f,S) \setminus \mathbb{B}_R$ for $R > 0$. Then we have $f|_{\Gamma_1} \equiv 0$, and so, $\lambda_1 = \lim_{t\to\infty} f|_{\Gamma_1} = 0$.

Note that $\Sigma(f, S) = \mathbb{R}^2_+$. Then we see that

$$\min_{(x,y)\in\Sigma(f,S)} f(x, y) = 0 = \lambda_1.$$

So, it follows from Theorem 4.4, f attains its infimum on S, that is, 0. Moreover, by Theorem 4.5, we see that the set of all optimal solutions of f on S is nonempty, but not compact.

References

1. V. G. Andronov, E. G. Belousov, and V. M. Shironin. On solvability of the problem of polynomial programming. *Izvestija Akadem. Nauk SSSR, Tekhnicheskaja Kibernetika (in Russian)*, (4):194–197, 1982.

2. E. G. Belousov and D. Klatte. A Frank–Wolfe type theorem for convex polynomial programs. *Comput. Optim. Appl.*, 22(1):37–48, 2002.

3. J. Bochnak, M. Coste, and M.-F. Roy. *Real Algebraic Geometry*, volume 36. Springer, Berlin, 1998.

4. J. M. Borwein. On the existence of pareto efficient points. *Math. Oper. Res*, 8:64–73, 1983.

5. S. T. Dinh, H. V. Hà, and T. S. Phạm. A Frank–Wolfe type theorem for nondegenerate polynomial programs. *Math. Program. Ser. A.*, 147(1–2):519–538, 2014.

6. M. Frank and P. Wolfe. An algorithm for quadratic programming. *Naval Res. Logist. Quart.*, 3:95–110, 1956.

7. H. V. Hà and T. S. Phạm. *Genericity in Polynomial Optimization*, volume 3 of *Ser. Optim. Appl.* World Scientific, Singapore, 2017.

8. A. D. Ioffe. Approximate subdifferentials and applications. i: The finite dimensional theory. *Trans. Amer. Math. Soc.*, 281:389–416, 1984.

9. D. Klatte. On a frank-wolfe type theorem in cubic optimization. *Optimization*, 68(2–3):539–547, 2019.

10. B. S. Mordukhovich. *Variational Analysis and Generalized Differentiation, I: Basic Theory; II: Applications*. Springer, Berlin, 2006.

11. B. S. Mordukhovich. *Variational Analysis and Applications*. Springer, New York, 2018.

12. T. S. Phạm. Optimality conditions for minimizers at infinity in polynomial programming. *Math. Oper. Res.*, 44(4):1381–1395, 2019

13. T. S. Phạm. Tangencies and polynomial optimization. *Math. Program. Ser. A.*, 199(1–2):1239–1272, 2023.

14. T. S. Phạm. Local minimizers of semi-algebraic functions from the viewpoint of tangencies. *SIAM J. Optim.*, 30(3):1777–1794, 2020.

15. R. T. Rockafellar and R. Wets. *Variational Analysis*, volume 317 of *Grundlehren Math. Wiss.* Springer, Berlin, 1998.

16. L. van den Dries and C. Miller. Geometric categories and o-minimal structures. *Duke Math. J.*, 84:497–540, 1996.