The Lévy-Khintchine formula

Infinitely divisible cylindrical measure

Definition.
A cylindrical measure $\mu\colon \mathfrak{Z}(U,U^*) \to [0,1]$ is called infinitely divisible if for each $k \in \mathbb{N}$ there exists a cylindrical measure μ_k such that $\mu = \mu_k^{*k}$.

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Equivalent:

μ infinitely divisible $\Leftrightarrow$ for each $k \in \mathbb{N}$ there exists cyl. measure μ_k:

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Example: if $(L(t) : t \geqslant 0)$ is a cylindrical Lévy process then the cylindrical distribution of $L(1)$ is infinitely divisible.

Cylindrical Lévy measure

Definition. Let

$$\mathfrak{Z}_*(U,U^*)$$

$$:= \Big\{ \big\{ u \in U : \big(\langle u, u_1^* \rangle, \dots, \langle u, u_n^* \rangle \big) \in B \big\} : u_i^* \in U^*,\ B \in \mathfrak{B}(\mathbb{R}^n \setminus \{0\}) \Big\}.$$

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A set function $\lambda \colon \mathfrak{Z}_*(U,U^*) \to [0,\infty]$ is called a *cylindrical Lévy measure* if for all $u_1^*, \dots, u_n^* \in U^*$ the map

$$\lambda_{u_1^*,\dots,u_n^*} \colon \mathfrak{B}(\mathbb{R}^n) \to [0,\infty], \qquad \lambda_{u_1^*,\dots,u_n^*}(B) = \lambda \circ \pi^{-1}_{u_1^*,\dots,u_n^*}(B \setminus \{0\})$$

defines a Lévy measure on $\mathfrak{B}(\mathbb{R}^n)$.

Infinitely divisible and cylindrical Lévy measure

Lemma. For each infinitely divisible cylindrical measure μ there exists a cylindrical Lévy measure λ on $\mathfrak{Z}_*(U,U^*)$ such that:

$$\lambda\circ\pi^{-1}_{u_1^*,\dots,u_n^*} = \text{Lévy measure of } \mu\circ\pi^{-1}_{u_1^*,\dots,u_n^*} \quad \text{on } \mathfrak{B}(\mathbb{R}^n\setminus\{0\})$$

for all $u_1^*,\dots,u_n^*\in U^*$ and $n\in\mathbb{N}$.

Lévy-Khintchine formula

Theorem. For every infinitely divisible cylindrical measure μ there exist cylindrical measures μ_1, μ_2 such that $\mu = \mu_1 * \mu_2$ and

$$\varphi_{\mu_1}(u^*) = \exp\big(-\tfrac{1}{2}\langle Qu^*, u^*\rangle\big),$$

for a non-negative, symmetric operator $Q\colon U^* \to U$, and

$$\varphi_{\mu_2}(u^*) = \exp\left(ia(u^*) + \int_U \left(e^{i\langle u,u^*\rangle} - 1 - i\langle u, u^*\rangle\, \mathbb{1}_{B_{\mathbb{R}}}(\langle u, u^*\rangle)\right) \lambda(\mathrm{d}u)\right),$$

for a function $a\colon U^* \to \mathbb{R}$, and a cylindrical Lévy measure λ on $\mathfrak{Z}_*(U, U^*)$.

Notation. Let L be a cylindrical Lévy process. The triplet (a, Q, λ) of $L(1)$ is called cylindrical characteristics of L.

Hedgehog processes

Theorem. Let H be a Hilbert space with an orthonormal basis $(e_k)_{k\in\mathbb{N}}$ and let $(\ell_k)_{k\in\mathbb{N}}$ be a sequence of independent, real valued Lévy processes with characteristics (b_k, r_k, λ_k) for $k \in \mathbb{N}$. TFAE:

(a) For each $(\alpha_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{R})$ we have

(i) $\displaystyle\sum_{k=1}^{\infty} \mathbb{1}_{B_{\mathbb{R}}}(\alpha_k)\,|\alpha_k| \left| b_k + \int_{1<|\beta|\leqslant|\alpha_k|^{-1}} \beta\,\lambda_k(\mathrm{d}\beta) \right| < \infty;$

(ii) $(r_k)_{k\in\mathbb{N}} \in \ell^\infty(\mathbb{R});$

(iii) $\displaystyle\sum_{k=1}^{\infty} \int_{\mathbb{R}} \left(|\alpha_k \beta|^2 \wedge 1 \right) \lambda_k(\mathrm{d}\beta) < \infty.$

(b) For each $t \geqslant 0$ and $h^* \in H^*$ the sum

$$L(t)h^* := \sum_{k=1}^{\infty} \langle e_k, h^* \rangle \ell_k(t)$$

converges P-a.s.

Hedgehog processes

continues. If in this case the set $\{\varphi_{\ell_k(1)} : k \in \mathbb{N}\}$ is equicontinuous at 0, then $(L(t) : t \geqslant 0)$ defines a cylindrical Lévy process in H with cylindrical characteristics (a, Q, μ) satisfying

$$a(h^*) = \sum_{k=1}^{\infty} \langle e_k, h^* \rangle \left(b_k + \int_{\mathbb{R}} \beta \big(\mathbb{1}_{B_{\mathbb{R}}}(\langle e_k, h^* \rangle \beta) - \mathbb{1}_{B_{\mathbb{R}}}(\beta) \big) \, \lambda_k(\mathrm{d}\beta) \right),$$

$$Qh^* = \sum_{k=1}^{\infty} \langle e_k, h^* \rangle r_k e_k, \qquad h^*(\mu)(\mathrm{d}\beta) = \sum_{k=1}^{\infty} \big(\lambda_k \circ m_k(h^*)^{-1} \big) \, (\mathrm{d}\beta),$$

for each $h^* \in H^*$, where $m_k(h^*) \colon \mathbb{R} \to \mathbb{R}$ is defined by $m_k(h^*)(\beta) = \langle e_k, h^* \rangle \beta$.

Subordination

Theorem. Let W be a cylindrical Wiener process in U with covariance operator $C\colon U^* \to U$ which factorises through a Hilbert space $\mathscr{H}$ by $C = ii^*$ for the embedding $i\colon \mathscr{H} \to U$. If ℓ is an independent, real valued subordinator with characteristics $(\alpha, 0, \varrho)$ then

$$L(t)u^* := W(\ell(t))u^* \qquad \text{for all } u^* \in U^*,\ t \geqslant 0,$$

defines a cylindrical Lévy process $(L(t) : t \geqslant 0)$ with

$$\varphi_{L(t)}\colon U^* \to \mathbb{C}, \qquad \varphi_{L(t)}(u^*) = \exp\left(-t\tau\left(\tfrac{1}{2}\langle Cu^*, u^*\rangle\right)\right),$$

where $\tau(\beta) = \alpha\beta + \int_0^\infty (1 - e^{-\beta s})\,\varrho(\mathrm{d}s)$, and with cylindrical characteristics $(0, Q, \mu)$ given by $Q = \alpha C$ and $\mu = (\gamma \otimes \varrho) \circ \kappa^{-1}$, where $\kappa\colon H_C \times \mathbb{R}_+ \to U$ is defined by $\kappa(h, s) := \sqrt{s}\; ih$ and γ denotes the canonical cylindrical Gaussian measure on $\mathscr{H}$

Stochastic integration

Cylindrical semi-martingale but ...

If L is a cylindrical Lévy process then $(L(t)u^* : t \geqslant 0)$ is a Lévy process in $\mathbb{R}$ for each $u^* \in U^*$ which satisfies

$$L(t)u^* = tr_{u^*} + \int_{|\beta|\leqslant 1} \beta\, \widetilde{N}_{u^*}(t, d\beta) + \int_{|\beta|>1} \beta\, N_{u^*}(t, d\beta),$$

where $r_{u^*} \in \mathbb{R}$ and, with $\Delta L(s)u^* := L(s)u^* - L(s-)u^*$,

$$N_{u^*}(t, B) := \sum_{s\in[0,t]} \mathbb{1}_B \big(\Delta L(s)u^*\big), \qquad t \geqslant 0, B \in \mathfrak{B}(\mathbb{R}).$$

But the map $u^* \mapsto N_{u^*}(t, B)$ is not linear in general.

Approaches to stochastic integration

- Semi-martingale approach: Meyer (1962, 1967, ..), Kunita and Watanabe (1967,..), Doleans-Dade (1970), Itô (1965)...
  - integrator = martingale + bounded variation process
  - martingales → rich structure, finite expectation
  - bounded variation processes → pathwise integration

  Reversed semi-martingale approach: Protter (1986)
  - Good integrators

- Vector-valued measure approach: Métivier and Pellaumail (1980), ...
  - stochastic integral as measure in the space of random variables

- Decoupling approach: Kwapien and Woyczyński (1991)
  - decoupling inequalities and tangent processes

- Daniell integration approach: Bichteler (2002)
  - mimics Daniell integration in calculus

Stochastic integration

w.r.t. cylindrical martingales:

- M. Métivier, J. Pellaumail, 1980
- G. Kallianpur, J. Xiong, 1996
- R. Mikulevicius, B. L. Rozovskii, 1998.

w.r.t. cylindrical Lévy processes:

- A. Jakubowski, M. Riedle, 2017
- G. Bodo, M. Riedle, 2022 (work in progress)

Induced cylindrical random variables

Example: induced cylindrical random variable

Example: Let $X : \Omega \to U$ be a (genuine) random variable. Then

$$Z : U^* \to L^0_P(\Omega; \mathbb{R}), \qquad Zu^* := \langle X, u^* \rangle$$

defines a cylindrical random variable.

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defines a cylindrical random variable.

But: not for every cylindrical random variable $Z : U^* \to L^0_P(\Omega; \mathbb{R})$ there exists a classical random variable $X : \Omega \to U$ satisfying

$$Zu^* = \langle X, u^* \rangle \qquad \text{for all } u^* \in U^*.$$

Theorem: induced cylindrical random variable

Theorem: For a cylindrical random variable $Z\colon U^* \to L^0_P(\Omega;\mathbb{R})$ the following are equivalent:

(a) there exists a random variable $X\colon \Omega \to U$ such that

$$Zu^* = \langle X, u^* \rangle \qquad \text{for all } u^* \in U^*;$$

(b) the cylindrical distribution of Z extends to a Radon measure on $\mathfrak{B}(U)$.

Definition: in this case, Z is called induced by X.

Induced random variables and operator theory

Let $T\colon U \to V$ be a linear, continuous operator and $Z\colon U^* \to L^0_P(\Omega;\mathbb{R})$ a cylindrical random variable. Then

$$TZ : V^* \to L^0_P(\Omega;\mathbb{R}), \qquad (TZ)v^* := Z(T^*v^*)$$

defines a cylindrical random variable TZ on V.

Induced random variables and operator theory

Let $T\colon U \to V$ be a linear, continuous operator and $Z\colon U^* \to L_P^0(\Omega;\mathbb{R})$ a cylindrical random variable. Then

$$TZ : V^* \to L_P^0(\Omega;\mathbb{R}), \qquad (TZ)v^* := Z(T^*v^*)$$

defines a cylindrical random variable TZ on V.

Definition. The cylindrical random variable TZ is induced by a genuine random variable $X\colon \Omega \to V$ if

$$(TZ)v^* = \langle X, v^* \rangle \ P\text{-a.s. for all } v^* \in V^*.$$

Radonified increments

Let G, H be Hilbert spaces,

L be a cylindrical Lévy process on G with characteristics (a, Q, λ),

$F\colon G \to H$ be a Hilbert-Schmidt operator.

Then for each $0 \leqslant s \leqslant t$ there exists a random variable $X\colon \Omega \to H$ such

$$\big(L(t) - L(s)\big)\big(F^* h^*\big) = \langle X, h^* \rangle \qquad \text{for all } h^* \in H^*.$$

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$$\big(L(t) - L(s)\big)\big(F^* h^*\big) = \langle X, h^* \rangle \qquad \text{for all } h^* \in H^*.$$

In this case, X is infinitely divisible with genuine characteristics $(t-s)(a_F, Q_F, \lambda_F)$:

$$\langle a_F, h^* \rangle = a(F^* h^*) + \int_H \langle h, h^* \rangle \big(\mathbb{1}_{B_H}(h) - \mathbb{1}_{B_{\mathbb{R}}}(\langle h, h^* \rangle)\big)\, (\lambda \circ F^{-1})(\mathrm{d}h),$$

$$Q_F = FQF^*,$$

$$\lambda_F = \lambda \circ F^{-1} \quad \text{on } \mathfrak{B}(H \setminus \{0\}).$$

Stochastic integral: definition

Let G, H be Hilbert spaces. A deterministic, simple function is of the form

$$\psi\colon [0,T] \to L_2(G,H), \qquad \psi(t) = \sum_{i=1}^{n-1} F_i \, \mathbb{1}_{(t_i,t_{i+1}]}(t),$$

where $0 = t_1 < \cdots < t_n = T$ and $F_i \in L_2(G,H)$.

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where $0 = t_1 < \cdots < t_n = T$ and $F_i \in L_2(G,H)$. Letting

$$I(\psi)(t) := \int_0^t \psi \, dL := \sum_{i=1}^{n-1} F_i\big(L(t_{i+1} \wedge t) - L(t_i \wedge t)\big),$$

define $I(\psi) := \big(I(\psi)(t) : t \in [0,T]\big)$.

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define $I(\psi) := \big(I(\psi)(t) : t \in [0,T] \big)$.

Definition. A function $\psi\colon [0,T] \to L_2(G,H)$ is integrable if there exists a sequence (ψ_n) of deterministic simple functions such that

(1) (ψ_n) converges to ψ Lebesgue a.e.;

(2) $(I(\psi_n))$ is Cauchy in the space of H-valued semi-martinagles.

In this case: $I(\psi) := \lim_{n\to\infty} I(\psi_n)$.

Stochastic integral: modular space

A modular m is a function such that

$$M_m := \left\{ \psi \colon [0,T] \to L_2(G,H) : m(\psi) < \infty \right\}$$

is a complete linear space with norm $\|\psi\|_m := \inf \left\{ \varepsilon > 0 : m(\frac{\psi}{\varepsilon}) < \varepsilon \right\}$.

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Example (1): Bochner spaces:

$$m(\psi) = \int_0^T \|\psi(t)\|^p \, dt.$$

Example (2): Musielak-Orlicz spaces:

$$m(\psi) = \int_0^T \kappa\big(t, \psi(t)\big) \, dt,$$

for a nice function $\kappa\colon [0,T] \times L_2(G,H) \to \mathbb{R}$.

Stochastic integral: deterministic integrands

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Proposition. For a cylindrical Lévy process L on G with characteristics (a, Q, λ) and $F \colon G \to H$ Hilbert-Schmidt define

$$\langle b(F), h^* \rangle := a(F^* h^*) + \int_H \langle h, h^* \rangle \big(\mathbb{1}_{B_H}(h) - \mathbb{1}_{B_{\mathbb{R}}}(\langle h, h^* \rangle)\big) (\lambda \circ F^{-1})(\mathrm{d}h).$$

Then, for $\Psi \colon [0,T] \to L_2(G,H)$,

$$\begin{aligned} m_L(\Psi) := &\int_0^T \sup_{T \colon H \to H} \|b(T\Psi(t))\| \, \mathrm{d}t + \int_0^T \operatorname{tr}[Q\Psi^*(t)] \, \mathrm{d}t \\ &+ \int_0^T \int_H \big(\|h\|^2 \wedge 1\big)(\lambda \circ \Psi^{-1}(t))(\mathrm{d}h) + \int_0^T \big(\|\Psi(t)\|_{HS} \wedge 1\big) \, \mathrm{d}t \end{aligned}$$

defines a modular.

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is a complete linear space with norm $\|\psi\|_m := \inf \left\{\varepsilon > 0 : m(\frac{\psi}{\varepsilon}) < \varepsilon\right\}$.

Example. Let L be the canonical α-stable cylindrical process with characteristic function $\varphi_{L(1)}(h) = \exp(-\|h\|^\alpha)$ for some $\alpha \in (0,2)$. Then

$$M_{m_L} = \left\{\Psi\colon [0,T] \to L_2(G,H) : \int_0^T \|\Psi(t)\|_{HS}^\alpha \,\mathrm{d}t < \infty\right\}.$$

Stochastic integral: deterministic integrands

Proposition. Let L be a cylindrical Lévy process with modular m_L. Then for every sequence (ψ_n) of deterministic, simple functions ψ_n the following are equivalent:

(1) (ψ_n) is Cauchy in the modular space M_{m_L};

(2) $(I(\psi_n))$ is Cauchy in the space S of semi-martingales on H.

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(2) $(I(\psi_n))$ is Cauchy in the space S of semi-martingales on H.

Theorem. A function $\psi\colon [0,T] \to L_2(G,H)$ is integrable w.r.t. L if and only if $\psi \in M_{m_L}$.