Lectures On Semialgebraic Geometry

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Real Algebraic Geometry
Lecture 1: Semi Algebraic sets

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0. Introduction

Roughly speaking, Algebraic Geometry on a field $\mathbb{K}$ studies algebraic sets in $\mathbb{K}^n$ i.e. the sets of the form

$$\{x \in \mathbb{K}^n : P_1(x) = \cdots = P_k(x) = 0\},$$

where P_i are polynomials with coefficients in $\mathbb{K}$.

When $\mathbb{K} = \mathbb{R}$.

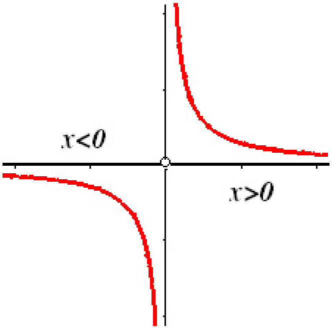
• One of the difficulties when studying real algebraic sets is that the field of real numbers $\mathbb{R}$ is not algebraically closed. Therefore the number of real zeros (counted with multiplicity) of a real polynomial can be not equal to its degree, e.g. the number of real zeros of $P(x) = x^2 + a$ depends on $a < 0, a = 0$, or $a > 0$ ($\mathbb{R}$ is an ordered field!).

• Besides, though the class of real algebraic sets is closed under taking finite unions and intersections, but not closed under taking complement.

• Moreover, in general, images of algebraic sets by polynomial functions and their connected components are not algebraic sets.

For example, the equation $xy - 1 = 0$ defines a hyperbola in $\mathbb{R}^2$ consisting of the connected components:

$$\{(x, y) \in \mathbb{R}^2 : xy - 1 = 0, x > 0\}, \ \{(x, y) \in \mathbb{R}^2 : xy - 1 = 0, x < 0\}$$



Its image under the projection on Ox coordinate is two intervals:

$$\{x \in \mathbb{R} : x < 0\}, \ \{x \in \mathbb{R} : x > 0\}.$$

These sets are given by equations and inequalities, but they can not be given by equations only.

This lecture deals with the class of semi-algebraic sets which are those defined by equalities and inequalities of real polynomials.

• This class is closed under Boolean operators: unions, intersections and taking complements.

• This class has a very interesting property: it is closed under projection (Tarski-Seidenberg's Theorem).

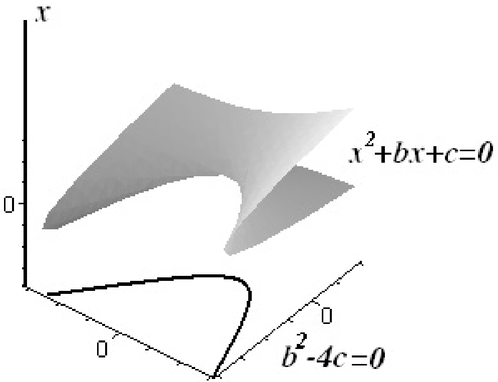
• A semi-algebraic set has only finitely many connected components, and the components are semi-algebraic (Łojasiewicz's Theorem).

These fundamental properties create great conveniences in studying semi-algebraic sets.
Usually, Real Algebraic Geometry is identified with Semialgebraic Geometry.

Real Algebraic Geometry = Semialgebraic Geometry

Example. Let $f(b, c, x) = x^2 + bx + c$ and $\Delta(b, c) = b^2 - 4c$.

	Algebraic	Geometry	Logic
$f = 0$	Equation	Algebraic set	Formula
In $\mathbb{R}$	Condition of $\exists$ sol. x	Projection	$\exists x, f = 0$
$0, 1, 2$ sol.	$\Delta < 0, = 0, > 0$	Semialgebraic set	Formula free of $\exists, \forall$



Definition. The class of semi-algebraic sets in $\mathbb{R}^n$ is the smallest class of subsets of $\mathbb{R}^n$ satisfying the following properties:
(SA1) It contains all sets of the form

$$\{x \in \mathbb{R}^n : P(x) > 0\} \ , \ P \in \mathbb{R}[X_1, \cdots, X_n].$$

(SA2) It is stable under taking finite unions, finite intersections and complements.
A mapping $f : X \to \mathbb{R}^m$ is called a semi-algebraic mapping if its graph is a semi-algebraic set.

It is equivalent to use the following definition for each semialgebraic set:

Proposition.

A subset of $\mathbb{R}^n$ is semi-algebraic if and only if it can be represented as the form:

$$\bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{(x_1, \cdots, x_n) \in \mathbb{R}^n : P_{ij}(x_1, \cdots, x_n) \ s_{ij} \ 0\},$$

where $p, q \in \mathbb{N}$, $P_{ij} \in \mathbb{R}[X_1, \cdots, X_n]$ và $s_{ij} \in \{=, >\}$, $0 \leq i \leq p, 0 \leq j \leq q$.

Proof. The class of sets of the above form satisfies (SA1) and (SA2), and it is contained in the class of semi-algebraic sets. By the condition of the smallest class the two classes are coincide. □

Note. On the real fields

$$P_1 = \cdots = P_k = 0 \ \Leftrightarrow \ P_1^2 + \cdots + P_k^2 = 0.$$

Therefore, all semialgebraic sets in $\mathbb{R}^n$ if and only if it can be represented as a finite union of sets of the form

$$\{x \in \mathbb{R}^n : P(x) = 0, Q_1(x) > 0, \cdots Q_k(x) > 0\},$$

where $P, Q_1, \cdots, Q_k \in \mathbb{R}[X_1, \cdots, X_n]$.

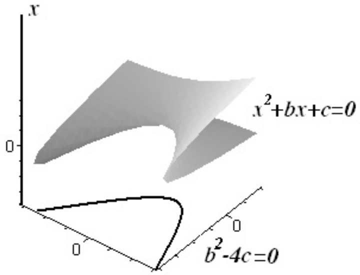
Example.

1) The class of real algebraic sets $\subsetneq$ the class of semi-algebraic sets. Moreover, every algebraic subset in $\mathbb{R}^n$ is of the form

$$\{x \in \mathbb{R}^n : P(x) = 0\}, \text{where } P \text{ is a polynomial.}$$

2) Polynomial functions are semi-algebraic.
3) A semi-algebraic set in $\mathbb{R}$ is a finite union of points and open intervals.

4) Let $f(b, c, x) = x^2 + bx + c$.
The set of the values of (b, c) in $\mathbb{R}^2$ such that f has a real solution is the projection of the set $\{(x, b, c) : f(b, c, x) = 0\}$ onto the plane (b, c).
It is the semi-algebraic set $\{(b, c) : b^2 - 4c \geq 0\}$.



5) The function

$$\xi : \{(b, c) : b^2 - 4c > 0\} \to \mathbb{R},\ \xi(b, c) = \frac{1}{2}(b + \sqrt{b^2 - 4c})$$

is semi-algebraic because its graph is given by:

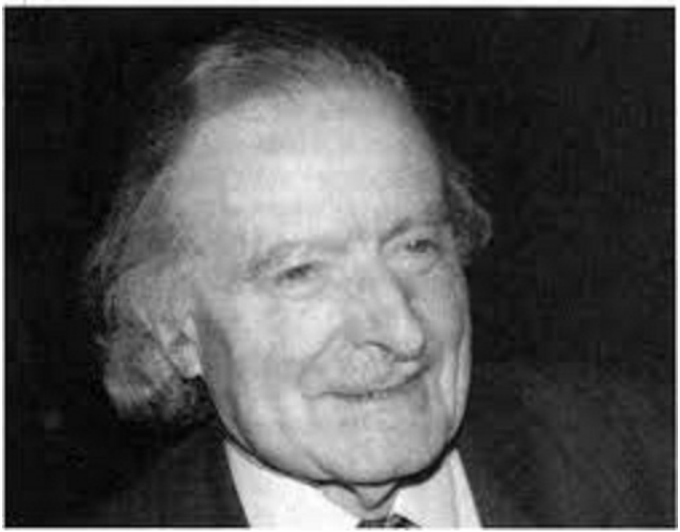
$$\{(b, c, x) :\ b^2 - 4c > 0, x^2 + bx + c = 0, x > \frac{b}{2}\}.$$

6) The following sets are not semi-algebraic:
$\{(x, y) \in \mathbb{R}^2 : y = [x]\}$,
$\{(x, y) \in \mathbb{R}^2 : y = \sin x\}$,
$\{(x, y) \in \mathbb{R}^2 : y = e^x \}$.

Exercise. Prove that:
1) $(f_1, \cdots, f_m) : X \to \mathbb{R}^m$ is semialgebraic if and only if f_i is emialgebraic for all $i \in \{1, \cdots, m\}$.
2) If $f : X \to \mathbb{R}$ is semialgebraic and $f(x) \neq 0$, for all $x \in X$, then $1/f$ is semialgebraic.
3) If $f : X \to \mathbb{R}$ is semialgebraic and $f \geq 0$, then $\sqrt{f}$ is semialgebraic.
4) If $f : X \to \mathbb{R}$ is semialgebraic, then there is a polynomial $P(X, Y) \neq 0$, such that $P(x, f(x)) = 0$, for all $x \in X$.
5) The class of constructible sets in $\mathbb{C}^n$, by definition, is the smallest Boolean algebra of subsets of $\mathbb{C}^n$ which contains all complex algebraic sets.
Prove that $X \subset \mathbb{C}^n$ is constructible if and only if $X = \bigcup_{i=1}^{p} V_i \setminus W_i$, where V_i, W_i are algebraic sets.
Prove that if we identify $\mathbb{C} \equiv \mathbb{R}^2$, then every constructible subset of $\mathbb{C}^n$ is semi-algebraic in $\mathbb{R}^{2n}$.

Alfred Tarski (1901-1983)

Stanisław Łojasiewicz (1926-2002)



2. Tarski-Seidenberg's Theorem - Łojasiewicz's Theorem.

Most of the basic properties of semi-algebraic sets are implied from the following two theorems:

Theorem (Tarski-Seidenberg).

Let S be a semi-algebraic subset of $\mathbb{R}^n \times \mathbb{R}^k$. Let $\pi : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ be the natural projection. Then $\pi(S)$ is a semi-algebraic set.

Theorem (Łojasiewicz).

The number of connected components of a semi-algebraic set is finite, and each of the components is also semi-algebraic.

Exercise.

The set of the form $\{(x_1, \cdots, x_n) \in \mathbb{R}^n : P(x_1, \cdots, x_n) > 0\}$ is the image via the projection of the algebraic set

$$\{(x_1, \cdots, x_n, t) \in \mathbb{R}^{n+1} : t^2 P(x_1, \cdots, x_n) = 1\}.$$

From that, we have:

1) Each semialgebraic sets is the image of an algebraic set through a projection.

From the Tarski-Seidenberg theorem, we have:

2) The class of semialgebraic sets in $\mathbb{R}^n, n \in \mathbb{N}$, is the smallest class of subsets that contains all algebraic sets and that is closed under Boolean operators and projections.

Semialgebraic sets and first-order formulas

Definition. A first-order formula $\Phi(x_1, \cdots, x_n)$ of n variables with parameters in $\mathbb{R}$ (precisely, a first-order formula of the language of ordered fields with parameters in $\mathbb{R}$) is a finite combination of

- atomic formulas:

$$(P(x_1, \cdots, x_n) > 0), \text{ where } P \text{ is a real polynomial,}$$

  joined with each others by logical operators
- $\vee$ (or), $\wedge$ (and), $\neg$ (not),
- qualifications $\exists$ (exists), $\forall$ (for all) with respect to variables.

Let $\Phi(x, y), \Psi(x, y)$ be formulas with variables $x = (x_1, \cdots, x_m)$ and $y = (y_1, \cdots, y_n)$.
When (x, y) take values in $X \times Y$, the formulas defines the following sets
$\Phi = \{(x, y) \in X \times Y : \ \Phi(x, y) \}, \quad \Psi = \{(x, y) \in X \times Y : \ \Psi(x, y) \}$.
Then: $\Phi(x, y) \vee \Psi(x, y)$ defines $\Phi \cup \Psi$,
$\Phi(x, y) \wedge \Psi(x, y)$ defines $\Phi \cap \Psi$,
$\neg\Phi(x, y)$ defines $X \times Y \setminus \Phi$,
$\exists y \Phi(x, y)$ defines $\pi(\Phi)$, với $\pi(x, y) = x$,
$\forall y \Phi(x, y) \equiv \neg(\exists y \neg\Phi(x, y))$ defines $X \setminus \pi(X \times Y \setminus \Phi)$.

Therefore, $X \subset \mathbb{R}^n$ is a semialgebraic set if and only if there is a quantifier-free formula $\Psi(x_1, \cdots, x_n)$ of the form

$$\bigvee_{i=1}^{p} \bigwedge_{j=1}^{q} (P_{ij}(x_1, \cdots, x_n)\ s_{ij}\ 0),$$

where $p, q \in \mathbb{N}$, $P_{ij} \in \mathbb{R}[X_1, \cdots, X_n]$ và $s_{ij} \in \{=, >\}$, $0 \leq i \leq p, 0 \leq j \leq q$, such that

$$X = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \Psi(x_1, \cdots, x_n)\ \}.$$

The Tarski-Seidenberg theorem has the following logical formulation:

Theorem (Tarski-Seidenberg).

For every first-order formula $\Phi(x_1, \cdots, x_n)$, there exists a quantifier-free formula $\Psi(x_1, \cdots, x_n)$, such that the following formula is always true in $\mathbb{R}$:

$$\forall x_1, \cdots, x_n (\Phi(x_1, \cdots, x_n) \Leftrightarrow \Psi(x_1, \cdots, x_n)).$$

In particular, the set $\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : \Phi(x) \}$ is semi-algebraic.

For example, the formula

$$\Phi = (\exists x, x^2 + bx + c = 0) \wedge (\exists y, y^2 + by + c = 0) \wedge \neg(x = y)$$

is equivalent to the qualifier-free formula

$$\Psi = (b^2 - 4c > 0).$$

From the definitions and the Tarski-Seidenberg theorem, we get

Proposition (Elementary properties).

(i) The closure, the interior, and the boundary of a semi-algebraic set are semi-algebraic.
(ii) Images and inverse images of semi-algebraic sets under semi-algebraic maps are semi-algebraic.
(iii) Compositions of semi-algebraic maps are semi-algebraic.

Proof.
(i) If A is a semi-algebraic subset of $\mathbb{R}^n$, then its closure is

$$\overline{A} = \{x \in \mathbb{R}^n : \forall \epsilon, \epsilon > 0, \exists y (y \in A) \wedge (\sum_{i=1}^{n} (x_i - y_i)^2 < \epsilon^2)\},$$

where $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots y_n)$. By the Tarski-Seidenberg theorem, $\overline{A}$ is semi-algebraic.
So $\mathrm{int}\,(A) = \mathbb{R}^n \setminus \overline{\mathbb{R}^n \setminus A}$ and $\mathrm{bd}\,(A) = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$ are semi-algebraic.
(ii) Let $f : X \to Y$ be a semi-algebraic function and $A \subset X, B \subset Y$ be semi-algebraic subsets.
Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be the natural projections.
Then $f(A) = \pi_Y(f \cap A \times Y)$ and $f^{-1}(B) = \pi_X(f \cap X \times B)$.
So they are semi-algebraic.

(iii) Let $f : X \to Y, g : Y \to Z$ be semi-algebraic maps. Then $g \circ f = \pi(f \times Z \cap X \times g)$, where $\pi : X \times Y \times Z \to X \times Z$ defined by $\pi(x, y, z) = (x, z)$. So $g \circ f$ is semi-algebraic. □

Exercise. Use Tarski-Seidenberg's Theorem to do the following:
1) Let $n \in \mathbb{N}, k \leq n$, and $i_1, \cdots, i_k \in \{1, \cdots, n\}$. Denote $\Gamma_{i_1 \cdots i_k} =$

$$\{(a_0, \cdots, a_n) \in \mathbb{R}^n : a_0 + \cdots + a_n T^n \text{ has } k \text{ zeros with mutiplicities } i_1, \cdots, i_k\}.$$

Prove that $\Gamma_{i_1 \cdots i_k}$ is a semi-algebraic set..
2) Let $f : A \to \mathbb{R}$ be a semialgebraic function and $p \in \mathbb{N}$. Prove that the set $C^p(f) = \{x \in A : f \text{ is of class } C^p \text{ at } x\}$ is semialgebraic, and the partial derivatives $\partial f / \partial x_i$ are semialgebraic functions on $C^p(f)$.
3) Let $f, g : X \to \mathbb{R}$ be semialgebraic. Prove that $|f|, \max(f, g), \min(f, g)$ are semialgebraic.
4) Let $f, g : X \to \mathbb{R}$ be semialgebraic. Prove that the functions defined by

$$M(t) = \sup\{f(x) : g(x) = t\}, \; m(t) = \inf\{f(x) : g(x) = t\}, \; t \in g(X),$$

are semialgebraic.

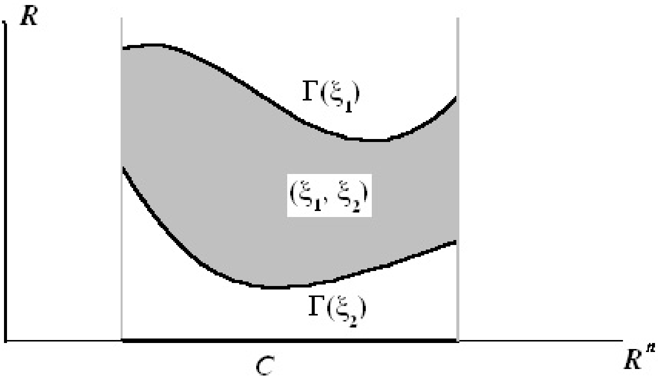
3. Cylindrical decomposition theorem.

Tarski (1931, see [T]) stated and proved T-S Theorem in logic form (the real closed field $\mathbb{R}$ admits quantifier elimination).
Later, Seidenberg (1954, see [S]) proved the theorem by using Sturm's sequences, which proved to be of great interest to other mathematicians.
In this lecture, the Tarski-Seidenberg theorem and the Łojasiewicz theorem are proved by Łojasiewicz's method (1964, see [Ł]). The proof is based on Cylindrical decomposition theorem and hence gives rather precise information on semi-algebraic sets.

Definition. Let $\xi_1, \xi_2 : C \longrightarrow \overline{\mathbb{R}}$, where $\xi_1 < \xi_2$. Write

$$\Gamma(\xi_1) = \{(x,t) : t = \xi_1(x)\} \quad \text{(the graph)},$$

$$(\xi_1, \xi_2) = \{(x,t) : x \in C,\ \xi_1(x) < t < \xi_2(x)\} \quad \text{(the band)}.$$



Theorem (Cylindrical decomposition - Łojasiewicz).

Let $f_1, \cdots, f_p \in \mathbb{R}[X][T]$, $X = (X_1, \cdots, X_n)$. Then there exist an augmentation $f_1, \cdots f_p, f_{p+1}, \cdots, f_{p+q} \in \mathbb{R}[X][T]$ and a partition of $\mathbb{R}^n$ into finitely many semi-algebraic sets $S_1, \cdots, S_k$ such that for each connected component C of each S_i there are continuous functions

$$-\infty = \xi_{C,0} < \xi_{C,1} < \cdots < \xi_{C,r(C)} < \xi_{C,r(C)+1} = +\infty$$

on C satisfying the following two properties:
(i) Each f_i $(1 \leq i \leq p+q)$ has a constant sign on each $\Gamma(\xi_{C,j})$ $(1 \leq j \leq r(C))$ and on each $(\xi_{C,j}, \xi_{C,j+1})$ $(0 \leq j \leq r(C))$.
(ii) Each of the set $\Gamma(\xi_{C,j})$, $(\xi_{C,j}, \xi_{C,j+1})$ is of the form

$$\{(x,t) \in C \times \mathbb{R} : f_i(x,t)\ s(i)\ 0,\ i = 1, \cdots, p+q\},$$

for a suitable $s : \{1, \cdots, p+q\} \longrightarrow \{<, =, >\}$.

A semialgebraic of the form

$$\bigcup_{i=1}^{p} \bigcap_{j=1}^{q} \{(x_1, \cdots, x_n) \in \mathbb{R}^n : P_{ij}(x_1, \cdots, x_n) \ s_{ij} \ 0\}, \text{ where } s_{ij} \in \{=, >\},$$

is said to be described by P_{ij}.

The Tarski-Seidenberg theorem and the Łojasiewicz theorem come from Cylindrical decomposition theorem by induction n as follows.

Proposition.

(T-S)$_n$ If $S \subset \mathbb{R}^n \times \mathbb{R}$ is semialgebraic, then $\pi(S)$ is semialgebraic.
(Ł)$_n$ If $S \subset \mathbb{R}^n \times \mathbb{R}$ is semialgebraic, then the number of the connected components of S is finite, and each of the components is also semi-algebraic

Proof. By induction on n.
It is trivial when $n = 0$.
Suppose (T-S)$_{n-1}$ and (Ł)$_{n-1}$. Let $S \subset \mathbb{R}^n \times \mathbb{R}$ be a semi-algebraic described by $f_1, \cdots, f_p \in \mathbb{R}[X_1, \cdots, X_n][X_{n+1}]$.
By Cylindrical decomposition theorem, there exist an augmentation of this family and a partition $\mathbb{R}^n = \bigcup_i S_i = \bigcup_i \bigcup_j C_{ij}$, where S_i is semi-algebraic and C_{ij} is a connected component of S_i.
By (Ł)$_{n-1}$, the number of the $C'_{ij}s$ is finite and C_{ij} is semi-algebraic.
Therefore, $\mathbb{R}^n \times \mathbb{R}$ is partitioned into graphs and bands of continuous functions on the $C'_{ij}s$, which are connected semi-algebraic sets.
Since S is a union of these sets, S satisfies (Ł)$_n$ and
$\pi(S) = \cup\{C_{ij} : C_{ij} \times \mathbb{R} \cap S \neq \emptyset\}$ is semi-algebraic, i.e. (T-S)$_n$. □

The proof of Cylindrical decomposition theorem.

Lemma 1 (Thom's Lemma).

Let $f_1, \cdots, f_k \in \mathbb{R}[T]$ be a finite family of polynomials which is stable under differentiation, i.e. if $f_i' \neq 0$ then $f_i' \in \{f_1, \cdots, f_k\}$.
For $s : \{1, \cdots, k\} \to \{<, =, >\}$, put

$$A_s = \{t \in \mathbb{R} : f_i(t)\ s(i)\ 0,\ i = 1, \cdots, k\}.$$

Then A_s is connected, i.e. empty, a point, or an interval.

Proof. By induction on k. It is trivial for $k = 0$.
Suppose the lemma is true for $k - 1$ $(k > 0)$. Order $f_1, \cdots, f_k$ such that $\deg(f_k) = \max\{\deg(f_i) : i = 1, \cdots, k\}$.
Let $A' = \{t : f_i(t)\ s(i)\ 0, i = 1, \cdots, k - 1\}$. By the inductive hypothesis A' is empty, a point, or an interval.
If A' is empty or a point, so is $A_s = A' \cap \{t : f_k(t)\ s(k)\ 0\}$.
If A' is an interval, then f_k' has a constant sign on A' and hence f_k is either strictly monotone or constant on A'. In each case one can easily check that A_s is connected. □

Exercise. Find $f \in \mathbb{R}[T]$, such that $\{t \in \mathbb{R} : f(t) > 0\}$ is not connected.

Example. Consider the general polynomial og degree 2
$G(a_0, a_1, a_2, T) = a_0 + a_1 T + a_2 T^2$.
Then the necessary and sufficient condition for:
G has 0 complex solution is $a_2 = a_1 = 0, a_0 \neq 0$,
G has 1 complex solutions is $(a_2 \neq 0, a_1^2 - 4a_0a_2 = 0) \vee (a_2 = 0, a_1 \neq 0)$,
G has 2 distinct complex solutions is $a_2(a_1^2 - 4a_0a_2) \neq 0$,
G has ∞ complex solutions is $a_0 = a_1 = a_2 = 0$.

In general, to count the number of distinct complex zeros of a polynomial, we have:

Lemma 2.

Let $G(A, T) = A_0 + A_1 T + \cdots + A_d T^d \in \mathbb{Z}[A, T], A = (A_0, \cdots, A_d)$, be a general polynomial of degree d, and $e \in \{0, \cdots, d, \infty\}$. Then the set

$$\{a \in \mathbb{R}^{d+1} : G(a, T) \text{ has exactly } e \text{ distinct complex zeros }\}$$

is a finite union of sets of the form

$$\{a \in \mathbb{C}^{d+1} : p_1(a) = \cdots = p_k(a) = 0,\ q(a) \neq 0\},$$

where $p_i, q \in \mathbb{Z}[A]$.

Corollary.

As a consequence, for every $f \in \mathbb{R}[X_1, \cdots, X_n, T] = \mathbb{R}[X][T]$,

$$f(X_1, \cdots, X_n, T) = a_0(X) + a_1(X)T + \cdots + a_d(X)T^d,$$

the set

$$\{x \in \mathbb{R}^n : f(x, T) \text{ has exactly } e \text{ distinct complex zeros}\}$$

is a semi-algebraic subset of $\mathbb{R}^n$.

The cases $d = 0$ or $e \in \{0, \infty\}$ are trivial.
Let $d > 0, e \in \{1, \cdots, d\}$, and $a = (a_0, \cdots, a_d) \in \mathbb{C}^{d+1}, a_d \neq 0$.
Let $g =$ degree of GCD$(G(a,T), \dfrac{\partial G}{\partial T}(a,T))$ in $\mathbb{C}[T]$.
Then the number of distinct complex zeros of $G(a,T)$ is $d - g$, and the degree of LCM$(G(a,T), \dfrac{\partial G}{\partial T}(a,T))$ is $2d - g - 1$.
Hence the condition is that $G(a,T)$ has at most e distinct zeros, which is equivalent to $d - g \leq e$, that is, to $2d - g - 1 \leq d + e - 1$.
The last condition is equivalent to the condition:
$(*)$ There exist $q(x,T) = x_0 + x_1 T + \cdots + x_{e-1} T^{e-1}$ and $r(x,T) = x_e + x_{e+1} T + \cdots + x_{2e} T^e$, with $x = (x_0, \cdots, x_{2e}) \in \mathbb{C}^{2e+1} \setminus 0$, such that
$G(a,T)q(x,T) = \dfrac{\partial G}{\partial T}(a,T)r(x,T)$
This equality can be rewritten as

$$G(a,T)q(x,T) - \frac{\partial f}{\partial T}(a,T)r(x,T) = \beta_0(a,x) + \beta_1(a,x)T + \cdots + \beta_{d+e-1}(a,x)T^{d+e}$$

where $\beta = (\beta_0, \cdots, \beta_{d+e-1}) : \mathbb{C}^{d+1} \times \mathbb{C}^{2e+1} \to \mathbb{C}^{d+e-1}$ is a bilinear function.

So $(*)$ is equivalent to the condition $\beta_0(a,x) = \cdots = \beta_{d+e-1}(a,x) = 0$ has nonzero solution $x \in \mathbb{C}^{2e+1}$.
The last condition is equivalent to the vanishing of all $(2e+1)$-minor of the matrix of the linear map $\beta(a, \dot{})$.
Note that each of the minors is a polynomial in $a_0, \cdots, a_d$ with coefficients in $\mathbb{Z}$.
Therefore, for each $d' \leq d$, the set $M_e^{d'} =$

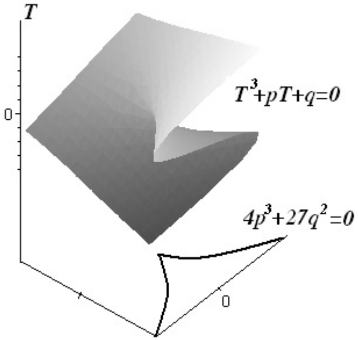
$\{a \in \mathbb{R}^{d+1}$: $G(a,T)$ is of degree d' and has at most e distinct complex zeros$\}$

is the intersection of the set $\{a \in \mathbb{R}^{d+1} : a_d = \cdots = a_{d'+1} = 0, a_{d'} \neq 0\}$ with the zero set of certain polynomials in $\mathbb{Z}[A]$.
So $\{a = (a_0, \cdots, a_d) \in \mathbb{R}^{d+1}$: $G(a,T)$ has exactly e complex zeros$\} =$

$$\bigcup_{d'=0}^{d} M_e^{d'} \setminus M_{e-1}^{d'}$$ a semi-algebraic set.

Since $f(x,T) = G(a_0(x), \cdots, a_d(x), T)$, the corrolary follows. □

Exercise. Use the method of proving the lemma to check:
1) The condition that $f(T) = T^2 + bT + c$ has ≤ 1 zero is $b^2 - 4c = 0$.
2)The condition that $f(T) = T^3 + pT + q$ has ≤ 2 zeros is $4p^3 + 27q^2 = 0$.



When the number of complex zeros is constant, the following connectedness ensures the number of real zeros is also constant.

Lemma 3.

Let $f = a_0 + \cdots + a_d T^d \in \mathbb{R}[X_1, \cdots, X_n][T]$ and $e \leq d$. Let C be a connected subset of $\mathbb{R}^n$. Suppose that $f(x, T) \in \mathbb{R}[T]$ has exactly e distinct complex zeros for each $x \in C$. Then the number of distinct real zeros of $f(x, T)$ is also constant as x ranges over C. If these zeros are ordered by $\xi_1(x) < \cdots < \xi_r(x)$, then the functions $\xi_j : X \longrightarrow \mathbb{R}$ are continuous.

Proof. Let $x_0 \in C$, and let $z_1, \cdots, z_e$ be the distinct zeros of $f(x_0, T)$. Take closed balls B_i centered at z_i in $\mathbb{C}$, such that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $B_i \cap \mathbb{R} = \emptyset$ if $z_i \notin R$. By continuity of roots (Rouché's theorem), there exists a neighborhood U of x_0 in C such that for each $x \in U$ the ball B_i contains at least one zero $\zeta_i(x)$ of $f(x, T)$. By the supposition, $\zeta_i(x)$ is the only zero of $f(x, T)$ in B_i.

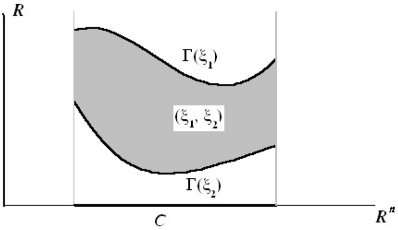
The graph of ζ_i on U is $\{(x,t) \in U \times B_i : f(x,t) = 0\}$, hence this graph is closed in $U \times B_i$, in combination with the compactness of B_i which implies that ζ_i is continuous on U. Since the coefficients of $f(x,T)$ are real, the set $\{\zeta_1(x), \cdots, \zeta_e(x)\}$ is closed under complex conjugation. Hence if $\zeta_i(x_0) \in \mathbb{R}$ then $\zeta_i(x) \in \mathbb{R}$ for all $x \in U$. This shows that the number of real zeros is locally constant. Since C is connected, this number is constant and the real zeros must keep their order as x runs through C. □

Exercise. Examine the lemma when $f(T) = T^2 + bT + c$, $(b,c) \in X = \mathbb{R}^2$.

Definition. Let $\xi_1, \xi_2 : C \longrightarrow \overline{\mathbb{R}}$, với $\xi_1 < \xi_2$. Write

$$\Gamma(\xi_1) = \{(x,t) : t = \xi_1(x)\} \quad \text{(the graph)},$$

$$(\xi_1, \xi_2) = \{(x,t) : x \in C,\ \xi_1(x) < t < \xi_2(x)\} \quad \text{(the band)}.$$



Theorem (Cylindrical decomposition - Łojasiewicz).

Let $f_1, \cdots, f_p \in \mathbb{R}[X][T]$, $X = (X_1, \cdots, X_n)$. Then there exist an augmentation $f_1, \cdots f_p, f_{p+1}, \cdots, f_{p+q} \in \mathbb{R}[X][T]$ and a partition of $\mathbb{R}^n$ into finitely many semi-algebraic sets $S_1, \cdots, S_k$ such that for each connected component C of each S_i there are continuous functions

$$-\infty = \xi_{C,0} < \xi_{C,1} < \cdots < \xi_{C,r(C)} < \xi_{C,r(C)+1} = +\infty$$

on C satisfying the following two properties:
(i) Each f_i $(1 \leq i \leq p+q)$ has a constant sign on each $\Gamma(\xi_{C,j})$ $(1 \leq j \leq r(C))$ and on each $(\xi_{C,j}, \xi_{C,j+1})$ $(0 \leq j \leq r(C))$.
(ii) Each of the sets $\Gamma(\xi_{C,j})$, $(\xi_{C,j}, \xi_{C,j+1})$ is of the form

$$\{(x,t) \in C \times \mathbb{R} : f_i(x,t)\ s(i)\ 0,\ i = 1, \cdots, p+q\},$$

for a suitable $s : \{1, \cdots, p+q\} \longrightarrow \{<, =, >\}$.

Proof. Let $d = \max\{\deg_T(f_i), i = 1, \cdots, p\}$.
Augment $f_1, \cdots, f_p$ to
$\{f_1, \cdots, f_{p+q}\} = \{\frac{\partial^\nu f_i}{\partial T^\nu} : 1 \le i \le p, 0 \le \nu \le d\}$.
For each $\Delta \subset \{1, \cdots, p\} \times \{0, \cdots, d\}$, and $e \in \{0, \cdots, pd^2\} \cup \{\infty\}$, put

$$f_\Delta(T) = \prod_{(i,\nu)\in\Delta} \frac{\partial^\nu f_i}{\partial T^\nu} \quad \in \mathbb{R}[X][T], \text{ and}$$

$$A_{\Delta,e} = \{x \in \mathbb{R}^n : f_\Delta(x, T) \text{ has exactly } e \text{ complex zeros}\}.$$

By Lemma 2, $A_{\Delta,e}$ is a semi-algebraic set.
For a given Δ the family $\{A_{\Delta,e} : e$ varies $\}$ forms a partition of $\mathbb{R}^n$. Since the class of semi-algebraic sets is a boolean algebra we can find a partition (the intersection of the partitions) $\mathbb{R}^n = S_1 \cup \cdots \cup S_k$, where each S_i is semi-algebraic such that each set $A_{\Delta,e}$ is a union of the $S_i's$.
We will prove that $f_1, \cdots, f_{p+q}$ and $S_1, \cdots, S_k$ satisfy the conclusion of the theorem.

For each connected component C of S_i put

$$\Delta(C) = \{(i,\nu) : \frac{\partial^\nu f_i}{\partial T^\nu} \not\equiv 0 \text{ on } C \times \mathbb{R}\}.$$

By Lemma 3, there exist continuous functions $\xi_{C,_1} < \cdots < \xi_{C,r(C)}$ on C such that $\{(x,t) \in C \times \mathbb{R} : f_{\Delta(C)} = 0\} = \Gamma(\xi_{C,1}) \cup \cdots \cup \Gamma(\xi_{C,r(C)})$.

Check (i): If $(i,\nu) \notin \Delta(C)$ then $\dfrac{\partial^\nu f_i}{\partial T^\nu} \equiv 0$ on the sets given in (i).
If $(i,\nu) \in \Delta(C)$, then $C \subset A_{\{(i,\nu)\},e}$, for certain $e \in \{0,\cdots,d\} \cup \{\infty\}$ and the number of real zeros of $\dfrac{\partial^\nu f_i}{\partial T^\nu}(x,T)$ is independent of $x \in C$.
Since $\dfrac{\partial^\nu f_i}{\partial T^\nu}$ is a factor of $f_{\Delta(C)}$, by Lemma 3, the zeros of $\dfrac{\partial^\nu f_i}{\partial T^\nu}(x,T)$, for $x \in C$, must be among the $\xi_{C,j}(x)'s$. Since C is connected, (i) is checked.

Check (ii): Let B be one of the sets in (i). By (i), $\epsilon(i,\nu) =$ sign$(\frac{\partial^\nu f_i}{\partial T^\nu} \mid_B)$ is well-defined. Put

$$B' = \{(x,t) \in C \times \mathbb{R} : \mathrm{sign}(\frac{\partial^\nu f_i}{\partial T^\nu}(x,t) = \epsilon(i,\nu),\ 1 \le i \le p, 0 \le \nu \le d\}.$$

Clearly $B \subset B'$. If $B \neq B'$ then exist $(x,t') \in B' \setminus B,\ (x,t) \in B$ (say $t < t'$). Thom's lemma implies that $\{r \in \mathbb{R} : (x,r) \in B'\}$ is connected, so $\{x\} \times [t,t'] \subset B'$. Since $(x,t) \in B,\ (x,t') \notin B$, $f_{\Delta(C)}$ must change sign on $\{x\} \times [t,t']$. But $f_{\Delta(C)}$ is a product of $\frac{\partial^\nu f_i}{\partial T^\nu}$, so $f_{\Delta(C)}$ cannot change sign on B', contradiction. Therefore $B = B'$. □

Exercise.

1) Contruct the augment family of polynomials and the partition of $\mathbb{R}^2 = \{(b, c)\}$ satisfying the theorem for $f(b, c, T) = T^2 + bT + c$.

2) Contruct the augment family of polynomials and the partition of $\mathbb{R}^2 = \{(p, q)\}$ satisfying the theorem for $f(p, q, T) = T^3 + pT + q$.

Exercise.

1) Suppose $X \subset \mathbb{R}^n$ is a semialgebraic set described by polynomial $f_1, \cdots, f_p$ of degree $\leq d$. Find an upper bound for the connected components of X as a function of n, p, d.

The following exercises are related to resultants (ref. [BR]).

Let A be a factorial commutative ring. Let

$P(T) = a_0 + \cdots + a_p T^p \in A[T],\ a_p \neq 0,$

$Q(T) = b_0 + \cdots + b_q T^q \in A[T],\ b_q \neq 0.$

For $0 \leq k \leq \min(p,q)$, the k-nd Sylvester's matrix of P, Q is defined by:

$$M_k(P,Q) = \underbrace{\begin{pmatrix} a_0 & \cdots & 0 & b_0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ & & a_0 & & & b_0 \\ a_p & & \vdots & b_q & & \vdots \\ & \ddots & & & \ddots & \\ 0 & & a_p & 0 & & b_q \end{pmatrix}}_{q-k \quad\quad p-k} \Bigg\} p+q-k$$

2) Prove that the following conditions are equivalent:

(a) The degree of $\mathrm{GCD}(P,Q)$ is $\geq k+1$.

(b) P, Q have $\geq k+1$ common zeros (counted with multiplicity) in the

3) From the above exercise, prove that the condition is that P, Q have k distinct zeros in $\overline{A}$, which is the condition given by equalities and inequalities of certain polynomials in $\mathbb{Z}[a_0, \cdots, a_p, b_0, \cdots, b_q]$.
4) When $A = \mathbb{C}$, prove that P has exactly k zeros if and only if the degree of GCD(P, P') is $p - k$.
This implies Lemma 1.9.
5) The resultant of P, Q is defined by Res$(P, Q) = \det(M_0(P, Q))$.
Therefore,

$$\mathrm{Res}(P, Q) = 0 \;\Leftrightarrow\; P, Q \;\text{ having GCD of degree } > 0.$$

6) The discriminant of P is defined by
Disc(P)=Res$(P, P') = \det(M_0(P, Q))$.
When $A = \mathbb{C}$, we have

$$\mathrm{Disc}(P) = 0 \;\Leftrightarrow\; P \;\text{ having zeros of multiplicity} > 0$$

7) Compute the discriminants of polynomials of degree 2, 3.

Further reading: Sturm's Theorem and Tarski-Seidenberg's Theorem (Ref. [BCR], [C]).

Further reading: Semialgebraic sets in general real closed fields (Ref. [BCR]).

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End of Lecture 1