Representing a Homogenous Polynomial as a Sum of Powers of Linear Forms

Thesis for the Degree of
Candidatus Scientiarum

Johannes Kleppe

Department of Mathematics
University of Oslo
Spring 1999

Abstract

The problem of expressing a homogenous polynomial f as a sum of powers of linear forms is investigated via apolarity and solved in the following cases: f is a binary form, f is a ternary cubic and f is a ternary quartic.

While a general binary form f of degree d is a sum of $\lceil \frac{d+1}{2} \rceil$ linear powers, we prove that there are special f which need d summands in a representation as a sum of d^{th}-powers of linear forms. Similarly, it is known that a general ternary cubic and quartic is a sum of 4, respectively 6, linear powers. We prove that there are some ternary cubics and quartics f which are a sum of 5, respectively 7, linear powers and no less. Furthermore, we find all such f.

Preface

This thesis has been written for the degree of Candidatus Scientiarum at the Department of Mathematics, University of Oslo. My supervisor has been associate professor Kristian Ranestad.

In this thesis I solve the problem of expressing *all* homogenous polynomials f as sums of powers of linear forms, when f is a binary form, a ternary cubic and a ternary quartic. For a *general* f, the minimal number of summands is known, but little is known for *all* f. In particular, the results in chapter 3 about ternary quartics are, to my knowledge, previously not known.

I have tried to emphasize the number of summands needed in a representation of a given form as a sum of powers of linear forms, and what the forms that are "exceptional" in some sense, look like. In order to obtain my results, several different methods have been used, and each one is described briefly at the end of the introduction.

Acknowledgments

I would like to thank my supervisor for introducing me to this field of mathematics. Every discussion we had gave me many new ideas and question to investigate. I am also grateful for the way he let me work at my own pace, and I feel I have learnt a good deal about research in general and algebraic geometry in particular.

A special thanks goes to my father, Jan O. Kleppe. Whenever I had a question, he was there for me with his knowledge of general theory and enthusiasm for my work. Also I want to thank the rest of my family and my friends for being there when I needed a break from my master thesis.

Finally, I would like to say that even though mathematics can be exciting, beautiful and important, when it comes to what gives life meaning, it cannot compete with faith, hope and love.

May, 1999 Johannes Kleppe

Contents

CHAPTER 0

Introduction

During the last decades of the 19^{th} century, a lot of work was done to determine how a homogenous polynomial f of degree d can be represented as a sum of powers of linear forms,

$$f = l_1^d + l_2^d + \cdots + l_s^d,$$

and the problem has regained interest in recent years. There are two main issues. One is to find the minimal number s of summands that are needed in such a representation. The other is, given the minimal s, to determine the size of the family of such representations. For a *general* form f the question about the minimal s was recently solved.

THEOREM 0.1 (Alexander-Hirschowitz)**:**
A general form f of degree d in $n+1$ variables is a sum of $s = \lceil \frac{1}{n+1}\binom{n+d}{n} \rceil$[†] *powers of linear forms, unless*

$d = 2$, where $s = n+1$ instead of $\lceil \frac{n+2}{2} \rceil$,
$d = 4$ and $n = 2, 3, 4$, where $s = 6, 10, 15$ instead of $5, 9, 14$, respectively,
$d = 3$ and $n = 4$, where $s = 8$ instead of 7.

Proof: This follows from a result of Alexander and Hirshowitz [1] and Terracini's Lemma [18]. The exceptions where classically known, see [3], [14], [17], [15], [12] and [5]. □

But even though the minimal s is found for a general f, little is known when *all* f are to be considered. For a *special* f the minimal value of s may be both smaller or bigger. In this thesis we will find the minimal s for all binary forms (chapter 1), ternary cubics (chapter 2) and ternary quartics (chapter 3). While the general f in these three cases is a sum of $\lceil \frac{d+1}{2} \rceil$, 4 or 6 linear power, respectively, we will see that this is not the case for all f.

[†] Recall that $\lceil x \rceil$ is the smallest integer greater than or equal to x.

0.1 Notation and terminology

Throughout this thesis k will be an algebraically closed field of characteristic 0, but many of the results obtains here, are valid in a broader context.

Whenever A is a graded ring, or a graded ideal in a graded ring, A_d will denote the component of A of degree d, and we will write $\langle a_1, \dots, a_m \rangle$ for the subvectorspace of A_d generated by the elements $a_1, \dots, a_m \in A_d$. As usual, $(a_1, \dots, a_m)$ denotes the ideal of A generated by these elements.

We will use the word *form* to mean a homogenous polynomial of positive degree. In particular, the words *quadric*, *cubic* and *quartic* will refer to a form of degree two, three and four, respectively. And by *binary* and *ternary* forms we will mean forms in two, respectively three, variables.

Now, for any two polynomials $f, g \in S = k[x_0, \dots, x_n]$, let $f \sim g$ if and only if there exists an invertible linear transformation φ of S_1 such that $\bar{\varphi}(f) = g$, where $\bar{\varphi}$ is the homomorphism $S \to S$ of k-algebras induced by φ. This defines an equivalence relation $\sim$ on S which represents a change of basis. Notice that if $f \sim g$ and $f = \sum_{i=1}^{s} \lambda_i \, l_i^d$ for some $l_i \in S_1$ and $\lambda_i \in k$, then $g = \bar{\varphi}(f) = \sum_{i=1}^{s} \lambda_i \, \varphi(l_i)^d$. Hence the minimal s such that f is a sum of s linear powers, is an invariant of the equivalence class of f under $\sim$. Since finding such s is the main problem in this thesis, we will often perform such invertible linear transformations of S_1. Therefore any linear transformation will be assumed invertible, unless otherwise specified. Moreover, when we say that some polynomial is unique with respect to some property, we will often mean unique up to a nonzero scalar.

When we want to emphasize that a form f is a sum of s linear powers, and no less than s linear powers, we sometimes say "f is a sum of exactly s linear powers". This does not mean that f cannot be written as a sum of more than s linear powers, of course.

0.2 Preliminary definitions and results

We want the two polynomial rings $S = k[x_0, \dots, x_n]$ and $T = k[\partial_0, \dots, \partial_n]$ to act on each other by differentiation. T acts on S in the usual way, i.e.

$$\left(\prod_{i=0}^{n} \partial_i^{\alpha_i} \right) \left(\prod_{i=0}^{n} x_i^{\beta_i} \right) = \prod_{i=0}^{n} \alpha_i! \binom{\beta_i}{\alpha_i} x_i^{\beta_i - \alpha_i}$$

or $\partial^{\alpha}(x^{\beta}) = \alpha! \binom{\beta}{\alpha} x^{\beta - \alpha}$ when we use multi-indices. Similarly we define the action of S on T by $x^{\beta}(\partial^{\alpha}) = \beta! \binom{\alpha}{\beta} \partial^{\alpha - \beta}$. In particular we notice that

$$f(D) = \sum_{\alpha} \alpha! \, a_{\alpha} b_{\alpha} = D(f)$$

for two forms $f = \sum_\alpha a_\alpha x^\alpha \in S$ and $D = \sum_\beta b_\beta \partial^\beta \in T$ of the same degree. Furthermore,

$$D\big(l_a^d\big) = e!\tbinom{d}{e} D(a)\, l_a^{d-e} \tag{0.1}$$

for any $D \in T_e$ and $l_a = \sum_{i=0}^n a_i x_i \in S_1$, where $a = (a_0, \dots, a_n) \in \mathbb{P}^n$. Of course, $D\big(l_a^d\big) = 0$ when $e > d$. In general, we say that two forms $f \in S$ and $D \in T$ are *apolar* if $f(D) = D(f) = 0$.

Definition 0.2: When $f \in S$ is a homogenous polynomial, define $f^\perp$ by

$$f^\perp = \{D \in T \mid D(f) = 0\}$$

and A^f by

$$A^f = T/f^\perp.$$

It is easily seen that $f^\perp$ is a homogenous ideal of T, $f = \oplus_{e\geq 0} f_e^\perp$. Moreover, A^f has a one-dimensional socle in degree $\deg f$, and is a graded Artinian Gorenstein ring. Furthermore, every graded Artinian Gorenstein ring arise this way for suitable f. This is the content of the following lemma.

***Lemma* 0.3** (Macaulay, [11])**:** *The map $f \mapsto A^f$ gives a bijection between hypersurfaces $F = \{f = 0\} \subseteq \mathbb{P}^n$ of degree d and graded Artinian Gorenstein quotient rings $A = T/I$ of T with socle in degree d.*

Proof: For Macaulay's result in terms of *inverse systems*, see [11, chapter IV], [6, theorem 21.6 and exercise 21.7], [7] or [10, lemma 1.2]. □

The polynomial f is called the *dual socle generator* of A^f, and is defined only up to a nonzero scalar. Also note that for any $f \in S_d$,

$$\bigcap_{i=0}^{n} (\partial_i f)^\perp = \{D \in T \mid \partial_i D(f) = 0 \,\forall\, i = 0, \dots, n\} = f^\perp \cup T_d \tag{0.2}$$

because $\partial_i g = 0$ for all i if and only if $g \in S_0 = k$.

For any $f \in S_d$, we define $\varphi_f : T_d \to k$ by $\varphi_f(D) = D(f) = f(D)$. This is obviously a k-linear homomorphism, and $\sum_i c_i \varphi_{f_i} = \varphi_g$ where $g = \sum_i c_i f_i$. Furthermore, $\varphi_f = 0$ if and only if $D(f) = 0$ for all $D \in T_d$, which means that $f = 0$. Hence $\mathrm{Hom}_k(T_d, k) = \{\varphi_f \mid f \in S_d\}$, because these two vectorspaces have equal dimensions. We are now ready to prove our main lemma.

***Lemma* 0.4:** *Let $f, g_1, \dots, g_s \in S_d$ be forms of the same degree d. Then the following statements are equivalent:*

(a) *there exist $\lambda_1, \dots, \lambda_s \in k$ such that* $f = \sum_{i=1}^{s} \lambda_i g_i$

(b) $\bigcap_{i=1}^{s} (g_i)^\perp \subseteq f^\perp$

(c) $\bigcap\limits_{i=1}^{s}(g_i)_d^\perp \subseteq f_d^\perp$

Proof: The first implication (a) $\Rightarrow$ (b) follows immediately from the fact that if $f = \sum_{i=1}^{s} \lambda_i g_i$, then $D(f) = \sum_{i=1}^{s} \lambda_i D(g_i)$. The second, (b) $\Rightarrow$ (c), is obvious. To prove that (c) $\Rightarrow$ (a), we proceed as follows:

First we claim that, for any $g_1, \dots, g_s \in S_d$, we have

$$\dim_k \bigcap\limits_{i=1}^{s}(g_i)_d^\perp = \dim_k T_d - \dim_k \langle g_1, \dots, g_s\rangle. \tag{0.3}$$

Let $t = \dim_k\langle g_1, \dots, g_s\rangle \leq s$. Possibly after renumbering the elements, we may assume that $\langle g_1, \dots, g_t\rangle = \langle g_1, \dots, g_s\rangle$. Furthermore, $\cap_{i=1}^{t}(g_i)_d^\perp = \cap_{i=1}^{s}(g_i)_d^\perp$. If we now look at $\mathrm{Hom}_k(V, k)$ where $V = S_d/\langle g_1, \dots, g_t\rangle$, we see that for any $D \in \cap_{i=1}^{t}(g_i)_d^\perp$, the map $h \mapsto D(h)$ defines a k-linear homomorphism $\psi_D : V \to k$. If $\psi_D = 0$, then $D(h + \langle g_1, \dots, g_t\rangle) = 0$ for all $h \in S_d$, hence $D = 0$. Therefore the map $\cap_{i=1}^{t}(g_i)_d^\perp \hookrightarrow \mathrm{Hom}_k(V, k)$ given by $D \mapsto \psi_D$ is injective, and it follows that

$$\dim_k \bigcap\limits_{i=1}^{t}(g_i)_d^\perp \leq \dim_k \mathrm{Hom}_k(V, k) = \dim_k V = \dim_k T_d - t. \tag{0.4}$$

On the other hand, we know that the following sequence is exact,

$$0 \to \bigcap\limits_{i=1}^{t}(g_i)_d^\perp \to T_d \to \bigoplus\limits_{i=1}^{t} T_d/(g_i)_d^\perp,$$

which gives that

$$\dim_k T_d - \dim_k \bigcap\limits_{i=1}^{t}(g_i)_d^\perp \leq \dim_k \bigoplus\limits_{i=1}^{t} T_d/(g_i)_d^\perp = \sum_{i=1}^{t} 1 = t. \tag{0.5}$$

The equations (0.4) and (0.5) implies that $\dim_k \cap_{i=1}^{t}(g_i)_d^\perp = \dim_k T_d - t$, which proves (0.3). Now we have

$$(c) \Leftrightarrow \bigcap\limits_{i=1}^{s}(g_i)_d^\perp \cap f_d^\perp = \bigcap\limits_{i=1}^{s}(g_i)_d^\perp \Leftrightarrow \dim_k \bigcap\limits_{i=1}^{s}(g_i)_d^\perp \cap f_d^\perp = \dim_k \bigcap\limits_{i=1}^{s}(g_i)_d^\perp$$

By (0.3) this is equivalent to $\dim_k\langle g_1, \dots, g_s, f\rangle = \dim_k\langle g_1, \dots, g_s\rangle$, which means that $\langle g_1, \dots, g_s, f\rangle = \langle g_1, \dots, g_s\rangle$. Therefore, $f \in \langle g_1, \dots, g_s\rangle$, and there exist $\lambda_1, \dots, \lambda_s \in k$ such that $f = \sum_{i=1}^{s} \lambda_i g_i$. □

REMARK 0.4.1: We can also prove the implication (c) $\Rightarrow$ (a) the following way: Consider the projections $T_d \to T_d/\cap_{i=1}^{s}(g_i)_d^\perp \to T_d/f_d^\perp$ and the induced injections

$$\mathrm{Hom}_k(T_d/f_d^\perp, k) \subseteq \mathrm{Hom}_k(T_d/\bigcap\limits_{i=1}^{s}(g_i)_d^\perp, k) \subseteq \mathrm{Hom}_k(T_d, k). \tag{0.6}$$

The proof of equation (0.3) actually proves that $\mathrm{Hom}_k(T_d/\cap_{i=1}^s (g_i)_d^\perp, k) = \langle \varphi_{g_1}, \ldots, \varphi_{g_s} \rangle$, because they have equal dimensions. Hence the first of the inclusions in (0.6) implies that $\varphi_f \in \langle \varphi_{g_1}, \ldots, \varphi_{g_s} \rangle$, that is, there exist $\lambda_1, \ldots, \lambda_s \in k$ such that $\varphi_f = \sum_{i=1}^s \lambda_i \varphi_{g_i}$. This means that $D(f) = \sum_{i=1}^s \lambda_i D(g_i) = D\big(\sum_{i=1}^s \lambda_i g_i\big)$ for all $D \in T_d$, and it follows that $f = \sum_{i=1}^s \lambda_i g_i$. Compare this proof with [13, paragraph 1.3] or [16, theorem 1.4].

Normally, we will use lemma 0.4 in situations where the g_i's are powers of linear forms, that is, $g_i = l_{a_i}^d$ where $l_{a_i} = \sum_{j=0}^n a_{ij} x_j$ and $a_i = (a_{i0}, \ldots, a_{in})$ is a point in $\mathbb{P}^n$. In this case we see that

$$\bigcap_{i=1}^s \big(l_{a_i}^d\big)_e^\perp = \{D \in T_e \mid D\big(l_{a_i}^d\big) = 0 \;\forall\, i = 1, \ldots, s\}$$

For $e > d$ this obviously equals all of T_e, but when $e \le d$ it follows from equation (0.1) that

$$\bigcap_{i=1}^s \big(l_{a_i}^d\big)_e^\perp = \{D \in T_e \mid D(a_i) = 0 \;\forall\, i = 1, \ldots, s\} = \bigcap_{i=1}^s \big(l_{a_i}^e\big)_e^\perp = I_e \qquad (0.7)$$

where $I = \cap_{i=1}^s \mathsf{m}_{a_i}$ and m_P is the "maximal" homogenous ideal through $P = (p_0, \ldots, p_n) \in \mathbb{P}^n$ generated by the 2×2 minors of $\left(\begin{smallmatrix} x_0 & x_1 & \cdots & x_n \\ p_0 & p_1 & \cdots & p_n \end{smallmatrix}\right)$. We see that we might describe I as $I = \oplus_{d \ge 0} \cap_{i=1}^s (l_{a_i}^d)_d^\perp$. Using equation (0.7), lemma 0.4 says that $f \in \langle l_{a_1}^d, \ldots, l_{a_s}^d \rangle$ if and only if $I \subseteq \cap_{i=1}^s (l_{a_i}^d)^\perp \subseteq f^\perp$. Thus we have proved the following corollary:

***Corollary* 0.5:** *If $f \in S$ is a form of degree d, then f is a sum of s linear powers $l_{a_1}^d, \ldots, l_{a_s}^d$, with $l_{a_i} \in S_1$, if and only if*

$$\bigcap_{i=1}^s \mathsf{m}_{a_i} \subseteq f^\perp.$$

Furthermore, $\cap_{i=1}^s (\mathsf{m}_{a_i})_e = \cap_{i=1}^s \big(l_{a_i}^d\big)_e^\perp$ for all degrees $e \le d$.

In general, we call a subscheme Γ of $\mathbb{P}^n$ *apolar* to f if the homogenous ideal I_Γ of Γ is contained in $f^\perp$. Hence the corollary says that $f = \sum_{i=1}^s \lambda_i l_{a_i}^d$ for some reduced set of points $\Gamma = \{a_1, \ldots, a_s\}$ if and only if the homogenous ideal $I_\Gamma = \cap_{i=1}^s \mathsf{m}_{a_i}$ is apolar to f.

REMARK 0.5.1: We have seen that if $f = \sum_{i=1}^s \lambda_i l_{a_i}^d$, then $\cap_{i=1}^s (l_{a_i}^d)^\perp \subseteq f^\perp$, and hence $\dim_k \cap_{i=1}^s (l_{a_i}^d)_e^\perp \le \dim_k f_e^\perp$ for all e. Now equation (0.3) says that $\dim_k \cap_{i=1}^s (l_{a_i}^d)_e^\perp = \dim_k T_e - \dim_k \langle l_{a_1}^e, \ldots, l_{a_s}^e \rangle \ge \dim_k T_e - s$, which gives

$$s \ge \dim_k T_e - \dim_k \bigcap_{i=1}^s (l_{a_i}^d)_e^\perp \ge \dim_k T_e - \dim_k f_e^\perp = \dim_k A_e^f. \qquad (0.8)$$

Hence any f cannot be a sum of less linear powers than the maximum of the Hilbert function of A^f.

One might think that equation (0.8) ought to be an equality for general f, but this is not true. Since A^f is Artinian Gorenstein with socledegree $d = \deg f$, the maximum of the Hilbert function of A^f is $\dim_k A^f_{e_0}$ where $e_0 = \lfloor \frac{d}{2} \rfloor$. Now $\dim_k A^f_{e_0} \le \dim_k T_{e_0} = \binom{n+e_0}{e_0}$ for all $f \in S_d$, while the s given in theorem 0.1 satisfies $s \ge \binom{n+e_0}{e_0}$, with equality only when n and d are very small. Therefore, equation (0.8) *cannot* be an equality for a general f, unless n and d are very small. However, (0.8) is an equality for $f = \sum_{i=1}^{s} l_i^d$ if the linear forms $l_1, \ldots, l_s$ are general enough, see [8].

Now we want to define two quantities that will play an important role in this thesis. They cover different needs, in fact, the first one will be used mostly in the last couple of sections, while the second will be used almost everywhere else.

Definition 0.6: For $f \in S_{2d}$ we define the *catalecticant matrix* of f to be

$$\mathrm{Cat}_{\mathcal{D}}(f) = \big(D_i D_j(f)\big)_{1 \le i,j \le N}$$

where $N = \binom{n+d}{n}$ and $\mathcal{D} = \{D_1, \ldots, D_N\}$ is a basis for T_d.

We may think of $\mathrm{Cat}_{\mathcal{D}}$ as a matrix with entries in T_{2d}, namely $\mathrm{Cat}_{\mathcal{D}} = v_{\mathcal{D}} v_{\mathcal{D}}^T$ where $v_{\mathcal{D}}$ is a $N \times 1$ matrix (a columnvector) and $v_{\mathcal{D}}^T = [D_0, \ldots, D_N]$ is the transpose of $v_{\mathcal{D}}$. In particular, $\operatorname{rank} \mathrm{Cat}_{\mathcal{D}}(f)$ is independent of the choice of basis for T_d. Indeed, if $\mathcal{E}$ is another basis, then $v_{\mathcal{E}} = A v_{\mathcal{D}}$ for some invertible $N \times N$ matrix A, and $\mathrm{Cat}_{\mathcal{E}}(f) = (v_{\mathcal{E}} v_{\mathcal{E}}^T)(f) = (A v_{\mathcal{D}} v_{\mathcal{D}}^T A^T)(f) = A\,\mathrm{Cat}_{\mathcal{D}}(f) A^T$. Moreover, since multiplication in A^f gives perfect pairings $A^f_d \times A^f_d \to A^f_{2d} \cong k$, we have

$$\operatorname{rank} \mathrm{Cat}_{\mathcal{D}}(f) = \dim_k A^f_d. \tag{0.9}$$

Definition 0.7: Given $f \in S_d$ such that $f_e^\perp = \langle D_0, \ldots, D_m \rangle$, we define the rational map $\pi_e^f : \mathbb{P}^n \to \mathbb{P}^m$ by $\pi_e^f(a) = \big(D_0(a), \ldots, D_m(a)\big)$ for $a = (a_0, \ldots, a_n) \in \mathbb{P}^n$.

We notice that the *base locus* of π_e^f, i.e.

$$\{a \in \mathbb{P}^n \mid D_i(a) = 0 \,\forall\, i = 0, \ldots, m\} = \bigcap_{i=0}^{m} V(D_i) = V(f_e^\perp),$$

is independent of the choice of basis for $f_e^\perp$. For $e \le d$ we might think of π_e^f as the composition of the e-uple embedding of $\mathbb{P}^n$ and the projection from the partials of f of order $d-e$, $\{D(f) \mid D \in T_{d-e}\}$, considered as points in $\mathbb{P}^{\binom{n+e}{e}-1}$. See [13, chapter 2] for more details about this construction and a general motivation for studying this map.

Throughout this thesis we will several times encounter a set of points Γ that is a complete intersection of two forms $f, g \in k[x_0, x_1, x_2]$, and we need to know the generators of the homogenous ideal I_Γ. In situations where f and g intersect nonsingularly, the following theorem will be useful.

Theorem 0.8 (Max Noether)**:**
Suppose that two forms $f, g \in k[x_0, x_1, x_2]$ meet transversely in a finite set of points Γ. If $h \in k[x_0, x_1, x_2]$ is any form that vanishes on Γ, then $h \in (f, g)$.

Proof: That f and g meet transversely means that their intersection Γ is nonsingular, and the result is a consequence of the unmixedness theorem, see [6, corollary 18.14 and exercise 18.10]. □

Before we start our investigations, we would like to say a few words about our methods: The most used notion in this thesis is *apolarity*, we usually use the ideal $f^\perp$ in conjunction with lemma 0.4 or corollary 0.5 to find powersum-representations for f. As long as the dimension of $f_e^\perp$ is large enough for small e, specifically $\dim_k f_2^\perp \geq 2$ when f is a ternary quartic, we prove some of our results by studying π_e^f and its base locus. When $\dim_k f_e^\perp$ is too small, we often try to subtract a suitable multiple of some linear power from f to get a new form $g = f - \lambda\, l^d$ where $\dim_k g_e^\perp > \dim_k f_e^\perp$, provided we can control $g_e^\perp / f_e^\perp$ sufficiently. Finally, in section 3.3.1 we use the Buchsbaum-Eisenbud structure theorem [2] to get a minimal free resolution for A^f, and further manipulations of this gives us what we want.

Chapter 1

Binary forms

In this chapter we deal with binary forms, that is, homogenous polynomials in two variables, and we will prove that any binary form f is a sum of $\deg f$ linear powers. But to do so, we need the following lemma. Note that we call a binary form f *squarefree* if no factor of f appears twice, i.e. $f = \prod_i l_i$ for some linear forms l_i where $l_i \nparallel l_j$ for all $i \neq j$.

Lemma 1.1: *Let $f_1, \dots, f_n \in k[x_0, x_1]$ be binary forms of the same degree such that any linear combination of them is not squarefree. Then there exists a linear form $l \in k[x_0, x_1]_1$ such that $l^2 | f_i$ for all i.*

Proof: We may assume that the f_i's are linearly independent, or else we could replace them by a linearly independent subset. Let $g = \sum_{i=2}^n c_i f_i$ be *any* linear combination of $f_2, \dots, f_n$, and consider the Jacobian matrix $J = \begin{pmatrix} \partial_0 f_1 & \partial_0 g \\ \partial_1 f_1 & \partial_1 g \end{pmatrix}$. This matrix has rank ≤ 1 at a point $P \in \mathbb{P}^1$ if and only if $(\det J)(P) = 0$. If $\det J$ equals 0 as an element of $k[x_0, x_1]$, then f_1 and g are equal, up to a scalar. This is impossible since the f_i's are linearly independent. Hence there are only finitely many P that makes $(\det J)(P)$ zero, which is equivalent to $\nabla f_1(P) \parallel \nabla g(P)$, where $\nabla f = (\partial_i f)$. For any such P, either both $\nabla f_1(P)$ and $\nabla g(P)$ equals zero, or there exists a unique point $(a, b) \in \mathbb{P}^1$ such that $\nabla(af_1 + bg)(P) = 0$. Therefore, there must exist $P \in \mathbb{P}^1$ such that $\nabla f_1(P) = \nabla g(P) = 0$, or else there would only be finitely many $(a, b) \in \mathbb{P}^1$ such that $af_1 + bg$ is not squarefree, which contradicts our assumptions. But this means that both f_1 and g has a double root in P, i.e. there exists $l \in k[x_0, x_1]$ of degree 1 such that $l^2 | f_1$ and $l^2 | g$.

This l might in general depend on the linear combination $g = \sum_{i=2}^n c_i f_i$. Since f_1 is a nonzero polynomial, we know that there are only finitely many l such that $l^2 | f_1$, say $l_1, \dots, l_m$. Let $V_j = \{(c_2, \dots, c_n) \,|\, l_j^2 \text{ divides } \sum_{i=2}^n c_i f_i\}$. Now $\cup_{j=1}^m V_j = k^{n-1}$ because any linear combination $\sum_{i=2}^n c_i f_i$ has a common square factor with f_1. Since every V_j is a vectorspace, there must be one j such that $V_j = k^{n-1}$. Hence for this j we have $l_j^2 | f_i$ for all i. $\square$

REMARK 1.1.1: Since $\operatorname{char} k = 0$, we can use Bertini's theorem [9, p. 274-275] to prove lemma 1.1. Given n binary forms $f_1, \ldots, f_n$ of the same degree such that any linear combination is not squarefree, let $h = \gcd(f_1, \ldots, f_n)$ and $\tilde{f}_i = f_i/h$. Then $\langle \tilde{f}_1, \ldots, \tilde{f}_n \rangle$ is a base-point-free linear system, and by Bertini's theorem the general member is nonsingular. Furthermore, if we let $a_1, \ldots, a_m$ be the zeroes of h, then $\{g \in \langle \tilde{f}_1, \ldots, \tilde{f}_n \rangle \,|\, g(a_i) = 0\}$ is a proper, closed subset for all i. Hence we can find $g \in \langle \tilde{f}_1, \ldots, \tilde{f}_n \rangle$ such that g is nonsingular and $g(a_i) \neq 0$ for all i. By assumption, $g \cdot h \in \langle f_1, \ldots, f_n \rangle$ is not squarefree. Then h cannot be squarefree, since g is squarefree and has no common factor with h. But this means that there exists a linear form l such that $l^2 | h$, and then $l^2 | f_i$ for all i.

THEOREM 1.2:
Every binary form $f \in k[x_0, x_1]$ of degree d is a sum of d or less linear powers.

Proof: We claim that

$$f \text{ is a sum of } s \text{ linear powers} \Leftrightarrow \exists\, D \in f_s^\perp \text{ such that } D \text{ is squarefree.}$$

Even though we only need one of the implications to prove theorem 1.2, we will prove the equivalence for later use.

Assume that $f = \sum_{i=1}^s \lambda_i \, l_{a_i}^d$ where $a_1, \ldots, a_s$ are distinct points in $\mathbb{P}^1$. Let $\delta_i = a_{i0}\partial_1 - a_{i1}\partial_0$ and $D = \prod_{i=1}^s \delta_i$. Then δ_i is the unique $\delta \in k[\partial_0, \partial_1]_1$ such that $\delta(l_{a_i}) = 0$, and D is squarefree. Furthermore, $D(l_{a_i}) = 0$ for all $i = 1, \ldots, s$, hence $D \in f_s^\perp$.

Conversely, assume that there exists $D \in f_s^\perp$ which is squarefree. Factorize this D into its linear factors, i.e. $D = \prod_{i=1}^s \delta_i$ for some $\delta_i = a_{i0}\partial_1 - a_{i1}\partial_0$, where $a_i = (a_{i0}, a_{i1})$ are points in $\mathbb{P}^1$. These points are all distinct since D is squarefree. Now, $\mathsf{m}_{a_i} = (\delta_i)$, which implies that $\cap_{i=1}^s \mathsf{m}_{a_i} = \left(\prod_{i=1}^s \delta_i\right) = (D)$. Then $\cap_{i=1}^s \mathsf{m}_{a_i} \subseteq f^\perp$ because $D \in f^\perp$, and by corollary 0.5 this means that there exist $\lambda_1, \ldots, \lambda_s$ such that $f = \sum_{i=1}^s \lambda_i \, l_{a_i}^d$, as claimed.

Now assume that $f \in k[x_0, x_1]_d$ is not a sum of d linear powers. Then no $D \in f_d^\perp$ is squarefree, and by lemma 1.1, they have a common square factor. This means that $f_d^\perp \subseteq (\delta^2)_d$ for some $\delta \in k[\partial_0, \partial_1]_1$, which obviously is impossible since $\dim_k f_d^\perp = \dim_k k[\partial_0, \partial_1]_d - 1 = d$ and $\dim_k (\delta^2)_d = d - 1$. □

REMARK 1.2.1: The proof of theorem 1.2 tells us exactly which forms $f \in k[x_0, x_1]_d$ that are not a sum of less than d linear powers. If $d = 1$ then this is obviously any nonzero f, and if $d = 2$ then f must be a product of two different linear forms. For $d \geq 3$ any such f must satisfy that no $D \in f_{d-1}^\perp$ is squarefree, which implies that $f_{d-1}^\perp \subseteq (\delta^2)_{d-1}$ for some $\delta \in k[\partial_0, \partial_1]_1$. Since

$$\dim_k f_{d-1}^\perp = \dim_k k[\partial_0, \partial_1]_{d-1} - \dim_k A_{d-1}^f = d - \dim_k A_1^f \geq d - 2$$

because $A^f = k[\partial_0, \partial_1]/f^\perp$ is Gorenstein by 0.3, and $\dim_k(\delta^2)_{d-1} = d-2$, it follows that $f^\perp_{d-1} = (\delta^2)_{d-1}$. Hence $D(\delta^2 f) = 0$ for all $D \in k[\partial_0, \partial_1]_{d-3}$, and therefore $\delta^2(f) = 0$. Consequently, $f = l_1 \cdot l_2^{d-1}$ where $\{l_1, l_2\}$ is a basis for $k[x_0, x_1]_1$ such that $\delta(l_1) \neq 0$ and $\delta(l_2) = 0$.

On the other hand, if $f = x_0 x_1^{d-1}$, then $f^\perp = (\partial_0^2, \partial_1^d)$. Now $f^\perp_e = (\partial_0^2)_e$ for all $e < d$, so the lowest degree for which there exists $D \in f^\perp$ that is squarefree, is d (take for instance $\partial_0^d - \partial_1^d \in f^\perp_d$). Therefore, this is an f which is a sum of d linear powers and no less, and up to linear transformations of $k[x_0, x_1]_1$, this is the only one.

Remark 1.2.2: Using the same notation as in the proof of theorem 1.2, we see that for all degrees $e \leq d$ we have $\cap_{i=1}^s \left(l_i^d\right)^\perp_e = \cap_{i=1}^s (\mathsf{m}_{a_i})_e = (D)_e$, which implies that $\dim_k \cap_{i=1}^s \left(l_i^d\right)^\perp_d = \max(d+1-s, 0)$. By (0.3), this is the same as saying that for any $i \leq s$, l_i^d is not contained in the subspace generated by $l_1^d, \dots, l_{i-1}^d$, since the dimension of the intersection drops by one for each l_i^d added. Therefore, the minimal s such that there exist different (nonproportional) $l_1, \dots, l_s \in k[x, y]_1$ with $\sum_{i=1}^s l_i^d = 0$, is $d+2$. This can of course be proven more directly.

Chapter 2

Ternary cubics

In this chapter we investigate how to write a ternary cubic $f \in k[x_0, x_1, x_2]_3$ as a sum of linear powers. Theorem 0.1 tells us that a *general* ternary cubic is a sum of four linear powers. Our main result states that every f is a sum of four or less linear powers, with one exception.

Our method will use the rational map π_e^f as defined in definition 0.7

$$\pi_2^f : P \mapsto (D_0(P), D_1(P), D_2(P))$$

with $e = 2$, and we will first motivate this. Note that we by a *fibre* of π_2^f at the point a will refer to the intersection $\{\sum_i c_i D_i = 0\} \cap \{\sum_i c_i' D_i = 0\}$ where $\sum_i c_i y_i$ and $\sum_i c_i' y_i$ are two lines that intersect in a. In particular, all base points are part of any fibre of π_2^f.

Now, assume that an $f \in k[x_0, x_1, x_2]_3$ with $f_1^\perp = 0$ is a sum of four linear powers, say $f = \sum_{i=1}^4 \lambda_i \, l_{a_i}^4$. Since $\dim_k f_1^\perp = 0$, the Hilbert function of A^f must equal $(1, 3, 3, 1)$, and all four points $a_1, \dots, a_4$ cannot lie on a line. Hence the vectorspace of quadrics passing through $\{a_1, \dots, a_4\}$ is two-dimensional. Therefore, we can find two linearly independent quadrics $D_0, D_1 \in T_2$ such that $D_j(a_i) = 0$ for all i and j, which implies that $D_0(f) = D_1(f) = 0$. Since $\dim_k f_2^\perp = 3$, $f_2^\perp = \langle D_0, D_1, D_2 \rangle$ for some $D_2 \in T_2$.

If no line passes through three of the points $\{a_i\}$, the intersection of D_0 and D_1 will be $\{a_1, \dots, a_4\}$. Hence $\pi_2^f : \mathbb{P}^2 \to \mathbb{P}^2$ has at least one fibre of degree four, namely $(\pi_2^f)^{-1}(0, 0, 1) = \{a_1, \dots, a_4\}$. On the other hand, if three of the points $a_1, \dots, a_4$ lie on a line $L = \{\delta_0 = 0\}$, say $\{a_1, a_2, a_3\} \subseteq L$, then $\delta_0 f = \lambda_4 l_{a_4}^2$. As we will prove in lemma 2.1, this means that a_4 is a base point of π_2^f. If a quadric contains three points on a line, then the quadric must be reducible with the line as one component. Hence $D_0 = \delta_0 \delta_1$ and $D_1 = \delta_0 \delta_2$ for some $\delta_i \in T_1$, and $\{\delta_1 = 0\} \cap \{\delta_2 = 0\} = \{a_4\}$. In this case, the fibre of π_2^f above $(0, 0, 1)$ consists of the line L (with two embedded points) and the isolated point a_4. In both cases, we see that the four points $\{a_i\}$ are contained within a fibre of π_2^f.

We will need the following lemmas, which we prove in the more general setting where $S = k[x_0, \dots, x_n]$ and $T = k[\partial_0, \dots, \partial_n]$.

Lemma 2.1: *Let $f \in S_d$ be any form such that $f_e^\perp = \langle D_0, \dots, D_m \rangle$ for some $e \le d$ and $D_i \in T_e$. Then a point $a \in \mathbb{P}^n$ is a base point of $\pi_e^f : \mathbb{P}^n \to \mathbb{P}^m$ if and only if there exists an $E \in T_{d-e}$ such that $E(f) = l_a^e$.*

Proof: By definition, a is a base point of π_e^f if and only if $D_i(a) = 0$ for all $i = 0, \dots, m$. If $\mathcal{E} = \{E_0, \dots, E_N\}$ is a basis for T_{d-e}, then

$$\bigcap_{i=0}^{N} (E_i f)_e^\perp = \{D \in T_e \mid E_i(D(f)) = 0 \,\forall\, i\} = \{D \in T_e \mid D(f) = 0\} = f_e^\perp.$$

Hence $D_i(a) = 0$ for all i if and only if

$$\left(l_a^e\right)_e^\perp = \{D \in T_e \mid D(a) = 0\} \supseteq f_e^\perp = \bigcap_{i=0}^{N} (E_i f)_e^\perp.$$

By lemma 0.4 this is equivalent to $l_a^e = \sum_{i=0}^N \lambda_i \, E_i f = (\sum_{i=0}^N \lambda_i \, E_i)(f)$ for some $\lambda_i \in k$. With $E = \sum_{i=0}^N \lambda_i \, E_i \in T_{d-e}$ we get the desired conclusion. □

Lemma 2.2: *Let $A = (a_{ij})$ be an $(n+1)\times(n+1)$ matrix where $a_{ij} \in k$ for all $0 \le i, j \le n$. For $i = 0, \dots, n$, define $\bar{x}_i \in S_1$ and $\bar{\partial}_i \in T_1$ by $\bar{x}_i = \sum_{j=0}^n a_{ij} x_j$ and $\bar{\partial}_i = \sum_{j=0}^n a_{ji} \partial_j$. Let $\varphi : S \to S$ be the homomorphism of k-algebras induced by $x_i \mapsto \bar{x}_i$. Similarly, let $\psi : T \to T$ be the homomorphism induced by $\partial_i \mapsto \bar{\partial}_i$. Then for all $D \in T$,*

$$D\big(\varphi(f)\big) = \varphi\big(\psi(D)(f)\big). \tag{2.1}$$

We note that the linear transformation of S_1 which induces φ, is given by $\bar{x} = Ax$, where x and $\bar{x}$ are columnvectors, $x^T = [x_0, \dots, x_n]$. Similarly, the linear transformation of T_1 is given by $\bar{\partial} = A^T \partial$. These linear transformation might not be invertible, since A is not assumed invertible, but A should always be invertible when we use this lemma in the following.

For convenience, we will adopt the analytical standard and write $f(\bar{x})$ for $\varphi(f)$ in this proof, and think of $f(\bar{x})$ as "$f(x)$ where x_i is replaced by $\bar{x}_i$".

Proof of Lemma 2.2. Since both φ and ψ are homomorphisms of k-algebras, we only need to verify (2.1) for all $D \in \{\partial_0, \dots, \partial_n\}$. And this is really a simple consequence of the chain rule.

$$\begin{aligned} \partial_i\big(\varphi(f)\big) = \partial_i\big(f(\bar{x})\big) &= \sum_{j=0}^{n} (\partial_j f)(\bar{x}) \cdot \partial_i \bar{x}_j = \sum_{j=0}^{n} a_{ji} (\partial_j f)(\bar{x}) \\ &= \varphi\Big(\sum_{j=0}^{n} a_{ji} \partial_j f\Big) = \varphi\big(\psi(\partial_i)(f)\big). \end{aligned}$$

□

REMARK 2.2.1: Notice that if $g \sim f$, then by definition $f = \varphi(g)$ where φ is a homomorphism of k-algebras $S \to S$ induced by an invertible linear transformation of S_1. By lemma 2.2 we have

$$D(f) = D\big(\varphi(g)\big) = \varphi\big(\psi(D)(g)\big).$$

Since $\varphi(h) = 0$ if and only if $h = 0$, it follows that

$$f^{\perp} = \{D \in T \mid D(f) = 0\} = \{D \in T \mid \psi(D)(g) = 0\} = \{\psi^{-1}(D) \mid D \in g^{\perp}\}.$$

Hence $g^{\perp} = \psi(f^{\perp})$, which implies that π_e^g and π_e^f have isomorphic base loci and $A^g = T/g^{\perp} \cong T/f^{\perp} = A^f$.

For the rest of this chapter, we let $S = k[x_0, x_1, x_2]$ and $T = k[\partial_0, \partial_1, \partial_2]$. Note that, in the proof of theorem 2.3, we will only consider the base locus of π_2^f as a *set* of points, not as a scheme. Also, recall that if $f \sim g$, then f is a sum of s linear powers if and only if g is. Hence we are allowed to perform automorphisms φ of S induced by linear transformations of S_1 and replace f by $\varphi(f)$. By lemma 2.2, a linear transformation of S_1 corresponds to a linear transformation of T_1, and we will normally use this fact without explicitly refering to the lemma.

THEOREM 2.3:
Every ternary cubic $f \in S_3$ is a sum of four or less linear powers, except $f \sim x_0x_1^2 + x_1x_2^2$. This is a sum of exactly five linear powers.

Proof: If $\dim_k f_1^{\perp} > 0$, then there exists $\delta \in T_1$ such that $\delta(f) = 0$. This means that after a suitable linear transformation, we can assume that $f \in k[x_0, x_1]$, and theorem 1.2 tells us that f is a sum of 3 or less linear powers.

When $f_1^{\perp} = 0$, the Hilbert function of the Artinian Gorenstein ring A^f must be $(1, 3, 3, 1)$. Hence $\dim_k f_2^{\perp} = 3$, and π_2^f is a rational map $\mathbb{P}^2 \to \mathbb{P}^2$. If the linear system $f_2^{\perp}$ is base-point-free, it follows by applying Bertini's theorem twice (once to get a nonsingular quadric, next to get another that intersects the first one properly), that there exist two linearly independent quadrics $D_0, D_1 \in f_2^{\perp}$ which intersect nonsingularly, i.e. in four distinct points $a_1, \dots, a_4 \in \mathbb{P}^2$. Then $\cap_{i=1}^4 \mathsf{m}_{a_i} \subseteq (D_0, D_1)$ by theorem 0.8. Since $(D_0, D_1) \subseteq f^{\perp}$, it follows that f is a linear combination of $l_{a_1}^3, \dots, l_{a_4}^3$ by corollary 0.5.

Now assume that π_2^f has base points, and let $a = (a_0, a_1, a_2) \in \mathbb{P}^2$ be one. Then by lemma 2.1, there exists $\delta \in T_1$ such that $\delta f = l_a^2$. Furthermore, by lemma 2.2, if we let φ be the automorphism of S induced by a linear transformation of S_1 corresponding to a linear transformation $\partial_0 \mapsto \delta$ of T_1, then

$$\partial_0\big(\varphi(f)\big) = \varphi\big(\delta(f)\big) = \varphi\big(l_a^2\big) = \varphi(l_a)^2.$$

Hence we may assume that $\partial_0 f = l_a^2$. If $a_0 \neq 0$, we can integrate this equation to get $f = \frac{1}{3a_0}\, l_a^3 + g$, where $g \in k[x_1, x_2]_3$. By theorem 1.2 this g is a sum of three or less linear powers, and f is therefore a sum of four or less linear powers.

We now assume that π_2^f has two distinct base points, a and b. Then there are $\delta, \delta' \in T_1$ such that $\delta f = l_a^2$ and $\delta' f = l_b^2$. Since a and b are distinct, δ and δ' are linearly independent, and we might assume that $\delta = \partial_0$ and $\delta' = \partial_1$ after a suitable linear transformation. Hence $\partial_0 f = l_a^2$ and $\partial_1 f = l_b^2$. Then $\partial_0\partial_1 f = 2a_1 l_a = 2b_0 l_b$, and since a and b are distinct, we must have $a_1 = b_0 = 0$. If now $a_0 = b_1 = 0$, then $l_a \parallel x_2 \parallel l_b$, a contradiction. Hence either $a_0 \neq 0$ or $b_1 \neq 0$, and we can integrate the corresponding equation and find a representation of f as a sum of four or less linear powers, following the ideas of the previous paragraph.

Assume that f is not a sum of four linear powers. Then by the previous paragraphs, π_2^f has exactly one base point, and we may assume that $\partial_0 f = l_a^2$ where $a_0 = 0$. In this case we can perform a linear transformation and get $l_a = x_1$. Then we have $\partial_0 f = x_1^2$, and by integrating this equation, we get

$$f = x_0 x_1^2 + \sum_{i=0}^{3} c_i x_1^{3-i} x_2^i = (x_0 + c_0 x_1 + c_1 x_2) x_1^2 + c_2 x_1 x_2^2 + c_3 x_2^3.$$

This reduces to $f = x_0 x_1^2 + s x_1 x_2^2 + t x_2^3$ after another linear transformation. Here s must be nonzero, because if $s = 0$, then $f - t x_2^3$ is a polynomial in two variables and hence a sum of three linear powers by theorem 1.2, which contradicts our assumption. By scaling x_2, we might assume that $s = 1$.

Then $f = x_0 x_1^2 + x_1 x_2^2 + t x_2^3$ and $f_2^\perp = \langle \partial_0^2, \partial_2^2 - \partial_0\partial_1 - 3t\partial_1\partial_2, \partial_0\partial_2 \rangle$. The base points of this system are $(0,1,0)$ and $(0,1,3t)$, and since π_2^f should have only one base point, t must equal 0. Hence the only f which might not be a sum of four or less linear powers, is $f = x_0 x_1^2 + x_1 x_2^2$, up to equivalence.

When $f = x_0 x_1^2 + x_1 x_2^2$, we get $f_2^\perp = \langle \partial_0^2, \partial_2^2 - \partial_0\partial_1, \partial_0\partial_2 \rangle$. Now π_2^f maps $\{\partial_0 = 0\}$ to $(0,1,0)$, except the base point $(0,1,0)$, and it is an isomorphism everywhere else. Since every fibre of π_2^f is contained within a line, f cannot be a sum of four linear powers, by the argument at the beginning of this chapter. But it is easily seen that f is a sum of five linear powers, just look at $f + x_2^3$ which is a sum of four linear powers. □

Remark 2.3.1: If an $f \in S_3$ with $f_1^\perp = 0$ is a sum of three linear powers, say $f = \sum_{i=1}^3 \lambda_i l_i^3$, then the l_i's must be linearly independent since $f_1^\perp = 0$. Hence $f = x_0^3 + x_1^3 + x_2^3$ after a linear transformation. Thus an $f \in S_3$ such that $f_1^\perp = 0$ and $f \not\sim x_0^3 + x_1^3 + x_2^3$ and $f \not\sim x_0 x_1^2 + x_1 x_2^2$, is a sum of exactly four linear powers.

Example 2.3.2: Let us have a closer look at the exception $f = x_0 x_1^2 + x_1 x_2^2$ of theorem 2.3. Since $f_2^\perp = \langle \partial_0^2, \partial_2^2 - \partial_0\partial_1, \partial_0\partial_2 \rangle$, we see that

$$\left(\pi_2^f\right)^2 (a_0, a_1, a_2) = \pi_2^f (a_0^2, a_2^2 - a_0 a_1, a_0 a_2) = a_0^3 (a_0, a_1, a_2).$$

This means that π_2^f is an isomorphism on $\{\partial_0 \neq 0\}$. Moreover, the line $\{\partial_0 = 0\}$ maps to $(0,1,0)$, except the base point $P_0 = (0,1,0)$. Hence the fibre of π_2^f above a point $P \in \mathbb{P}^2$ is

$$\begin{array}{ccc} 3P_0 + \text{a point} \in \{\partial_0 \neq 0\} & \text{if} & P \in \{\partial_0 \neq 0\} \\ 3P_0 + \text{the line } \{\partial_0 = 0\} & \text{if} & P = P_0 \\ 4P_0 & \text{if} & P \in \{\partial_0 = 0\} \setminus P_0 \end{array}$$

In particular we notice that every fibre of π_2^f is contained within a line.

We have used the birational map π_2^f to study the linear system $f_2^\perp$, but we can also look at $f_2^\perp$ more algebraically. Let $I = (f_2^\perp)$ be the homogenous ideal generated by $f_2^\perp$. We notice that I is a primary ideal with $\operatorname{rad} I = (x, y)$, and that I is generated by the 2×2 minors of $\left(\begin{smallmatrix} \partial_0 & \partial_2 & \partial_1 \\ 0 & \partial_0 & \partial_2 \end{smallmatrix}\right)$. The reason why this f is an exception, is related to the fact that $\dim_k (T/I)_{(x,z)} = \deg I = 3$. Hence the intersection multiplicity at $(0,1,0)$ of any pair of quadrics in $f_2^\perp$ must be at least three, as we realized from the discussion of π_2^f.

Example 2.3.3: $f = x_0x_1^2 + x_1x_2^2 + tx_2^3$ where $t \neq 0$ provides us with another quite interesting example. After the linear transformation given by

$$x_0 \mapsto x_0 - \tfrac{2}{27}t^{-2}x_1 + \tfrac{1}{3}t^{-4/3}x_2, \quad x_1 \mapsto x_1, \quad x_2 \mapsto -\tfrac{1}{3}t^{-1}x_1 + t^{-1/3}x_2,$$

we may assume that $f = x_0x_1^2 + x_2^3$. In this case $f_2^\perp = \langle \partial_0\partial_2, \partial_1\partial_2, \partial_0^2 \rangle$, and

$$\left(\pi_2^f\right)^2 (a_0, a_1, a_2) = \pi_2^f (a_0a_2, a_1a_2, a_0^2) = a_0^2 a_2 (a_0, a_1, a_2).$$

Hence, π_2^f is an isomorphism on $\{\partial_0\partial_2 \neq 0\}$, and it maps $\{\partial_0 = 0\}$ to $(0,1,0)$ and $\{\partial_2 = 0\}$ to $(0,0,1)$, except the base points $(0,1,0)$ and $(0,0,1)$.

We know from the proof of theorem 2.3 that f is a sum of four or less linear powers, since π_2^f has two base points in this case. Moreover, since π_2^g has three base points when $g = x_0^3 + x_1^3 + x_2^3$, and only one base point when $g = x_0x_1^2 + x_1x_2^2$, it follows by remarks 2.3.1 and 2.2.1, that f is a sum of four linear powers and no less.

Notice that the homogenous ideal $J = (f_2^\perp)$ in this case is generated by the 2×2 minors of $\left(\begin{smallmatrix} \partial_0 & \partial_1 & 0 \\ 0 & \partial_0 & \partial_2 \end{smallmatrix}\right)$, and that $J = (x, y) \cap (x^2, z)$ is a primary decomposition of J. Hence every pair of quadrics in $f_2^\perp$ intersect singularly in $(0,1,0)$. In this case, however, we can find a pair that intersect in a line and a point outside, for instance $\partial_0\partial_2$ and $\partial_1\partial_2$. Hence we have a fibre of π_2^f that is *not* contained in a line.

Chapter 3

Ternary quartics

We will now turn our attention to our main objects of study, polynomials of degree four in three variables, $f \in S_4$, where $S = k[x_0, x_1, x_2]$. If $\dim_k f_1^\perp > 0$, we get $f \in k[x_0, x_1]_4$ after a suitable linear transformation, and these cases were treated in general in chapter 1. Therefore, in this chapter we will only consider f where $f_1^\perp = 0$. This implies that A^f has Hilbert function $(1, 3, s, 3, 1)$ where $3 \leq s \leq 6$, and the *general* f is a sum of six linear powers, by theorem 0.1. It is not true that every $f \in S_4$ is a sum of six or less linear powers, as we will see. However, seven linear powers always suffice. We will treat each possible value of $\dim_k f_2^\perp$ separately, and the methods used will vary quite a bit. Notice that the assumption $f_1^\perp = 0$ above is automatically satisfied when $\dim_k f_2^\perp < 3$.

For any $l_{a_1}, \ldots l_{a_s} \in S_1$, it follows from (0.3) and (0.7) that

$$\begin{aligned} \dim_k \bigcap_{i=1}^{s} \left(l_{a_i}^d\right)_e^\perp &= \dim_k \bigcap_{i=1}^{s} \left(l_{a_i}^e\right)_e^\perp \\ &= \dim_k T_e - \dim_k \langle l_{a_1}^e, \ldots, l_{a_s}^e \rangle \geq \binom{e+2}{2} - s, \end{aligned} \tag{3.1}$$

for all $e \leq d$ and with equality if and only if $l_{a_1}^e, \ldots, l_{a_s}^e$ are linearly independent. Because of this inequality, an $f \in S_4$ cannot be a sum of less than $\dim_k A_2^f$ linear powers, see remark 0.5.1.

3.1 Base points

Having our methods from chapter 2 fresh in memory, we start by investigating π_3^f. Since $\dim_k f_3^\perp = 7$, π_3^f is now a rational map $\mathbb{P}^2 \to \mathbb{P}^6$ defined by $P \mapsto (D_i(P))$ where $f_3^\perp = \langle D_0, \ldots, D_6 \rangle$. Proposition 3.1 tells us what happens if this map has base points. But first we look at some examples.

Example 3.0.1: Let $f = x_0^4 + x_1^4 + x_2^4$. By computing the partials of f, we see that $f^\perp = (\partial_0\partial_1, \partial_0\partial_2, \partial_1\partial_2, \partial_0^4 - \partial_1^4, \partial_0^4 - \partial_2^4)$. Hence $\dim_k f_2^\perp = 3$. Furthermore, $f_3^\perp = (f_2^\perp)_3$ since the ideal $f^\perp$ has no generators of degree three. This implies that π_2^f and π_3^f have the same base points, which are $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. f is evidently a sum of three linear powers and no less, since $f_1^\perp = 0$.

Now let g be *any* ternary quartic such that $g_1^\perp = 0$ and g is a sum of three linear powers, say $g = \sum_{i=1}^3 \lambda_i \, l_{a_i}^4$. Then the l_{a_i}'s must be linearly independent, since $g_1^\perp = 0$, and therefore $g \sim f$.

Example 3.0.2: Let $f = x_0x_1^3 + x_2^4$. Then $f_2^\perp = \langle \partial_0^2, \partial_0\partial_2, \partial_1\partial_2 \rangle$ and

$$f_3^\perp = \langle \partial_0^3, \partial_0^2\partial_1, \partial_0^2\partial_2, \partial_0\partial_1\partial_2, \partial_0\partial_2^2, \partial_1^2\partial_2, \partial_1\partial_2^2 \rangle = (f_2^\perp)_3.$$

We notice that π_2^f and π_3^f have equal base loci since $f_3^\perp = (f_2^\perp)_3$, and that $f_2^\perp$ equals $f_2^\perp$ of example 2.3.3. Hence π_3^f has two base points, $(0,1,0)$ and $(0,0,1)$, where the first one is a double point.

It is obvious that f is a sum of five linear powers, since $x_0x_1^3 \in k[x_0,x_1]_4$. In order to prove that f is not a sum of less than five linear powers, we assume that $f = \sum_{i=1}^4 \lambda_i \, l_{a_i}^4$ for some distinct points $a_i \in \mathbb{P}^2$. Then $I = \cap_{i=1}^4 (l_{a_i}^4)^\perp \subseteq f^\perp$, and in particular $I_2 \subseteq f_2^\perp = \langle \partial_0^2, \partial_0\partial_2, \partial_1\partial_2 \rangle$. If $\partial_0^2 \in I$, then $\partial_0(a_i) = 0$ for all i, and hence $\partial_0(f) = 0$. Since this contradicts the fact that $f_1^\perp = 0$, we have $\partial_0^2 \notin I$, and therefore $\dim_k I_2 \leq 2$. Now (3.1) implies that $\dim_k I_2 = 2$, and that the $l_{a_i}^2$'s are linearly independent. But this is impossible since $0 = \partial_0^2(f) = 12 \sum_{i=1}^4 \lambda_i \, \partial_0(a_i)^2 \, l_{a_i}^2$.

The following proposition solves the base point case completely. Notice that once again, we only consider the base locus of π_e^f to be a set of points.

Proposition 3.1: *Given $f \in S_4$ such that $f_1^\perp = 0$ and π_3^f has at least one base point. Then π_3^f has less than four base points, and:*

(a) *if π_3^f has three base points, then $f \sim x_0^4 + x_1^4 + x_2^4$, and f is a sum of exactly three linear powers.*

(b) *if π_3^f has exactly two base points, then $f \sim x_0x_1^3 + x_2^4$, which is a sum of exactly five linear powers.*

(c) *if π_3^f has only one base point, then f is a sum of exactly four or exactly six linear powers, except $f \sim x_0x_1^3 + x_1^2x_2^2$, which is a sum of exactly seven linear powers.*

Furthermore, $\dim_k f_2^\perp = 2$ if f is a sum of exactly four or six linear powers, and $\dim_k f_2^\perp = 3$ if f is a sum of exactly three, five or seven linear powers.

Proof: First we assume that π_3^f has at least two distinct base points, say a and b. Then by lemma 2.1, there are $\delta_0, \delta_1 \in T_1$ such that $\delta_0 f = l_a^3$ and $\delta_1 f = l_b^3$. Now, δ_0 and δ_1 are linearly independent, because a and b are distinct points in $\mathbb{P}^2$, and hence by lemma 2.2 we might perform a linear transformation such that $\delta_i = \partial_i$. Then $\partial_0 f = l_a^3$ and $\partial_1 f = l_b^3$, which implies that $\partial_0\partial_1 f = 3a_1 l_a^2 = 3b_0 l_b^2$. Since a and b are distinct, we must have $a_1 = b_0 = 0$ and either $a_0 \neq 0$ or $b_1 \neq 0$. By integrating the corresponding equation and performing a linear transformation, we might assume that $f = x_2^4 + g$ where $g \in k[x_0, x_1]_4$.

Now, $(0,0,1)$ is obviously a base point, since $\partial_2 f = 4x_2^3$. Let $p \in \mathbb{P}^2$ be a second base point, hence $\delta_q f = l_p^3$ for some $\delta_q = \sum_{i=0}^2 q_i\partial_i \in T_1$. Then $3p_2 l_p^2 = \partial_2 l_p^3 = \delta_q\partial_2 f = \delta_q(4x_2^3) = 12q_2x_2^2$, which requires that $p_2 = q_2 = 0$ since $p \neq (0,0,1)$. By performing a suitable linear transformation of $k[\partial_0, \partial_1]$, we can assume that $\delta_q = \partial_0$, hence $\partial_0 f = \partial_0 g = l_p^3$. If $p_0 = 0$, then $f \sim x_0x_1^3 + x_2^4$. By example 3.0.2 this f is a sum of five linear powers, and no less than five. Moreover, $\dim_k f_2^\perp = 3$ and π_3^f has exactly two base points. If $p_0 \neq 0$, we can integrate to get $g = \frac{1}{4p_0} l_p^4 + cx_1^4$. Since $\dim_k f_1^\perp = 0$, we must have $c \neq 0$, and then $f = x_0^4 + x_1^4 + x_2^4$ after a linear transformation. By example 3.0.1 this f is a sum of exactly three linear powers, $\dim_k f_2^\perp = 3$ and π_3^f has three base points. Thus so far we have proved (a) and (b) and that π_3^f has less than four base points.

Consider the case where π_3^f has only one base point, $a = (a_0, a_1, a_2)$. Then there exists $\delta \in T_1$ such that $\delta f = l_a^3$. Furthermore, by lemma 2.2, we may assume that $\delta = \partial_0$, i.e. $\partial_0 f = l_a^3$. If $a_0 \neq 0$, then we can integrate this equation to get $f = \frac{1}{4a_0} l_a^4 + g$ where $g \in k[x_1, x_2]_4$. Moreover, we may assume that $f = x_0^4 + g$ after a linear transformation mapping l_a to $\sqrt[4]{4a_0}\, x_0$.

Now if $g \sim x_1x_2^3$, then $f \sim x_0^4 + x_1x_2^3 \sim x_0x_1^3 + x_2^4$, but this is impossible since π_3^f has only one base point. Then by theorem 1.2 and remark 1.2.1, g is a sum of at most three linear powers. Hence f is a sum of four or less linear powers. But f cannot be a sum of three linear powers, by example 3.0.1 and the fact that π_3^f has only one base point. Hence $g \sim x_1^4 + x_2^4 + (c_1x_1 + c_2x_2)^4$, and we may assume that $f = x_0^4 + x_1^4 + x_2^4 + (c_1x_1 + c_2x_2)^4$ where both $c_i \neq 0$. Hence f is a sum of exactly four linear powers, and $f_2^\perp = \langle \partial_0\partial_1, \partial_0\partial_2 \rangle$ so $\dim_k f_2^\perp = 2$.

If $a_0 = 0$, then we can assume that $\partial_0 f = x_1^3$. Integration gives

$$f = x_0x_1^3 + g(x_1, x_2) = (x_0 + c_0x_1 + c_1x_2)x_1^3 + c_2x_1^2x_2^2 + c_3x_1x_2^3 + c_4x_2^4$$

where $g = \sum_{i=0}^4 c_i x_1^{4-i} x_2^i \in k[x_1, x_2]_4$. Since $f_1^\perp = 0$, not all of c_2, c_3, c_4 can be zero. Hence after a linear transformation, we might assume that

$$f = x_0x_1^3 + rx_1^2x_2^2 + sx_1x_2^3 + tx_2^4$$

where $(r, s, t) \in \mathbb{P}^2$. If we look at all second derivatives of f,

$$\partial_0^2 f = \partial_0\partial_2 f = 0 \quad \partial_0\partial_1 f = 3x_1^2 \quad \partial_1^2 f = 6x_0x_1 + 2rx_2^2$$
$$\partial_1\partial_2 f = 4rx_1x_2 + 3sx_2^2 \quad (\partial_2^2 - \tfrac{2}{3}r\partial_0\partial_1)f = 6sx_1x_2 + 12tx_2^2$$

we see that $\langle \partial_0^2, \partial_0\partial_2 \rangle \subseteq f_2^\perp$, and the dimension of $f_2^\perp$ depends on whether $\partial_1\partial_2 f$ and $(\partial_2^2 - \frac{2}{3}r\partial_0\partial_1)f$ are proportional or not. In terms of the quantity

$\Delta = 8rt - 3s^2$,

$$\dim_k f_2^\perp = \begin{cases} 2, & \Delta \neq 0 \\ 3, & \Delta = 0. \end{cases}$$

First we consider the case $\Delta = 0$. Then $t = 0$, because if $t \neq 0$, then π_3^f has two base points. Indeed, $f - \frac{1}{2^8 t^3}(sx_1 + 4tx_2)^4 = (x_0 - \frac{s^4}{2^8 t^3}x_1 - \frac{s^3}{2^4 t^2}x_2)x_1^3$, hence $f \sim x_0 x_1^3 + x_2^4$ and we conclude by example 3.0.2. With $t = 0$, we have $f \sim x_0 x_1^3 + x_1^2 x_2^2$. Now example 3.1.2 tells us that π_3^f has exactly one base point and that f is the sum of seven linear powers, and no less.

Finally we suppose $\Delta \neq 0$. Assume that f is a sum of five linear powers, i.e. $f = \sum_{i=1}^5 \lambda_i\, l_{a_i}^4$. This is equivalent to $I = \cap_{i=1}^5 (l_{a_i}^4)^\perp \subseteq f^\perp$, and hence $I_2 \subseteq f_2^\perp = \langle \partial_0^2, \partial_0\partial_2 \rangle$. If $\partial_0^2 \in I$, then $\partial_0(a_i) = 0$ for all i, hence $\partial_0 \in I$. This implies that $\partial_0 \in f^\perp$, which is a contradiction. Hence $\partial_0^2 \notin I$ and $\dim_k I_2 \leq 1$. By (3.1), $\dim_k I_2 = 1$ and $l_{a_1}^2, \ldots, l_{a_5}^2$ are linearly independent. But since $\partial_0^2 \in f^\perp$, we have

$$0 = \partial_0^2 f = 12 \sum_{i=1}^5 \lambda_i\, \partial_0(a_i)^2\, l_{a_i}^2,$$

and this is now a contradiction. Hence f is not a sum of five linear powers. On the other hand, to prove that f is a sum of six linear powers, we consider

$$g = \alpha f + (x_2 + \beta x_1)^4 = (\alpha x_0 + \beta^4 x_1 + 4\beta^3 x_2)x_1^3 + r' x_1^2 x_2^2 + s' x_1 x_2^3 + t' x_2^4$$

where $r' = \alpha r + 6\beta^2$, $s' = \alpha s + 4\beta$ and $t' = \alpha t + 1$. If we can choose α and β such that $\Delta' = 8r't' - 3(s')^2 = 0$ and $t' \neq 0$, it follows that π_3^g has two base points, and g is a sum of five linear powers, as solved above. Furthermore, if $\alpha \neq 0$, then $f = \frac{1}{\alpha}(g - (x_2 + \beta x_1)^4)$ is a sum of six linear powers.

To prove that such choices for α and β are possible, just pick β such that $r - 3s\beta + 6t\beta^2 \neq 0$ and $s - t\beta \neq 0$. This is possible since $\Delta \neq 0$ implies that not both s and t are zero. Now let $\alpha = \frac{8}{\Delta}(-r + 3s\beta - 6t\beta^2)$. Then $\alpha \neq 0$, $\Delta' = \alpha(\alpha\Delta + 8(r - 3s\beta + 6t\beta^2)) = 0$ and $t' = \frac{3}{\Delta}(s - t\beta)^2 \neq 0$. □

REMARK 3.1.1: In the proof of proposition 3.1 we showed that π_3^f has less than four base points indirectly, we proved that if it has more than one base point, then it has either two or three. This fact is true in a more general setting, and it is possible to prove it directly.

Let $f \in S_d$ and suppose that π_{d-1}^f has four distinct base points, $a_1, \ldots, a_4$. Then there exist $\delta_i \in T_1$ such that $\delta_i f = l_{a_i}^{d-1}$. But the δ_i's must be linearly dependent, say $\sum_{i=1}^4 c_i \delta_i = 0$ for some $c_i \in k$. Then $\sum_{i=1}^4 c_i\, l_{a_i}^{d-1} = 0$ also, but four distinct $(d-1)^{\text{th}}$-powers are linearly independent for $d \geq 4$.

EXAMPLE 3.1.2: Let $f = \frac{1}{3}x_0 x_1^3 + \frac{1}{2}x_1^2 x_2^2$. Note that $\partial_1 f = x_0 x_1^2 + x_1 x_2^2$ is the polynomial studied in example 2.3.2. Furthermore, $f_2^\perp = \langle \partial_0^2, \partial_2^2 - \partial_0\partial_1, \partial_0\partial_2 \rangle$

and

$$f_3^\perp = \langle \partial_0^3, \partial_0^2\partial_1, \partial_0^2\partial_2, \partial_0\partial_1\partial_2, \partial_0\partial_2^2, \partial_2^3, \partial_1(\partial_2^2 - \partial_0\partial_1)\rangle = (f_2^\perp)_3.$$

In this case we see that π_3^f has only one base point, namely $(0,1,0)$. We notice that $f_2^\perp$ equals $f_2^\perp$ of example 2.3.2, and that π_2^f and π_3^f have the same base locus. The scheme corresponding to $T/(f_3^\perp)$ is a triple point, and it is this fact that makes f so exceptional.

Assume that $f = \sum_{i=1}^6 \lambda_i\, l_{a_i}^4$, where $a_1,\ldots,a_6$ are six distinct points in $\mathbb{P}^2$. Let $I = \cap_{i=1}^6 (l_{a_i}^4)^\perp \subseteq f^\perp$, then in particular, $I_2 \subseteq f_2^\perp$. Since $\partial_0^2 \notin I$ and $0 = \partial_0^2(f) = 12\sum_{i=1}^6 \lambda_i\, \partial_0(a_i)^2\, l_{a_i}^2$, we must have $\dim_k I_2 \in \{1,2\}$. If $\dim_k I_2 = 2$, then there are two quadrics in $f_2^\perp$ passing through all six points. Then these points should be contained in a fibre of π_2^f, which is impossible (see example 2.3.2). Hence $\dim_k I_2 = 1$, i.e. $I_2 = \langle D_0\rangle$ for some $D_0 \in f_2^\perp$.

Since $\dim_k I_3 \geq 4$ by (3.1), there exists $D_1 \in I_3 \setminus (D_0)_3$. Now $(D_0, D_1) \subseteq I \subseteq f^\perp$, and $\{a_1,\ldots,a_6\} \subseteq X$ where $X = V(D_0)\cap V(D_1)$. We notice that any D in $f_2^\perp$ or $f_3^\perp$ intersects $\{\partial_0 = 0\}$ singularly in $(0,1,0)$. Hence D_0 and D_1 also intersect singularly in $(0,1,0)$. If they have no common factor, then X consists of less than six distinct points, which is a contradiction.

When D_0 and D_1 have a common factor, we want to prove that X is contained within a line and a point. Then at least five of the points a_i must be contained in the line, and hence $\dim_k I_2 \geq 2$, which is a contradiction.

Assume that D_0 and D_1 have a common factor, then D_0 must be reducible since $D_1 \notin (D_0)$, and therefore $D_0 \in \langle \partial_0^2, \partial_0\partial_2\rangle$. Since we know that $D_0 \neq \partial_0^2$, it follows that $D_0 = \partial_0(\partial_2 + c\partial_0)$ for some $c \in k$. Then D_1 must have ∂_0 or $\partial_2 + c\partial_0$ as a factor, and therefore $D_1 \in (\partial_0^2, \partial_0\partial_2)_3 + \langle \partial_2^3\rangle$. Since $(\partial_0^2, \partial_0\partial_2) = (\partial_0^2, D_0)$, we might assume that $D_0 \in (\partial_0^2)_3 + \langle \partial_2^3\rangle$ without changing X. Then $D_1 = \delta_a\partial_0^2 + b\partial_2^3$ for suitable $\delta_a = \sum_{i=0}^2 a_i\partial_i \in T_1$ and $b \in k$. We see that D_0 and D_1 have ∂_0 as a common factor if and only if $b = 0$, but then $X \subseteq \{\partial_0 = 0\} \cup \{\text{a point}\}$. Hence $b \neq 0$, and we may assume that $b = 1$. Then D_0 and D_1 must have $\partial_2 + c\partial_0$ as a common factor, which means that the point $(1, c', -c) \in \{D_1 = 0\}$ for all $c' \in k$. This implies that $a_1 = 0$. Then $D_1 \in k[\partial_0, \partial_2]$, which means that $\{D_1 = 0\}$ consists of three lines through $(0,1,0)$. Then $X = \{\partial_2 + c\partial_0\}$, and we have the contradiction we sought.

Since $f + x_2^4$ obviously is a sum of six linear powers, the conclusion is that f is a sum of seven linear powers, and no less.

Remark 3.1.3: Let us have another look at $f = x_0x_1^3 + rx_1^2x_2^2 + sx_1x_2^3 + tx_2^4$ where $(r,s,t) \in \mathbb{P}^2$ and $\Delta = 8rt - 3s^2$. When $\Delta \neq 0$, it is possible to perform linear transformations such that $(r,s,t) = (0,1,0)$ or $(1,0,1)$. Indeed, if $t \neq 0$, then we can assume $t = 1$, and by performing $x_2 \mapsto x_2 - \frac{s}{4}x_1$, we might assume $s = 0$. Now r must be nonzero, since f is not a sum of less than six linear powers. Hence by scaling x_1 and x_0, we may assume $r = 1$. If

$t = 0$ then $\Delta \neq 0$ implies $s \neq 0$. By scaling x_2 we might assume $s = 1$, and then the linear transformation $x_2 \mapsto x_2 - \frac{r}{3}x_1$ permits us to assume $r = 0$. Hence the following is true:

$$\begin{array}{llll} \Delta = 0 \wedge t \neq 0 & \Rightarrow & f \sim x_0x_1^3 + x_2^4 & \text{a sum of 5 linear powers} \\ \Delta \neq 0 \wedge t = 0 & \Rightarrow & f \sim x_0x_1^3 + x_1x_2^3 & \text{a sum of 6 linear powers} \\ \Delta \neq 0 \wedge t \neq 0 & \Rightarrow & f \sim x_0x_1^3 + x_1^2x_2^2 + x_2^4 & \text{a sum of 6 linear powers} \\ \Delta = 0 \wedge t = 0 & \Rightarrow & f \sim x_0x_1^3 + x_1^2x_2^2 & \text{a sum of 7 linear powers} \end{array}$$

3.2 $\dim_k f_2^{\perp} \geq 2$

First we will look at $f \in S_4$ such that $A^f = T/f^{\perp}$ has Hilbert function $(1, 3, 4, 3, 1)$. This means that $f_2^{\perp} = \langle D_0, D_1\rangle$ for some linearly independent $D_i \in T_2$. We know that $f = \sum_{i=1}^{s} \lambda_i \, l_{a_i}^4 \Leftrightarrow \cap_{i=1}^{s}(l_{a_i}^4)^{\perp} \subseteq f^{\perp}$ and the right inclusion implies that $\cap_{i=1}^{s}(l_{a_i}^4)_2^{\perp} = \cap_{i=1}^{s}(l_{a_i}^2)_2^{\perp} \subseteq f_2^{\perp} = \langle D_0, D_1\rangle$. Thus if f is a sum of four linear powers, then $\cap_{i=1}^{4}(l_{a_i}^2)_2^{\perp} = \langle D_0, D_1\rangle$ by (3.1). Hence $D_j(a_i) = 0$ for all i and j. If we let $X_f = \{D_0 = 0\} \cap \{D_1 = 0\}$, this means that $a_i \in X_f$ for all i. The following result tells us that f is a sum of either four or six linear powers, depending on how X_f looks like.

Theorem 3.2:
Given $f \in S_4$ such that $f_2^{\perp} = \langle D_0, D_1\rangle$, let $X_f = V(D_0) \cap V(D_1)$.

(a) *If X_f consists of four distinct points, then f is a sum of the corresponding four linear powers.*

(b) *If X_f is supported at less that four points, then f is a sum of six linear powers, and no less.*

(c) *If X_f is supported at a line and a point outside the line, then f is a sum of four linear powers.*

(d) *If X_f has support on a line only, then f is a sum of exactly six linear powers.*

Proof: We recall that by remark 0.5.1, f cannot be a sum of less than four linear powers, since $\dim_k A_2^f = 4$. To prove (a), let $a_1, \ldots, a_4$ be the four points of X_f, and let $I = \cap_{i=1}^{4} \mathfrak{m}_{a_i}$. Then $I = (D_0, D_1)$ by theorem 0.8. Hence $I \subseteq f^{\perp}$, and by corollary 0.5, $f = \sum_{i=1}^{4} \lambda_i \, l_{a_i}^4$ for suitable $\lambda_i \in k$.

To prove (c) and (d), we recall that by lemma 2.1 a point a is a base point of π_3^f if and only if $\delta_0 f = l_a^3$ for some $\delta_0 \in T_1$. Moreover, this is equivalent to $\delta_0\delta_1 f = \delta_0\delta_2 f = 0$ for some $\delta_1, \delta_2 \in T_1$ such that $\{\delta_1 = 0\} \cap \{\delta_2 = 0\} = \{a\}$, because $f_1^{\perp} = 0$. Hence, since $\dim_k f_2^{\perp} = 2$, we see that a point $a \in \mathbb{P}^2$ is a base point of π_3^f if and only if $f_2^{\perp} = \langle \delta_0\delta_1, \delta_0\delta_2\rangle$ for some $\delta_i \in T_1$ such that $\{\delta_1 = 0\} \cap \{\delta_2 = 0\} = \{a\}$. Note that π_3^f cannot have more than one base point by proposition 3.1 since $\dim_k f_2^\perp = 2$. We realize that both (c) and (d) correspond to base point cases of proposition 3.1. In (c) the base point a lies outside the line $\{\delta_0 = 0\}$, hence $\partial_0 f = x_0^3$ after a linear transformation, and from the proof of proposition 3.1 we see that f is a sum of exactly four linear powers. In (d) the base point a lies on the line $\{\delta_0 = 0\}$, hence $\partial_0 f = x_1^3$ after a linear transformation, and from the proof of proposition 3.1 we see that f is a sum of exactly six linear powers.

In (b), X_f has support on less than four points. Then it follows from the argument given prior to this theorem, that f cannot be a sum of four linear powers. Assume that f is a sum of five linear powers, say $f = \sum_{i=1}^5 \lambda_i \, l_{a_i}^4$, $\lambda_i \neq 0$, and let $I = \cap_{i=1}^5 (l_{a_i}^4)^\perp$. By lemma 0.4, $I \subseteq f^\perp$, and hence $\dim_k I_2 \leq 2$. If $\dim_k I_2 = 2$, then $I_2 = f_2^\perp = \langle D_0, D_1 \rangle$, and thus $D_j(a_i) = 0$ for all i and j. This means that $a_i \in X_f$ for all i, which is impossible since X_f is supported at less than four points. Hence $\dim_k I_2 \leq 1$, and by (3.1), $\dim_k I_2 = 1$ and the l_{a_i}'s are linearly independent. Moreover, $I_2 = \langle D_0' \rangle$ for some $D_0' \in f_2^\perp$. Pick $D_1' \in f_2^\perp$ such that $f_2^\perp = \langle D_0', D_1' \rangle$. Then

$$0 = D_1'(f) = \sum_{i=1}^5 \lambda_i \, D_1'\big(l_{a_i}^4\big) = 12 \sum_{i=1}^5 \lambda_i \, D_1'(a_i) \, l_{a_i}^2$$

which is a contradiction since $D_1' \notin I$ implies that $D_1'(a_i)$ is not 0 for all i.

In order to prove that f is a sum of six linear powers, we consider π_3^f. Since D_0 and D_1 are relative prime, we realize that π_3^f is base-point-free, compare with the paragraph concerning (c) and (d). Therefore, by Bertini's theorem, the general member of the linear system $f_3^\perp$ is nonsingular. Since D_0 and D_1 have no common factor, it follows that $\dim_k (D_0, D_1)_3 = 6$. Thus $f_3^\perp = (D_0, D_1)_3 + \langle D_2 \rangle$ for some $D_2 \in T_3$. Since $\{c = 0\}$ is a proper closed subset of $f_3^\perp = \{cD_2 + D_3 \,|\, D_3 \in (D_0, D_1)_3\}$, we can find $D \in f_3^\perp \setminus (D_0, D_1)_3$ that is nonsingular. Now define $\varphi : \{D = 0\} \to \mathbb{P}^1$ by $\varphi(P) = (D_0(P), D_1(P))$. This map is base-point-free, since $f_3^\perp = (D_0, D_1)_3 + \langle D \rangle$ and any base point of φ would be a base point of π_3^f as well. Now Bertini's theorem implies that the general member of $\langle D_0, D_1 \rangle$ is nonsingular when considered as a subscheme of $\{D = 0\}$. Let D' be one such member. Hence we have $D \in f_3^\perp$ and $D' \in f_2^\perp$ that intersect nonsingularly, i.e. in six distinct points, say $a_1, \dots, a_6$. Let $\Gamma = \{a_1, \dots, a_6\}$. Then $I_\Gamma = (D, D')$ by theorem 0.8, and by corollary 0.5, there exist λ_i such that $f = \sum_{i=1}^6 \lambda_i \, l_{a_i}^4$. □

Example 3.2.1: Let $f = x_0 x_1^3 + 3x_0^2 x_1 x_2$. We start by observing that since $f_2^\perp = \langle \partial_1^2 - \partial_0\partial_2, \partial_2^2 \rangle$, we have $X_f = \{(1,0,0)\}$. Then by theorem 3.2, f should be a sum of exactly six linear powers. To find such an representation explicitly, we look at $f_3^\perp$. We see that

$$f_3^\perp = \langle \partial_0^3, \partial_1^2\partial_2, \partial_0\partial_2^2, \partial_1\partial_2^2, \partial_2^3, \partial_0(\partial_1^2 - \partial_0\partial_2), \partial_1(\partial_1^2 - \partial_0\partial_2) \rangle,$$

and hence $f_3^\perp = \langle \partial_0^3 \rangle + (f_2^\perp)_3$. In the proof of theorem we used a $D \in f_2^\perp$ and a $D' \in f_3^\perp$ which intersected nonsingularly. We can achieve this by choosing $D = \partial_1^2 - \partial_0\partial_2$ and $D' = \partial_0^3 - \partial_2^3$. In fact, the intersection is now

$$\{D = 0\} \cap \{D' = 0\} = \{(1, \pm 1, 1), (1, \pm\epsilon, \epsilon^2), (1, \pm\epsilon^2, \epsilon)\}$$

where $\epsilon = e^{2\pi i/3}$ is a third-root of 1. Now f should be a sum of six linear powers corresponding to these six points. Indeed, we see that

$$f = \tfrac{1}{24}\big((x_0 + x_1 + x_2)^4 - (x_0 - x_1 + x_2)^4 + (x_0 + \epsilon x_1 + \epsilon^2 x_2)^4 \\ - (x_0 - \epsilon x_1 + \epsilon^2 x_2)^4 + (x_0 + \epsilon^2 x_1 + \epsilon x_2)^4 - (x_0 - \epsilon^2 x_1 + \epsilon x_2)^4\big).$$

Remark 3.2.2: We note that if f is a sum of four "general enough" linear powers, indeed any $f \sim x_0^4 + x_1^4 + x_2^4 + l_a^4$ where $l_a \neq x_i$, then $\dim_k f_2^\perp = 2$, compare with remark 0.5.1. If $a_i \neq 0$ for all i, this f belongs to category (a) of theorem 3.2, while it belongs to (c) if $a_i = 0$ for exactly one i. From the proof of proposition 3.1, we know that category (d) is nonempty, and together with example 3.2.1 this proves that all four categories of theorem 3.2 are nonempty.

Now we turn to $f \in S_4$ such that $\dim_k f_2^\perp = 3$. This case is rather simple, as the following theorem tells us.

Theorem 3.3:
Given $f \in S_4$ such that A^f has Hilbert function $(1, 3, 3, 3, 1)$. Then either

(a) $f \sim x_0^4 + x_1^4 + x_2^4$, *a sum of three linear powers, or*

(b) $f \sim x_0 x_1^3 + x_2^4$, *a sum of five linear powers, or*

(c) $f \sim x_0 x_1^3 + x_1^2 x_2^2$, *a sum of seven linear powers.*

Proof: Note that $\dim_k f_2^\perp = 3$, and assume that π_2^f is base-point-free. By Bertini's theorem there are two quadrics $D_0, D_1 \in f_2^\perp$ which intersect nonsingularly, i.e. in four distinct points. Let $\Gamma = V(D_0) \cap V(D_1) = \{a_1, \ldots, a_4\}$. By theorem 0.8, $I_\Gamma = (D_0, D_1)$. Since $D_i \in f^\perp$, we have $I_\Gamma \subseteq f^\perp$, and by corollary 0.5, $f = \sum_{i=1}^4 \lambda_i l_{a_i}^4$ for suitable $\lambda_i \in k$. Since $f_1^\perp = 0$, then $\{l_{a_i}\}$ must be a basis for S_1, and either $f \sim x_0^4 + x_1^4 + x_2^4$ or $f \sim x_0^4 + x_1^4 + x_2^4 + l_a^4$. The first case is impossible since we assumed that π_2^f had no base points. Hence after a linear transformation, we get $f = x_0^4 + x_1^4 + x_2^4 + l_a^4$ where $a \neq (1,0,0), (0,1,0), (0,0,1)$. But then $\dim_k A_2^f = 4$, which is a contradiction.

Hence π_2^f has at least one base point, and so has π_3^f, since $f_3^\perp = (f_2^\perp)_3$. Note that the ideal $f^\perp$ has three generators of degree two and two of degree four, see for instance [16] or [4]. Now the conclusion follows easily from proposition 3.1, since $\dim_k f_2^\perp = 3$. $\square$

3.3 $\dim_k f_2^\perp = 1$

We now turn our attention to the case where $A^f = T/f^\perp$ has Hilbert function $(1,3,5,3,1)$. We know from (3.1) that such an f cannot be a sum of less than five linear powers. Furthermore, $f_2^\perp = \langle D_0 \rangle$ for some $D_0 \in T_2$. Here D_0 might be nonsingular or not, and our discussion will deal with each case quite differently. In both cases however, if f is a sum of six or less linear powers, then the six points must lie on $\{D_0 = 0\}$, i.e.

$$f = \sum_{i=1}^{s} \lambda_i \, l_{a_i}^4 \text{ where } s \le 6 \quad \Rightarrow \quad \{a_1, \dots, a_s\} \subseteq \{D_0 = 0\}. \tag{3.2}$$

To prove this, look at $I = \cap_{i=1}^s (l_{a_i}^4)^\perp \subseteq f^\perp$. We know that $\dim_k I_2 \le \dim_k f_2^\perp = 1$ and $\dim_k I_2 = \dim_k T_2 - \dim_k \langle l_{a_1}^2, \dots, l_{a_s}^2 \rangle \ge 6 - s$. For $s = 6$, assume that $\dim_k I_2 = 0$. Then $\dim_k \langle l_{a_1}^2, \dots, l_{a_6}^2 \rangle = 6$, hence the $l_{a_i}^2$'s are linearly independent by (3.1). But $0 = D_0(f) = 12 \sum_{i=1}^s \lambda_i \, D_0(a_i) \, l_{a_i}^2$, which is a contradiction. Hence $\dim_k I_2 = 1$ in all cases, which means that $I_2 = \langle D_0 \rangle$ and $D_0(a_i) = 0$ for all i.

3.3.1 D_0 nonsingular

In this case D_0 is a nonsingular quadric, and the discussion of this case is inspired by the one found in [16].

Theorem 3.4:
Any f such that $f_2^\perp = \langle D_0 \rangle$ with D_0 nonsingular, is a sum of exactly five linear powers.

Proof: Since A^f is Gorenstein of codimension 3, the structure theorem of Buchsbaum-Eisenbud [2] applies. Hence A^f has the following minimal free resolution

$$0 \to T(-7) \to F_2 \to F_1 \to T \to A^f \to 0$$

where $F_1 = T(-3) \oplus T(-2)^{\oplus 4}$, $F_2^* \cong F_1(7)$, ϕ is a skew-symmetric 5×5 matrix and $f^\perp$ is generated by the 4×4 pfaffians of ϕ. Thus

$$\phi = \begin{bmatrix} 0 & q_1 & q_2 & q_3 & q_4 \\ -q_1 & 0 & \delta_1 & \delta_2 & \delta_3 \\ -q_2 & -\delta_1 & 0 & \delta_4 & \delta_5 \\ -q_3 & -\delta_2 & -\delta_4 & 0 & \delta_6 \\ -q_4 & -\delta_3 & -\delta_5 & -\delta_6 & 0 \end{bmatrix} \tag{3.3}$$

where $q_i \in T_2$ and $\delta_i \in T_1$. Now we want to perform "symmetrical" row- and column-operations on ϕ to obtains a skew-symmetrical matrix with more zeroes. Such operations do not change the ideal generated by the pfaffians.

If the forms δ_3, δ_5 and δ_6 are linearly dependent, we get a matrix with zeroes at the positions where $\pm\delta_6$ are now by performing the suitable row- and column-operations. In particular, if $\delta_3 = 0$, then just interchange the second and the fourth row. If $\delta_5 = a\delta_3$ for some $a \in k$, then subtract a times the second row from the third row, and then interchange row number three and four. If $\delta_6 = a\delta_3 + b\delta_5$ for some scalars $a, b \in k$, then replace the fourth row with itself minus a times the second row minus b times the third row.

Next we assume that δ_3, δ_5 and δ_6 are linearly independent. By interchanging the last two columns, we may assume that $\{\delta_2, \delta_4, \delta_6\}$ is a basis for T_1. Hence $\delta_3 = a_2\delta_2 + a_4\delta_4 + a_6\delta_6$ for some $a_i \in k$, and by subtracting a_6 times the fourth row from the second, we get $\delta_3 \in \langle \delta_2, \delta_4 \rangle$. Similarly we may assume that $\delta_5 \in \langle \delta_2, \delta_4 \rangle$. Hence $\delta_i = a_{i2}\delta_2 + a_{i4}\delta_4$ for $i = 3, 5$, where $a_{ij} \in k$. Then by subtracting μ times the fourth row from the fifth, where μ is a solution of $\left| \begin{smallmatrix} a_{32}-\mu & a_{52} \\ a_{34} & a_{54}-\mu \end{smallmatrix} \right| = 0$, it follows that $\dim_k \langle \delta_3, \delta_5 \rangle = 1$. Thus the δ's in the last column are linearly dependent, and hence we can, without loss of generality, assume that $\delta_6 = 0$.

Now the 4×4 pfaffian that gives D_0, the quadric generator of $f^\perp$, equals $\delta_3\delta_4 - \delta_2\delta_5$. This is irreducible by assumption, hence in particular, δ_2 and δ_3 must be nonproportional. Then either $\delta_4 \in \langle \delta_2, \delta_3 \rangle$, or $\{\delta_2, \delta_3, \delta_4\}$ is a basis for T_1. In the first case, by subtracting a suitable multiple of the second row from the third and then scale the third row, we can assume that $\delta_4 = \delta_3$. In the second case, by expressing δ_5 as a linear combination of δ_2, δ_3 and δ_4, and subtracting its components along δ_3 and δ_4, we might assume that $\delta_5 = \delta_2$. Then interchange the last two rows and columns to get $\delta_4 = \delta_3$.

If $\delta_5 \in \langle \delta_2, \delta_3 \rangle$, then $D_0 = \delta_3^2 - \delta_2\delta_5$ is a polynomial in two variables, which contradicts the fact that D_0 is irreducible. Therefore, $\{\delta_2, \delta_3, \delta_5\}$ is a basis for T_1, and after a linear transformation, we get $\delta_2 = \partial_0$, $\delta_3 = \partial_1$ and $\delta_5 = \partial_2$. Furthermore, we get $\delta_1 = 0$ by subtracting suitable multiples of the fourth and fifth column from the third, and a suitable multiple of the fifth row from the second. All in all, we have proved that after several row- and column-operations, we can assume that

$$\phi = \begin{bmatrix} 0 & q_1 & q_2 & q_3 & q_4 \\ -q_1 & 0 & 0 & \partial_0 & \partial_1 \\ -q_2 & 0 & 0 & \partial_1 & \partial_2 \\ -q_3 & -\partial_0 & -\partial_1 & 0 & 0 \\ -q_4 & -\partial_1 & -\partial_2 & 0 & 0 \end{bmatrix} \tag{3.4}$$

where $q_i \in T_2$. The pfaffians of this matrix are

$$\partial_1^2 - \partial_0\partial_2 \quad q_2\partial_0 - q_1\partial_1 \quad q_2\partial_1 - q_1\partial_2 \quad q_4\partial_0 - q_3\partial_1 \quad q_4\partial_1 - q_3\partial_2$$

and we see that $D_0 = \partial_1^2 - \partial_0\partial_2$ is irreducible, as it should be.

Since we below will need to make a linear combination of the q_i's, we notice that by adding c times the second column to the fourth, and c times the third to the fifth, and performing the "symmetrical" row-operations, we get a matrix that is equal to the one we have, except for q_3 and q_4. The new entries are $q_3' = q_3 + cq_1$ and $q_4' = q_4 + cq_2$. Hence the two new pfaffians are $q_4'\partial_i - q_3'\partial_{i+1} = (q_4\partial_i - q_3\partial_{i+1}) + c(q_4\partial_i - q_3\partial_{i+1})$ for $i = 0, 1$.

Now, $P \in \{\partial_1^2 - \partial_0\partial_2 = 0\} \subseteq \mathbb{P}^2$ if and only if $P = (s^2, st, t^2)$ for some $(s,t) \in \mathbb{P}^1$. Hence the three pfaffians D_0, $q_2\partial_0 - q_1\partial_1$ and $q_2\partial_1 - q_1\partial_2$ are all zero if and only if $\bar{q}_2 s - \bar{q}_1 t = 0$, where $\bar{q}_i = q_i(s^2, st, t^2) \in k[s,t]_4$. Similarly, D_0 and the two last pfaffians are zero if and only if $\bar{q}_4 s - \bar{q}_3 t = 0$.

If every linear combination of $\bar{q}_2 s - \bar{q}_1 t$ and $\bar{q}_4 s - \bar{q}_3 t$ have multiple zeroes, then by lemma 1.1, they must have a common (double) zero $(s,t) \in \mathbb{P}^1$. But then $(s^2, st, t^2) \in \mathbb{P}^2$ would be a common zero for all five generator of $f^\perp$, which is impossible. Hence there is a linear combination of $\bar{q}_2 s - \bar{q}_1 t$ and $\bar{q}_4 s - \bar{q}_3 t$ that has five distinct roots. As we noticed above, we may assume that this is $\bar{q}_4 s - \bar{q}_3 t$. Now let ψ be the upper right 3×2 submatrix of ϕ, i.e.

$$\psi = \begin{bmatrix} q_3 & q_4 \\ \partial_0 & \partial_1 \\ \partial_1 & \partial_2 \end{bmatrix}.$$

Then the three pfaffians D_0, $D_1 = q_4\partial_0 - q_3\partial_1$ and $D_2 = q_4\partial_1 - q_3\partial_2$ of ϕ equals the 2×2 minors of ψ, and they intersect in five distinct points, say $\Gamma = \{a_1, \dots, a_5\}$. By Hilbert-Burch [6, theorem 20.15] the homogenous ideal I_Γ of these points is generated by the 2×2 minors of ψ. Hence $I_\Gamma = (D_0, D_1, D_2) \subseteq f^\perp$, and we get $f = \sum_{i=1}^5 \lambda_i\, l_{a_i}^4$ for suitable $\lambda_i \in k$ by corollary 0.5. □

3.3.2 D_0 singular

We now turn our attention to the case where D_0 is singular, and since D_0 is a quadric, this means that D_0 is reducible. If an f with $f_2^\perp = \langle D_0 \rangle$ is a sum of five linear powers, then one way to find such a representation, is to subtract a suitable linear power from f. Hopefully, we get a new polynomial g with $\dim_k g_2^\perp = 2$, and by theorem 3.2 we know when such a polynomial is a sum of four linear powers. This is essensially what we are going to do, but first we need to do some preparations, starting with the following lemma, in which $\operatorname{tr} M$ is the trace of M and id_n is the $n \times n$ identity matrix.

Lemma 3.5: *Let $M_n = (m_{ij})$ be an $n \times n$ matrix of rank ≤ 1. Then*

$$\det\bigl(\mathrm{id}_n - \lambda\, M_n\bigr) = 1 - \lambda \cdot \operatorname{tr} M_n. \tag{3.5}$$

Proof: If $n = 1$, the statement is obviously true. Assume that equation (3.5) is true for some $n \geq 1$, and let $\vec{m}_i$ and $\vec{e}_i$ be the i^{th} column of M_{n+1} and

id_{n+1}, respectively. Then

$$\begin{aligned}
&\left|\vec{e}_1 - \lambda\vec{m}_1 \quad \vec{e}_2 - \lambda\vec{m}_2 \quad \cdots \quad \vec{e}_n - \lambda\vec{m}_n \quad \vec{m}_{n+1}\right| \\
&\qquad = \left|\vec{e}_1 \quad \vec{e}_2 - \lambda\vec{m}_2 \quad \cdots \quad \vec{e}_n - \lambda\vec{m}_n \quad \vec{m}_{n+1}\right| \\
&\qquad\quad - \lambda\left|\vec{m}_1 \quad \vec{e}_2 - \lambda\vec{m}_2 \quad \cdots \quad \vec{e}_n - \lambda\vec{m}_n \quad \vec{m}_{n+1}\right| \\
&\qquad = \left|\vec{e}_1 \quad \vec{e}_2 - \lambda\vec{m}_2 \quad \cdots \quad \vec{e}_n - \lambda\vec{m}_n \quad \vec{m}_{n+1}\right| - \lambda\cdot 0 \\
&\qquad = \ldots = \left|\vec{e}_1 \quad \vec{e}_2 \quad \cdots \quad \vec{e}_n \quad \vec{m}_{n+1}\right| = m_{n+1,n+1}
\end{aligned}$$

and therefore

$$\begin{aligned}
\det\bigl(\mathrm{id}_{n+1} - \lambda\, M_{n+1}\bigr) &= \left|\vec{e}_1 - \lambda\vec{m}_1 \quad \cdots \quad \vec{e}_n - \lambda\vec{m}_n \quad \vec{e}_{n+1} - \lambda\vec{m}_{n+1}\right| \\
&= \left|\vec{e}_1 - \lambda\vec{m}_1 \quad \cdots \quad \vec{e}_n - \lambda\vec{m}_n \quad \vec{e}_{n+1}\right| \\
&\quad - \lambda\left|\vec{e}_1 - \lambda\vec{m}_1 \quad \cdots \quad \vec{e}_n - \lambda\vec{m}_n \quad \vec{m}_{n+1}\right| \\
&= \det\bigl(\mathrm{id}_n - \lambda\, M_n\bigr) - \lambda\cdot m_{n+1,n+1} \\
&= 1 - \lambda\cdot \operatorname{tr} M_{n+1}
\end{aligned}$$

and the lemma follows by induction on n. □

Assume we have an $f \in S_4$ with $f_2^\perp = \langle D_0\rangle$ for some $D_0 \in T_2$. Our goal is now to find a method that determines when it is possible to subtract a linear power from f and get a new form g which is a sum of four linear powers. First choose $D_1, \ldots, D_5 \in T_2$ such that $\mathcal{D} = \{D_0, \ldots, D_5\}$ is a basis for T_2. Observe that with this basis we have

$$\mathrm{Cat}_{\mathcal{D}}\bigl(f\bigr) = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix}$$

where A is a symmetrical 5×5 matrix. In fact, in terms of the vector $v^T = [D_1, \ldots, D_5]$, we see that $A = (vv^T)(f)$. Futhermore, A is invertible since $\operatorname{rank}\mathrm{Cat}_{\mathcal{D}}(f) = \dim_k A_2^f = 5$ by (0.9). For any $a \in \{D_0 = 0\}$ we have

$$\mathrm{Cat}_{\mathcal{D}}\bigl(l_a^4\bigr) = \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \quad \text{and} \quad \mathrm{Cat}_{\mathcal{D}}\bigl(g\bigr) = \begin{bmatrix} 0 & 0 \\ 0 & A - \lambda C \end{bmatrix}$$

where $C = (vv^T)(l_a^4)$ and $g = f - \lambda\, l_a^4$. Since $D_iD_j(l_a^4) = 24\, D_i(a)D_j(a)$, it follows that $C = 24\, v(a)v(a)^T$. Hence $\operatorname{rank} C = 1$, and

$$\operatorname{rank}\mathrm{Cat}_{\mathcal{D}}(g) = \operatorname{rank}(A - \lambda C) \geq \operatorname{rank} A - \operatorname{rank} C = 4$$

with equality if and only if $\det(A - \lambda C) = 0$. If we let $B = A^{-1}$, then this is equivalent to $\det(\mathrm{id}_5 - \lambda BC) = 0$. As we have proved in lemma 3.5, this determinant equals $1 - \lambda\cdot\operatorname{tr}(BC)$. Using the fact that $\operatorname{tr}(u_1u_2^T) = \operatorname{tr}(u_2^Tu_1) = u_2^Tu_1$ holds for any columnvectors u_1 and u_2, we get

$$\operatorname{tr}(BC) = 24\operatorname{tr}(Bv(a)v(a)^T) = 24\, v(a)^T B\, v(a) = 24\,\eta(a)$$

where we define $\eta \in T_4$ by

$$\eta = v^T B\, v. \tag{3.6}$$

Hence if $\eta(a) = 0$, the rank of $\mathrm{Cat}_{\mathcal{D}}(g)$ always equals 5. However, if $\eta(a) \neq 0$, then there exists a unique $\lambda_a \in k$ such that $\operatorname{rank} \mathrm{Cat}_{\mathcal{D}}(f - \lambda_a\, l_a^4) = 4$, namely $\lambda_a = \frac{1}{24\,\eta(a)}$.

Since $\dim_k (f - \lambda_a\, l_a^4)_2^\perp = 2$ and $D_0(f - \lambda_a\, l_a^4) = D_0(f) - 12\,\lambda_a D_0(a)\, l_a^2 = 0$, it follows that $(f - \lambda_a\, l_a^4)_2^\perp = \langle D_0, \rho_a \rangle$ for some $\rho_a \in \langle D_1, \ldots, D_5 \rangle$. This ρ_a is unique up to a scalar. Now $0 = \rho_a(f - \lambda_a\, l_a^4) = \rho_a(f) - 12\,\lambda_a\, \rho_a(a)\, l_a^2$, hence $\rho_a(a) \neq 0$ because $\rho_a(f) \neq 0$. Scale ρ_a such that $\rho_a(a) = \eta(a)$, and let $\rho_a = \sum_{i=1}^5 c_i D_i = c^T v$. For all j we now have

$$0 = \rho_a D_j(f - \lambda_a l_a^4) = \rho_a D_j(f) - 24\lambda_a \rho_a(a) D_j(a) = \sum_{i=1}^{5} c_i D_i D_j(f) - D_j(a)$$

Written as a matrix equation, this says $Ac = v(a)$. Hence $c = B\, v(a)$ and

$$\rho_a = v(a)^T B\, v = v^T B\, v(a). \tag{3.7}$$

In particular, we notice that $\rho_a \neq 0$ since $v(a) \neq 0$.

Notice that both η and ρ_a are independent of the basis $\mathcal{D}$, as long as D_0 is one of the basisvectors. If $\mathcal{E} = \{D_0, E_1, \ldots, E_5\}$ is another basis, then $v_{\mathcal{E}} = M v_{\mathcal{D}}$ for some invertible 5×5 matrix M. Here $v_{\mathcal{E}}^T = [E_1, \ldots, E_5]$ and $v_{\mathcal{D}}$ is the columnvector v above. Now $A_{\mathcal{E}} = (v_{\mathcal{E}} v_{\mathcal{E}}^T)(f) = (M v_{\mathcal{D}} v_{\mathcal{D}}^T M^T)(f) = M A_{\mathcal{D}} M^T$ and $B_{\mathcal{E}} = A_{\mathcal{E}}^{-1} = (M^T)^{-1} B_{\mathcal{D}} M^{-1}$. Hence

$$\eta_{\mathcal{E}} = v_{\mathcal{E}}^T B_{\mathcal{E}}\, v_{\mathcal{E}} = \big(v_{\mathcal{D}}^T M^T\big)\big((M^T)^{-1} B_{\mathcal{D}} M^{-1}\big)\big(M v_{\mathcal{D}}\big) = v_{\mathcal{D}}^T B_{\mathcal{D}}\, v_{\mathcal{D}} = \eta_{\mathcal{D}},$$

and similarly for ρ_a. If $\mathcal{E}$ is a basis that does not have D_0 as a basisvector, then our method to define η and ρ_a fails, and we just define η and ρ_a to be the same polynomials as in the cases where $\mathcal{E}$ contains D_0. Also notice that η is nonzero as an element of T, since

$$\eta(f) = \operatorname{tr}\big(Bvv^T\big)(f) = \operatorname{tr}\big(Bvv^T(f)\big) = \operatorname{tr}(BA) = \operatorname{tr}(\mathrm{id}_5) = 5. \tag{3.8}$$

Now, if there exists an $a \notin \{\eta = 0\}$ such that D_0 and ρ_a intersect in either four distinct points or a line and a point outside the line, then theorem 3.2 tells us that $f - \lambda_a l_a^4$ is a sum of four linear powers, and f is then a sum of five. On the other hand, if $f = \sum_{i=1}^5 \lambda_i\, l_{a_i}^4$, then $\{a_1, \ldots, a_5\} \subseteq \{D_0 = 0\}$, as we proved in the beginning of section 3.3. Futhermore, $g_j = f - \lambda_j\, l_{a_j}^4$ is a sum of four linear powers, hence $\operatorname{rank} \mathrm{Cat}_{\mathcal{D}}(g_j) \leq 4$ by (0.9) and (0.8). Since we have already shown that this rank is ≥ 4, it follows that $\operatorname{rank} \mathrm{Cat}_{\mathcal{D}}(g_j) = 4$. Then we must have $a_j \notin \{\eta = 0\}$, $\lambda_j = \lambda_{a_j}$ and $(g_j)_2^\perp = \langle D_0, \rho_{a_j} \rangle$. Since g_j is a sum of four linear powers, it follows from theorem 3.2 that D_0 and ρ_{a_j} intersect in either four distinct points or a line and a point outside. The other four points, $\{a_i \,|\, i \neq j\}$, are contained in this intersection.

EXAMPLE 3.5.1: As an example of the quantities we have just introduced, let us look at $f = \frac{1}{24}\big((c_0x_0+4x_2)x_0^3+6x_1^2x_2^2+c_1x_2^4\big)$, where $c_j \in k$. We see that $f_2^\perp = \langle \partial_0\partial_1 \rangle$, hence our discussion above applies. As a basis for T_2, choose $\mathcal{D} = \{\partial_0\partial_1, \partial_0^2, \partial_1^2, \partial_2^2, \partial_0\partial_2, \partial_1\partial_2\}$, and let $v^T = [\partial_0^2, \partial_1^2, \partial_2^2, \partial_0\partial_2, \partial_1\partial_2]$. Then

$$\mathrm{Cat}_{\mathcal{D}}(f) = \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \quad \text{where} \quad A = (vv^T)(f) = \begin{bmatrix} c_0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & c_1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

In particular, we see that A is invertible, and its inverse B is given by

$$B = A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & -c_1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -c_0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using $\eta = v^T B\, v$ (3.6) and $\rho_a = v^T B\, v(a)$ (3.7), we see that

$$\begin{aligned} \eta &= 2\partial_0^3\partial_2 - c_0\partial_0^2\partial_2^2 - c_1\partial_1^4 + 3\partial_1^2\partial_2^2, \\ \rho_a &= a_0a_2\partial_0^2 + (a_2^2 - c_1a_1^2)\partial_1^2 + a_1^2\partial_2^2 + a_0(a_0 - c_0a_2)\partial_0\partial_2 + a_1a_2\partial_1\partial_2, \end{aligned}$$

and simple calculations now gives $\eta(f) = 5$, as expected.

For any $a \in Y_0 := \{\partial_0 = 0\}$, we have $\rho_a = (a_2^2 - c_1a_1^2)\partial_1^2 + a_1^2\partial_2^2 + a_1a_2\partial_1\partial_2$. Hence the intersection between $C_a := \{\rho_a = 0\}$ and $Y_1 := \{\partial_1 = 0\}$ is given by $a_1^2\partial_2^2 = 0$. If $a_1 \neq 0$, this intersection is just the (double) point $(1,0,0)$, and $X_f := C_a \cap C_f$, where $C_f := \{\partial_0\partial_1 = 0\} = Y_0 \cup Y_1$, is supported at less than four points. If $a_1 = 0$, then $a = (0,0,1)$ and $\rho_a = \partial_1^2$. In this case X_f has support on a line only.

Next we notice that $\rho_a = a_0a_2\partial_0^2 + a_2^2\partial_1^2 + a_0(a_0 - c_0a_2)\partial_0\partial_2$ for any $a \in Y_1$. Therefore, $C_a \cap Y_0 = \{a_2^2\partial_1^2 = 0\}$, which is a double point if $a_2 \neq 0$. On the other hand, if $a_2 = 0$, then $a = (1,0,0)$ and $\rho_a = \partial_0\partial_2$. Hence X_f has support on a line and a point outside the line. But now $\eta(1,0,0) = 0$.

Since for any $a \in C_f$, either $a \in \{\eta = 0\}$ or $X_f = C_a \cap C_f$ has support in less than four points or on a line, it follows from the discussion preceding this example, that f cannot be a sum of five linear powers.

REMARK 3.5.2: The polynomial f studied in example 3.5.1 is not the only polynomial that is not a sum of five linear powers. Both h_1 and h_2 where

$$\begin{aligned} h_1 &= \tfrac{1}{24}\left((c_0x_0 + 4x_2)x_0^3 + x_1^4 + (4x_1 + c_1x_2)x_2^3\right) \\ h_2 &= \tfrac{1}{24}\left((c_0x_0 + 4x_2)x_0^3 + (c_1x_1 + 4x_2)x_1^3 + x_2^4\right) \end{aligned}$$

share this property with f. In both cases, $(h_i)_2^\perp = \langle \partial_0 \partial_1 \rangle$. With h_1 we have

$$\eta = 2\partial_0^3\partial_2 - c_0\partial_0^2\partial_2^2 + \partial_1^4 - c_1\partial_1^2\partial_2^2 + 2\partial_1\partial_2^3,$$
$$\rho_a = a_0a_2\partial_0^2 + a_1^2\partial_1^2 + a_1a_2\partial_2^2 + a_0(a_0 - c_0a_2)\partial_0\partial_2 + a_2(a_2 - c_1a_1)\partial_1\partial_2,$$

while using h_2 we get

$$\eta = 2\partial_0^3\partial_2 - c_0\partial_0^2\partial_2^2 + 2\partial_1^3\partial_2 - c_1\partial_1^2\partial_2^2 + \partial_2^4,$$
$$\rho_a = a_0a_2\partial_0^2 + a_1a_2\partial_1^2 + a_2^2\partial_2^2 + a_0(a_0 - c_0a_2)\partial_0\partial_2 + a_1(a_1 - c_1a_2)\partial_1\partial_2.$$

In both cases it is easy to check that $C_a \cap C_{h_i}$ is supported at less than four points or at just a line for all $a \in C_{h_i}$ such that $\eta(a) \neq 0$.

Theorem 3.6 tells us that up to linear transformations, these polynomials are the only ones with $f_2^\perp = \langle D_0 \rangle$ and D_0 singular, which are not a sum of five linear powers.

We are now ready to prove the following theorem, which tells us how many linear powers are needed to express an f with $\dim_k f_2^\perp = 1$ as a linear combination of them.

Theorem 3.6:
Given $f \in S_4$ such that $f_2^\perp = \langle D_0 \rangle$ and D_0 is singular. If $\{D_0 = 0\}$ is a product of two distinct lines, then f is a sum of five linear powers, except when

1. $f \sim (c_0x_0 + x_2)x_0^3 + x_1^2x_2^2 + c_1x_2^4$
2. $f \sim (c_0x_0 + x_2)x_0^3 + x_1^4 + (x_1 + c_1x_2)x_2^3$
3. $f \sim (c_0x_0 + x_2)x_0^3 + (c_1x_1 + x_2)x_1^3 + x_2^4$

In these three cases, f is a sum of six linear powers. If $\{D_0 = 0\}$ is a double line, then f is a sum of exactly seven linear powers.

Proof: The simlest case is when $\{D_0 = 0\}$ is a double line, that is, $D_0 = \delta^2$ for some $\delta \in T_1$. If f is a linear combination of $l_{a_1}^4, \dots, l_{a_s}^4$ where $s \leq 6$, then $\{a_1, \dots, a_s\} \subseteq \{D_0 = 0\}$, as we proved in the beginning of section 3.3. Hence $\delta(a_i) = 0$ for all i, and therefore $\delta(f) = 0$. But this contradicts the fact that $f_1^\perp = 0$. To prove that f is indeed a sum of seven linear powers, we just have to notice that since $\eta \neq 0$ by (3.8), we can find $a \notin \{\eta = 0\}$. Then $\dim_k(f - \lambda_a l_a^4)_2^\perp = 2$, and by theorem 3.2, $f - \lambda_a l_a^4$ is a sum of six linear powers.

When $\{D_0 = 0\}$ is a product of two distinct lines, we may assume that $D_0 = \partial_0\partial_1$ after a linear transformation. Hence $f = h_0(x_0, x_2) + h_1(x_1, x_2)$ where both h_0 and h_1 are binary forms. First we want to prove that f necessarily is a sum of six linear powers. If not, at least one of the h_i's cannot be a sum of three linear powers. In fact, since $f = (h_0 + cx_2^4) + (h_1 - cx_2^4)$, we can assume that $h_0 + cx_2^4$ is not a sum of three linear powers for infinitely many $c \in k$. By remark 1.2.1, this means that $h_0 + cx_2^4 = l_1 l_2^3$ for some $l_i = l_i(c) \in S_1$ for these c. Hence $h_0 + cx_2^4$ is not squarefree for infinitely many c, and by the proof of lemma 1.1, we conclude that they must have a common multiple root. Hence $h_0 = lx_2^3$ for some $l \in S_1$, but then $\partial_0^2 f = \partial_0^2 h_0 = 0$, which contradicts the fact that $f_2^\perp = \langle \partial_0 \partial_2 \rangle$.

Consider $\partial_0 f = \partial_0 h_0$. If $\partial_0 h_0 \not\sim x_0 x_2^2$, then by remark 1.2.1 $\partial_0 h_0 = l_a^3 + l_b^3$ for some $l_a, l_b \in k[x_0, x_2]_1$. Here a and b must be distinct points in $\mathbb{P}^1$, or else $\dim_k f_2^\perp > 1$. Assume that $a_0, b_0 \neq 0$, and let $g = f - \frac{1}{4a_0} l_a^4$ and $\delta = b_0 \partial_2 - b_2 \partial_0$. Now $\delta \partial_0 g = \delta(\partial_0 f - l_a^3) = \delta(l_b^3) = 0$, and furthermore $\partial_0 \partial_1 g = \partial_0(\partial_1 f - 0) = 0$. Also $\operatorname{rank} \operatorname{Cat}(g) \geq 4$ since $\operatorname{rank} \operatorname{Cat}(f) = 5$. Therefore, $g_2^\perp = \langle \partial_0 \partial_1, \partial_0 \delta \rangle$. We see that $\{\partial_1 = 0\} \cap \{\delta = 0\} = b \notin \{\partial_0 = 0\}$, and by theorem 3.2 g is a sum of four fourth-powers of linear forms. Hence f is a sum of five linear powers.

Now assume that f is *not* a sum of five linear powers. Then, for both $i = 0$ and $i = 1$, the previous paragraph shows that $\partial_i h_i$ must equal either $l_1^3 + x_2^3$ or $l_2 l_3^2$ for some $l_j \in k[x_i, x_2]_1$. If $\partial_i h_i = l_1^3 + x_2^3$ with $\partial_i l_1 \neq 0$, then $h_i = \frac{1}{4\partial_i l_1} l_1^4 + x_i x_2^3 + cx_2^4$ for some $c \in k$. Thus $h_i = x_i^4 + l_a x_2^3$ where $a_i \neq 0$ after a linear transformation of $k[x_i, x_2]_1$ that leaves x_2 fixed. If $\partial_i h_i = l_1 l_2^2$, then $12(\partial_i l_2)^2 h_i = (4(\partial_i l_2) l_1 - (\partial_i l_1) l_2) l_2^3 + cx_2^4$ for some $c \in k$. If $\partial_i l_2 \neq 0$, it follows that $h_i = l_b x_i^3 + cx_2^4$ with $b_2 \neq 0$ after a linear transformation that leaves x_2 fixed. If $\partial_i l_2 = 0$, then $\partial_i h_i = l_1 x_2^2$ where $\partial_i l_1 \neq 0$. Hence $h_i = \frac{1}{2\partial_i l_1} l_1^2 x_2^2 + cx_2^4$ for some $c \in k$, and $h_i = x_i^2 x_2^2 + cx_2^4$ after the linear transformation given by $l_1 \mapsto \sqrt{2\partial_i l_1}\, x_i$ and $x_2 \mapsto x_2$.

Now we notice that $\partial_i^2(x_i^2 x_2^2 + cx_2^4) = 2x_2^2$ and $\partial_i \partial_2(x_i^4 + l_a x_2^3) = 3a_i x_2^2$. Hence if both h_i's are of one of these two "forms", then $\dim_k f_2^\perp > 1$. For instance, if $h_0 = x_0^2 x_2^2 + cx_2^4$ and $h_1 = x_1^4 + l_a x_2^3$, then $(3a_1 \partial_0^2 - 2\partial_1 \partial_2)(f) = 0$. Since $f_2^\perp = \langle \partial_0 \partial_1 \rangle$ by assumption, it follows that one of the h_i's must equal the last "form". Hence we may assume that $h_0 = l_b x_0^3 + cx_2^4$.

Then $f = h_0 + h_1$ must look like either $l_b x_0^3 + x_1^4 + l_a x_2^3$ or $l_b x_0^3 + x_1^2 x_2^2 + cx_2^4$ or $l_b x_0^3 + l_{b'} x_1^3 + c' x_2^4$. In the last case, $c' \neq 0$, or else $\partial_2^2 f$ would equal zero. Moreover, we know that $b_2, a_1, b'_2 \neq 0$. By scaling the variables suitably, we can assume that these constants all equal 1. Hence we have proved that if f is not a sum of five linear powers, then after a linear transformation f must be equal to one of the following three forms:

1. $f = (c_0 x_0 + x_2) x_0^3 + x_1^2 x_2^2 + c_1 x_2^4$
2. $f = (c_0 x_0 + x_2) x_0^3 + x_1^4 + (x_1 + c_1 x_2) x_2^3$
3. $f = (c_0 x_0 + x_2) x_0^3 + (c_1 x_1 + x_2) x_1^3 + x_2^4$

where $c_j \in k$. On the other hand, these f are not a sum of five linear powers, as proven in example 3.5.1 and remark 3.5.2. The only difference between these f and those in examble 3.5.1 and remark 3.5.2 is that those f are scaled differently, as is easily verified. This concludes the proof of the theorem. □

REMARK 3.6.1: In the first case we considered in this chapter, where $f_2^\perp = \langle D_0 \rangle$ and D_0 was nonsingular, we quite successfully used the structure theorem of Buchsbaum-Eisenbud [2] to prove that f was a sum of five linear powers. Of course, we could use this method for any $f \in S$ since A^f always is Gorenstein of codimension 3. In fact, when $D_0 = \partial_0\partial_1$, it is not hard to prove that ϕ can be chosen to look like this,

$$\phi = \begin{bmatrix} 0 & q_1 & q_2 & q_3 & q_4 \\ -q_1 & 0 & 0 & \partial_0 & \partial_2 \\ -q_2 & 0 & 0 & 0 & \partial_1 \\ -q_3 & -\partial_0 & 0 & 0 & 0 \\ -q_4 & -\partial_2 & -\partial_1 & 0 & 0 \end{bmatrix}$$

but from here on it gets more complicated. One reason for this is the fact that not all of these f are a sum of five linear powers.

In the next section, where we look at f such that $\dim_k f_2^\perp = 0$, we will see that all f are "general" in the sense that they are a sum of six linear powers. But even in this case we do not use the structure theorem, since the matrix ϕ in that case will be 7×7 and hence more difficult to work with.

3.4 $\dim_k f_2^\perp = 0$

In this section we will look at $f \in S_4$ such that A^f has Hilbert function $(1, 3, 6, 3, 1)$, and as we soon will see, any such f is *general* in the sense that it is a sum of six linear powers. But before we turn to the proof of this, there are a couple of things we want to point out.

Prior to theorem 3.6 we introduced several quantities. Of course, we could do the same things in this case, with the insignificant difference that whenever $\mathcal{D} = \{D_0, \ldots, D_5\}$ is any basis for T_2, we let $A = \mathrm{Cat}_{\mathcal{D}}(f)$ since this matrix is invertible. However, we want to start at the other end this time.

Whenever $\mathcal{D} = \{D_0, \ldots, D_5\}$ is a basis for T_2, let $\mathcal{B} = \{h_0, \ldots, h_5\}$ be the "dual" basis for S_2 in the sense that

$$D_i(h_j) = h_j(D_i) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Such a basis always exists. Indeed, for any $j = 1, \ldots, 6$, we see that

$$\dim_k \bigcap_{i \neq j} (D_i)_2^\perp = \dim_k S_2 - \dim_k \langle \{D_i\}_{i \neq j} \rangle = 1$$

by (0.3), hence $\cap_{i\neq j}(D_i)_2^\perp = \langle h'_j\rangle$ for some nonzero $h'_j \in S_2$. Since $\mathcal{D}$ is a basis and $h'_j(D_i) = 0$ for all $i \neq j$, it follows that $h'_j(D_j) \neq 0$. Hence, if we let $h_j = h'_j/h'_j(D_j)$, we get the desired "dual" basis. Note that with $u^T = [h_0, \dots, h_5]$ and $v^T = [D_0, \dots, D_5]$, $\mathcal{D}$ and $\mathcal{B}$ are dual bases if and only if $uv^T = vu^T = \mathrm{id}_6$.

Given $f \in S_4$ with $\dim_k f_2^\perp = 0$, we let $A = \mathrm{Cat}_{\mathcal{D}}(f) = (vv^T)(f)$ and $B = A^{-1}$. Then for *any* $a \in \mathbb{P}^2$, we define $\rho_a \in T_2$ by

$$\rho_a = v(a)^T B\, v.$$

It is immediate from this definition that $\rho_a \neq 0$ for all a and $\rho_a \neq \rho_b$ for $a \neq b$. Since $v(f)$ is a columnvector with entries in S_2, it equals Cu for some 6×6 matrix C. But then $C = Cuv^T = v(f)v^T = (vv^T)(f) = A$, and hence

$$v(f) = Au.$$

With $\eta = v^T B\, v$ as before, we see that $\eta \neq 0$, because

$$\eta(f) = (v^T B\, v)(f) = v^T B\,(v(f)) = v^T BAu = v^T u = \mathrm{tr}(\mathrm{id}_6) = 6. \tag{3.9}$$

Furthermore, $\rho_a(f) = v(a)^T B\, v(f) = v(a)^T u$. Since ρ_a is independent of the choice of basis $\mathcal{D}$, we might assume that $\mathcal{D}$ is the standard basis for T_2, $\mathcal{D} = \{\partial_0^2, \partial_1^2, \partial_2^2, \partial_0\partial_1, \partial_0\partial_2, \partial_1\partial_2\}$. Then the dual basis equals $\mathcal{B} = \{\frac{1}{2}x_0^2, \frac{1}{2}x_1^2, \frac{1}{2}x_2^2, x_0x_1, x_0x_2, x_1x_2\}$, and hence $\rho_a(f) = \frac{1}{2}(\sum a_i x_i)^2 = \frac{1}{2}l_a^2$.

Since $\rho_a(l_a^4) = 12\,\rho_a(a)\,l_a^2$, it follows that if $\rho_a(a) = \eta(a) \neq 0$, then $\rho_a(f - \lambda_a l_a^4) = 0$ where $\lambda_a = \frac{1}{24\rho_a(a)}$, as in the discussion prior to example 3.5.1. Now $\dim_k(f - \lambda_a l_a^4)_2^\perp = \mathrm{rank}\,\mathrm{Cat}_{\mathcal{D}}(f - \lambda_a l_a^4) \geq \mathrm{rank}\,A - \mathrm{rank}\,\mathrm{Cat}_{\mathcal{D}}(l_a^4) = 5$ by (0.9). Hence we have proved that for any $a \notin \{\eta = 0\}$, we have

$$(f - \lambda_a l_a^4)_2^\perp = \langle \rho_a\rangle. \tag{3.10}$$

Furthermore, if $D(f)$ is a square for some $D \in T_2$, say $D(f) = l_a^2$, then $(D - 2\rho_a)(f) = 0$. Since $f_2^\perp = 0$, it follows that $D = 2\rho_a$. Thus the ρ_a's are precisely those $D \in T_2$ such that $D(f)$ is a square.

Theorem 3.7:
Any $f \in S_4$ such that $f_2^\perp = 0$ is a sum of six linear powers.

Proof: The idea behind the proof is this: Since $\eta \neq 0$ by (3.9), we can find $a \in \mathbb{P}^2$ such that $\eta(a) \neq 0$. Then we consider $g = f - \lambda_a l_a^4$, and by (3.10) we have $g_2^\perp = \langle\rho_a\rangle$. Therefore, theorems 3.4 and 3.6 tell us when g is a sum of five linear powers. But we need to know what ρ_a looks like. Since ρ_a is a quadric, the rank of $H_a = (x_ix_j(\rho_a))_{i,j=0,1,2}$ decides whether ρ_a is nonsingular, a product of two distinct lines or a square. Thus both $Z_0 = \{a \mid \rho_a \text{ is reducible}\}$ and $Z_1 = \{a \mid \rho_a \text{ is a square}\}$ are closed subsets of $\mathbb{P}^2$, since $Z_0 = \{a \mid \det H_a = 0\}$ and $Z_1 = \{a \mid \mathrm{rank}\, H_a \leq 1\}$.

Any a such that ρ_a is a square, is of no use when we try to write f as a sum of six linear powers, hence we need to control the size of Z_1. Assume that Z_1 contains four distinct points $a_1, \dots, a_4$ on a line, i.e. $\delta(a_i) = 0$ for some $\delta \in T_1$. Now ρ_{a_i} is a square, hence $\rho_{a_i} = \delta_{b_i}^2$, and $b_1, \dots, b_4$ are four distinct points in $\check{\mathbb{P}}^2$. The $l_{a_i}^2$'s are linearly dependent because the points a_i lie on a line, hence $\sum_{i=1}^4 c_i l_{a_i}^2 = 0$ for some $c_i \in k$, not all zero. It follows that $\sum_{i=1}^4 c_i \delta_{b_i}^2 = 0$, since $(\sum_{i=1}^4 c_i \rho_{a_i})(f) = \sum_{i=1}^4 c_i \cdot \frac{1}{2} l_{a_i}^2 = 0$ and $f_2^\perp = 0$. But the only way four squares can be linearly dependent, is that the corresponding points $\{b_1, \dots, b_4\}$ contained in a line. Hence we might assume that $\delta_{b_i} \in k[\partial_0, \partial_1]$ for all i after a linear transformation. Then $\{\delta_{b_i}^2\}_{i=1}^4$ is a basis for $k[\partial_0, \partial_1]_2$, and since $\delta_{b_i}^2(f) = \frac{1}{2} l_{a_i}^2$, it follows that $D(f) \in \langle l_{a_1}^2, \dots, l_{a_4}^2 \rangle$ for all $D \in k[\partial_0, \partial_1]_2$. This implies that $\delta\partial_0^2(f) = \delta\partial_0\partial_1(f) = \delta\partial_1^2(f) = 0$ because $\delta(a_i) = 0$ for all i. Thus we get $\partial_0\delta(f), \partial_1\delta(f) \in \langle x_2^2 \rangle$ and $\delta'\delta(f) = 0$ for a suitable $\delta' \in \langle \partial_0, \partial_1 \rangle$. This contradicts the fact that $f_2^\perp = 0$, and it proves that Z_1 cannot contain four distinct points on a line.

Let $C_a = \{\rho_a = 0\}$ and $X = \{\eta = 0\}$. If there exists an $a \notin X$ such that ρ_a is nonsingular, then by theorem 3.4, $f - \lambda_a l_a^4$ is a sum of five linear powers. Now assume that f is *not* a sum of six linear powers. Then ρ_a must be singular for all $a \notin X$. Since Z_0 is a closed subset of $\mathbb{P}^2$ and $\eta \neq 0$ as an element of T_4, it follows that ρ_a is reducible for all $a \in \mathbb{P}^2$.

We notice that for any $a, b \notin X$ such that $\rho_a(b) = \rho_b(a) = 0$, C_a and C_b cannot intersect in neither four distinct points nor a line and a point outside. Indeed, if we could find such points a and b, then $g_2^\perp = \langle \rho_a, \rho_b \rangle$ where $g = f - \lambda_a\, l_a^4 - \lambda_b\, l_b^4$, and by theorem 3.2, g would be a sum of four linear powers. This is impossible, since f is not a sum of six linear powers by assumption.

From the definition of ρ_a, we see that $\rho_a(b) = \rho_b(a)$. It follows that the set of points a such that C_a passes through a given point p, is exactly C_p. Furthermore, X and C_a have a common linear component L for some a if and only if $a \in C_b$ for all $b \in L$, i.e. $a \in \cap_{b \in L} C_b$. Hence $Y = \{a \in \mathbb{P}^2 \mid X$ and C_a has a common component$\}$ is a proper, closed subset of $\mathbb{P}^2$, since X cannot have more than four linear components.

Since X, Y and Z_1 are proper, closed subsets of $\mathbb{P}^2$, we may pick an $a \in \mathbb{P}^2$ such that $\eta(a) \neq 0$, ρ_a is not a square and X and C_a have no common components. Let p be the singular point of C_a. Since $\rho_a \neq \rho_p$, we can find a linear component L of C_a that is not contained in C_p. Now the "bad" points on L are the points b such that $\rho_p(b) = 0$ or $\eta(b) = 0$, and we know that there are only finitely many of them. If b_1 is any other point, then $\rho_{b_1}(a) = 0$ and $\rho_{b_1}(p) \neq 0$. Hence C_a and C_{b_1} has no common component. Since we have assumed that f is not a sum of six linear powers, C_a and C_{b_1} must intersect singularly, and the only way this can happen, is that C_a contains the singular point q_1 of C_{b_1}.

Let $b_2 \in L$ be another "good" point. Then the singular points q_1 and q_2

of C_{b_1} and C_{b_2}, respectively, cannot be distinct and lie on the same linear component of C_a. Indeed, if they did, then C_{b_1} and C_{b_2} would intersect in four distinct points, because any linear component of C_{b_i} which contained q_j, $i \neq j$, would necessarily pass through p. Since C_a has only two components, then the C_{b_i}'s must have a common singular point $q \in C_a$ for infinitely many "good" points $b_i \in L$. Since every C_{b_i} necessarily contains a, it follows that they all have a common linear component, that is, the unique line $\{\delta = 0\}$ through a and q.

Let b_1, b_2 and b_3 be three such points. Then $\rho_{b_i} = \delta_i \delta$ for some $\delta_i \in T_1$, and the δ_i's are linearly dependent because $\delta_i(q) = 0$ for all i. Hence $\sum_{i=1}^{3} c_i \delta_i = 0$ for some $c_i \in k$, not all of them zero. Then $\sum_{i=1}^{3} c_i \rho_{b_i} = 0$ and $\sum_{i=1}^{3} c_i l_{b_i}^2 = 2(\sum_{i=1}^{3} c_i \rho_{b_i})(f) = 0$. But this is impossible for distinct points $b_i \in \mathbb{P}^2$, hence we have our desired contradiction. □

Bibliography

[1] J. Alexander and A. Hirschowitz, *Polynomial interpolation in several variables*, Journal of Algebraic Geometry **1** (1995), 201–222.

[2] D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for codimension three*, Amer. J. Math. **99** (1977), 447–485.

[3] A. Clebsch, *Über Curven vierter 0rdnung*, J. Reine Angew. Math. **59** (1861), 125–145.

[4] S. J. Diesel, *Some irreducibility and dimension theorems for families of height 3 gorenstein algebras*, Pacific J. Math. **172** (1996), 363–395.

[5] A. Dixon, *The canonical forms of the ternary sextic and the quarternary quartic*, Proc. London Math. Soc. (2) **4** (1906), 223–227.

[6] D. Eisenbud, *Commutative Algebra, with a View Toward Algebraic Geometry*, GTM†, no. 150, Springer-Verlag, New York, 1995.

[7] J. Emsalem, *Géomètrie des Points Epais*, Bull. Soc. Math. France **106** (1978), 399–416.

[8] J. Emsalem and A. Iarrobino, *Some zero-dimensional generic singularities; finite algebras having small tangent space*, Compositio Math. **36** (1978), 145–188.

[9] R. Hartshorne, *Algebraic Geometry*, GTM†, no. 52, Springer-Verlag, New York, 1977, Full address: Springer-Verlag New York Inc., 175 Fifth Avenue, New York, NY 10010, USA.

[10] A. Iarrobino, *Associated graded algebra of a Gorenstein Artin algebra*, AMS Memoirs Vol. 107 (1994), no. 514.

[11] F. H. S. Macaulay, *The Algebraic Theory of Modular Systems*, Cambridge University Press, London, 1916.

†Graduate Texts in Mathematics

[12] F. Palatini, *Sulla rappresentazione delle forme ternaire mediante la somma di potenze di forme lineari*, Rom. Acc. L. Rend. (5) **12** (1903), 378–384.

[13] K. Ranestad and F.-O. Schreyer, *Varieties of sums of powers*, eprint, `http://www.math.uio.no/~ranestad/papers.html`.

[14] T. Reye, *Darstellung quarternärer biquadratischer Formen als Summen von zehn Biquadraten*, J. Reine Angew. Math. **78** (1874), 123–129.

[15] H. W. Richmond, *On canonical forms*, Quart. J. Pure Appl. Math. **33** (1904), 331–340.

[16] F.-O. Schreyer, *Algebra and geometry of Fano 3-folds of genus 12 and index 1*, a preliminary manuscript, `http://btm8x5.mat.uni-bayreuth.de/~schreyer/Papers.html`.

[17] J. J. Sylvester, *Sur une extension d'un théorème de Clebsch relatif aux courbes de quartième degré*, Compte Rendus de l'Acad. de Science **102** (1886), 1532–1534, (Collected Math. Works IV, p. 527-530).

[18] A. Terracini, *Sulle* V_k *per cui la varietà degli* $S_h(h+1)$*-seganti ha dimensione minore dell'ordinario*, Rend. Circ. Mat. Palermo **31** (1911), 392–396.