MEK4350
Karsten Trulsen
17 August 2022

1 Elementary description of waves

1.1 Monochromatic waves

A monochromatic, or regular, or simple-harmonic wave is a sinusoidal wave with a unique period T and a unique wavelength λ. The *frequency* f, *angular frequency* ω and *wavenumber* k are given by

$$f = \frac{1}{T}, \qquad \omega = \frac{2\pi}{T}, \qquad k = \frac{2\pi}{\lambda}. \tag{1}$$

Even though the wave itself is real, it can be convenient to represent it by a complex amplitude A which can be expressed by a modulus $|A|$ and an argument $\arg A$, with $A = |A|\mathrm{e}^{\mathrm{i}\arg A}$,

$$\eta(\boldsymbol{r}, t) = \mathrm{Re}\{A\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\} \tag{2}$$

$$= \frac{A}{2}\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \text{c.c.} \tag{3}$$

$$= |A|\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \arg A). \tag{4}$$

Here $\boldsymbol{k} = k_x\boldsymbol{i}_x + k_y\boldsymbol{i}_y + k_z\boldsymbol{i}_z$ is the *wavenumber vector*, and $\boldsymbol{r} = x\boldsymbol{i}_x + y\boldsymbol{i}_y + z\boldsymbol{i}_z$ is the position vector. The wavenumber is the magnitude of the wavenumber vector $k = |\boldsymbol{k}| = \sqrt{k_x^2 + k_y^2 + k_z^2}$. The notation c.c. means the complex conjugate of the previous expression.

We use the symbols $\{\boldsymbol{i}_x, \boldsymbol{i}_y, \boldsymbol{i}_z\}$ to denote the unit vectors along the $\{x, y, z\}$-axes. The common convention with $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ being unit vectors along the $\{x, y, z\}$-axes is problematic since we want to reserve the symbol $\boldsymbol{k}$ for the wavenumber vector.

We denote the *phase function* by

$$\chi(\boldsymbol{r}, t) = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \arg A. \tag{5}$$

A *phase surface* in $(\boldsymbol{r}, t)$ space is an equiscalar surface of the phase function, $\chi =$ constant. The wavenumber vector is orthogonal to the phase surface since $\boldsymbol{k} = \nabla\chi$. Some phase surfaces are so special that they have their own names. With reference to the use of $\cos\chi$ in (4) a *crest* (Norwegian *kam*) is a local maximum and is achieved for $\chi = 2\pi n$, a *trough* (Norwegian *buk*) is a local minimum and is achieved for $\chi = \pi + 2\pi n$, and *zero-crossings* are achieved for $\chi = \pi/2 + \pi n$, where n is an arbitrary integer.

In the expression for the phase function (5) we have chosen opposite signs for the term with the wavenumber vector $\boldsymbol{k}$ and the term with the angular frequency ω. In this case, and provided $\omega > 0$, the phase surfaces move in the direction of the wavenumber vector $\boldsymbol{k}$ as time increases.

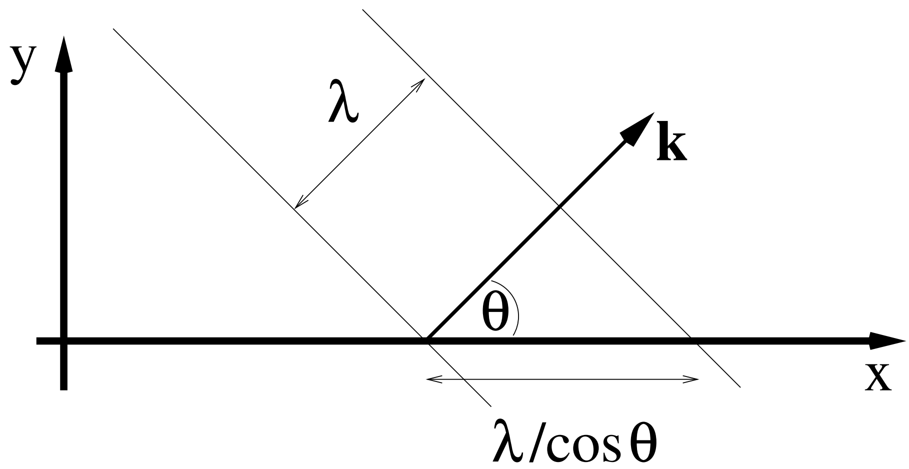

Figure 1: The distance between two phase surfaces along a direction different from that of the wavenumber vector is longer than the wavelength.

The *phase speed* is the ratio between the wave length and the wave period, or equivalently, the ratio between the angular frequency and the wavenumber

$$c = \frac{\lambda}{T} = \frac{\omega}{k}. \tag{6}$$

The phase speed is the speed that the phase surface moves in its normal direction, *i.e.* in the direction of the wavenumber vector $\boldsymbol{k}$. The wavelength is by default supposed to be measured along the direction of the wavenumber vector.

If we want to "project" the phase speed onto a direction different from the wavenumber vector $\boldsymbol{k}$, we will have to find the distance between two phase surfaces along that direction different from $\boldsymbol{k}$. This gives a longer "projected" wavelength, see figure 1, thus the projected phase speed will be greater than the original value c. This is not how a vector projection works. We shall adopt the convention that *velocity* (Norwegian *hastighet*) is a vector and *speed* (Norwegian *fart*) is a scalar. As the phase speed does not obey the rules for vector projection, it should not be called a velocity. This curiosity arises from the fact that the phase surface is a geometrical rather than a material location. The phase speed is not the speed of translation of a material property like mass, momentum, energy, etc.

1.2 Free and bound waves, dispersion relation

A wave is said to be *free* if it can exist on its own without being forced, or equivalently, if its amplitude can be freely chosen. Otherwise, if the amplitude cannot be freely chosen, we say it is *bound* or *forced.*

Usually, in order for waves to be free, a certain relationship needs to be satisfied between the wavenumber vector $\boldsymbol{k}$ and the angular frequency ω, this is the *dispersion relation*, $\omega = \omega(\boldsymbol{k})$.

When we refer to a *linear dispersion relation* we mean that the dispersion relation is independent of the amplitude A, otherwise the dispersion relation is said to be nonlinear. This refers to the linearity of the equations governing the wave motion, not a linear relationship between $\boldsymbol{k}$ and ω.

If the dispersion relation specifies a proportionality between the wavenumber k and the angular frequency ω the waves are said to be *non-dispersive*, otherwise we say they are *dispersive*. For non-dispersive waves the phase speed does not depend on the frequency, for dispersive waves it depends on the frequency.

If the dispersion relation depends only on the wavenumber k and not on the direction of the wavenumber vector $\boldsymbol{k}$ the waves are said to be *isotropic*, otherwise we say they are *anisotropic*. Isotropic waves have the same properties in all directions, anisotropic waves have different properties in different directions.

For example, let us consider the two-dimensional wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) = 0. \tag{7}$$

If we try a solution $\eta = A\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$, with $\boldsymbol{k} = k_x \boldsymbol{i}_x + k_y \boldsymbol{i}_y$, we get

$$\left(-\omega^2 + c^2(k_x^2 + k_y^2)\right) A\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} = 0$$

We recognize that the amplitude can be freely chosen provided we satisfy the dispersion relation

$$\omega^2 = c^2(k_x^2 + k_y^2) = c^2 k^2 \tag{8}$$

where $k = |\boldsymbol{k}| = \sqrt{k_x^2 + k_y^2}$. Our solution is a free non-dispersive and isotropic wave with phase speed c.

The group velocity is the gradient $\boldsymbol{c}_g = \frac{\partial \omega}{\partial \boldsymbol{k}}$. For the example above we find

$$\boldsymbol{c}_g = \frac{\partial \omega}{\partial k} \frac{\partial k}{\partial \boldsymbol{k}} = c \frac{\boldsymbol{k}}{k}.$$

We recognize that for non-dispersive isotropic waves the magnitude of the group velocity is equal to the phase speed. We also recognize that for isotropic waves the direction of the group velocity is given by $\boldsymbol{k}$.

Consider another example, the forced wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \left(\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) = b \cos(k_1 x - \omega_1 t) \tag{9}$$

where b and k_1 and ω_1 are constants pertaining to the forcing. It is natural to assume a particular solution similar to the forcing,

$$\eta = a \cos(k_1 x - \omega_1 t). \tag{10}$$

Plugging in we find the amplitude of the forced wave

$$a = \frac{b}{k_1^2 c^2 - \omega_1^2}. \tag{11}$$

It should be noticed that this wave can only exist while the forcing is on ($b \neq 0$) and the forced wave does not satisfy the dispersion relation.

1.3 Irregular waves as superposition of monochromatic waves

If the governing equations are linear (in η) and unforced, we can limit attention to free waves like the monochromatic solutions of the form $\eta(\boldsymbol{r}, t) = A\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$ and use the principle of linear superposition to synthesize irregular wave solutions

$$\eta(\boldsymbol{r}, t) = \eta_L(\boldsymbol{r}, t) \equiv \sum_n A_n \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_n\cdot\boldsymbol{r}-\omega_n t)} \tag{12}$$

where each term is a free monochromatic wave satisfying the dispersion relation $\omega_n = \omega(\boldsymbol{k}_n)$ with complex amplitude A_n. In this setting there is no reason why the individual terms should interact. If the amplitudes A_n are independent stochastic variables, then the Central Limit Theorem predicts that the statistics of the resulting wave field should be Gaussian.

If the governing equations are nonlinear, the principle of linear superposition is not valid. However, if the governing equations are only weakly nonlinear, with the nonlinear contribution characterized by a small parameter $\epsilon \ll 1$, it may be possible to express an irregular wave solution by

$$\eta(\boldsymbol{r}, t) = \eta_L(\boldsymbol{r}, t) + \epsilon\eta_{NL}(\boldsymbol{r}, t) \tag{13}$$

where the leading order linear contribution $\eta_L(\boldsymbol{r}, t)$ is given by (12). However, in this setting there are two reasons why the Central Limit Theorem will not be valid any more: (I) The nonlinear contribution $\eta_{NL}(\boldsymbol{r}, t)$ will depend on the linear contribution $\eta_L(\boldsymbol{r}, t)$, and (II) the coefficients A_n within the linear contribution will depend on each other.

The ultimate goal of this course is to understand how the statistical properties of the wave field differ between the linear solution (12) and the nonlinear solution (13). First we will look at some examples of weakly nonlinear problems, in particular the case of water surface waves.

1.4 Exercises

1. **Isotropic waves.**

   Show that for isotropic waves, the group velocity $\boldsymbol{c}_g$ is in the direction of the wavenumber vector $\boldsymbol{k}$.

2. **Free waves.**

   Find the dispersion relations, phase speeds and group velocities of the following equations, and characterize the free waves as being dispersive/non-dispersive and isotropic/anisotropic:

   (a) The wave equation

   $$\frac{\partial^2\eta}{\partial t^2} - c^2\nabla^2\eta = 0$$

   (b) The heat equation

   $$\frac{\partial\eta}{\partial t} = \kappa\nabla^2\eta$$

(c) The linear Schrödinger equation

$$\mathrm{i}\frac{\partial \eta}{\partial t} + \alpha \frac{\partial^2 \eta}{\partial x^2} + \beta \frac{\partial^2 \eta}{\partial y^2} = 0$$

(d) The advection equation

$$\frac{\partial \eta}{\partial t} + \boldsymbol{v} \cdot \nabla \eta = 0$$

3. **Some solutions of the wave equation.**

Consider the wave equation

$$\frac{\partial^2 \eta}{\partial t^2} - c^2 \nabla^2 \eta = 0$$

(a) If $\boldsymbol{n}$ is a unit vector in any fixed direction, show that the wave equation has solutions

$$\eta(\boldsymbol{r}, t) = f(\boldsymbol{n} \cdot \boldsymbol{r} - ct) + g(\boldsymbol{n} \cdot \boldsymbol{r} + ct)$$

where f and g are any twice differentiable functions.

(b) For three-dimensional space, and if r is the radial distance in spherical coordinates, show that the wave equation has solutions

$$\eta(\boldsymbol{r}, t) = \frac{f(r - ct) + g(r + ct)}{r}$$

where f and g are any twice differentiable functions.

(c) For two-dimensional space, with r and θ being plane polar coordinates such that $x = r\cos\theta$ and $y = r\sin\theta$, and seeking solutions by separation of variables $\eta(\boldsymbol{r}, t) = R_n(r)\mathrm{e}^{\mathrm{i}(n\theta - \omega t)}$ for integer n, show that the radial functions $R_n(r)$ are solutions of the Bessel equation

$$r^2 R_n'' + r R_n' + (k^2 r^2 - n^2) R_n = 0$$

where $k = \frac{\omega}{c}$. The solution can be expressed as a combination of the Bessel functions of the first and second kind, $J_n(kr)$ and $Y_n(kr)$. Alternatively, we can employ the Hankel functions $H_n^{(1)}(kr) = J_n(kr) + \mathrm{i}Y_n(kr)$ and $H_n^{(2)}(kr) = J_n(kr) - \mathrm{i}Y_n(kr)$, which have asymptotic behavior for large arguments $H_n^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kr}}\mathrm{e}^{\mathrm{i}(kr - \frac{1}{2}n\pi - \frac{1}{4}\pi)}$ and $H_n^{(1)}(kr) \sim \sqrt{\frac{2}{\pi kr}}\mathrm{e}^{-\mathrm{i}(kr - \frac{1}{2}n\pi - \frac{1}{4}\pi)}$. We see that $H_n^{(1)}(kr)$ represents an outgoing wave and $H_n^{(2)}(kr)$ represents an incoming wave.

(d) For three-dimensional space in spherical coordinates $\{r, \theta, \varphi\}$, assuming radial symmetry such that η is only a function of r and t, set $\eta(r, t) = u(r)\mathrm{e}^{-\mathrm{i}\omega t}$ and show that u satisfies the Bessel equation

$$\xi^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + 2\xi \frac{\mathrm{d}u}{\mathrm{d}\xi} + (\xi^2 - n(n+1))u = 0$$

where $\xi = \omega r/c$ and where $n = 0$.

Two independent solutions are the spherical Bessel functions of the first and second kind, $\mathrm{j}_n(\xi)$ and $\mathrm{y}_n(\xi)$, or alternatively the two spherical Hankel functions $\mathrm{h}_n^{(1)}(\xi) = \mathrm{j}_n(\xi) + \mathrm{i}\mathrm{y}_n(\xi)$ and $\mathrm{h}_n^{(2)}(\xi) = \mathrm{j}_n(\xi) - \mathrm{i}\mathrm{y}_n(\xi)$. In particular we have

$$\mathrm{j}_0(\xi) = \frac{\sin(\xi)}{\xi}$$

$$\mathrm{y}_0(\xi) = -\frac{\cos(\xi)}{\xi}$$

Show that $\mathrm{j}_0(\xi)$ and $\mathrm{y}_0(\xi)$ correspond to standing waves, $\mathrm{h}_0^{(1)}(\xi)$ corresponds to an outgoing wave, and $\mathrm{h}_0^{(2)}(\xi)$ corresponds to an incoming wave.

(e) Including the angular dependence among the spherical coordinates show how the case $n \neq 0$ appears.

4. **Free and bound waves.**

Suppose waves in one horizontal direction x behave according to the equation

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} = f(x - c_1 t)$$

where c_0 and c_1 are two unequal constants, and where f is some function. The left-hand side of the equation is called the homogeneous part while the right-hand side is called a forcing or inhomogeneous part.

Show that without forcing, $f = 0$, the equation supports free waves.

Show that with forcing, $f \neq 0$, the equation has a bound wave solution that does not satisfy the linear dispersion relation.

Show that the full solution is a linear superposition of a bound wave (the particular or inhomogeneous solution) and an infinite number of free waves (the homogeneous solution).

5. **"Projection" of the phase speed.**

Next time you go to the coast or a lake, find a long pier that does not obstruct the water surface waves below. Suppose the waves are long-crested and monochromatic and have wave vector pointing in the x-direction, i.e., crests aligned in the y-direction. Suppose the pier is oriented at an angle θ with the x-axis. You want to run along the pier such that you follow one particular crest. Show that you need to run at a speed $c/\cos\theta$ where c is the phase speed of the waves. How fast would you have to run in the limit that the crests become parallel to the pier?

6. **Resonant growth.**

Starting from rest, with $\eta(x, y, t) = 0$ and $\frac{\partial \eta}{\partial t}(x, y, t) = 0$ at $t = 0$, and with a forcing that satisfies the dispersion relation, $k_1 c = \omega_1$, find a solution of the forced wave equation (9) for $t > 0$.