# Lecture 3:
# Curve Selection Lemma
# The Łojasiewicz inequalities

Tạ Lê Lợi

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## Contents

The first part of this lecture we present Curve Selection Lemma, which has many applications in geometry and Singularity theory.
Roughly speaking, the lemma states that in a semialgebraic subset the existence of a sequence convergent to a point $a$ is equivalent to the existence of a curve in the subset starting from $a$. Moreover, one can choose the curve to be the restriction of an analytic one.
Therefore, considering a sequence in a semialgebraic set, we can transfer to an analytic curve. It is convenient to use analysis tools.
We can also use the lemma to transfer a several-variable problems to a one-variable one.

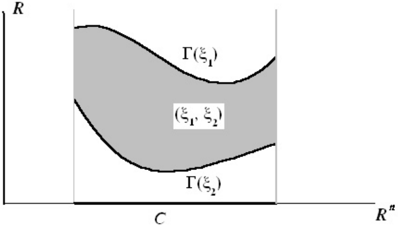
In the second part, basing on the lemma, we present some important inequalities in Semialgebraic geometry: Łojasiewicz's inequalities.

**Curve Selection Lemma (weak form).**

Let $X \subset \mathbb{R}^n$ be a semialgebraic subset. Let $a$ be an accumulate point of $X$. Then there exists $\gamma : [0, \epsilon) \to \mathbb{R}^n$ which is continuous, analytic on $(0, \epsilon)$, $\gamma(0) = a$ and $\gamma((0, \epsilon)) \subset X \setminus \{a\}$.

**Proof**. By Cell decomposition theorem, in a neighborhood of $a$, $X$ is a finite union of sets of the forms $\Gamma(\xi)$ or $(\xi_1, \xi_2)$, where $\xi, \xi_1, \xi_2 : C \to \mathbb{R}$ are analytic on semialgebraic submanifold $C \subset \mathbb{R}^{m-1}$, and can be continuous extended to $\overline{C}$.
By induction on $n$, the curve $\gamma$ in $X$ is constructed by lifting the curve in $C$ by $\xi_i$.

Percisely, put $a = (a', a_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, by induction, there exists a continuous curve $\varphi : [0, \epsilon) \to \mathbb{R}^{n-1}$, such that $\varphi(0) = a'$, $\varphi|_{(0,\epsilon)}$ is analytic and $\varphi((0, \epsilon)) \subset C \setminus \{a'\}$. There are two cases to consider:




- Case $X = \Gamma(\xi)$ and $a \in \overline{X}$:
Define $\gamma(t) = (\varphi(t), \xi(\varphi(t)), t \in [0, \epsilon)$.
- Case $X = (\xi_1, \xi_2)$ and $a = (a', a_n) \in \overline{X}$,
$a_n = \xi_1(a') + \theta_0(\xi_2(a') - \xi_1(a'))$, where $\theta_0 \in [0, 1]$.
Define $\gamma(t) = \xi_1(\varphi(t)) + \theta(t)(\xi_2(\varphi(t)) - \xi_1(\varphi(t))), t \in [0, \epsilon)$,
where $\theta$ is an affine function satisfying $\theta(0) = \theta_0, \theta(\epsilon) = \frac{1}{2}$.

□

**Exercise.** Find examples of sets which do not have the 'curve selection' property as the conclusion of above lemma.

By the Puiseux theorem, the lemma can be made stronger.

**Theorem (Puiseaux).**

Let

$$F(t,y) = y^n + a_1(t)y^{n-1} + \cdots + a_n(t),$$

be a polynomial in $y$, where each $a_i$ is complex analytic function at $0$. Then there exists a positive integer $k$ and complex analytic functions at $0$, $h_1, \cdots, h_n$ such that

$$F(t^k, y) = (y - h_1(t)) \cdots (y - h_n(t)).$$

**Curve Selection Lemma (strong form).**

Let $X \subset \mathbb{R}^n$ be a semialgebraic subset. Let $a$ be an accumulate point of $X$. Then there exists $\gamma : (-\epsilon, \epsilon) \to \mathbb{R}^n$ which is analytic, $\gamma(0) = a$ and $\gamma((0, \epsilon)) \subset X \setminus \{a\}$.

**Proof**. In duction on $n$. We can suppose $a = 0$, and we only need to prove the case where $X$ is a semialgebraic subset of the form $\Gamma(\xi)$ given in the above lemma.

By the lemma about the boundaries of semialgebraic subsets, there exists

$$f(x,y) = y^d + a_1(x)y^{d-1} + \cdots + a_0(x) \in \mathbb{R}[x][y],$$

where $x \in \mathbb{R}^{n-1}, \overline{C} \subset \mathbb{R}^{n-1}$, such that $f(x,\xi(x)) = 0$ for all $x \in \overline{C}$.
By induction, there exists a curve $x : (-\epsilon, \epsilon) \to \mathbb{R}^{n-1}$ which is analytic, $x(0) = 0$ and $x((0,\epsilon)) \subset C \setminus \{0\}$.
Put $F(t,y) = y^d + a_1(x(t))y^{d-1} + \cdots + a_0(x(t))$.
Then $F \in \mathcal{O}((-\epsilon,\epsilon))[y]$, and $F(t,\xi(x(t)) = 0$, when $t \in [0,\epsilon)$.
Extending $F$ to complex space ($t, y \in \mathbb{C}$), by Puiseux's theorem, shrinking $\epsilon$ if needed, we get $k \in \mathbb{N}$ such that:

$$F(t^k, y) = \prod_{i=1}^{d}(y - h_i(t)), \text{ where } h_i\text{'s analytic on the disk } |t| < \sqrt[k]{\epsilon}.$$

Since $F(t^k, \xi(x(t^k))) = 0$ when $t \in [0, \sqrt[k]{\epsilon})$ and the uniqueness of analytic functions, there exists $\nu \in \{1, \cdots, d\}$ such that $\xi(x(t^k)) = h_\nu(t),\ t \in [0, \sqrt[k]{\epsilon})$.
Then $\gamma(t) = (x(t^k), h_\nu(t)), t \in (-\sqrt[k]{\epsilon}, \sqrt[k]{\epsilon})$, is the desired curve. $\square$

**Exercise.**

1) Let $X = \{(x,y) \in \mathbb{R}^2 : x > 0,\ e^{-\frac{1}{x}} < y < e^{-\frac{1}{2x}}\ \}$.
Prove that one can choose a curve from $0$ into $X$ as Curve selection lemma in weak form, but the curve can not analytical extended to a larger interval as in the above lemma.
2) Prove that the function $\varphi(t) = t^\alpha,\ t \in [0,1]$, is continuous semialgebraic if and only if $\alpha$ is a nonnegative rational number.
3) Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be smooth nonnegative semialgebraic funtions and $f(0) = g(0) = 0$. Use Curve selection lemma to prove that there is a neighborhood of cận $0$ in which the derivatives $df(x), dg(x)$ can not be in the opposite directions but one of them vanishes.
4) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth semialgebraic function. Prove that if $a$ is either a regular point or a isolated singular point of $f$, then there exists $\varepsilon_0 > 0$ such that for any sphere $S_\varepsilon$ of center $a$ and radius $\varepsilon < \varepsilon_0$ is transversal with hypersurface $Z = \{x : f(x) = 0\}$, i.e. $\mathrm{grad}\, f(x)$ and $x - a$ are linearly independent for all $x \in Z \cap S_\epsilon$.

# 2. The Łojasiewicz inequalities

Stanisław Łojasiewicz (1926-2002)

# 2. The Łojasiewicz inequalities

Łojasiewicz inequalities have many interesting relations to various branches of mathematics: Differential Analysis, Dynamical Systems, Algebraic Geometry, Optimization, ...
In this part, we present the inequalities for the semialgebraic class. Some applications are given.

# The Hörmander-Łojasiewicz inequality

**Theorem (The Hörmander-Łojasiewicz inequality).**

Let $f, g : K \to \mathbb{R}$ be continuous semialgebraic functions on compact set $K$. Suppose $f^{-1}(0) \subset g^{-1}(0)$. Then there exist $\alpha > 0,\ C > 0$ such that

$$|f(x)| \geq C|g(x)|^{\alpha}, \quad \forall x \in K.$$

**Proof**. Since $K$ is compact and $g$ is continuous semialgebraic, $T = \{t \in \mathbb{R} : \exists x(x \in K, t = |g(x)|\}$ is a compact semialgebraic subset. Put

$$\varphi(t) = \inf\{|f(x)| : x \in K, |g(x)| = t\},\ t \in T.$$

The inequality is equivalent to: $\varphi(t) \geq Ct^{\alpha}, \forall t \in T$.
The graph$\varphi$ can be represented in the form

$$= \{(t,u) : \exists x, y \in K, t = |g(x)|, u = |f(x)|, |g(y)| = t \Rightarrow |f(y)| \geq |f(x)|\}.$$

Hence it is a compact, semialgebraic subset.

Case 1: $0$ is not a limit point of của $T$.
Then there exists $\delta > 0$: $T^* = \{t \in T : t \neq 0\} = \{t \in T : t \geq \delta\}$.
Therefore $K^* = \{x \in K : g(x) \neq 0\}$ is compact.
Since $f^{-1}(0) \subset g^{-1}(0)$, $f(x) \neq 0$ when $x \in K^*$, and hence $\min_{K^*} |f| > 0$.
This implies that for all $\alpha \geq 0$, there exists $C > 0$ such that
$\varphi(t) \geq \min_{K^*} |f| \geq C(\max_K |g|)^\alpha \geq Ct^\alpha, \forall t \in T^*$.
Hence, we get the desired inequality.
Case 2: $0$ is a limit point of $T$.
Since $f^{-1}(0) \subset g^{-1}(0)$, $\varphi(t) > 0$ when $t > 0$.
By Curve selection lemma, there exists a analytic parameterization for $\varphi$:

$$t = t(s) = as^k + o(s^k), \varphi(t(s)) = bs^l + o(s^l),$$

where $a, b \neq 0, k, l \in \mathbb{N}, k \neq 0$.
Then there exist $C_1, \delta > 0$, such that $\varphi(t) \geq C_1 |t|^{\frac{l}{k}}$ when $|t| < \delta$.
By Case 1, on compact set $\{t \in T : t \geq \delta\}$, we have $C_2 > 0$, such that $\varphi(t) \geq C_2 t^{\frac{l}{k}}$.
Therefore, put $\alpha = \dfrac{l}{k}, C = \min(C_1, C_2)$, we get $\varphi(t) \geq Ct^\alpha, \forall t \in T$. $\square$

**Exercise.**

1) Find an example to show that the compactness in the theorem is necessary.
2) In the theorem, the supposition that $f, g$ are continuous is necessary?
3) Find an example of functions $f, g$ of class $C^\infty$ which satisfy the suppositions of the theorem, but the conclusion inequality does not hold for any $C, \alpha > 0$.

More information on the exponents.

**Definition.** By the Hörmander-Łojasiewicz inequality, the infimum

$$L_K(f,g) = \inf\{\alpha : \exists C, |f(x)| \geq C|g(x)|^{\alpha}, \quad \forall x \in K\}$$

is well defined and called the Łojasiewicz exponent of $g$ with respect to $f$ on $K$.

**Theorem (Bochnak and Risler).**

Under the supposition of the Hörmander-Łojasiewicz inequality theorem, $L_K(f,g)$ is a rational number.
Moreover, there exists $C > 0$ such that

$$|f(x)| \geq C|g(x)|^{L_K(f,g)}, \quad \forall x \in K.$$

**Proof**. See, J. Bochnak and J,J. Risler, '*Sur les exposants de Łojasiewicz*', Comment. Math. Helv. 50(1975), 493-507. □

We have the following forms of the above inequality.
Let $d$ denote the Euclidean distance in $\mathbb{R}^n$.

**Łojasiewicz's inequality 1.**

Let $f : K \to \mathbb{R}$ be continuous semialgebraic functions on compact $K \subset \mathbb{R}^n$. Put $Z = \{x \in K : f(x) = 0\}$. Then there exist $\alpha, C > 0$ such that

$$|f(x)| \geq Cd(x, Z)^{\alpha}, \quad \forall x \in K.$$

**Proof**. Apply the above theorem for $g(x) = d(x, Z),\ x \in K$. □

**Exercise.** Prove that if $X \subset \mathbb{R}^n$ is semialgebraic subset, then the function $\mathbb{R}^n \ni x \mapsto d(x, X)$ is semialgebraic.

An application of Łojasiewicz's inequality 1.

• **L. Schwarz's Division Problem (1957-58)**: Let $f$ be an analytic function and $T$ be a distribution. Prove that there exists a distribution $S$: $fS = T$.

At the same time with Hörmander, Łojasiewicz (1959) proved the problem basing on stratification and an inequality of the form

$$|f(x)| \geq Cd(x, f^{-1}(0))^{\alpha}, \text{ when } x \text{ is in a compact neighborhood.}$$

See, S. Ł ojasiewicz, *Sur le problèm de la division*, Studia Math. (1959).

## Łojasiewicz's inequality 2.

Let $X, Y \subset \mathbb{R}^n$ be closed semialgebraic subsets. Then the pair $(X, Y)$ satisfies the regular separation condition: For any compact set $K$, there exists $\alpha > 0, C > 0$ such that

$$d(x, X) + d(x, Y) \;\geq\; Cd(x, X \cap Y)^{\alpha},\; \forall x \in K.$$

**Proof**. Apply the above theorem to
$f(x) = d(x, X) + d(x, Y), \;\; g(x) = d(x, X \cap Y).$ □

**Exercise.** Pove that the following subsets do not satisfy the regular separation condition

$$X = \{(x, y) \in \mathbb{R}^2 : y = e^{-\frac{1}{x}}, 0 < x \leq 1\} \cup 0,\; Y = \{(x, y) : y = 0, 0 \leq x \leq 1\}.$$

(Note that $y = e^{-\frac{1}{x}}$ is flat at $0$).

One of the applications of Łojasiewicz inequality 2 is the following 'cluing' differentiable functions theorem.
Let $X$ be a subset of $\mathbb{R}^n$. Put

$$\mathcal{E}(X) = \{ f : X \to \mathbb{R} \ : \exists \tilde{f} \in C^\infty(\mathbb{R}^n),\ \tilde{f}|_X = f \ \},$$

called the class of Whitney's fields on $X$, i.e. the class of functions on $X$ that can be extended to be of class $C^\infty$ on the whole $\mathbb{R}^n$.

**Theorem.**

The followings are equivalent:
(i) $X, Y \subset \mathbb{R}^n$ are closed subset which satisfy the regular separate condition:
$\forall x_0 \in X \cap Y, \exists U$ a neighborhood of $x_0$, $\exists \alpha, C > 0$ such that

$$d(x, X) + d(x, Y) \ \geq \ C\ d(x, X \cap Y)^\alpha, \quad \forall x \in U.$$

(ii) If $f \in \mathcal{E}(X),\ g \in \mathcal{E}(Y)$ and $f = g$ on $X \cap Y$, then there exists the extension $F \in \mathcal{E}(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ such that $F|_X = f, \quad F|_Y = g$.

**Proof**. See B. Malgrange, Ideals of Differentiable Functions, Oxford Univ. Press, 1966, Ch.I, Th.5.5. □

# The Bochnak-Łojasiewicz inequality

**Theorem (The Bochnak-Łojasiewicz inequality).**

Let $f : U \to \mathbb{R}$ be a semialgebraic function of class $C^1$ on an open subset $U \subset \mathbb{R}^m$ containg $0$ and $f(0) = 0$. Then there exist a neighborhood $V$ of $0$ and $C > 0$ such that

$$\|x\|\|\text{grad}\, f(x)\| \geq C|f(x)|, \quad \forall x \in V.$$

**Proof**. Suppose the opposition. Then, by the curve selection lemma, there is an analytic curve $\gamma : (-\varepsilon, \varepsilon) \to U$, $\gamma(0) = 0$, such that

$$(*) \qquad \frac{\|\gamma(t)\|\|\text{grad}\, f(\gamma(t))\|}{|f(\gamma(t))|} \to 0, \text{ when } t \to 0.$$

Moreover, the curve can be parametrized so that $f \circ \gamma$ is analytic.
Then $\gamma(t) = at^p + o(t^p), a \neq 0, p \geq 1, f(\gamma(t) = bt^q + o(t^q), b \neq 0, q \geq 1$.
From $|(f \circ \gamma)'(t)| = |\langle \text{grad}\, f(\gamma(t), \gamma'(t) \rangle| \leq \|\text{grad}\, f(\gamma(t)\|\gamma'(t)\|$, we have

$$\frac{\|\gamma(t)\|\|\text{grad}\, f(\gamma(t)\|}{|f(\gamma(t)|} \geq \frac{\|\gamma(t)\||(f \circ \gamma)'(t)|}{\|\gamma'(t)\||f(\gamma(t)|} = \frac{\|at^p + o(t^p)\||qbt^{q-1} + o(t^{q-1})|}{\|pat^{p-1} + o(t^{p-1})\||bt^q + o(t^q)|} \to \frac{q}{p}, \text{ when } t \to 0$$

This contradicts $(*)$. $\square$

An application of the above inequality in Singularity Theory:

• **The problem of finitely determined germs:** Let $f : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a germ of differentiable of class $C^k$. Find conditions for that there exist a homeomorphism germ $h$ such that

$$f \circ h = T_0^k f \text{ (the Taylor polynomial of degree } k \text{ of } f \text{ at } 0).$$

**Theorem (Kuipier-Kuo-Bochnak-Łojasiewicz).**

Let $f$ be a $C^k$ function on a neighborhood of $0$. Then the followings are equivalent:
(i) There exists $C > 0$ such that $\|\mathrm{grad}\, f(x)\| \geq C\|x\|^{k-1}$, when $x$ is in a neiborhood of $0$.
(ii) There exists a homeomorphism $h$ between neighborhoods of $0$ such that $f \circ h = T_0^k(f)$,

**Proof**. See:
T.C.Kuo, *On $C^0$-sufficiency of jets of potential function*, Topology 8(1969), 167-171.
J.Bochnak và S.Łojasiewicz, *A converse of the Kuipier-Kuo theorem*, Springer Lecture Notes 192 (1971), 254-261. □

# The gradient inequality

**Theorem (The gradient inequality).**

Let $f : U \to \mathbb{R}$ be a semialgebraic function of class $C^1$ on an open subset $U \subset \mathbb{R}^n$ containing $0$ and $f(0) = 0$. Then there exist a neighborhood $V$ of $0$, $C > 0$ and $0 < \rho < 1$ such that

$$\|\mathrm{grad}\, f(x)\| \geq C|f(x)|^\rho, \quad \forall x \in V.$$

**Proof**. We need the following lemma:

**Lemma.**

Under the assumptions of the theorem. Suppose that $U$ is bounded. Then for any sequence $(x_k)$ in $U \setminus f^{-1}(0)$ such that $f(x_k) \to 0$ when $k \to \infty$, we have $\dfrac{\|\mathrm{grad}\, f(x_k)\|}{|f(x_k)|} \to \infty$, when $k \to \infty$.

Indeed, contrary to the conclusion, there exists a consequence $(x_k)$ in $U \setminus f^{-1}(0)$ such that $f(x_k) \to 0$ but $\dfrac{\|\mathrm{grad}\, f(x_k)\|}{|f(x_k)|} \not\to \infty$, when $k \to \infty$.
Then, by the boundedness, $(x_k)$ has a accumulate point $\bar{x} \in \overline{U}$.
By Curve selection lemma there exists an analytic curve
$\gamma : (-\varepsilon, \varepsilon) \to U \setminus f^{-1}(0)$, $\gamma(0) = \bar{x}$, $f(\gamma(t)) \to 0$, when $t \to 0^+$,
and $\dfrac{\|\mathrm{grad}\, f(\gamma(t))\|}{|f(\gamma(t))|} < M$.
Moreover, reparametrize if needed, we can suppose that $f \circ \gamma$ ia analytic.
Then $f(\gamma(t)) = at^m + o(t^m)$, where $a \neq 0$ và $m \geq 1$.
Therefore,

$$(f \circ \gamma)'(t) = mat^{m-1} + o(t^{m-1}) = \langle \mathrm{grad}\, f(\gamma(t)), \gamma'(t) \rangle \leq \|\mathrm{grad}\, f(\gamma(t))\| \|\gamma'(t)\|.$$

This comes to contradiction:

$$M\|\gamma'(t)\| \geq \frac{|\langle \mathrm{grad}\, f(\gamma(t)), \gamma'(t) \rangle|}{|f(\gamma(t))|} \geq \frac{|(f \circ \gamma)'(t)|}{|f(\gamma(t))|} = \frac{|mat^{m-1} + o(t^{m-1})|}{|at^m + o(t^m)|} \to +$$

To prove the theorem, we consider the function

$$\varphi(t) = \inf\{\|\mathrm{grad}\, f(x)\| : x \in U, |f(x)| = t\}, \ t \geq 0 \text{ sufficiently small.}$$

Then $\varphi$ is a semialgebraic function.
By the above lemma, there exists a compact neighborhood $V$ of $0$ such that $\mathrm{grad}\, f^{-1}(0) \cap V \subset f^{-1}(0) \cap V$, and $\dfrac{\varphi(t)}{t} \to \infty$, when $t \to 0^+$.
By the proof of Theorem on the H-Ł inequality, there exist $k, l \in \mathbb{N}$, such that $\varphi(t) = O(t^{\frac{l}{k}})$, and hence $\frac{l}{k} < 1$.
Therefore, there exist $C > 0, \rho = \frac{l}{k} < 1$, such that

$$\|\mathrm{grad}\, f(x)\| \geq C|f(x)|^{\rho}, \quad \forall x \in V.$$

□

Some applications of the gradient inequality:

• **The Whitney's conjecture (1960)**: Let $f$ be an analytic function on an open subset $U \subset \mathbb{R}^n$. Then there exists an open neighborhood of $Z = f^{-1}(0)$ which is a deformation retraction of $Z$.

Łojasiewicz (1963), *Une propriété topologie des sous-ensembles analytiques réel*, Colloques Internationaux du CNRS, proved this conjecture was true, basing on the gradient inequality.

- **The gradient conjecture of Thom.**

Let $f$ be an analytic function on an open subset $U \subset \mathbb{R}^n$.
Consider the orbit of $\nabla f$, i.e. maximal curves $x(t)$ satisfying

$$x'(t) = -\nabla f(x(t)), t \in [0, \beta).$$

Łojasiewicz (1963) proved that $\beta = +\infty$ and there exists $\lim\limits_{t\to\infty} x(t) = x_0$.
Thom's conjecture (1988-99): $x(t)$ admits a tangent at $x_0$,
i.e. $\lim_{t\to\infty} \dfrac{x(t) - x_0}{\|x(t) - x_0\|}$ exists.
Kurdyka-Parusiński-Mostowski prove this conjecture is true. Their proof mainly based on the Łojasiewicz inequalities.

**Theorem (Kurdyka-Parusiński-Mostowski, 2000).**

Let $\tilde{x}(t) = \dfrac{x(t) - x_0}{\|x(t) - x_0\|}$ be the radial projection of $x(t)$ onto the unit sphere. Then $\tilde{x}$ is of finite length.
In particular $x(t)$ admits a tangent at $x_0$.

**Proof**. See K. Kurdyka, T. Mostowski and A. Parusiński, '*Proof of Gradient conjecture of R. Thom*', Ann. Math. 52 (2000), 763-792. □

# End of Lecture 3