Specialization techniques and stable rationality

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Lecture 1:
Birational invariants and specialization

These talks will revolve around a paper written with Johannes Nicaise:

J. Nicaise, J.C. Ottem. *Tropical degenerations and stable rationality* (2020).

In the paper we give a quite general method for the (stable) rationality problem for hypersurfaces and complete intersections in toric varieties.

We work over a field k of characteristic 0.
(Usually $k = \mathbb{C}$).

Two varieties X and Y are *stably birational* if $X \times \mathbb{P}^m \sim_{bir} Y \times \mathbb{P}^l$ for some $m, l \geq 0$.

X is *stably rational* if it is stably birational to $\mathbb{P}^n$.

The Rationality Problem
Determine whether a given variety is (stably) rational or not.

The Rationality problem for hypersurfaces
For which d, n is a general degree d-hypersurface in $\mathbb{P}^{n+1}$ (stably) irrational?

Two of the main applications

Theorem (Nicaise-O.)

The very general complex quartic fivefold in $\mathbb{P}^6$ is not stably rational.

Theorem (Nicaise-O.)

A very general complete intersection of a quadric and a cubic in $\mathbb{P}^6$ is not stably rational.

The goal of the lectures is to explain the proofs of these theorems.

Other results

- New proofs for hypersurfaces of higher degree or lower dimension
- Many new classes of complete intersections in $\mathbb{P}^n$.
- Many new classes of hypersurfaces in other toric varieties.

Theorem

Consider a very general ample hypersurface X of bidegree (a, b) in $\mathbb{P}^1 \times \mathbb{P}^4$

$$x_0^a f_0 + x_0^{a-1} x_1 f_1 + \ldots + x_1^a f_a = 0$$

Then X is stably rational if and only if

- $a = 1$; or
- $b \leq 2$

Overview of the lectures

Monday

Rationality problems, basic birational invariants, specialization methods.

Tuesday

The Grothendieck ring of varieties, Nicaise–Shinder's motivic volume

Wednesday

First applications: Quartic fivefolds, $(2, 3)$-complete intersections, ..

Thursday

Toric degenerations

Friday

Further applications

Ingredients

The proof uses

- Specialization of birational types (Nicaise–Shinder, Kontsevich–Tschinkel)
- Tropical geometry, toric degenerations
- Stable irrationality of known lower-dimensional varieties

General strategy for rationality problems

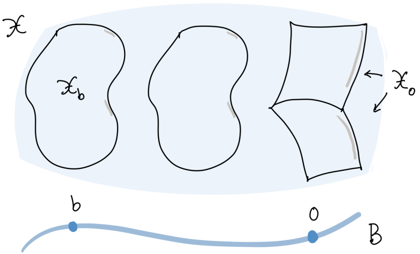
There are two basic steps:

(1) Look for **obstructions to rationality** (birational invariants) e.g., the Brauer group.

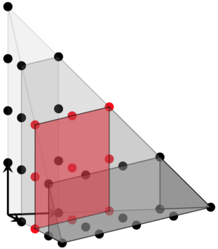
(2) Show that the obstruction is **non-trivial**.

Two themes in the lectures

Verify (2) by *specialization* to a simpler, but sometimes singular variety.



Construct suitable degenerations combinatorially:



Preliminaries

Cohomology of blow-ups

If X is a smooth complex variety,

$W = Bl_Z X$, the blow-up in a smooth center $Z \subset X$ of codimension c,

Then there is a natural isomorphism

$$H^p(W,\mathbb{Z}) = H^p(X,\mathbb{Z}) \oplus H^{p-2}(Z,\mathbb{Z})[E] \oplus \cdots H^{p-2(c-1)}(Z,\mathbb{Z})[E]^{c-1} \quad (1)$$

where $E \subset W$ is the exceptional divisor.

Chow groups of blow-ups

There is a similar description for Chow groups:

$$CH^p(W,\mathbb{Z}) = CH^p(X,\mathbb{Z}) \oplus CH^{p-1}(Z,\mathbb{Z})[E] \oplus \cdots \oplus CH^{p-(c-1)}(Z,\mathbb{Z})[E]^{c-1} \qquad (2)$$

Birational invariants

Hypersurfaces in $\mathbb{P}^n$

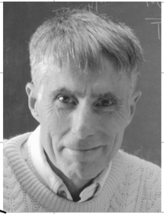
d	curves	surfaces	3-folds	4-folds	5-folds	6-folds	7-folds	8-folds
2					Rational			
3								
4								
5								
6								
7			Easy cases					
8								
9								

Obstruction to rationality: Differential forms $H^0(X, \Omega_X^p)$

The obstruction is non-trivial when $d \geq n+1$.

Hypersurfaces in $\mathbb{P}^n$

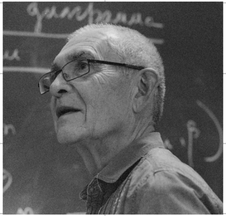
d	curves	surfaces	3-folds	4-folds	5-folds	6-folds	7-folds	8-folds
2					Rational			
3			Clemens—Griffiths					
4			Iskovskikh-Manin					
5								
6								
7			Easy cases					
8								
9								



Obstruction to rationality: The intermediate jacobian $H^{1,2}(X)/H^3(X,\mathbb{Z})$.
The obstruction is non-trivial: Analyse the singularities of the theta divisor.

Hypersurfaces in $\mathbb{P}^n$

d	curves	surfaces	3-folds	4-folds	5-folds	6-folds	7-folds	8-folds
2					Rational			
3			Clemens—Griffiths					
4			Iskovskikh-Manin					
5								
6								
7			Easy cases					
8								
9								



Obstruction to rationality: The birational automorphism group $Bir(X)$ is finite.
The obstruction is non-trivial: Use the Noether-Fano inequalities.

The example of Artin–Mumford

Stable birational invariant: $H^3(X, \mathbb{Z})_{\text{tors}}$

This is 0 for $X = \mathbb{P}^n$.

$H^3(X, \mathbb{Z})_{\text{tors}}$ is clearly invariant under taking products with $\mathbb{P}^m$.

If $\pi : W \to X$ is a blow-up in a smooth center $Z \subset X$, then

$$H^3(W, \mathbb{Z}) = H^3(X, \mathbb{Z}) \oplus H^1(Z, \mathbb{Z})[E]$$

and by the Universal Coefficient Theorem,

$$H^1(Z, \mathbb{Z})_{\text{tors}} = H_0(Z, \mathbb{Z})_{\text{tors}} = 0$$

$\rightsquigarrow$ $H^3(W, \mathbb{Z})$ and $H^3(X, \mathbb{Z})$ have the same torsion.

The invariant is non-trivial for rather special varieties:

Proposition (Artin-Mumford)

There exist (resolutions of) double quartic solids $X \to \mathbb{P}^3$ given by

$$w^2 = f(x, y, z, t)$$

for which $H^3(X, \mathbb{Z})_{\text{tors}} \neq 0$.

These are unirational threefolds.

This invariant is closely related to the *Brauer group*.

Hypersurfaces in $\mathbb{P}^n$

d	curves	surfaces	3-folds	4-folds	5-folds	6-folds	7-folds	8-folds
2					Rational			
3			Clemens—Griffiths					
4			Colliot-Thelene—Pirutka					
5				Birational rigidity				
6					Birational rigidity	Kollár		
7			Easy cases			Birational rigidity		
8							Birational rigidity	Kollár
9								Birational rigidity

Kollár's strategy

Obstruction to rationality: Rational varieties are *ruled* (=birational to $\mathbb{P}^1 \times Y$)

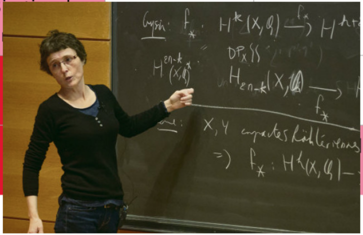
The obstruction is non-trivial: Specialize X modulo p such that:
for a resolution $Y \to X_p$, Ω_Y^{n-1} contains a positive line subbundle.

$\rightsquigarrow$ X_p is not ruled.
$\rightsquigarrow$ X is not ruled *(Ruledness specializes in families [Matsusaka])*.
$\rightsquigarrow$ X is not rational.

Decomposition of the diagonal

Recent developments

d	curves	surfaces	3-folds	4-folds	5-folds	6-folds	7-folds	8-folds
2					Rational			
3			Clemens—Griffiths					
4			Colliot-Thelene—Pirutka					
5				Birational rigidity				
6					Birational rigidity			
7			Easy cases			Bi		
8								
9								



Recent developments

2					Rational			
3			Clemens—Griffiths					
4			Colliot-Thelene—Pirutka	Totaro				
5				Birational rigidity				
6					Birational rigidity	Kollár	Totaro	
7			Easy cases			Birational rigidity	Totaro	
8							Birational rigidity	Kollár
9								Birational rigidity

Recent developments

d	curves	surfaces	3-folds	4-folds	5-folds	6-folds	7-folds	8-folds
2					Rational			
3			Clemens—Griffiths					
4			Colliot-Thelene—Pirutka	Totaro				
5				Birational rigidity	Schreieder			
6					Birational rigidity	Kollár	Totaro	
7			Easy cases			Birational rigidity	Totaro	
8							Birational rigidity	Kollár
9								Birational rigidity

Recent developments

9-folds	10-folds	11-folds	12-folds	13-folds	14-folds	15-folds	16-folds	17-folds	18-folds	19-folds
				$d \geq \log_2(n) + 2$						
				Schreieder						
Kollár	Totaro									
Kollár										

Recent developments

d	curves	surfaces	3-folds	4-folds	5-folds	6-folds	7-folds	8-folds
2					Rational			
3			Clemens—Griffiths	???				
4			Colliot-Thelene—Pirutka	Totaro	Quartic fivefolds			
5				Birational rigidity	Schreieder			
6					Birational rigidity	Kollár	Totaro	
7			Easy cases			Birational rigidity	Totaro	
8							Birational rigidity	Kollár
9								Birational rigidity

Decomposition of the diagonal

Consider the diagonal embedding of X

$$\Delta \subset X \times X$$

We say that X admits a *decomposition of the diagonal* if there is an equality

$$\Delta = [X \times x] + Z \text{ in } CH_n(X \times X) \tag{3}$$

where $Z \subset X \times X$ is a subvariety which does not dominate X via the first projection.

Decomposition of the diagonal

Obstruction to Rationality: Any stably rational variety has a decomposition of Δ.

For $X = \mathbb{P}^n$, we have a decomposition (in $CH^n(\mathbb{P}^n \times \mathbb{P}^n)$):

$$\Delta = \sum_{i=0}^{n} p_1^* h^i \cdot p_2^* h^{n-i}$$

Here $p_2^* h^n \sim [\mathbb{P}^n \times x]$ and the other terms are supported on $D \times X$ for some $D \subset X$.

Stable birational invariance follows from the formulas for the Chow groups of blow-ups.

Main point: Δ acts as a correspondence in a special way (the identity map).

Example

Let X be a smooth projective curve of genus ≥ 1.

Claim: X does not have a decomposition of Δ:

Let $\omega \in H^0(X, \Omega_X^1)$ denote a global holomorphic 1-form. Then

$$[X \times x]^*\omega = pr_{2*}(pr_2^*[x] \cdot pr_1^*\omega) = 0$$

and

$$Z^*\omega = pr_{2*}(Z \cdot pr_1^*\omega) = pr_{2*}(0) = 0$$

$\rightsquigarrow$ $\Delta \neq [X \times x] + Z$, because $\Delta^*\omega = \omega$.

Example

A similar argument shows that a variety with a decomposition of Δ satisfies

- $H^0(X, \Omega_X^p) = 0$ for $p > 0$
- $H^3(X, \mathbb{Z})_{\text{tors}} = 0$

How to prove that X admits no decomposition of Δ? This is a delicate matter.

Voisin's specialization method:

Degenerate to a variety X_0 with *mild singularities*.

Show that (some resolution of) X_0 does not admit a decomposition of the diagonal.

Deduce from this that X does not admit a decomposition of the diagonal either.

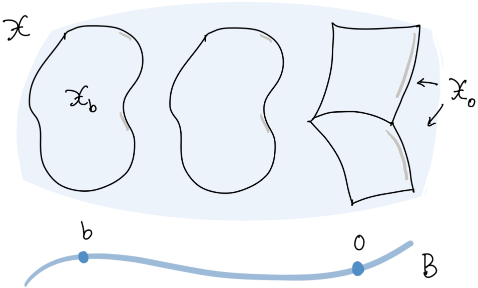
$\rightsquigarrow$ X is not stably rational.

Families of varieties and specialization

A *family* of varieties is a flat morphism

$$f : \mathcal{X} \to B$$

of k-varieties; we will usually require f to be projective.



In this situation, it is natural to ask how the following vary in the fibers of f:

- The (stable) rationality of $\mathcal{X}_b$
- The Chow groups $CH^p(\mathcal{X}_b)$
- The cohomology groups $H^i(\mathcal{X}_b, \mathbb{Z})$

Example

If $\mathcal{X} \to B$ is smooth, and we are over $k = \mathbb{C}$, then all the fibers $\mathcal{X}_b$ are diffeomorphic (Ehresmann's fibration theorem). Hence $H^i(\mathcal{X}_b, \mathbb{Z})$ are all isomorphic.

However, the two first items can vary drastically in the family.

For instance, in a smooth family $\mathcal{X} \to \mathbb{A}^1$, it can happen that there are exactly countably infinitely many fibers $\mathcal{X}_t$ which admit a decomposition of Δ.

Specialization of Rationality

The behaviour of rationality in families can be subtle:

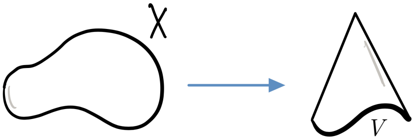
Example (Rational specializing to irrational)

Consider the family

$$\mathcal{X} = \{x_0^3 + x_1^3 + x_2^3 + tx_3^3 = 0\} \subset \mathbb{P}^3 \times \mathbb{A}^1$$

For $t \neq 0$, the fiber $\mathcal{X}_t$ is a cubic surface, hence rational.

But the fiber over $t = 0$ is a cone $C(V)$ over the elliptic curve $V := \{x_0^3 + x_1^3 + x_2^3 = 0\}$, which is irrational.



The last example shows that rationality behaves strangely in families with very singular fibers.

Example

If $\mathcal{X} \to B$ is a family of *smooth* projective surfaces, and $\mathcal{X}_b$ is rational for some $b \in B$, then every fiber is rational.

This follows by Castelnuovo's criterion, because the groups

$$H^1(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}), \qquad H^0(\mathcal{X}_b, \mathcal{O}(2K_{\mathcal{X}_b}))$$

are constant in the family

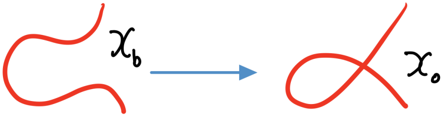
Example (Irrational specializing to rational)

Consider the family

$$\mathcal{X} = \{x_0^3 + x_1^2 x_2 + t x_2^3 = 0\} \subset \mathbb{P}^2 \times \mathbb{A}^1$$

For $t \neq 0$, the fiber $\mathcal{X}_t$ is a smooth cubic curve, hence irrational.

But the fiber over $t = 0$ is a nodal cubic, which is rational.



Classical question: Can this happen in families of *smooth* varieties?

Example (Irrational specializing to rational II)

Consider a smooth $(2,2)$-divisor $X \subset \mathbb{P}^2 \times \mathbb{P}^3$.

If X is very general, it is known to be stably irrational [Hassett-Pirutka-Tschinkel].

However, if the equation of X is of the form

$$y_0 F_0 + y_1 F_1 + y_2 F_2 = 0$$

where F_i are generic $(2,1)$-forms, then X is smooth, and rational.

(X contains the $\mathbb{P}^2$ given by $\{y_0 = y_1 = y_2 = 0\}$, which defines a section of the quadric bundle $X \to \mathbb{P}^2$, so X is rational.)

The last example is in fact rather wild:

"Most" $(2,2)$-divisors are stably irrational.
But there are also infinitely many divisors in the parameter space of smooth $(2,2)$-divisors parametrizing rational hypersurfaces.

In general, for a family $f : \mathcal{X} \to B$, we define the *Rational locus* as

$$Rat(f) = \{b \in B \,|\, \mathcal{X}_b \text{ is rational}\}$$

Proposition

$Rat(f)$ is a countable union of locally closed subsets of B.

Main idea of the proof.

Let n denote the relative dimension of n and let $P = \mathbb{P}^n_B$.

Let $Z \subset X \times_B P$ be a closed subvariety. If $Z_b \to X_b$ and $Z_b \to P_b$ are both birational, then we obtain a birational map $X_b \dashrightarrow P_b$.
Conversely, any such birational map arises in this way.

We reduce to looking at certain subvarieties of $X \times_B P$.

There is a *relative Hilbert scheme* $Hilb(X \times_B P/B)$ paramterizing subvarieties in the fibers of $X \times_B P \to B$.

This Hilbert scheme has only countably many components $\rightsquigarrow$ OK. □

Definition

A property is said to hold for $b \in B$ *very general*, if it is holds outside a countable union of closed subsets in B.

Important observation:

Proposition

For $b \in B$ very general, the fiber $\mathcal{X}_b$ is isomorphic (as a scheme) to the geometric generic fiber $\mathcal{X}_{\overline{K}}$, where $K = k(B)$.

More precisely, there is a field isomorphism $\overline{K} \to k(b)$, and isomorphisms $\mathcal{X}_b \to \mathcal{X}_{\overline{K}}$ making the following diagram commute:

$$\begin{array}{ccc} \mathcal{X}_b & \longrightarrow & \mathcal{X}_{\overline{K}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k(b) & \longrightarrow & \operatorname{Spec} \overline{K} \end{array}$$

Therefore, if we only care about the very general member of some family of varieties (e.g., the very general hypersurface), this is the same thing as the geometric generic fiber.

Specialization

Let R be a DVR, and let $\mathcal{X}$ be an integral R-scheme.

We will often be in the situation where we have a diagram of the form

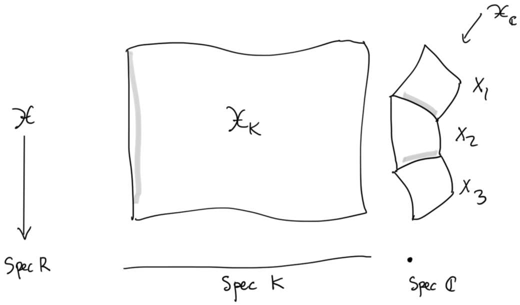
$$\begin{array}{ccccc} \mathcal{X}_K & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{X}_k \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} K & \hookrightarrow & \operatorname{Spec} R & \hookleftarrow & \operatorname{Spec} k \end{array}$$

$K = Frac(R)$ is the fraction field;
$k = R/m$ is the residue field.

Definition

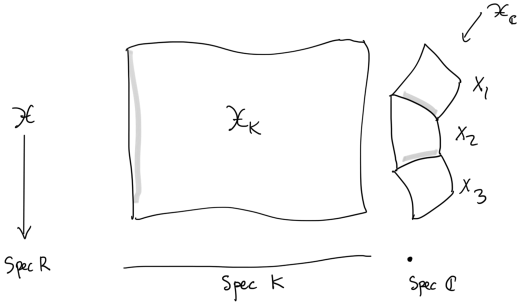
$X = \mathcal{X}_K$ is called the *generic fiber*, wheras $Y = X_k$ is the *special fiber*.

Specialization



We say that a variety X/K *specializes* to a variety Y/k if there exists a scheme $\mathcal{X}/R$ as above, with $\mathcal{X}_K \simeq X$ and $\mathcal{X}_k \simeq Y$.

Specialization of cycles



For a codimension p subvariety $Z \subset \mathcal{X}_K$, we can take its Zariski closure in $\mathcal{X}$ and obtain a subvariety $\mathcal{Z}$ of $\mathcal{X}$. Intersecting with the special fiber, we get a codimension p-cycle Z_k on $\mathcal{X}_k$.

This is compatible with rational equivalence, which gives the *specialization map of Chow groups*

$$CH^p(\mathcal{X}_K) \to CH^p(\mathcal{X}_k)$$

Obstructing rationality via specialization/degeneration

In general, birational invariants such as $Br(X)$ are hard to compute.
For our purposes, it is also enough to know that they are non-zero.

Common strategy: specialize to a simpler, perhaps singular, variety X_0,
and hope that X_0 contains enough information to deduce that the generic fiber is non-stably rational.

The rational obstruction needs to be suffiently sophisticated for this to work:

The "cone over an elliptic curve"-example shows that one also needs to consider families with "controlled" singularities.

Quartic threefolds (sketch)

Construct a degeneration $\mathcal{X} \to B$ of quartic threefolds, so that $\mathcal{X}_0$ is birational to the Artin-Mumford example Y.

$\rightsquigarrow$ $\mathcal{X}_0$ carries a non-trivial unramified Brauer class $\alpha_0 \in Br(k(\mathcal{X}_0))[2]$.

$\rightsquigarrow$ some resolution $\widetilde{\mathcal{X}}_0$ has non-trivial $Br(\widetilde{\mathcal{X}_0})[2]$.

$\rightsquigarrow$ $\widetilde{\mathcal{X}}_0$ does not admit a decomposition of Δ

$\rightsquigarrow$ $\mathcal{X}_b$ does not admit a decomposition of Δ, for $b \in B$ very general

$\rightsquigarrow$ the very general $\mathcal{X}_b$ is not stably rational.

Lecture 2:
The motivic volume

The Grothendieck ring

Let F be a field. The Grothendieck group $\mathbf{K}(\mathrm{Var}_F)$ of F-varieties is the abelian group with the following presentation:

- *Generators:* isomorphism classes $[X]$ of F-schemes X of finite type;
- *Relations:* whenever X is an F-scheme of finite type, and Y is a closed subscheme of X, then $[X] = [Y] + [X - Y]$.

Ring structure: induced by $[X] \cdot [X'] = [X \times_F X']$ for all F-schemes X and X' of finite type.

Identity element: $1 = [\mathrm{Spec}\, F]$, the class of the point.

Lefschetz motive: $\mathbb{L} = [\mathbb{A}^1_F] \in \mathbf{K}(\mathrm{Var}_F)$.

Example

$[\mathbb{A}^n] = [\mathbb{A}^1 \times \cdots \times \mathbb{A}^1] = \mathbb{L} \times \cdots \times \mathbb{L} = \mathbb{L}^n$

Example

Partitioning $\mathbb{P}^n_F$ into the hyperplane at infinity and its complement, we find

$$[\mathbb{P}^n_F] = [\mathbb{P}^{n-1}_F] + [\mathbb{A}^n_F] = [\mathbb{P}^{n-1}_F] + \mathbb{L}^n.$$

Now it follows by induction on n that

$$[\mathbb{P}^n_F] = 1 + \mathbb{L} + \ldots + \mathbb{L}^n$$

in $\mathbf{K}(\mathrm{Var}_F)$.

The Grothendieck ring $\mathbf{K}(\mathrm{Var}_F)$ is insensitive to non-reduced structures: if X is an F-scheme of finite type, then the complement of X_{red} in X is empty, so that $[X] = [X_{\mathrm{red}}]$.

$\mathbf{K}(\mathrm{Var}_F)$ can be generated by smooth and proper F-varieties:

Theorem (Bittner 2004)

Let F be a field of characteristic zero. Then $\mathbf{K}(\mathrm{Var}_F)$ has also the following presentation:

- *Generators:* isomorphism classes $[X]$ of connected smooth and proper F-schemes X;
- *Relations:* $[\emptyset] = 0$, and, whenever X is a connected smooth and proper F-scheme and Y is a connected smooth closed subscheme of X,

$$[\mathrm{Bl}_Y X] - [E] = [X] - [Y] \tag{4}$$

where $\mathrm{Bl}_Y X$ denotes the blow-up of X along Y, and E is the exceptional divisor.

Question: When do X and X' define the same class in $\mathbf{K}(\mathrm{Var}_F)$?

Obvious sufficient condition: X and X' be piecewise isomorphic,
(i.e., they can be partitioned into subschemes that are pairwise isomorphic)
$\rightsquigarrow$ $[X] = [X']$ (by scissor relations).

Example

Let $C \subset \mathbb{A}^2_F$ be the affine plane cusp given by

$$y^2 - x^3 = 0.$$

Then C is piecewise isomorphic to $\mathbb{A}^1_F$:
$C - \{(0,0)\} \simeq \mathbb{A}^1_F - \{0\}$.
So $[C] = \mathbb{L}$ in $\mathbf{K}(\mathrm{Var}_F)$.

However, this condition is not necessary:

Example (Borisov)

There exist two smooth varieties X and X' over $\mathbb{C}$ such that $[X] = [X']$ but X and X' are not birational, and therefore not piecewise isomorphic.

This is due to issues of cancellation:

X and X' can be embedded into a common $\mathbb{C}$-variety W such that $W - X$ and $W - X'$ can be partitioned into pairwise isomorphic subschemes $W_1, \ldots, W_r$ and $W_1', \ldots, W_r'$, respectively.

It follows that

$$[X] = [W] - \sum_{i=1}^{r} [W_i] = [W] - \sum_{i=1}^{r} [W_i'] = [X'],$$

even though X and X' are not piecewise isomorphic.

Remark

The varieties X and X' in Borisov's example are smooth, but not proper.

The ring of stable birational types

SB_F = set of stable birational equivalence classes of integral F-varieties

$[X]_{\mathrm{sb}}$ = equivalence class of X.

We consider the free abelian group $\mathbb{Z}[\mathrm{SB}_F]$.

For any F-scheme X of finite type, we set

$$[X]_{\mathrm{sb}} = [X_1]_{\mathrm{sb}} + \ldots + [X_r]_{\mathrm{sb}} \qquad \text{in } \mathbb{Z}[\mathrm{SB}_F]$$

where $X_1, \ldots, X_r$ are the irreducible components.

In particular, $[X_{\mathrm{red}}]_{\mathrm{sb}} = [X]_{\mathrm{sb}}$ in this group.

Ring product: $[X]_{\mathrm{sb}} \cdot [Y]_{\mathrm{sb}} = [X \times_F Y]_{\mathrm{sb}}$.

The Larsen–Lunts theorem

Theorem (Larsen & Lunts 2003)

Let F be a field of characteristic zero. Then there exists a unique map

$$\mathrm{sb}\colon \mathbf{K}(\mathrm{Var}_F) \to \mathbb{Z}[\mathrm{SB}_F]$$

that maps $[X]$ to $[X]_{\mathrm{sb}}$ for every smooth and proper F-scheme X.

The morphism sb is a surjective ring morphism, and its kernel is the ideal in $\mathbf{K}(\mathrm{Var}_F)$ generated by $\mathbb{L}$.

Therefore,

$$\mathbf{K}(\mathrm{Var}_F)/(\mathbb{L}) \simeq \mathbb{Z}[\mathrm{SB}_F]$$

Sketch of proof.

The morphism sb maps $\mathbb{L} = [\mathbb{P}^1_F] - [\operatorname{Spec} F]$ to 0, because $\operatorname{Spec} F$ is stably birational to $\mathbb{P}^1_F$. Thus sb induces

$$\overline{\mathrm{sb}}\colon \mathbf{K}(\mathrm{Var}_F)/\mathbb{L}\mathbf{K}(\mathrm{Var}_F) \to \mathbb{Z}[\mathrm{SB}_F].$$

Here is the inverse:
By resolution of singularities, every class in SB_F has a representative X that is a connected smooth proper F-scheme.
For every $m \geq 0$, we have

$$[X \times_F \mathbb{P}^m_F] - [X] = [X](\mathbb{L} + \mathbb{L}^2 + \ldots + \mathbb{L}^m)$$

in $\mathbf{K}(\mathrm{Var}_F)$ by the scissor relations.
Thus $[X \times_F \mathbb{P}^m_F]$ and $[X]$ are congruent modulo $\mathbb{L}$.

Sketch of proof.

Moreover, the class of $[X \times_F \mathbb{P}^m_F]$ modulo $\mathbb{L}$ is independent under blow-ups of smooth closed subschemes of $X \times_F \mathbb{P}^m_F$, because the exceptional divisor of such a blow-up is a projective bundle over the center.

Weak Factorization Theorem $\implies$ the class of X in $\mathbf{K}(\mathrm{Var}_F)/\mathbb{L}\mathbf{K}(\mathrm{Var}_F)$ only depends on the stable birational equivalence class of X.

This yields a ring map

$$\mathbb{Z}[\mathrm{SB}_F] \to \mathbf{K}(\mathrm{Var}_F)/\mathbb{L}\mathbf{K}(\mathrm{Var}_F)$$

that is inverse to $\overline{\mathrm{sb}}$. □

Beware: $\mathrm{sb}([X])$ is usually different from $[X]_{\mathrm{sb}}$ when X is not smooth and proper.

Example

In $\mathbf{K}(Var_F)$, we have $[\mathbb{A}^1] = [\mathbb{P}^1] - [\operatorname{Spec} F]$,so

$$\mathrm{sb}(\mathbb{A}^1) = \mathrm{sb}(\mathbb{P}^1) - \mathrm{sb}[\operatorname{Spec} F] = 0$$

So $\mathrm{sb}(\mathbb{A}^1) = 0 \neq \left[\mathbb{A}^1\right]_{\mathrm{sb}}$.

Example

If X is a nodal cubic in $\mathbb{P}^2_F$, then it follows from the scissor relations that

$$[X] = \mathbb{L}$$

in $\mathbf{K}(\mathrm{Var}_F)$. Thus $\mathrm{sb}([X]) = 0$.

Corollary

Let F be a field of characteristic zero, and let X and X' be smooth and proper F-schemes.
Then X and X' are stably birational if and only if $[X] \equiv [X']$ modulo $\mathbb{L}$ in $\mathbf{K}(\mathrm{Var}_F)$.

In particular, $[X] \equiv c$ modulo $\mathbb{L}$ for some integer c if and only if every connected component of X is stably rational; in that case, c is the number of connected components of X.

Remark

Again the corollary is false without the assumption that X and X' are smooth and proper (Borisov's example).

Some notation

Field of Puiseux series:

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m}))$$

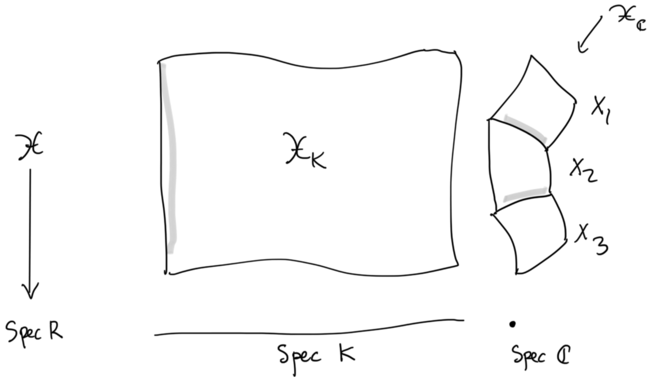
Valuation ring:

$$R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]]$$

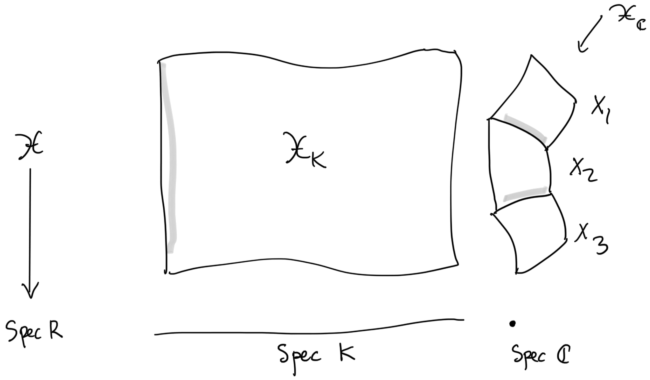
An R-scheme is *strictly semi-stable* if, Zariski locally, it admits an étale morphism to a scheme of the form

$$\operatorname{Spec} R[z_1, \ldots, z_s]/(z_1 \cdots z_r - t^q)$$

where $s \geq r \geq 0$ and q is a positive rational number.



In short, we will consider families $\mathcal{X} \to \operatorname{Spec} R$, and want to compare the rationality properties of the generic fiber $\mathcal{X}_K$, to that of the special fiber, $\mathcal{X}_{\mathbb{C}}$.



Note however that $\mathcal{X}_{\mathbb{C}}$ may have several irreducible components, so it makes most sense to do this comparison in $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$.

The theorem of Nicaise–Shinder

Definition

If $\mathcal{X}$ is strictly semi-stable, then a *stratum* of the special fiber $\mathcal{X}_k$ is a connected component E of an intersection of irreducible components of $\mathcal{X}_k$.

$$\mathcal{S}(\mathcal{X}) := \text{the set of strata of } \mathcal{X}_k.$$

Theorem (Nicaise–Shinder)

There exists a unique ring morphism

$$\mathrm{Vol}\colon \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_k]$$

such that, for every strictly semistable proper R-scheme $\mathcal{X}$ with smooth generic fiber $X = \mathcal{X}_K$, we have

$$\mathrm{Vol}([X]_{\mathrm{sb}}) = \sum_{E \in \mathcal{S}(X)} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}. \tag{5}$$

Let us make the following observations:

- Vol sends $[\operatorname{Spec} K]_{\mathrm{sb}}$ to $[\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}}$.
- If $\mathcal{X} \to \operatorname{Spec} R$ is *smooth and proper*, then $\operatorname{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = [\mathcal{X}_{\mathbb{C}}]_{\mathrm{sb}}$.

These two in conjunction have an important consequence, namely that if $\mathcal{X} \to \operatorname{Spec} R$ is smooth and proper, and the generic fiber $\mathcal{X}_K$ is geometrically stably rational, then so is the special fiber.

Theorem

Stable rationality specializes in smooth and proper families.

This was a long-standing open question, solved by Nicaise–Shinder (and Kontsevich–Tschinkel with 'stable rationality' replaced by 'rationality').

More generally:

Corollary

Let S be a Noetherian $\mathbb{Q}$-scheme, and let $X \to S$ and $Y \to S$ be smooth and proper morphisms.
Then the set

$$\{s \in S \,|\, X \times_S \overline{s} \text{ is stably birational to } Y \times_S \overline{s} \text{ for any geometric point } \overline{s} \text{ based at } s\}$$

is a countable union of **closed** subsets of S.

In particular, the set

$$\{s \in S \,|\, X \times_S \overline{s} \text{ is stably rational, for any geometric point } \overline{s} \text{ based at } s\}$$

is a countable union of **closed** subsets of S.

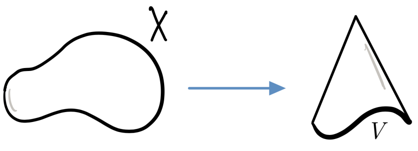
Example (Rational specializing to irrational)

Consider the family

$$\mathcal{X} = \{x_0^3 + x_1^3 + x_2^3 + t^3 x_3^3 = 0\} \subset \mathbb{P}^3 \times \mathbb{A}^1$$

The fiber over $t = 0$ is a cone $C(V)$ over the elliptic curve $V := \{x_0^3 + x_1^3 + x_2^3 = 0\}$,

which is irrational.



What goes wrong in this example?

Example (Rational specializing to irrational)

Issue: The family $\mathcal{X}$ is not strictly semi-stable.
Consider the blow-up $\mathcal{Y} \to \mathcal{X}$ of the vertex of the cone $\mathcal{X}_0 = C(V)$:

$$\mathcal{Y} \to \mathbb{A}^1$$

This is now strictly semi-stable.
The fiber $\widetilde{Y}_0$ has two components $\widetilde{X}_0$ and the exceptional divisor E.
We have $E \simeq$ cubic surface, so

$$\begin{aligned} \mathrm{Vol}(\mathcal{X}_K) = \mathrm{Vol}(\mathcal{Y}_K) &= \left[\widetilde{X}_0\right]_{\mathrm{sb}} + [E]_{\mathrm{sb}} - \left[E \cap \widetilde{X}_0\right]_{\mathrm{sb}} \\ &= \left[\mathbb{P}^1 \times V\right]_{\mathrm{sb}} + \left[\mathbb{P}^2\right]_{\mathrm{sb}} - [V]_{\mathrm{sb}} \\ &= [\operatorname{Spec} F]_{\mathrm{sb}} \end{aligned}$$

So there is no contradiction.

Toroidal models

For our main applications, we need a more flexible notion than semi-stability:

Definition

A monoid M is called *toric* if it is isomorphic to the monoid of lattice points in a strictly convex rational polyhedral cone.

To any monoid M we can attach its monoid R-algebra $R[M]$; the monomial associated with an element $m \in M$ will be denoted by x^m.

Definition

Let $\mathcal{X}$ be a flat separated R-scheme of finite presentation.
We say that $\mathcal{X}$ is *strictly toroidal* if, Zariski-locally on $\mathcal{X}$, we can find a smooth morphism

$$\mathcal{X} \to \operatorname{Spec} R[M]/(x^m - t^q)$$

for some toric monoid M, some positive rational number q, and some element m in M such that $k[M]/(x^m)$ is reduced.

Example

Consider the scheme

$$\operatorname{Spec} R[x, y, z, w]/(t - xy, t - zw),$$

which is clearly strictly toroidal.

The special fiber has four irreducible components of dimension 2 intersecting at the origin, which never happens for strictly semi-stable schemes.

The following schemes will be important when degenerating complete intersections:

Example

Let r and s be positive integers, and let $a = (a_1, \ldots, a_r)$ and $b = (b_1, \ldots, b_s)$ be tuples of positive integers. Consider the R-schemes

$$\begin{aligned} \mathcal{X}_a &= \operatorname{Spec} R[x_{i,j} \mid i = 1, \ldots, r;\ j = 1, \ldots, a_i]/(t - \prod_{j=1}^{a_1} x_{1,j}, \ldots, t - \prod_{j=1}^{a_r} x_{r,j}), \\ \mathcal{Y}_b &= \operatorname{Spec} R[y_{i,j} \mid i = 1, \ldots, s;\ j = 0, \ldots, b_i]/(ty_{1,0} - \prod_{j=1}^{b_1} y_{1,j}, \ldots, ty_{s,0} - \prod_{j=1}^{b_s} y_{s,j}). \end{aligned}$$

Then $\mathcal{X}_a$, $\mathcal{Y}_b$ and $\mathcal{X}_a \times_R \mathcal{Y}_b$ are strictly toroidal.

Note that $\mathcal{X}$ is *strictly semi-stable* if it admits Zariski-locally a smooth morphism to a scheme of the form $\mathcal{X}_a$ with $r = 1$.

Advantages of toroidal singularities

- The product of two strictly toroidal R-schemes is again strictly toroidal. This is no longer true for strictly-semistable.
- The condition of strict semi-stability is quite restrictive, and producing a semi-stable model often leads to many blow-ups which which are hard to analyze. The toroidal condition is much more flexible, and reduces the computations substantially.
- Strictly toroidal degenerations also arise naturally when we break up projective hypersurfaces into pieces of smaller degrees:

Example

Let $f_0, \ldots, f_r \in k[z_0, \ldots, z_{n+1}]$ be general homogeneous polynomials of positive degrees $d_0, \ldots, d_r$ such that $d_0 = d_1 + \ldots + d_r$.
Then

$$\mathcal{X} = \operatorname{Proj} R[z_0, \ldots, z_{n+1}]/(tf_0 - f_1 \cdot \ldots \cdot f_r)$$

is strictly toroidal.

$\mathcal{X}$ is not strictly semi-stable at the points of $\mathcal{X}_k$ where f_0 and at least two among $f_1, \ldots, f_r$ vanish.

The theorem of Nicaise-Shinder (toroidal version)

Recall:

$$\mathcal{S}(\mathcal{X}) = \text{the set of strata of the special fiber } \mathcal{X}_k.$$

Theorem (Nicaise-Shinder)

There exists a unique ring morphism

$$\mathrm{Vol}\colon \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_k]$$

such that, for every strictly toroidal proper R-scheme $\mathcal{X}$ with smooth generic fiber $X = \mathcal{X}_K$, we have

$$\mathrm{Vol}([X]_{\mathrm{sb}}) = \sum_{E \in \mathcal{S}(X)} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}. \tag{6}$$

Lecture 3:
First applications

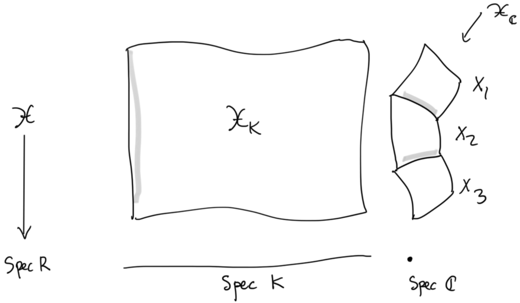
A quick summary so far

SB_F = set of stable birational equivalence classes of integral F-varieties

The ring of stable birational types: $\mathbb{Z}[\mathrm{SB}_F]$.

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m})), \qquad R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]].$$

We consider schemes $\mathcal{X}/R$ which are either semistable, or more generally, toroidal.



The theorem of Nicaise–Shinder

Theorem (Nicaise–Shinder)

There exists a unique ring morphism

$$\mathrm{Vol}\colon \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_k]$$

such that, for every strictly semistable (or toroidal) proper R-scheme $\mathcal{X}$ with smooth generic fiber $X = \mathcal{X}_K$, we have

$$\mathrm{Vol}([X]_{\mathrm{sb}}) = \sum_{E \in \mathcal{S}(X)} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}.$$

Here $\mathcal{S}(\mathcal{X})$ denotes the set of strata of $\mathcal{X}_k$.

Important observation: Vol maps $\mathrm{Spec}\, K$ to $\mathrm{Spec}\, k$.

A key idea in [NO20], is to use this an obstruction to stable rationality of $\mathcal{X}_K$:

Corollary

1. Let X be a smooth and proper K-scheme. If

$$\mathrm{Vol}([X]_{\mathrm{sb}}) \neq [\operatorname{Spec} k]_{\mathrm{sb}}$$

in $\mathbb{Z}[\mathrm{SB}_k]$, then X is not stably rational.

2. Let $\mathcal{X}$ be a strictly semistable proper R-scheme with smooth generic fiber $X = \mathcal{X}_K$. If

$$\sum_{E \in \mathcal{S}(\mathcal{X})} (-1)^{\operatorname{codim}(E)} [E]_{\mathrm{sb}} \neq [\operatorname{Spec} k]_{\mathrm{sb}}$$

in $\mathbb{Z}[\mathrm{SB}_k]$, then X is not stably rational.

Proof.

If X is stably rational, then $[X]_{\mathrm{sb}} = [\operatorname{Spec} K]_{\mathrm{sb}}$ so that $\mathrm{Vol}([X]_{\mathrm{sb}}) = [\operatorname{Spec} k]_{\mathrm{sb}}$.
The second part of the statement follows immediately from the formula for Vol. □

Example (Voisin)

A very general double quartic threefold is irrational.

Sketch of proof.

Let $f, g \in \mathbb{C}[x, y, z, w]$ denote quartics, so that f appears in the Artin-Mumford example

$$w^2 = f(x, y, z, w) \subset \mathbb{P}(1, 1, 1, 1, 2).$$

Consider the family

$$\mathcal{X} = \{w^2 = f(x, y, z, w) + tg(x, y, z, w)\} \subset \mathbb{P}(1, 1, 1, 1, 2) \times \mathbb{A}^1$$

Note: $\mathcal{X}_0$ is the Artin-Mumford threefold.

Sketch of proof.

The family $\mathcal{X}/\mathbb{A}^1$ becomes semi-stable after blowing up the 10 nodes in the special fiber $\mathcal{X}_0$.
Let $\mathcal{Y} \to \mathbb{A}^1$ denote the resulting family.
As the blow-ups only introduce rational varieties in the special fiber, we get

$$\begin{aligned} \mathrm{Vol}(\mathcal{X}_K) &= \mathrm{Vol}(\mathcal{Y}_K) \\ &= [\widetilde{X_0}]_{\mathrm{sb}} + a[\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} \text{ for some } a \in \mathbb{Z} \\ &\neq [\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} \qquad \text{in } \mathbb{Z}[\mathrm{SB}_{\mathbb{C}}] \end{aligned}$$

because $[\widetilde{X_0}]$ is not stably rational.
$\rightsquigarrow$ $\mathcal{X}_K$ is not stably rational.
$\rightsquigarrow$ the very general double quartic solid is not stably rational. $\square$

For our main applications, we get better results using degenerations with many components.

Main strategy in [NO20]:
Look for suitable degenerations

$$\mathcal{X} \to \operatorname{Spec} R$$

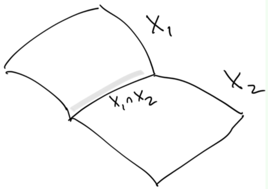
with $\mathcal{X}_K \subset \mathbb{P}_K^{n+1}$ smooth hypersurface, with the property that
stably irrational strata of low dimension do not cancel out in the alternating sum

$$\mathrm{Vol}([X]_{\mathrm{sb}}) = \sum_{E \in \mathcal{S}(X)} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}.$$

$\therefore$ We deduce irrationality of $\mathcal{X}_K$ from that of varieties of lower dimension.

Example (Two components in the special fiber)

Suppose the special fiber $\mathcal{X}_{\mathbb{C}} = X_0 \cup X_1$, intersecting along X_{01}.



The motivic volume takes the form

$$\mathrm{Vol}(\mathcal{X}_K) = [X_0]_{\mathrm{sb}} + [X_1]_{\mathrm{sb}} - [X_{01}]_{\mathrm{sb}}$$

From this, we deduce that either of the following conditions guarantee that the generic fiber $\mathcal{X}_K$ is not stably rational:

i) Exactly one of X_0, X_1, X_{01} is stably irrational.

ii) X_0 and X_1 are both stably irrational.

iii) X_0 and X_{01} are stably irrational, but they are not stably birational to each other.

iv) X_0, X_1, X_{01} are all stably irrational.

Quartic fivefolds

Quartic fivefolds

Let $F \in \mathbb{C}[x_0, \ldots, x_6]$ be a very general homogeneous polynomial of degree 4.

Consider the following R-scheme

$$\mathcal{X} = \operatorname{Proj} R[x_0, \ldots, x_6, y]/(x_5 x_6 - ty, y^2 - F) \tag{7}$$

where the variable y has weight 2.

Note that the generic fiber $\mathcal{X}_K$ is isomorphic to a smooth quartic hypersurface in $\mathbb{P}^6_K$ (inverting t allows us to eliminate y using the first equation).

Moreover, $\mathcal{X}$ is strictly toroidal.

The special fiber has two components:

$$\begin{aligned} X_0 &= \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_6, y]/(x_5, y^2 - F) \\ X_1 &= \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_6, y]/(x_6, y^2 - F). \end{aligned}$$

Note that these are both very general quartic double fivefolds.

We do not know whether these are stably rational or not.

However, their intersection,

$$X_{01} = \operatorname{Proj} \mathbb{C}[x_0, \ldots, x_4, y]/(y^2 - F)$$

is a very general quartic double fourfold, and thus stably irrational [Hassett–Pirutka–Tschinkel].

In either case, we get

$$\begin{aligned} \operatorname{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) &= [X_0]_{\mathrm{sb}} + [X_1]_{\mathrm{sb}} - [X_{01}]_{\mathrm{sb}} \\ &\neq [\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}} \end{aligned}$$

On (2,3)-complete intersections

Theorem

Very general complete intersections of a quadric and a cubic in $\mathbb{P}^n$ are stably irrational for $n \leq 6$.

Our main contribution is stable irrationality for $n = 6$.

History related to the *Lüroth problem*:

- Fano (1908): (Incorrect) proof of irrationality for $n = 5$
- Enriques (1912): Proof of unirationality for $n = 5$
- Hassett–Tschinkel (2018): Stable irrationality for $n = 5$.
- Morin (1955), Conte–Murre (1998): Unirationality for $n = 6$.

The above result settles the rationality problem for all complete intersections of dimension ≤ 4 - except cubic fourfolds.

The proof for $(2,3)$-complete intersections

Let $\mathbb{P}^6 = \text{Proj } k[x_0, \ldots, x_6]$ and let $P = \{x_0 = \ldots = x_3 = 0\} \simeq \mathbb{P}^2$.

$$Y = \{q = c = 0\} \subset \mathbb{P}^6$$

for q and c very general of degree 2 and 3.
We assume Y contains P and is very general with respect to this property.

Blow up the plane P:

$$\begin{array}{ccc} X \subset Bl_P\mathbb{P}^6 & \xrightarrow{\pi} & \mathbb{P}^6 \\ \downarrow{\scriptstyle p} & & \\ \mathbb{P}^3 & & \end{array}$$

$X = Q \cap C$ where $Q \in |2H - E|$ and $C \in |3H - E|$.

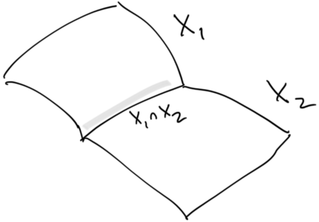
It suffices to show that generic intersections

$$X = Q \cap C \subset Bl_P\mathbb{P}^6$$

where $Q \in |2H - E|$ and $C \in |3H - E|$ are stably irrational.

Now degenerate Q to $Q_0 + E$ where $Q_0 \in |2H - 2E| = |2p^*h|$.

This induces a degeneration of $\mathcal{X} \to \mathbb{A}^1$ with special fiber $\mathcal{X}_0 = X_1 \cup X_2$:



There are three strata:

- $X_1 = Q_0 \cap C$
- $X_2 = E \cap C$
- $X_{12} = Q_0 \cap E \cap C$

The stratum $X_1 = Q_0 \cap C$:

$$\begin{array}{ccccc} Q_0 = \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}^3 \oplus \mathcal{O}(1,1)) & \hookrightarrow & \mathbb{P}(\mathcal{O}^3 \oplus \mathcal{O}(1)) & \xrightarrow{\pi} & \mathbb{P}^6 \\ \downarrow & & \downarrow p & & \\ \mathbb{P}^1 \times \mathbb{P}^1 & \hookrightarrow & \mathbb{P}^3 & & \end{array}$$

$C|_{Q_0}$ is a very general divisor in $|\mathcal{O}(2) \otimes p^*\mathcal{O}(1,1)|$ in $\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}^3 \oplus \mathcal{O}(1,1))$.

$\rightsquigarrow$ X_1 is stably irrational by [Schreieder 2017].

The strata $X_2 = E \cap C$ and $X_{12} = E \cap Q_0 \cap C$

C restricts to a $(1,2)$-divisor on $E \simeq \mathbb{P}^2 \times \mathbb{P}^3$

Q_0 restricts to a $(0,2)$-divisor on $E \simeq \mathbb{P}^2 \times \mathbb{P}^3$.

$\rightsquigarrow$ X_2 and X_{12} are both rational.

By the motivic volume formula:

$$\begin{aligned} \mathrm{Vol}([\mathcal{X}]_{\mathrm{sb}}) &= [X_1]_{\mathrm{sb}} + [X_2]_{\mathrm{sb}} - [X_{12}]_{\mathrm{sb}} \\ &= [X_1]_{\mathrm{sb}} + [\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} - [\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} \\ &= [X_1]_{\mathrm{sb}} \\ &\neq [\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} \end{aligned}$$

This implies that a very general X is stably irrational. $\square$

Improvements

Remark

[Pavic–Schreieder 2021] extended this proof to show that a very general quartic fivefold does not admit a decomposition of Δ.

Remark

The result on $(2,3)$ complete intersections was extended by [Skauli 2021], who:

- Showed that these fourfolds do not admit a decomposition of Δ.
- Gave explicit examples (over $\mathbb{Q}$) of stably irrational $(2,3)$-fourfolds.

Here the decomposition of the Δ-technique leads to more computations, but has the advantage it also works in positive characteristic.

Lecture 4:
Toric degenerations

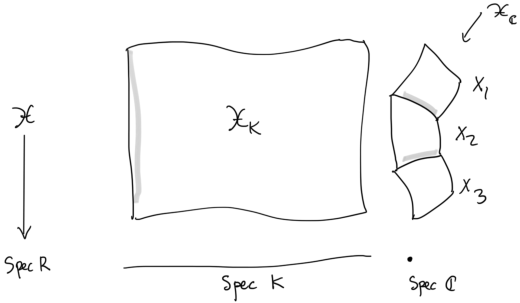
A quick summary so far

SB_F = set of stable birational equivalence classes of integral F-varieties

The ring of stable birational types: $\mathbb{Z}[\mathrm{SB}_F]$.

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m})), \qquad R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]].$$

We consider schemes $\mathcal{X}/R$ which are either semistable, or more generally, toroidal.



The theorem of Nicaise–Shinder

Theorem (Nicaise–Shinder)

There exists a unique ring morphism

$$\mathrm{Vol}\colon \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_k]$$

such that, for every strictly semistable (or toroidal) proper R-scheme $\mathcal{X}$ with smooth generic fiber $X = \mathcal{X}_K$, we have

$$\mathrm{Vol}([X]_{\mathrm{sb}}) = \sum_{E \in \mathcal{S}(X)} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}.$$

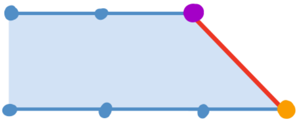
Here $\mathcal{S}(\mathcal{X})$ denotes the set of strata of $\mathcal{X}_k$.

Important observation: Vol maps $\operatorname{Spec} K$ to $\operatorname{Spec} k$.

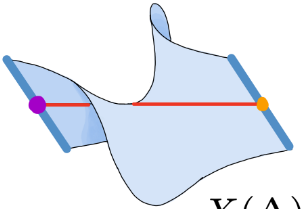
Projective toric varieties

$$\left\{ \begin{array}{c} \text{projective toric varieties } (X, L), \\ L \text{ basepoint free ample line bundle} \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{lattice polytopes } \Delta \subset \mathbb{R}^n \\ L \text{ defined up to translation} \end{array} \right\}$$

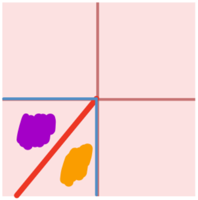
1-1 inclusion preserving correspondence between faces of Δ and toric strata of X:



Δ



$X(\Delta)$



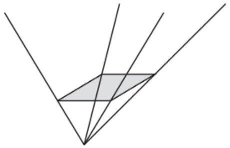
Σ_Δ

We use the standard notations M, N, $M_{\mathbb{R}}$, $N_{\mathbb{R}}$ from toric varieties.

Let $\Delta \subset M_{\mathbb{R}}$ be a lattice polyhedron.

Consider the *cone* over Δ:

$$C(\Delta) = \text{closure of } \{(rm, r) | m \in \Delta, r \geq 0\} \subset M_{\mathbb{R}} \oplus \mathbb{R}$$



This cone is rational polyhedral, with asymptotic cone

$$C(\Delta) \cap (M_{\mathbb{R}} \oplus 0) = \text{Asym}(\Delta)$$

(asymptotic cone of Δ = Hausdorff limit of $r\Delta$ as $r \to 0$).

The finitely generated k-algebra

$$S_\Delta := k[C(\Delta) \cap (M \oplus \mathbb{Z})]$$

has a grading given by $\deg z^{(m,d)} = d$.

Degree 0 part:

$$(S_\Delta)_0 = k[\mathrm{Asym}(\Delta) \cap M]$$

The toric variety

$$X(\Delta) := \operatorname{Proj} S_\Delta$$

is projective over $\operatorname{Spec} k[\mathrm{Asym}(\Delta) \cap M]$.

Projective embedding: (if Δ is "very ample"):

If $m_i = (m_{i1}, \dots, m_{in}) \in \mathbb{Z}^n$ $i = 0, \dots, r$ are the integral points of Δ, we get a map

$$\begin{aligned} \phi : (\mathbb{C}^*)^n &\rightarrow \mathbb{P}^r \\ x &\mapsto [x^{m_0}, \dots, x^{m_r}] \end{aligned}$$

where we (as usual) write

$$x^{m_i} := x_1^{m_{i1}} \cdots x_n^{m_{in}}$$

Then $X(\Delta)$ is the closure of the image of ϕ.

Facts

- There is a 1-1 inclusion preserving correspondence between faces of Δ and toric strata of $X(\Delta)$.

- Since $X(\Delta)$ is defined as a Proj, there is a natural line bundle $L = \mathcal{O}(1)$.

  $H^0(\Sigma_\Delta, \mathcal{O}(1))$ has a basis corresponding to the integral points of Δ.

Example (Projective space)

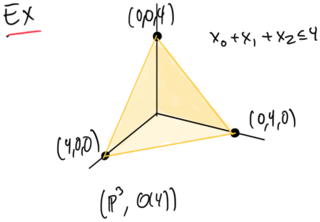
$(\mathbb{P}^n, \mathcal{O}(1))$ is given by the *n-dimensional simplex*

$$\Delta = \left\{ \sum x_i \leq 1,\ x_i \geq 0 \right\}$$

More generally, $(\mathbb{P}^n, \mathcal{O}(d))$ is given by the *dialated simplex*

$$d\Delta = \left\{ \sum x_i \leq d,\ x_i \geq 0 \right\}$$

This is the d-th Veronese embedding of $\mathbb{P}^n$.



Example (Product polytopes)

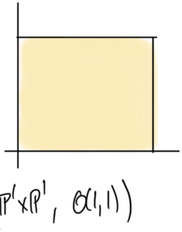
If (X, L) and (Y, M) correspond to polytopes $P_X \subset \mathbb{R}^n$ and $P_Y \subset \mathbb{R}^m$, then the product

$$(X \times Y, L \boxtimes M)$$

is given by the product polytope $P_X \times P_Y \subset \mathbb{R}^{n+m}$.

For instance $(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b))$ is given by the rectangle

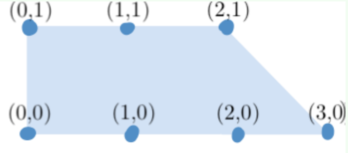
$$P_{a,b} = \{(x,y) \,|\, 0 \leq x \leq a, 0 \leq y \leq b\}$$



Example (Blow-up)

Consider the trapezoid

$$T_{a,b} = \{(x,y)|0 \leq x, 0 \leq y \leq b, x+y \leq a\}$$

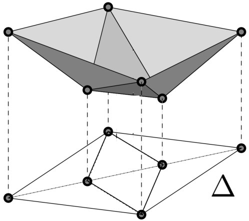


$T_{3,1}$

The corresponding toric variety is $X = Bl_p\mathbb{P}^2$ polarized by the line bundle $L = aH - (a-b)E$.

In general, one obtains the polytope of a blow-up X of a variety Y by "chopping off a corner" of the polytope Δ_Y.

Regular subdivisions



A subdivision $\mathscr{P}$ of Δ is called *regular* if there is a piecewise linear function $\phi : \Delta \to \mathbb{R}_{\geq 0}$ such that

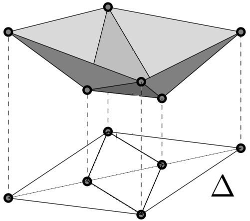
(i) The polytopes of $\mathscr{P}$ are the orthogonal projections on the hyperplane $z = 0$ of $\mathbb{R}^{n+1}$ of the faces of the *upper convex hull*

$$\widetilde{\Delta} := \{(x, z) \in \Delta \times \mathbb{R} | 0 \leq z \leq \phi(x)\}$$

which are not vertical nor equal to Δ.

(ii) The function ϕ is strictly convex, i.e., the hyperplanes determined by each of the faces of $\widetilde{\Delta}$ intersect $\widetilde{\Delta}$ only along that face.

The Mumford Degeneration



Given a regular subdivision $\mathscr{P}$, we can construct a (flat) degeneration

$$\mathcal{X} \to \mathbb{A}^1,$$

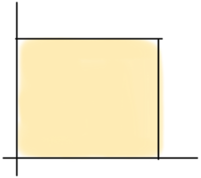
satisfying:

- $\mathcal{X} - \mathcal{X}_0 \simeq X(\Delta) \times \mathbb{C}^*$.
- The special fiber $\mathcal{X}_0$ is a union of toric varieties

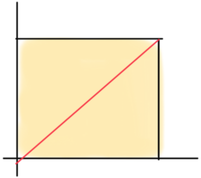
$$\mathcal{X}_0 = \bigcup_{P \in \mathscr{P}} X(P)$$

- The components intersect according to the combinatorics of the subdivision: If $P, Q \in \mathscr{P}$ share a common face R, then $X(P) \cap X(Q)$ can be identified with the toric variety $X(R)$ (which is a subvariety of both).

Ex



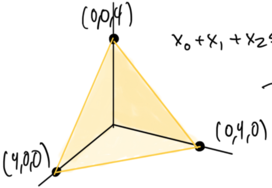
subdivide $\rightsquigarrow$



$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)) \rightsquigarrow (\mathbb{P}^2, \mathcal{O}(1)) \cup (\mathbb{P}^2, \mathcal{O}(1))$

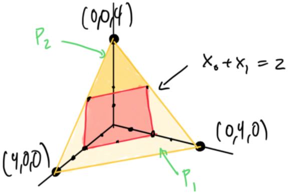
intersecting along a $(\mathbb{P}^1, \mathcal{O}(1))$.

Ex



$x_0 + x_1 + x_2 \leq 4$

$\rightsquigarrow$



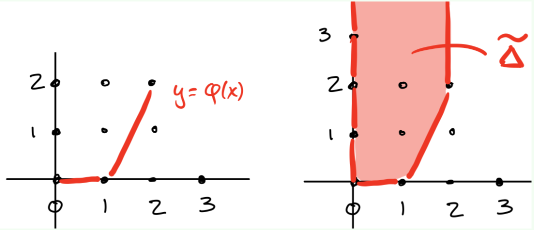
$(\mathbb{P}^3, \mathcal{O}(4)) \rightsquigarrow$ union of two toric 3-folds intersecting along $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\phi : \Delta \to \mathbb{R}_{\geq 0}$ be a piecewise linear function taking integer values on $\Delta \cap M$.

$$\widetilde{\Delta} = \{(m, r) | m \in \Delta, r \geq \phi(m)\} \subset M_{\mathbb{R}} \oplus \mathbb{R}$$

Example

$\Delta = [0, 2] \longleftrightarrow (\mathbb{P}^1, \mathcal{O}(2))$.
Define ϕ by $\phi(0) = \phi(1) = 0$, $\phi(2) = 2$.



Subdivision: $\mathscr{P} = \{[0, 1], [1, 2], \{1\}\}$

Asymptotic cone of $\widetilde{\Delta}$:

$$\mathrm{Asym}(\Delta) = 0 \oplus \mathbb{R}_{\geq 0}$$

$\rightsquigarrow$ $k[C(\widetilde{\Delta}) \cap (M \oplus \mathbb{Z} \oplus \mathbb{Z})]$ is a $k[\mathbb{N}]$-algebra.

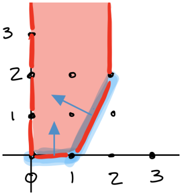
$\rightsquigarrow$ $X(\widetilde{\Delta})$ is a toric variety with a projective morphism

$$\pi : X(\widetilde{\Delta}) \to \mathbb{A}^1_k$$

This is the *Mumford degeneration* associated to Δ and ϕ.

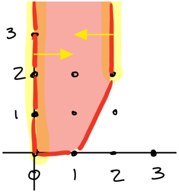
$\widetilde{\Delta}$ has two types of faces:

- Horizontal faces: mapping homeomorphically to elements of $\mathscr{P}$.



For a maximal face δ for which $\phi|_\delta$ has slope $n_\delta \in N$ has normal cone = ray generated by $(-n_\delta, 1)$.

- Vertical faces: mapping non-homeomorphically to faces of Δ.



If δ is a vertical face, the normal cone $N_{\widetilde{\Delta}}(\delta)$ lies in $N_{\mathbb{R}} \times 0$ (and is a cone in the normal fan to Δ).

The projection

$$\pi : X(\widetilde{\Delta}) \to \mathbb{A}^1_k$$

is given by the monomial z^ρ, where $\rho = (0,1) \in \mathrm{Asym}(\widetilde{\Delta}) \subset M_{\mathbb{R}} \oplus \mathbb{R}$.

The primitive generators for the rays of $\Sigma(\widetilde{\Delta})$ are either of the form $(n,0)$ or $(n,1)$ for $n \in N$.

$\rightsquigarrow$ z^ρ does not vanish on divisors corresponding to rays of the first type, and vanishes with order 1 along the divisors corresponding to the second type.

Hence (scheme-theoretically),

$$\pi^{-1}(0) = \bigcup_{\delta \in \mathscr{P}_{\max}} X(\delta)$$

$X(\widetilde{\Delta}) - \pi^{-1}(0)$ is isomorphic to $X(\Delta) \times \mathbb{C}^*$:

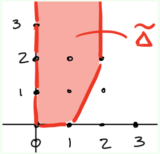
Reason:

Localize $k[C(\widetilde{\Delta}) \cap (M \oplus \mathbb{Z} \oplus \mathbb{Z})]$ at $z^{(0,1,0)}$.

This is the same thing as replacing $\widetilde{\Delta}$ with $\Delta \times \mathbb{R}$

$X(\Delta \times \mathbb{R}) = X(\Delta) \times \operatorname{Spec} k[\mathbb{Z}] = X(\Delta) \times \mathbb{C}^*$.

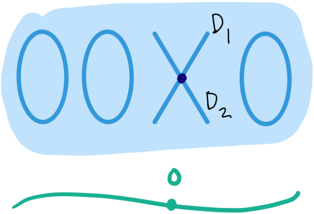
Example



$\mathscr{P}$ has two maximal faces, so

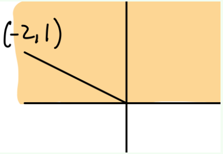
$$\pi^{-1}(0) = D_1 \cup D_2$$

$D_1 \cap D_2$ is defined by the vertex $v = (1, 0) \in \widetilde{\Delta}$.



Example

The normal fan:



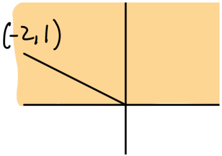
The monoid $K_v\widetilde{\Delta} \cap \mathbb{Z}^2$ has generators $(-1,0), (1,2), (0,1)$.

$$k[K_v\widetilde{\Delta} \cap \mathbb{Z}^2] \simeq k[z_1, z_2, t]/(z_1z_2 - t^2)$$

where $z_1 = z^{(-1,0)}$, $z_2 = z^{(-1,0)}$, $t = z^{(0,1)}$.

This is a local model of the smoothing of a node.

In this example, the total space has an A_1-singularity.



We can understand this from the normal fan:

Start with $\mathbb{A}^1 \times \mathbb{P}^1$ and perform a *weighted blow-up* by adding the ray $(-2, 1)$.

This gives another $\mathbb{P}^1$ and an A_1 singularity.

Newton subdivision

Let

$$f = \sum_m c_m x^m \in K[M]$$

be a Laurent polynomial with Newton polytope $\Delta \subset \mathbb{R}^{n+1}$.

$\phi : \Delta \to \mathbb{R}$ given by the lower convex envelope of the function

$$m \mapsto \mathrm{ord}_t(c_m).$$

$\rightsquigarrow$ regular subdivision $\mathscr{P}$ + corresponding degeneration of $X(\Delta)$.

For every face δ of $\mathscr{P}$, set

$$f_\delta = \sum_{\mathbb{Z}^{n+1}\cap\delta} c_m x^m$$

Non-degeneracy condition: We assume that $Z(f_\delta)$ is smooth for all δ.

Let $\mathcal{X} = X(\Delta) \times_{k[t]} R$.

$\rightsquigarrow$ $\mathcal{X}_K = X_K(\Delta)$ and $\mathcal{X}_k = \bigcup_{P\in\mathscr{P}_{\max}} X(P)$.

Taking the Zariski closure of $Z(f)$ in $\mathcal{X}_K$, we also get a degeneration

$$\mathcal{Y} \to \mathbb{A}^1_k$$

with $\mathcal{Y}_K = Z(f)$.

Proposition

Assuming that f is non-degenerate in the above sense, the corresponding degeneration has toroidal singularities. Hence we can apply the motivic volume formula.

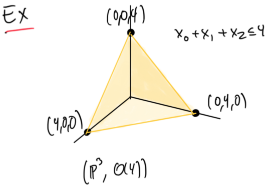
Definition

A polytope Δ is called *stably irrational* if:
for every algebraically closed field F of characteristic 0, and every very general polynomial $g \in F[M]$ with Newton polytope Δ, the hypersurface $Z(g)$ is stably irational.

Otherwise we say Δ is stably rational.

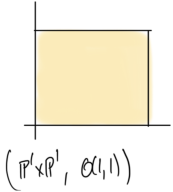
Example

The dilated $(n+1)$-simplex $d\Delta \subset \mathbb{R}^{n+1}$ is stably irrational if and only if the very general degree d hypersurface in $\mathbb{P}^{n+1}$ is not stably rational.



Example

The product polytope $2\Delta_2 \times 2\Delta_3 \subset \mathbb{R}^5$ is stably irrational
(by Hassett-Pirutka-Tschinkel).



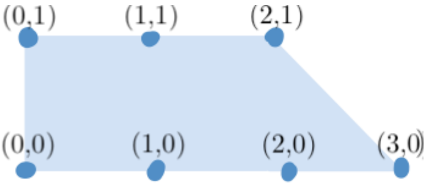
Degenerating a hypersurface

Example (Lattice width 1)

If Δ is a polytope with *lattice width 1*, then Δ is stably rational.

Reason: A polynomial f with that Newton polytope is linear in one variable (after a change of coordinates).

e.g., $1 + 2x + x^3 + xy + x^2y$ has Newton polytope:



A nodal cubic curve

Example

f_0 := general homogeneous polynomial of degree d in $k[z_1, \ldots, z_{n+1}]$
f_1 :=general homogeneous polynomial of degree $d-1$ in $k[z_0, \ldots, z_{n+1}]$

Let

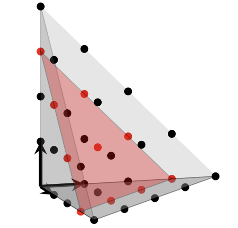
$$f = tf_0 + z_0 f_1$$

Newton polytope:

$$\Delta = \{(u_0, \ldots, u_{n+1}) | u_0 + \ldots + u_{n+1} = d\} \subset \mathbb{R}^{n+2}_{\geq 0}$$

Example

The subdivision is induced by $\phi = \max\{0, 1 - u_0\}$:



Two maximal cells:

$$\delta_{\leq} = \{(u_0, \ldots, u_{n+1}) | u_0 \leq 1\}$$

$$\delta_{\geq} = \{(u_0, \ldots, u_{n+1}) | u_0 \geq 1\}$$

With intersection

$$\delta_{=} = \{(u_0, \ldots, u_{n+1}) | u_0 = 1\}$$

Example

The toric $k[t]$-scheme $X(\widetilde{\Delta})$ defined by ϕ is the blow-up of

$$\mathbb{P}^{n+1}_{k[t]} = \operatorname{Proj} k[t][z_0, \ldots, z_{n+1}]$$

in $H = \{z_0 = t = 0\} \subset \mathbb{P}^{n+1}_k$.

For the R-scheme $\mathcal{X} = X(\widetilde{\Delta}) \times_{k[t]} R$, we have

$$\mathcal{X}_k = D_1 + D_2$$

where
$D_1 \simeq \mathbb{P}^{n+1}_k$ (strict transform);
$D_2 \simeq \mathbb{P}(\mathcal{O}_H \oplus \mathcal{O}_H(1))$ (exceptional divisor).
$D_1 \cap D_2 \simeq \mathbb{P}^n_k$.

Example

The Zariski closure

$$\mathcal{Y} \to \operatorname{Spec} R$$

of $Z(f) \subset X_K = \mathbb{P}_K^{n+1}$ in $\mathcal{X}$ gives a proper and *semistable* R-model of $Z(f)$.

Two components in the special fiber:

$E_1 = \mathcal{Y} \cap D_1 =$ degree $(d-1)$-hypersurface defined by $f_1 = 0$.
$E_2 = \mathcal{Y} \cap D_2 =$ section of $\mathcal{O}(1) \oplus \pi^*\mathcal{O}(d-1)$ in $\mathbb{P}(\mathcal{O}_H \oplus \mathcal{O}_H(1)) \rightsquigarrow$ rational.
Also,
$E_1 \cap E_2 =$ degree $(d-1)$-hypersurface defined by $f_1(0, z_1, \ldots, z_{n+1}) = 0$.

Conslusion:

Theorem

Suppose that a very general hypersurface of degree $d-1$ in $\mathbb{P}^n$ is stably irrational.

Then at least one of the following must hold:

(i) a very general hypersurface of degree d in $\mathbb{P}^{n+1}$ is stably irrational;

(ii) a very general hypersurface of degree d in $\mathbb{P}^n$ is stably irrational

We will improve this result in the next example.

Example

The result for quartic 5-folds implies that we also get stable irrationality for

- Quintic 6-folds
- Sextic 7-folds
- ...

Lecture 5:
Further applications

Recap

The ring of stable birational types: $\mathbb{Z}[\mathrm{SB}_F]$.

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m})), \qquad R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]].$$

Theorem (Nicaise–Shinder)

There exists a unique ring morphism

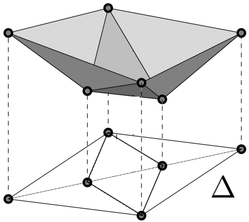
$$\mathrm{Vol}\colon \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_k]$$

such that, for every strictly semistable (or toroidal) proper R-scheme $\mathcal{X}$ with smooth generic fiber $X = \mathcal{X}_K$, we have

$$\mathrm{Vol}([X]_{\mathrm{sb}}) = \sum_{E \in \mathcal{S}(X)} (-1)^{\mathrm{codim}(E)} [E]_{\mathrm{sb}}.$$

Here $\mathcal{S}(\mathcal{X})$ denotes the set of strata of $\mathcal{X}_k$.

Obstruction to rationality: Vol maps $[\mathrm{Spec}\, K]_{\mathrm{sb}}$ to $[\mathrm{Spec}\, k]_{\mathrm{sb}}$.



A regular subdivision $\mathscr{P}$ $\rightsquigarrow$ degeneration of $X(\Delta)$

$$\mathcal{X} \to \mathbb{A}^1,$$

satisfying:

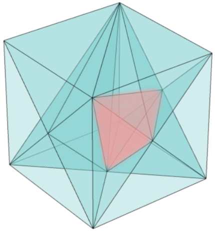
$$\mathcal{X}_0 = \bigcup_{P \in \mathscr{P}} X(P)$$

and if $P, Q \in \mathscr{P}$ share a common face R, then $X(P) \cap X(Q)$ can be identified with the toric variety $X(R)$ (which is a subvariety of both).

Further applications

General strategy for hypersurfaces in a toric variety $X(\Delta)$:

Construct a subdivision $\mathscr{P}$ of Δ, so that all but one lower-dimensional polytope is stably rational (or make sure that the various intersections do not cancel out in the alternating formula for Vol).



Theorem (Increasing degree / decreasing dimension)

Suppose that a very general hypersurface of degree d in $\mathbb{P}^{n+1}$ is stably irrational.

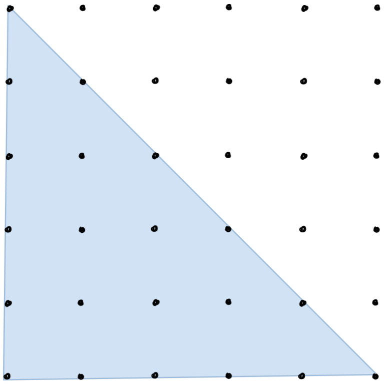
Then we also have that:

(i) A very general hypersurface of degree $d+1$ in $\mathbb{P}^{n+1}$ is stably rational.

(ii) A very general hypersurface of degree d in $\mathbb{P}^{n}$ is stably rational.

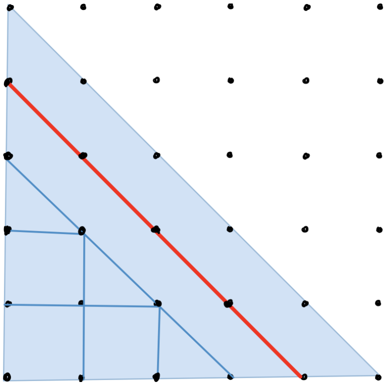
Proof of (i)

Consider the following subdivision of $(d+1)\Delta_{n+1}$:

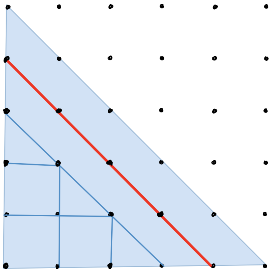


Proof of (i)

Consider the following subdivision of $(d+1)\Delta_{n+1}$:



Proof of (i)



The red polytope corresponds to a degree d hypersurface $Y \subset \mathbb{P}^n$.

All other polytopes have lattice width 1 (hence they are rational).

We get a degeneration $\mathcal{X} \to \operatorname{Spec} R$ of degree $(d+1)$-hypersurfaces in $\mathbb{P}^{n+1}$ with

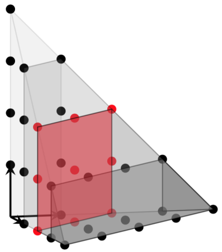
$$\begin{aligned} \operatorname{Vol}(\mathcal{X}_K) &= [Y]_{\mathrm{sb}} + a[\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} \\ &\neq [\operatorname{Spec}\mathbb{C}]_{\mathrm{sb}} \end{aligned}$$

□

The Quartic fivefold again

Newton polytope: $\Delta = \{(x_1, \dots, x_6) \in \mathbb{R}^6_{\geq 0} | \sum_i x_i \leq 4\}$

Subdivision below $\rightsquigarrow$ degeneration with special fiber $X_1 \cup X_2 \cup X_3 \cup X_4$.



Red polytope = $(2,2)$-divisor $Y \subset \mathbb{P}^2 \times \mathbb{P}^3$

$\rightsquigarrow$ stably irrational by [Hassett–Pirutka–Tschinkel 2016].

All other polytopes have *lattice width 1*, hence rational.

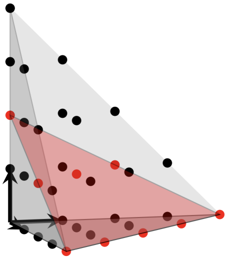
Thus

$$\mathrm{Vol}(\mathcal{X}_K) = [Y]_{\mathrm{sb}} + a[\mathrm{Spec}\,\mathbb{C}]_{\mathrm{sb}} \neq [\mathrm{Spec}\,\mathbb{C}]_{\mathrm{sb}}$$

$\square$

The Quartic fivefold again

Here is the previous degeneration:



Red polytope = double quartic 4-fold.

Variation of birational types

Question: In a family of hypersurfaces

$$\mathcal{X} \to B,$$

how does the stable rationality types vary in the fibers $\mathcal{X}_b$?

Intuition: If some fiber is stably irrational, then the stable birational types should vary.

Theorem

Let W be a variety over k.

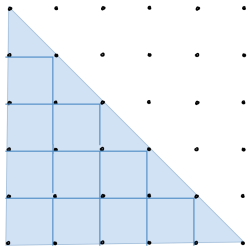
Let Δ be a polytope such that

- Δ is stably irrational.
- Δ admits a regular subdivision $\mathscr{P}$ such that every face of $\mathscr{P}$ which is not contained in $\partial\Delta$ is stably rational.

Then for every very general polynomial $g \in k[M]$ with Newton polytope Δ, the hypersurface

$$Z(g) = \{g = 0\} \subset (\mathbb{C}^*)^n$$

is not stably birational to W.



Corollary (Shinder)

Let W be a k-variety.

If a very general degree-d hypersurface in $\mathbb{P}^n$ is stably irrational,
then a very general degree-d hypersurface in $\mathbb{P}^n$ is not birational to W.

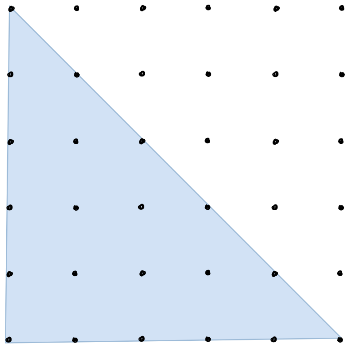
Proposition

Let H be a hyperplane in $\mathbb{P}_k^{n+1}$.

Let X be a degree d hypersurface in $\mathbb{P}_k^{n+1}$ that is very general with respect to H.

If X is stably irrational, then X is not stably birational to $X \cap H$.

Proof:



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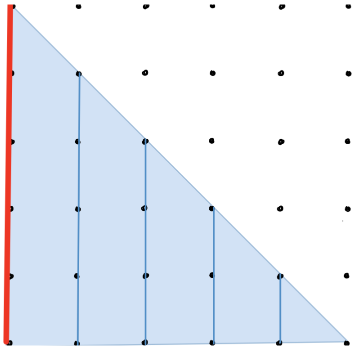


There is a more general result for other polytopes Δ.

Results for complete intersections

Many new classes of complete intersections in $\mathbb{P}^n$

(i) Logarithmic bounds à la Schreieder

(ii) Complete intersections of r quadrics in $\mathbb{P}^n$ are stably irrational if $r \geq 3$ and $2r \geq n-1$.

(iii) In dimension 4:

$$(4), (5), (\mathbf{2}, \mathbf{3}), (2,4), (3,3), (2,2,2), (2,2,3), (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2})$$

(iv) In dimension 5:

$$(\mathbf{4}), (5), (6), (\mathbf{2}, \mathbf{4}), (2,5), (\mathbf{3}, \mathbf{3}), (3,4), (\mathbf{2}, \mathbf{2}, \mathbf{3}), (2,2,4), (2,3,3),$$
$$(\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}), (2,2,2,3), (\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{2}).$$

Proposition

A very general intersection of a **quadric and a quartic** in $\mathbb{P}^8$ is not stably rational.

Let $q, f \in k[x_0, \dots, x_8]$ be very general of degrees $2, 4$.

$$\mathcal{X} := \operatorname{Proj} R[x_0, \dots, x_8]/(f, tq - x_7 x_8)$$

Then $\mathcal{X}_k = E_1 \cup E_2$ where

- $E_1 = \{f = x_7 = 0\}$
- $E_2 = \{f = x_8 = 0\}$
- $E_{12} = \{f = x_7 = x_8 = 0\}$ (stably irrational)

In any case,

$$\begin{aligned} \operatorname{Vol}(\mathcal{X}_K) &= [E_1]_{\mathrm{sb}} + [E_2]_{\mathrm{sb}} - [E_{12}]_{\mathrm{sb}} \\ &\neq [\operatorname{Spec} \mathbb{C}]_{\mathrm{sb}} \end{aligned}$$

Proposition

A very general intersection of **two cubics** in $\mathbb{P}^7$ is not stably rational.

Let $q_1, q_2, c_1, c_2 \in k[x_0, \ldots, x_7]$ be very general forms of degrees $2, 2, 3, 3$.

$$\mathcal{X} := \operatorname{Proj} R[x_0, \ldots, x_7]/(c_1, tc_2 - x_7 q_2)$$

Then $\mathcal{X}_k = E_1 \cup E_2$, where

- $E_1 = \{c_1(x_0, \ldots, x_6, 0) = 0\}$
- $E_2 = \{c_1 = q_2 = 0\}$
- $E_{12} = E_1 \cap E_2 = \{c_1 = q_2 = x_7 = 0\}$ (stably irrational)

As

$$\operatorname{Vol}(\mathcal{X}_K) = [E_1]_{\mathrm{sb}} + [E_2]_{\mathrm{sb}} - [E_1 \cap E_2]_{\mathrm{sb}}$$

it suffices to prove:

Claim 1. E_1 is not stably birational to E_{12}

Claim 2. E_2 is not stably birational to E_{12}

Claim 1. E_1 is not stably birational to E_{12}

$E_1 = \{c_1(x_0, \ldots, x_6, 0) = 0\}$
$E_{12} = E_1 \cap E_2 = \{c_1 = q_2 = x_7 = 0\}$ (stably irrational)

Consider the family

$$\mathcal{Y} = \operatorname{Proj} k[t][x_0, \ldots, x_7]/(tc_1 - x_6 q_2, x_7)$$

We have

$$\mathcal{Y}_k = (x_6 = x_7 = 0) \cup (q_2 = x_7 = 0)$$

a union of two rational varieties intersecting along a rational subvariety, so

$$\operatorname{Vol}(\mathcal{Y} \times_{k[t]} K) = [\operatorname{Spec} k]_{\text{sb}} \neq [E_1 \cap E_2]_{\text{sb}}$$

Hence $\mathcal{Y} \times_{k[t]} K$ is not birational to $E_{12} \times_k K$.

The proof of Claim 2 is very similar.

Theorem

Let $d_1, \ldots, d_r$ be positive integers such that $d_r \geq d_i$ for all i.
Assume that

$$n + r \geq \sum_{i=1}^{r-1} d_i + 2$$

and that there exists a stably irrational smooth hypersurface of degree d_r in $\mathbb{P}_k^{n+r-\sum_{i=1}^{r-1} d_i}$.
Then a very general complete intersection in $\mathbb{P}_k^{n+r}$ of multidegree $(d_1, \ldots, d_r)$ is not stably rational.

Corollary

Let $d_1, \ldots, d_r$ be positive integers such that $d_r \geq 4$ and $d_r \geq d_i$ for all i. Assume that

$$\sum_{i=1}^{r-1} d_i + 2 \leq n + r \leq 2^{d_r - 2} + \sum_{i=1}^{r} d_i - 3.$$

Then a very general complete intersection in $\mathbb{P}_k^{n+r}$ of multidegree $(d_1, \ldots, d_r)$ is not stably rational.

Proposition

Let n and r be integers such that

$$n \geq 3, \quad r \geq 3, \quad r \geq n-1.$$

Then a very general complete intersection of r quadrics in $\mathbb{P}_k^{n+r}$ is stably irrational.

For

$$X = (q_1, \ldots, q_r) \subset \mathbb{P}^{n+r},$$

degenerate $q_r \rightsquigarrow x_{n+r}x_{n+r-1}$ and use induction on r.

Products of projective spaces

Theorem

A very general $(2,3)$-divisor $X \subset \mathbb{P}^1 \times \mathbb{P}^4$ is not stably rational.

Subdivisions of the polytope $a\Delta_1 \times b\Delta_n$ allows us to raise degree/dimension:

(a,b) in $\mathbb{P}^m \times \mathbb{P}^n$ stably irrational $\implies$ $(a,b+1)$ and $(a+1,b)$ also stably irrational in $\mathbb{P}^m \times \mathbb{P}^n$ and $\mathbb{P}^m \times \mathbb{P}^{n+1}$.

$\therefore$ we get all bidegrees corresponding to rational/irrational hypersurfaces.

The Hassett–Pirutka–Tschinkel quartic

Consider $Y \subset \mathbb{P}^2 \times \mathbb{P}^3$, bidegree $(2,2)$, defined by

$$xyU^2 + xzV^2 + yzW^2 + (x^2 + y^2 + z^2 - 2(xy + xz + yz))T^2 = 0$$

Hassett–Pirutka–Tschinkel/Schreieder:
Anything that specializes to Y does not admit a decomposition of Δ (hence is stably irrational).

$(2,3)$-divisors in $\mathbb{P}^1 \times \mathbb{P}^4$

$$\begin{aligned} P &= \text{the Newton polytope of the HPT quartic.} \\ &= \text{convex hull of column vectors of } \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \end{aligned}$$

Starting observation: P is contained in the Newton polytope of a general $(2,3)$-divisor:

$$2\Delta_1 \times 3\Delta_4 = \{(u,v) \in \mathbb{R}^{1+4}_{\geq 0} \,|\, u \leq 2,\ v_1 + \ldots + v_4 \leq 3\}.$$

In concrete terms, the following bidegree $(2,3)$ polynomial

$$\begin{gathered} x_0^2y_0^3 - 2x_0x_1y_0^3 + x_1^2y_0^3 - 2x_0^2y_0^2y_1 - 2x_0x_1y_0^2y_1 \\ + x_0^2y_0y_1^2 + x_0x_1y_1y_2^2 + x_0^2y_1y_3^2 + x_0x_1y_0y_4^2 \end{gathered}$$

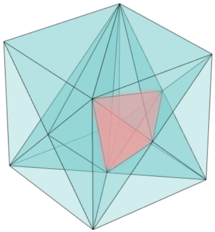
dehomogenizes to the HPT quartic.

Let $\mathscr{P}$ denote the regular subdivision of the polytope $2\Delta_1 \times 3\Delta_4$ induced by the convex function

$$f\colon \mathbb{R}^5 \to \mathbb{R},\ x \mapsto \min_{z \in P} \|x - z\|^2$$

The cells in $\mathscr{P}$:

dim δ	0	1	2	3	4	5
number	43	192	353	323	146	26



$\rightsquigarrow$ degeneration of $\mathbb{P}^1 \times \mathbb{P}^4$ into a union of 26 toric varieties.

Going through the cells of dimension 2 and 4 reveals that any face δ of even dimension either

- has lattice width one (rational, as the equation is linear with respect to a variable)
- corresponds to a quadric bundle over $\mathbb{P}^1_k$ (rational).
- defines a conic bundle over $\mathbb{A}^3$ with a section (rational)

In $\mathbb{Z}[\mathrm{SB}_{\mathbb{C}}]$ we have

$$\mathrm{Vol}([\mathcal{X}]_{\mathrm{sb}}) = [HPT] + \sum_{\#I \text{ odd}} [X_I] + a[\operatorname{Spec}\mathbb{C}] \qquad \text{for some } a \in \mathbb{Z}$$

As this is $\neq [\operatorname{Spec}\mathbb{C}]$, a very general X is stably irrational. $\square$