Stochastic integral: random integrands

Let G, H be Hilbert spaces.. An adapted, simple process is of the form

$$\Psi\colon \Omega \times [0,T] \to L_2(G,H), \qquad \Psi(\omega,t) = \sum_{i=1}^{n-1} \left(\sum_{k=1}^{N_i} F_{i,k}\, \mathbb{1}_{A_{i,k}}(\omega) \right) \mathbb{1}_{(t_i,t_{i+1}]}(t),$$

where $0 = t_1 < \cdots < t_n = T$, $F_{i,k} \in L_2(G,H)$, and $A_{i,k} \in \mathscr{F}_{t_i}$.

Stochastic integral: random integrands

Let G, H be Hilbert spaces.. An adapted, simple process is of the form

$$\Psi \colon \Omega \times [0,T] \to L_2(G,H), \qquad \Psi(\omega,t) = \sum_{i=1}^{n-1} \left(\sum_{k=1}^{N_i} F_{i,k}\, \mathbb{1}_{A_{i,k}}(\omega) \right) \mathbb{1}_{(t_i,t_{i+1}]}(t),$$

where $0 = t_1 < \cdots < t_n = T$, $F_{i,k} \in L_2(G,H)$, and $A_{i,k} \in \mathscr{F}_{t_i}$. Letting

$$I(\Psi)(t) = \int_0^t \Psi \, dL := \sum_{i=1}^{n-1} \sum_{k=1}^{N_i} \mathbb{1}_{A_{i,k}} F_{i,k} \big(L(t_{i+1} \wedge t) - L(t_i \wedge t) \big)$$

define $I(\Psi) := \big(I(\Psi)(t) : t \in [0,T] \big)$.

Stochastic integral: random integrands

Let G, H be Hilbert spaces.. An adapted, simple process is of the form

$$\Psi\colon \Omega \times [0,T] \to L_2(G,H), \qquad \Psi(\omega,t) = \sum_{i=1}^{n-1} \left(\sum_{k=1}^{N_i} F_{i,k}\, \mathbb{1}_{A_{i,k}}(\omega) \right) \mathbb{1}_{(t_i,t_{i+1}]}(t),$$

where $0 = t_1 < \cdots < t_n = T$, $F_{i,k} \in L_2(G,H)$, and $A_{i,k} \in \mathscr{F}_{t_i}$. Letting

$$I(\Psi)(t) = \int_0^t \Psi\, dL := \sum_{i=1}^{n-1} \sum_{k=1}^{N_i} \mathbb{1}_{A_{i,k}} F_{i,k}\big(L(t_{i+1} \wedge t) - L(t_i \wedge t)\big)$$

define $I(\Psi) := \big(I(\Psi)(t) : t \in [0,T]\big)$.

Definition. A predictable, stochastic process $\Psi\colon \Omega \times [0,T] \to L_2(G,H)$ is integrable if there exists a sequence (Ψ_n) of adapted, simple processes such that

(1) (Ψ_n) converges to Ψ $P \otimes$Leb a.e.;

(2) $(I(\Psi_n))$ is Cauchy in the space of H-valued semi-martinagles.

In this case: $I(\Psi) := \lim_{n\to\infty} I(\Psi_n)$.

Stochastic integral: final result

Letting M_{m_L} be the modular space, define

$$L^0_P(\Omega; M_{m_L}) := \{X : \Omega \to M_{m_L} : \text{ measurable}\}$$

with metric $d(X,0) = E\big[\,\|X\|_{m_L} \wedge 1\big]$.

Stochastic integral: final result

Letting M_{m_L} be the modular space, define

$$L^0_P(\Omega; M_{m_L}) := \big\{ X : \Omega \to M_{m_L} : \text{ measurable} \big\}$$

with metric $d(X,0) = E\big[\, \|X\|_{m_L} \wedge 1\big]$.

Proposition. For every sequence (Ψ_n) of adapted, simple processes Ψ_n the following are equivalent:

(1) (Ψ_n) is Cauchy in $L^0_P(\Omega; M_{m_L})$;

(2) $(I(\Psi_n))$ is Cauchy in the space of semi-martingales on H.

Stochastic integral: final result

Letting M_{m_L} be the modular space, define

$$L^0_P(\Omega; M_{m_L}) := \{X : \Omega \to M_{m_L} : \text{ measurable}\}$$

with metric $d(X, 0) = E\big[\,\|X\|_{m_L} \wedge 1\big]$.

Proposition. For every sequence (Ψ_n) of adapted, simple processes Ψ_n the following are equivalent:

(1) (Ψ_n) is Cauchy in $L^0_P(\Omega; M_{m_L})$;

(2) $(I(\Psi_n))$ is Cauchy in the space of semi-martingales on H.

Theorem. A predictable, stochastic process $\Psi : \Omega \times [0, T] \to L_2(G, H)$ is integrable w.r.t. L if and only if $\Psi \in L^0_P(\Omega; M_{m_L})$.

Key ingredient: decoupled tangent sequences

Decoupling inequality for tangent sequences:

- Jakubowski (1982),
- Kwapień and Woyczyński (1988, 1992)
- Peña and Giné (1999)

Decoupled tangent sequence

Definition. Let $\big(\Omega, \mathscr{F}, P, (\mathscr{F}_n)_{n\in\mathbb{N}}\big)$ be a filtered probability space and $(X_n)_{n\in\mathbb{N}}$ an $(\mathscr{F}_n)$-adapted sequence of H-valued random variables. If $\big(\Omega', \mathscr{F}', P', ({\mathscr{F}'}_n)_{n\in\mathbb{N}}\big)$ is another filtered probability space, then a sequence $(Y_n)_{n\in\mathbb{N}}$ of H-valued random variables defined on $\big(\Omega\times\Omega', \mathscr{F}\otimes \mathscr{F}', P\otimes P', (\mathscr{F}_n\otimes{\mathscr{F}'}_n)_{n\in\mathbb{N}}\big)$ is said to be a decoupled tangent sequence to $(X_n)_{n\in\mathbb{N}}$ if

(1) for each $\omega \in \Omega$, we have that $(Y_n(\omega,\cdot))_{n\in\mathbb{N}}$ is a sequence of independent random variables on $(\Omega', \mathscr{F}', P')$;

(2) the sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ satisfy for each $n \in \mathbb{N}$ that

$$\mathscr{L}(X_n|\mathscr{F}_{n-1}\otimes{\mathscr{F}'}_{n-1}) = \mathscr{L}(Y_n|\mathscr{F}_{n-1}\otimes{\mathscr{F}'}_{n-1}) \quad P\otimes P' - \text{a.s.}$$

Here: $X_n(\omega,\omega') := X_n(\omega)$ for all $(\omega,\omega') \in \Omega\times\Omega'$.

Decoupled tangent sequence inequality

Theorem. (Kwapień and Woyczyński). Let $(X_n)_{n\in\mathbb{N}}$ an $(\mathscr{F}_n)$-adapted sequence of H-valued random variables on a probability space $\big(\Omega, \mathscr{F}, P, (\mathscr{F}_n)_{n\in\mathbb{N}}\big)$ with a decoupled tangent sequence $(Y_n)_{n\in\mathbb{N}}$ of H-valued random variables defined on $\big(\Omega\times\Omega', \mathscr{F}\otimes\mathscr{F}', P\otimes P', (\mathscr{F}_n\otimes\mathscr{F}'_n)_{n\in\mathbb{N}}\big)$ for another filtered probability space $\big(\Omega', \mathscr{F}', P', (\mathscr{F}'_n)_{n\in\mathbb{N}}\big)$.

Then there exists a constant $c>0$ (independent of (X_n) and (Y_n)) such that

$$E_P\left[\left\|\sum_{k=1}^N X_k\right\|\wedge 1\right] \leqslant c\,E_{P\otimes P'}\left[\left\|\sum_{k=1}^N Y_k\right\|\wedge 1\right]$$

for all $N\in\mathbb{N}$.

Decoupled tangent sequence for cylindrical Lévy

Lemma. Let L be a cylindrical Lévy process on G, $0 = t_0 \le \ldots \le t_N = T$ be a partition of $[0,T]$ and $\Theta_n \colon \Omega \to L_2(G,H)$ an $\mathscr{F}_{t_{n-1}}$-measurable simple random variable. By defining cylindrical random variables

$$\widetilde{L}(t) \colon G \to L^0_{P\otimes P}(\Omega \times \Omega; \mathbb{R}), \qquad \Big(\widetilde{L}(t)g\Big)(\omega,\omega') = \Big(L(t)g\Big)(\omega'),$$

it follows that $(\widetilde{L}(t) : t \ge 0)$ is a cylindrical Lévy process on G and the sequence of its Radonified increments

$$\Big(\Theta_n\big(\widetilde{L}(t_n) - \widetilde{L}(t_{n-1})\big)\Big)_{n\in\{1,\ldots,N\}}$$

defined on $\big(\Omega\times\Omega, \mathscr{F}\otimes\mathscr{F}, P\otimes P, (\mathscr{F}_{t_n}\otimes\mathscr{F}_{t_n})_{n\in\{0,\ldots,N\}}\big)$ is a decoupled tangent sequence to the sequence of Radonified increments

$$\Big(\Theta_n\big(L(t_n) - L(t_{n-1})\big)\Big)_{n\in\{1,\ldots,N\}}$$

defined on $\big(\Omega, \mathscr{F}, P, (\mathscr{F}_{t_n})_{n\in\{0,\ldots,N\}}\big)$.

Stochastic integral: final result

Letting M_{m_L} be the modular space, define

$$L^0_P(\Omega; M_{m_L}) := \{X : \Omega \to M_{m_L} : \text{ measurable}\}$$

with metric $d(X,0) = E\big[\, \|X\|_{m_L} \wedge 1\big]$.

Proposition. For every sequence (Ψ_n) of adapted, simple processes Ψ_n the following are equivalent:

(1) (Ψ_n) is Cauchy in $L^0_P(\Omega; M_{m_L})$;

(2) $(I(\Psi_n))$ is Cauchy in the space of semi-martingales on H.

Proof simplified: (1) implies $(I(\Psi_n)(t))_{n\in\mathbb{N}}$ Cauchy in $L^0_P(\Omega; H)$ for each $t \in [0, T]$.

Stochastic integral: key ingredient

Let (Ψ_n) be a sequence of adapted, simple processes in $L^0(\Omega, M_{m_L})$ of the form

$$\Psi_n(\omega,t) = \sum_{i=1}^{N_n-1} \mathbb{1}_{A_i^n}(\omega) F_i^n \, \mathbb{1}_{(t_i^n, t_{i+1}^n]}(t) \qquad \text{for } A_i^n \in \mathscr{F}_{t_i^n}, F_i^n \in L_2(G,H).$$

$$E\left[\left\| \int_0^T \Psi_n \, dL \right\| \wedge 1 \right]$$

Stochastic integral: key ingredient

Let (Ψ_n) be a sequence of adapted, simple processes in $L^0(\Omega, M_{m_L})$ of the form

$$\Psi_n(\omega,t) = \sum_{i=1}^{N_n-1} \mathbb{1}_{A_i^n}(\omega) F_i^n \, \mathbb{1}_{(t_i^n, t_{i+1}^n]}(t) \qquad \text{for } A_i^n \in \mathscr{F}_{t_i^n}, F_i^n \in L_2(G,H).$$

$$E\left[\left\|\int_0^T \Psi_n \, dL\right\| \wedge 1\right]$$
$$= \int_\Omega \left(\left\| \sum_{i=1}^{N_n-1} \mathbb{1}_{A_i^n}(\omega) F_i^n \left(L(t_{i+1}^n) - L(t_i^n) \right)(\omega) \right\| \wedge 1 \right) P(\mathrm{d}\omega)$$

Stochastic integral: key ingredient

Let (Ψ_n) be a sequence of adapted, simple processes in $L^0(\Omega, M_{m_L})$ of the form

$$\Psi_n(\omega,t) = \sum_{i=1}^{N_n-1} \mathbb{1}_{A_i^n}(\omega) F_i^n \, \mathbb{1}_{(t_i^n, t_{i+1}^n]}(t) \qquad \text{for } A_i^n \in \mathscr{F}_{t_i^n}, F_i^n \in L_2(G,H).$$

$$\begin{aligned}
&E\left[\left\| \int_0^T \Psi_n \, dL \right\| \wedge 1 \right] \\
&\quad = \int_\Omega \left(\left\| \sum_{i=1}^{N_n-1} \mathbb{1}_{A_i^n}(\omega) F_i^n \left(L(t_{i+1}^n) - L(t_i^n) \right)(\omega) \right\| \wedge 1 \right) P(\mathrm{d}\omega) \\
&\quad \leqslant c \int_{\Omega\times\Omega} \left(\left\| \sum_{i=1}^{N_n-1} \mathbb{1}_{A_i^n}(\omega) F_i^n \big(\widetilde{L}(t_{i+1}^n) - \widetilde{L}(t_i^n) \big)(\omega') \right\| \wedge 1 \right) (P\otimes P)(\mathrm{d}\omega, \mathrm{d}\omega')
\end{aligned}$$

Stochastic integral: key ingredient

Let (Ψ_n) be a sequence of adapted, simple processes in $L^0(\Omega, M_{m_L})$ of the form

$$\Psi_n(\omega,t) = \sum_{i=1}^{N_n-1} \mathbb{1}_{A_i^n}(\omega) F_i^n \, \mathbb{1}_{(t_i^n, t_{i+1}^n]}(t) \qquad \text{for } A_i^n \in \mathscr{F}_{t_i^n}, F_i^n \in L_2(G,H).$$

$$\begin{aligned}
&E\left[\left\| \int_0^T \Psi_n \, dL \right\| \wedge 1\right] \\
&\quad = \int_\Omega \left(\left\| \sum_{i=1}^{N_n-1} \mathbb{1}_{A_i^n}(\omega) F_i^n \left(L(t_{i+1}^n) - L(t_i^n) \right)(\omega) \right\| \wedge 1 \right) P(\mathrm{d}\omega) \\
&\quad \leqslant c \int_{\Omega\times\Omega} \left(\left\| \sum_{i=1}^{N_n-1} \mathbb{1}_{A_i^n}(\omega) F_i^n \big(\widetilde{L}(t_{i+1}^n) - \widetilde{L}(t_i^n)\big)(\omega') \right\| \wedge 1 \right) (P\otimes P)(\mathrm{d}\omega, \mathrm{d}\omega') \\
&\quad = c \int_\Omega E_P \left[\left\| \int_0^T \Psi_n(\omega) \, d\widetilde{L} \right\| \wedge 1 \right] P(\mathrm{d}\omega).
\end{aligned}$$

Stochastic evolution equations

Stochastic evolution equations

$$dX(t) = AX(t)\,dt + \Sigma\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in H;
- $\Sigma : G \to H$ linear and continuous; $S(t)\Sigma$ Hilbert-Schmidt
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in G.

Definition: A stochastic process $(X(t) : t \in [0,T])$ in H is called a **weak solution** if it satisfies for all $h \in \mathsf{D}(A^*)$ and $t \in [0,T]$:

$$\langle X(t), h\rangle = \langle X(0), h\rangle + \int_0^t \langle X(s), A^*h\rangle\, ds + L(s)(\Sigma^* h).$$

Stochastic evolution equations

$$dX(t) = AX(t)\,dt + \Sigma\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in H;
- $\Sigma : G \to H$ linear and continuous; $S(t)\Sigma$ Hilbert-Schmidt
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in G.

Theorem: The following are equivalent:

(a) $t \mapsto S(t)\Sigma$ is stochastically integrable;

(b) there exists a weak solution $(X(t) : t \in [0,T])$.

In this case, the weak solution is given by

$$X(t) = S(0)X(0) + \int_0^t S(t-s)\Sigma\,dL(s) \qquad \text{for all } t \in [0,T].$$

Stochastic evolution equations

$$dX(t) = AX(t)\,dt + \Sigma\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in H;
- $\Sigma : G \to H$ linear and continuous; $S(t)\Sigma$ Hilbert-Schmidt
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in G.

Example: Let L be the canonical α-stable cylindrical Lévy process with characteristic function $\varphi_{L(t)}(g^*) = e^{-t\|g^*\|^\alpha}$. Then there exists a weak solution if and only if

$$\int_0^T \|S(s)\Sigma\|_{\mathrm{HS}}^\alpha \,\mathrm{d}s < \infty.$$

that is $S(\cdot)\Sigma \in M_{m_L} = L^\alpha\big([0,T]; L_2(G,H)\big)$.

Temporal regularity

$$dX(t) = AX(t)\,dt + \Sigma\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in H;
- $\Sigma : G \to H$ linear and continuous; $S(t)\Sigma$ Hilbert-Schmidt
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in G.

Theorem: Let ν be the cylindrical Levy measure of L and $(g_k)_{k\in\mathbb{N}}$ an ONB of G. If there exists a constant $K > 0$ such that

$$\lim_{n\to\infty} \nu\left(\left\{g \in G : \sum_{k=1}^{n} \langle g, e_k\rangle^2 > K\right\}\right) = \infty,$$

then there does not exist any modification $\widetilde{X}$ of the solution X such that $\big(\langle \widetilde{X}(t), h^*\rangle : t \in [0,T]\big)$ has càdlàg paths for each $h^* \in H^*$.

Temporal regularity

$$dX(t) = AX(t)\,dt + \Sigma\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in H;
- $\Sigma : G \to H$ linear and continuous; $S(t)\Sigma$ Hilbert-Schmidt
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in G.

Example: (Peszat, Zabczyk, Imkeller,....Liu, Zhai)
Let $(L(t) : t \geqslant 0)$ be of the form

$$L(t)g = \sum_{k=1}^{\infty} \langle e_k, g\rangle \sigma_k h_k \qquad \text{for all } g \in G,$$

where h_k are real-valued, α-stable processes and $(\sigma_k) \in \ell^{(2\alpha)/(2-\alpha)} \setminus \ell^{\alpha}$.
Then the solution does not have a **càdlàg modification**.

Temporal regularity

$$dX(t) = AX(t)\,dt + \Sigma\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in H;
- $\Sigma : G \to H$ linear and continuous; $S(t)\Sigma$ Hilbert-Schmidt
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in G.

Example: (Brzezniak, Zabczyk)
Let $(L(t) : t \geqslant 0)$ be of the form

$$L(t)v^* = W(\ell(t))g \qquad \text{for all } g \in G,$$

where W is a cylindrical but not a classical Wiener process in G and ℓ a real-valued Lévy subordinator. Then the solution has not a **càdlàg** modification.

Literature

- D. Applebaum, M. Riedle, Cylindrical Lévy processes in Banach spaces, *Proc. of Lond. Math. Soc. 101, (2010), 697 - 726*.
- M. Griffiths, M. Riedle, Modelling Lévy space-time white noises, *Journal of the Lond. Math. Soc. 104, (2021), 1452 - 1474*.
- A. Jakubowski, M. Riedle, Stochastic integration with respect to cylindrical Lévy processes, *Annals of Probability*, 45, (2017), 4273-4306.
- U. Kumar, M. Riedle, The stochastic Cauchy problem driven by a cylindrical Lévy processes, *Electronic Journal of Probability*, 25, (2020).
- M. Riedle, Cylindrical Wiener processes, *Séminaire de Probabilités XLIII, Lecture Notes in Mathematics, Vol. 2006, (2011), 191 – 214*.
- M. Riedle, Infinitely divisible cylindrical measures on Banach spaces, *Studia Mathematica* 207, (2011), 235-256.
- M. Riedle, Ornstein-Uhlenbeck processes driven by cylindrical Lévy processes, *Potential Analysis*, 42, (2015), 809-838.