Lecture 2: Cell Decomposition - Stratification

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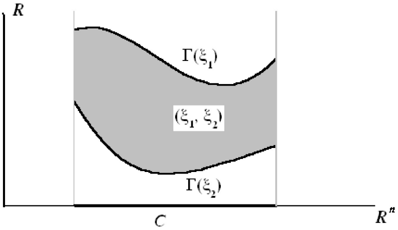
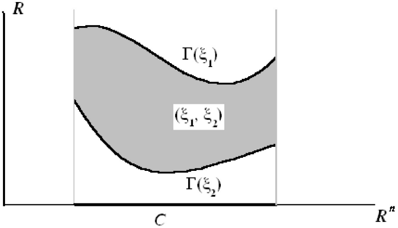
Contents

1. Cell decomposition

From Cylindrical decomposition theorem, a semialgebraic subset of $\mathbb{R}^n$ has an especially simple form - it splits into finitely many cells. Each cell is similar to a curving box.



Moreover, each semialgebraic function is 'cellwise' analytic. In this lecture we show these results and their consequences.



Definition. The semialgebraic cells in $\mathbb{R}^n$ are defined by induction on n as follows:

- A semialgebraic cell in $\mathbb{R}$ is a point or an open interval.
- If $C \subset \mathbb{R}^n$ is a cell and $f, g : C \to \mathbb{R}$ are continuous semialgebraic functions such that $f < g$, then the sets:

$$\Gamma(f) = \{(x,t) : t = f(x)\},\ (f,g) = \{(x,t) : f(x) < t < g(x)\},$$

$C \times \mathbb{R}, (-\infty, f) = \{(x,t) : t < f(x)\}$ and $(f, +\infty) = \{(x,t) : f(x) < t\}$.

are semialgebraic cells in $\mathbb{R}^{n+1}$.

Let $k \in \mathbb{N} \cup \{\omega\}$. A C^k cell is a cell with the basis set C being a C^k-manifold and the functions f, g being of class C^k.

Exercise. Prove that each cell is homeomorphic to a box $(0,1)^d$.

A C^p cell decomposition of $\mathbb{R}^n$ is defined by induction on n:
- A C^p decomposition of $\mathbb{R}$ is a finite collection of intervals and points

$$\{(-\infty, a_1), \cdots, (a_p, +\infty), \{a_1\}, \cdots, \{a_p\}\},$$

where $a_1 < \cdots < a_p$, $p \in \mathbb{N}$.
- A C^p decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into C^p cells C, such that the collection of all the projections $\pi(C)$ is a C^p decomposition of $\mathbb{R}^n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first n coordinates.

We say that a decomposition compatible with a class $\mathcal{A}$ of subsets of $\mathbb{R}^n$, if each $S \in \mathcal{A}$ is a union of some cells of the decomposition.

Theorem (C^ω semialgebraic cell decomposition).

Let $A_1, \cdots, A_p$ be semialgebraic subsets of $\mathbb{R}^n$. Then there exists a C^ω semialgebraic cell decomposition of $\mathbb{R}^n$ compatible with $\{A_1, \cdots, A_p\}$.

Proof. Induction on n and basing on Cylindrical decomposition theorem.
For $n = 1$: Let $\mathcal{G}$ be the family of polynomials which describes $A_1, \cdots, A_p$.
Augment $\mathcal{G}$ to $\mathcal{F}$ by all non null partial derivatives of all degree of polynomials in $\mathcal{G}$.
Then, by Thom's Lemma, $\mathcal{F}$ defines a cell decomposition of $\mathbb{R}$ such that each A_i is a union of cells defined by sign conditions of polynomials in $\mathcal{F}$.

For $n > 1$: For each $k = 1, \cdots, n$, let $\pi_k : \mathbb{R}^n \to \mathbb{R}^k$ denote the projection the first k coordinates. From Cylindrical decomposition theorem and induction, we can construct a family of polynomials $\mathcal{F} = \mathcal{F}_n \cup \cdots \cup \mathcal{F}_1$, which satisfies the following properties for each $k \in \{1, \cdots, n\}$:

- $\mathcal{F}_k = \{f_{k,j} : j = 1, \cdots l_j\} \subset \mathbb{R}[X_1, \cdots, X_k]$ is constructed from the polynomials describing $\pi_k(A_1), \cdots, \pi_k(A_p)$, then augmented to be closed under derivative operator $\frac{\partial}{\partial X_k}$.
- The family $\mathcal{F}_{k-1} \cup \cdots \cup \mathcal{F}_1$ defines decomposition $\mathcal{C}_{k-1}$ of $\mathbb{R}^{k-1}$ consisting of cells, each of the cells is given by the sign condition of the polynomial in the family.
- For each $C \in \mathcal{C}_{k-1}$ there exists semialgebraic functions $\xi_{C,1} < \cdots < \xi_{C,r(C)} : C \to \mathbb{R}$, such that each $x \in C$, $\{\xi_{C,1}(x), \cdots, \xi_{C,r(C)}(x)\}$ is the zeros of $f_{k,1}(x, X_k), \cdots, f_{k,l_k}(x, X_k)$, and
- Each of $\pi_k(A_1), \cdots, \pi_k(A_p)$ is an union of cells such that each of the cells has the form $\Gamma(\xi_{C,i}), (\xi_{C,i}, \xi_{C,i+1}), (-\infty, \xi_{C,1})$ or $(\xi_{C,r(C)}, +\infty)$, where $C \in \mathcal{C}_{k-1}$, and the polynomials in $\mathcal{F}_k$ do not change sign on that cell.

Therefore, to prove the cells are submanifolds of class C^{ω}, we need to prove $\xi_C = \xi_{C,i} : C \to \mathbb{R}$ is of class C^{ω}, for each cell $C \in \mathcal{C}_{k-1}$.
By induction, C is a cell of class C^{ω}. Let $f \in \mathcal{F}_k$ be the polynomial in X_k of smallest degree (wrt. X_k) and $f(x, \xi_C(x)) = 0$, for all $x \in C$.
By the closeness under the derivative operator $\frac{\partial}{\partial X_k}$ of the family $\mathcal{F}_k$ and by the unchange sign of $\frac{\partial f}{\partial X_k}$ on $\Gamma(\xi_C)$, we get $\frac{\partial f}{\partial X_k}(x, \xi_C(x)) \neq 0$, for all $x \in C$.
By the Implicit function theorem, ξ_C is an analytic function. □

From the above theorem, we have the following basic properties:

Theorem (The piecewise analytic property).

Let $A \subset \mathbb{R}^n$ be a semialgebraic subset and $f : A \to \mathbb{R}$ be a semialgebraic function. Then there exists a C^ω semialgebraic decomposition $\mathcal{C}$ of $\mathbb{R}^n$, compatible with A, such that $f|_C$ is of class C^ω, for all $C \in \mathcal{C}$ and $C \subset A$.

Proof. Applying the cell decomposition theorem to $A_1 = \Gamma(f) \subset \mathbb{R}^{n+1}$, we get the result. $\square$

Proposition (Monotonnicity theorem).

Let $f : \mathbb{R} \to \mathbb{R}$ be a semialgebraic function. Then there exist points $-\infty = a_0 < a_1 < \cdots < a_N = +\infty$ such that on each interval (a_i, a_{i+1}) the function is either constant, or strictly monotone and analytic.
As a consequence, for all $a \in \mathbb{R} \cup \{\pm\infty\}$, the limits $\lim_{x \to a^+} f(x), \lim_{x \to a^-} f(x)$ exist (in $\mathbb{R} \cup \{\pm\infty\}$).

Proof. From the above theorem, there exists a decomposition of $\mathbb{R}$ into finite points or intervals on which f is analytic.
Each of the intervals can be decomposed into finte points or intervals compatible with the conditions $f' = 0, f' > 0, f' < 0$.
The result follows. □

Proposition (Uniformly finiteness).

Let $A \subset \mathbb{R}^n$ be a semialgebraic set. Let $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be the natural projection. Suppose that $\# A \cap \pi^{-1}(x) < +\infty$, $\forall x \in \mathbb{R}^{n-1}$.
Then there exists $N \in \mathbb{N}$, such that $\# A \cap \pi^{-1}(x) \leq N$, $\forall x \in \mathbb{R}^{n-1}$.

Proof. Decompose $\mathbb{R}^n$ into cells compatible with A.
Since$\# A \cap \pi^{-1}(x) < +\infty$, for all $x \in \mathbb{R}^{n-1}$, A is a finite union of sets of the graph form $\Gamma(\xi_C)$, for $C \subset \pi(A)$ being cells in $\mathbb{R}^{n-1}$.
Therefore, $N =$ maximum of the numbers of graphs over C, for $C \subset \pi(A)$, is the desired uniform bound. □

2. Dimension.

Dimension is a basic notion of topology and geometry.
Since semialgebraic sets are finite union of manifolds, the following definition of dimension of semialgebraic sets is suitabe.

Definition. The dimension of a semialgebraic subset $X \subset \mathbb{R}^n$ is defined by

$$\dim X = \sup\{\dim \Gamma : \ \Gamma \subset X, \ \Gamma \text{ is a analytic submanifold of } \mathbb{R}^n\}.$$

Note. If $X = \cup_{i \in I} C_i$ is a finite union of analytic cells, then

$$\dim X = \max\{\dim C_i : i \in I\}.$$

In fact, let $\Gamma \subset X$ be a submanifold such that $\dim \Gamma = \dim X$. Since $\Gamma = \cup_{i \in I} \Gamma \cap C_i$, by Baire's property of manifolds, there exists $i_0 \in I$ such that $\Gamma \cap C_{i_0}$ has nonempty interior in Γ. So $\dim C_{i_0} \geq \dim \Gamma$. From this we get the conclusion of the note.

Basing on results of dimension of manifolds, the dimension of semialgebraic has following natural properties:

Proposition.

(i) Let $X, Y \subset \mathbb{R}^n$ be semialgebraic subsets. Then

$$\dim X < n \Leftrightarrow \overset{\circ}{X} = \emptyset.$$
$$X \subset Y \Rightarrow \dim X \leq \dim Y.$$

(ii) If $X = \cup_{i=1}^{p} X_i$, where X_i are semialgebraic, then

$$\dim X = \max_{1 \leq i \leq p} \dim X_i.$$

(iii) Let $X \subset \mathbb{R}^m \times \mathbb{R}^n$ be semialgebraic. Let $\pi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ be the natural projection. Suppose $\dim \pi^{-1}(x) \cap X \leq k, \ \forall x \in \pi(X)$. Then

$$\dim \pi(X) \leq \dim X \leq \dim \pi(X) + k.$$

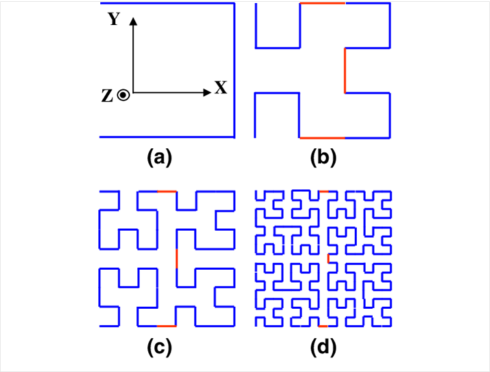
(iv) Let $f : M \to N$ be a semialgebraic mapping, and $X \subset M$ be a semialgebraic subset. Then $\dim f(X) \leq \dim X$.

Proof. (i) is clear.
(ii) Let $\Gamma \subset X$ be a submanifolsd such that $\dim \Gamma = \dim X$. Represent $X_i = \cup_j \Gamma_{i,j}$ as a finite union of manifolds. Arguing as in the above note for $\Gamma = \bigcup_{i,j}(\Gamma \cap \Gamma_{i,j})$, we get i_0, j_0 such that the interior of Γ_{i_0,j_0} trong Γ is not empty. From that we have (ii).
(iii) By Cell decomposition theorem, $X = \cup_i \Gamma_i$ is a finite union of analytic cells such that π_{Γ_i} is of constant rank and $\pi(\Gamma_i)$ is an analytic cell.
Then $\pi(X) = \cup_i \pi(\Gamma_i)$ and $\dim \Gamma_i \geq$ rankπ_{Γ_i}. Therfore,

$$\dim X = \max_i \dim \Gamma_i \geq \max_i \operatorname{rank} \pi_{\Gamma_i} = \max_i \dim \pi(\Gamma_i) = \dim \pi(X).$$

Besides, let $\Gamma \subset X$ be an analytic cell in the above decomposition with dimension $\dim X$. Then each fiber $\pi_\Gamma^{-1}(x) = \pi^{-1}(x) \cap \Gamma,\ x \in \pi(\Gamma)$ is a submanifold of dimension $\dim \Gamma -$ rankπ_Γ. From the supposition, we have $\dim X = \dim \Gamma \leq \dim \pi(X) + k$.
(iv) is followed from (iii) with the note that
$X = \pi_1(f_X),\ f(X) = \pi_2(f_X)$, where $f_X = f \cap X \times N$ and π_1, π_2 are the projections from $M \times N$ to M, N respectively. □

Exercise. Learn the construction of Peano's curves, which are continuous maps $f : [0,1] \to [0,1]^2$, with image $f([0,1]) = [0,1]^2$..

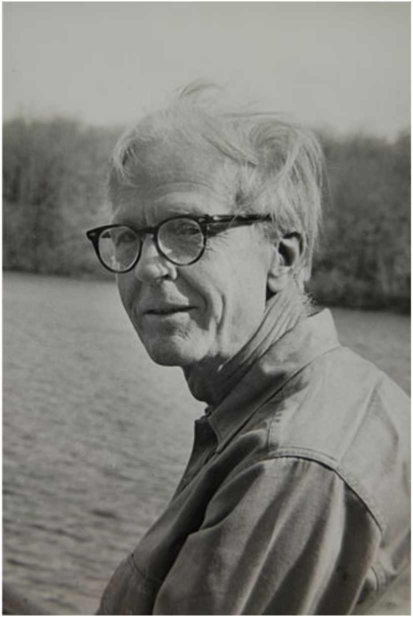


(a) (b)
(c) (d)

Therefore, in general, continuous mappings do not have the property (iv) of the above proposition.

3. Stratification.

Hassler Whitney (1907-1989)



René Thom (1923-2002)

3. Stratification.

Stratification theory was initialled by Whitney. In 1957 Whitney showed that every algebraic set in $\mathbb{R}^n$ can be partitioned into finitely many connected semialgebraic submanifolds, which are fitted to each other along their boundaries and satisfy some certain 'good' condition (called Whitney regular conditions). Such a partition is called stratification.
Many problems, in many different fields, were solved by basing on the property that the involved sets are stratified.
For example, equi-singularity problems proposed by Thom (see [GWPL] and the examples below).
Another example: To solved the Schwartz's division problem, Łojasiewicz (1959) (see [Ł]), constructed the stratification of semi-analytic sets and certain metric properties (called Łojasiewicz's inequalities). From that he obtained the solution for the problem.
In this part, we present a result of Łojasiewicz that any semialgebraic set can be stratified.

Definition. A stratification of $\mathbb{R}^n$ is a partition $\{\Gamma_\alpha\}_{\alpha\in\Lambda}$ of $\mathbb{R}^n$ into finitely many subsets, called strata, such that:

(S1) Each stratum Γ_α is a connected submanifold of $\mathbb{R}^n$.

(S2) Boundary condition:

if $\overline{\Gamma}_\alpha \cap \Gamma_\beta \neq \emptyset$, then $\Gamma_\beta \subset \overline{\Gamma}_\alpha$ and $\dim\Gamma_\beta < \dim\Gamma_\alpha$.

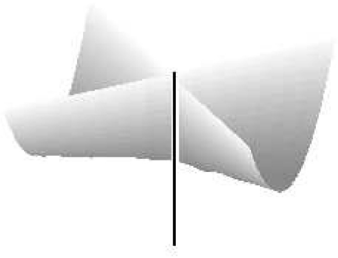
i.e. $\overline{\Gamma}_\alpha \setminus \Gamma_\alpha$ is a union of some of the strata with $\dim\Gamma_\beta < \dim\Gamma_\alpha$.

A strafication is called compatible with a subset $X \subset \mathbb{R}^n$ iff $\Gamma_\alpha \cap X \neq \emptyset$, then $\Gamma_\alpha \subset X$, i.e. X is a union of some Γ_α.

A strafication is called semialgebraic iff each stratum is semialgebraic.

Example.

In $\mathbb{R}^3$, let V be a subset defined by $x^2 - zy^2 = 0$ (Whitney's umbrela).



Partition 1: $\mathbb{R}^3 \setminus V, V \setminus Oz, Oz$,
is not a stratification because it does not satisfy (S2).
Partition 2: $\mathbb{R}^3 \setminus V, V \setminus Oz, \{(0.0, z) : z > 0\}, \ \{(0, 0, z) : z < 0\}, O$,
is a stratification compatible with V.

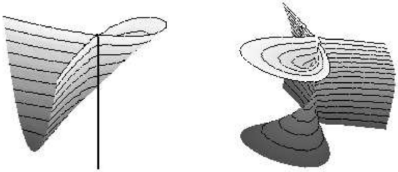
Note. For Partition 2, the topo types of $V \cap B_a$, where B_a is a ball with center a of sufficiently small radius, are the same when a ranges in a stratum (the topological equisingularity).

Exercise.

1) Find a stratification compatible with X given by the following equation:

a) $x^3 + zx^2 - y^2 = 0$ b) $x^3 + y^2 - z^2x^2 = 0$.

Find a stratification which satisfies the topological equisingularity for X.



2) Prove that $\mathbb{R}^{n+1}$ is stratified by the family

$\Gamma_{i_1\cdots i_k} = \{(a_0, \cdots, a_n) \in \mathbb{R}^{n+1}:$

$a_0 + a_1T + \cdots + a_nT^n$ has exactly k complex zeros with multiplicities $i_1, \cdots, i_k\}$,

where $k \in \mathbb{N}, (i_1, \cdots, i_k) \in \{0, \cdots, n\}^k$.

Concretize when $n = 2$.

Theorem (Semi-algebraic stratification).

Let $X_1, \cdots, X_k$ be semialgebraic subsets of $\mathbb{R}^n$. Then there exists a stratification of $\mathbb{R}^n$, compatible with each X_i, and the strata are semialgebraic.

To prove the theorem we prepare some tools and lemmas.

To generalize Thom's Lemma to polynomial of several variables we have the following notion:

Definition. A family of polynomial functions $f_1, \cdots, f_N : \mathbb{R}^n \to \mathbb{R}$ is called separating iff for any $s : \{1, \cdots, N\} \to \{<, =, >\}$, the semialgebraic subset of the form

$$A_s = \{x \in \mathbb{R}^n : \ f_i(x)\ s(i)\ 0,\ i = 1, \cdots, N\},$$

satisfies:
(i) A_s is either empty or connected.
(ii) If $A_s \neq \emptyset$, then the closure of A_s has the algebraic description

$$\overline{A_s} = \{x \in \mathbb{R}^n : \ f_i(x)\ \underline{s}(i)\ 0,\ i = 1, \cdots, N\}$$

where $\underline{s}(i)$ is $\leq$, $=$ or $\geq$ according as $s(i)$ is $<$, $=$ or $>$.
(i.e. the closure is obtained by relaxing all strict inequalities to weak inequalities).

To construct a separating family from a finite set of polynomial functions, we need to pay attention to the boundedness of the roots of a polynomial with respect to its coefficient of the highest degree.

Lemma 1.

Put $g(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_0 \in \mathbb{C}[t]$, where $a_d \neq 0$. Then if $\xi \in \mathbb{C},\ g(\xi) = 0$, then $|\xi| \leq \max_{0 \leq k \leq d-1} \left(d \frac{|a_k|}{|a_d|} \right)^{\frac{1}{d-k}}$.

Proof. When $|t| > M$, we have

$$|g(t)| > |a_d| M^d - (|a_{d-1}| M^{d-1} + \cdots + |a_0|).$$

Therefore, when we choose M such that $\frac{1}{d}|a_d| M^d \geq |a_k| M^k$, for all $k = 0, \cdots d-1$, ta có $|g(t)| > 0$. From that we get the estimate. $\square$

For several-variable polynomials, the following notion gives the boundedness of the zeros.

Definition. A polynomial $f \in \mathbb{R}[X_1, \cdots, X_k]$ is called quasi-monic with respect to X_k iff

$$f = a_d X_k^d + a_{d-1}(X_1, \cdots, X_{k-1}) X_k^{d-1} + \cdots + a_0(X_1, \cdots, X_{k-1}),$$

where the leading coefficient is a constant $a_d \neq 0$.

Example. $f(x, y) = xy - 1$.

The equation $f(x, y) = 0$ has the root $x = \dfrac{1}{y} \to \infty$, when $y \to 0$.
Change the coordinates: $x = X + \lambda Y, y = X - \lambda Y$, where $\lambda \neq 0$.
Then $f = X^2 - \lambda^2 Y^2 - 1$ is monic with respect to X
and $f = 0$ has the root $X = \pm\lambda\sqrt{Y^2 + 1}$ which is locally bounded.

Lemma 2.

Let $g_1, \cdots, g_p$ be real polynomials of k variables. Then there exists a linear change of coordinates $\varphi : \mathbb{R}^k \to \mathbb{R}^k$, such that $g_1 \circ \varphi, \cdots, g_p \circ \varphi$ are monic with respect to X_k.

Proof. Represent a polynomial in the form $g = \sum_{j=0}^{d} p_j$, where p_j is the homogenous of degree j and $p_d \neq 0$.
Then the set of directions $Q_g = \{e \in S^{k-1} : p_d(e) \neq 0\}$ is open and dense subset of the unit sphere S^{k-1}.
For $p_d(e) \neq 0$, let φ be a linear transfomation of $\mathbb{R}^k$ such that $\varphi(e_k) = e$, where $e_k = (0, \cdots, 1)$. Then

$$g \circ \varphi(X_k e_k) = \sum_{j=0}^{d} p_j(\varphi(X_k e_k)) = \sum_{j=0}^{d} p_j(X_k e) = \sum_{j=0}^{d} p_j(e) X_k^j = p_d(e) X_k^d + \cdots +$$

Hence, $g \circ \varphi$ is quasi-monic wrt. X_k.
By the densense and open property, there exists $e \in Q_{g_1} \cap \cdots \cap Q_{g_p}$, and hence there exists a linear change of coordinates φ which satisfies the demand of the lemma. $\square$

Theorem (Separating family).

Any finite set of polynomials on $\mathbb{R}^n$ can be augmented to a separating family.

Proof. Induction on n. Let $f_1, \cdots, f_p : \mathbb{R}^n \to \mathbb{R}$ be polynomial functions.
By the above lemma, after changing of coordinates, we get $f_i(x,t) \in \mathbb{R}[x][t]$, where $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $i = 1, \cdots, p$, are quasi-monic wrt. t.
For $n = 1$, by Thom's lemma, augmenting all derivatives of the polynomials we get the separating family.

For $n > 1$, we add all non null derivatives of all orders wrt. t of $f_1, \cdots, f_p$ to get $f_1, \cdots, f_p, f_{p+1}, \cdots, f_{p+q}$.
By Cylindrical decomposition theorem, $\mathbb{R}^{n-1}$ can be partitioned to semialgebraic sets $S_1, \cdots, S_k$, which are described by a finite set of polynomial on $\mathbb{R}^{n-1}$.
By the induction, that set can be added to get a separating family on $\mathbb{R}^{n-1}$, say $f_{p+q+1}, \cdots, f_{p+q+r}$.
We prove that $f_1, \cdots, f_{p+q+r}$ is a separating family on $\mathbb{R}^{n-1} \times \mathbb{R}$.
Consider $A = \{(x,t) : \ f_i(x,t) \ s(i) \ 0, \ i = 1, \cdots, p+q+r\}$.
By Cylindrical decomposition theorem, A is either $\emptyset$ or of the forms $\Gamma(\xi_j)$ or (ξ_j, ξ_{j+1}), where $\xi_j, \xi_{j+1} : C \to \mathbb{R}$ are continuous semialgebraic functions on the set
$C = \{x : \ f_i(x) \ s(i) \ 0, \ i = p+q+1, \cdots, p+q+r\}$.
By the induction, C is connected, and hence A is connected.
When $A \neq \emptyset$, put $A' = \{(x,t) : \ f_i(x,t) \ \underline{s}(i) \ 0, \ i = 1, \cdots, p+q+r\}$.
Clearly, $\overline{A} \subset A'$.
We need to prove that $A' \subset \overline{A}$. By induction,
$\overline{C} = \{x : \ f_i(x) \ \underline{s}(i) \ 0, \ i = p+q+1, \cdots, p+q+r\}$.
Let $x_0 \in \overline{C}$. Since $f_i(x,t), \ i = 1, \cdots, p+q$, are monic wrt. t, by Lemma 1, their zeros ξ_j are locally bounded at x_0 on $\overline{C}$.

Therefore, $\overline{A} \cap \pi^{-1}(x_0) \neq \emptyset$. By Thom's lemma, there are two posibilities for the fiber $A' \cap \pi^{-1}(x_0)$:
(1) A point: that fiber coincides with the fiber $\overline{A} \cap \pi^{-1}(x_0)$.
(2) A closed interval J: When $(x_0, t) \in \overset{\circ}{J}$, we have
$f_i(x_0, t)\ s(i)\ 0,\ i = 1, \cdots, p+q$. This implies $(x_0, t) \in \overline{A}$. Thus $J \subset \overline{A}$.
In the two cases $\overline{A} \cap \pi^{-1}(x_0) \supset A' \cap \pi^{-1}(x_0)$. Therefore, $A' \subset \overline{A}$. $\square$

Lemma 3.

Let $X \subset \mathbb{R}^n$ be a semialgebraic subset. Then $\dim(\overline{X} \setminus X) < \dim X$.

Proof. By Cylindrical theorem and the separating family theorem, change the coordinates if needed, X is a finite of sets of the form $\Gamma(\xi)$ or (ξ_1, ξ_2), where $C \subset \mathbb{R}^{n-1}$ is semialgebraic and $\xi, \xi_1, \xi_2 : C \to \mathbb{R}$ are continuous semialgebraic and zeros of a quasi-monic wrt. t

$$f(x,t) = a_0(x) + \cdots + a_{d-1}(x)t^{d-1} + a_d t^d \in \mathbb{R}[x][t].$$

Moreover, the above set is of the form

$$A = \{(x,t) : \ f_i(x,t) \ \epsilon(i) \ 0, \ i = 1, \cdots, N\}$$

where $f_1, \cdots, f_N$ is a separating family containing f.

Since the coefficient of t^d is not 0, ξ is locally bounded at each $y \in \overline{C} \setminus C$ (see Lemma 1). Therefore, there exist $l = \liminf_{x \to y} \xi(x)$ and $L = \limsup_{x \to y} \xi(x)$.

Since $(x, \xi(x)) \in A$,
$(y, l), (y, L) \in \{(x,t) : \ f_i(x,t) \ \underline{\epsilon}(i) \ 0, \ i = 1, \cdots, k\} = \overline{A}$.
By Thom's lemma $y \times [l, L] \subset \overline{A}$. Hence, $f(y,t) = 0$, for all $t \in [l, L]$.
Since $f(y,t)$ is a polynomial not zero (wrt. t), $l = L$. Therefore, there exists $\lim_{x \to y} \xi(x)$, for all $y \in \overline{C} \setminus C$. So $y \times \mathbb{R} \cap \overline{\Gamma(\xi)}$ is a point.
From this, by induction and dimension property (iii), we get

$$\dim(\overline{\Gamma}(\xi) \setminus \Gamma(\xi)) \leq \dim((\overline{C} \setminus C) \times \mathbb{R} \cap \overline{\Gamma(\xi)}) < \dim \Gamma(C) = \dim \Gamma(\xi).$$

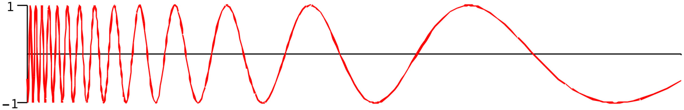
The above arguments give

$$\dim(\overline{(\xi_1, \xi_2)} \setminus (\xi_1, \xi_2)) < \dim(\xi_1, \xi_2).$$

□

Exercise. Let $X = \{(x, y) \in \mathbb{R}^2 : y = \sin \frac{1}{x}, x \neq 0\}$.

Prove that $\dim(\overline{X} \setminus X) = \dim X$.



Lemma 4.

Let $X \subset \mathbb{R}^n$ be a semialgebraic subset of $\dim X = k$. Then there exists a closed semialgebraic subset $F \subset \mathbb{R}^n$ with $\dim F < k$ such that $X \setminus F$ is a submanifold of dimension k.

Proof. By Cell decomposition theorem, X is a finite union of analytic cells. Let F be the union of the cells of dimension $\leq k-1$. By the above lemma, $\dim F < k$ and $X \setminus F$ is empty or a analytic submanifold of dimension k. □

Theorem (Semi-algebraic stratification).

Let $X_1, \cdots, X_k$ be semialgebraic subsets of $\mathbb{R}^n$. Then there exists a stratification of $\mathbb{R}^n$, compatible with each X_i, and the strata are semialgebraic.

Proof. For $A, W \subset \mathbb{R}^n$, let $b(A, W)$ denote the (relative) boundary of A in W. We will construct a sequence of semialgebraic subsets:

$$\mathbb{R}^n = Z_n \supset Z_{n-1} \supset \cdots \supset Z_0 \supset Z_{-1} = \emptyset$$

such that $\dim Z_j \leq j$, $Z_j \setminus Z_{j-1}$ is either empty or a connected semialgebraic submanifold of dimension j.

Suppose $Z_n \supset \cdots \supset Z_k$ are constructed. By Lemma 4, there exists a closed semialgebraic subset $F_k \subset Z_k$ of dimension $< k$ such that $W_k = Z_k \setminus F_k$ is either empty or a submanifold of dimension k.

Let $\{\Gamma^j_\sigma\}$ be the family of connected components of $Z_j \setminus Z_{j-1}$. Put

$$Z_{k-1} = F_k \bigcup_{j>k} \bigcup_{\sigma} b(\overline{\Gamma}^j_\sigma, W_k) \bigcup_i b(X_i, W_k),$$

Then Z_{k-1} is semialgebraic. By Lemma 3, $\dim Z_{k-1} < k$. Clearly, $Z_k \setminus Z_{k-1}$ is either empty or a k-dimensional submanifold.

Exercise. Check that the family $\{\Gamma^j_\sigma\}_{j,\sigma}$ is a desired stratification. $\square$

Exercise.

Use the above theorem to prove Sard's Theorem for semialgebraic functions:

1) Let $M \subset \mathbb{R}^n$ be a semialgebraic submanifold and $f : M \to \mathbb{R}$ be a differentiable semialgebrac function. Put $C = \{x : Df(x) = 0\}$. Prove that $f(C)$ is a finite set.

2) Let M, N be semialgebraic submanifolds and $f : M \to N$ be a differentiable semialgebraic map. Put $C = \{x : \mathrm{rank}_x f < \dim N\}$. Prove that The critical values set $f(C)$ is semialgebraic and $\dim f(C) < \dim N$.

3) Let X be a semialgebraic subset of $\mathbb{R}^n$. Leti $\Sigma(X)$ be the set of points where X is not a submanifold. Suppose $a \in X$ is either a smooth point of X or a isolated point of $\Sigma(X)$. Use the sard theorem for the function $f(x) = \|x - a\|^2$ restricted to a neighborhood of $X \setminus \{a\}$, to prove the existence of $\varepsilon_0 > 0$ such that every sphere S_ε of center a and radius $\varepsilon < \varepsilon_0$, we have $X \cap S_\varepsilon$ is a submanifold.

Further Reading: Stratification Theory - Regular conditions of Whitney.
Ref.

C. B. Gibson, K. Wirthmuller, A. A. du Plessis và E. J. N. Loojenga, *Topological Stability of Smooth Mappings*, Lecture Notes in Mathematics 552, Springer-Verlag, 1976.

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End of Lecture 2