COHOMOLOGY OF PROJECTIVE SYSTEMS ON RANKED POSETS

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Abstract. We consider different cohomology theories for a ranked poset with a local orientation. Under certain condidtions on the poset we show that the cohomology theories coincide.

Introduction

In spline theory we consider piecewise polynomial functions on some space, which possesses a certain degree of smoothness in the subspaces where the polynomial pieces connect. The general framwork for this set-up is a partial ordered set Λ and a projective system F of abelian groups on Λ. The partial ordered set gives the combinatorial structure of a grid or a triangulation of the underlying space. The projective system is a system of polynomial functions of bounded degree, where smoothness is encoded algebraically in the restriction of the system to the connection subspaces. The problem of determine the space of piecewise polynomial functions with the given degree of smoothness, in particular the dimension of the space, has been discussed by several authors, by various methods. See e.g. [2], [3], [4], [5], [7], [8] and [9].

The idea of using cellular (co-)homology theory to determine the dimension of the space of picewise polynomial functions is not new, it was already introduced by Billera ([**?**, B2] in the late 80ś. Our approach to the problem is to use a general theory of limits of projective systems on partial ordered sets. This theory was thoroughly explored in an early paper by Laudal [6]. In his treatise he considers mainly constant projective systems. For our purpose more general projective systems is required.

The partial ordered structures of grids or triangulations have an additional structure given by a rank function, corresponding to the geometrical dimension of the actual facet. Under certain conditions on the ranked partial ordered set, including the existence of a local orientation, we show that cellular (co-)homology of these abstract cell complexes coincide with the theory of inverse limits of projective systems developed in [6]. The correspondance between the cellular (co-)homology theory and the invers limit approach was also treated by Basak in [1], but limited to constant projective systems. The advantage of our approach is that the general theory of invers limits of projective systems provides us with usefull tools to attack more applied problems in e.g. spline theory. We will discuss the applied aspect in a forthcoming paper.

1. Ranked posets

A **ranked poset** Λ is a finite, connected poset, equipped with a strict order-preserving dimension function $d : \Lambda \to \mathbb{N}_0$. The maximum value d_Λ of the dimension function is called the **geometric dimension** of the poset, and the minimum value is called the **minimal rank** of Λ, denoted m_Λ. We say that Λ is of **pure dimension** if all maximal elements of Λ have the same dimension, and of **pure minimal rank** if all minimal elements have the same dimension. An element $\lambda \in \Lambda$ of dimension $q = d(\lambda)$ is called a **q-cell** of Λ, and the set $\Lambda_q \subset \Lambda$ of all q-cells is the **q-skeleton** of Λ. If $\lambda_1 < \lambda_2$ we say that λ_1 is a **facet** of λ_2 of codimension $d(\lambda_2) - d(\lambda_1)$. A facet $\lambda_1 < \lambda_2$ of codimension 1 is called a **face** of λ_2. For any cell λ in Λ, $\Delta\lambda$ denotes the set of faces of λ and for any subset $S \subset \Lambda$, $\Delta S = \{\tau \in \Delta\sigma \,|\, \text{for some } \sigma \in S\}$. If λ_1 is a face of λ_2, then the pair $\lambda_1 < \lambda_2$ is called a **cover** in Λ, and the set of all covers in Λ is denoted $Cov(\Lambda)$. The set of facets of codimension k is denoted $\Delta^k\lambda$. Dually, a cell that have λ as one of its faces is called a **co-face** of λ. The set of cofaces is denoted $\nabla\lambda$.

Definition 1.1. *An **abstract cell complex** is a ranked poset satisfying the following condition: For any $\lambda_1 \in \Delta^2\lambda_2$, $\Delta\lambda_2 \cap \nabla\lambda_1 = \{\tau_1, \tau_2\}$, i.e. there exists exactly two elements in $\Delta\lambda_2$, τ_1 and τ_2, such that $\lambda_1 < \tau_i < \lambda_2$, $i = 1, 2$.*

An abstract cell complex does in general not have a geometrical realization. Consequently there is no natural orientation of the cell complex. In order to define (co-)homology on an abstract cell complex we need to impose a local orientation on Λ.

Definition 1.2. *A **local orientation** of an abstract cell complex Λ is a map $\epsilon : Cov(\Lambda) \to \{\pm 1\}$ such that for $\lambda_1 \in \Delta^2\lambda_2$, with $\Delta\lambda_2 \cap \nabla\lambda_1 = \{\tau_1, \tau_2\}$ we have*

$$\epsilon_{\lambda_1<\tau_1}\epsilon_{\tau_1<\lambda_2} + \epsilon_{\lambda_1<\tau_2}\epsilon_{\tau_2<\lambda_2} = 0$$

where we use the notation $\epsilon_{\lambda<\tau} = \epsilon(\lambda < \tau)$.

Two local orientations ϵ, ϵ' are equivalent if there exists a map $\mu : \Lambda \to \{\pm 1\}$ such that $\epsilon_{\lambda<\tau} = \mu(\tau)\mu(\lambda)^{-1}\epsilon'_{\lambda<\tau}$.

In general, the existence of a local orientation is a rather strict condition on the ranked poset. Notice also that if $d(\lambda_2) = d(\lambda_1) + 2$, there may not exist any $\lambda_1 < \tau < \lambda_2$. In that case the condition of Definition 1.2 is vacuous.

An abstract cell complex Λ of pure dimension n is called **non-singular** if each $(n-1)$-cell of Λ is a face of at most two maximal cells and dually, each 1-cell of Λ has at most two vertices. A local orientation of an non-singular abstract cell complex is an **orientation** of Λ if for any $(n-1)$-cell τ with $\nabla\tau = \{\sigma_1, \sigma_2\}$, $\epsilon_{\tau<\sigma_1} = -\epsilon_{\tau<\sigma_2}$, and for each 1-cell τ with $\Delta\tau = \{\gamma_1, \gamma_2\}$, $\epsilon_{\gamma_1<\tau} = -\epsilon_{\gamma_2<\tau}$. The abstract cell complex is **orientable** if there exist an orientation.

The **dual poset** Λ^* of a poset Λ has the same objects as Λ, but with reversed ordering. We use the notation $\lambda^* \in \Lambda^*$ for the corresponding object of $\lambda \in \Lambda$ in Λ^*. The dual S^* of a subset $S \subset \Lambda$ is the set $S^* = \{\lambda^* \,|\, \lambda \in S\}$ of dual elements. If Λ is ranked, there is also a natural ranking of Λ^*; the dimension of the dual element λ^* is the codimension of λ, i.e. $d(\lambda^*) = d_\Lambda - d(\lambda)$, where d_Λ is the geometric dimension of Λ. Thus the q-skeleton of Λ corresponds to the $(d_\Lambda - q)$-skeleton of Λ^*. Notice that for $\lambda \in \Lambda_p$

$$\nabla\lambda^* = \{\sigma^* \in (\Lambda^*)_{d-p-1} \,|\, \lambda^* < \sigma^*\} = \{\sigma \in \Lambda_{p-1} \,|\, \sigma < \lambda\}^* = (\Delta\lambda)^*$$

The dual complex Λ^* has geometrical dimension $d_{\Lambda^*} = d_\Lambda - m_\Lambda$ and minimal rank $m_{\Lambda^*} = 0$. There is an isomorphism $\Lambda^{**} \simeq \Lambda$, given by a dimension shift m_Λ, with equality $\Lambda^{**} = \Lambda$ if and only if Λ has minimal rank $m_\Lambda = 0$.

The **order dimension** $r(\Lambda)$ of a finite poset Λ is the maximum length of ordered chains of elements, i.e.

$$r(\Lambda) = max\{p \,|\, \lambda_0 < \lambda_1 < \cdots < \lambda_p \in \Lambda\}$$

Since the dimension function of Λ is strict order-preserving, it follows that $d_\Lambda \geq r(\Lambda)$. For a poset Λ we denote by $\Lambda^{(1)}$ the induced **order complex** of Λ. It is a ranked poset of geometric dimension $d_{\Lambda^{(1)}} = r(\Lambda)$ and minimal rank $m_{\Lambda^{(1)}} = 0$. The p-skeleton of $\Lambda^{(1)}$ consists of sequences

$$\lambda_0 < \lambda_1 < \cdots < \lambda_p$$

and the ordering $\prec$ is by inclusion. One can show that the order complex $\Lambda^{(1)}$ has the structure of a simplicial complex, i.e. a cell complex where every q-cell, $q \geq 1$, has exactly $q+1$ faces, $\Delta\lambda = \{\tau_0, \tau_1, \ldots, \tau_q\}$, such that if $q \geq 2$, $\tau_i \cap \tau_j$ is a $(q-2)$-cell for any pair $i \neq j$. The order complex $\Lambda^{(1)}$ is equipped with the local orientation ϵ given by

$$\epsilon(\lambda_0 < \cdots \hat{\lambda}_i \cdots < \lambda_p \prec \lambda_0 < \cdots < \lambda_i < \cdots < \lambda_p) = (-1)^i$$

2. Cellular (co-)homology

Definition 2.1. *Let Λ be a poset, and let $\mathcal{A}$ be the category of abelian groups (or any abelian category). A **projective system** with values in $\mathcal{A}$ on Λ is a contravariant functor $F : \Lambda \to \mathcal{A}$, where Λ is considered as a category, i.e. for any $\lambda \in \Lambda$ we associate an object $F(\lambda)$, and for a relation $\lambda < \sigma$, a morphism $F_{\lambda<\sigma} : F(\sigma) \to F(\lambda)$ in $\mathcal{A}$.*

Let Λ be a locally oriented ranked poset of geometric dimension d_Λ, with local orientation ϵ, and let F be a projective system on Λ. We define a chain complex

$$C_p(\Lambda, F) = \coprod_{\lambda\in\Lambda_p} F(\lambda)[\lambda] \quad p \geq 0$$

with differential $\delta : C_p(\Lambda, F) \to C_{p-1}(\Lambda, F)$ given by

$$\delta(f_\sigma[\sigma]) = \sum_{\lambda\in\Delta\sigma} \epsilon_{\lambda<\sigma} F_{\lambda<\sigma}(f_\sigma)[\lambda]$$

for any $f_\sigma \in F(\sigma)$. By the local orientation property, $\delta^2 = 0$. In fact,

$$\delta^2(f_\sigma[\sigma]) = \sum_{\tau\in\Delta\sigma} \sum_{\lambda\in\Delta\tau} \epsilon_{\lambda<\tau}\epsilon_{\tau<\sigma} F_{\lambda<\sigma}(f_\sigma)[\lambda] = \sum_{\lambda\in\Delta^2\sigma} \Big(\sum_{\lambda<\tau<\sigma} \epsilon_{\lambda<\tau}\epsilon_{\tau<\sigma} \Big) F_{\lambda<\sigma}(f_\sigma)[\lambda] = 0$$

Definition 2.2. *The homology groups of the complex $(C_\bullet(\Lambda, F), \delta)$*

$$H_p(\Lambda, F) = H_p(C_*(\Lambda, F)), \quad p \geq 0$$

*are called the **cellular homology groups** of the locally oriented ranked poset Λ with coefficients in the projective system F.*

Notice that equivalent local orientations give isomorphic cellular homology groups.

Let F^* be a projective system of abelian groups on Λ^*. Define a cochain complex

$$C^p(\Lambda, F^*) = \prod_{\lambda\in\Lambda_p} F^*(\lambda^*)$$

with differential $\partial : C^p(\Lambda, F^*) \to C^{p+1}(\Lambda, F^*)$ given by

$$\partial\xi(\sigma) = \sum_{\lambda\in\Delta\sigma} \epsilon_{\lambda<\sigma} F^*_{\sigma^*<\lambda^*} \xi(\lambda)$$

Again we have $\partial^2 = 0$ by the local orientation property.

Definition 2.3. *The cohomology groups of the complex $(C^\bullet(\Lambda, F^*), \partial)$*

$$H^p(\Lambda, F^*) = H^p(C^*(\Lambda, F^*), \partial), \quad p \geq 0$$

*are called the **cellular cohomology groups** of the locally oriented ranked poset Λ with coefficients in the projective system F^*.*

A projective system F on Λ induces a projective system $F^* = \mathcal{H}om_k(F, k)$ on Λ^* by

$$F^*(\lambda^*) = Hom_k(F(\lambda), k) = F(\lambda)^*$$

and $F^*_{\sigma^*<\lambda^*} : F^*(\lambda^*) \to F^*(\sigma^*)$ given by

$$F^*_{\sigma^*<\lambda^*}(\phi(\lambda))(f_\sigma) = \phi(\lambda) \circ F_{\lambda<\sigma}(f_\sigma)$$

where $\phi \in F^*$, and $\phi(\lambda) : F(\lambda) \to k$. Thus we have

$$C^p(\Lambda, F^*) = \prod_{\lambda\in\Lambda_p} F^*(\lambda^*) = \prod_{\lambda\in\Lambda_p} Hom_k(F(\lambda), k)$$

$$= Hom_k(\coprod_{\lambda\in\Lambda_p} F(\lambda), k) = Hom_k(C_p(\Lambda, F), k) = C_p(\Lambda, F)^*$$

The differential $\partial : C^p(\Lambda, F^*) \to C^{p+1}(\Lambda, F^*)$ of the cochain complex is induced by the differential $\delta : C_{p+1}(\Lambda, F) \to C_p(\Lambda, F)$. In fact we have

$$\begin{aligned}\partial\phi(\sigma)(f_\sigma) &= \sum_{\lambda\in\Delta\sigma} \epsilon_{\lambda<\sigma} F_{\sigma^*<\lambda^*}\phi(\lambda)(f_\lambda) = \sum_{\lambda\in\Delta\sigma} \epsilon_{\lambda<\sigma}\big(\phi(\lambda)\circ F_{\lambda<\sigma}\big)(f_\lambda)\\ &= \phi\big(\sum_{\lambda\in\Delta\sigma} \epsilon_{\lambda<\sigma} F_{\lambda<\sigma}\big)(f_\lambda[\lambda]) = \phi(\delta(f_\sigma[\sigma])\end{aligned}$$

for any $\sigma \in \Lambda_{p+1}$. It follows that

$$H^q(\Lambda, F^*) \simeq H_q(\Lambda, F)^*, \quad q \geq 0$$

For the constant projective system $F = \mathbb{Z}$ on Λ we write $H_p(\Lambda, \mathbb{Z}) = H_p(\Lambda)$ (resp. $H^p(\Lambda, \mathbb{Z}^*) = H^p(\Lambda)$).

Notice also that for finite Λ and a projective system F on Λ we have an isomorphism

$$C^p(\Lambda^*, F) = \prod_{\lambda\in(\Lambda^*)_p} F(\lambda) \xrightarrow{\beta} \coprod_{\lambda\in\Lambda_{d-p}} F(\lambda)[\lambda] = C_{d-p}(\Lambda, F)$$

given by

$$\beta(\xi) = \sum_{\lambda\in\Lambda_{d-p}} \xi(\lambda^*)[\lambda]$$

It is easily seen that $\beta\partial(\xi) = \delta\beta(\xi)$. Thus we have,

Proposition 2.4. *Let Λ be a locally oriented finite poset, and F a projective system of abelian groups on Λ. Then there are natural isomorphisms*

$$H^p(\Lambda^*, F) \simeq H_{d-p}(\Lambda, F)\ , \quad p \geq 0$$

where $d = d_\Lambda$ is the geometric dimension of Λ.

3. Limits of projective systems

Let Λ be a finite poset, and $\Lambda_1 \subset \Lambda$ a closed subposet. Let $\mathcal{C}_\Lambda$ be the abelian category of projective systems on Λ with values in the category $\mathcal{A}$ of abelian groups. The category $\mathcal{C}_\Lambda$ has enough injectives and projectives. As in [6] we define a covariant functor

$$\varprojlim_{\Lambda/\Lambda_1} : \mathcal{C}_\Lambda \longrightarrow \mathcal{A}$$

as follows; for any projective system F, **the projective limit** $\varprojlim_{\Lambda/\Lambda_1} F$ is an abelian group, together with a family of group homomorphisms $\Pi_\lambda : \varprojlim_{\Lambda/\Lambda_1} F \longrightarrow F(\lambda)$. It is unique, up to isomorphism, and universal with the above property. The homomorphisms define a natural transformation $\varprojlim_{\Lambda/\Lambda_1} F \longrightarrow F$, of the constant projective system $\varprojlim_{\Lambda/\Lambda_1} F$ into F. For all $\lambda \in \Lambda_1$, we have $\Pi_\lambda = 0$. One can show that the limit exist and that the functor is left exact. If $\Lambda_1 = \emptyset$, we set $\varprojlim_{\Lambda/\emptyset} = \varprojlim_{\Lambda}$.

Denote by $\mathbb{Z}_{\Lambda/\Lambda_1}$ the projective system on Λ given by $\mathbb{Z}_{\Lambda/\Lambda_1}(\lambda) = \mathbb{Z}$ if $\lambda \notin \Lambda_1$ and $\mathbb{Z}_{\Lambda/\Lambda_1}(\lambda) = 0$ if $\lambda \in \Lambda_1$. Then we have [6]

$$\varprojlim_{\Lambda/\Lambda_1}{}^{(\bullet)} F = R^\bullet \varprojlim_{\Lambda/\Lambda_1} F \simeq \mathrm{Ext}^\bullet_{\mathcal{C}_\Lambda}(\mathbb{Z}_{\Lambda/\Lambda_1}, F)$$

The functor $\varprojlim_{\Lambda/\Lambda_1}$ is left-exact, thus an exact sequence

$$0 \to F'' \longrightarrow F \longrightarrow F' \to 0$$

of projective systems of abelian groups on Λ induces a long-exact sequence of abelian groups

$$\cdots \to \varprojlim_{\Lambda/\Lambda_1}{}^{(j)} F'' \longrightarrow \varprojlim_{\Lambda/\Lambda_1}{}^{(j)} F \longrightarrow \varprojlim_{\Lambda/\Lambda_1}{}^{(j)} F' \longrightarrow \varprojlim_{\Lambda/\Lambda_1}{}^{(j+1)} F'' \to \cdots$$

for $j \geq 0$.

In [6] it is shown that the complex

$$D^p(\Lambda, F) = \prod_{\lambda_0<\lambda_1<\cdots<\lambda_p\in\Lambda} F(\lambda_0)$$

with differential $\delta : D^p(\Lambda, F) \to D^{p+1}(\Lambda, F)$ given by

$$\delta\xi(\lambda_0 < \cdots < \lambda_{p+1}) = F_{\lambda_0<\lambda_1}\xi(\lambda_1 < \cdots < \lambda_{p+1}) + \sum_{i=1}^{p+1}(-1)^i\xi(\lambda_0 < \cdots \hat{\lambda}_i \cdots < \lambda_{p+1})$$

is a resolving complex for the inverse limit functor, i.e. we have

$$H^p(D^\bullet(\Lambda, F)) = \varprojlim_{\Lambda}{}^{(p)} F \ , \quad p \geq 0$$

There is also a relative version of the D-complex. Let $\Lambda_1 \subset \Lambda$ be a closed subposet. Define

$$D^p(\Lambda, \Lambda_1, F) = \{\xi \in D^p(\Lambda, F) \,|\, \xi(\lambda_0 < \lambda_1 < \cdots < \lambda_p) = 0 \,\text{if}\, \lambda_p \in \Lambda_1\}$$

In this case we have

$$H^p(D^\bullet(\Lambda, \Lambda_1, F)) = \varprojlim_{\Lambda/\Lambda_1}{}^{(p)} F \quad p \geq 0$$

A projective system F on Λ induces a projective system $\tilde{F}$ on the dual of the order complex, $(\Lambda^{(1)})^*$, given by

$$\tilde{F}(\lambda_0 < \cdots < \lambda_p) = F(\lambda_0)$$

and

$$\tilde{F}(\lambda_0 < \cdots \hat{\lambda}_i \cdots < \lambda_p \prec \lambda_0 < \cdots < \lambda_p) = \begin{cases} F_{\lambda_0<\lambda_1} & \text{if } i = 0 \\ Id_{F(\lambda_0)} & \text{if } i \neq 0 \end{cases}$$

By similarity of the definitions we have $D^p(\Lambda, F) = C^p(\Lambda^{(1)}, \tilde{F})$. Denote by $D^p(F)$, for $p \geq 0$, the projective system on Λ given by

$$D^p(F)(\lambda) = C^p(\hat{\lambda}^{(1)}, \tilde{F}) = \prod_{\lambda_0<\lambda_1<\cdots<\lambda_p\leq\lambda} F(\lambda_0)$$

where, for $\lambda < \sigma$, $D^p(F)(\sigma) \to D^p(F)(\lambda)$ is the projection. The differential $\delta : D^p(F)(\lambda) \to D^{p+1}(F)(\lambda)$ is given by

$$\delta\xi(\lambda_0 < \cdots < \lambda_{p+1} \leq \lambda) = F_{\lambda_0<\lambda_1}\xi(\lambda_1 < \cdots < \lambda_{p+1} \leq \lambda) + \sum_{i=1}^{p+1}(-1)^i\xi(\lambda_0 < \cdots \hat{\lambda}_i \cdots < \lambda_{p+1} \leq \lambda)$$

In fact, $(D^\bullet(F), \delta)$ defines a projective system of cochain complexes on Λ and we have

$$D^\bullet(\Lambda, F) = \varprojlim_{\Lambda} D^\bullet(F)$$

We define a double complex by considering the projective system $D^\bullet(F)$ as a projective system on $\Lambda^{**} \simeq \Lambda$. The double complex is given by

$$C^{p,q} = C^p(\Lambda^*, D^q(F))$$

The differential $\delta : C^{p,q} \to C^{p,q+1}$ is the differential of the complex $D^\bullet(\hat{\lambda}, F)$, for any $\lambda \in \Lambda$ and $\partial : C^{p,q} \to C^{p+1,q}$ is the differential of the cell complex. It is obvious that the two differentials commute, i.e. $\partial\delta = \delta\partial$. The total differential of the double complex $D = \partial + (-1)^p\delta$ satisfies $D^2 = 0$. It is a first quadrant double complex, and the two spectral sequences

$${}'E_2^{p,q} = H^p(\Lambda^*, \varprojlim_{\hat{\lambda}}{}^{(q)} \tilde{F})$$

and

$$''E_2^{p,q} = H^q(H^p(\Lambda^*, D^\bullet(F)))$$

both converge to the cohomology of the total complex.

Lemma 3.1. *For any $\lambda \in \Lambda$ we have*

$$\varprojlim_{\hat{\lambda}}{}^{(q)} \tilde{F} = \begin{cases} F(\lambda) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

Proof. We define a contracting cochain homotopy $s : C^{q+1}(\hat{\lambda}^{(1)}, F) \to C^q(\hat{\lambda}^{(1)}, F)$ by

$$s\xi(\lambda_0 < \cdots < \lambda_q \leq \lambda) = \begin{cases} 0 & \text{if } \lambda_q = \lambda \\ \xi(\lambda_0 < \cdots < \lambda_q < \lambda \leq \lambda) & \text{else} \end{cases}$$

For $\lambda_q \neq \lambda$, we have

$$\begin{aligned}
(\partial s - s\partial)&\xi(\lambda_0 < \cdots < \lambda_q \leq \lambda) \\
&= F_{\lambda_0<\lambda_1}(\xi(\lambda_1 < \cdots < \lambda_q < \lambda \leq \lambda)) + \sum_{i=1}^{q} (-1)^i \xi(\lambda_0 < \cdots \hat{\lambda_i} \cdots < \lambda_q < \lambda \leq \lambda) \\
&\quad - F_{\lambda_0<\lambda_1}(\xi(\lambda_1 < \cdots < \lambda_q < \lambda)) - \sum_{i=1}^{q} (-1)^i \xi(\lambda_0 < \cdots \hat{\lambda_i} \cdots < \lambda_q < \lambda \leq \lambda) \\
&\quad - (-1)^{q+1}\xi(\lambda_0 < \cdots < \lambda_q \leq \lambda) = (-1)^q \xi(\lambda_0 < \cdots < \lambda_q \leq \lambda)
\end{aligned}$$

For $\lambda_q = \lambda$ we have

$$(\partial s - s\partial)\xi(\lambda_0 < \cdots < \lambda_q \leq \lambda) = (-1)^q \xi(\lambda_0 < \cdots < \lambda_{q-1} < \lambda_q \leq \lambda)$$

□

It follows that the first spectral sequences degenerates and we have,

$$'E_2^{p,q} = H^p(\Lambda^*, \varprojlim_{\hat{\lambda}}{}^{(q)} \tilde{F}) = \begin{cases} H^p(\Lambda^*, F) & \text{for } q = 0 \\ 0 & \text{for } q \neq 0 \end{cases}$$

Denote by F^j the projective system on $(\Lambda^{(1)})^*$ given by

$$F^j(\lambda_0 < \cdots < \lambda_q) = C^j(\hat{\lambda_q^*}, F(\lambda_0))$$

where $F(\lambda_0)$ denotes the constant projective system on $\hat{\lambda_q^*}$. An inclusion of sequences $\lambda_0 < \cdots \hat{\lambda_i} \cdots < \lambda_q \prec \lambda_0 < \cdots < \lambda_q$ gives rise to a map

$$F^j(\lambda_0 < \cdots \hat{\lambda_i} \cdots < \lambda_q) \longrightarrow F^j(\lambda_0 < \cdots < \lambda_q)$$

which is the projection for $i = q$, the map $F(\lambda_1) \to F(\lambda_0)$ for $i = 0$, and the identity for $0 < i < q$. Let $\xi \in C^p(\Lambda^*, D^q(F))$. For a q-cell $\lambda_0 < \cdots < \lambda_q$ of $\Lambda^{(1)}$ define $\overline{\xi}_{[\lambda_0,\ldots,\lambda_q]} \in C^p(\hat{\lambda_q^*}, F(\lambda_0))$ by

$$\overline{\xi}_{[\lambda_0,\ldots,\lambda_q]}(\lambda^*) = \xi(\lambda^*)(\lambda_0 < \cdots < \lambda_q \leq \lambda)$$

for $\lambda^* \in (\hat{\lambda_q^*})_p$.

Proposition 3.2. *Fix $q \geq 0$. The map $\xi \mapsto \overline{\xi}$ induces an isomorphism of cochain complexes*

$$C^p(\Lambda^*, D^q(F)) \longrightarrow \prod_{\lambda_0<\cdots<\lambda_q} C^p(\hat{\lambda_q^*}, F(\lambda_0))$$

Proof. We have an isomorphism of cochain groups

$$C^p(\Lambda^*, D^q(F)) = \prod_{\lambda^* \in (\Lambda^*)_p} \prod_{\lambda_0 < \cdots < \lambda_q < \lambda} F(\lambda_0) = \prod_{\lambda_0 < \cdots < \lambda_q} \prod_{\substack{\lambda^* < \lambda_q^* \\ \lambda^* \in (\Lambda^*)_p}} F(\lambda_0)$$
$$= \prod_{\lambda_0 < \cdots < \lambda_q} C^p(\hat{\lambda_q^*}, F(\lambda_0))$$

where $F(\lambda_0)$ is the constant projective system. For $\sigma ** \in \hat{\lambda_q^*}$, i.e. for $\lambda_q \leq \sigma$ we have

$$\overline{\partial \xi}_{[\lambda_0 \ldots \lambda_q]}(\sigma^*) = \partial \xi(\sigma^*)(\lambda_0 < \cdots < \lambda_q \leq \sigma)$$
$$= \sum_{\lambda^* \in \Delta\sigma^*} \epsilon_{\sigma<\lambda} D^q(F)_{\sigma<\lambda} \xi(\lambda^*)(\lambda_0 < \cdots < \lambda_q \leq \sigma)$$
$$= \sum_{\lambda^* \in \Delta\sigma^*} \epsilon_{\sigma<\lambda} \xi(\lambda^*)(\lambda_0 < \cdots < \lambda_q \leq \lambda)$$

and

$$\partial \overline{\xi}_{[\lambda_0 \ldots \lambda_q]}(\sigma^*) = \sum_{\lambda^* \in \Delta\sigma^*} \epsilon_{\sigma<\lambda} \overline{\xi}_{[\lambda_0 \ldots \lambda_q]}(\lambda^*)$$
$$= \sum_{\lambda^* \in \Delta\sigma^*} \epsilon_{\sigma<\lambda} \xi(\lambda^*)(\lambda_0 < \cdots < \lambda_q \leq \lambda)$$

which shows that the map is a map of cochains. □

Thus, for the second spectral sequence we have

$$''E_2^{p,q} = H^q\left(H^p\left(C^\bullet\left(\Lambda^*, D^\bullet\left(F\right)\right)\right)\right) = H^q(H^p(\prod_{\lambda_0 < \cdots < \lambda_\bullet} C^\bullet(\hat{\lambda_\bullet^*}, F(\lambda_0))))$$
$$= H^q\left(\prod_{\lambda_0 < \cdots < \lambda_\bullet} H^p\left(\hat{\lambda_\bullet^*}, F\left(\lambda_0\right)\right)\right) = \varprojlim_{\Lambda}{}^{(q)} H^p\left(\hat{\lambda_\bullet^*}, F\left(\lambda_0\right)\right)$$

Thus, we have proved the following theorem,

Theorem 3.3. *There is a first quadrant spectral sequence*

$$E_2^{p,q} = \varprojlim_{\Lambda}{}^{(q)} H^p(\hat{\lambda_\bullet^*}, F(\lambda_0))$$

converging to cellular cohomology $H^n(\Lambda^*, F)$

If Λ^* has acyclic closed cells, i.e.

$$H^p(\hat{\lambda_\bullet^*}, F(\lambda_0)) = \begin{cases} F(\lambda_0) & \text{for } p = 0 \\ 0 & \text{for } p \neq 0 \end{cases}$$

the spectral sequence degenerates, and we obtain the main result,

Corollary 3.4. *Let* Λ *be a locally oriented combinatorial cell complex of geometrical dimension* d*, such that* Λ^* *has acyclic closed cells, and let* F *be projective system of* A*-modules on* Λ*. Then we have*

$$H_p(\Lambda, F) = \varprojlim_{\Lambda}{}^{(d-p)} F, \quad p \geq 0$$

where d *is the geometric dimension of* Λ.

Proof. In case of acyclic closed cells, the two spectral sequences degenerate. Combining this with Proposition 2.4 the result follows. □

An important tool in computations of inverse limits is the use of κ-functors. Let Γ and Λ be two posets. A order-preserving map $\kappa : \Gamma \longrightarrow \Lambda^{(1)}$ such that $\kappa(\gamma) \subset \Lambda^{(1)}$ is closed for all $\gamma \in \Gamma$ is called a κ-functor of Γ in Λ. For a projective system F on Λ we define a projective system on Γ by

$$\gamma \mapsto \varprojlim_{\kappa(\gamma)} F$$

Denote by $im\,\kappa = \bigcup_{\gamma\in\Gamma} \kappa(\gamma) \subset \Lambda$. Then for any $\gamma \in \Gamma$ there is a canonical homomorphism

$$\varprojlim_{im\,\kappa} F \longrightarrow \varprojlim_{\kappa(\gamma)} F$$

and this family of homomorphisms define a canonical homomorphism

$$\varprojlim_{im\,\kappa} F \longrightarrow \varprojlim_{\Gamma} \varprojlim_{\kappa(\gamma)} F$$

Theorem 3.5. *Let $\kappa : \Gamma \to \Lambda^{(1)}$ be a κ-functor with $im\,\kappa = \Lambda$, and such tht if $\gamma_1, \gamma_2 \in \Gamma$, and $\lambda \in \kappa(\gamma_1) \cap \kappa(\gamma_2)$, then there exixts $\gamma < \gamma_1, \gamma_2$ such that $\lambda \in \kappa(\gamma)$. Then there is a spectral sequence*

$$E_2^{p,q} = \varprojlim_{\Gamma}{}^{(p)} \varprojlim_{\kappa(\gamma)}{}^{(q)} F$$

converging to

$$\varprojlim_{\Lambda}{}^{(\bullet)} F$$

Proof. See [6] for a proof. □

References

[1] Basak, Tathagata: *Combinatorial cell complexes and Poincaré duality*, Geom. Dedicata 147 (2010) pp: 357-387

[2] Billera, Louis J.: *Homology of smooth splines: generic triangulations and a conjecture of Strang*, Trans. Amer. Math. Soc. 310 (1988), no. 1, pp: 325340.

[3] Deng, Jiansong, Chen, Falai, Feng, YuYu: *Dimensions of spline spaces over ??-meshes*, J. Comput. Appl. Math. 194 (2006), no. 2, pp: 267283

[4] Deng, Jiansong, Chen, Falai, Jin, Liangbing: *Dimensions of biquadratic spline spaces over T-meshes*, J. Comput. Appl. Math. 238 (2013), pp: 6894.

[5] Dokken, Tor, Lyche, Tom, Pettersen, Kjell Fredrik: *Polynomial splines over locally refined box-partitions*, Comput. Aided Geom. Design 30 (2013), no. 3, pp: 331356.

[6] Laudal, Olav Arnfinn: *Sur les limites projectives et inductives*, Ann. sci. de l'École Normale Supérieure 82.2 (1965) pp: 241-296.

[7] Mourrain, Bernard: *On the dimension of spline spaces on planar T-meshes*, Math. Comp. 83 (2014), pp: 847-871

[8] Mourrain, Bernard, Villamizar, Nelly: *Homological techniques for the analysis of the dimension of triangular spline spaces* Journal of Symbolic Computation 50, (2013) pp: 564-577.

[9] Schenck, Hal: *A Spectral Sequence for Splines* Adv. in Appl. Math. 19, (1997) pp: 183-199

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