**MEK4350**
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# 3 Fourier analysis

Here we discuss how arbitrary oscillations can be represented as superpositions of simple harmonic oscillations in time $\mathrm{e}^{-\mathrm{i}\omega t}$ or along a line $\mathrm{e}^{\mathrm{i}kx}$ or in space $\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}$, and how arbitrary waves can be represented as superpositions of simple harmonic waves in one dimension $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ or in several dimensions $\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$. Here $t$ is time, the position vector can be in two-dimensional plane $\boldsymbol{r} = x\boldsymbol{i}_x + y\boldsymbol{i}_y$ or three-dimensional space $\boldsymbol{r} = x\boldsymbol{i}_x + y\boldsymbol{i}_y + z\boldsymbol{i}_z$, $\omega$ is the angular frequency, $k$ is the wavenumber, and $\boldsymbol{k}$ is the wavenumber vector in two-dimensional plane $\boldsymbol{k} = k_x\boldsymbol{i}_x + k_y\boldsymbol{i}_y$ or three-dimensional space $\boldsymbol{k} = k_x\boldsymbol{i}_x + k_y\boldsymbol{i}_y + k_z\boldsymbol{i}_z$.

We start by giving a brief review of some of the tools that will be most useful in the following, the inner product and the corresponding norm defined on a linear space of complex functions or complex sequences.

## 3.1 Inner product spaces

Consider a complex vector space $\Omega$ with an inner product $\langle f, g\rangle$ and with a norm $\|f\|$, where $f$ and $g$ are vectors in $\Omega$. Often the vectors can be complex functions $f(x)$ on a finite or infinite interval of $x$, or they can be finite or infinite complex sequences $\{f_j\}$.

Let us recall the requirements to be an inner product: Let $f, g, h \in \Omega$ and let $\alpha, \beta \in \mathbb{C}$, we have $\langle f, g\rangle : \Omega \times \Omega \to \mathbb{C}$, where

1. $\langle f, g\rangle = \overline{\langle g, f\rangle}$
2. $\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$
3. $\langle f, f\rangle \geq 0$
4. $\langle f, f\rangle = 0$ if and only if $f = 0$

Two vectors $f$ and $g$ are orthogonal if their inner product is zero $\langle f, g\rangle = 0$.

Similarly, let us recall the requirements to be a norm: Let $f, g \in \Omega$ and let $\alpha \in \mathbb{C}$, we have $\|f\| : \Omega \to \mathbb{R}$, where

1. $\|f + g\| \leq \|f\| + \|g\|$
2. $\|\alpha f\| = |\alpha|\,\|f\|$
3. $\|f\| \geq 0$
4. $\|f\| = 0$ if and only if $f = 0$

We shall only be interested in the particular norm that is associated with the inner product:

$$\|f\| = \sqrt{\langle f, f\rangle}.$$

### 3.1.1 Exercises

1. Show that the Cauchy–Schwartz inequality holds:

$$|\langle f, g\rangle| \leq \|f\|\,\|g\|$$

Hint: Start with $\langle f - \alpha g, f - \alpha g\rangle \geq 0$, where $\alpha$ is a complex scalar, and set $\alpha = \langle f, g\rangle/\langle g, g\rangle$.

2. Show that the triangle inequality, also known as the Minkowski inequality, holds for the norm associated with the inner product $\|f\| = \sqrt{\langle f, f\rangle}$:

$$\|f + g\| \leq \|f\| + \|g\|$$

Hint: Start with $\|f + g\|^2 = \langle f + g, f + g\rangle$, recall that the real value of a complex number is smaller than or equal to the absolute value of the number, and use the Cauchy–Schwartz inequality.

3. Show that for the space of complex functions on the interval $a \leq x \leq b$, the integral

$$\langle f, g\rangle = \int_a^b f(x)g^*(x)\,dx \tag{1}$$

is an inner product.

The norm $\|f(x)\| = \sqrt{\int_a^b |f(x)|^2\,\mathrm{d}x}$ is often called the $L^2$-norm.

4. Show that for the space of complex sequences with index $j$, the sum

$$\langle f, g\rangle = \sum_j f_j g_j^* \tag{2}$$

is an inner product.

The norm $\|f_j\| = \sqrt{\sum_j |f_j|^2}$ is often called the $l^2$-norm.

5. Show that the functions $\{1, \cos(nx), \sin(nx); n = 1, 2, \ldots\}$ on the interval $0 \leq x \leq 2\pi$ are orthogonal subject to the inner product (1).

6. Show that the functions $\{1, \exp(inx); n = \pm 1, \pm 2, \ldots\}$ are orthogonal on the interval $0 \leq x \leq 2\pi$ subject to the inner product (1).

7. Show that the functions $\{\sin(nx); n = 1, 2, \ldots\}$ are orthogonal on the interval $0 \leq x \leq \pi$ subject to the inner product (1).

8. Show that the functions $\{1, \cos(nx); n = 1, 2, \ldots\}$ are orthogonal on the interval $0 \leq x \leq \pi$ subject to the inner product (1).

9. Show that the sequences or $N$-tuples $\phi_n = (\phi_{n,1}, \phi_{n,2}, \ldots, \phi_{n,N})$ where $\phi_{n,j} = \exp \frac{2\pi i n j}{N}$, for $n = 1, 2 \ldots N$, are orthogonal subject to the inner product (2).

10. Show that the sequences or $N$-tuples $\phi_n = (\phi_{n,J}, \phi_{n,J+1}, \ldots, \phi_{n,J+N-1})$ where $\phi_{n,j} = \exp \frac{2\pi i n j}{N}$, for $n = M, M+1 \ldots M+N-1$, where $M$ and $J$ are arbitrary integers, are orthogonal subject to the inner product (2).

## 3.2 Generalized Fourier series

If we have a set of $N$ linearly independent basis vectors $\{\phi_n; n = 1, 2, \ldots, N\}$ spanning a subspace of $\Omega$, and if we have some vector $f \in \Omega$ not necessarily within that subspace, we can project $f$ into the subspace

$$f \sim \sum_{n=1}^{N} \hat{f}_n \phi_n.$$

We use the notation $\sim$ since $f$ may not lie entirely within the subspace. This expansion is known as a generalized Fourier series. The generalized Fourier coefficients $\hat{f}_n$ are determined by insisting on equality when taking the inner product of both sides with $\phi_m$

$$\langle f, \phi_m \rangle = \sum_{n=1}^{N} \hat{f}_n \langle \phi_n, \phi_m \rangle \qquad \text{for } m = 1, 2, \ldots, N. \tag{3}$$

These are $N$ linear equations in $N$ unknowns, which can be solved in $O(N^3)$ operations.

The system of equations (3) minimizes the error in the norm, see exercise below.

A great computational as well as analytical advantage is achieved if the set of basis vectors is orthogonal, i.e. $\langle \phi_n, \phi_m \rangle = 0$ for $n \neq m$. In that case the right-hand side simplifies such that we can write the explicit solution

$$\hat{f}_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} \qquad \text{for } n = 1, 2, \ldots, N.$$

Typically, it takes $O(N)$ operations to compute each inner product, so the set of all the coefficients can be computed in $O(N^2)$ operations. The so-called Fast Fourier Transform (FFT) is an algorithm that reduces the computational burden further to only $O(N \log N)$ operations.

### 3.2.1 Exercises

1. Show that the system of equations (3), for the determination of the Fourier coefficients $\hat{f}_n$, minimizes the error in the norm $\|f - P_N f\|$, where

$$P_N f = \sum_{n=1}^{N} \hat{f}_n \phi_n.$$

   Hint: Minize the square of the norm, and treat $\hat{f}_n$ and $\hat{f}_n^*$ as independent variables.

2. Show that Bessel's inequality holds for a generalized Fourier series:

   If we represent $f$ by a projection $P_N f$ into a subspace spanned by orthogonal basis vectors $\phi_n$

$$f \sim P_N f = \sum_{n=1}^{N} \hat{f}_n \phi_n$$

   where the Fourier coefficients are $\hat{f}_n = \langle f, \phi_n \rangle / \|\phi_n\|^2$, then

$$\sum_{n=1}^{N} |\hat{f}_n|^2 \|\phi_n\|^2 \leq \|f\|^2.$$

   Hint: Start with $\|f - P_N f\|^2 = \langle f - P_N f, f - P_N f \rangle \geq 0$.

   Remark: If the error in the norm vanishes, $\|f - P_N f\| \to 0$, as $N \to \infty$, we say that we have convergence in the mean. This corresponds to Bessel's inequality becoming an equality.

## 3.3 Generalized functions, distributions

The subsequent analysis is greatly simplified if we extend our notion of functions to the so-called generalized functions (also known as distributions). Care will be needed since these generalized functions cannot always be used in nonlinear expressions, the product of two generalized functions of the same argument is not necessarily well defined.

We will define a generalized function by its action on test functions under an integral. Suppose $g(x)$ is a generalized function and $\phi(x)$ is a test function, the generalized function is defined by the value of the integral

$$\int_{-\infty}^{\infty} g(x)\phi(x)\,\mathrm{d}x. \tag{4}$$

Test functions are ordinary functions that satisfy conditions of regularity. Different regularity conditions for the test functions give rise to different classes of generalized functions. In general the test functions are real $\phi(x) : \mathbb{R} \to \mathbb{R}$ and are infinitely smooth, i.e. are infinitely differentiable.

In the following we employ for simplicity infinitely smooth test functions with compact support, i.e. they are identical to zero outside a finite domain in $x$.

Generalized functions can also be introduced as the limit of sequences of ordinary functions, as discussed in the book "An introduction to Fourier analysis and generalized functions" by Lighthill (1958).

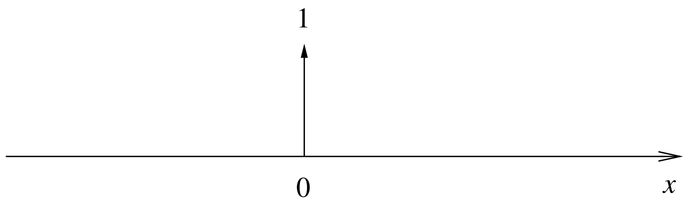
### 3.3.1 Dirac delta

The Dirac delta $\delta(x)$ is defined by the requirement

$$\int_{-\infty}^{\infty} \delta(x)\phi(x)\,\mathrm{d}x = \phi(0). \tag{5}$$

No ordinary function has this property, therefore we take the above integral as a defining property of $\delta(x)$ rather than an integral in the ordinary sense. We say that this generalized function is *singular* rather than *regular*.

Sometimes it is stated that the Dirac delta is zero everywhere except at the origin. However, such a requirement is not only unnecessary, it is not even true for an important representation that we shall use for $\delta(x)$. In any case, as a graphical representation for the Dirac delta we draw a function that is zero everywhere except at the origin where we draw an arrow with unit length:




**Example:** Compute $a\delta(bx + c)$ where $\delta$ is the Dirac delta, and where $a$, $b$ and $c$ are three non-zero constants: We look at the action of the expression above on a test function $\phi(x)$ under an integral. With the substitution $y = bx + c$ we get

$$\int_{-\infty}^{\infty} a\delta(bx + c)\phi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \delta(y)\frac{a}{b}\phi(\frac{y-c}{b})\,\mathrm{d}y = \frac{a}{b}\phi(-\frac{c}{b}).$$

Based on the previous example we deduce that the physical dimension of the Dirac delta is equal to the inverse of the physical dimension of its argument.

**Example:** Compute the derivative of the Dirac delta $\delta'(x)$. We do this by finding the action of this derivative on a test function under an integral. Using integration by parts we get

$$\int_{-\infty}^{\infty} \delta'(x)\phi(x)\,\mathrm{d}x = [\delta(x)\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)\phi'(x)\,\mathrm{d}x = -\phi'(0)$$

where the fact that $[\delta(x)\phi(x)]_{-\infty}^{\infty} = 0$ is due to the test function having compact support, i.e. it vanishes outside a finite domain in $x$.

We define the Heaviside step function by

$$H(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1/2 & \text{for } x = 0 \\ 1 & \text{for } x > 0 \end{cases}$$

The value at the origin is sometimes taken to be 0 or 1 instead of 1/2, but for our purposes it does not matter which finite value we take at the origin.

**Example:** Compute the derivative of the Heaviside step function $H'(x)$. In this case it is obvious that the derivative is zero for any non-zero $x$. However, we want to be careful in order to check what happens at the discontinuity at the origin, thus we check the action of $H'(x)$ on a test function under an integral. Using integration by parts we get

$$\int_{-\infty}^{\infty} H'(x)\phi(x)\,\mathrm{d}x = [H(x)\phi(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} H(x)\phi'(x)\,\mathrm{d}x = \int_{0}^{\infty} \phi'(x)\,\mathrm{d}x = [\phi(x)]_0^{\infty} = \phi(0)$$

where the fact that $[H(x)\phi(x)]_{-\infty}^{\infty} = 0$ is due to the test function having compact support (it vanishes outside a finite domain in $x$), and the fact that the value of $H(0)$ does not matter is due to the test function being infinitely smooth.

Thus we conclude that $H'(x) = \delta(x)$.

**Example:** Compute the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

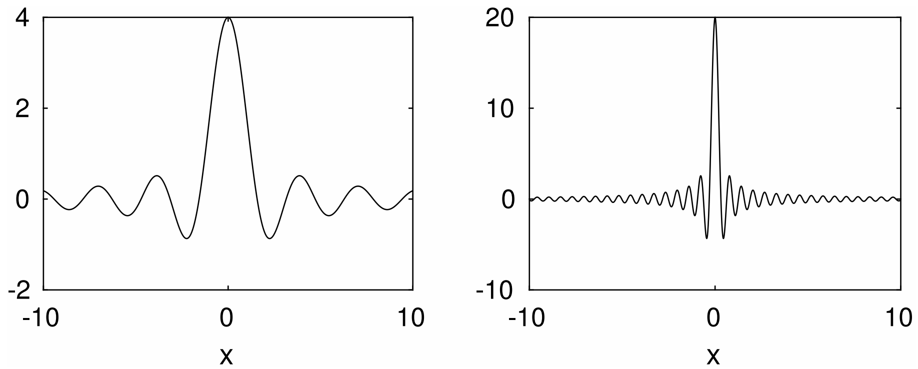
First we notice that the integrand is not absolutely integrable. In order to make this integral meaningful we insist that the upper and lower limits go equally fast to infinity

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k \equiv \lim_{a\to\infty} I(x,a)$$

where

$$I(x,a) = \int_{-a}^{a} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k = \frac{2\sin(xa)}{x}.$$

It is easy to see that $I(0,a) = 2a$. It can also be found that $\int_{-\infty}^{\infty} I(x,a)\,\mathrm{d}x = 4\,\mathrm{Si}(\infty) = 2\pi$ (we show this as an exercise below). It is interesting to plot $I(x,a)$ for some values of $a$, on the left $I(x,2)$ and on the right $I(x,10)$:



As $a \to \infty$ the value at the origin becomes infinitely large at the same time as the oscillations become infinitely fast everywhere except at the origin. This prompts us to employ the method of stationary phase when we check the action of $I(x,a)$ on a test function under an integral. In the limit of large $a$ the contributions to the integral will cancel out everywhere except in a small region around the origin $-\epsilon < x < \epsilon$ such that the following approximation can be used

$$\begin{aligned}
\int_{-\infty}^{\infty} I(x,a)\phi(x)\,\mathrm{d}x &\approx \int_{-\epsilon}^{\epsilon} I(x,a)\phi(x)\,\mathrm{d}x \\
&\approx \phi(0)\int_{-\epsilon}^{\epsilon} I(x,a)\,\mathrm{d}x \\
&\approx \phi(0)\int_{-\infty}^{\infty} I(x,a)\,\mathrm{d}x \;=\; 2\pi\phi(0)
\end{aligned}$$

In the limit $a \to \infty$ this result becomes exact. Thus we conclude that

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k = 2\pi\delta(x)$$

The above example shows that the essential behavior $\int_{-\infty}^{\infty}\delta(x)\phi(x)\,\mathrm{d}x = \phi(0)$ does not depend on an additional assumption or requirement that $\delta(x) = 0$ for $x \neq 0$. Indeed, as $a \to \infty$ we see that $I(x,a)$ does not vanish for non-zero $x$, but its action on a test function under an integral is as if it had been zero for non-zero $x$.

### 3.3.2 Products of generalized functions

It is well defined to multiply a singular generalized function $g(x)$ by a regular function $r(x)$, since the action on a test function $\phi(x)$ under the integral is to consider the generalized function $g(x)$ acting on the new test function $r(x)\phi(x)$,

$$\int_{-\infty}^{\infty} [g(x)r(x)]\,\phi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} g(x)\,[r(x)\phi(x)]\,\mathrm{d}x,$$

where we may consider the product $r(x)\phi(x)$ as another test function.

**Example:** Compute the product of a regular function $r(x)$ and the Dirac delta $\delta(x)$:

$$\int_{-\infty}^{\infty} [\delta(x)r(x)]\,\phi(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \delta(x)\,[r(x)\phi(x)]\,\mathrm{d}x = r(0)\phi(0).$$

The product of two generalized functions with two different arguments is also well defined by the following obvious extension: Let $g(x,y)$ be such a product, i.e. a generalized function of two arguments, and let $\phi_x(x)$ and $\phi_y(y)$ be test functions of each of the two arguments, we define the generalized function by its action on the test functions under the double integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\phi_x(x)\phi_y(y)\,\mathrm{d}x\,\mathrm{d}y.$$

We use the short-hand notation $\delta(\boldsymbol{r}) = \delta(x)\delta(y)$ for a two-dimensional vector $\boldsymbol{r} = x\boldsymbol{i}_x + y\boldsymbol{i}_y$. Similarly we use the short-hand notation $\delta(\boldsymbol{r}) = \delta(x)\delta(y)\delta(z)$ for a three-dimensional vector $\boldsymbol{r} = x\boldsymbol{i}_x + y\boldsymbol{i}_y + z\boldsymbol{i}_z$.

**Example:** Compute the product of the two Dirac deltas $\delta(x)\delta(y)$ of the two Cartesian coordinates $\{x, y\}$: We look at the action of this product on the test functions $\phi_x(x)$ and $\phi_y(y)$ under a double integral,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\phi_x(x)\phi_y(y)\,\mathrm{d}x\,\mathrm{d}y = \phi_x(0)\phi_y(0).$$

**Example:** Compute $\nabla^2 \ln r$ where $\nabla^2$ is the two-dimensional Laplace operator and where $r = |\boldsymbol{r}|$ is the distance from the origin in the two-dimensional plane: Let the position vector relative to the origin be $\boldsymbol{r} = x\boldsymbol{i}_x + y\boldsymbol{i}_y = r\boldsymbol{i}_r$ with polar coordinates $\{r, \theta\}$. Straightforward computation reveals that the result is zero at all non-singular points of the logarithm

$$\nabla^2 \ln r = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \ln r}{\partial r}\right) = 0 \qquad \text{for } r \neq 0,$$

but in order to take a careful look at what happens at the singular point at the origin we look at the action of $\nabla^2 \ln r$ on a test function $\phi(\boldsymbol{r})$ under a double integral. Using the identity $\phi\nabla^2 g = \nabla\cdot(\phi\nabla g) - \nabla\phi\cdot\nabla g$, and integrating over the domain $\Omega = \{\boldsymbol{r} : |\boldsymbol{r}| \leq R\}$ with boundary $\partial\Omega = \{\boldsymbol{r} : |\boldsymbol{r}| = R\}$, we have

$$\int_\Omega \phi\nabla^2 \ln r\,\mathrm{d}\sigma = \int_\Omega \nabla\cdot(\phi\nabla\ln r) - \nabla\phi\cdot\nabla\ln r\,\mathrm{d}\sigma = \int_\Omega \nabla\cdot\left(\phi\frac{\boldsymbol{i}_r}{r}\right) - \frac{1}{r}\frac{\partial\phi}{\partial r}\,\mathrm{d}\sigma.$$

Using Gauss theorem we get

$$\int_\Omega \phi\nabla^2 \ln r\,\mathrm{d}\sigma = \int_{\partial\Omega}\left(\phi\frac{\boldsymbol{i}_r}{r}\right)\cdot\boldsymbol{n}\,\mathrm{d}s - \int_\Omega \frac{1}{r}\frac{\partial\phi}{\partial r}\,\mathrm{d}\sigma.$$

Upon recognizing that $\boldsymbol{n}\,\mathrm{d}s = \boldsymbol{i}_r\,r\mathrm{d}\theta$ on the boundary, and that $\mathrm{d}\sigma = r\mathrm{d}r\mathrm{d}\theta$, we get

$$\int_\Omega \phi(\boldsymbol{r})\nabla^2 \ln r\,\mathrm{d}\sigma = 2\pi\phi(\boldsymbol{0})$$

Thus we conclude that for two-dimensional $\boldsymbol{r}$ and $r = |\boldsymbol{r}|$ we have

$$\nabla^2 \ln r = \nabla\cdot\left(\frac{\boldsymbol{i}_r}{r}\right) = 2\pi\delta(\boldsymbol{r})$$

Finally, we note that the product of two singular generalized functions of the same argument is not well defined. In particular, the square of the Dirac delta, $\delta^2(x) = \delta(x)\delta(x)$ is not well defined.[1]Thus it may not be straightforward to employ generalized functions to solve nonlinear problems.

[1]The problem of multiplication of singular generalized functions of the same argument is related to the so-called "Schwartz' impossibility result": all the natural requirements for being an associative algebra cannot be simultaneously satisfied (Schwartz 1954).

### 3.3.3 Exercises

1. **Derivative of an arbitrary step**

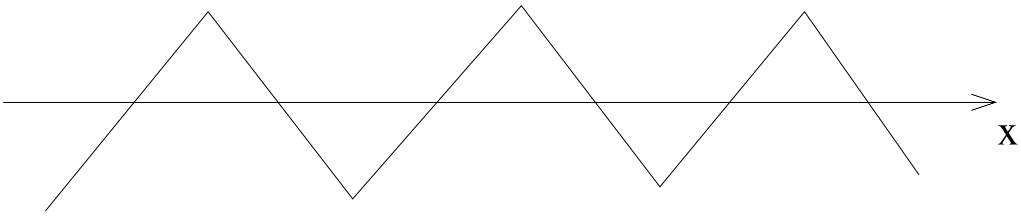
   Let the function $h(x)$ be given by

$$h(x) = \begin{cases} h_1 & \text{for} \quad x < a \\ h_2 & \text{for} \quad x \geq a \end{cases}$$

   for arbitrary constants $h_1$, $h_2$ and $a$. Compute the derivative $h'(x)$.

   Hint: Compute $h'(x)$ for $x \neq a$, and compute $\int_{-\infty}^{\infty} h'(x)\phi(x)\,\mathrm{d}x$ for a test-function $\phi(x)$. Use integration by parts.

2. Here is the graph of a function $g(x)$:




   Sketch the graphs of the first derivative $g'(x)$ and the second derivative $g''(x)$.

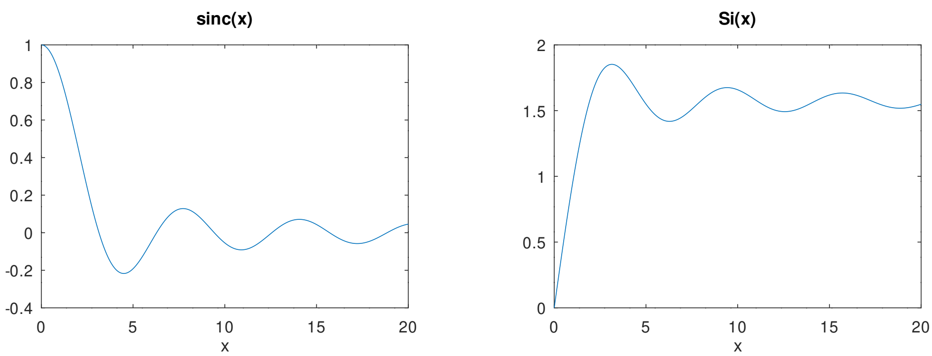
3. Compute $\delta''(x)$.

   Hint: Integration by parts twice.

4. **The sine integral Si**$(x)$ **— the integral of the sinc function**[2] $\operatorname{sinc} x = \frac{\sin x}{x}$

   The sine integral is defined as

$$\operatorname{Si}(x) = \int_0^x \operatorname{sinc} \xi \, \mathrm{d}\xi$$




[2]The sinc function is also known as the cardinal sine function. There are two common definitions, we use the unnormalized sinc function adopted by Mathematica. The normalized sinc function $\operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x}$ is adopted by Python and Matlab and Octave and by DLMF. Wikipedia recognizes both definitions. The Spherical Bessel function of the first kind is a sinc, $j_0(x) = \frac{\sin x}{x}$, see DLMF and MathWorld.

(a) Show using l'Hôpital's rule that $\operatorname{sinc} 0 = 1$.

(b) Show that $\mathrm{Si}(x)$ has:

   i. local extrema for $x = \pm\pi, \pm 2\pi, \pm 3\pi, \dots$

   ii. $\mathrm{Si}(0) = 0$

   iii. $\mathrm{Si}(-x) = -\mathrm{Si}(x)$

(c) Show that $\mathrm{Si}(\infty) = \frac{\pi}{2}$.

   Hint: There are several ways to show this, one way is as follows:

   i. Observe that the integrand is even, integrate only from 0 to $\infty$.

   ii. Rewrite as a double integral using $\int_0^\infty e^{-xt}\,\mathrm{d}t = \frac{1}{x}$.

   iii. Reverse the order of integration.

   iv. Perform two integrations by parts.

   v. Recognize the integral that defines $\arctan\infty = \pi/2$.

5. **Three-dimensional Dirac delta $\delta(\boldsymbol{r})$**

   Show that $\nabla^2 \frac{1}{|\boldsymbol{r}|}$ is proportional to the Dirac delta $\delta(\boldsymbol{r})$ for a three-dimensional position vector $\boldsymbol{r}$.

   Hint: Use Gauss theorem and spherical coordinates.

## 3.4 Fourier transform on infinite interval

Let $f(x)$ be defined on the entire real axis $-\infty < x < \infty$. We would like to express $f(x)$ as a superposition of the basis functions $\mathrm{e}^{\mathrm{i}kx}$. In this case, with no boundary conditions along the $x$-axis, we allow all real $k$ so we express this as an integral

$$f(x) \sim \int_{-\infty}^{\infty} F(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

Using the inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g^*(x)\,\mathrm{d}x$, the basis functions are orthogonal $\langle \mathrm{e}^{\mathrm{i}kx}, \mathrm{e}^{\mathrm{i}lx}\rangle = 2\pi\delta(k-l)$ where $\delta(k)$ is the Dirac delta, and we have the expression for the Fourier transform

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x.$$

Differentiation of $f(x)$ can now be expressed by a multiplication of $F(k)$ by some power of $(\mathrm{i}k)$,

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} f(x) \sim \int_{-\infty}^{\infty} (\mathrm{i}k)^\alpha F(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k.$$

**Example: Fractional derivatives.**

In the above expression for the $\alpha$-th derivative of $f(x)$, there is no reason why $\alpha$ needs to be an integer. Suppose we consider $f(x) = \sin x$ we have

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha}(x) = (-\mathrm{i})^\alpha \frac{i}{2}\mathrm{e}^{-\mathrm{i}x} - (\mathrm{i})^\alpha \frac{i}{2}\mathrm{e}^{\mathrm{i}x} = \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(x+\frac{\pi\alpha}{2})} - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(x+\frac{\pi\alpha}{2})} = \sin(x + \frac{\pi\alpha}{2})$$

This gives a smooth transition from one integral derivative to the next!

### 3.4.1 Exercises

1. Let $f(x) = \cos(3x)$. Compute the Fourier transform $F(k)$. Let $g(x) = f'(x)$, compute $G(k)$. Show that the differentiation consists of carrying out a multiplication of $F(k)$ by some power of $ik$.

2. If we suppose that the differentiation formula can also be used for fractional $\alpha$, compute $\frac{\mathrm{d}^{1/2}}{\mathrm{d}x^{1/2}} \cos(3x)$.

3. Let $f(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Compute the Fourier transform $F(k)$.

## 3.5 Fourier series on finite interval

Let $f(x)$ be defined on the finite interval $0 \leq x < L$. We consider that $f(x)$ can be extended as a periodic function over the entire real axis, thus $f(x) = f(x+L)$. We would like to express $f(x)$ as a superposition of the basis functions $\mathrm{e}^{\mathrm{i}k_n x}$ subject to the condition that the basis functions should be both continuous and periodic on the interval. Thus we must impose the condition $\mathrm{e}^{\mathrm{i}k_n L} = \mathrm{e}^{\mathrm{i}k_n 0} = 1$ which requires $k_n = n\Delta k$ with $\Delta k = 2\pi/L$ and $n \in \mathbb{Z}$, so we express this as a sum

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}_n \mathrm{e}^{\mathrm{i}k_n x}.$$

Using the inner product $\langle f, g\rangle = \int_0^L f(x) g^*(x)\,\mathrm{d}x$, the basis functions are orthogonal $\langle \mathrm{e}^{\mathrm{i}k_n x}, \mathrm{e}^{\mathrm{i}k_m x}\rangle = L\delta_{n,m}$ where $\delta_{n,m}$ is the Kronecker delta, $\delta_{n,m} = 0$ for $n \neq m$ and $\delta_{n,n} = 1$, and we have the expression for the Fourier coefficients

$$\hat{f}_n = \frac{1}{L}\int_0^L f(x)\mathrm{e}^{-\mathrm{i}k_n x}\,\mathrm{d}x.$$

**Example:** Compute the Fourier series of the function

$$f(x) = a\mathrm{e}^{\frac{2\pi \mathrm{i}x}{L}} \qquad \text{for } 0 \leq x < L$$

on the interval $0 \leq x < L$ and compute the Fourier transform of the periodic extension

$$f(x) = a\mathrm{e}^{\frac{2\pi \mathrm{i}x}{L}} \qquad \text{for } -\infty < x < \infty.$$

Computing the Fourier coefficients of the Fourier series, we find

$$\hat{f}_n = a\delta_{n,1}$$

where $\delta_{n,1}$ is the Kronecker delta and the discrete wavenumber is $k_n = 2\pi n/L$.

Computing the Fourier transform we find

$$F(k) = a\delta(k - \frac{2\pi}{L})$$

where $\delta(k)$ is the Dirac delta.

Finally we see that both representations give exact reconstruction of the original function

$$\int_{-\infty}^{\infty} F(k)\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k = \sum_{n=-\infty}^{\infty} \hat{f}_n \mathrm{e}^{\mathrm{i}k_n x} = a\mathrm{e}^{\frac{2\pi \mathrm{i}x}{L}}.$$

This illustrates that a Fourier series can be written as a Fourier integral provided we allow the Fourier transform to be a generalized function.

**Example:**
Compute the Fourier series of the function

$$f(x) = \begin{cases} 1 & \text{for } 0 \leq x < \pi \\ 0 & \text{for } \pi \leq x < 2\pi \end{cases}$$

on the interval $0 \leq x < 2\pi$.

With the basis functions $\mathrm{e}^{\mathrm{i}nx}$ for $n \in \mathbb{Z}$, we have the representation

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}_n \mathrm{e}^{\mathrm{i}nx}$$

Using the inner product $\langle f, g\rangle = \int_0^{2\pi} f(x)g^*(x)\,\mathrm{d}x$, the basis functions are orthogonal $\langle \mathrm{e}^{\mathrm{i}nx}, \mathrm{e}^{\mathrm{i}mx}\rangle = 2\pi\delta_{n,m}$, and

$$\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{e}^{-\mathrm{i}nx}\,\mathrm{d}x = \begin{cases} \frac{1}{2} & \text{for } n = 0 \\ \frac{-\mathrm{i}}{2\pi n}\left[1-(-1)^n\right] & \text{for } n \neq 0 \end{cases}$$

Thus we may write

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi n}\sin(nx)$$

**Example: The Gibbs phenomenon.**
Suppose we limit the above sum to a finite number of terms $N$

$$s_N(x) = \frac{1}{2} + \sum_{n=1}^{N} \frac{1-(-1)^n}{\pi n}\sin(nx)$$

it is interesting to study how $s_N(x)$ behaves as $N$ increases. Plot this function on the interval $0 \leq x < 2\pi$ for some values $N = 0, 1, 2, 5, 10, 100, 1000$. Notice the behavior near the discontinuities at $x = 0$ and $x = \pi$ and $x = 2\pi$.

In order to study the convergence of a Fourier series more closely, we first define

$$P_N f(x) = \sum_{n=-N}^{N} \hat{f}_n \mathrm{e}^{\mathrm{i}nx}$$

Several types of convergence can be considered. For example:

1. Pointwise convergence. It can be shown that

$$\lim_{N\to\infty} P_N f(x) = \frac{1}{2}\left(f(x+) + f(x-)\right) \qquad \text{for each fixed } x$$

2. Uniform convergence. It can be shown that near a point of discontinuity there is an overshoot of approximately 0.089 times the discontinuous jump, thus we do not have uniform convergence for discontinuous $f(x)$.

3. Convergence in norm, or convergence in the mean,

$$\lim_{N\to\infty} \|P_N f - f\| = 0$$

We shall not deal further with convergence here. For more thorough discussion on convergence the reader is referred to other textbooks.

### 3.5.1 Differentiation of Fourier series term by term

Given the representation

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}_n \mathrm{e}^{\mathrm{i}k_n x}$$

we may hope that the derivatives of $f(x)$ can be obtained by differentiating the series term by term

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha}(x) \sim \sum_{n=-\infty}^{\infty} (\mathrm{i}k_n)^\alpha \hat{f}_n \mathrm{e}^{\mathrm{i}k_n x}.$$

This works fine for sufficiently smooth functions, e.g. consider $\sin x = \frac{i}{2}\mathrm{e}^{-\mathrm{i}x} - \frac{i}{2}\mathrm{e}^{\mathrm{i}x}$ which gives $\frac{\mathrm{d}}{\mathrm{d}x}\sin x = (-\mathrm{i})\frac{i}{2}\mathrm{e}^{-\mathrm{i}x} - (\mathrm{i})\frac{i}{2}\mathrm{e}^{\mathrm{i}x} = \frac{1}{2}\mathrm{e}^{-\mathrm{i}x} + \frac{1}{2}\mathrm{e}^{\mathrm{i}x} = \cos x$, however there are some challenges if the function is not smooth as suggested in the following example:

**Example:**
For the discontinuous function considered above,

$$f(x) = \begin{cases} 1 & \text{for } 0 \le x < \pi \\ 0 & \text{for } \pi \le x < 2\pi \end{cases}$$

on the interval $0 \le x < 2\pi$, we found the infinite Fourier series

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi n} \sin(nx)$$

If we differentiate this series term by term we get

$$f'(x) \sim \sum_{n=1}^{\infty} \frac{1-(-1)^n}{\pi} \cos(nx),$$

which is a divergent series for $x = 0$ and $x = \pi$.

On the other hand, if we compute the derivative of $f(x)$ in the sense of generalized functions we get

$$f'(x) = \delta(x) - \delta(x-\pi)$$

and upon computing the Fourier series for this representation of $f'(x)$ we get precisely the divergent series given above. The perfectly acceptable generalized function $f'(x)$ has a divergent Fourier series expansion which can be obtained by termwise differentiation of the Fourier series for $f(x)$. Thus the concept of generalized functions allows us to ascribe meaning to divergent series expansions!

**Example: Fractional derivatives — again.**

In the above expression for the $\alpha$-th derivative of $f(x)$, there is no reason why $\alpha$ needs to be an integer. Suppose we consider $f(x) = \sin x$ we have

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha}(x) = (-\mathrm{i})^\alpha \frac{i}{2}\mathrm{e}^{-\mathrm{i}x} - (\mathrm{i})^\alpha \frac{i}{2}\mathrm{e}^{\mathrm{i}x} = \frac{\mathrm{i}}{2}\mathrm{e}^{-\mathrm{i}(x+\frac{\pi\alpha}{2})} - \frac{\mathrm{i}}{2}\mathrm{e}^{\mathrm{i}(x+\frac{\pi\alpha}{2})} = \sin(x + \frac{\pi\alpha}{2})$$

This gives a smooth transition from one integral derivative to the next!

### 3.5.2 Exercises

1. Compute the Fourier series of $f(x) = x$ for $0 \le x < 2\pi$. Differentiate the Fourier series for $f(x)$ term by term and show that the result is divergent at the point of discontinuity $x = 0$. Find $g(x) = f'(x)$ by direct differentiation and find the Fourier series for $g(x)$. Show that the divergent series for $f'(x)$ found by termwise differentiation is equal to the Fourier series for the direct derivative $g(x)$. Show graphically on the computer that the Fourier series for $f(x)$ suffers from the Gibbs phenomenon.

## 3.6 Discrete Fourier Transform (DFT)

We consider complex sequences with $N$ elements, $\{f_j; j = 1, 2, \ldots, N\}$. We would like to express $f_j$ as a superposition of the basis functions $\{\mathrm{e}^{\frac{2\pi\mathrm{i}jn}{N}}; j = 1, 2, \ldots, N\}$ for $n = 1, 2, \ldots, N$ so we express this as a sum

$$f_j = \sum_{n=1}^{N} \tilde{f}_n e^{\frac{2\pi inj}{N}}.$$

Using the inner product $\langle f, g\rangle = \sum_{j=1}^N f_j g_j^*$ the basis functions are orthogonal $\langle \mathrm{e}^{\frac{2\pi\mathrm{i}jn}{N}}, \mathrm{e}^{\frac{2\pi\mathrm{i}jm}{N}}\rangle = N\delta_{n,m}$ where $\delta_{n,m}$ is the Kronecker delta, and we have the expression for the Discrete Fourier Transform

$$\tilde{f}_n = \frac{1}{N}\sum_{j=1}^{N} f_j e^{-\frac{2\pi inj}{N}}.$$

The DFT has the property that it allows cyclic permutation of the sequences $\{f_j\}$ and $\{\tilde{f}_n\}$. To this end we suppose that $f_j$ and $\tilde{f}_n$ are continued periodically with $f_{j+N} = f_j$ and $\tilde{f}_{n+N} = \tilde{f}_n$ for any $j$ and $n$.

We can start either of the above sums at arbitrary integer coefficients and still get the same result, thus for arbitrary integers $r$ and $s$ we have

$$f_j = \sum_{n=r}^{r+N-1} \tilde{f}_n e^{\frac{2\pi ijn}{N}}$$

and

$$\tilde{f}_n = \frac{1}{N} \sum_{j=s}^{s+N-1} f_j e^{-\frac{2\pi ijn}{N}}.$$

For practical applications of DFT it is often useful to consider the index $j$ to denote a discrete set of values for the position $x$, and to consider the index $n$ to denote a discrete set of values for the wavenumber $k$. For a function $f(x)$ on the interval $0 \leq x < L$ we set $x_j = j\Delta x = \frac{Lj}{N}$ for $j = 0, 1, \ldots, N-1$, and consider the samples $f_j = f(x_j)$. Similarly we set $k_n = n\Delta k = \frac{2\pi n}{L}$ for $n = 0, 1, \ldots, N-1$. The DFT transform pair then becomes

$$f_j \equiv f(x_j) = \sum_{n=0}^{N-1} \tilde{f}_n e^{ik_n x_j}$$

and

$$\tilde{f}_n = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-ik_n x_j}$$

where both of these sums can be subject to arbitrary cyclic permutation.

### 3.6.1 Exercises

1. Show that arbitrary cyclic permutations for any integers $r$ and $s$ are allowed for the above DFT transform pair.

2. On the computer there are functions `fftshift` and `ifftshift` that perform cyclic permutations for particular values of $r$. For which values of $r$? Check the behavior for even and odd $N$. Under which conditions are `fftshift` and `ifftshift` inverses? Under which conditions is `fftshift` the inverse of itself?

3. On the computer we can perform the DFT and its inverse with the functions `fft` and `ifft`. These are often defined differently for different systems, and differently from the definitions we have used so far. Usually they are defined by

$$\text{ifft:} \qquad f_j = A \sum_{n=r}^{r+N-1} \tilde{f}_n e^{\pm\frac{2\pi \mathrm{i}(n-p)(j-q)}{N}}$$

and

$$\text{fft:} \qquad \tilde{f}_n = B \sum_{j=s}^{s+N-1} f_j e^{\mp\frac{2\pi \mathrm{i}(n-p)(j-q)}{N}}$$

Choose your favorite computer system and determine the appropriate values for $A$, $B$, $r$, $s$, $p$, $q$ and the appropriate signs in the exponents for `fft` and `ifft`.

For the pair {`fft`, `ifft`} to be exact inverses it is necessary that $ABN = 1$ and that the signs are opposite. This turns out not to be satisfied on all computer systems! Is it satisfied on your system?

Hint: If the documentation of your computer system does not tell you the answer, set $N$ to some small values and compute some sample transforms.

## 3.7 Folding or aliasing

Suppose we have a function $f(x)$ on the interval $0 \le x < L$. We can represent the function as a Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{\mathrm{i}k_n x}, \qquad \text{with} \qquad \hat{f}_n = \frac{1}{L}\int_0^L f(x) e^{-\mathrm{i}k_n x}\, dx,$$

where $k_n = \frac{2\pi n}{L}$. Alternatively, we can sample $f(x)$ at the discrete points $x_j = j\Delta x = \frac{Lj}{N}$ for $j = 0, 1, \ldots, N-1$, and represent the sample values $f_j \equiv f(x_j)$ by means of a DFT

$$f_j = \sum_{n=0}^{N-1} \tilde{f}_n e^{\mathrm{i}k_n x_j}, \qquad \text{with} \qquad \tilde{f}_n = \frac{1}{N}\sum_{j=0}^{N-1} f_j e^{-\mathrm{i}k_n x_j}.$$

We want to know the relationship between the continuous and the discrete Fourier coefficients $\hat{f}_n$ and $\tilde{f}_n$.

If we evaluate the infinite Fourier series at the discrete sampling points, and take the inner product with $e^{\frac{2\pi ijm}{N}}$, and make use of the identity

$$\sum_{j=0}^{N-1} e^{\frac{2\pi ijn}{N}} = N \sum_{l=-\infty}^{\infty} \delta_{n,lN}$$

where $n$ is an arbitrary integer, we get the formula for aliasing or folding

$$\tilde{f}_n = \sum_{l=-\infty}^{\infty} \hat{f}_{n+lN}. \tag{6}$$

This formula shows that if $\hat{f}_n$ is zero outside an interval for $n$ of length $N$, then the values of $\tilde{f}_n$ will be equal to the values of $\hat{f}_n$ within that interval.

The most common way to employ the above result is that if $\hat{f}_n$ is zero for $|n| \ge \frac{N}{2}$, or correspondingly for $|k_n| \ge k_{\frac{N}{2}}$, then the values of $\tilde{f}_n$ will be equal to the values of $\hat{f}_n$ within that interval. The particular wavenumber $k_{\frac{N}{2}}$ is called the Nyquist wavenumber, and this criterion is called the Nyquist sampling criterion.

For practical application we should make sure that $N$ is chosen sufficiently large such that $\hat{f}_n$ is sufficiently small outside an appropriate interval for $n$ of length $N$, then $\tilde{f}_n$ will be approximately equal to $\hat{f}_n$ within that interval.

### 3.7.1 Exercises

1. Show that

$$\sum_{j=r}^{r+N-1} e^{\frac{2\pi i j n}{N}} = N \sum_{l=-\infty}^{\infty} \delta_{n,lN}$$

for arbitrary starting index $r$.

Note: The result is an infinite row of Kronecker delta functions.

2. Consider the complex function

$$f(x) = 2\mathrm{e}^{5\mathrm{i}x} + 12\mathrm{e}^{7\mathrm{i}x}$$

for $0 \leq x < 2\pi$.

We wish to sample this function at the points $x_j = \frac{2\pi j}{N}$ for $j = 0, 1, \ldots, N-1$, and to represent the function by its DFT Fourier coefficients $\tilde{f}_n$ such that the values of $\tilde{f}_n$ are not contaminated by aliasing.

Determine the smallest value of $N$ needed to avoid aliasing, and also determine the smallest value of $N$ according to the standard Nyquist sampling criterion $|k_n| < k_{\frac{N}{2}}$.

This problem demonstrates that with a clever use of cyclic permutation, and a thorough understanding of aliasing, it is sometimes possible to use a substantially smaller value for $N$ than the value suggested by the standard Nyquist sampling criterion.

## 3.8 Interpolation and differentiation using DFT

Suppose the function $f(x)$ is sampled at $N$ points $x_j = \frac{Lj}{N}$, $f_j = f(x_j)$, for $j = 0, 1, 2, \ldots, N-1$, and suppose we have computed the DFT coefficients $\tilde{f}_n$ for $n = 0, 1, 2, \ldots, N-1$. We certainly have the reconstruction formula at the sampling points with arbitrary cyclic permutation

$$f_j = \sum_{n=r}^{r+N-1} \tilde{f}_n e^{\mathrm{i}k_n x_j},$$

with $k_n = \frac{2\pi n}{L}$. However, suppose we remove the index $j$ in order to use this formula as an interpolation between the sampling points

$$g(x; r) = \sum_{n=r}^{r+N-1} \tilde{f}_n e^{\mathrm{i}k_n x},$$

we recognize that the interpolation will depend dramatically on the value of $r$. The question becomes: For which value of $r$ of the cyclic permutation does this interpolation best resemble the original function $f(x)$? Without knowing anything about the function $f(x)$ the optimal choice is usually considered to be the interpolation with the slowest possible oscillations, corresponding to $r = -N/2$. On the other hand, if we know something about $f(x)$, the optimal value for $r$ can be deduced from the aliasing formula relating $\tilde{f}_n$ with $\hat{f}_n$ such that the appropriate range for $n$ is such that $\tilde{f}_n$ is most exactly equal to $\hat{f}_n$.

After the appropriate value of $r$ has been determined, and for $N$ sufficiently large to minimize aliasing, we can approximate the derivative

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha}(x_j) \approx \sum_{n=r}^{r+N-1} (\mathrm{i}k_n)^\alpha \tilde{f}_n e^{\mathrm{i}k_n x_j}$$

### 3.8.1 Exercises

1. Consider again the complex function

$$f(x) = 2\mathrm{e}^{5\mathrm{i}x} + 12\mathrm{e}^{7\mathrm{i}x}$$

for $0 \leq x < 2\pi$ sampled at the points $x_j = \frac{2\pi j}{N}$ for $j = 0, 1, \ldots, N-1$ with the smallest value of $N$ needed to avoid aliasing. Determine the required value of $r$ to correctly compute the derivative at the sampling points, $f'(x_j)$.

## 3.9 The smoothness of a function

We have seen that if we differentiate a discontinuous function like the Heaviside step, we get a Dirac delta after only one differentiation. If we differentiate a continuous function with a discontinuous derivative, such as $|x|$, we get a step function after the first differentiation and a Dirac delta after two differentiations.

Taking inspiration from the above, we may define an index $\alpha$ to be the number of times needed to differentiate a function $f(x)$ such that a Dirac delta appears. For the Heaviside step we have $\alpha = 1$, for the absolute value we have $\alpha = 2$, while for a smooth function like $f(x) = \sin x$ we have $\alpha = \infty$. We shall employ the value of $\alpha$ as the degree of smoothness of $f(x)$.

Then let us consider how this affects the Fourier transform. Suppose we let $f(x)$ be defined on the entire real axis, let us apply an integration by parts on the integral for the Fourier transform $F(k)$

$$F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x = \left[\frac{\mathrm{i}}{2\pi k} f(x)\mathrm{e}^{-\mathrm{i}kx}\right]_{x=-\infty}^{\infty} - \frac{\mathrm{i}}{2\pi k}\int_{-\infty}^{\infty} f'(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$

Since we suppose that $f(x) \to 0$ as $x \to \pm\infty$, only the last integral remains. Two possibilities exist: Either $f'(x)$ contains a Dirac delta, and we have $|F(k)| \sim \beta|k|^{-1}$ as $k \to \pm\infty$. Here $\beta$ is some constant and the symbol $\sim$ means "behaves asymptotically as". Or else we can carry out one more integration by parts. Repeating this procedure multiple times, we realize that the procedure stops after $\alpha$ integrations by part, and we have

$$|F(k)| \sim \beta|k|^{-\alpha} \qquad \text{as } k \to \pm\infty$$

If $\alpha = \infty$ we say that $F(k)$ goes to zero faster than algebraically.

A nice way to plot this graphically is to use a log-log plot

$$\log |F(k)| \sim \log \beta - \alpha \log |k| \qquad \text{as } k \to \pm\infty$$

so the determination of the index $\alpha$ amounts to determine the slope of a straight line.

As in the case of the fractional derivative, there is no reason to limit the value of $\alpha$ to be an integer, thus we allow the degree of smoothness of a function $f(x)$ to be any real value for the index $\alpha$. It will be fun to discover that the ocean surface typically exhibits such a "fractional" smoothness.

**Example:** We have previously seen that the step function has Fourier coefficients proportional to $k^{-1}$. We have previously also seen that the trigonometric functions $\sin x$ and $\cos x$ have Fourier coefficients identical to zero for $|k| > 1$ which means they go to zero faster than algebraically.

### 3.9.1 Exercises

1. Show that if we draw the graph of $y = ax^b$ in a doubly logarithmic coordinate system, then the result will be a straight line, and the values of $a$ and $b$ can be determined by respectively the point of intersection with the second axis and the slope of the line.

   Choose suitable values for $a$ and $b$ and demonstrate that this actually works by plotting the graph on the computer with both `plot(x,y)` and `loglog(x,y)`.

2. Look at the three functions

   $$f(x) = x \qquad g(x) = |x - \pi| \qquad h(x) = \cos(x) \qquad \text{for } 0 \le x < 2\pi$$

   a) Plot the graphs of the three functions and explain how fast the Fourier coefficients $\hat{f}_n$, $\hat{g}_n$ and $\hat{h}_n$ are expected to approach zero when $n \to \pm\infty$.

   Use the computer to show that the expectation above is correct: First discretize $x$ with $N$ equally spaced values $x_j = 2\pi j/N$. Then compute the Fourier coefficients by means of `fft(y)`, where $y_j$ is the value of the function evaluated at $x_j$. Finally plot `abs(fft(y))` in a doubly logarithmic coordinate system.

   b) Explain why the command `x = linspace(0,2*pi,N)` is not correct, and show how it must be done correctly.

   c) Use `plot` to plot the graphs of `abs(fft(y))` for the three functions in the same linear coordinate system. Plot the graph of `abs(fftshift(fft(y)))` in the same linear coordinate system. Explain why we only need to study half of the Fourier transform.

d) Use `loglog` to plot the graph of $\epsilon +$ `abs(fft(y))` in the same doubly logarithmic coordinate system. Plot straight lines with the expected slopes for the $f$ and $g$. Show that you achieve the expected result for all three functions!

Hint 1: The constant $0 < \epsilon \ll 1$ has been included to avoid that the computer complains about taking the logarithm of non-positive numbers. A useful value could be $\epsilon = 10^{-15}$.

Hint 2: In order to avoid aliasing $N$ must be chosen sufficiently large, $N = 1024$ is probably sufficient.

e) Show that if we commit the error associated with the `linspace` command above, then we ruin the nice behavior of $\hat{h}$.

## 3.10 Parseval's theorem

Parseval's theorem is a special case of Plancherel's theorem. Suppose we look at functions $f(x)$ and $g(x)$ on the real line $x \in \mathbb{R}$, and suppose we consider the two Fourier integrals

$$f(x) = \int_{-\infty}^{\infty} F(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x$$

and

$$g(x) = \int_{-\infty}^{\infty} G(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}x.$$

Then we have

$$\int_{-\infty}^{\infty} f(x) g^*(x) \, \mathrm{d}x = 2\pi \int_{-\infty}^{\infty} F(k) G^*(k) \, \mathrm{d}k.$$

In particular, we have

$$\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = 2\pi \int_{-\infty}^{\infty} |F(k)|^2 \, \mathrm{d}k.$$

The integral $\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x$ is sometimes denoted the "energy" of the "signal" $f(x)$, even though the units or the interpretation of $f(x)$ may not necessarily correspond to mechanical energy. We recognize that Parseval's theorem is only meaningful if the energy of the signal is finite.

The following three examples show that a function can be absolutely integrable without having finite energy, a function can have finite energy without being absolutely integrable, and a function can have a well defined Fourier transform in terms of a generalized function without neither having finite energy nor being absolutely integrable.

**Example:** Consider the function

$$f(x) = \begin{cases} \frac{1}{\sqrt{x}} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

Notice that $\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x = 2$ while $\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x$ is infinite.

**Example:** Consider the function

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } x > 1 \\ 0 & \text{elsewhere} \end{cases}$$

Notice that $\int_{-\infty}^{\infty} |f(x)| \, \mathrm{d}x$ is infinite while $\int_{-\infty}^{\infty} |f(x)|^2 \, \mathrm{d}x = 1$.

**Example:** Consider the function $f(x) = \sin x$ on the real line $x \in \mathbb{R}$. Any attempt to apply Parseval's theorem will fail for two reasons: Firstly because the energy of $f(x)$ is infinite on the real line. Secondly because $F(k)$ is a generalized function containing a couple of Dirac delta, and the multiplication of a Dirac delta with itself is not well defined.

Also notice that even though $f(x) = \sin x$ neither is absolutely integrable nor has finite energy, it still has a well defined Fourier transform in terms of generalized functions.

The above example suggests that a different approach should be taken. One possible approach is to consider the "power" of the signal rather than the "energy". By the "power" of the signal $f(x)$ we mean the energy divided by the length of the interval along the $x$-axis, taking inspiration from the mechanical power being energy divided by time. In the present case we have an infinite interval and we can employ the following as the "power" of $f(x)$

$$\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, \mathrm{d}x.$$

On the other hand, if we can represent

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n \mathrm{e}^{\mathrm{i}k_n x}$$

for some fixed values of $k_n$ not depending on $L$, we get

$$\begin{aligned}
\lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} |f(x)|^2 \, \mathrm{d}x &= \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \left( \sum_{n=-\infty}^{\infty} \hat{f}_n \mathrm{e}^{\mathrm{i}k_n x} \right) \left( \sum_{m=-\infty}^{\infty} \hat{f}_m \mathrm{e}^{\mathrm{i}k_m x} \right)^* \mathrm{d}x \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \hat{f}_n \hat{f}_m^* \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \mathrm{e}^{\mathrm{i}(k_n - k_m)x} \, \mathrm{d}x \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \hat{f}_n \hat{f}_m^* \delta_{n,m} \\
&= \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2
\end{aligned}$$

This demonstrates that functions with infinite energy need not be problematic provided they have finite power.

For Fourier series representation of functions $f(x)$ on a finite interval $0 \le x < L$ Parseval's theorem follows immediately from Bessel's inequality in the case that we have convergence in the mean

$$\int_0^L |f(x)|^2 \, \mathrm{d}x = L \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2$$

For DFT we have simply

$$\sum_{j=1}^{N} |f_j|^2 = N \sum_{n=1}^{N} |\tilde{f}_n|^2$$

### 3.10.1 Exercises

1. Show the slightly more general relation

$$\sum_{j=1}^{N} f_j g_j^* = N \sum_{n=1}^{N} \tilde{f}_n \tilde{g}_n^*$$

## 3.11 Convolution

Suppose $f(x)$ and $g(x)$ are functions on the entire real axis. By the convolution of $f(x)$ and $g(x)$ we mean a new function $c(x)$ defined by

$$c(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(x-y) g(y) \, \mathrm{d}y.$$

Suppose we have the Fourier transforms $F(k)$ and $G(k)$,

$$f(x) = \int_{-\infty}^{\infty} F(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k$$

and

$$g(x) = \int_{-\infty}^{\infty} G(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k,$$

substitution into the convolution integral gives

$$c(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(k) \mathrm{e}^{\mathrm{i}k(x-y)} \, \mathrm{d}k \int_{-\infty}^{\infty} G(l) \mathrm{e}^{\mathrm{i}ly} \, \mathrm{d}l \, \mathrm{d}y = 2\pi \int_{-\infty}^{\infty} F(k) G(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k.$$

Thus we have that the Fourier transform of the convolution is equal to $2\pi$ times the product of the Fourier transforms.

### 3.11.1 Exercises

1. Suppose we have a function $f(x)$ on the interval $0 \leq x < L$ and we consider the truncated Fourier sum

$$P_N f(x) = \sum_{n=-N}^{N} \hat{f}_n \mathrm{e}^{\mathrm{i}k_n x}$$

where $k_n = \frac{2\pi n}{L}$ and where

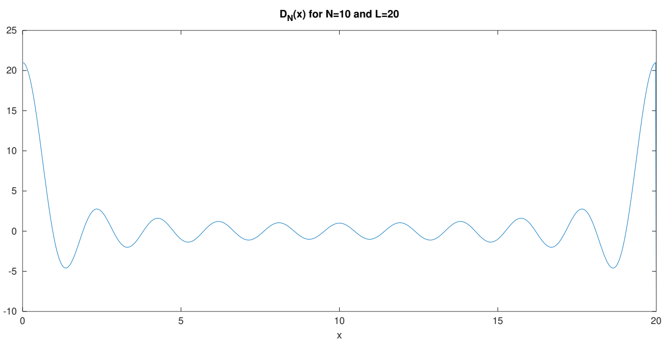
$$\hat{f}_n = \frac{1}{L}\int_0^L f(x)\mathrm{e}^{-\mathrm{i}k_n x}\,\mathrm{d}x.$$

Show that we can write $P_N f(x)$ as a convolution of $f(x)$ and the Dirichlet kernel $D_N(x)$

$$P_N f(x) = \frac{1}{2\pi}\int_0^L D_N(x-y) f(y)\,\mathrm{d}y$$

where the Dirichlet kernel is

$$D_N(x) = \frac{\sin\frac{2\pi(N+\frac{1}{2})x}{L}}{\sin\frac{\pi x}{L}}.$$




Hint: Change the order of the sum and the integral, and compute the geometric series of the complex exponentials!

2. Consider multiplication of two polynomials

$$p(x) = \sum_{n=0}^{N} p_n x^n$$

and

$$q(x) = \sum_{n=0}^{N} q_n x^n,$$

and let the product be another polynomial

$$c(x) = p(x)q(x) = \sum_{n=0}^{2N} c_n x^n.$$

Show that the coefficients of the product polynomial are given by the discrete convolution

$$c_n = \sum_{m=0}^{N} p_{n-m} q_m \qquad \text{for } n = 0, 1, 2, \ldots, 2N.$$

Show that the DFT of $c_n$ is given by the product of the DFTs of $p_n$ and $q_n$.

Select your favorite polynomials $p(x)$ and $q(x)$ and show that the coefficients of the product polynomial $c(x)$ can be found by two calls to `fft` and one call to `ifft` on the computer!

Notice that the direct computation of the coefficients $c_n$ above requires the order of $N^2$ operations ($2N + 1$ coefficients, each requiring $N + 1$ multiplications), while the use of the Fast Fourier Transform requires the order of $N \log N$ operations, thus for large $N$ the polynomial multiplication with FFT is much faster! You should time your program and find out exactly how large $N$ needs to be on your computer in order that the FFT-method for polynomial multiplication is faster than direct evaluation of the convolution!

## 3.12 Poisson's summation formula

Recall that in the derivation of the formula for aliasing (6) was obtained by combining the DFT with the infinite Fourier series. We can also combine the continuous Fourier transform with the infinite Fourier series for another useful result.

We start by considering the Dirac delta at the origin, $\delta(x)$, repeated as a periodic function with period $L$. This can be represented as an infinite Fourier series

$$g(x) \equiv \sum_{j=-\infty}^{\infty} \delta(x - jL) = \sum_{n=-\infty}^{\infty} \hat{g}_n \mathrm{e}^{\frac{2\pi i n x}{L}}$$

where the coefficients are determined by integrating over one period $L$ within which there is exactly one Dirac delta,

$$\hat{g}_n = \frac{1}{L} \int_0^L \sum_{j=-\infty}^{\infty} \delta(x - jL) \mathrm{e}^{-\frac{2\pi i n x}{L}} \, \mathrm{d}x = \frac{1}{L}$$

Then we consider the Fourier transform of a (non-periodic) function $f(x)$ defined on the entire real axis

$$f(x) = \int_{-\infty}^{\infty} F(k) \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k$$

If we sample this function at even intervals $2\pi/L$, and take the sum, we get

$$\sum_{n=-\infty}^{\infty} f(\frac{2\pi n}{L}) = \int_{-\infty}^{\infty} F(k) \sum_{n=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi i k n}{L}} \, \mathrm{d}k = \int_{-\infty}^{\infty} F(k) L \sum_{j=-\infty}^{\infty} \delta(k - jL) \, \mathrm{d}k$$

Thus we get the final result

$$\sum_{n=-\infty}^{\infty} f(\frac{2\pi n}{L}) = L \sum_{j=-\infty}^{\infty} F(jL)$$

### 3.12.1 Exercises

1. Let $f(x) = \mathrm{e}^{-|x|}$ and compute the sum $\displaystyle\sum_{j=1}^{\infty} \frac{1}{1 + j^2}$.