FINITE DIMENSIONAL REPRESENTATIONS OF THE PROJECTIVE MODULAR GROUP

ARNE B. SLETSJØE

ABSTRACT. We study finite dimensional representations of the projective modular group. Various explicit dimension formulas are given.

1. INTRODUCTION

1.1. **Finite-dimensional modules.** Let k be an algebraically closed field of characteristic zero. An n-dimensional left module of a k-algebra A is a vector space M of dimension n over k, equipped with a structure map $\rho \in Hom_{k\text{-}alg}(A, End_k(M))$. The (left) module M is simple if it has no non-trivial submodules, or equivalently, if the structure map ρ is surjective. A module which does not split into a direct sum of submodules is called indecomposable. To each indecomposable module M we associate a unique, up to permutations of the components, semi-simple module $\overline{M}$, given as the direct sum of the composition factors of M.

Two structure maps ρ, ρ' define equivalent modules if they are conjugates, i.e. there exists an invertible $n \times n$-matrix P such that for every $a \in A$, $\rho'(a) = P^{-1}\rho(a)P$. The set of conjugation classes of modules has in general no structure of an algebraic variety, but restricting to the semi-simple modules of dimension n there exist an algebraic quotient, denoted by $\mathcal{M}_n$. The variety $\mathcal{M}_n$ can also be considered as a quotient of the variety of all n-dimensional A-modules under the weaker equivalence relation, where two modules are equivalent if they have conjugate composition factors. The Zariski open subset of $\mathcal{M}_n$ of simple modules is denoted $Simp_n(A)$. This subvariety is independent of the choice of equivalence relation.

1.2. **The modular group.** The modular group $SL(2, \mathbf{Z})$ is generated by the two matrices

$$u = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \qquad v = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

where $\bar{u}^3 = \bar{v}^2 = I$ in the projectivised group $PSL(2, \mathbf{Z}) \simeq SL(2, \mathbf{Z})/\{\pm I\}$. There are no relations between the two generators and $PSL(2, \mathbf{Z})$ is clearly isomorphic to the free product $\mathbf{Z}_3 * \mathbf{Z}_2$.

Finite-dimensional representations of a group G correspond bijectively to finite-dimensional modules of the group algebra $k[G]$. Thus the representation theory of $PSL(2, \mathbf{Z})$ is equivalent to the representation theory of the group algebra

$$A = k[\mathbf{Z}_3 * \mathbf{Z}_2] \simeq k[\mathbf{Z}_3] * k[\mathbf{Z}_2] \simeq k\langle x, y\rangle/(x^3 - 1, y^2 - 1)$$

The algebra $k[\mathbf{Z}_p]$ is hereditary, and free products of hereditary algebras are hereditary. Thus the algebra A is hereditary.

Let M be some finite dimensional representation of the non-commutative model $B = k\langle x, y\rangle/(x^3 - y^2)$ of the ordinary cusp. The inclusion of the center $Z(B) \simeq k[x^3 = y^2] \hookrightarrow B$ induces a fibration

$$Simp_n(B) \longrightarrow \mathbf{A}^1$$

for which the fibres outide the origin are constant, isomorphic to $Simp_n(A)$.

1.3. **Low-dimensional modules.** We shall give an explicit description of the low-dimensional modules of $A = k\langle x, y\rangle/(x^3 - 1, y^2 - 1)$. For further details we refer to [12].

The variety $Simp_1(A)$ consists of 6 points, denoted $k(\omega, \alpha)$, and defined by $x \mapsto \omega, \omega^3 = 1$ and $y \mapsto \alpha = \pm 1$. We have $Ext_A^1(k(\omega, \alpha), k(\omega', \alpha')) \simeq k$ if $\omega \neq \omega'$ and $\alpha \neq \alpha'$, and zero elsewhere.

The variety of two-dimensional simple modules $Simp_2(A)$ has three disjoint components, and each component is an affine line with two distinct closed points removed. We observ that there exist non-conjugate indecomposable modules E_1 and E_2, satisfying $Ext_A^1(E_1, E_2) \neq 0$. Thus it is not possible to complete $Simp_2(A)$ to an ordinary commutative scheme, under the strong conjugation relation. To illustrate, for $s \in k$ we have a 2-dimensional A-module

$$M_s: \quad x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & \omega \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} 1 & 0 \\ s & -1 \end{pmatrix}$$

where $\omega^3 = 1$, $\omega \neq 1$. For $s \neq 0, 2(\omega - 1)$ the modules are simple, while M_0 and $M_{2(\omega-1)}$ are indecomposable, non-simple, corresponding to extensions of $k(\omega, -1)$ by $k(1, 1)$, respectively $k(\omega, 1)$ by $k(1, -1)$. The trivial extensions are the respective associated semi-simple modules. In fact all simple modules have a representation of this form.

There is yet another family, given by

$$N_t: \quad x \mapsto \begin{pmatrix} 1 & 0 \\ 1 & \omega \end{pmatrix}, \qquad y \mapsto \begin{pmatrix} 1 & t \\ 0 & -1 \end{pmatrix}$$

For $t \neq 0, 2(\omega - 1)$ the modules are simple and $N_t \simeq M_t$. As above the two exeptional modules N_0 and $N_{2(\omega-1)}$ are indecomposable and non-simple, but not isomorphic to any M_s. The module N_0 corresponds to an extension of $k(1, -1)$ by $k(\omega, 1)$, the opposite direction of M_0, and $Ext_A^1(M_0, N_0) \neq 0$.

For $n = 3$ the situation is as follows. Two matrices P and Q are given by

$$P = \begin{pmatrix} \lambda_1 & \frac{\lambda_1\lambda_3}{\lambda_2} + \lambda_2 & \lambda_2 \\ 0 & \lambda_2 & \lambda_2 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad Q = \begin{pmatrix} \lambda_3 & 0 & 0 \\ -\lambda_2 & \lambda_2 & 0 \\ \lambda_2 & -\frac{\lambda_1\lambda_3}{\lambda_2} - \lambda_2 & \lambda_1 \end{pmatrix}$$

where $(\lambda_1\lambda_2\lambda_3)^2 = 1$. The variety $Simp_3(A)$ has two components, given by $\lambda_1\lambda_2\lambda_3 = \pm 1$. For the component $\lambda_1\lambda_2\lambda_3 = 1$ we put

$$x \mapsto PQ = \begin{pmatrix} 0 & 0 & \lambda_1\lambda_2 \\ 0 & -\lambda_1\lambda_3 & \lambda_1\lambda_2 \\ \lambda_2\lambda_3 & -\frac{\lambda_1\lambda_3^2}{\lambda_2} - \lambda_2\lambda_3 & \lambda_1\lambda_3 \end{pmatrix} \qquad y \mapsto PQP = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

defining a 3-dimensional module of A. A consequence of the Cayley-Hamilton Theorem is that since $x^3 = y^2 = 1$, the images of the 9 monomials

$$1, x, y, x^2, xy, yx, x^2y, xyx, x^2yx$$

form a linear basis for $M_3(k)$ if and only if the module is simple. A computation shows that this set of matrices is linearly independent if and only if

$$(\lambda_1^3 + 1)(\lambda_2^3 + 1)(\lambda_3^3 + 1) \neq 0$$

One can show that in this case the trace ring ([3]) is generated by

$$t_{xy} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}$$
$$t_{xyx} = \lambda_1 + \lambda_2 + \lambda_3$$

For any choice of cube root λ of -1 we get the relation

$$t_{xy} - \lambda t_{xyx} + 2\lambda^2 = 0$$

corresponding to a straight line in the (t_{xy}, t_{xyx})-plane. Thus the non-simple locus is given by the three lines

$$t_{xy} - \lambda t_{xyx} + 2\lambda^2 = 0$$

for $\lambda^3 = -1$. The three intersections of the three lines correspond to semi-simple modules which are sums of three one-dimensional simple modules.

Similar analysis for $Simp_n(A)$ for $n = 4, 5$, can be found in [12].

1.4. **Main results.** In this paper we investigate the finite dimensional modules of the projective modular group $PSL(2, \mathbf{Z})$. Unfortunately the litterature is somewhat limited on this subject. We have mentioned the paper of Tuba and Wenzl ([12]). Other important papers are due to Westbury ([13]) and Le Bruyn and Adriaenssen ([9]). These papers give nice desriptions of the finite-dimensional modules, especially in low dimensions, i.e. up to dimension 5. Le Bruyn and Adriaenssens also give modules of dimension $6n$. Our aim in this paper is to give more explicit results on the varieties of modules of arbitrary dimensions.

The paper is organised as follows. In section 2 we recall a result of Wesbury [13], proving a dimension formulae for the variety of simple n-dimensional A-modules of given dimension vector $\alpha = (\alpha_1, \alpha_2, \alpha_3; \alpha_1', \alpha_2')$, denoted $Simp_\alpha(A)$. Suppose α is a Schur root, i.e. there exist a module M of dimension vector α such that $End_A(M) \simeq k$. Westburys result tells us that in that case

$$dim\, Simp_\alpha(A) = n^2 - \sum_{i=1}^{3} \alpha_i^2 - \sum_{j=1}^{2} (\alpha_j')^2 + 1$$

In section 3 we look at the locus $\mathcal{D}_\alpha$ of weak equivalence classes of non-simple modules, represented by the associated semi-simple module. Using deformation theory we prove that $\overline{E} \in D_\alpha$, corresponding to an extension of two non-isomorphic simple modules of dimension vector β and γ, is a smooth point of a component of D_α of codimension $-\langle \beta, \gamma, \rangle$.

In section 4 we show that there exist $\frac{1}{2}(n+d+1)$ distinct points of $\mathcal{D}_\alpha$ corresponding to the maximally iterated extensions in the given component. Each of these points in the weak equivalence relation corresponds to a set of strong equivalence classes. In section 5 we study this collection more carefully. The weak equivalence classes split up in several equivalence classes under the strong relation. The number of strong classes depends not only on the dimension vector α and the composition factors, but also on the order in which the iterated extensions are composed, described by the representation graph Γ. We show that the non-commutative scheme $Ind_\Gamma(A)$ can be described as a quotient of a certain vector space by a group of conjugations.

In section 6 we consider the component of $\mathcal{M}_n$ of highest dimension, and inside this the subvariety of non-simple modules. It turns out that the minimal codimension of this non-simple locus is the same as the dimension of the space of modular forms of weight $2n$.

2. The Quiver $K(3,2)$

2.1. **Quiver representation.** Let $K(3,2)$ be the quiver with vertices v_1, v_2, v_3 and w_1, w_2 and with an arrow from v_i to w_j for $i = 1, 2, 3$ and $j = 1, 2$. The variety of equivalence classes of (semi-)simple representations of $\mathbf{Z}_3 * \mathbf{Z}_2$ corresponds to the variety of equivalence classes of θ-(semi)stable representations of $K(3,2)$ ([13]), where θ is the character assigning to the dimension vector $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_1', \alpha_2')$ the dimension difference

$$\theta(\alpha) = (\alpha_1' + \alpha_2') - (\alpha_1 + \alpha_2 + \alpha_3)$$

The correspondance goes via the category $\mathcal{C}(\mathbf{Z}_3 * \mathbf{Z}_2)$, whose objects are triples (M, N, ϕ), where M is a left $k[\mathbf{Z}_3]$-module, N is a left $k[\mathbf{Z}_2]$-module and $\phi : N \to M$ is a k-linear map. To a semi-simple representation M of $\mathbf{Z}_3 * \mathbf{Z}_2$ we associate the triple $(M, M, id) \in \mathcal{C}(\mathbf{Z}_3 * \mathbf{Z}_2)$, where the $\mathbf{Z}_3$- and $\mathbf{Z}_2$-module structures of M are induced by the two inclusions

$$\mathbf{Z}_3 \hookrightarrow \mathbf{Z}_3 * \mathbf{Z}_2 \hookleftarrow \mathbf{Z}_2$$

The triple (M, M, id) is mapped to the $K(3,2)$-module $M \oplus M$ where $v_i(m, m') = (e_i m, 0)$ for an idempotent $e_i = \frac{1}{3}(1 + \omega^i x + \omega^{2i} x^2) \in k[\mathbf{Z}_3] \simeq k[x]/(x^3 - 1)$, $w_j(m, m') = (0, f_j m')$, with $f_j = \frac{1}{2}(1 + (-1)^j y) \in k[\mathbf{Z}_2] \simeq k[y]/(y^2 - 1)$ and $\phi_{ij}(m, m') = (0, f_j e_i m)$. Here $i = 1, 2, 3$, $j = 1, 2$ and ω is a primitive cube root of unity. Thus the eigenspaces of the generators x and y correspond to the components of the module $M \oplus M$ over the quiver.

Let M be an indecomposable A-module with dimension vector denoted by $\alpha = (\alpha_1, \alpha_2, \alpha_3; \alpha_1', \alpha_2')$ such that $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = \alpha_1' + \alpha_2' = n$. The dimension of $Ext_A^1(M, M)$ is given by the formula ([11])

$$dim\, Ext_A^1(M, M) = dim\, Hom_A(M, M) - \langle \alpha, \alpha \rangle$$

where

$$\langle \alpha, \beta \rangle = \sum_{i=1}^{3} \alpha_i \beta_i + \sum_{j=1}^{2} \alpha_j' \beta_j' - (\sum_{i=1}^{3} \alpha_i)(\sum_{i=1}^{3} \beta_i)$$

Thus if $End_A(M) \simeq k$, we get

$$d = dim\, Ext_A^1(M, M) = 1 + n^2 - \sum_{i=1}^{3} {\alpha_i}^2 - \sum_{j=1}^{2} (\alpha_j')^2$$

2.2. **Westburys theorem.** The following theorem is due to Westbury ([13]).

Theorem 2.1. *Let $\alpha = (\alpha_1, \alpha_2, \alpha_3; \alpha_1', \alpha_2')$ be a dimension vector of non-negative integers, satisfying*

  i) $\alpha_1 + \alpha_2 + \alpha_3 = \alpha_1' + \alpha_2' = n$
  ii) $\alpha_i + \alpha_j' \leq n$ *for all $i = 1, 2, 3$ and $j = 1, 2$*

Then the component $Simp_\alpha(A)$ of $Simp_n(A)$ corresponding to the dimension vector α is a non-empty smooth affine variety of dimension

$$d = dim\, Simp_\alpha(A) = 1 + n^2 - \sum_{i=1}^{3} {\alpha_i}^2 - \sum_{j=1}^{2} (\alpha_j')^2$$

Corollary 2.2. *The dimension of $Simp_n(A)$, i.e. the largest dimension of its components is given by*

$$dim\, Simp_n(A) = \begin{cases} 6m^2 + 2sm + s - 1 & n = 6m + s, s = 1, \ldots, 5, m \geq 0 \\ 6m^2 + 1 & n = 6m, m \geq 1 \end{cases}$$

Proof. It is easy to see that maximal dimension is obtained when the partition of n is most evenly distributed. Using this general principle the computation of the dimensions is straight-forward. $\square$

Remark 2.3. *The generating function of the sequence of maximal dimensions is given by*

$$\sum_{n=1}^{\infty} dim\,(Simp_n(A))t^n = \frac{t^2 + 2t^6 - 2t^7 + t^8}{(1-t)^2(1-t^6)}$$

3. Deformation theory

3.1. **Extensions for the modular group.** The variety $\mathcal{M}_\alpha$ of finite dimensional modules of A of given dimension vector α, up to weak equivalence, where two modules are equivalent if they have conjugate composition factors, is a coarse moduli space. The α-component of the variety of simple modules $Simp_\alpha(A) \subset \mathcal{M}_\alpha$ is a Zariski open subset of $\mathcal{M}_\alpha$. The complement in $\mathcal{M}_\alpha$ of weak equivalence classes of non-simple modules is denoted $\mathcal{D}_\alpha$. The dimension of the tangent space of $\mathcal{M}_\alpha$ at an indecomposable module M is given in section 2 by the formula for the dimension of $Ext_A^1(M, M)$. The ring A is hereditary so there are no obstructions and the coarse moduli is a smooth space of the given dimension.

Let E be an indecomposable, non-simple representation and suppose

$$0 \to W \longrightarrow E \longrightarrow V \to 0$$

is a non-split exact sequence, presenting the indecomposable module E as an extension of simple A-modules V and W. At the point $\overline{E} \in \mathcal{D}_\alpha$ the tangent space of $\mathcal{M}_\alpha$ contains directions corresponding to simple deformations as well as directions correponding to non-simple modules. To study the non-simple deformation we consider deformations of E under the extension constrain, i.e. liftings of E that preserves the exact sequence presentation. This is the topic of the next subsections.

3.2. **Deformations of presheaves.** Let $\underline{a}$ be the category of commutative pointed artinian k-algebras, i.e. objects are commutative artinian k-algebras R with a fixed quotient $\pi : R \to k$ and morphisms preserving the point.

Let $R \in obj(\underline{a})$ and let $\mathcal{F} : \Lambda \to A\text{-}\mathbf{mod}$ be a presheaf of left A-modules on a poset Λ, i.e. $\mathcal{F}$ is a contravariant functor on Λ with values in the category of left A-modules. A **lifting** of the presheaf $\mathcal{F}$ to R is a presheaf $\mathcal{F}_R$ of left $A \otimes_k R^{op}$-modules

$$\mathcal{F}_R : \Lambda \longrightarrow A \otimes_k R^{op}\text{-}\mathbf{mod}$$

together with an isomorphism of presheaves

$$\eta : \mathcal{F}_R \otimes_R k \stackrel{\cong}{\longrightarrow} \mathcal{F}$$

and such that $\mathcal{F}_R \simeq \mathcal{F} \otimes_k R$ as presheaves of right R-modules.

Two liftings $\mathcal{F}_R'$ and $\mathcal{F}_R''$ are said to be **equivalent** if there exists an isomorphism

$$\phi : \mathcal{F}_R' \longrightarrow \mathcal{F}_R''$$

of presheaves such that $\eta'' \circ (\phi \otimes id_k) = \phi \circ \eta'$. Denote by

$$Def_{\mathcal{F}}(R) = \{liftings\ of\ \mathcal{F}\ to\ R\}/\backsim$$

the set of equivalence classes of liftings of $\mathcal{F}$ to R. We shall refer to such equivalence classes as **deformations** of $\mathcal{F}$ to R. This construction is functorial, defining a covariant functor

$$Def_{\mathcal{F}} : \underline{a} \longrightarrow \underline{sets}$$

One can show that there exist an element $\psi = (\psi^2, \psi^1, \psi^0)$ representing a class of a certain cohomology group $Ext_\Lambda^2(\mathcal{F}, \mathcal{F} \otimes ker\,\pi)$ such that $[\psi] = 0$ in $Ext_\Lambda^2(\mathcal{F}, \mathcal{F} \otimes$

$ker\,\pi)$ is equivalent to the existence of a lifting of $\mathcal{F}$ to R. If $[\psi] = 0$, then the set of deformations is given as a principal homogenous space over $Ext_\Lambda^1(\mathcal{F}, \mathcal{F} \otimes ker\,\pi)$.

Applying the deformation functor to the k-algebra $R = k[x]/(x^2)$, we have $ker\,\pi \simeq k$, and the tangent space of the deformation functor $Def_\mathcal{F}$ is given by $Ext_\Lambda^1(\mathcal{F}, \mathcal{F})$. In the next section we shall study $Ext_\Lambda^i(\mathcal{F}, \mathcal{G})$ for $i \geq 0$ in more details.

3.3. **A double complex.** Let $\mathcal{F}, \mathcal{G} : \Lambda \to A\text{-}\mathbf{mod}$ be presheaves of left A-modules on a poset Λ. Define a covariant functor

$$Hom_k(\mathcal{F}, \mathcal{G}) : Mor(\Lambda) \longrightarrow A\text{-}bimod$$

by $Hom_k(\mathcal{F}, \mathcal{G})(\lambda' \leq \lambda) = Hom_k(\mathcal{F}(\lambda), \mathcal{G}(\lambda'))$. There is a double complex

$$K^{\bullet\bullet} = D^\bullet(\Lambda, C^\bullet)$$

given by

$$K^{p,q} = \prod_{\Lambda_0 \leq .. \leq \lambda_p} C^q(A, Hom_k(\mathcal{F}(\lambda_p), \mathcal{G}(\lambda_0)))$$

where $C^q(A, \mathcal{H})$ is the Hochschild cochain complex, with Hochschild differential

$$d : K^{p,q} \longrightarrow K^{p,q+1}$$

and where

$$\delta : K^{p,q} \longrightarrow K^{p+1,q}$$

is the differential of the $D^\bullet$-complex given by

$$
\begin{aligned}
\delta\psi(\lambda_0 \leq \ldots \leq \lambda_{p+1}) = & \mathcal{G}(\lambda_0 \leq \lambda_1)\psi(\lambda_1 \leq \ldots \leq \lambda_{p+1}) \\
& + \sum_{i=1}^{p} (-1)^i \psi(\lambda_0 \leq \ldots \leq \hat{\lambda}_i \leq \ldots \leq \lambda_{p+1}) \\
& + (-1)^{p+1}\psi(\lambda_0 \leq \ldots \leq \lambda_p)\mathcal{F}(\lambda_p \leq \lambda_{p+1})
\end{aligned}
$$

Let $Tot(K^{\bullet\bullet})$ be the total complex with total differential $\partial = d + (-1)^p\delta$, satisfying $\partial^2 = 0$. Define

$$Ext_\Lambda^n(\mathcal{F}, \mathcal{G}) = H^n(Tot(K^{\bullet\bullet})) \qquad n \geq 0$$

As indicated in the last section we are interested in $Ext_\Lambda^n(\mathcal{F}, \mathcal{F})$ for $n = 0, 1, 2$. Our main tool will be the spectral sequence of the next proposition. Denote by s, t the source and target functors on $Mor(\Lambda)$.

Proposition 3.1. *Let A be a k-algebra, Λ a poset and $\mathcal{F}, \mathcal{G}$ two presheaves of left A-modules on Λ. Then there exists a first quadrant spectral sequence*

$$E_2^{p,q} = \varprojlim_{Mor(\Lambda)} {}^{(p)} Ext_A^q(\mathcal{F}(t), \mathcal{G}(s))$$

converging to the total cohomology $Ext_\Lambda^{p+q}(\mathcal{F}, \mathcal{G})$.

Proof. The spectral sequence is obtained by computing cohomology of the double complex $K^{\bullet\bullet}$ with respect to the Hochschild differential and then with respect to the $D^\bullet$-differential. $\square$

3.4. **Application to the poset** $\Lambda_2 = \{0 \leq 1 \leq 2\}$**.** A short excact sequence

$$\mathcal{F}: \qquad 0 \to W \xrightarrow{\epsilon} E \xrightarrow{\pi} V \to 0$$

of left A-modules can be considered as a presheaf on Λ_2 with certain obvious properties. We put $\mathcal{F}(2) = W$, $\mathcal{F}(1) = E$, $\mathcal{F}(0) = V$, $\mathcal{F}(0 \leq 1) = \pi$ and $\mathcal{F}(1 \leq 2) = \epsilon$. The smooth moduli of deformations of the A-module E has a tangent space $\mathcal{T}_{\mathcal{M},E}$ given by $Ext_A^1(E, E)$ and the subspace $\mathcal{T}_{\mathcal{M},E}^{\mathcal{F}} \subset \mathcal{T}_{\mathcal{M},E}$ coresponding to directions in $\mathcal{T}_{\mathcal{M},E}$ subject to the extension constrain, is given by the image of the projection map

$$Ext_\Lambda^1(\mathcal{F}, \mathcal{F}) \longrightarrow Ext_A^1(E, E)$$

Denote this image by $Ext_A^1(E, E)_0$. Observe that the longest sequence of non-trivial elements of $Mor(\Lambda_2)$ has length 2, inducing vanishing of $E_2^{p,q}$ for $p \geq 3$. On the other extreme we have $E_2^{p,q} = 0$ for $q \geq 2$ since A is hereditary.

Furthermore, the higher derivatives of the invers limit functor can be computed as cohomology of the complex

$$End(\mathcal{F}(0)) \oplus End(\mathcal{F}(1)) \oplus End(\mathcal{F}(2))$$

$$\left\downarrow \begin{pmatrix} \pi^* & -\pi_* & 0 \\ (\pi \circ \epsilon)^* & 0 & (\pi \circ \epsilon)_* \\ 0 & \epsilon^* & -\epsilon_* \end{pmatrix}\right.$$

$$Hom(\mathcal{F}(1), \mathcal{F}(0)) \oplus Hom(\mathcal{F}(2), \mathcal{F}(0)) \oplus Hom(\mathcal{F}(2), \mathcal{F}(1))$$

$$\left\downarrow \begin{pmatrix} \epsilon^* & -1 & \pi_* \end{pmatrix}\right.$$

$$Hom(\mathcal{F}(2), \mathcal{F}(0))$$

We have $\pi \circ \epsilon = 0$ and in fact $ker\,\pi = im\,\epsilon$. It is obvious that the second cohomology group vanishes,

$$\varprojlim_{Mor(\Lambda_2)} {}^{(2)} Hom_A(\mathcal{F}(t), \mathcal{F}(s)) = 0$$

Similar argument proves that

$$\varprojlim_{Mor(\Lambda_2)} {}^{(2)} Ext_A^1(\mathcal{F}(t), \mathcal{F}(s)) = 0$$

Thus the spectral sequence reduces to the exact sequence

$$0 \to E_2^{1,0} \longrightarrow Ext_{\Lambda_2}^1(\mathcal{F}, \mathcal{F}) \longrightarrow E_2^{0,1} \to 0$$

and the isomorphism

$$0 \to E_2^{1,1} \longrightarrow Ext_{\Lambda_2}^2(\mathcal{F}, \mathcal{F}) \to 0$$

Proposition 3.2. *Let A be as above and let*

$$\mathcal{F}: \qquad 0 \to W \xrightarrow{\epsilon} E \xrightarrow{\pi} V \to 0$$

be a non-split short exact sequence of A-modules. Suppose $Hom_A(V, W) = Hom_A(W, V) = 0$ and that $End_A(V) \simeq End_A(W) \simeq k$ Then

$$Ext_A^1(E, E)_0 \simeq ker\{\epsilon^* \pi_* : Ext_A^1(E, E) \to Ext_A^1(W, V)\}$$

Proof. The map $Ext_{\Lambda_2}^1(\mathcal{F}, \mathcal{F}) \to Ext_A^1(E, E))$ factors through the surjective map $Ext_{\Lambda_2}^1(\mathcal{F}, \mathcal{F}) \to E_2^{0,1}$ and it is enough to consider the image of the map $E_2^{0,1} \to Ext_A^1(E, E))$. Pick $(a, b, c) \in Ext_A^1(W, W) \oplus Ext_A^1(E, E) \oplus Ext_A^1(V, V)$ such that $\epsilon_*(a) = \epsilon^*(b) \in Ext_A^1(W, E)$ and $\pi_*(b) = \pi^*(c) \in Ext_A^1(E, V)$. Thus $\pi_* \epsilon^*(b) = \epsilon^* \pi_*(b) = 0 \in Ext_A^1(W, V)$ and the image of $E_2^{0,1}$ lies in the given kernel.

On the other hand, for $b \in ker\{Ext^1_A(E,E) \to Ext^1_A(W,V)\}$, we have $\epsilon^*(b) \in ker\, \pi_*$ and $\pi_*(b) \in ker\, \epsilon^*$. By exactness there exists $a \in Ext^1_A(W,W)$ such that $\epsilon_*(a) = \epsilon^*(b)$ and $c \in Ext^1_A(V,V)$ such that $\pi^*(c) = \pi_*(b)$. This proves that $E^{0,1}_2$ maps surjectively onto the kernel. $\square$

Similarly the obstruction space for deformations under the extension constrain is given by the image of the map

$$Ext^2_{\Lambda_2}(\mathcal{F},\mathcal{F}) \longrightarrow Ext^2_A(E,E)$$

which is trivial since A is hereditary.

Notice that if $Ext^1_A(W,V) = 0$ and E is a non-trivial extension of V by W, then by the completition theorem [3] there are no simple deformations of E, all deformations of E preserve the extension structure. This fits with the fact that $Ext^1_A(E,E)_0 \simeq Ext^1_A(E,E)$ and there are no obstructions. The map $\epsilon^* \pi_*$ of Proposition 3.2 is a composition of two surjective maps, and therefore itself surjective. It follows that the moduli space of extensions is smooth of dimension

$$dim\, Ext^1_A(E,E) - dim\, Ext^1_A(W,V)$$

Proposition 3.3. *Let $\mathcal{F}$ be an non-split exact sequence as above, with dimension vector β and γ for W and V respectively. Then*

$$dim\, Ext^1_{\Lambda_2}(\mathcal{F},\mathcal{F}) = 1 - \langle \beta + \gamma, \beta + \gamma \rangle + \langle \beta, \gamma \rangle$$

Proof. Follows immedeately from the dimension formula and

$$dim\, Ext^1_{\Lambda_2}(\mathcal{F},\mathcal{F}) = dim\, Ext^1_A(E,E)_0 = dim\, Ext^1_A(E,E) - dim\, Ext^1_A(W,V)$$

$\square$

Notice that the codimension of the non-simple locus at $\mathcal{F}$ is given by $-\langle \beta, \gamma \rangle$ where $\beta + \gamma = \alpha$.

3.5. **An example.** Consider the dimension vector $\alpha = (2,2,2;3,3)$, $n = 6$. Then $dim\, Simp_\alpha(A) = 1 + 6^2 - (2^2 + 2^2 + 2^2) - (3^2 + 3^2) = 7$. The minimum value of the codimension $-\langle \beta, \gamma \rangle$ is obtained for $\beta = (2,2,1;3,2)$, $\gamma = (0,0,1;0,1)$ which gives $-\langle \beta, \gamma \rangle = 2$. This shows that $\mathcal{M}_\alpha$ in this case is a 7-dimensional smooth variety with a 5-dimensional closed subvariety of non-simple modules.

Remark 3.4. *Notice that the non-simple locus does not have to be a smooth subvariety, even if the obstruction space of deformations under the extension constrain is trivial. A non-simple module can be written as an extension in many different ways.*

4. Maximally iterated extensions

4.1. **Maximally iterated extensions.** Let M be a left A-module. The structure map $\rho : A \to End_k(M)$ of the module M is determined by two $n \times n$-matrices $X = \rho(x)$ and $Y = \rho(y)$, up to simultanous conjugation, satisfying $X^3 = Y^2 = I$. The eigenspaces of X and Y corresponding to the three cube roots and the two square roots of unity, denoted X_i, $i = 1,2,3$, resp. Y_j, $j = 1,2$, have dimensions given by the dimension vector $\alpha = (\alpha_1, \alpha_2, \alpha_3; \alpha'_1, \alpha'_2)$.

Let

$$M = M_r \supset M_{r-1} \supset \cdots \supset M_1 \supset M_0 = 0$$

be any composition series of M. We say that M is a maximally iterated extension if all factors $\overline{M}_i = M_i/M_{i-1}$ have dimension 1.

4.2. **Counting Maximally iterated extensions.** There exists a set of distinct points in $\mathcal{M}_\alpha$, corresponding to the weak classes of maximally iterated extensions of A-modules.

Proposition 4.1. *Let α be a Schur root. The set of maximally iterated extensions of dimension vector α is finite of cardinality $N = \frac{1}{2}(d + n + 1)$.*

Proof. The associated semi-simple module $\overline{M} = \oplus_{i=1}^n \overline{M}_i$ of a maximally iterated extension M can be written as a sum

$$\overline{M} = k(1,1)^{a_1} \oplus k(\omega, 1)^{a_2} \oplus k(\omega^2, 1)^{a_3}$$
$$\oplus k(1,-1)^{\alpha_1 - a_1} \oplus k(\omega, -1)^{\alpha_2 - a_2} \oplus k(\omega^2, -1)^{\alpha_3 - a_3}$$

for non-negative integers a_1, a_2, a_3, satisfying $a_i \leq \alpha_i$, $i = 1, 2, 3$. The generating function for the problem of finding all integers fullfilling this condition is given by

$$g(x) = (1 + x + x^2 + \ldots + x^{\alpha_1})(1 + x + x^2 + \ldots + x^{\alpha_2})(1 + x + x^2 + \ldots + x^{\alpha_3})$$
$$= \frac{1 - x^{\alpha_1 + 1}}{1 - x} \frac{1 - x^{\alpha_2 + 1}}{1 - x} \frac{1 - x^{\alpha_3 + 1}}{1 - x}$$
$$= (1 - x^{\alpha_1 + 1})(1 - x^{\alpha_2 + 1})(1 - x^{\alpha_3 + 1}) \frac{1}{(1 - x)^3}$$
$$= (1 - x^{\alpha_1 + 1})(1 - x^{\alpha_2 + 1})(1 - x^{\alpha_3 + 1}) \sum_{i=0}^{\infty} \binom{i + 2}{2} x^i$$

Let $m = a_1 + a_2 + a_3$. The coefficient of the x^m-term is given by

$$N = \binom{m + 2}{2} - \binom{m - \alpha_1 + 1}{2} - \binom{m - \alpha_2 + 1}{2} - \binom{m - \alpha_3 + 1}{2}$$
$$+ \binom{m - \alpha_1 - n_2}{2} + \binom{m - \alpha_1 - \alpha_3}{2} + \binom{m - \alpha_2 - \alpha_3}{2}$$
$$- \binom{m - \alpha_1 - \alpha_2 - \alpha_3}{2}$$

By definition $m = \alpha_1'$ and $n - m = \alpha_2'$. The conditions for existence of a Schur root give

$$\alpha_i + m = \alpha_i + \alpha_1' \leq \alpha_i + \alpha_j + \alpha_k \quad \Rightarrow \quad m \leq \alpha_j + \alpha_k$$

and

$$\alpha_i + \alpha_1 + \alpha_2 + \alpha_3 - m = \alpha_i + \alpha_2' \leq \alpha_i + \alpha_j + \alpha_k \quad \Rightarrow \quad m \geq \alpha_i$$

for $\{i, j, k\} = \{1, 2, 3\}$.

Thus $m - \alpha_i - \alpha_j \leq 0$ and $m - \alpha_i + 1 \geq 1$ and we get

$$N = \binom{m + 2}{2} - \binom{m - \alpha_1 + 1}{2} - \binom{m - \alpha_2 + 1}{2} - \binom{m - \alpha_3 + 1}{2}$$
$$= \frac{1}{2} \left((m^2 + 3m + 2) - (m^2 - 2m\alpha_1 + m + \alpha_1{}^2 - \alpha_1) \right.$$
$$\left. - (m^2 - 2m\alpha_2 + m + \alpha_2{}^2 - \alpha_2) - (m^2 - 2m\alpha_3 + m + \alpha_3{}^2 - \alpha_3) \right)$$
$$= \frac{1}{2} (-2m^2 + 2 + 2mn - \sum_i \alpha_i{}^2 + n)$$
$$= m(n - m) + \frac{1}{2} (n - \sum_i \alpha_i{}^2) + 1$$

Changing back to $m = \alpha'_1$, $n - m = \alpha'_2$, this takes the form

$$N = \alpha'_1 \alpha'_2 + \frac{1}{2}(n - \sum_i \alpha_i{}^2) + 1$$

But $n = \alpha'_1 + \alpha'_2$, so $\alpha'_1 \alpha'_2 = \frac{1}{2}(n^2 - \sum_j (\alpha'_j)^2)$ and we get

$$N = \frac{1}{2}(n + n^2) - \frac{1}{2}\sum_i \alpha_i{}^2 - \frac{1}{2}\sum_j (\alpha'_j)^2 + 1$$

From Theorem 2.1 we know that

$$d = \dim \mathrm{Ext}^1_A(E, E) = n^2 - \sum_i \alpha_i{}^2 - \sum_j (\alpha'_j)^2 + 1$$

showing that

$$N = \frac{1}{2}(n + n^2) - \frac{1}{2}n^2 - \frac{1}{2} + \frac{d}{2} + 1 = \frac{1}{2}(n + d + 1)$$

$\square$

Notice that a consequence of this formula is that the number $n + d$ must be odd for any dimension vector α.

5. Parametrisation of Maximally iterated extensions

5.1. **Strong vs. weak equivalence relation.** A point in the subvariety $\mathcal{D}_\alpha$ corresponds to an equivalence class of indecomposable representations, all with conjugate composition factors. Under the stronger conjugation equivalence relation this class splits up in several conjugacy classes. Since indecomposable modules deform into simples, but not vice versa, the most extreme modules are the maximally iterated extensions, where all composition factors are of dimension 1. Let $\overline{E} \in \mathcal{D}_\alpha$ be the weak equivalence class of a maximally iterated extension E. The problem is to understand how this equivalence class splits into different classes under the stronger conjugation equivalence relation.

More precisely, given a dimension vector α we choose one of the $\frac{1}{2}(n + d + 1)$ points of $\mathcal{M}_\alpha$ of the last section, corresponding to a maximally iterated extension E. The corresponding semi-simple module $\overline{E}$ defines a set of 1-dimensional simple representations, possible of multiplicity greater than 1. Let Γ be a representation graph for this set, i.e. an ordering of the given simple factors. Following Laudal ([8]) we define $Ind_\Gamma(A)$ as the non-commutative scheme of indecomposable Γ-representations. Our task is to describe this non-commutative scheme for the modular group ring.

5.2. **Some usefull lemmas.** For an upper triangular $n \times n$-matrix $Z = (z_{ij})$ we introduce the notation

$$i \underset{Z}{\sim} j$$

to indicate that the diagonal elements $z_{ii} = z_{jj}$. This is obviously an equivalence relation on the set $\{1, \ldots, n\}$. The subscript Z, referring to the matrix, will be omitted if the reference to the matrix is unambiguous.

Lemma 5.1. *Let X be an upper triangular $n \times n$-matrix on Jordan block form, satisfying the equation $X^3 = I$. Then X is diagonal.*

Proof. Suppose $X = (x_{ij})$ is on Jordan form, i.e. the three eigenvalues on the diagonal is sequentially ordered and all off-diagonal entries are zero, except possible for $x_{i,i+1}$ if $x_{ii} = x_{i+1,i+1}$. But $(X^3)_{i,i+1} = 3(x_{ii})^2 x_{i,i+1} = 0$, showing that X is diagonal. $\square$

Lemma 5.2. *Let Z be a diagonalisable upper triangular $n \times n$-matrix. Then there exists an upper triangular invertible matrix U such that $U^{-1}ZU$ is diagonal.*

Proof. Let P be an invertible $n \times n$-matrix. By the LU-decomposition Theorem $P = \Gamma L U$ where Γ is a permutation matrix, L is lower triangular and U is upper triangular. Suppose P^{-1} diagonalize Z, i.e. $PZP^{-1} = D$, D is diagonal. Then

$$UZU^{-1} = L^{-1}\Gamma^{-1}D\Gamma L$$

The left-hand side of this equation is an upper triangular matrix, while the right-hand side is lower triangular since conjugation of a diagonal matrix by a permutation matrix again is diagonal. Thus the matrix UZU^{-1} is diagonal. $\square$

The outcome of these two lemmas is that given a pair of upper triangular matrices (X, Y), where X satisfies the equation $X^3 = I$, we can always diagonalise X, without disturbing the upper triangular form of Y.

The next lemma gives an explicit form for the upper triangular matrices Y, satisfying $Y^2 = I$.

Let $\mathcal{P}(i, k)$, $i < k$ be given by

$$\{i = i_0 < i_1 < \cdots < i_m < i_{m+1} = k\} \in \mathcal{P}(i, k)$$

if m is odd and $y_{i_r i_r} = (-1)^r y_{ii}$ for $r = 0, 1, 2, \ldots, m + 1$. Define the sequence $\chi : \mathbf{Z}_+ \to \mathbf{Z}_+$ recursively by $\chi(1) = 1$ and $\chi(n) = \sum_{i=1}^{n-1} \chi(i)\chi(n - i)$.

Lemma 5.3. *Let $Y = (y_{ij})$ be an upper triangular $n \times n$-matrix, satisfying $Y^2 = I$. Then $y_{ii} = \pm 1$ and for $y_{ii} = y_{kk}$, $i < k$ we have*

$$y_{ik} = \sum_{\mathcal{P}(i,k)} (-y_{ii}) \frac{\chi(\nu)}{2^m} \prod_{s=0}^{m} y_{i_s i_{s+1}}$$

where $m = 2\nu + 1$. The entry y_{ik} can be choosen freely whenever $y_{ii} \neq y_{kk}$.

An immedeate consequence of this lemma is that the space of upper triangular $n \times n$-matrices, satisfying $Y^2 = I$ with eigenspaces of dimension α'_1, α'_2 is an affine space of dimension $\alpha'_1 \cdot \alpha'_2 = \frac{1}{2}(n^2 - (\alpha_1')^2 - (\alpha_2')^2)$.

Proof. Since Y is upper triangular and $Y^2 = I$ the diagonal entries obviously satisfy the equation $y_{ii}^2 = 1$.

The relation $Y^2 = I$ can be written as a system of $\binom{n}{2}$ quadratic equations in the $\binom{n}{2}$ off-diagonal entries y_{ik}, $1 \leq i < k \leq n$

(1) $$(y_{ii} + y_{kk})y_{ik} + \sum_{j=i+1}^{k-1} y_{ij}y_{jk} = 0$$

The proof of the lemma has two parts. First we show that the given vector (y_{ik}) is a solution of the system (1). Then we show that all solutions are of this form. We proceed by induction on the difference $\delta = k - i$. If $\delta = 1$ the defining equation reduces to

$$(y_{ii} + y_{i+1,i+1})y_{i,i+1} = 0$$

If $y_{ii} + y_{i+1,i+1} = 0$ there are no conditions on $y_{i,i+1}$. If $y_{ii} + y_{i+1,i+1} \neq 0$ it follows that $y_{i,i+1} = 0$. This fits with the given formula since in that case $\mathcal{P}(i, i + 1) = \emptyset$. Thus the lemma is true for $\delta = 1$.

Now suppose the formula is valid for $\delta < t$ and let (i, k) satisfy $k - i = t$. If $y_{ii} + y_{kk} = 0$ we have to show that

$$\sum_{j=i+1}^{k-1} y_{ij}y_{jk} = 0$$

Since $y_{ii} \neq y_{kk}$ the sum splits into two parts

$$\sum_{j=i+1}^{k-1} y_{ij}y_{jk} = \sum_{i \sim j} y_{ij}y_{jk} + \sum_{j \sim k} y_{ij}y_{jk}$$

$$= \sum_{i \sim j} \Big(\sum_{\mathcal{P}(i,j)} (-y_{ii})\frac{\chi(p)}{2^p} \prod_{s=0}^{p} y_{i_s i_{s+1}} \Big) y_{jk}$$

$$+ \sum_{j \sim k} y_{ij} \Big(\sum_{\mathcal{P}(j,k)} (-y_{jj})\frac{\chi(q)}{2^q} \prod_{s=0}^{q} y_{j_t j_{t+1}} \Big)$$

For each sequence

$$i = i_0 < i_1 < \ldots < i_{m+1} = k$$

there are two apperances of the term $y_{i_0 i_1}y_{i_1 i_2}\ldots y_{i_{m-1}k}$, one in each sum. In the first sum $j = i_m$ and in the second sum $j = i_1$. Thus the two terms add up to

$$(-y_{ii})\frac{\chi(m-1)}{2^{m-1}} \prod_{s=0}^{m-1} y_{i_s i_{s+1}}y_{jk} + y_{ij}(-y_{jj})\frac{\chi(m-1)}{2^{m-1}} \prod_{s=1}^{m} y_{j_t j_{t+1}}$$

$$= (-y_{i_0 i_0})\frac{\chi(m-1)}{2^{m-1}} \prod_{s=0}^{m-1} y_{i_s i_{s+1}}y_{i_m k} + y_{ii_1}(-y_{i_1 i_1})\frac{\chi(m-1)}{2^{m-1}} \prod_{t=1}^{m} y_{i_t i_{t+1}}$$

$$= (-y_{i_0 i_0} - y_{i_1 i_1})\frac{\chi(m-1)}{2^m} \prod_{t=0}^{m-1} y_{i_t i_{t+1}} = 0$$

For $y_{ii} = y_{kk} = \pm 1$ the equation (1) takes the form

$$y_{ik} = -\frac{1}{2}y_{ii} \sum_{j=i+1}^{k-1} y_{ij}y_{jk}$$

and we have to show that

$$-\frac{1}{2}y_{ii} \sum_{j=i+1}^{k-1} y_{ij}y_{jk} = \sum_{\mathcal{P}(i,k)} (-y_{ii})\frac{\chi(m-1)}{2^{m-1}} \prod_{s=0}^{m} y_{i_s i_{s+1}}$$

We need a lemma.

Lemma 5.4. *Let $i < j < k$ and suppose $y_{ii} = y_{kk}$. Then*

$$\Big[\bigcup_{i \sim j} (\mathcal{P}(i,j) \times \mathcal{P}(j,k)) \Big] \cup \{i < j < k \,|\, y_{ii} \neq y_{jj}\} = \mathcal{P}(i,k)$$

Proof. Let $\mathcal{I} = \{i = i_0 < i_1 < \cdots < i_p = j\} \in \mathcal{P}(i,j)$ and $\mathcal{J} = \{j = j_0 < j_1 < \cdots < j_q = k\} \in \mathcal{P}(j,k)$. Define the join

$$\mathcal{I} \cdot \mathcal{J} = \{i = i_0 < i_1 < \cdots < i_p = j_0 < j_1 < \ldots j_q = k\}$$
$$= \{i = i_0 < i_1 < \cdots < i_p < i_{p+1} < \ldots i_{p+q} = k\}$$

where we have put $i_{p+t} = j_t$ for $t = 0, 1, \ldots, q$. If p, q are even, so is $p + q$, and for $r > p$ we have

$$y_{i_r i_r} = y_{j_{r-p} j_{r-p}} = (-1)^{r-p}y_{jj} = (-1)^{r-p}y_{i_p i_p} = (-1)^{r-p+p}y_{i_0 i_0} = (-1)^r y_{ii}$$

Thus $\mathcal{I} \cdot \mathcal{J} \in \mathcal{P}(i,k)$. The set $\{i < j < k \,|\, y_{ii} \neq y_{jj}\}$ is easily seen to be in $\mathcal{P}(i,k)$.

Now pick some $\mathcal{K} \in \mathcal{P}(i, k)$ and choose $j \in \mathcal{K}$ such that $y_{ii} = y_{jj} = y_{kk}$. Then $\mathcal{K} = \mathcal{I} \cdot \mathcal{J}$ where

$$\mathcal{I} = \{i = k_0 < \cdots < k_p = j\} \quad \text{and} \quad \mathcal{J} = \{j = k_p < k_{p+1} \cdots < k_{p+q} = k\}$$

and $y_{ii} = y_{k_p, k_p} = (-1)^p y_{ii}$ is possible only if $p = j - i$ is an even number. $\qquad \square$

Proof of Lemma 5.3 continued.
Our assumption is that

$$y_{ik} = \sum_{\mathcal{P}(i,k)} (-y_{ii}) \frac{\chi(\nu)}{2^m} \prod_{s=0}^{m} y_{i_s i_{s+1}}$$

for $k - i = \delta < t$. For $k - i = t$ we get

$$(y_{ii} + y_{kk})y_{ik} + \sum_{j=i+1}^{k-1} y_{ij} y_{jk}$$

$$= (y_{ii} + y_{kk})y_{ik} + \sum_{i \not\sim j} y_{ij} y_{jk}$$

$$+ \sum_{j=i+1}^{k-1} \Big(\sum_{\mathcal{P}(i,j)} (-y_{ii}) \frac{\chi(n-1)}{2^{n-1}} \prod_{s=0}^{n} y_{i_s i_{s+1}} \Big) \Big(\sum_{\mathcal{P}(j,k)} (-y_{jj}) \frac{\chi(m-1)}{2^{m-1}} \prod_{t=0}^{m} y_{i_t i_{t+1}} \Big)$$

$$= (y_{ii} + y_{kk})y_{ik} + \sum_{i \not\sim j} y_{ij} y_{jk}$$

$$+ \sum_{j=i+1}^{k-1} \Big(\sum_{\mathcal{P}(i,j) \times \mathcal{P}(j,k)} (-y_{ii}) \frac{\chi(n-1)\chi(m-1)}{2^{n+m-2}} \prod_{s=0}^{n+m} y_{i_s i_{s+1}} \Big)$$

$$= (y_{ii} + y_{kk})y_{ik} + \sum_{i \not\sim j} y_{ij} y_{jk} + \sum_{j=i+1}^{k-1} \Big(\sum_{\mathcal{P}(i,k)} (-y_{ii}) \frac{\chi(n+m-2)}{2^{n+m-2}} \prod_{s=0}^{n+m} y_{i_s i_{s+1}} \Big)$$

where $s_{n+j} = t_j$ for $j = 0, 1, \ldots, m$.

For $y_{ii} + y_{kk} \neq 0$ we get

$$y_{ik} = - \sum_{j=i+1}^{k-1} \Big(\sum_{\mathcal{P}(i,k)} (-y_{ii}) \frac{\chi(u-1)}{2^{u-1}} \prod_{s=0}^{u} y_{i_s i_{s+1}} \Big)$$

and for $y_{ii} + y_{kk} = 0$ this relation gives no new condition on y_{ik}. $\qquad \square$

In the next lemma we establish the explicit form for the stabilising GL_n-subgroup of the diagonal matrix X, satisfying $X^3 = I$, with eigenspaces of dimension $\alpha_1, \alpha_2, \alpha_3$.

Lemma 5.5. *The stabiliser subgroup $G_X \subset GL_n$, stabilising the diagonal matrix X, is given by*

$$G_X = \{(g_{ij}) \in GL_n \,|\, g_{ij}(x_{ii} - x_{jj}) = 0\}$$

It is a linear group of dimension

$$dim\, G_X = \alpha_1^2 + \alpha_2^2 + \alpha_3^2$$

isomorphic to the product group $M_{\alpha_1}(k) \times M_{\alpha_2}(k) \times M_{\alpha_3}(k)$.

Proof. Let $g \in G_X$. The relation $gX = Xg$ gives for the entries $g_{ij}x_{jj} = x_{ii}g_{ij}$ or $g_{ij}(x_{jj} - x_{ii}) = 0$. Thus we must have $g_{ij} = 0$ for all pairs (i, j) such that $x_{jj} - x_{ii} \neq 0$, i.e. $g_{ij} = 0$ whenever $x_{ii} \neq x_{jj}$. There are n^2 entries in the whole matrix, n are on the diagonal and $\sum_{i=1}^{3} \alpha_i{}^2 - \alpha_i$ entries correspond to off-diagonal pairs (i, j) with $x_{ii} = x_{jj}$, adding up to the given dimension.

Since X is diagonal with at most three different eigenvalues, conjugation by a certain permutation matrix shows that the stabiliser subgroup is isomorphic to the product of three full matrix groups, of given dimension. $\qquad\square$

Denote by

$$\mathcal{Y}(\Gamma) = \{Y \in M_n(k) \,|\, Y \text{ upper triangular, } Y^2 = I, \, Y \in \Gamma_Y\}$$

By $Y \in \Gamma_Y$ we mean that $y_{11}, ..., y_{nn}$ is choosen in accordance with the representation graph Γ.

We have $dim\,\mathcal{Y}(\Gamma) = \alpha_1'\alpha_2' = \frac{1}{2}(n^2 - \sum_j(\alpha_j')^2)$ with a natural basis given by $\{(i,j) \,|\, i < j,\, i \nsim_Y j\}$. The subgroup of G_X for $X \in \Gamma_X$ stabilising $\mathcal{Y}(\Gamma)$ is denoted $G_{X,\mathcal{Y}(\Gamma)}$.

Recall that the original problem of this section was the following: Given a weak equivalence class $\overline{E} \in \mathcal{D}_\alpha$ of a maximally iterated extension E. This equivalence class splits up in a bunch of equivalence classes under the stronger conjugation relation, described by the non-commutative scheme $Ind_\Gamma(A)$. Then we have the following

Proposition 5.6. *Then there is a one-to-one correspondance*

$$Ind_\Gamma(A)) \longleftrightarrow \mathcal{Y}(\Gamma)/G_{X,\mathcal{Y}(\Gamma)}$$

Proof. A representation in $Ind_\Gamma(A)$ is given by two upper triangular matrices X and Y, satisfying $X^3 = Y^2 = I$. By Lemma 5.1 we can assume that X is diagonalisable and by Lemma 5.2 we can in fact assume that X is diagonal and Y is upper triangular. The parameter space $\mathcal{Y}(\Gamma)$ is described in Lemma 5.3. $\qquad\square$

6. Relation to modular forms

The algebra of modular forms is freely generated as a commutative ring by the the two forms E_4 and E_6. Let $f(n)$ be the dimension of the space of modular forms of dimension $2n$, $n \geq 2$. The generating function for $f(n)$ is given by

$$\sum_{n=0}^{\infty} f(n)x^n = \frac{1}{1-x^2} \cdot \frac{1}{1-x^3}$$

We have seen that $dim\,Simp_n(A) = dim\,Simp_\alpha(A)$ for $|\alpha| = n$, where the partitions $n = \alpha_1 + \alpha_2 + \alpha_3 = \alpha_1' + \alpha_2'$ are at most evenly distributed. Let $\alpha = \beta + \gamma$. We are interested in the number $-\langle\beta,\gamma\rangle = |\beta| \cdot |\gamma| - \sum_{i=1}^3 \beta_i\gamma_i - \sum_{j=1}^2 \beta_j'\gamma_j'$. It can be proven that the minimal value of this number is obtained when $|\gamma| = 1$ (or $|\beta| = 1$).

For fixed $|\beta|$ and $|\gamma|$, the smallest value of $-\langle\beta,\gamma\rangle$ is again achieved when β and γ are at most evenly distributed. On the other hand, $|\beta| \cdot |\gamma|$ has its minimal positive value when one of the factors equals 1. Thus the minimal value for $-\langle\beta,\gamma\rangle$ is given by $n - 1 - \beta_i - \beta_j'$ for maximal $\beta_i = \alpha_i - 1$ and $\beta_j' = \alpha_j' - 1$. Since α is at most evenly distributed we have $\beta_i = [\frac{n-1}{3}]$ and $\beta_j' = [\frac{n-1}{2}]$, where $[q]$ denotes the greatest integer, less than q.

The generating function for the number $[\frac{n-1}{k}]$ is given by

$$(x + x^2 + x^3 + \dots)(x^k + x^{2k} + x^{3k} + \dots) = \frac{x^{k+1}}{(1-x)(1-x^k)}$$

Thus the generating function for the sequence $n - 1 - [\frac{n-1}{3}] - [\frac{n-1}{2}]$ is given by

$$P(x) = x(x + 2x^2 + 3x^3 + \dots) - \frac{x^4}{(1-x)(1-x^3)} - \frac{x^3}{(1-x)(1-x^2)}$$

$$= \frac{x^2}{(1-x)^2} - \frac{x^4}{(1-x)(1-x^3)} - \frac{x^3}{(1-x)(1-x^2)}$$

$$= \frac{1}{(1-x^2)(1-x^3)} - 1$$

Thus except for the one of weight 0, the number of modular forms of weight $2n$ equals the codimension of the non-simple stratum of the highest-dimensional component of the variety of semi-simple representations of A of dimension n.

References

[1] *Eriksen, E.*, Noncommutative deformations of sheaves and presheaves of modules, math. AG/04052344

[2] *Formanek, E.*, Central polynomials for matrix rings. J. of Alg. 23, (1972), pp. 129-132

[3] *Jøndrup, S., Laudal, O.A., Sletsjøe, A.B.*, Noncommutative plane curves, Institut Mittag-Leffler Report No. 20 (2003/2004), math. AG/0405350.

[4] *King, A.*, Moduli of representations of finite dimensional algebras, Quart. J. Math. Oxford Ser. (2) 45 (1994) 515-530

[5] *Laudal, O.A*,Sur les limites projectives et injectives, Ann. Scient. Èc. Norm. Sup. (3) 82 (1965), 241296.

[6] *Laudal, O.A*, Formal moduli of algebraic structures, Lecture Notes in Mathematics, no. 754, Springer-Verlag, 1979.

[7] *Laudal, O.A.*, Noncommutative algebraic geometry, Rev. Mat. Iberoamericana 19 (2003), no. 2, 509580.

[8] *Laudal, O.A.*, The structure of $Simp_{<\infty}(A)$ for finitely generated k-algebras A. Manuscript (2004)

[9] *Le Bruyn, L., Adriaenssens, J.*, Non-commutative covers and the modular group, math.RA/0307139

[10] *McConnell, J.C, Robson, J.C*, Noncommutative Neotherian Rings. Wiley-Interscience New York 1987

[11] *Schofield, A*, Representations of rings over skew fields, London Mathematical Society Lecture Notes Series 92 Cambridge University Press (1985)

[12] *Tuba, I., Wenzl, H*, Representations of the braid group $B3$ and of $SL(2, \mathbb{Z})$, math.RT/9912013 (1999)

[13] *Westbury, B.*, On the character varieties of the modular group, preprint Nottingham (1995)

Matematisk institutt,
University of Oslo,
Pb. 1053, Blindern,
N-0316 Oslo, Norway
  E-mail address: arnebs@math.uio.no