Tropical degenerations and stable rationality

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Joint work with with Johannes Nicaise.

We work over a field of characteristic 0.

Two varieties X and Y are *stably birational* if $X \times \mathbb{P}^m \sim_{bir} Y \times \mathbb{P}^l$ for some $m, l \geq 0$.

X is *stably rational* if it is stably birational to $\mathbb{P}^n$

The paper [NO19] gives a quite general method for the (stable) rationality problem for complete intersections in toric varieties.

Hypersurfaces in $\mathbb{P}^n$

Theorem

A very general quartic fivefold $X \subset \mathbb{P}^6$ is not stably rational.

Also: New proofs of hypersurfaces of higher degree or lower dimension (eg quartic fourfolds, quintic fivefolds, ..)

Hypersurfaces in $\mathbb{P}^n$

d	curves	surfaces	3-folds	4-folds	5-folds	6-folds	7-folds	8-folds	9-folds	10-folds
2					Rational					
3										
4										
5										
6										
7			Stably irrational							
8										
9										

Hypersurfaces in $\mathbb{P}^n$

d	curves	surfaces	3-folds	4-folds	5-folds	6-folds	7-folds	8-folds	9-folds	10-folds
2					Rational					
3			Clemens-Griffiths							
4			Colliot-Thélène-Pirutka	Totaro						
5				Kollár	Schreieder					
6					Kollár	Kollár	Totaro			
7			Stably irrational			Kollár	Totaro			
8							Kollár	Kollár	Kollár	Totaro
9								Kollár	Kollár	Totaro

Hypersurfaces in $\mathbb{P}^n$

9-folds	10-folds	11-folds	12-folds	13-folds	14-folds	15-folds	16-folds	17-folds	18-folds	19-folds
				$d \ge \log_2(n) + 2$						
				Schreieder						
Kollár	Totaro									
Kollár										

Complete intersections

Theorem

Very general complete intersections of a quadric and a cubic in $\mathbb{P}^n$ are stably irrational for $n \leq 6$.

Our main contribution is stable irrationality for $n = 6$.

History related to the *Lüroth problem*:

- Fano (1908): (Incorrect) proof of irrationality for $n = 5$
- Enriques (1912): Proof of unirationality for $n = 5$
- Hassett–Tschinkel (2018): Stable irrationality for $n = 5$.
- Morin (1955), Conte–Murre (1998): Unirationality for $n = 6$.

The above result settles the rationality problem for all complete intersections of dimension ≤ 4 - except cubic fourfolds.

Other results

Many new classes of complete intersections in $\mathbb{P}^n$

- Logarithmic bounds à la Schreieder for stable irrationality.
- Complete intersections of r quadrics in $\mathbb{P}^n$ are stably irrational if $r \geq 3$ and $2r \geq n - 1$.
- In dimension 5:

$$(\mathbf{4}), (5), (6), (\mathbf{2}, \mathbf{4}), (2, 5), (\mathbf{3}, \mathbf{3}), (3, 4), (\mathbf{2}, \mathbf{2}, \mathbf{3}), (2, 2, 4), (2, 3, 3), (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}), (2, 2, 2, 3), (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}).$$

Many new cases for hypersurfaces in $\mathbb{P}^\ell \times \mathbb{P}^m$.
A sample:

Theorem

A very general ample hypersurface X of bidegree (a, b) in $\mathbb{P}^1 \times \mathbb{P}^n$ $(n \leq 4)$ is stably rational if and only if

- $a = 1$; *or*
- $b \leq 2$

Ingredients

The proof uses

- Specialization of birational types (Nicaise–Shinder, Kontsevich–Tschinkel)
- Tropical geometry, toric degenerations
- Stable irrationality of known lower-dimensional varieties

Stable birational types

SB_F = set of stable birational equivalence classes of integral F-varieties

$[X]_{\mathrm{sb}}$ = equivalence class of X.

We consider $\mathbb{Z}[\mathrm{SB}_F]$.

For any F-scheme X of finite type, we set

$$[X]_{\mathrm{sb}} = [X_1]_{\mathrm{sb}} + \ldots + [X_r]_{\mathrm{sb}} \qquad \text{in } \mathbb{Z}[\mathrm{SB}_F]$$

where $X_1, \ldots, X_r$ are the irreducible components.

Ring product: $[X]_{\mathrm{sb}} \cdot [Y]_{\mathrm{sb}} = [X \times_F Y]_{\mathrm{sb}}$.

Larsen–Lunts (2003): There is a natural isomorphism

$$\mathbf{K}(Var/F)/(\mathbb{L}) \simeq \mathbb{Z}[\mathrm{SB}_F].$$

induced by $[X] \mapsto [X]_{\mathrm{sb}}$ for X smooth and proper.

Some notation

Field of Puiseux series:

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m}))$$

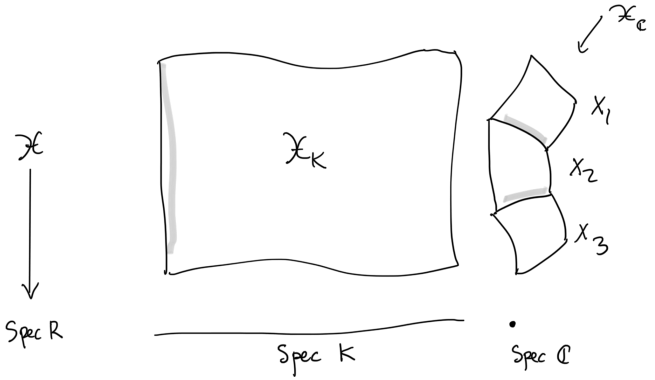
Valuation ring:

$$R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]]$$

An R-scheme is *strictly semi-stable* if, Zariski locally, it admits an étale morphism to a scheme of the form

$$\operatorname{Spec} R[z_1, \ldots, z_s]/(z_1 \cdots z_r - t^q)$$

where $s \geq r \geq 0$ and q is a positive rational number.



The limits of rationality

$\mathcal{X}$ = a proper semi-stable model over R, with special fiber

$$\mathcal{X}_{\mathbb{C}} = \sum_{i \in I} X_i$$

Theorem (Nicaise–Shinder 2019)

There exists a unique ring homomorphism

$$\mathrm{Vol} : \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$

such that

$$\mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}}$$

for any $\mathcal{X}$ *as above, where* $X_J = X_{j_1} \cap \ldots \cap X_{j_r}$

Basic consequences

- Vol sends $[\operatorname{Spec} K]_{\mathrm{sb}}$ to $[\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}$

$\rightsquigarrow$ obstruction to stable rationality of $\mathcal{X}_K$:

If

$$\sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}} \neq [\operatorname{Spec} \mathbb{C}] \qquad \text{in } \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$$

then $\mathcal{X}_K$ is stably irrational.

- If $\mathcal{X}$ is a smooth and proper R-scheme, then the formula simplifies to

$$\operatorname{Vol}([X_K]_{\mathrm{sb}}) = [\mathcal{X}_{\mathbb{C}}]_{\mathrm{sb}}$$

$\therefore$ Stable rationality specializes in smooth and proper families in characteristic 0.

Example (Voisin)

A very general double quartic threefold is irrational.

The proof involves degenerating to the Artin–Mumford example.

For our applications, we get better results using degenerations with many components.

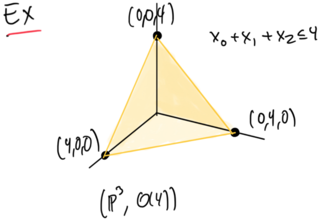
Key point: *irrational strata of low dimension may be shown to not cancel out in the alternating sum*

$$\mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}}.$$

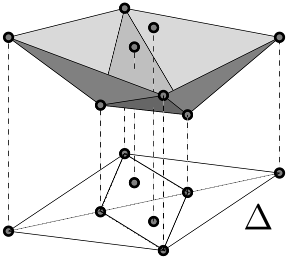
To carry this out we need a more powerful way of constructing degenerations.

Tropical degenerations

Consider a lattice polytope $\Delta \subset \mathbb{R}^n$ corresponding to a toric variety Y.



Let $\mathscr{P}$ be a *regular subdivision* of Δ into lattice polytopes.



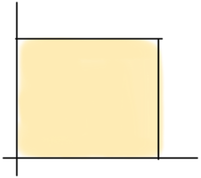
$\mathscr{P}$ induces a degeneration of Y into a union of toric varieties

$$\mathcal{Y}_0 = \bigcup_{P \in \mathscr{P}} Y_P$$

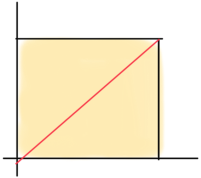
If $P_1, P_2 \in \mathscr{P}$ intersect along a common face Q, then

$$Y_{P_1} \cap Y_{P_2} = Y_Q$$

Ex



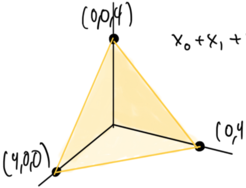
subdivide $\rightsquigarrow$



$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)) \rightsquigarrow (\mathbb{P}^2, \mathcal{O}(1)) \cup (\mathbb{P}^2, \mathcal{O}(1))$

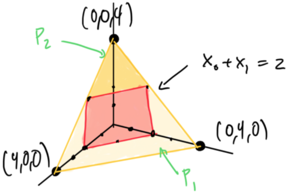
intersecting along a $(\mathbb{P}^1, \mathcal{O}(1))$.

Ex



$x_0 + x_1 + x_2 \leq 4$

$\rightsquigarrow$



$(\mathbb{P}^3, \mathcal{O}(4)) \rightsquigarrow$ union of two toric 3-folds intersecting along $\mathbb{P}^1 \times \mathbb{P}^1$.

Let f be a general Laurent polynomial with Newton polytope $\Delta \subset \mathbb{R}^{n+1}$.

For every face δ of $\mathscr{P}$, set

$$f_\delta = \sum_{\mathbb{Z}^{n+1} \cap \delta} c_m x^m$$

Non-degeneracy: We assume that $Z(f_\delta)$ is smooth for all δ.

Theorem

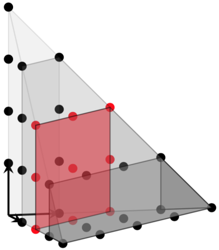
If

$$\sum_{\delta \subsetneq \partial \Delta} (-1)^{\dim \delta} [Z(f_\delta)]_{\mathrm{sb}} \neq (-1)^n [\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}$$

in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$, *then a very general hypersurface in* $(\mathbb{C}^*)^{n+1}$ *with Newton polytope* Δ *is not stably rational.*

The Quartic fivefold is stably irrational

Newton polytope: $\Delta = \left\{(x_1, \dots, x_6) \in \mathbb{R}^6_{\geq 0} | \sum_i x_i \leq 4\right\}$
Subdivision below $\rightsquigarrow$ degeneration with special fiber $X_1 \cup X_2 \cup X_3 \cup X_4$.



Red polytope $= (2,2)$-divisor in $\mathbb{P}^2 \times \mathbb{P}^3$
$\rightsquigarrow$ stably irrational by [Hassett–Pirutka–Tschinkel 2016].
All other polytopes have *lattice width one*, hence rational.
Thus

$$\sum_{\delta \subsetneq \partial\Delta} (-1)^{\dim \delta} [Z(f_\delta)]_{\mathrm{sb}} \neq (-1)^n [\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}$$

□

Products of projective spaces

Theorem

A very general $(2,3)$-divisor $X \subset \mathbb{P}^1 \times \mathbb{P}^4$ is not stably rational.

Subdivisions of the polytope $a\Delta_1 \times b\Delta_n$ allows us to raise degree/dimension:

(a,b) in $\mathbb{P}^m \times \mathbb{P}^n$ stably irrational $\implies$ $(a,b+1)$ and $(a+1,b)$ also stably irrational in $\mathbb{P}^m \times \mathbb{P}^n$ and $\mathbb{P}^m \times \mathbb{P}^{n+1}$.

$\therefore$ we get all bidegrees corresponding to rational/irrational hypersurfaces.

The Hassett–Pirutka–Tschinkel quartic

Consider $Y \subset \mathbb{P}^2 \times \mathbb{P}^3$, bidegree $(2,2)$, defined by

$$xyU^2 + xzV^2 + yzW^2 + (x^2 + y^2 + z^2 - 2(xy + xz + yz))T^2 = 0$$

Hassett–Pirutka–Tschinkel/Schreieder: Anything that specializes to Y does not admit a decomposition of the diagonal (hence is stably irrational).

$(2,3)$-divisors in $\mathbb{P}^1 \times \mathbb{P}^4$

$$\begin{aligned} P &= \text{the Newton polytope of the HPT quartic.} \\ &= \text{convex hull of column vectors of } \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

Starting observation: P is contained in the Newton polytope of a general $(2,3)$-divisor:

$$2\Delta_1 \times 3\Delta_4 = \{(u,v) \in \mathbb{R}^{1+4}_{\geq 0} \mid u \leq 2,\ v_1 + \ldots + v_4 \leq 3\}.$$

In concrete terms, the following bidegree $(2,3)$ polynomial

$$\begin{gathered} x_0^2 y_0^3 - 2x_0x_1y_0^3 + x_1^2y_0^3 - 2x_0^2y_0^2y_1 - 2x_0x_1y_0^2y_1 \\ + x_0^2y_0y_1^2 + x_0x_1y_1y_2^2 + x_0^2y_1y_3^2 + x_0x_1y_0y_4^2 \end{gathered}$$

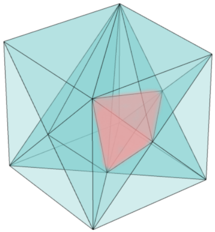
dehomogenizes to the HPT quartic.

Let $\mathscr{P}$ denote the regular subdivision of the polytope $2\Delta_1 \times 3\Delta_4$ induced by the convex function

$$f\colon \mathbb{R}^5 \to \mathbb{R},\ x \mapsto \min_{z\in P} \|x - z\|^2$$

The cells in $\mathscr{P}$:

dim δ	0	1	2	3	4	5
number	43	192	353	323	146	26



$\rightsquigarrow$ degeneration of $\mathbb{P}^1 \times \mathbb{P}^4$ into a union of 26 toric varieties.

Going through the cells of dimension 2 and 4 reveals that any face δ of even dimension either

- has lattice width one (rational, as the equation is linear with respect to a variable)
- corresponds to a quadric bundle over $\mathbb{P}^1_k$ (rational).
- defines a conic bundle over $\mathbb{A}^3$ with a section (rational)

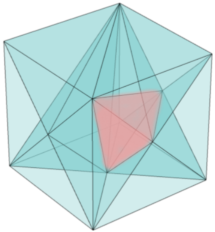
In $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$ we have

$$\mathrm{Vol}([\mathcal{X}]_{\mathrm{sb}}) = [HPT] + \sum_{\#I \text{ odd}} [X_I] + a[\operatorname{Spec}\mathbb{C}] \qquad \text{for some } a \in \mathbb{Z}$$

As this is $\neq [\operatorname{Spec}\mathbb{C}]$, a very general X is stably irrational. $\square$

End remarks

General strategy: Construct subdivisions $\mathscr{P}$ so that all but one lower-dimensional polytope is stably rational (or make sure that the various intersections do not cancel out).



If time permits: $(2,3)$-complete intersections

Let $\mathbb{P}^6 = \text{Proj } k[x_0, \ldots, x_6]$ and let $P = \{x_0 = \ldots = x_3 = 0\} \simeq \mathbb{P}^2$.

$$Y = \{q = c = 0\} \subset \mathbb{P}^6$$

for q and c very general of degree 2 and 3. We assume X contains P.

Blow-up:

$$\begin{array}{ccc} X \subset Bl_P\mathbb{P}^6 & \xrightarrow{\pi} & \mathbb{P}^6 \\ \big\downarrow p & & \\ \mathbb{P}^3 & & \end{array}$$

$X = Q \cap C$ where $Q \in |2H - E|$ and $C \in |3H - E|$.

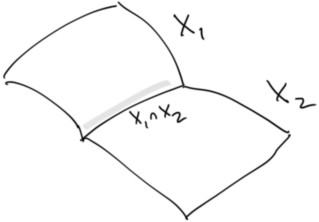
It suffices to show that generic intersections

$$X = Q \cap C \subset Bl_P\mathbb{P}^6$$

where $Q \in |2H - E|$ and $C \in |3H - E|$ are stably irrational.

Now degenerate Q to $Q_0 + E$ where $Q_0 \in |2H - 2E| = |2p^*h|$.

This induces a degeneration of $\mathcal{X} \to \mathbb{A}^1$ with special fiber $\mathcal{X}_0 = X_1 \cup X_2$:



There are three strata:

- $X_1 = Q_0 \cap C$
- $X_2 = E \cap C$
- $X_{12} = Q_0 \cap E \cap C$

The stratum $X_1 = Q_0 \cap C$:

$$\begin{array}{ccccc} Q_0 = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^3 \oplus \mathcal{O}(1,1)) & \hookrightarrow & \mathbb{P}(\mathcal{O}^3 \oplus \mathcal{O}(1)) & \xrightarrow{\pi} & \mathbb{P}^6 \\ \downarrow & & \downarrow p & & \\ \mathbb{P}^1 \times \mathbb{P}^1 & \hookrightarrow & \mathbb{P}^3 & & \end{array}$$

$C|_{Q_0}$ is a very general divisor in $|\mathcal{O}(2) \otimes p^*\mathcal{O}(1,1)|$ in $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^3 \oplus \mathcal{O}(1,1))$.

$\rightsquigarrow$ X_1 is stably irrational by [Schreieder 2017].

The strata $X_2 = E \cap C$ and $X_{12} = E \cap Q_0 \cap C$

C restricts to a $(1,2)$-divisor on $E \simeq \mathbb{P}^2 \times \mathbb{P}^3$

Q_0 restricts to a $(0,2)$-divisor on $E \simeq \mathbb{P}^2 \times \mathbb{P}^3$.

$\rightsquigarrow$ X_2 and X_{12} are both rational.

By the motivic volume formula:

$$\begin{aligned} \mathrm{Vol}([\mathcal{X}]_{\mathrm{sb}}) &= [X_1]_{\mathrm{sb}} + [X_2]_{\mathrm{sb}} - [X_{12}]_{\mathrm{sb}} \\ &= [X_1]_{\mathrm{sb}} + [\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} - [\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} \\ &\neq [\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} \end{aligned}$$

This implies that a very general X is stably irrational. □