Approximately inner derivations

To the memory of Ola Bratteli

The first paper Ola and I wrote was on generation results pertinent to approximately inner derivations; the last we did with Derek Robinson was still on approximately inner derivations (known as being generators) addressing a difficulty surrounding them 30 years later in 2008. I will explain them and talk on how we were stuck for these years.

Aki Kishimoto at Lysebu, January 2017

1 When I first met him

It was 1977 when I met Ola Bratteli for the first time. Though he was just one year my senior, he was already an established mathematician; I was just starting to write a few papers.

This took place in Copenhagen in September. While I was there he came from Oslo and gave a talk. I do not remember what he talked on exactly; what I distinctively recall is his 'one' sounded like 'bon' to me (while he was explaining a Bratteli diagram, I think). Once I recognized it a couple of moments later, that has never happened again. But similar misunderstandings continued; anyway my hearing and speaking ability of English was, and has been, limited; so our communication greatly depended on his patience. He never failed to be kind and patient in this respect or other.

Ola was on the way back to Marseille by car; I was supposed to be there by the first of October. So he took me to Marseille, stopping at Cologne and Paris. (A couple of days in Paris delayed my arrival for a day or so.) This was the start of our friendship; after Marseille, we were going to meet in Sydney, Sendai, Trondheim, Canberra, Swansea, Oslo, Sapporo, etc. You can guess who invited us there if not by ourselves.

Allow me to indulge myself in remembering some other people I met on this occasion or before and after.

- G.A. Elliott: I met him first at Araki's office in Kyoto and he kindly played the role of a conductor from there to Copenhagen. I remember I asked him to take me to a Japanese restaurant after getting sick of steaks on Aeroflot etc. Not a good omen for my stay in Europe.

- G.K. Pedersen et al. All the members of the University of Copenhagen (except for George) and all what I experienced there were new. Very simple things were yet confusing.

- S. Sakai: He was giving a series of lectures on unbounded derivations and the resulting lecture note was incorporated into a book titled Operator algebras in dynamical systems published in 1991.

- M. Takesaki: I met him twice before in Kyoto; he was becoming a regular visitor there. A scene I recall is he and Sakai were talking on their future in USA with me listening at a coffee shop.

- H. Hasegawa: He was my physics teacher in Kyoto and was a mentor who would not stint sparing time with students. I met him at George's home in one evening and walked around downtown with him the next day.

- D.W. Robinson: Ola's desk was in his sunny carpeted office of Marseille where I met him soon after my arrival.

So 1977 is the year I met the three big names in unbouded derivations, O.B., D.W.R. and S.S. for the first time.

2 The first joint paper with Ola

Derek and Ola then were in the middle of writing the second volume of *Operator algebras and quantum statistical mechanics*. Two papers emerged from the problems they encountered in this process and shared with me. Another appeared as a joint paper with Ola which I want to pick up as the starting point of this talk (the title: *Generation of semigroups, and two-dimensional quantum lattice systems* which appeared in 1980 but written in 1977).

The chapter 3 of the first volume of Bratteli-Robinson's book is dedicated to *Groups, Semigroups, and Generators*, where the aforementioned result is already presented. (So they had not yet finished the first volume either then.) It starts with Banach space theory and presents the results pertinent to operator algebras (which were mostly obtained by Ola and Derek).

Let me be more specific. We all know the generator of a flow on a C*-algebra is a *closed derivation* with nice properties. If a closed derivation δ generates a flow let us call δ an *f-generator*. If δ or $-\delta$ generates a semiflow (or a one-parameter semigroup of injective endomorphisms but not a flow) let us call δ an *s-generator*. Let us say our main concern is determining when a closed derivation is a *generator*, an f-generator or s-generator. My impression is we somehow ignored the case of s-generators as irrelevant then.

Let A be an approximately finite-dimensional C*-algebra or AF algebra, which was of course introduced by Ola and probably considered as general enough in some physical context. Let (A_n) be an increasing sequence of finite-dimensional $*$-subalgebras of A such that $A_\infty = \bigcup_n A_n$ is dense. Let δ be a derivation from A_∞ into A, i.e., a $*$-linear map satisfying

$$\delta(xy) = \delta(x)y + x\delta(y).$$

Then there is a sequence (h_n) in A_{sa} such that

$$\delta(x) = \mathrm{ad}\, ih_n(x), \quad x \in A_n.$$

(A canonical way of defining h_n is:

$$ih_n = \sum \delta(e_{ij}^{(k)})e_{ji}^{(k)}$$

in terms of matrix units of A_n.) Hence δ is *approximately inner* (or AI) and is *well-behaved* (and closeable) as such. Note that A_∞ defined this way is unique as a $*$-algebra (as showned by Ola).

Let $Der(A_\infty, A)$ be the linear space of derivations from A_∞ into A. What Sakai showed is

Theorem 2.1 (*S. Sakai*) *Any closed derivation on A is an extension of a derivation in $Der(A_\infty, A)$ up to conjugacy.*

So considering closed derivations on A amounts to considering closed extensions of derivations in $Der(A_\infty, A)$.

There arise some questions naturally.

- For any $\delta \in Der(A_\infty, A)$ is its closure an f-generator?

  No. You may have to take a proper extension of the closure of δ to obtain an f-generator (even if it is AI). The recent result of Matsui and Sato shows there is a flow which is not AI; so A_∞ is never be a core for such a generator.

- Does every $\delta \in Der(A_\infty, A)$ extend to an f-generator?

  No (probably). There is a $\delta \in Der(A_\infty, A)$ whose closure is an s-generator but not an f-generator at least for the UHF algebra A of type 2^∞.

- Is there a $\delta \in Der(A_\infty, A)$ such that δ does not extend to a generator, an f-generator or s-generator?

  I believe everybody says Yes! But I do not know for sure.

- Are there $\delta \in Der(A_\infty, A)$ whose closure is an f-generator?

  Yes, e.g., $\delta = 0$ or bounded and many more. Anyway this is what people were working on.

Our main concern then was obtaining sufficient conditions on δ (described in terms of (h_n)) which ensure that the closure of δ is an f-generator. There is a satisfactory result for quantum spin systems due to Derek Robinson, so what we aimed at was mathematical sophistication (which we thought might give more insights).

Theorem 2.2 *(Bratteli – K) Let $\delta \in Der(A_\infty, A)$ and suppose there is a subsequence (n_k) and $h_k \in (A_{n_{k+1}})_{sa}$ and $C > 0$ such that*

$$\delta | A_{n_k} = \mathrm{ad}\, ih_k | A_{n_k}$$

and

$$\mathrm{dist}(h_k, A_{n_k}) < Ck$$

then the closure of δ is an f-generator.

Actually this was what Ola showed to me and he suggested to write a joint paper combining it with some of my small results. Later I managed to extend the above result slightly (eliminating $h_k \in A_{n_{k+1}}$ but introducing more conditions like $\mathrm{dist}(h_k, A_{n_k}) < Cn$) so I felt I could return a debt slightly.

The proof relies on a perturbation method, a rare example in this context. (Most of other results resort to constructing analytic elements.) It was formulated as a semi-flow theory on a Banach space, but applying it to derivations yields flows after all (the estimates in the proof works for both δ and $-\delta$).

There is an example where $\mathrm{dist}(h_k, A_{h_k}) = O(k^3)$ and the closure of δ is not an f-generator (applying a Hilbert space example by Palle Jorgensen to the CAR algebra). Unfortunately in this example the closure of δ is a proper s-generator (so proving it is not an f-generator). Hence this is not satisfactory for two reasons. The discrepancy between $O(k)$ and $O(k^3)$; the fact that the closure of δ is still a generator.

There are examples of δ whose closure is not an s-generator for sure. But there might be a generator extension, approximately inner or not.

Conclusion: Our present knowledge on $Der(A_\infty, A)$ is as murky as in 40 years ago! Matsui and Sato's is certainly a great result (answering Powers and Sakai's problem or conjecture in negative of whether every flow on UHF algebras are approximately inner or not) but it was shown in the way that does not seem to shed light on derivations as we tacitly expected.

Digressions: Having a friend whose father was a prime minister gave me a chance to see the consul general of Japan's modern posh residence in Marseille. Ola was invited to dinner by the consul general as well as some of his Japanese acquaintances including me on a warm evening. Ola appeared in shorts and sandals while the host was in suit. (The attire of the rest of us was in between.)

Drunk driving (while saying he could be sent to jail in Norway). I was complicit as I was dependent on him for survival totally for the first month and to a lesser degree for the rest of my one-year stay. We had dinner inevitably accompanied by a bottle of wine together downtown Marseille every weekday evening for the first month as I stayed at a bed and breakfast near Ola's apartment at Casis next to Marseille. (Less often but still quite often after I moved to an apartment in Marselle, which Ola helped me to find.) Ola's favorite are fish restaurants facing the vieux port followed by Vietnamese.

With close connection forged during Marseille Ola was my de facto mentor of both English and mathematics (following H. Hasegawa and H. Araki).

3 An example

I cannot resist the temptation to give a toy model which might give the desired examples appropriate to our first result.

Let $A = \bigotimes_{k=1}^{\infty} A_k$ with $A_k = M_2$. Let

$$\Phi = \sum_{i=1}^{3} \sigma_i \otimes \sigma_i \in M_2 \otimes M_2,$$

where σ_i's are Pauli matrices. We let Φ_k be the Φ sitting at $A_k \otimes A_{k+1}$. We define a derivation $\delta^{(\lambda)}$ on the $*$-algebra A_∞ generated by $A_k,\ k \in \mathbb{N}$

$$\delta^{(\lambda)}(x) = \sum_{k=1}^{\infty} ik^\lambda [\Phi_k, x]$$

for $\lambda \geq 0$ where there are only a finite number of non-zero terms. Then the above theorem (or direct calculations) shows that the closure of $\delta^{(\lambda)}$ is a generator if $\lambda \leq 1$. We expect if $\lambda > 1$ then $\delta^{(\lambda)}$ does not extend to a generator. (A reason for that is the propagation speed at k is proportional to k^λ, as suggested by the result of Lieb and Derek, and so the information at 0 arrives at infinity in a finite time, which indicates no time flow generated.)

We must have entertained this kind of examples but have never come to any conclusions. (Note Φ was chosen so that Φ_k and Φ_{k+1} is highly non-commutative so that any element of A_∞ is not analytic for $\delta^{(\lambda)}$ if $\lambda > 1$. If $\Phi \in M_2 \otimes M_2$ is chosen so that Φ_k's commute with each other then all $\delta^{(\lambda)}$ extends to an f-generator due to Sakai's result on commutative derivations.)

4 The last paper

The convergence of inner flows $t \mapsto \operatorname{Ad} e^{ith_n}$ to the flow $t \mapsto \alpha_t = e^{t\delta_\alpha}$ is equivalent to the graph convergence of generators: $\operatorname{ad} ih_n$ to δ_α. ($x \in A$ in the domain of δ_α if and only if there is a sequence (x_n) in A such that $x_n \to x$ and $\operatorname{ad} ih_n(x_n)$ converges, say to y; in this case $y = \delta_\alpha(x)$.) What we showed in *Approximately inner derivations, 2008* is

Theorem 4.1 (*Bratteli – Robinson – K*) *In the above situation let*

$$D = \{x \in D(\delta_\alpha) \mid \delta_\alpha(x) = \lim_n \operatorname{ad} ih_n(x)\}.$$

Under some assumption D can never contain all the elements of compact α-spectra.

The assumption required above is weak and is satisfied in a physical situation. For example: A is unital, simple, and separable and α is AI with non-zero Connes spectrum.

Hence in particular $D \neq D(\delta_\alpha)$ for any choice of (h_n).

So this is a kind of negative result. We were rather loose about which to use, pointwise limit or graph limit, when discussing AI in the early days. But of course we had to use the graph limit to keep the theory as general as possible. The above result shows we cannot make a convenient assumption on D.

We do not know if we could make D a core for δ by choosing a suitable (h_n). (There are many examples where D is a core but I have no idea how we could prove it.)

The proof: what we actually show is each non-zero α-spectral subspace (for an open subset) contains a non-trivial central sequence; which the conclusion follows easily from. (We know this does not always follow from the full Connes spectrm.)

5 In between and after

We dreamt of solving the problem of characterizing approximately inner derivations, say a characterization of such in terms of *local* properties.

Well at least I did and we sometimes discussed it in early years. But actually what we could do as well as others was something like: You assume something on $D(\delta)$ (with something extra); then you conclude that δ is a generator (satisfying some extra condition).

For example if δ is a generator then $D(\delta)$ is maximal among the domains of closed derivations and a derivation $\delta_1 : D(\delta) \to A$ is $\lambda\delta$ up to bounded derivations. Not always true but mostly.

Ola came to Sendai for a month or so while he stayed in Kyoto in 1984. (Sendai is a city prone to rattle; they were still talking about the 1978 quake then by which the building I had an office in was damaged. But the repaired one could stand the 2011 quake remarkably though many others could not.) What we did then was considering derivations defined on $D(\delta) = A_F^\gamma$ where γ is an action of G, compact or abelian, on A. This is about the last time we discussed derivations before suddenly reminded by Derek. (His stay in Kyoto produced a book entitled Derivations, Dissipations and Group Actions on C*-algebras in 1986.)

I would like to state a specific problem related to the above observations. As noted before the domain $D(\delta)$ may decide δ up to scales and bounded derivations. The problem is characterize such a Banach $*$-algebra B, which has a dynamical information on the C*-closure of B, a somewhat reminiscent of the Tomita-Takaseka theory!

The existence of non-AI flows on AF algebras may indicate this was not feasible with the knowledge we had then (and even now). Let me state formally the definition of AI.

Definition 5.1 *A derivation δ on A is approximately inner if there is a sequence (h_n) in A_{sa} satisfying: for any $x \in D(\delta)$ there is a sequence (x_n) in A such that $x_n \to x$ and $\mathrm{ad}\, ih_n(x_n) \to \delta(x)$ (in the separable case).*

The following examples of AI derivations are generators.

- If a derivation is bounded (or defined everywhere) then it is AI.
- If δ generates a flow which is universally weakly inner then it is AI (due to George Elliott).
- If δ generates a flow which is almost AF then it is AI. (Almost AF: There is an increasing sequence (A_n) of finite-dimensional C*-subalgebras of A such that $\bigcup_n A_n$ is dense and
$$\sup_{t\in[0,1]} \mathrm{dist}(A_n, \alpha_t(A_n)) \to 0$$
as $n \to \infty$ where dist is the distance defined by Erik Christensen.) As a matter of fact the proof is show that δ is just a bounded perturbation of an AF generator in this case (or $\alpha_t(A_n) = A_n$ follows after a bounded perturbation).

As the above examples indicate we succeed to show that δ is AI only if we impose a much stronger condition.

I think we do not know if all closed derivation on the compact operators are AI or not, let alone universally weakly inner derivations on any A (i.e., derivations whose adjoint on A^* is norm-densely defined). This at least is the first problem we have to crack. (If δ is a closed derivation on $\mathcal{K}$ then $\delta|D(\delta)\cap F$ is AI where F is the finite-rank operators.)

Except for the above three examples, if we know δ is AI then that is because we are given (h_n) such that δ is defined in terms of $\mathrm{ad}\, ih_n$. If we start with (h_n) we can modify (h_n) only in two obvious ways as far as I know.

- Replace (h_n) by $(h_n + b_n)$ where (b_n) in A_{sa} satisfies $\|b_n\| \to 0$ or (b_n) is a central sequence more generally.

- Replace (h_n) by $u_n h_n u_n^*$ where (u_n) is a central sequence of unitaries in A.

There are two types of conditions for closed derivations to be generators. One for the range of $\mathrm{id} \pm \delta$, $\mathrm{Ran}(\mathrm{id} \pm \delta) = A$, and the other for the norm estimates $\|(\mathrm{id} \pm \delta)(x)\| \geq \|x\|$. I collect some definitions or conditions for closed derivations which I thought might be relevant to the latter and were introduced at various stages of development. (I do not know any other conditions related to the first one; except the one that $D(\delta)$ has a nice approximate identity in case A has no unit.)

- δ is well-behaved; $\phi\delta(x) = 0$ for any $x \in D(\delta)_{sa}$ and some state ϕ with $|\phi(x)| = \|x\|$. (The terminology is sue to Sakai; 'some state' can be replaced by 'any state' due to Charles Batty.)

- δ is transversal if $\phi\delta(x) = 0$ for any $x \in D(\delta)$, any $\lambda \in \mathrm{Sp}(x)$, and some state $\phi(x^k) = \lambda^k, k \in \mathbb{N}$. (I cannot replace 'some state' by 'any state' here.)

- δ is strongly transversal if the set of $x \in D(\delta)$ satisfying the above condition with 'any state' in place of 'some state' is dense in $D(\delta)$. (This condition implies transversality. I do not know if this is equivalent to it or not.)

- The 'complete' version of the above. It does not look obvious the above property extends to $\delta \otimes \mathrm{id}_n$ on $A \otimes M_n$.

- $\tau\delta = 0$ for any tracial state τ. ($\tau\delta$ is a trace on $D(\delta)$ if it is non-zero. This could be equivalent to tranversality.)

- δ is implementable if it has a faithful representation π and a self-adjoint operator H such that
$$\pi\delta(x) = i[H, \pi(x)].$$
(If we substitute 'symmetric' for self-adjoint, this would follow from being 'well-behaved'.)

- δ is approximately implementable if there are (π_n, H_n) such that $\|\pi_n(x)\| \to \|x\|$ and $H_n = H_n^*$,
$$\|\pi_n\delta(x) - i[H_n, \pi_n(x)]\| \to 0$$
for $x \in D(\delta)$. (Then actually δ is implementable.)

- δ is extendable to an f-generator (with the C*-algebra allowed to expand). (This is equivalent to the above and then δ is completely well-behaved.)

- δ is MF (i.e., embeddable into δ_β on $(\prod_{n=1}^\infty M_{k_n})_\beta/I$ where $\beta_t = \prod_{n=1}^\infty \beta_t^{(n)}$ so on). (Following the notion of MF flows, an extension of quasi-diagonal flows introduced by Derek and me.)

- δ is AI as explained above. (AI derivations on MF are MF. All the MF flows are AI or unknown.)

- δ is continuously AI if there is a continuous function $h : [0, \infty) \to A_{sa}$ such that δ is the graph limit of $\mathrm{ad}\, ih(s)$. (All the known examples of AI flows are continuously AI or unknown. A continuously AI flow is liftable.)

An easy consequence is: AI derivations and MF derivations are extendable to f-generators. More generally

Proposition 5.2 *The set of extendable closed derivations are closed under graph limit.*

The same is true for the set of well-behaved derivations etc. But we still cannot decide if the following are equal or not; the set of extendable closed derivations, the completely well-behaved closed derivations, the set of well-behaved closed derivations. (The strategy I am taking here is to first characterize a condition for δ to be extendable; later if we denote the extension by (B, δ_1) consider when $B = A$ or if not the case which π gives $\pi(B)'' = \pi(A)''$, a problem attacked by Ola and Derek et al. before.)

6 In future

I have started to think semiflows could be more important than flows for understanding physical processes if we leave the realm of equilibrium quantum statistical mechanics. After all quantum processes do not look reversible (as the universe does not). Probably measurement processes (which are irreversible) are somewhat incorporated as part of physical process without a human intervention (so semiflows may naturally arise). I also remember my physics teacher's obsession of entropy and endeavor of interpreting apparent irreversiblity from the principle of quantum mechanics. And non-type I C^*-algebras seem to allow semi-flows of injective endomorphisms. (It has as many approximately inner endomorphisms as representations if A is nuclear and separable at least.)

But I do not really know how to construct AI semi-flows on C^*-algebras except for quasi-free semi-flows on the CAR algebra. (Non-trivial AI flows can be constructed on a non-type I separable C^*-algebras. The construction is based on the existence of a certain non-trivial central sequence (h_n) and showing the analytic elements exist abundantly for the graph limit of $\mathrm{ad}\, i(h_1 + \cdots + h_n)$; so it is no good for semi-flows.) Problem: Is there a closed derivation δ such that δ generates a semiflow but $-\delta$ not?

In equilibrium quantum mechanics there is a parallel phenomenon: δ has equilibrium states but $-\delta$ not (for positive temperatures). This seems to suggest there must be a situation where a non-unital C^*-algebra admitting an unbounded trace plays a role as an observable algebra, a generalization of $\mathcal{K}$ for quantum mechanics.

The problem is I do not have any physical models! I entertain an idea of macroscopic limit or so in place of thermodynamic limit, but is there a such?

What a joyous coincidence it was Reiko and I saw Ola in a cafe hidden from the Duomo in Florence many years ago! (This was 1999; Ola and I visited Florence in 1978 too on the way to Naples.) If happened again I would ask him such a question as above casually and vaguely, which would trigger discussions somehow and likely lead us to something unexpected if it does not answer the original question; this was our modus operandi which I miss most.