Preface

This thesis has been written for the degree of Doctor Scientiarum (dr. scient.) at the Department of Mathematics, University of Oslo. My supervisor has been professor Kristian Ranestad.

I would like to thank the University of Oslo for financial support. Special thanks go to my supervisor, professor Kristian Ranestad, for helping me through this period of time, and to professor Frank-Olaf Schreyer who first introduced me to the problem of "additive splittings" during my stay at the University of Bayreuth, Germany, the fall of 2000. I would also like to thank my father, professor Jan Oddvar Kleppe, for interesting discussions.

The problem professor Schreyer originally gave me is the following. Consider a homogeneous polynomial f of degree 3 (we were looking at double hyperplane sections of canonical curves) that is a sum of two polynomials in separate variables, that is $f = g + h$ with $g \in k[x_1, \dots x_s]$ and $h \in k[x_{s+1}, \dots, x_r]$ up to base change. The minimal resolution of the ideal

$$(\{\partial_i \partial_j \mid i = 1, \dots, s,\ j = s+1, \dots, r\}) \subseteq R = k[\partial_1, \dots, \partial_r]$$

will be part of any resolution of $\operatorname{ann} f$. Therefore the graded Betti number $\beta_{r-1,r}$ of $R/\operatorname{ann} f$ will be nonzero. He asked if I could prove that this was an equivalence.

After computing some examples, I realized degree three did not matter much, and I wondered if something stronger might be true. Could $1 + \beta_{r-1,r}$ be the maximal length of an "additive splitting" of f? It was also clear that I had to allow degenerations of such splittings. I decided to take the simple approach of definition 2.7 and restrict my attention to "deformations" defined over a polynomial ring. In the end it turned out that $1 + \beta_{r-1,r}$ does not always count the length of a maximal degenerate splitting.

Chapter 1 contains a brief discussion of background material. In chapter 2 I define precisely want I mean by regular and degenerate additive splittings. I also define a matrix algebra M_f, which probably is the most important new object in this thesis, and I give some basic results about M_f and additive splittings.

In chapter 3 I effectively determine all regular splittings, and I use this to calculate the minimal free resolution of $R/\operatorname{ann} f$ and its graded Betti numbers. I also discuss some consequences for $\mathbf{PGor}(H)$, the scheme parameterizing all graded Artinian Gorenstein quotients of R. Chapter 4 studies degenerate splittings. The central question is whether we can use all of M_f to construct generalizations of f that splits $\beta_{r-1,r}$ times. I give some conditions that implies a positive answer, and I construct several counter examples in general. Finally, chapter 5 generalizes M_f and some results about it.

June, 2005 Johannes Kleppe

Contents

Chapter 1

Introduction

It is well known that any homogeneous polynomial f of degree two in r variables over a field of characteristic $\neq 2$ can be written as a linear combination of $n = \operatorname{rank} H \leq r$ squares. Here $H = (\partial_i \partial_j f)$ is the Hessian matrix of f. The usual way to generalize this to higher degrees is to ask how to write a form f of degree d as a sum of powers of linear forms, $f = \sum_{i=1}^{n} c_i l_i^d$, and how small n can be. This is usually called Waring's problem, and has been studied by many people and has been solved for general f.

There is, however, a different way to generalize the sum of squares theorem. If we write $f = \sum_{i=1}^{n} c_i l_i^2$ with n minimal, then $l_1, \dots, l_n$ are necessarily linearly independent. For higher degrees, when $f = \sum_{i=1}^{n} c_i l_i^d$ and $d \geq 3$, the l_i's can no longer be linearly independent, except for very special f. With this in mind, we see that there is another question that naturally generalizes of the sum of squares theorem: When is it possible to write f as a sum of several homogeneous polynomials in independent sets of variables? We will call this a *regular splitting* of f (definition 2.4). Some examples of polynomials that split regularly are $f = x_1^3 + x_2 x_3 x_4$, $f = x_1 x_2^6 + x_3^2 x_4^5 + x_4^3 x_5^4$ and $f = (x_1 + x_2)^8 + x_2^3 (x_2 + x_3)^5$. Sometimes there exist more than one regular splitting of the same polynomial, like $f = x_1^2 + x_2^2 = \frac{1}{2}\big((x_1 + x_2)^2 + (x_1 - x_2)^2\big)$.

To make the theory work in positive characteristics we assume that f is a homogeneous polynomial in the divided power algebra $\mathcal{R} = k[x_1, \dots, x_r]^{DP}$. The polynomial ring $R = k[\partial_1, \dots, \partial_r]$ acts on $\mathcal{R}$ by partial differentiation. An important ideal in R will be $\operatorname{ann}_R f$, the set of $D \in R$ that annihilates f, i.e. $D(f) = 0$. It is well known that $R/\operatorname{ann}_R f$ is a Gorenstein ring of dimension zero, and furthermore that every graded Artinian Gorenstein quotient arises this way, cf. lemma 1.3.

To study the splitting behavior of a homogeneous polynomial f of degree d, we associate to f the following set of $r \times r$-matrices.

Definition 2.14: Given $f \in \mathcal{R}_d$, define

$$M_f = \{A \in \operatorname{Mat}_k(r, r) \mid I_2(\partial\ A\partial) \subseteq \operatorname{ann}_R f\}.$$

Here $\partial = [\partial_1, \ldots, \partial_r]^\mathsf{T}$ is a column vector, thus $(\partial\ A\partial)$ is the $r \times 2$ matrix consisting of the two columns ∂ and $A\partial$, and $I_2(\partial\ A\partial)$ is the ideal generated by its 2×2 minors. The study of M_f has a central position in this paper. One goal is figure out what M_f can tell us about f. To transfer matrices $A \in M_f$ back into polynomials $g \in \mathcal{R}$, we also define a k-linear map

$$\gamma_f : M_f \to \mathcal{R}_d$$

sending $A \in M_f$ to the unique $g \in \mathcal{R}_d$ that satisfies $\partial g = A\partial f$ (definition 2.16). An important property of M_f is the following.

Proposition 2.21: *Let $d \geq 3$ and $f \in \mathcal{R}_d$. M_f is a k-algebra, and all commutators belong to $\ker \gamma_f$. In particular, M_f is commutative if $\operatorname{ann}(f)_1 = 0$.*

In chapter 3 we analyze the situation of regular splittings completely. In particular, we prove that the idempotents in M_f determine all regular splittings of f in the following precise way.

THEOREM 3.7:
Assume $d \geq 2$, $f \in \mathcal{R}_d$ and $\operatorname{ann}_R(f)_1 = 0$. Let $\operatorname{Coid}(M_f)$ be the set of all complete sets $\{E_1, \ldots, E_n\}$ of orthogonal idempotents in M_f, and let

$$\operatorname{Reg}(f) = \{\{g_1, \ldots, g_n\} \mid f = g_1 + \cdots + g_n \textit{ is a regular splitting of } f\}.$$

The map $\{E_i\}_{i=1}^n \mapsto \{g_i = \gamma_f(E_i)\}_{i=1}^n$ defines a bijection

$$\operatorname{Coid}(M_f) \to \operatorname{Reg}(f).$$

In particular, there is a unique maximal regular splitting of f when $d \geq 3$.

We also give an extended version of this theorem. In the generalization (theorem 3.18) we also prove that, loosely speaking, $M_f = \oplus_{i=1}^n M_{g_i}$, if these algebras are computed inside the appropriate rings. Note in particular the uniqueness when $d = 3$, which is not there when $d = 2$.

In the last two sections of chapter 3 we examine a regular splitting $f = \sum_{i=1}^n g_i$ more carefully. For each i, the additive component g_i is a polynomial in some divided power subring $\mathcal{S}_i \subseteq \mathcal{R}$. The definition of a regular splitting requires that these subrings are independent in the sense that $(\mathcal{S}_i)_1 \cap \sum_{j \neq i} (\mathcal{S}_j)_1 = 0$ for all i. We let S_i be a polynomial subring of R dual to $\mathcal{S}_i$. Assuming the minimal free resolutions of every $S_i / \operatorname{ann}_{S_i}(g_i)$ is known, then we are able to compute the minimal free resolution of $R/\operatorname{ann}_R f$. Theorem 3.33 does this for the case $n = 2$. The induction process to get $n \geq 2$ is carried out for the shifted graded Betti numbers (see equation (1.1) below), culminating in the following theorem.

THEOREM 3.35:
Let $d \geq 2$ and $f, g_1, \dots, g_n \in \mathcal{R}_d$. Suppose $f = g_1 + \cdots + g_n$ is a regular splitting of f. Let $s_i = \dim_k R_{d-1}(g_i)$ for every i. Let $s = \sum_{i=1}^n s_i$, and define

$$\nu_{nk} = (n-1)\binom{r}{k+1} + \binom{r-s}{k+1} - \sum_{i=1}^n \binom{r-s_i}{k+1}.$$

Denote by $\hat{\beta}_{kj}^f$ and $\hat{\beta}_{kj}^{g_i}$ the shifted graded Betti numbers of $R/\operatorname{ann}_R(f)$ and $R/\operatorname{ann}_R(g_i)$, respectively. Then

$$\hat{\beta}_{kj}^f = \sum_{i=1}^n \hat{\beta}_{kj}^{g_i} + \nu_{nk}\delta_{1j} + \nu_{n,r-k}\delta_{d-1,j}$$

for all $0 < j < d$ and all $k \in \mathbb{Z}$. Here the symbol δ_{ij} is defined by $\delta_{ii} = 1$ for all i, and $\delta_{ij} = 0$ for all $i \neq j$.

We proceed to study some consequences for $\mathbf{PGor}(H)$, the quasi-projective scheme parameterizing all graded Artinian Gorenstein quotients R/I with Hilbert function H. We define a subset $\operatorname{PSplit}(H_1, \dots, H_n) \subseteq \mathbf{PGor}(H)$ that parametrizes all quotients $R/\operatorname{ann}_R f$ such that f has a regular splitting $f = \sum_{i=1}^n g_i$ such that the Hilbert function of $R/\operatorname{ann}_R(g_i)$ is H_i for all i, and we are able to prove under some conditions that its closure $\overline{\mathrm{PSplit}(H_1,\dots,H_n)}$ is an irreducible, generically smooth component of $\mathbf{PGor}(H)$ (theorem 3.47).

In chapter 4 we turn our attention to degenerate splittings, i.e. polynomials that are specializations of polynomials that split regularly. A simple example is $f = x^{(2)}y = \frac{1}{t}\big((x+y)^{(3)} - x^{(3)}\big)$. The main question that we are trying to shed some light upon, is the following.

***Question* 4.1:** *Given $f \in \mathcal{R}_d$, $d \geq 3$, is it possible to find $f_t \in \mathcal{R}_d[t_1,\dots,t_n]$ such that $f_0 = f$ and f_t splits regularly $\dim_k M_f - 1$ times over $k(t_1,\dots,t_n)$?*

By lemma 4.2, $\dim_k M_f$ is an upper bound for the length of a regular splitting of f_t. Thus the question asks when this upper bound is achieved. This would mean that M_f not only determines the regular splittings of f, but that we are able to use all of M_f to construct degenerate splittings as well.

We first prove that we can construct an f_t with the desired properties using all powers of a single nilpotent matrix A. This is theorem 4.5. In particular it gives a positive answer to question 4.1 in case M_f is generated by A alone as a k-algebra.

Theorem 4.5:
Let $d \geq 3$ and $f \in \mathcal{R}_d$. Assume that M_f contains a non-zero nilpotent matrix $A \in \mathrm{Mat}_k(r,r)$, and let $n = \mathrm{index}(A) - 1 \geq 1$. Then f is a specialization of some $f_t \in \mathcal{R}_d[t_1,\dots,t_n]$ that splits regularly n times inside $\mathcal{R}_d(t_1,\dots,t_n)$.

We later give a generalized version of this theorem. A careful analysis shows that this covers most cases with $r \leq 4$, and we are able to solve the rest by hand. Hence we get the following result.

Theorem 4.9:
Assume that $r \leq 4$ and $\bar{k} = k$. Let $f \in \mathcal{R}_d$, $d \geq 3$, satisfy $\mathrm{ann}_R(f)_1 = 0$. Then for some $n \geq 1$ there exists $f_t \in \mathcal{R}_d[t_1,\dots,t_n]$ such that $f_0 = f$ and f_t splits regularly $\dim_k M_f - 1$ times over $k(t_1,\dots,t_n)$.

The rest of chapter 4 is devoted to constructing examples where question 4.1 has a negative answer. We are able to do this for all (r,d) with $r \geq 5$ and $d \geq 3$, except the six pairs $(5,3), (6,3), (7,3), (8,3), (5,4)$ and $(6,4)$.

Finally, in chapter 5, we consider some generalizations of M_f. We do not yet have a particular use for these generalizations. However, M_f proved very useful to us, and we show how to define two similar algebras and prove some basic results about them.

1.1 Polynomials and divided powers

Let $R = k[\partial_1, \dots, \partial_r]$ be a polynomial ring in r variables with the standard grading over a field k. As usual, we denote by R_d the k-vector space spanned by all monomials of total degree d. Then $R = \oplus_{d \ge 0} R_d$, and elements in $\cup_{d\ge 0} R_d$ are called homogeneous. An ideal I in R is homogeneous if $I = \oplus_d I_d$ where $I_d = I \cap R_d$. The unique maximal homogeneous ideal in R is $\mathsf{m}_R = (\partial_1, \dots, \partial_r)$.

The graded Betti numbers β_{ij} of a homogeneous ideal I are the coefficients that appear in a graded minimal free resolution of I. We will often speak of the "shifted" graded Betti numbers, by which we mean $\hat{\beta}_{ij} = \beta_{i,i+j}$. So if $0 \to F_c \to \cdots \to F_1$ is a graded minimal free resolution of I, then the i^{th} term is

$$F_i \cong \bigoplus_{j \ge i} \beta_{ij} R(-j) = \bigoplus_{j\ge 0} \hat{\beta}_{ij} R(-i-j) \tag{1.1}$$

In particular, β_{1j} is the minimal number of generators of I of degree j.

Let $\mathcal{R} = \oplus_{d\ge 0} \mathcal{R}_d$ be the graded dual of R, i.e. $\mathcal{R}_d = \operatorname{Hom}_k(R_d, k)$. It is called a ring of divided powers, and we write $\mathcal{R} = k[x_1, \dots, x_r]^{DP}$. Let $\mathbb{N}_0$ denote the non-negative integers. The divided power monomials

$$\left\{ x^{(\alpha)} = \prod_{i=1}^{r} x_i^{(\alpha_i)} \;\middle|\; \alpha \in \mathbb{N}_0^r \text{ and } |\alpha| = \sum_{i=1}^{r} \alpha_i = d \right\}$$

form a basis for $\mathcal{R}_d$ as a k-vector space. This basis is dual to the standard monomial basis for R_d, i.e. $\{\partial^\beta = \Pi_i\, \partial_i^{\alpha_i} \mid \beta \in \mathbb{N}_0^r \text{ and } |\beta| = d\}$, in the sense that $x^{(\alpha)}\big(\partial^\alpha\big) = 1$ and $x^{(\alpha)}\big(\partial^\beta\big) = 0$ for $\alpha \ne \beta$. The ring structure of $\mathcal{R}$ is the natural one generated by

$$x_i^{(a)} \cdot x_i^{(b)} = \tbinom{a+b}{a}\, x_i^{(a+b)},$$

see [Eis95, Section A2.4] or [IK99, Appendix A] for details. We will refer to elements of $\mathcal{R}_d$ simply as homogeneous polynomials or forms of degree d. If

$\operatorname{char} k = 0$, we may identify $\mathcal{R}$ with the regular polynomial ring $k[x_1, \dots, x_r]$ by letting $x_i^{(d)} = x_i^d/d!$

Let R act on $\mathcal{R}$ by

$$\partial^\beta\big(x^{(\alpha)}\big) = x^{(\alpha-\beta)},$$

i.e. the action generated by $\partial_i\big(x_i^{(d)}\big) = x_i^{(d-1)}$ and $\partial_j\big(x_i^{(d)}\big) = 0$ for all $i \neq j$. The reason for our notation is that ∂_i is indeed a derivation, which follows by bilinearity from

$$\begin{aligned}
&\partial_i\big(x_i^{(a)}\big) \cdot x_i^{(b)} + x_i^{(a)} \cdot \partial_i\big(x_i^{(b)}\big) \\
&\qquad\qquad = x_i^{(a-1)} \cdot x_i^{(b)} + x_i^{(a)} \cdot x_i^{(b-1)} \\
&\qquad\qquad = \tbinom{a+b-1}{a-1} x_i^{(a+b-1)} + \tbinom{a+b-1}{a} x_i^{(a+b-1)} \\
&\qquad\qquad = \tbinom{a+b}{a} x_i^{(a+b-1)} = \tbinom{a+b}{a} \partial_i\big(x_i^{(a+b)}\big) \\
&\qquad\qquad = \partial_i\big(x_i^{(a)} \cdot x_i^{(b)}\big).
\end{aligned}$$

Under the identification $x_i^{(d)} = x_i^d/d!$ when $\operatorname{char} k = 0$, the action of ∂_i becomes normal partial differentiation with respect to x_i.

Arrange the elements of the standard monomial bases for $\mathcal{R}_d$ and R_d into column vectors h and D using the same ordering. The fact that they are dual can then be expressed as $Dh^\mathsf{T} = I$, the identity matrix. If $\{f_1, \dots, f_N\}$ is any basis for $\mathcal{R}_d$, $N = \dim_k \mathcal{R}_d = \binom{r-1+d}{d}$, then there is a dual basis for R_d. Indeed, there exists an $N \times N$ invertible matrix P such that $f = [f_1, \dots, f_N]^\mathsf{T} = P^\mathsf{T} h$. Let $E = P^{-1}D$. Then $Ef^\mathsf{T} = P^{-1}Dh^\mathsf{T}P = I$, hence E is the dual basis of f (as column vectors).

If S is any ring, let $\operatorname{Mat}_S(a,b)$ be the set of $a \times b$ matrices defined over S, and let $\mathrm{GL}_r(S)$ be the invertible $r \times r$ matrices. When $S = k$, we usually just write GL_r. We will frequently make use of the following convention.

> If $v \in S^b$ is any vector and $A \in \operatorname{Mat}_S(a,b)$ any matrix, we denote by v_i the i^{th} entry of v and by A_{ij} the $(i,j)^{\text{th}}$ entry of A.

In particular, $(Av)_i = \sum_{j=1}^b A_{ij} v_j$ is the i^{th} entry of the vector Av, and the $(i,j)^{\text{th}}$ entry of the rank one matrix $(Av)(Bv)^\mathsf{T}$ is $(Avv^\mathsf{T}B^\mathsf{T})_{ij} = (Av)_i(Bv)_j$.

For any $P \in \mathrm{GL}_r$, define $\phi_P : \mathcal{R} \to \mathcal{R}$ to be the k-algebra homomorphism induced by $x_i \mapsto \sum_{j=1}^r P_{ji} x_j$ for all i. We usually let x denote the column vector

$x = [x_1, \dots x_r]^\mathsf{T}$, thus ϕ_P is induced by $x \mapsto P^\mathsf{T}x$. The "dual" map $R \to R$, which we also denote by ϕ_P, is induced by $\partial \mapsto P^{-1}\partial$, where $\partial = [\partial_1, \dots, \partial_r]^\mathsf{T}$. For any $D \in R$ and $f \in \mathcal{R}$, it follows that

$$\phi_P(Df) = (\phi_P D)(\phi_P f),$$

and in particular, $\operatorname{ann}_R(\phi_P f) = \phi_P(\operatorname{ann}_R f)$.

If $D \in \operatorname{Mat}_R(a,b)$ and $h \in \operatorname{Mat}_\mathcal{R}(b,c)$, then Dh denotes the $a \times c$ matrix whose $(i,j)^{\text{th}}$ entry is $(Dh)_{ij} = \sum_{k=1}^b D_{ik}(h_{kj}) \in \mathcal{R}$. Of course, this is nothing but the normal matrix product, where multiplication is interpreted as the action of R and $\mathcal{R}$. We already used this notation when discussing dual bases. Also, for any $f \in \mathcal{R}$, we let $D(f)$ (or simply Df) denote the $a \times b$ matrix whose $(i,j)^{\text{th}}$ entry is $(Df)_{ij} = D_{ij}(f) \in \mathcal{R}$. It follows that if $E \in \operatorname{Mat}_R(a', a)$, then $E(D(f)) = (ED)(f)$.

If $A \in \operatorname{Mat}_R(a,b)$ and $v_i \in R^a$ is the i^{th} column vector in A, then we let $I_k(A) = I_k(v_1 \ \cdots \ v_b)$ be the ideal generated by all $k \times k$ minors of A ($k \le a, b$). Of course, this only depends on $\operatorname{im} A = \langle v_1, \dots, v_b \rangle = \{\sum_{i=1}^b c_i v_i \,|\, c_1, \dots, c_b \in k\}$.

1.2 Annihilator ideals and Gorenstein quotients

Given any k-vector subspace $V \subseteq \mathcal{R}_d$, define its orthogonal $V^\perp \subseteq R_d$ by

$$V^\perp = \{D \in R_d \,|\, Df = 0 \;\forall\, f \in V\}.$$

Similarly, if $U \subseteq R_d$, define $U^\perp = \{f \in \mathcal{R}_d \,|\, Df = 0 \;\forall\, D \in U\}$.

Let $n = \dim_k V$ and $N = \dim_k \mathcal{R}_d = \dim_k R_d$. Pick a basis $\{f_1, \dots, f_n\}$ for V, and expand it to a basis $\{f_1, \dots, f_N\}$ for $\mathcal{R}_d$. Let $\{D_1, \dots, D_N\}$ be the dual basis for R_d. Clearly, $V^\perp = \langle D_{n+1}, \dots, D_N \rangle$, the k-vector subspace of R_d spanned by $D_{n+1}, \dots, D_N$. Therefore,

$$\dim_k V + \dim_k V^\perp = \dim_k R_d.$$

By symmetry, this equation is true also when applied to $V^\perp$, that is, we get $\dim_k V^\perp + \dim_k V^{\perp\perp} = \dim_k R_d$. Hence it follows that $\dim_k V^{\perp\perp} = \dim_k V$. Since $V^{\perp\perp} = \{g \in \mathcal{R}_d \,|\, Dg = 0 \;\forall\, D \in V^\perp\}$ obviously contains V, we have in fact

$V^{\perp\perp} = V$. Note in particular that $\mathcal{R}_d^{\perp} = 0$ and $R_d^{\perp} = 0$. This says precisely that the pairing (k-bilinear map) $R_d \times \mathcal{R}_d \to k$ defined by $(D, f) \mapsto D(f)$ is non-degenerate.

Definition 1.1: For any $f \in \mathcal{R}_d$, $d \geq 0$, the *annihilator ideal* in R of f is defined to be

$$\operatorname{ann}_R(f) = \{D \in R \mid Df = 0\}.$$

Since f is homogeneous, $\operatorname{ann}_R(f)$ is a homogeneous ideal in R. We notice that its degree d part $\operatorname{ann}_R(f)_d$ is equal to $\langle f \rangle^{\perp}$ as defined above. The annihilator ideals have several nice properties.

First, consider the homomorphism $R_e \to \mathcal{R}_{d-e}$ defined by $D \mapsto D(f)$. We denote its image by

$$R_e(f) = \{D(f) \mid D \in R_e\},$$

and its kernel is by definition $\operatorname{ann}_R(f)_e$. We observe that if $R_e(f) = 0$ for some $e < d = \deg f$, then $R_d(f) = 0$ because $R_d = R_{d-e} \cdot R_e$. Since $R_d \times \mathcal{R}_d \to k$ is non-degenerate, this implies $f = 0$. Thus the contraction map $R_e \times \mathcal{R}_d \to \mathcal{R}_{d-e}$ is also non-degenerate. The R-module $R(f) = \oplus_{e \geq 0} R_e(f)$ is called the module of contractions.

Lemma 1.2: *Let $d, e \geq 0$ and $f \in \mathcal{R}_d$. The ideal $\operatorname{ann}_R(f) \subseteq R$ satisfies:*

(a) *If $0 \leq k \leq e \leq d$, then the degree k part $\operatorname{ann}_R(f)_k$ is determined by the degree e part $\operatorname{ann}_R(f)_e$ by "saturation", that is, $D \in \operatorname{ann}_R(f)_k$ if and only if $ED \in \operatorname{ann}_R(f)_e$ for all $E \in R_{e-k}$.*

(b) *$R_e(f) \cong R_e / \operatorname{ann}_R(f)_e$ and $R_e(f)^{\perp} = \operatorname{ann}_R(f)_{d-e}$.*

(c) *$\dim_k(R/\operatorname{ann}_R(f))_e = \dim_k R_e(f) = \dim_k(R/\operatorname{ann}_R(f))_{d-e}$.*

(d) *$\cap_{D \in R_e} \operatorname{ann}_R(Df) = \operatorname{ann}_R(f) + R_d + \cdots + R_{d-e+1}$.*
In particular, $\cap_{D \in R_e} \operatorname{ann}_R(Df)_{d-e} = \operatorname{ann}_R(f)_{d-e}$.

Proof: To prove (a), let $D \in R_k$. Since $R_{d-e} \times \mathcal{R}_{d-e} \to k$ is non-degenerate, it follows for any $E \in R_{e-k}$ that $ED(f) = 0$ if and only if $E'ED(f) = 0$ for all $E' \in R_{d-e}$. Therefore, $ED(f) = 0$ for all $E \in R_{e-k}$ if and only if $E''D(f) = 0$ for all $E'' \in R_{d-k}$, which is equivalent to $D(f) = 0$ since $R_{d-k} \times \mathcal{R}_{d-k} \to k$ is non-degenerate. Thus

$$\operatorname{ann}_R(f)_k = \{D \in R_k \mid R_{e-k} \cdot D \subseteq \operatorname{ann}_R(f)_e\},$$

i.e. $\operatorname{ann}_R(f)_k$ is determined by $\operatorname{ann}_R(f)_e$ by "saturation".

The first part of (b) follows immediately from the exact sequence

$$0 \to \operatorname{ann}_R(f)_e \to R_e \to R_e(f) \to 0.$$

Since $R_e(f) \subseteq \mathcal{R}_{d-e}$, it follows from (a) that

$$\begin{aligned} R_e(f)^\perp &= \{D \in R_{d-e} \mid D(Ef) = 0 \text{ for all } E \in R_e\} \\ &= \{D \in R_{d-e} \mid D(f) = 0\} = \operatorname{ann}_R(f)_{d-e}. \end{aligned}$$

And (c) follows from (b) by taking dimensions of the two equalities. Note that

$$\begin{aligned} \bigcap_{D \in R_e} \operatorname{ann}_R(Df)_{d-e} &= \{E \in R_{d-e} \mid E(Df) = 0 \text{ for all } D \in R_e\} \\ &= R_e(f)^\perp = \operatorname{ann}_R(f)_{d-e}. \end{aligned}$$

Now (d) follows by "saturating downwards" due to (a). (Obviously, it is enough to use a basis for R_e in the intersection.) □

Let $f \in \mathcal{R}_d$. The *Hilbert function* $H_f = H(R/\operatorname{ann}_R f)$ of $R/\operatorname{ann}_R(f)$ computes the dimensions of the graded components of $R/\operatorname{ann}_R(f)$, i.e.

$$H_f(e) = \dim_k(R/\operatorname{ann}_R f)_e \text{ for all } e \geq 0.$$

Note that (c) implies that the Hilbert function of $R/\operatorname{ann}_R(f)$ is symmetric about $d/2$. Since $H_f(e) = 0$ for all $e > d$, we will often abuse notation and write $H_f = (h_0, \dots, h_d)$ where $h_e = H_f(e)$. Written this way, H_f is sometimes called the *h-vector* of $R/\operatorname{ann}_R f$.

A finitely generated k-algebra A is *Artinian* if and only if it has finite dimension as a k-vector space. Let $I \subseteq R$ be a homogeneous ideal. Then $A = R/I$ is Artinian if and only if $I_e = R_e$ for all $e \gg 0$. Its *socle* is defined by $\operatorname{Socle}(R/I) = (0 : \mathsf{m}_R)$, i.e. $\operatorname{Socle}(R/I) = \oplus_{e \geq 0} \operatorname{Socle}_e(R/I)$ where $\operatorname{Socle}_e(R/I) = \{D \in R_e \mid \partial_i D \in I_{e+1} \text{ for all } i = 1, \dots, r\}/I_e$. Furthermore, $\operatorname{Hom}_k(-, k)$ is a *dualizing functor* for A, hence its *canonical module* is

$$\omega_A = \operatorname{Hom}_k(A, k) = \bigoplus_{e \geq 0} \operatorname{Hom}_k(A_e, k).$$

A is called *Gorenstein* if $\omega_A \cong A$ (up to a twist). By [Eis95, proposition 21.5], $A = R/I$ is Gorenstein if and only if its socle is simple, i.e. $\dim_k \operatorname{Socle}(R/I) = 1$. By [Eis95, proposition 21.16] this is equivalent to the minimal free resolution of A being self-dual.

Lemma 1.3 (Macaulay): *There is a one-to-one correspondence between graded Artinian Gorenstein quotients R/I having socle degree d, and non-zero polynomials $f \in \mathcal{R}_d$ up to a scalar multiplum. The correspondence is given by $I = \operatorname{ann}_R f$ and $\langle f \rangle = (I_d)^\perp$.*

Proof: See [Eis95, Theorem 21.6 and Exercise 21.7] or [IK99, Lemma 2.14]. Macaulay's original proof in [Mac16, chapter IV] uses *inverse systems.* □

Note that it is customary to call $\operatorname{ann}_R(f)$ a *Gorenstein ideal* since the quotient $R/\operatorname{ann}_R f$ is Gorenstein. We conclude these preliminaries with the following fundamental lemma. It expresses the effect of dualizing ($V \mapsto V^\perp$) an inclusion $U \subseteq V$ in terms of annihilator ideals.

Lemma 1.4 (Apolarity): *Let $f_1, \dots, f_n, g_1, \dots, g_m \in \mathcal{R}_d$ be forms of the same degree d. Then the following statements are equivalent:*

(a) $\langle f_1, \dots, f_n \rangle \subseteq \langle g_1, \dots, g_m \rangle$

(b) $\cap_{i=1}^n \operatorname{ann}_R(f_i) \supseteq \cap_{i=1}^m \operatorname{ann}_R(g_i)$

(c) $\cap_{i=1}^n \operatorname{ann}_R(f_i)_d \supseteq \cap_{i=1}^m \operatorname{ann}_R(g_i)_d$

Proof: (a) just says that all f_i can be written as $f_i = \sum_{j=1}^m c_{ij} g_j$ for suitable $c_{ij} \in k$. So if $D \in R$ annihilates all g_j, it necessarily kills all f_i, which proves (a) $\Rightarrow$ (b). (b) $\Rightarrow$ (c) is trivial, and (c) $\Rightarrow$ (a) follows from $V^{\perp\perp} = V$ and

$$\langle f_1, \dots, f_n \rangle^\perp = \{D \in R_d \mid D(f_i) = 0 \,\forall\, i\} = \bigcap_{i=1}^{n} \operatorname{ann}_R(f_i)_d.$$ □

Remark 1.5: What is more often called the *apolarity lemma*, for example [IK99, Lemma 1.15], follows from lemma 1.4 by letting $n = 1$ and $g_i = l_{p_i}^{(d)}$, $l_{p_i} = \sum_j p_{ij} x_j$, with the additional observation that $D(l_{p_i}^{(d)}) = D(p_i) l_{p_i}^{(d-e)}$ for all $D \in R_e$.

Chapter 2

Additive splitting

2.1 What is an additive splitting?

We would like to say that a polynomial like $f = x_1^{(2)} x_2^{(2)} + x_3^{(4)}$ *splits* since it is a sum of two polynomials, $x_1^{(2)} x_2^{(2)}$ and $x_3^{(4)}$, that do not share any variable. Of course, we want to allow a change of variables. Therefore, we need to make the idea of "polynomials in separate variables" more precise.

Definition 2.1: Let $g_1, \dots, g_n \in \mathcal{R}$ be homogeneous polynomials, and for all i let $d_i = \deg g_i$. We say that $g_1, \dots, g_n$ are polynomials in (linearly) *independent sets of variables* if

$$R_{d_i-1}(g_i) \cap \Bigl(\sum_{j \neq i} R_{d_j-1}(g_j) \Bigr) = 0$$

as subspaces of $\mathcal{R}_1$ for all $i = 1, \dots, n$.

Remark 2.2: Let $f \in \mathcal{R}_d$. It is natural to say that $R_{d-1}(f)$ contains the "native" variables of f for the following reason. If $V \subseteq \mathcal{R}_1$ is a k-vector subspace, denote by $k[V]^{DP}$ the k-subalgebra of $\mathcal{R}$ generated by V. If $v_1, \dots, v_n$ is any basis for V, then $k[V]^{DP} = k[v_1, \dots, v_n]^{DP}$. In particular, $k[V]_0^{DP} = k$ and $k[V]_1^{DP} = V$. For all $\delta \in R_{d-1}(f)^\perp \subseteq R_1$ and all $D \in R_{d-1}$, it follows that $D\delta f \in \delta(R_{d-1}(f)) = 0$. Hence $\delta f = 0$ for all $\delta \in R_{d-1}(f)^\perp$, and therefore

$$f \in k[R_{d-1}(f)]^{DP}.$$

Thus definition 2.1 simply requires that the sets of native variables of $g_1, \dots, g_n$

are linearly independent, that is, if $\sum_{i=1}^n c_i v_i = 0$ for some $v_i \in R_{d_i-1}(g_i)$ and $c_i \in k$, then $c_i = 0$ for all i.

Remark 2.3: We note that definition 2.1 implies that

$$R_{d_i-e}(g_i) \cap \Big(\sum_{j \neq i} R_{d_j-e}(g_j) \Big) = 0$$

for all $i = 1, \dots, n$ and all $e > 0$. Indeed, if $h \in R_{d_i-e}(g_i) \cap \big(\sum_{j\neq i} R_{d_j-e}(g_j)\big)$, then $D(h) \in R_{d_i-1}(g_i) \cap \big(\sum_{j\neq i} R_{d_j-1}(g_j)\big) = 0$ for all $D \in R_{e-1}$, hence $h = 0$.

Definition 2.4: Let $f \in \mathcal{R}_d$. We say that f *splits regularly* $n-1$ times if f is a sum of n non-zero forms of degree d in independent sets of variables. That is, if there exist non-zero $g_1, \dots, g_n \in \mathcal{R}_d$ such that

$$f = g_1 + \cdots + g_n,$$

and for all i, $R_{d-1}(g_i) \cap \big(\sum_{j\neq i} R_{d-1}(g_j)\big) = 0$ as subspaces of $\mathcal{R}_1$. In this situation, we call the g_i's *additive components* of f, and we say that the expression $f = g_1 + \cdots + g_n$ is a *regular splitting of length* n.

Clearly, this concept is uninteresting for $d = 1$. For $d = 2$ and $\operatorname{char} k \neq 2$ it is well known that any $f \in \mathcal{R}_2$ can be written as a sum of $n = \operatorname{rank}(\partial\partial^{\mathsf{T}} f)$ squares. (When $\operatorname{char} k = 2$ it is in general only a limit of a sum of n squares). Consequently, we will concentrate on $d \geq 3$.

Example 2.5: Let $\operatorname{char} k \neq 2$ and $f = x^{(3)} + xy^{(2)} \in k[x,y]^{DP}$. Then

$$f = \tfrac{1}{2}\big((x+y)^{(3)} + (x-y)^{(3)}\big)$$

is a regular splitting of f of length 2. Indeed, $R_2\big((x+y)^{(3)}\big) = \langle x+y \rangle$ and $R_2\big((x-y)^{(3)}\big) = \langle x-y \rangle$, and their intersection is zero.

Remark 2.6: When f splits regularly, it is possible to separate the variables of its components by a suitable "rectifying" automorphism. More precisely, $f \in \mathcal{R}_d$ splits regularly $n-1$ times if and only if there exists $\mathcal{J}_1, \dots, \mathcal{J}_n \subseteq \{1, \dots, r\}$ such that $\mathcal{J}_i \cap \mathcal{J}_j = \varnothing$ for all $i \neq j$, a graded automorphism $\phi : \mathcal{R} \to \mathcal{R}$ and nonzero polynomials $h_i \in \mathcal{S}_d^i$ where $\mathcal{S}^i = k[\{x_j \mid j \in \mathcal{J}_i\}] \subseteq \mathcal{R}$, such that $\phi(f) = h_1 + \cdots + h_n$.

To prove this, assume that $f = g_1 + \cdots + g_n$ is a regular splitting of f. By definition, $R_{d-1}(g_i) \cap \big(\sum_{j \neq i} R_{d-1}(g_j)\big) = 0$ for all i. This simply means that $R_{d-1}(g_1), \dots, R_{d-1}(g_n)$ are linearly independent subspaces of $\mathcal{R}_1$, that is, if $\sum_{i=1}^n c_i v_i = 0$ for some $v_i \in R_{d-1}(g_i)$ and $c_i \in k$, then $c_i = 0$ for all i. Let $s_i = \dim_k R_{d-1}(g_i)$. Then in particular, $\sum_{i=1}^n s_i = \dim_k\big(\sum_{i=1}^n R_{d-1}(g_i)\big) \leq r$. Hence we may choose $\mathcal{J}_1, \dots \mathcal{J}_n \subseteq \{1, \dots, r\}$ such that $|\mathcal{J}_i| = s_i$ and $\mathcal{J}_i \cap \mathcal{J}_j = \varnothing$ for all $i \neq j$. Now, choose a graded automorphism $\phi : \mathcal{R} \to \mathcal{R}$ such that $\{\phi^{-1}(x_j) \,|\, j \in \mathcal{J}_i\}$ is a basis for $R_{d-1}(g_i)$ for all i, and let $h_i = \phi(g_i) \in \mathcal{R}_d$ and $\mathcal{S}^i = k[\{x_j \,|\, j \in \mathcal{J}_i\}]^{DP}$. Obviously, $h_i \neq 0$ and $\phi(f) = \sum_i \phi(g_i) = \sum_i h_i$. Thus we only have to prove that $h_i \in \mathcal{S}^i$ for all i. We note that

$$\mathcal{S}_1^i = \phi(R_{d-1}(g_i)) = (\phi R_{d-1})(\phi g_i) = R_{d-1}(h_i).$$

Therefore, for all $j \notin \mathcal{J}_i$ and $D \in R_{d-1}$, we have $\partial_j D(h_i) \in \partial_j(\mathcal{S}_1^i) = 0$. This implies that $\partial_j h_i = 0$ for all i and $j \notin \mathcal{J}_i$, and we are done.

For the converse, we immediately get $f = \sum_{i=1}^n g_i$ with $g_i = \phi^{-1}(h_i)$. Note that $R_{d-1}(g_i) = \phi^{-1}(R_{d-1}(h_i))$. Since $R_{d-1}(h_i) \subseteq \mathcal{S}_1^i$, and $\mathcal{S}_1^1, \dots, \mathcal{S}_1^n$ obviously are linearly independent subspaces of $\mathcal{R}_1$, so are $R_{d-1}(g_1), \dots, R_{d-1}(g_n)$. Thus $f = \sum_{i=1}^n g_i$ is a regular splitting.

We will also investigate how the regular splitting property specializes. For this purpose we give the following definition.

Definition 2.7: Let $f \in \mathcal{R}_d$. We say that f has a *degenerate splitting of length* m if there for some $n \geq 1$ exists an $f_t \in \mathcal{R}_d[t_1, \dots, t_n]$ such that $f_0 = f$ and f_t splits regularly $m-1$ times inside $\mathcal{R}_d(t_1, \dots, t_n) = k(t_1, \dots, t_n)[\partial_1, \dots, \partial_r]_d$.

Example 2.8: Let $f = x^{(d-1)}y \in k[x,y]^{DP}$, $d \geq 3$. Clearly $\operatorname{ann}_R f = (\partial_y^2, \partial_x^d)$. If f splits regularly, then it must be in the GL_2 orbit of $g = x^{(d)} + y^{(d)}$, and this implies that $\operatorname{ann}_R g$ is in the GL_2 orbit of $\operatorname{ann}_R f$. But $\operatorname{ann}_R(f)_2 = \langle \partial_y^2 \rangle$ and $\operatorname{ann}_R(g)_2 = \langle \partial_x \partial_y \rangle$, hence this is impossible.

Still, even though f does not split regularly, f has a degenerate splitting. For example, f is a specialization of

$$f_t = \tfrac{1}{t}\big[(x+ty)^{(d)} - x^{(d)}\big] = x^{(d-1)}y + tx^{(d-2)}y^{(2)} + \dots,$$

and f_t splits inside $k(t)[x,y]^{DP}$.

2.2 The associated algebra M_f

The starting point of this section is the definition of a regular splitting. We will see how this naturally leads to the definition of a k-vector space M_f associated to $f \in \mathcal{R}_d$. M_f consists of $r \times r$-matrices, and we prove that M_f is closed under multiplication when $d = \deg f \geq 3$. We start with a fundamental observation.

Lemma 2.9: *Let $f = g_1 + \cdots + g_n$ be a regular splitting of some $f \in \mathcal{R}_d$. Then*

$$\operatorname{ann}_R(f)_e = \bigcap_{i=1}^{n} \operatorname{ann}_R(g_i)_e \text{ for all } e < d.$$

Proof: Let $e < d$. Lemma 1.4 gives the inclusion $\operatorname{ann}_R(f)_e \supseteq \cap_{i=1}^n \operatorname{ann}_R(g_i)_e$, so we only need to prove that $\operatorname{ann}_R(f)_e \subseteq \operatorname{ann}_R(g_i)_e$ for all i. To do this, let $D \in \operatorname{ann}_R(f)_e$. Applying D to $f = \sum_{i=1}^n g_i$ gives $D(g_1) + \cdots + D(g_n) = 0$. Since $D(g_1), \ldots, D(g_n)$ are homogeneous polynomials of positive degree in separate rings, this implies $D(g_i) = 0$ for all i. Indeed, $D(g_i) = -\sum_{j \neq i} D(g_j)$ is an element of both $R_e(g_i)$ and $\sum_{j \neq i} R_e(g_j)$, and since their intersection is zero by remark 2.3, it follows that $D(g_i) = 0$. This proves that $\operatorname{ann}_R(f)_e \subseteq \operatorname{ann}_R(g_i)_e$ for all i and all $e < d$, and we are done. □

At first sight, one might think that there exist additional regular splittings of a polynomial $f \in \mathcal{R}_d$ if we allow "dummy" variables, i.e. if $\operatorname{ann}_R(f)_1 \neq 0$. But it is not so when $d \geq 2$, as we prove next. For this reason, we may freely assume $\operatorname{ann}_R(f)_1 = 0$ when studying regular splittings.

Corollary 2.10: *Let $d \geq 2$ and $f \in \mathcal{R}_d$. Every regular splitting of f takes place inside the subring $k[R_{d-1}(f)]^{DP} \subseteq \mathcal{R}$.*

Proof: Let $f = g_1 + \cdots + g_n$ be a regular splitting of f. By remark 2.2, $g_i \in k[R_{d-1}(g_i)]^{DP}$. Lemma 2.9 tells us that $\operatorname{ann}_R(f)_1 \subseteq \operatorname{ann}_R(g_i)_1$, and by duality (lemma 1.4) we get $R_{d-1}(g_i) \subseteq R_{d-1}(f)$. Thus each additive component is an element of $k[R_{d-1}(f)]^{DP}$. □

Remark 2.11: Let $f = g_1 + \cdots + g_n$ be a regular splitting of $f \in \mathcal{R}_d$. Lemma 2.9 tells us that $\operatorname{ann}_R(f)_e = \cap_{i=1}^n \operatorname{ann}_R(g_i)_e$ for all $e < d$. Using duality, this is equivalent to $R_{d-e}(f) = \sum_{i=1}^n R_{d-e}(g_i)$ for all $e < d$. In particular, we have $R_{d-1}(f) = R_{d-1}(g_1) + \cdots + R_{d-1}(g_n)$ when $d \geq 2$.

Let $\mathcal{S} = k[R_{d-1}(f)]^{DP}$ and $\mathcal{S}^i = k[R_{d-1}(g_i)]^{DP}$ for $i = 1, \ldots, n$. Since $R_{d-1}(g_i) \cap \big(\sum_{j \neq i} R_{d-1}(g_j)\big) = 0$ and $\sum_i R_{d-1}(g_i) = R_{d-1}(f)$, we get

$$\mathcal{S}^1 \otimes_k \cdots \otimes_k \mathcal{S}^n = \mathcal{S} \subseteq \mathcal{R}.$$

Obviously, $f \in \mathcal{S}^1_d \oplus \ldots \oplus \mathcal{S}^n_d$. Hence we have another characterization of a regular splitting: An $f \in \mathcal{R}_d$ splits regularly $n-1$ times if and only if there exist non-zero k-vector subspaces $V_1, \ldots, V_n \subseteq \mathcal{R}_1$ such that $V_i \cap \big(\sum_{j \neq i} V_j\big) = 0$ for all i and $\sum_{i=1}^n V_i = R_{d-1}(f)$, and $f \in \mathcal{S}^1_d \oplus \ldots \oplus \mathcal{S}^n_d$ where $\mathcal{S}^i = k[V_i]^{DP}$.

By lemma 2.9, if we want to split an $f \in \mathcal{R}_d$, we have to look for $g \in \mathcal{R}_d$ such that $\operatorname{ann}(f)_e \subseteq \operatorname{ann}(g)_e$ for all $e < d$. The next lemma investigates this relationship. Recall that ∂ denotes the column vector $\partial = [\partial_1, \ldots, \partial_r]^\mathsf{T}$, thus $\partial f = [\partial_1 f, \ldots, \partial_r f]^\mathsf{T}$.

Lemma 2.12: *Given $f, g \in \mathcal{R}_d$, the following are equivalent:*

(a) $\operatorname{ann}(f)_e \subseteq \operatorname{ann}(g)_e$ *for all* $e < d$,

(b) $\operatorname{ann}(f)_{d-1} \subseteq \operatorname{ann}(g)_{d-1}$,

(c) *there exists a matrix* $A \in \operatorname{Mat}_k(r, r)$ *such that* $\partial g = A \partial f$,

(d) $R_1 \cdot \operatorname{ann}(f)_{d-1} \subseteq \operatorname{ann}(g)_d$,

(e) $\mathsf{m} \cdot \operatorname{ann}(f) \subseteq \operatorname{ann}(g)$.

Proof: (a) $\Leftrightarrow$ (b) is immediate by lemma 1.2. The same lemma also tells us that $\cap_{i=1}^r \operatorname{ann}(\partial_i f) = \operatorname{ann}(f) + R_d$, which means that (b) just says that

$$\bigcap_{i=1}^{r} \operatorname{ann}(\partial_i f)_{d-1} \subseteq \bigcap_{i=1}^{r} \operatorname{ann}(\partial_i g)_{d-1}.$$

By lemma 1.4, this is equivalent to $\langle \partial_1 g, \ldots \partial_r g \rangle \subseteq \langle \partial_1 f, \ldots \partial_r f \rangle$, and (c) just expresses this in vector form. (b) $\Leftrightarrow$ (d) since $R_1^{-1} \operatorname{ann}_R(g)_d = \operatorname{ann}_R(g)_{d-1}$, again by lemma 1.2a. Finally, lemma 1.2a also shows that (d) $\Leftrightarrow$ (e), since $(\mathsf{m} \cdot \operatorname{ann}(f))_e = \sum_k \mathsf{m}_k \cdot \operatorname{ann}(f)_{e-k} = R_1 \cdot \operatorname{ann}(f)_{e-1}$. □

Let $f \in \mathcal{R}_d$. Both the previous lemma and the next lemma study the equation $\partial g = A \partial f$. In the previous we gave equivalent conditions on $g \in \mathcal{R}_d$ for $A \in \operatorname{Mat}_k(r, r)$ to exist. The next lemma tells us when g exists given A. Recall that is B is any matrix, then $I_k(B)$ denotes the ideal generated by all $k \times k$-minors of B.

Lemma 2.13: *Let $f \in \mathcal{R}_d$ and $A \in \mathrm{Mat}_k(r,r)$. The following are equivalent:*

(a) *There exists $g \in \mathcal{R}_d$ such that $\partial g = A\partial f$,*

(b) *$A\partial\partial^{\mathsf{T}}(f)$ is a symmetric matrix,*

(c) *$I_2(\partial\, A\partial) \subseteq \operatorname{ann} f$.*

Furthermore, if $d > 0$, then a $g \in \mathcal{R}_d$ satisfying $\partial g = A\partial f$ is necessarily unique.

Proof: It is well known that a set $\{g_i\}_{i=1}^r$ can be lifted to a g such that $\partial_i g = g_i$ if and only if $\partial_j g_i = \partial_i g_j$ for all i, j. This condition simply says that $\partial[g_1, \dots, g_r]$ is a symmetric matrix. Let $g_i = (A\partial f)_i$, that is, g_i is the i^{th} coordinate of the column vector $A\partial f$. Then the existence of g is equivalent to $A\partial\partial^{\mathsf{T}} f$ being a symmetric matrix. Thus (a) $\Leftrightarrow$ (b).

Since $(A\partial\partial^{\mathsf{T}})^{\mathsf{T}} = \partial\partial^{\mathsf{T}} A^{\mathsf{T}}$, it follows that $A\partial\partial^{\mathsf{T}}(f)$ is symmetric if and only if $(A\partial\partial^{\mathsf{T}} - \partial\partial^{\mathsf{T}} A^{\mathsf{T}})(f) = 0$. Thus (b) $\Leftrightarrow$ (c), since the $(i,j)^{\text{th}}$ entry of the matrix $(A\partial)\partial^{\mathsf{T}} - \partial(A\partial)^{\mathsf{T}}$ is $(A\partial)_i \partial_j - \partial_i (A\partial)_j$, the 2×2 minor of the $2 \times r$ matrix $(\partial\, A\partial)$ corresponding to the i^{th} and j^{th} row (up to sign). The last statement is trivial. □

Note that the 2×2 minors of $(\partial\, A\partial)$ are elements of R_2, so (c) is really a condition on $\operatorname{ann}(f)_2$. Combining lemma 2.9 with lemmas 2.12 and 2.13, we see that a regular splitting $f = g_1 + \dots, g_n$ implies the existence of matrices A satisfying $I_2(\partial\, A\partial) \subseteq \operatorname{ann}_R f$. These matrices will in fact enable us to find both regular and degenerate splittings. Thus we are naturally lead to the following definition.

Definition 2.14: Given $f \in \mathcal{R}_d$, define

$$M_f = \{A \in \mathrm{Mat}_k(r,r) \mid I_2(\partial\, A\partial) \subseteq \operatorname{ann}_R f\}.$$

Example 2.15: The notation $I_2(\partial\, A\partial)$ might be confusing, so we will consider an example with $r = 2$. Let $\mathcal{R} = k[x,y]^{DP}$ and $f = x^{(3)} + xy^{(2)} \in \mathcal{R}_3$. A quick calculation of the partials of f proves that $\operatorname{ann}_R f = (\partial_x^2 - \partial_y^2, \partial_y^3)$. We will show that the 2×2 matrix $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ belongs to M_f. Obviously,

$$A\partial = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \partial_y \\ \partial_x \end{pmatrix}.$$

The matrix $(\partial\ A\partial)$ has ∂ as its first column and $A\partial$ as its second column, so

$$(\partial\ A\partial) = \begin{pmatrix} \partial_x & \partial_y \\ \partial_y & \partial_x \end{pmatrix}.$$

Its only 2×2 minor is its determinant, $D = \partial_x^2 - \partial_y^2$, and since $D \in \operatorname{ann}_R f$, it follows by definition that $A \in M_f$.

Let us determine M_f. We need to find all matrices $B = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $I_2(\partial\ B\partial) \subseteq \operatorname{ann}_R f$. Since

$$(\partial\ B\partial) = \begin{pmatrix} \partial_x & a\partial_x + b\partial_y \\ \partial_y & c\partial_x + d\partial_y \end{pmatrix},$$

we get $I_2(\partial\ B\partial) = \big(c\partial_x^2 + (d-a)\partial_x\partial_y - b\partial_y^2\big)$. Hence $\partial_x^2 - \partial_y^2$ must divide $c\partial_x^2 + (d-a)\partial_x\partial_y - b\partial_y^2$, which is equivalent to $a = d$ and $b = c$. Therefore, M_f consists of all matrices B with $a = d$ and $b = c$, that is, $M_f = \langle I, A \rangle$.

Almost everything that we are going to study in this paper is connected to M_f. One goal is to find out what M_f can tell us about f. Before we can do this, we need investigate what properties M_f itself possesses. We will in particular show that M_f is closed under matrix multiplication when $\deg f \geq 3$. Obviously, $d \leq 1$ implies $\operatorname{ann}_R(f)_2 = R_2$, and therefore $M_f = \operatorname{Mat}_k(r, r)$. The case $d = 2$ is different, and not all of our results will apply to this case. We start with another definition.

Definition 2.16: Suppose $d > 0$ and $f \in \mathcal{R}_d$. Define a map

$$\gamma_f : M_f \to \mathcal{R}_d$$

by sending $A \in M_f$ to the unique $g \in \mathcal{R}_d$ satisfying $\partial g = A\partial f$, cf. lemma 2.13.

Note that $\partial\gamma_f(A) = A\partial f$ by definition. If $\operatorname{char} k \nmid d$, then the Euler identity ($x^{\mathsf{T}}\partial f = df$) implies that $\gamma_f(A) = \frac{1}{d}x^{\mathsf{T}}A\partial f$. By lemmas 2.9 and 2.12, the image of γ_f contains in particular all additive components of f. We will in chapter 3 see how to extract the regular splitting properties of f from M_f explicitly.

Lemma 2.17: *Let $d > 0$ and $f \in \mathcal{R}_d$, $f \neq 0$. Let β_{1e} be the minimal number of generators of $\operatorname{ann}_R(f)$ of degree e.*

(a) *M_f is a k-vector space containing the identity matrix I.*

(b) *$\gamma_f : M_f \to \mathcal{R}_d$ is k-linear.*

(c) $\dim_k \ker \gamma_f = r \cdot \beta_{11}$ *and* $\dim_k \operatorname{im} \gamma_f = 1 + \beta_{1d}$.

(d) $\dim_k M_f = 1 + \beta_{1d} + r \cdot \beta_{11}$.

Proof: Obviously, $I \in M_f$, so M_f is nonempty. And since the determinant is linear in each column, it follows that M_f is a k-vector space. Alternatively, let $A, B \in M_f$. Since $\partial\gamma_f(A) = A\partial f$, it follows for any $a, b \in k$ that

$$\partial\big(a\gamma_f(A) + b\gamma_f(B)\big) = a\partial\gamma_f(A) + b\partial\gamma_f(B) = (aA + bB)\partial f.$$

This implies that $aA+bB \in M_f$ for all $a, b \in k$, which proves (a), and furthermore that $\gamma_f(aA + bB) = a\gamma_f(A) + b\gamma_f(B)$, thus γ_f is k-linear.

Of course, $\gamma_f(A) = 0$ if and only if $A\partial f = 0$. For any $A \in \operatorname{Mat}_k(r, r)$, the equation $A\partial f = 0$ implies that $A \in M_f$, hence the kernel of γ_f consists of all such A. Recall that $(A\partial)_i$ denotes the i^{th} coordinate of the column vector $A\partial$, that is, $(A\partial)_i = a_i^{\mathsf{T}}\partial$ where a_i^{T} is the i^{th} row of A. Thus

$$\ker \gamma_f = \{A \in M_f \mid A\partial f = 0\} = \{A \in \operatorname{Mat}_k(r, r) \mid (A\partial)_i \in \operatorname{ann}_R(f)_1 \ \forall\, i\},$$

and therefore $\dim_k \ker \gamma_f = r \cdot \dim_k \operatorname{ann}(f)_1 = r\beta_{11}$.

Furthermore, by lemma 2.12, the image of γ_f are precisely those $g \in \mathcal{R}_d$ that satisfy $R_1 \cdot \operatorname{ann}(f)_{d-1} \subseteq \operatorname{ann}(g)_d$, which is equivalent to $\langle g \rangle \subseteq (R_1 \cdot \operatorname{ann}(f)_{d-1})^{\perp}$ by lemma 1.4. Since $\dim_k(R/\operatorname{ann}(f))_d = 1$, and $R_1 \cdot \operatorname{ann}(f)_{d-1}$ is a subspace of $\operatorname{ann}(f)_d$ of codimension $\dim_k(\operatorname{ann}(f)_d/R_1 \cdot \operatorname{ann}(f)_{d-1}) = \beta_{1d}$, it follows that $\dim_k \operatorname{im} \gamma_f = \operatorname{codim}_k(\mathsf{m} \cdot \operatorname{ann}(f))_d = 1 + \beta_{1d}$. This finishes part (c). (d) follows immediately. $\square$

Remark 2.18: We would like to point out that M_f is "large" only for special f. In fact, when $k = \bar{k}$ and $d \geq 4$, a general $f \in \mathcal{R}_d$ will satisfy $\beta_{11} = \beta_{1d} = 0$ (see for example [IK99, Proposition 3.12]), which implies $M_f = \langle I \rangle$. In particular, $M_f = M_g$ does not say very much by itself.

Example 2.19: Let us reconsider example 2.15. Since $\operatorname{ann}_R f = (\partial_x^2 - \partial_y^2, \partial_y^3)$, we see that $\beta_{11} = 0$ and $\beta_{13} = 1$. Lemma 2.17 implies that $\dim_k M_f = 1 + 1 = 2$.

As before, $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in M_f$ since $I_2(\partial\, A\partial) = (\partial_x^2 - \partial_y^2) \subseteq \operatorname{ann}_R f$. It follows that $M_f = \langle I, A \rangle$.

Let us also determine $\operatorname{im} \gamma_f$. Letting $g = x^{(2)}y + y^{(3)} \in \mathcal{R}_3$, we see that

$$\partial g = \begin{pmatrix} xy \\ x^{(2)} + y^{(2)} \end{pmatrix} = A\partial f.$$

Thus $A \in M_f$ and $\gamma_f(A) = g$. Obviously, $\gamma_f(I) = f$, hence $\operatorname{im} \gamma_f = \langle f, g \rangle$. This image consists of all $h \in \mathcal{R}_3$ such that $\operatorname{ann}_R(f)_2 \subseteq \operatorname{ann}_R(h)_2$. Thus another way to compute $\operatorname{im} \gamma_f$ is $(R_1 \cdot \operatorname{ann}_R(f)_2)^\perp = \langle \partial_x^3 - \partial_x\partial_y^2, \partial_x^2\partial_y - \partial_y^3 \rangle^\perp = \langle f, g \rangle$.

Remark 2.20: Before we move on, we would like to point out that there are several ways to compute M_f. One is to use the definition directly and find all $A \in \operatorname{Mat}_k(r, r)$ such that every 2×2 minor of $(\partial\, A\partial)$ is contained in $\operatorname{ann}_R(f)_2$. This can be effective when $\operatorname{ann}_R(f)_2$ is simple enough, as in example 2.15. In particular, if $\dim_k \operatorname{ann}_R(f)_2 < r - 1$, then $M_f = \langle I \rangle$. Another direct approach is to solve the system of linear equations that is contained in the statement "$A\partial\partial^\mathsf{T} f$ is symmetric". We will do this when we prove proposition 4.17.

Alternatively, we can find $\dim_k M_f$ by computing $\operatorname{ann}_R f$ and counting the number of generators of degree d, and then explicitly find the correct number of linearly independent matrices A satisfying $I_2(\partial\, A\partial) \subseteq \operatorname{ann}_R(f)$. In fact, most examples in this paper are constructed by first choosing $M \subseteq \operatorname{Mat}_k(r, r)$ and then finding $f \in \mathcal{R}$ such that $M \subseteq M_f$. Having done so, if we thereafter are able to show that $\operatorname{ann}_R f$ has no generators of degree 1 and $\dim_k M - 1$ generators of degree d, then it follows that $M_f = M$.

Note in particular that the M_f in example 2.19 is closed under matrix multiplication. This is in fact always true when $\deg f \geq 3$. We will now prove this important and a bit surprising fact about M_f.

Proposition 2.21: *Let $d \geq 3$ and $f \in \mathcal{R}_d$. M_f is a k-algebra, and all commutators belong to* $\ker \gamma_f$. *In particular, M_f is commutative if* $\operatorname{ann}(f)_1 = 0$.

Proof: We use lemmas 2.12 and 2.13 several times. Let $A, B \in M_f$. Since $B \in M_f$, there exists $g \in \mathcal{R}_d$ such that $\partial g = B\partial f$. Now $I_2(\partial\, A\partial) \subseteq R\operatorname{ann}(f)_2$, and $\operatorname{ann}(f)_2 \subseteq \operatorname{ann}(g)_2$ since $d \geq 3$. Hence $A \in M_g$, and there exists $h \in \mathcal{R}_d$ such that $\partial h = A\partial g$. Then $\partial h = AB\partial f$, thus $AB \in M_f$. Furthermore, since $A\partial\partial^{\mathsf{T}}(f)$, $B\partial\partial^{\mathsf{T}}(f)$ and $AB\partial\partial^{\mathsf{T}}(f)$ are all symmetric, we get

$$AB\partial\partial^{\mathsf{T}}(f) = \partial\partial^{\mathsf{T}}(f)(AB)^{\mathsf{T}} = \partial\partial^{\mathsf{T}}(f)B^{\mathsf{T}}A^{\mathsf{T}} = B\partial\partial^{\mathsf{T}}(f)A^{\mathsf{T}} = BA\partial\partial^{\mathsf{T}}(f).$$

Hence $(AB - BA)\partial\partial^{\mathsf{T}}f = 0$. Note that $C\partial\partial^{\mathsf{T}}f = 0 \Leftrightarrow (C\partial)_i\partial_j f = 0$ for all $i, j \Leftrightarrow C\partial f = 0$. Thus $(AB - BA)\partial f = 0$, and therefore $\gamma_f(AB - BA) = 0$. If $\operatorname{ann}(f)_1 = 0$, then it follows that $AB = BA$. □

Remark 2.22: When $d \geq 3$ it also follows for all $A, B \in M_f$ that

$$A\partial\partial^{\mathsf{T}}(f)B^{\mathsf{T}} = AB\partial\partial^{\mathsf{T}}(f) = \partial\partial^{\mathsf{T}}(f)B^{\mathsf{T}}A^{\mathsf{T}} = B\partial\partial^{\mathsf{T}}(f)A^{\mathsf{T}}.$$

Thus $(A\partial)(B\partial)^{\mathsf{T}}(f)$ is symmetric, which implies that $I_2(A\partial\ B\partial) \subseteq \operatorname{ann} f$, cf. lemma 2.13.

Example 2.23: Let $r = 3$, $d \geq 3$ and $f = x_1^{(d-1)}x_3 + x_1^{(d-2)}x_2^{(2)}$. First, let us determine $\operatorname{ann}_R f$. Clearly, $\operatorname{ann}_R(f)_1 = 0$, and a straightforward computation shows that $\operatorname{ann}_R(f)_2 = \langle \partial_3^2, \partial_2\partial_3, \partial_1\partial_3 - \partial_2^2 \rangle$. We note that these polynomials are the maximal minors of

$$\begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ \partial_2 & \partial_3 & 0 \end{pmatrix}.$$

By Hilbert-Burch the ideal $J = R\operatorname{ann}_R(f)_2$ defines a scheme of length 3 in $\mathbb{P}^2$. Indeed, $\partial_2^3 = \partial_1(\partial_2\partial_3) - \partial_2(\partial_1\partial_3 - \partial_2^2) \in J$, and this implies for every $e \geq 2$ that $(R/J)_e$ is spanned by (the images of) ∂_1^e, $\partial_1^{e-1}\partial_2$ and $\partial_1^{e-2}\partial_2^2$. Since $\partial_1^e(f)$, $\partial_1^{e-1}\partial_2(f)$ and $\partial_1^{e-2}\partial_2^2(f)$ are linearly independent for all $2 \leq e < d$, it follows that $\dim_k(R/J)_e = 3$ for all $e > 1$, and that $\operatorname{ann}_R(f)_e = J_e$ for all $1 < e < d$. Thus $\operatorname{ann}_R f$ needs exactly two generators of degree d, and we get

$$\operatorname{ann}_R f = (\partial_3^2, \partial_2\partial_3, \partial_1\partial_3 - \partial_2^2, \partial_1^{d-1}\partial_2, \partial_1^d).$$

Let

$$A = \left(\begin{smallmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{smallmatrix}\right).$$

We have just seen that $\operatorname{ann}_R(f)_2$ is generated by the 2×2 minors of $(\partial\, A\partial)$, hence $A \in M_f$. Because M_f is closed under multiplication, we also have $A^2 \in M_f$. By looking at $\operatorname{ann}_R f$, we see that $\beta_{11} = 0$ and $\beta_{1d} = 2$. Thus $\dim_k M_f = 3$, and it follows that $M_f = \langle I, A, A^2 \rangle$.

Remark 2.24: The "formula" for the annihilator ideal $\operatorname{ann}_R f$ in example 2.23 is true even for $d = 2$. In this case $\operatorname{ann}_R f$ has five generators of degree 2, thus M_f will be 6-dimensional. In fact, since in this case

$$\partial\partial^{\mathsf{T}} f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

it follows that M_f consists of all matrices that are symmetric about the anti-diagonal. Thus M_f is no longer closed under multiplication.

Remark 2.25: With A as in example 2.23, it is easy to determine all $g \in \mathcal{R}_d$ such that $A \in M_g$. Indeed, if $I_2(\partial\, A\partial) \subseteq \operatorname{ann}_R g$ for some $g \in \mathcal{R}_d$, then $\operatorname{ann}_R(g)_e \supseteq \operatorname{ann}_R(f)_e$ for all $e < d$ since the 2×2 minors of $(\partial\, A\partial)$ are the only generators of $\operatorname{ann}_R f$ of degree less than d. It follows that

$$\{g \in \mathcal{R}_d \mid A \in M_g\} = \operatorname{im} \gamma_f = \{af + bx_1^{(d-1)}x_2 + cx_1^{(d)} \mid a, b, c \in k\}.$$

If in addition $\operatorname{ann}_R(g)_1 = 0$, then $a \neq 0$, implying that g is in the GL_3 orbit of f ($\operatorname{char} k \nmid d$).

One natural question to ask is the following:

> Which subalgebras of $\operatorname{Mat}_k(r, r)$ arise as M_f for different $f \in \mathcal{R}_d$?

We have not been able to determine this in general, but we will in the remainder of this chapter point out some restrictions on M_f. We start with the following result, which holds even for $d < 3$.

Proposition 2.26: *Suppose $d \geq 0$ and $f \in \mathcal{R}_d$. Let $A, B \in \operatorname{Mat}_k(r, r)$ and $C \in M_f$. Assume that $AC, BC \in M_f$ and $BAC = ABC$. Then $A^iB^jC \in M_f$ for all $i, j \geq 0$. In particular, M_f is always closed under exponentiation.*

Proof: Lemma 2.13 says that $A \in M_f$ if and only if $A\partial\partial^{\mathsf{T}} f$ is symmetric. Thus all three matrices $C\partial\partial^{\mathsf{T}} f$, $AC\partial\partial^{\mathsf{T}} f$ and $BC\partial\partial^{\mathsf{T}} f$ are symmetric. It follows that

$$ABC\partial\partial^{\mathsf{T}} f = A\partial\partial^{\mathsf{T}} f C^{\mathsf{T}} B^{\mathsf{T}} = AC\partial\partial^{\mathsf{T}} f B^{\mathsf{T}} = \partial\partial^{\mathsf{T}} f C^{\mathsf{T}} A^{\mathsf{T}} B^{\mathsf{T}} = \partial\partial^{\mathsf{T}} f (ABC)^{\mathsf{T}},$$

hence $ABC \in M_f$, and we are done by induction. The last statement follows by letting $B = C = I$. Note that we have not assumed $d \geq 3$ here. $\square$

When $d \geq 3$ one might wonder if the assumptions $C, AC \in M_f$ actually implies that $A \in M_f$. If so, the conclusion of the previous proposition would immediately follow from the fact that M_f is closed under multiplication when $d \geq 3$. But M_f does not support division, in the sense that $C, AC \in M_f$ does not generally imply $A \in M_f$, as seen in the following example.

Example 2.27: Let $r = 4$ and $f = x_1^{(d-1)}x_4 + x_1^{(d-2)}x_2x_3 + x_2^{(d)}$. Then

$$\operatorname{ann}_R f = (\partial_1\partial_4 - \partial_2\partial_3, \partial_2\partial_4, \partial_3^2, \partial_3\partial_4, \partial_4^2, \partial_1\partial_2^2, \partial_1^{d-2}\partial_3 - \partial_2^{d-1}, \partial_1^d, \partial_1^{d-1}\partial_2).$$

This implies that $\dim_k M_f = 3$ when $d \geq 4$. Let

$$A = \begin{pmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}.$$

It is easy to verify that $A^2, A^3 \in M_f$, thus $M_f = \langle I, A^2, A^3 \rangle$ when $d \geq 4$. In particular, $A \notin M_f$, even though $A^2, A^3 \in M_f$.

We will finish this section with a result computing some special elements of $\operatorname{ann}_R f$. We start with a lemma.

Lemma 2.28: *Let $d \geq 0$ and $f \in \mathcal{R}_d$. Pick $A_1, \dots, A_m, B_1, \dots, B_n \in M_f$, and let $u \in \sum_{k=1}^m \operatorname{im} A_k^\mathsf{T} + \sum_{k=1}^n \ker B_k^\mathsf{T}$ and $v \in \big(\cap_{k=1}^m \ker A_k^\mathsf{T}\big) \cap \big(\cap_{k=1}^n \operatorname{im} B_k^\mathsf{T}\big)$. Then*

$$(u^\mathsf{T}\partial) \cdot (v^\mathsf{T}\partial) \in \operatorname{ann}_R f.$$

Proof: The proof rests on the following equation. If $A \in M_f$ and $b = [b_1, \dots, b_r]^\mathsf{T}$ and $c = [c_1, \dots, c_r]^\mathsf{T}$ are two vectors, then

$$\begin{vmatrix} b^\mathsf{T}\partial & b^\mathsf{T}A\partial \\ c^\mathsf{T}\partial & c^\mathsf{T}A\partial \end{vmatrix} = \begin{vmatrix} \sum_i b_i\partial_i & \sum_i b_i(A\partial)_i \\ \sum_j c_j\partial_j & \sum_j c_j(A\partial)_j \end{vmatrix} = \sum_{i,j=1}^r b_ic_j \begin{vmatrix} \partial_i & (A\partial)_i \\ \partial_j & (A\partial)_j \end{vmatrix},$$

and therefore

$$(b^\mathsf{T}\partial) \cdot (c^\mathsf{T}A\partial) - (b^\mathsf{T}A\partial) \cdot (c^\mathsf{T}\partial) \in \operatorname{ann}_R f. \tag{2.1}$$

By definition of u there exist $a_1, \dots a_m, b_1, \dots b_n \in k^r$ such that $B_k^\mathsf{T} b_k = 0$ and $u = \sum_{k=1}^m A_k^\mathsf{T} a_k + \sum_{k=1}^n b_k$. Furthermore, $A_k^\mathsf{T} v = 0$ and $v = B_1^\mathsf{T} c_1 = \cdots = B_n^\mathsf{T} c_n$ for some $c_1, \dots, c_n \in k^r$. Putting $(A, b, c) = (A_k, a_k, v)$ in (2.1), and using $A_k^\mathsf{T} v = 0$, implies $(a_k^\mathsf{T} A_k\partial)(v^\mathsf{T}\partial) \in \operatorname{ann}_R f$. Letting $(A, b, c) = (B_k, b_k, c_k)$ gives $(b_k^\mathsf{T}\partial)(v^\mathsf{T}\partial) \in \operatorname{ann}_R f$ since $B_k^\mathsf{T} b_k = 0$ and $B_k^\mathsf{T} c_k = v$. Adding these equations together proves that $(u^\mathsf{T}\partial) \cdot (v^\mathsf{T}\partial) \in \operatorname{ann}_R f$. □

The next proposition gives us a restriction on M_f when $\operatorname{ann}_R(f)_1 = 0$. We will use this in chapter 4.

Proposition 2.29: *Let $d \geq 2$ and $f \in \mathcal{R}_d$. Pick $A_1, \dots, A_m, B_1, \dots, B_n \in M_f$, and define*

$$U = \sum_{k=1}^{m} \operatorname{im} A_k^{\mathsf{T}} + \sum_{k=1}^{n} \ker B_k^{\mathsf{T}} \quad \text{and} \quad V = \Big(\bigcap_{k=1}^{m} \ker A_k^{\mathsf{T}}\Big) \cap \Big(\bigcap_{k=1}^{n} \operatorname{im} B_k^{\mathsf{T}}\Big).$$

Assume that (a) $U+V = k^r$ and $U \cap V \neq 0$, or (b) $\dim_k U = r-1$ and $\dim_k V \geq 2$. Then $\operatorname{ann}_R(f)_1 \neq 0$.

Proof: (a) Let $u \in U \cap V$. Since $u \in U$, lemma 2.28 implies for all $v \in V$ that $(u^{\mathsf{T}}\partial) \cdot (v^{\mathsf{T}}\partial) \in \operatorname{ann}_R f$. Because $u \in V$, we get $(u^{\mathsf{T}}\partial) \cdot (v^{\mathsf{T}}\partial) \in \operatorname{ann}_R f$ for all $v \in U$ by the same lemma. Now $U + V = k^r$ implies that $(u^{\mathsf{T}}\partial) \cdot R_1 \in \operatorname{ann}_R f$, hence $(u^{\mathsf{T}}\partial) \in \operatorname{ann}_R f$.

(b) If $V \nsubseteq U$, then $U + V = k^r$, and we are done by part (a). Thus we assume that $V \subseteq U$. Choose $u_1, u_2 \in V$, $u_1 \nparallel u_2$. Expand this to a basis $\{u_1, \dots, u_{r-1}\}$ for U, and choose $u_r \notin U$. Then $\{u_1^{\mathsf{T}}\partial, \dots, u_r^{\mathsf{T}}\partial\}$ is a basis for R_1. Let $\{l_1, \dots, l_r\}$ be the dual basis for $\mathcal{R}_1$. Since $(u^{\mathsf{T}}\partial)(u_1^{\mathsf{T}}\partial) \in \operatorname{ann}_R f$ for all $u \in U$, it follows that $u_1^{\mathsf{T}}\partial f = c_1 l_r^{(d-1)}$ for some $c_1 \in k$. Similarly, $u_2^{\mathsf{T}}\partial f = c_2 l_r^{(d-1)}$. Thus $(c_2u_1 - c_1u_2)^{\mathsf{T}}\partial f = 0$, and $\operatorname{ann}_R(f)_1 \neq 0$. □

Example 2.30: We will give an example of each of the two cases of proposition 2.29. In both cases, let $r = 3$, $d \geq 2$ and $f \in \mathcal{R}_d$.

1. Let $B_1 = \left(\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix}\right)$ and $B_2 = \left(\begin{smallmatrix} 0&0&0\\0&0&1\\0&0&0 \end{smallmatrix}\right)$, and assume that $B_1, B_2 \in M_f$. Then

   $$\operatorname{ann}_R f \supseteq I_2(\partial\, B_1\partial) + I_2(\partial\, B_2\partial) = (\partial_1\partial_3, \partial_2\partial_3, \partial_3^2) = \partial_3 \cdot \mathsf{m}_R.$$

   Hence $\partial_3 \in \operatorname{ann}_R(f)_1$, and $\operatorname{ann}_R(f)_1 \neq 0$. This belongs to case (a) of proposition 2.29 (with $A_i = 0$ for all i).

2. Let $A_1 = \left(\begin{smallmatrix} 0&1&0\\0&0&0\\0&0&0 \end{smallmatrix}\right)$ and $A_2 = \left(\begin{smallmatrix} 0&0&1\\0&0&0\\0&0&0 \end{smallmatrix}\right)$, and assume that $A_1, A_2 \in M_f$. Then

   $$\operatorname{ann}_R f \supseteq I_2(\partial\, A_1\partial) + I_2(\partial\, A_2\partial) = (\partial_2^2, \partial_2\partial_3, \partial_3^2) = (\partial_2, \partial_3)^2.$$

   Thus $f = c_1x_1^{(d)} + c_2x_1^{(d-1)}x_2 + c_3x_1^{(d-1)}x_3$, and therefore, $\operatorname{ann}_R(f)_1 \neq 0$. This is case (b) of proposition 2.29 (with $B_i = I$ for all i).

M_f has other properties that further restrict the subalgebras that arise as M_f, and we will say a little more about this in the next section.

2.3 Determinantal ideals

We mentioned in remark 2.20 that most examples in this paper are constructed by first choosing a subset (usually a subalgebra) $M \subseteq \mathrm{Mat}_k(r,r)$. Having chosen M, we can compute $X_d = \{f \in \mathcal{R}_d \mid M_f \supseteq M\}$, and finally choose one of these f to present as the example.

We now take a closer look at this method. Given a subset $M \subseteq \mathrm{Mat}_k(r,r)$, we will define an ideal $I(M)$ and an R-module $X(M)$. Studying $I(M)$ and $X(M)$ can be thought of as an alternative to studying all M_f that contain M, and we will make this connection precise. However, the first half of this section will only deal with $I(M)$ and a related ideal $\check{I}(M)$.

Definition 2.31: Let M be any subset of $\mathrm{Mat}_k(r,r)$. Let $I(M)$ and $\check{I}(M)$ be the ideals in R defined by

$$I(M) = \sum_{A \in M} I_2(\partial\, A\partial) \quad \text{and} \quad \check{I}(M) = \sum_{A,B \in M} I_2(A\partial\, B\partial).$$

Note that the ideal $I(M_f)$ is the part of $\mathrm{ann}_R f$ that determines M_f. Obviously, if M is a k-vector space, and $A_1, \dots, A_n$ is a basis for M, then

$$\check{I}(M) = \sum_{i<j} I_2(A_i\partial\, A_j\partial) = I_2(A_1\partial\, A_2\partial\, \dots\, A_n\partial).$$

Thus $\check{I}(M)$ is the ideal generated by the 2×2 minors of a matrix of linear forms. Conversely, if φ is any matrix of linear forms, then $I_2(\varphi) = \check{I}(M)$ for suitable M. We realize that $\check{I}(M)$ is a very general object. In this section we will usually require that the identity matrix I is in M. (Actually, it would be enough to assume that M contains an invertible matrix, but this is not important to us.) We start with a result relating $\check{I}(M)$ and the simpler object $I(M)$.

Lemma 2.32: *Assume $I \in M \subseteq \mathrm{Mat}_k(r,r)$. Then $I(M) \subseteq \check{I}(M) = I(M^2)$ and $I(M)_e = \check{I}(M)_e$ for all $e \geq 3$. In particular, if M is closed under matrix multiplication, then $I(M) = \check{I}(M)$.*

Proof: $I(M) \subseteq \check{I}(M)$ is immediate when $I \in M$. Let $A, B \in M$, and consider the determinant

$$D = \begin{vmatrix} \partial_i & (A\partial)_i & (B\partial)_i \\ \partial_j & (A\partial)_j & (B\partial)_j \\ \partial_k & (A\partial)_k & (B\partial)_k \end{vmatrix}.$$

By expanding along the third column, we get $D \in I(M)$. Thus expansion along the first row shows that

$$\partial_i \cdot \begin{vmatrix} (A\partial)_j & (B\partial)_j \\ (A\partial)_k & (B\partial)_k \end{vmatrix} \in I(M) \text{ for all } i, j \text{ and } k.$$

Therefore, $\mathsf{m}_R \check{I}(M) \subseteq I(M)$. Since $\check{I}(M)$ is generated in degree 2, it follows that $\check{I}(M)_e = I(M)_e$ for all $e \geq 3$. Furthermore, since $(A\partial)_j = \sum_{k=1}^r A_{jk}\partial_k$, we get

$$\sum_{k=1}^r A_{jk} \begin{vmatrix} \partial_i & (B\partial)_i \\ \partial_k & (B\partial)_k \end{vmatrix} = \begin{vmatrix} \partial_i & (B\partial)_i \\ (A\partial)_j & (AB\partial)_j \end{vmatrix} = \partial_i \cdot (AB\partial)_j - (A\partial)_j \cdot (B\partial)_i$$

and therefore,

$$\sum_{k=1}^r A_{jk} \begin{vmatrix} \partial_i & (B\partial)_i \\ \partial_k & (B\partial)_k \end{vmatrix} - \sum_{k=1}^r A_{ik} \begin{vmatrix} \partial_j & (B\partial)_j \\ \partial_k & (B\partial)_k \end{vmatrix} = \begin{vmatrix} \partial_i & (AB\partial)_i \\ \partial_j & (AB\partial)_j \end{vmatrix} + \begin{vmatrix} (A\partial)_i & (B\partial)_i \\ (A\partial)_j & (B\partial)_j \end{vmatrix}. \quad (2.2)$$

Hence, if $B \in M$, then $I_2(A\partial\, B\partial) \subseteq I(M)$ if and only if $I_2(\partial\, AB\partial) \subseteq I(M)$. In particular, $\check{I}(M) = I(M^2)$, since $I \in M$ implies $M \subseteq M^2$. If M is closed under multiplication, then also $M^2 \subseteq M$, implying $I(M) = \check{I}(M)$. □

We note that $\check{I}(M) = I(M)$ when M is closed under multiplication. If M is not closed, it is natural to ask if we can close M and not change the ideal $\check{I}(M)$. This is true, as the following proposition shows.

Proposition 2.33: *Assume $I \in M \subseteq \mathrm{Mat}_k(r, r)$. Let M' be the k-subalgebra of $\mathrm{Mat}_k(r, r)$ generated by M. Then $I(M') = \check{I}(M)$.*

Proof: We have not assumed that M is a k-vector space. It is just any subset of $\mathrm{Mat}_k(r, r)$ containing the identity matrix I. Therefore, its powers are defined as $M^k = \{\Pi_{i=1}^k A_i \mid A_i \in M \text{ for all } i\}$, and not the linear span. Note that $M^k \subseteq M^{k+1}$ since $I \in M$. Because $\mathrm{Mat}_k(r, r)$ is a finite-dimensional vector space, it follows that $M' = \langle M^k \rangle$, the linear span of M^k, for large k. Since a minor is linear in each column, we get $I(\langle M^k \rangle) = I(M^k)$. Thus to prove that $I(M') = \check{I}(M)$, it is enough to show that $I_2(\partial\, A\partial) \subseteq \check{I}(M)$ for all $A \in M^k$ for all $k \gg 0$.

For every $A, B \in \mathrm{Mat}_k(r, r)$ and all $1 \leq i < j \leq r$, define $(A, B)_{ij} \in R_2$ by $(A, B)_{ij} = (A\partial)_i \cdot (B\partial)_j$. We will usually suppress the subscripts. Note that

$$\begin{vmatrix} (AC\partial)_i & (AD\partial)_i \\ (BC\partial)_j & (BD\partial)_j \end{vmatrix} = \sum_{k,l=1}^r A_{ik}B_{jl} \begin{vmatrix} (C\partial)_k & (D\partial)_k \\ (C\partial)_l & (D\partial)_l \end{vmatrix}.$$

Thus $(AC, BD) - (AD, BC) \in I_2(C\partial\, D\partial)$, and if $I_2(C\partial\, D\partial) \subseteq \check{I}(M)$, then

$$(AC, BD) = (AD, BC) \mod \check{I}(M). \tag{2.3}$$

Assume that $I_2(X\partial\, Y\partial) \subseteq \check{I}(M)$ for all $X, Y \in \{I, A, B, C\}$. We want to show that $I_2(\partial\, ABC\partial) \subseteq \check{I}(M)$. This is equivalent to $(ABC, I)_{ij} = (I, ABC)_{ij}$ mod $\check{I}(M)$ for all i and j. To prove this, we will use equation (2.3) eight times, and each time one of the matrices will be I. Indeed, modulo $\check{I}(M)$ we have

$$\begin{aligned}(ABC, I) &= (AB, C) = (A, CB) = (B, CA)\\ &= (BA, C) = (BC, A) = (B, AC) = (C, AB) = (I, ABC).\end{aligned}$$

The rest is a simple induction. We know that $I_2(\partial A\partial) \subseteq \check{I}(M)$ for all $A \in M^2$. Assume for some $k \geq 2$ that $I_2(\partial\, A\partial) \subseteq \check{I}(M)$ for all $A \in M^k$. Then by equation (2.2) also $I_2(A\partial\, B\partial) \subseteq \check{I}(M)$ for all $A \in M^i$ and $B \in M^j$ as long as $i + j \leq k$. Pick $A' = \Pi_{i=1}^{k+1} A_i \in M^{k+1}$. Let $A = A_1$, $B = \Pi_{i=2}^{k} A_i$ and $C = A_{k+1}$ so that $ABC = A'$. The induction hypothesis and the previous paragraph imply that $I_2(\partial\, A'\partial) \subseteq \check{I}(M)$. Hence we are done by induction on k. □

One consequence of lemma 2.32 and proposition 2.33 is that $\{I(M)\}$ does not change much if we restrict our attention to subsets $M \subseteq \mathrm{Mat}_k(r, r)$ that are k-algebras. Indeed, if $M \subseteq \mathrm{Mat}_k(r, r)$ is any subset containing the identity matrix I, and M' is the k-algebra generated by M, then $I(M)_e = I(M')_e$ for all $e \geq 3$. Thus these ideals can only be different in degree two.

Another consequence is the following corollary.

Corollary 2.34: *Let $A_1, \ldots, A_n \in \mathrm{Mat}_k(r, r)$ and $M = k[A_1, \ldots, A_n]$. Then*

$$I(M) = I_2(\partial\, A_1\partial\, \cdots\, A_n\partial).$$

Proof: M is the k-algebra generated by $\{I, A_1, \ldots, A_n\} \subseteq \mathrm{Mat}_k(r, r)$, and the result follows from proposition 2.33. □

We now associate to any subset $M \subseteq \mathrm{Mat}_k(r, r)$ a graded R-module $X(M)$. When we defined $M_f = \{A \in \mathrm{Mat}_k(r, r) \mid I_2(\partial\, A\partial) \subseteq \mathrm{ann}_R f\}$ in definition 2.14, we required f to a homogeneous polynomial. To simplify the following definition and results, we will allow any $f \in \mathcal{R}$. Of course, if $f = \sum_{k\geq 0} f_k$ and $f_k \in \mathcal{R}_k$, then $M_f = \cap_{k\geq 0} M_{f_k}$, since $I_2(\partial\, A\partial)$ is a homogeneous ideal.

Definition 2.35: Let $M \subseteq \operatorname{Mat}_k(r,r)$. Define the graded R-module $X(M)$ by

$$X(M) = \{f \in \mathcal{R} \,|\, M \subseteq M_f\}.$$

The discussion before the definition explains why $X(M)$ is a graded k-vector subspace of $\mathcal{R}$. Note that $\operatorname{ann}_R(f) \subseteq \operatorname{ann}_R(Df)$ for any $D \in R$. This implies that $M_f \subseteq M_{Df}$, thus $X(M)$ is indeed an R-module. $X(M)$ is closely connected to $I(M)$, as seen in the following lemma.

***Lemma* 2.36:** *Let $M \subseteq \operatorname{Mat}_k(r,r)$ be any subset. Then*

(a) $M \subseteq M_f$ *if and only if* $I(M) \subseteq \operatorname{ann}_R f$,

(b) $X_d(M) = \{f \in \mathcal{R}_d \,|\, R_{d-2}(f) \subseteq X_2(M)\}$ *for all* $d \geq 3$,

(c) $I(M)_d^\perp = X_d(M)$ *for all* $d \geq 0$,

(d) $I(M) = \cap_{f\in X(M)} I(M_f) = \cap_{f \in X(M)} \operatorname{ann}_R f$.

In particular, $I_2(\partial\, A\partial) \subseteq I(M)$ *if and only if* $A \in M_f$ *for all* $f \in X(M)$.

Proof: Clearly, $I(M) \subseteq \operatorname{ann}_R f$ if and only if $I_2(\partial\, A\partial) \subseteq \operatorname{ann}_R f$ for all $A \in M$, which is equivalent to $M \subseteq M_f$. This is (a).

Let $X = X(M)$. Pick $f \in \mathcal{R}_d$, $d \geq 3$. Since $I(M)$ is generated in degree two and $\operatorname{ann}_R(f)_{d-1} = \cap_{i=1}^r \operatorname{ann}_R(\partial_i f)_{d-1}$, it follows that $M_f = \cap_{i=1}^r M_{\partial_i f}$. Hence $f \in X_d$ if and only if $\partial_i f \in X_{d-1}$ for all i, and by induction this is equivalent to $Df \in X_2$ for all $D \in R_{d-2}$. This proves (b).

For all $d \geq 0$ we have $I(M)_d^\perp = \{f \in \mathcal{R}_d \,|\, Df = 0 \;\forall\, D \in I(M)\}$, which equals X_d by (a). For any $f \in X$ we note that $I(M) \subseteq I(M_f) \subseteq \operatorname{ann}_R f$, hence $I(M) \subseteq \cap_{f\in X} I(M_f) \subseteq \cap_{f\in X} \operatorname{ann}_R f$. Furthermore, by (c),

$$I(M)_d = X_d^\perp = \{D \in R_d \,|\, Df = 0 \;\forall\, f \in X_d\} = \bigcap_{f \in X_d} \operatorname{ann}_R(f)_d.$$

Thus $I(M)_d \supseteq (\cap_{f\in X} \operatorname{ann}_R f)_d$, which implies (d). In particular, it follows that $I_2(\partial\, A\partial) \subseteq I(M)$ if and only if $I_2(\partial\, A\partial) \subseteq \operatorname{ann}_R f$ for all $f \in X$, and this is equivalent to $A \in M_f$ for all $f \in X$. □

Remark 2.37: A consequence of lemma 2.36 is that results about M_f often correspond to results about $I(M)$. For example, we know that M_f is a k-algebra for all $f \in \mathcal{R}_d$, $d \geq 3$ (proposition 2.21). This corresponds to the fact that $I(M^2)_d \subseteq I(M)_d$ for all $d \geq 3$ when $I \in M$ (lemma 2.32).

To prove this, let $d \geq 3$ and $f \in \mathcal{R}_d$, and pick $A, B \in M_f$. Consider $M = \{I, A, B\} \subseteq M_f$. We have $I(M^2)_d \subseteq I(M)_d \subseteq \operatorname{ann}_R(f)_d$. Since $\operatorname{ann}_R(f)_2$ is determined by $\operatorname{ann}_R(f)_d$ by lemma 1.2a, and $I_2(\partial\, AB\partial) \subseteq I(M^2)$, we get $I_2(\partial\, AB\partial) \subseteq \operatorname{ann}_R f$. Hence $AB \in M_f$.

Conversely, let $A, B \in M$. Then $A, B \in M_f$ for all $f \in X = X(M)$, implying $AB \in M_f$ for all $f \in X_d$, $d \geq 3$. Hence $I_2(\partial\, AB\partial)_d \subseteq \cap_{f \in X_d} \operatorname{ann}_R(f)_d = I(M)_d$ for all $d \geq 3$, that is, $I(M^2)_d \subseteq I(M)_d$. Thus even though the proofs of these two results look very different, they actually imply each other.

As promised, we give another result that restricts which algebras that arise as M_f. The conclusion of this proposition does not in general follow from the other results we have proven about M_f.

Proposition 2.38: *Suppose $A_0, \dots, A_n \in M_f$. Let a_{ij} be the j^{th} column of A_i^T. (So $A_i = [a_{i1}, \dots, a_{ir}]^\mathsf{T}$, i.e. $(A_i)_{jk} = (a_{ij})_k$ for all i, j, k.) Let $s < r$. Assume that $a_{ij} = 0$ for all $i \geq 1$ and $j \leq s$, and that $a_{0j} \in \langle a_{1j}, \dots, a_{nj} \rangle$ for all $j > s$. Then $B = [a_{01}, \dots, a_{0s}, 0 \dots, 0]^\mathsf{T} \in M_f$.*

Proof: Let $M = \{A_0, \dots, A_n\}$. We want to prove that $I(M)$ contains every 2×2 minor of $(\partial\, B\partial)$. If $i, j \leq s$, then

$$\begin{vmatrix} \partial_i & (B\partial)_i \\ \partial_j & (B\partial)_j \end{vmatrix} = \begin{vmatrix} \partial_i & (A_0\partial)_i \\ \partial_j & (A_0\partial)_j \end{vmatrix} \in I(M).$$

If $i, j > s$, then this minor is obviously zero. So we are left with the case $i \leq s$ and $j > s$. By assumption $a_{0j} \in \langle a_{1j}, \dots, a_{nj} \rangle$, thus $a_{0j} = \sum_{k=1}^n c_{kj} a_{kj}$ for suitable $c_{kj} \in k$. It follows that

$$\begin{aligned} \begin{vmatrix} \partial_i & (B\partial)_i \\ \partial_j & (B\partial)_j \end{vmatrix} = \begin{vmatrix} \partial_i & a_{0i}^\mathsf{T}\partial \\ \partial_j & 0 \end{vmatrix} &= \begin{vmatrix} \partial_i & a_{0i}^\mathsf{T}\partial \\ \partial_j & a_{0j}^\mathsf{T}\partial \end{vmatrix} - \sum_{k=1}^n c_{kj} \begin{vmatrix} \partial_i & 0 \\ \partial_j & a_{kj}^\mathsf{T}\partial \end{vmatrix} \\ &= \begin{vmatrix} \partial_i & (A_0\partial)_i \\ \partial_j & (A_0\partial)_j \end{vmatrix} - \sum_{k=1}^n c_{kj} \begin{vmatrix} \partial_i & (A_k\partial)_i \\ \partial_j & (A_k\partial)_j \end{vmatrix} \in I(M). \end{aligned}$$

Therefore, $I_2(\partial\, B\partial) \subseteq I(M)$. Since $I(M) \subseteq \operatorname{ann}_R f$, this implies $B \in M_f$. □

Example 2.39: The assumptions in proposition 2.38 might seem a bit strange. One situation where it can be used, is the following. Let $c_1 + c_2 + c_3 = r$. For $i = 1, 2, 3$, pick $C_i \in \mathrm{Mat}_k(c_i, r)$, and define $B_i \in \mathrm{Mat}_k(r, r)$ by

$$B_1 = \begin{pmatrix} C_1 \\ 0 \\ 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} C_1 \\ C_2 \\ 0 \end{pmatrix} \qquad B_3 = \begin{pmatrix} C_1 \\ 0 \\ C_3 \end{pmatrix}.$$

Assume that $B_2, B_3 \in M_f$. If we apply proposition 2.38 with $A_0 = B_2$ and $A_1 = B_2 - B_3$, we get $B_1 \in M_f$. A special case when $r = 6$ is

$$B_1 = \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B_2 = \begin{pmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B_3 = \begin{pmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C \end{pmatrix},$$

where $C = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. As above, $B_2, B_3 \in M_f$ implies $B_1 \in M_f$.

We will end this section with a result identifying $Z(I(M)) \subseteq \mathbb{P}^{r-1}$, the set of closed points of the projective scheme determined by $I(M)$.

Proposition 2.40: *Suppose $M \subseteq \mathrm{Mat}_k(r, r)$. Then*

$$Z(I(M)) = \{v \in k^r \setminus \{0\} \mid v \textit{ is an eigenvector for every } A \in M\}/k^*.$$

Proof: By definition, $I(M) = \sum_{A \in M} I_2(\partial\, A\partial)$. Thus a $v \in k^r$ satisfies $D(v) = 0$ for all $D \in I(M)$ if and only if

$$\begin{vmatrix} v_i & (Av)_i \\ v_j & (Av)_j \end{vmatrix} = 0 \text{ for all } i < j \text{ and all } A \in M.$$

This is equivalent to v being an eigenvector for every $A \in M$. Thus $Z(I(M))$ is simply the projectivization of the union of the eigenspaces. □

Chapter 3

Regular splittings

This chapter covers our work on regular splittings. The first half deals with how to find such splittings. Then in section 3.3 we study how a regular splitting affects the Artinian Gorenstein quotient $R/\operatorname{ann}_R f$. In fact, if $f = \sum_{i=1}^n g_i$ is a regular splitting of f, then we express the Hilbert function and, more generally, the (shifted) graded Betti numbers of $R/\operatorname{ann}_R f$ in terms of those for $R/\operatorname{ann}_R(g_i)$, $i = 1, \dots, n$. To get there, we calculate the minimal free resolution of $R/\operatorname{ann}_R f$.

Section 3.4 concerns $\mathbf{PGor}(H)$, the space parameterizing all graded Artinian Gorenstein quotients R/I with Hilbert function H. We define a subset parameterizing those $R/\operatorname{ann}_R f$ where f splits regularly, and we compute its dimension and the dimension of the tangent space to $\mathbf{PGor}(H)$ at the point $R/\operatorname{ann}_R f$.

One goal of this paper is to study what M_f can tell us about $f \in \mathcal{R}_d$, and in section 3.2 we show how to extract from M_f the regular splitting properties of f. By corollary 2.10, any regular splitting of f happens inside the subring $k[R_{d-1}(f)]^{DP} \subseteq \mathcal{R}$. Thus we may assume that $\operatorname{ann}_R(f)_1 = 0$ by performing a suitable base change and reducing the number of variables, if necessary. If in addition $d \geq 3$, proposition 2.21 tells us that M_f is a commutative k-algebra. This will allow us to find all regular splittings. It turns out that the idempotents in M_f determine the regular splittings, so we start by studying these.

3.1 Idempotents and matrix algebras

This section discusses idempotents in general, and in particular how they relate to matrix algebras. We will see how eigenvalues and eigenspaces are connected to idempotents. We start with some elementary definitions.

Let A be a ring with unity. A nonzero element e in A is called an *idempotent* if $e^2 = e$. A subset $\{e_1, \ldots, e_n\} \subseteq A$ is a set of *orthogonal idempotents* in A if $e_i^2 = e_i \neq 0$ for all i and $e_i e_j = 0$ for all $i \neq j$. The set is *complete* if in addition $\sum_{i=1}^n e_i = 1$. If $\{e_1, \ldots, e_n\}$ is not complete, let $e_0 = 1 - \sum_{i=1}^n e_i \neq 0$. Then

$$e_i e_0 = e_i - e_i \sum_{j=1}^n e_j = e_i - e_i^2 = 0 = e_0 e_i$$

for all $i > 0$, and $e_0^2 = (1 - \sum_{i=1}^n e_i) e_0 = e_0$. Thus e_0 is an idempotent, and $\{e_0, \ldots, e_n\}$ is a complete set of orthogonal idempotents.

We define a *coid* to be a set $\mathcal{E} = \{e_1, \ldots, e_n\}$ of nonzero elements of A such that $e_i e_j = 0$ for all $i \neq j$ and $\sum_{i=1}^n e_i = 1$. This implies $e_i = e_i \sum_{j=1}^n e_j = e_i^2$, thus $\mathcal{E}$ is a complete set of orthogonal idempotents (hence the name coid). We define its *length* to be $l(\mathcal{E}) = n$, the size of $\mathcal{E}$ as a set.

Assume in addition that A is a commutative ring. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ and $\mathcal{E}' = \{e'_1, \ldots, e'_m\}$ be two coids. For all $1 \leq i \leq n$ and $1 \leq j \leq m$, let $e_{ij} = e_i e'_j$. Then $\sum_{i,j} e_{ij} = (\sum_{i=1}^n e_i)(\sum_{j=1}^m e'_j) = 1$, and for all $(i,j) \neq (k,l)$, we have $e_{ij} e_{kl} = e_i e'_j e_k e'_l = (e_i e_k)(e'_j e'_l) = 0$. Thus, if e_{ij} and e_{kl} are nonzero, then they are orthogonal idempotents. In particular, they are not equal. This shows that

$$\mathcal{E} \otimes \mathcal{E}' = \{e_{ij} \mid e_{ij} \neq 0\}$$

is another coid, which we call the *product* coid. This product has the following properties.

Lemma 3.1: *Suppose A is a commutative ring with unity. Let $\mathcal{E} = \{e_1, \ldots, e_n\}$ and $\mathcal{E}' = \{e'_1, \ldots, e'_m\}$ be two coids. Then $l(\mathcal{E} \otimes \mathcal{E}') \geq l(\mathcal{E})$, and $l(\mathcal{E} \otimes \mathcal{E}') = l(\mathcal{E})$ if and only if $\mathcal{E} \otimes \mathcal{E}' = \mathcal{E}$. Furthermore, if $\mathcal{E} \otimes \mathcal{E}' = \mathcal{E}$, then $\mathcal{E}$* refines *$\mathcal{E}'$ in the sense that there exists a partition $\{\mathcal{J}_1, \ldots, \mathcal{J}_m\}$ of $\{1, \ldots, n\}$ such that $e'_j = \sum_{i \in \mathcal{J}_j} e_i$.*

Proof: For each $i = 1, \ldots, n$, at least one of $e_i e'_1, \ldots, e_i e'_m$ must be nonzero, since $\sum_{j=1}^m e_i e'_j = e_i \neq 0$. This proves that $l(\mathcal{E} \otimes \mathcal{E}') \geq l(\mathcal{E})$. It also shows that, if

$l(\mathcal{E}\otimes\mathcal{E}') = l(\mathcal{E})$, then for every i there exists a unique j_i such that $e_ie'_{j_i} \neq 0$. Then $e_i = \sum_{j=1}^{m} e_ie'_j = e_ie'_{j_i}$, hence $\mathcal{E}\otimes\mathcal{E}'$ and $\mathcal{E}$ are equal. For every $j = 1,\dots,m$, let $\mathcal{J}_j = \{i \mid j_i = j\}$. Then $\mathcal{J}_j \cap \mathcal{J}_k = \varnothing$ for all $j \neq k$, and $\mathcal{J}_1 \cup \ldots \cup \mathcal{J}_m = \{1,\dots,n\}$. Thus $\{\mathcal{J}_j\}$ is a partition of $\{1,\dots,n\}$, and $e'_j = \sum_{i=1}^{n} e_ie'_j = \sum_{i\in\mathcal{J}_j} e_i$. □

The next proposition contains what we will need to know about idempotents. First, note the following. Let V be any k-vector space, and $V_1,\dots,V_n \subseteq V$ be subspaces. When we write $V = \oplus_{i=1}^{n} V_i$, we mean that the natural map $\oplus_{i=1}^{n} V_i \to V$ defined by $(v_i) \mapsto \sum_{i=1}^{n} v_i$ is an isomorphism. This is equivalent to $\sum_{i=1}^{n} V_i = V$ and $V_i \cap \big(\sum_{j\neq i} V_j\big) = 0$ for all i.

We say that A contains a unique maximal coid if it contains a coid $\mathcal{E}$ of maximal length and every coid refines into $\mathcal{E}$, cf. lemma 3.1.

Proposition 3.2: *Let A be a commutative ring with unity.*

(a) *For every coid $\{e_1,\dots,e_n\}$, the natural map $A \to e_1A \oplus \ldots \oplus e_nA$ is an isomorphism of rings. Furthermore, every ring-isomorphism $A \to \oplus_{i=1}^{n} A_i$ arise this way up to isomorphisms of the summands A_i.*

(b) *Assume in addition that A is Noetherian. Then A contains a unique maximal coid $\mathcal{E} = \{e_1,\dots,e_n\}$. In particular, the idempotents in A are precisely the elements $e = \sum_{i\in I} e_i$ with $\varnothing \neq I \subseteq \{1,\dots,n\}$.*

(c) *Let A also be Artinian, and let $\{e_1,\dots,e_n\}$ be the unique maximal coid. For every i, the ring $A_i = e_iA$ is local Artinian, and its maximal ideal is $A_i^{\mathrm{nil}} = \{a \in A_i \mid a^k = 0 \text{ for some } k\}$, the set of nilpotent elements in A_i. In particular, A contains exactly n prime ideals.*

Proof: We note that if $e \in A$ is an idempotent, then the ideal

$$eA = \{ea \mid a \in A\} \subseteq A$$

is itself a commutative ring, with identity e. The map $a \mapsto (e_1a,\dots,e_na)$ is obviously a homomorphism of rings. Since $(a_1,\dots,a_n) \mapsto \sum_{i=1}^{n} a_i$ is an inverse, it is an isomorphism.

Assume that $A \to \oplus_{i=1}^{n} A_i$ is an isomorphism of rings. For every $i = 1,\dots,n$, let $e_i \in A$ be the element mapped to $1 \in A_i$ and $0 \in A_j$ for all $j \neq i$. Then for all $i \neq j$, we have $e_ie_j \mapsto 0$ in every component, thus $e_ie_j = 0$. And $\sum_{i=1}^n e_i = 1$ since $1 \mapsto (1, \dots, 1)$. Hence $\{e_1, \dots, e_n\}$ is a coid, and $A \to \oplus_{i=1}^n A_i$ restricts to isomorphisms $e_iA \to A_i$. Thus the map $A \to \oplus_{i=1}^n A_i$ factors through the natural map $A \to \oplus_{i=1}^n e_iA \cong \oplus_{i=1}^n A_i$. This proves (a).

We will now prove (b) in several steps. First, suppose that A contains an idempotent $e \neq 1$. Then $1 - e$ is also idempotent. Let

$$\Upsilon = \Upsilon(A) = \{a \in A \mid a^2 = a\},$$

and note that $\Upsilon(eA) = \{ea \mid ea^2 = ea\} = e\Upsilon$. It follows that the isomorphism $A \to eA \oplus (1-e)A$ restricts to a bijection $\Upsilon \to e\Upsilon \times (1-e)\Upsilon$.

Assume that A contains infinitely many idempotents. Thus Υ is infinite, and for every idempotent e, at least one of $e\Upsilon$ and $(1-e)\Upsilon$ must be infinite. Pick $e_1 \in \Upsilon \setminus \{0, 1\}$ such that $(1-e_1)\Upsilon$ is infinite. Since $(1-e_1)A$ has infinitely many idempotents, we may choose $e_2 \in (1-e_1)\Upsilon \setminus \{0, 1-e_1\}$ such that $(1-e_2)(1-e_1)\Upsilon$ is infinite. Since $e_2 \in (1-e_1)\Upsilon$, we get $e_1e_2 = 0$. We may repeat this process as many times as we like, producing elements $e_1, e_2, \dots \in A$ such that $e_i^2 = e_i \neq 0$ for all i and $e_ie_j = 0$ for all $i \neq j$. If $e_k = \sum_{i<k} a_ie_i$ for some $a_i \in A$, then $e_k^2 = \sum_{i<k} a_ie_ie_k = 0$, which is a contradiction. Hence we have produced a non-terminating, ascending sequence of ideals

$$(e_1) \subsetneq (e_1, e_2) \subsetneq (e_1, e_2, e_3) \subsetneq \dots,$$

contradicting the Noetherian hypothesis.

Since A has only finitely many idempotents, there is a coid $\mathcal{E}$ of maximal length. If $\mathcal{E}'$ is any coid, we know that $l(\mathcal{E} \otimes \mathcal{E}') \geq l(\mathcal{E})$. By the maximality of $\mathcal{E}$, it must be an equality, implying $\mathcal{E} \otimes \mathcal{E}' = \mathcal{E}$. Furthermore, $l(\mathcal{E}') \leq l(\mathcal{E} \otimes \mathcal{E}') = l(\mathcal{E})$, with equality if and only if $\mathcal{E}' = \mathcal{E}$. Hence $\mathcal{E}$ is the unique coid of maximal length. Moreover, $\mathcal{E}$ is a refinement of $\mathcal{E}'$, so any coid is obtained from $\mathcal{E}$ by "grouping" some of its elements as in lemma 3.1. In particular, if $e \neq 1$ is any idempotent, then $\{e, 1-e\}$ can be refined to $\mathcal{E} = \{e_1, \dots, e_n\}$, implying that there is a non-empty subset $I \subseteq \{1, \dots, n\}$ such that $e = \sum_{i \in I} e_i$.

To prove (c), assume that A is Artinian, and let $a \in A$. Since

$$(1) \supseteq (a) \supseteq (a^2) \supseteq (a^3) \supseteq \dots$$

becomes stationary, there is an $n \geq 0$ such that $(a^n) = (a^{n+1})$. Hence there exists $b \in A$ such that $a^n = ba^{n+1}$. It follows that $a^{n+k}b^k = a^n$ for all $k \geq 1$, and therefore, $(ab)^{2n} = (ab)^n$. If $(ab)^n = 0$, then $a^n = a^{2n}b^n = 0$. Thus either a is nilpotent, or $(ab)^n \neq 0$ is idempotent.

The ring $A_i = e_iA$ contains no non-trivial idempotents because $\{e_1, \dots, e_n\}$ is maximal. Let $P \subseteq A_i$ be a prime ideal. Obviously, P contains all nilpotents. But if $a \in A_i$ is not nilpotent, then we have just proven that a must be invertible. Thus

$$P = A_i^{\text{nil}} = \{a \in A_i \,|\, a \text{ is nilpotent}\}.$$

Clearly, an ideal $P \subseteq A = \oplus_{i=1}^n A_i$ is prime if and only if $P = P_1 \oplus \ldots \oplus P_n$ and there exists j such that P_j is a prime ideal in A_j and $P_i = A_i$ for all $i \neq j$. Since A_j has a unique prime ideal, it follows that $P_j = A_j^{\text{nil}}$. Thus A has exactly n prime ideals. □

Remark 3.3: Continuing with the notation of the proof of part (c), we see that $A_P \cong (A_j)_{P_j} \cong A_j$. Hence the decomposition $A = \oplus_{i=1}^n A_i$ is the one that is obtained in [Eis95, section 2.4] using filtrations and localizations.

Note that the ideal A_i^{nil} is nilpotent. Since A_i is Noetherian, A_i^{nil} is finitely generated, say by $a_1, \dots, a_q$. Since every a_k is nilpotent, there exists m_k such that $a_k^{m_k} = 0$. The ideal $(A_i^{\text{nil}})^m$ is generated by products $\Pi_{j=1}^m \big(\sum_{k=1}^q c_{jk}a_k\big)$. When $m > \sum_{k=1}^q (m_k - 1)$, every monomial in the expansion is necessarily zero. Thus the product is zero, proving that A_i^{nil} is a nilpotent ideal.

Remark 3.4: Note that the commutativity of A in (b) is necessary. Indeed, $\text{Mat}_k(r, r)$ contains infinitely many idempotents when $r \geq 2$ and k is infinite. For instance, $A = \left(\begin{smallmatrix} 1 & a \\ 0 & 0 \end{smallmatrix}\right)$ is idempotent for all $a \in k$.

In this paper, when we apply proposition 3.2, the ring A will usually be a matrix algebra M. In this case, the idempotents in M are closely related to the eigenspaces of M. Before we prove this, we give some definitions.

Let M be a commutative subalgebra $M \subseteq \text{Mat}_k(r, r)$, and assume that M contains the identity matrix I. We say that $v \in V = k^r$ is an *eigenvector for M* if it is an eigenvector for all $A \in M$, that is, if for every $A \in M$ there exists $\lambda_A \in k$ such that $Av = \lambda_A v$. Obviously, $v = 0$ is an eigenvector.

Fix an eigenvector $v \neq 0$. Then $Av = \lambda_A v$ determines λ_A uniquely. Consider the map $\lambda : M \to k$ defined by $\lambda(A) = \lambda_A$. Let $A, B \in M$. Since M is a k-algebra, we get $aA + bB \in M$ for all $a, b \in k$, and $AB \in M$. It follows that

$$\lambda(aA + bB)v = (aA + bB)v = aAv + bBv = \big(a\lambda(A) + b\lambda(B)\big)v.$$

Since $v \neq 0$, this implies that $\lambda(aA + bB) = a\lambda(A) + b\lambda(B)$, and similarly, $\lambda(AB) = \lambda(A)\lambda(B)$. Moreover, $\lambda(I) = 1$. Thus λ is a homomorphism of k-algebras. We call λ an *eigenvalue function* for M.

For every homomorphism $\lambda : M \to k$ of k-algebras, we define

$$U_\lambda = \{v \in V \mid Av = \lambda(A)v \text{ for all } A \in M\}.$$

Clearly, λ is an eigenvalue function for M if and only if $U_\lambda \neq 0$. U_λ is the *eigenspace* associated to λ. Let $U_\lambda^0 = 0$. Define U_λ^k for $k \geq 1$ inductively by

$$U_\lambda^k = \big\{v \in V \bigm| \big(A - \lambda(A)I\big)v \in U_\lambda^{k-1} \text{ for all } A \in M\big\}.$$

In particular, $U_\lambda^1 = U_\lambda$, the eigenspace associated to λ. Obviously, the sequence $0 \subseteq U_\lambda^1 \subseteq U_\lambda^2 \subseteq \dots$ must stabilize since V is of finite dimension. Define $V_\lambda = \sum_{k \geq 0} U_\lambda^k$, that is, $V_\lambda = U_\lambda^k$ for all $k \gg 0$. We call V_λ the *generalized eigenspace* associated to λ.

The following proposition is a spectral theorem for M.

Proposition* 3.5*: *Let $M \subseteq \operatorname{Mat}_k(r, r)$ be a commutative subalgebra containing the identity matrix I.*

(a) *M has a unique maximal complete set of orthogonal idempotents $\{E_i\}_{i=1}^n$.*

(b) *$M_i = E_i M$ is local Artinian, and its unique prime ideal is*

$$M_i^{\text{nil}} = \{A \in M_i \mid A \text{ is nilpotent}\}.$$

(c) *$M = M_1 \oplus \dots \oplus M_n$.*

(d) *$k^r = \operatorname{im} E_1 \oplus \dots \oplus \operatorname{im} E_n$.*

(e) *Let $I = \{i \mid M_i = \langle E_i \rangle \oplus M_i^{\text{nil}}\}$. There are exactly $|I|$ homomorphism $M \to k$ of k-algebras. Indeed, for each $i \in I$, the k-linear map $\lambda_i : M \to k$ defined by $\lambda_i(E_i) = 1$ and $\lambda_i(A) = 0$ for all $A \in M_i^{\text{nil}} \oplus \big(\oplus_{j \neq i} M_j\big)$ is a homomorphism of k-algebras, and there are no others. Each λ_i is an eigenvalue function, and $V_{\lambda_i} = \operatorname{im} E_i$.*

(f) $M_i = \langle E_i \rangle \oplus M_i^{\mathrm{nil}}$ *for all* i *if and only if* k *contains every eigenvalue of each* $A \in M$.

Proof: Since M has finite dimension as a k-vector space, it is Artinian. Hence (a), (b) and (c) follow immediately from proposition 3.2.

To prove (d), that is, $k^r = \operatorname{im} E_1 \oplus \ldots \oplus \operatorname{im} E_n$, it is enough to note that $v \mapsto (E_1 v, \ldots, E_n v)$ and $(v_1, \ldots, v_n) \mapsto \sum_{i=1}^n v_i$ are k-linear maps and inverses of each other.

Clearly, each λ_i is a homomorphism of k-algebras. If $\lambda : M \to K$ is any k-algebra homomorphism onto some subfield K of $\bar{k}$, then $\ker \lambda$ must be a maximal ideal in M. Thus $\ker \lambda = M_i^{\mathrm{nil}} \oplus \big(\oplus_{j \neq i} M_j\big)$ for some i. If $K = k$, then this kernel must have codimension one as a k-vector subspace of M, which implies that $M_i = \langle E_i \rangle \oplus M_i^{\mathrm{nil}}$. Obviously, $\lambda(E_i) = \lambda\big(\sum_j E_j\big) = 1$, hence $\lambda = \lambda_i$.

To prove that λ_i is an eigenvalue function, we must find a nonzero $v \in k^r$ such that $Av = \lambda_i(A)v$ for all $A \in M$. We shall in fact prove that $V_{\lambda_i} \neq 0$, since this implies $U_{\lambda_i} \neq 0$. Since $E_i \neq 0$, it is enough to prove that $V_{\lambda_i} = \operatorname{im} E_i$.

Let $v \in U_{\lambda_i}^k$. For every $j \neq i$ we have $\lambda_i(E_j) = 0$, and thus $E_j v \in U_{\lambda_i}^{k-1}$. Then $E_j^k v \in U_{\lambda_i}^0 = 0$ by induction. But $E_j^k = E_j$, hence $v \in \ker E_j$. From $v \in \cap_{j \neq i} \ker E_j$, it follows that $v = \sum_j E_j v = E_i v \in \operatorname{im} E_i$. We also note for all $j \neq i$ and $A \in M_j$ that $Av = AE_j v = 0$. Thus

$$
\begin{aligned}
U_{\lambda_i}^k &= \left\{ v \in k^r \,\middle|\, \prod_{j=1}^k \big(A_j - \lambda_i(A_j) I\big) v = 0 \text{ for all } A_1, \ldots, A_k \in M \right\} \\
&= \left\{ v \in \operatorname{im} E_i \,\middle|\, \left(\prod_{j=1}^k A_j\right)(v) = 0 \text{ for all } A_1, \ldots, A_k \in M_i^{\mathrm{nil}} \right\} \\
&= \left\{ v \in \operatorname{im} E_i \,\middle|\, Av = 0 \text{ for all } A \in \big(M_i^{\mathrm{nil}}\big)^k \right\}.
\end{aligned}
$$

Since M_i^{nil} is nilpotent, this implies $V_{\lambda_i} = \operatorname{im} E_i$, and finishes the proof of (e).

To prove (f), assume that $M_i = \langle E_i \rangle \oplus M_i^{\mathrm{nil}}$ for all i. Pick $A \in M$. For all i, since $E_i A \in M_i$, there exists $\lambda_i \in k$ such that $E_i A - \lambda_i E_i \in M_i^{\mathrm{nil}}$. Hence there exists $m_i \geq 1$ such that $(E_i A - \lambda_i E_i)^{m_i} = 0$. It follows that $E_j \Pi_{i=1}^n (A - \lambda_i I)^{m_i} = 0$ for all j. Therefore, $\Pi_{i=1}^n (A - \lambda_i I)^{m_i} = 0$. Thus the minimal polynomial of A divides $\Pi_{i=1}^n (\lambda - \lambda_i)^{m_i}$. Hence $\lambda_1, \ldots, \lambda_n$ are all of A's eigenvalues, and they are all in k.

Conversely, let $A \in M_i$. A has at least one eigenvalue $\lambda \in \bar{k}$, and by assumption, $\lambda \in k$. Thus $A - \lambda E_i \in M_i$ is not invertible. Since M_i is local, $A - \lambda E_i$ must be nilpotent, i.e. $A \in \langle E_i \rangle \oplus M_i^{\text{nil}}$. Since this is true for every $A \in M_i$, it follows that $M_i = \langle E_i \rangle \oplus M_i^{\text{nil}}$. □

Remark 3.6: If $\{E_1, \dots, E_n\}$ is a coid in $\operatorname{Mat}_k(r,r)$, then $E_1, \dots, E_n$ can easily be diagonalized simultaneously. Indeed, let $s_i = \operatorname{rank} E_i$ for all i, and

$$\mathcal{J}_i = \Big\{ j \in \mathbb{Z} \,\Big|\, \sum_{k<i} s_k < j \le \sum_{k \le i} s_k \Big\}.$$

Choose a basis $\{v_j \,|\, j \in \mathcal{J}_i\}$ for $\operatorname{im} E_i$. Since $k^r = \oplus_{i=1}^n \operatorname{im} E_i$, it follows that $\sum_{i=1}^n s_i = r$, and that $\{v_1, \dots, v_r\}$ is a basis for k^r. Hence $\{\mathcal{J}_1, \dots, \mathcal{J}_n\}$ is a partition of $\{1, \dots, r\}$, and $P = [v_1, \dots, v_r]$ is invertible.

Note that $E_i^2 = E_i$ is equivalent to $E_i v = v$ for all $v \in \operatorname{im} E_i$. Hence $E_i v_j = v_j$ for all $j \in \mathcal{J}_i$. Similarly, since $E_i E_j = 0$ for all $i \neq j$, we get $E_i v_j = 0$ for all $j \notin \mathcal{J}_i$. It follows that

$$P^{-1} E_i P = \big[P^{-1} E_i v_1, \dots, P^{-1} E_i v_1\big] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where I is the $s_i \times s_i$ identity matrix. Thus every $E_i' = P^{-1} E_i P$ is a diagonal matrix, with diagonal entries $(E_i')_{jj} = 1$ if $j \in \mathcal{J}_i$ and $(E_i')_{jj} = 0$ otherwise.

Also note that a matrix $A \in \operatorname{Mat}_k(r,r)$ commutes with every E_i', $i = 1, \dots, n$, if and only if A can be written in block diagonal form

$$A = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A_n \end{pmatrix},$$

where A_i is an $s_i \times s_i$ matrix. Furthermore,

$$E_i' \operatorname{Mat}_k(r,r) E_i' = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & A_i & 0 \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, A_i \in \operatorname{Mat}_k(s_i, s_i) \right\},$$

hence $\big\{A \in \operatorname{Mat}_k(r,r) \,\big|\, A E_i' = E_i' A \text{ for all } i\big\} = \oplus_{i=1}^n E_i' \operatorname{Mat}_k(r,r) E_i'$.

3.2 Idempotents and regular splittings

We are now ready to prove that the idempotents in M_f determine the regular splittings of f, and how they do it. The bridge between M_f and the additive components of f is the map γ_f. Recall that $\gamma_f = M_f \to \mathcal{R}_d$ sends $A \in M_f$ to the unique $g \in \mathcal{R}_d$ that satisfies $\partial g = A\partial f$ ($d > 0$). This map connects the idempotents in M_f to the additive components of f, and establishes a bijection between the complete sets of orthogonal idempotents in M_f and the regular splittings of f.

Theorem 3.7:
Assume $d \geq 2$, $f \in \mathcal{R}_d$ and $\operatorname{ann}_R(f)_1 = 0$. Let $\operatorname{Coid}(M_f)$ be the set of all complete sets $\{E_1, \dots, E_n\}$ of orthogonal idempotents in M_f, and let

$$\operatorname{Reg}(f) = \big\{\{g_1, \dots, g_n\} \,\big|\, f = g_1 + \cdots + g_n \text{ is a regular splitting of } f\big\}.$$

The map $\{E_i\}_{i=1}^n \mapsto \{g_i = \gamma_f(E_i)\}_{i=1}^n$ defines a bijection

$$\operatorname{Coid}(M_f) \to \operatorname{Reg}(f).$$

In particular, there is a unique maximal regular splitting of f when $d \geq 3$.

Similar to our usage in the last section, when we here say that there is a unique maximal regular splitting of f, we mean that there is a unique regular splitting of maximal length and that every other regular splitting is obtained from the maximal one by "grouping" some of its summands, cf. proposition 3.2b.

Proof: First, note that $\operatorname{ann}_R(f)_1 = 0$ is equivalent to $R_{d-1}(f) = \mathcal{R}_1$, that is, $\{\partial Df \mid D \in R_{d-1}\} = k^r$. Hence, if $\partial g_i = E_i \partial f$, then

$$\{\partial D g_i \mid D \in R_{d-1}\} = \{E_i \partial D f \mid D \in R_{d-1}\} = \operatorname{im} E_i.$$

Since $\partial(v^{\mathsf{T}} x) = v$, this implies that

$$R_{d-1}(g_i) = \{v^{\mathsf{T}} x \mid v \in \operatorname{im} E_i\} \subseteq \mathcal{R}_1. \tag{3.1}$$

(Recall that x denotes the column vector $x = [x_1, \dots, x_r]^{\mathsf{T}}$.)

Assume that $\{E_1, \dots, E_n\} \subseteq M_f$ is a complete set of orthogonal idempotents. For each i, let $g_i = \gamma_f(E_i) \in \mathcal{R}_d$, that is, $\partial g_i = E_i \partial f$. Note that $g_i \neq 0$ because

$E_i \neq 0$ and $\operatorname{ann}_R(f)_1 = 0$. Since $\sum_{i=1}^n E_i = I$, we get $\sum_{i=1}^n g_i = f$. Furthermore, for all i, equation (3.1) implies that

$$R_{d-1}(g_i) \cap \Bigl(\sum_{j\neq i} R_{d-1}(g_j)\Bigr) = \Bigl\{ v^{\mathsf{T}} x \Bigm| v \in \operatorname{im} E_i \cap \Bigl(\sum_{j \neq i} \operatorname{im} E_j\Bigr)\Bigr\} \tag{3.2}$$

But the E_i's are orthogonal idempotents, thus $\operatorname{im} E_i \cap \bigl(\sum_{j\neq i} \operatorname{im} E_j\bigr) = 0$ by proposition 3.5d. Hence $f = g_1 + \cdots + g_n$ is a regular splitting of f.

Conversely, assume that f splits regularly as $f = g_1 + \cdots + g_n$. By lemmas 2.9 and 2.12 there exists for every i a matrix $E_i \in M_f$ such that $\partial g_i = E_i \partial f$. E_i is unique since $\operatorname{ann}_R(f)_1 = 0$, and $\gamma_f(E_i) = g_i$ by definition of γ_f. Furthermore, $\partial f = \sum_{i=1}^n \partial g_i = \sum_{i=1}^n E_i \partial f$ implies $\sum_{i=1}^n E_i = I$.

Because $f = \sum_i g_i$ is a regular splitting, we know for all i that

$$R_{d-1}(g_i) \cap \Bigl(\sum_{j\neq i} R_{d-1}(g_j)\Bigr) = 0.$$

Combined with equation (3.2), this implies $\operatorname{im} E_i \cap \bigl(\sum_{j\neq i} \operatorname{im} E_j\bigr) = 0$. For all $v \in k^r$ and all j we know that $E_j v = \sum_{k=1}^n E_k E_j v$. For any $i \neq j$, we rearrange this equation and get $E_i E_j v = E_j(v - E_j v) - \sum_{k\neq i,j} E_k E_j v$. This is an element of $\operatorname{im} E_i \cap \bigl(\sum_{j\neq i} \operatorname{im} E_j\bigr)$, and must therefore be zero. Hence $E_i E_j v = 0$ for all $v \in k^r$, implying $E_i E_j = 0$ for all $i \neq j$. This proves that $\{E_1, \dots, E_n\}$ is a complete set of orthogonal idempotents in M_f.

When $d \geq 3$, M_f is a commutative k-algebra, and has therefore a unique maximal complete set of orthogonal idempotents, by proposition 3.2. It follows that f has a unique regular splitting of maximal length, and that every other regular splitting of f is obtained from the maximal one by "grouping" some of the summands. □

Remark 3.8: To sum up, theorem 3.7 tells us that there is a correspondence between regular splittings $f = g_1 + \cdots + g_n$ and complete sets of orthogonal idempotents $\{E_1, \dots, E_n\} \subseteq M_f$ given by the equation $\partial g_i = E_i \partial f$. The correspondence is one-to-one because $\partial g_i = E_i \partial f$ determines g_i uniquely given E_i since $d > 0$, and it determines E_i uniquely given g_i because $\operatorname{ann}_R(f)_1 = 0$.

Remark 3.9: We want to point out that $d \geq 3$ is very different from $d = 2$ when we work with regular splittings. If $f \in \mathcal{R}_d$ and $d \geq 3$, then M_f contains a unique maximal complete set of orthogonal idempotents, and f has therefore a unique maximal splitting. This is in stark contrast to $d = 2$, when the representation of f as a sum of squares is far from unique. The explanation for this difference is that M_f does not have a unique maximal complete set of orthogonal idempotents when $d = 2$, and the reason for this is that M_f is not closed under multiplication.

Theorem 3.7 is not as complete as we would like it to be. It tells us how to find a regular splitting $f = \sum_{i=1}^{n} g_i$, but it does not say how M_{g_i} is related to M_f. This is something we would like to know, since M_f can contain matrices that are not idempotent. If these matrices are not found in one of the M_{g_i}'s, it would mean that we loose some information about f (contained in M_f) when we pass to the additive components $\{g_1, \dots, g_n\}$.

Fortunately, this is not the case, as theorem 3.18 will tell us. It would be nice if the relationship between M_f and the M_{g_i}'s was as simple as $M_f = \oplus_{i=1}^{n} M_{g_i}$. But it is not, because there is an important difference between f and the g_i's. In theorem 3.7 we assumed $\operatorname{ann}_R(f)_1 = 0$, an assumption which was justified by corollary 2.10. But if $f = g_1 + \cdots + g_n$ is a non-trivial regular splitting (i.e. $n \geq 2$), then necessarily $\operatorname{ann}_R(g_i)_1 \neq 0$ for all i. This affects M_{g_i}, and we have to adjust for this effect. Thus in order to state and prove theorem 3.18, we need to understand what happens to M_f if $\operatorname{ann}_R(f)_1 \neq 0$. After the adjustment, the simple relationship between M_f and the M_{g_i}'s is in fact restored.

Remark 3.10: In the following we will often choose a subspace $W \subseteq \mathcal{R}_1$ and consider the divided power subalgebra $\mathcal{S} = k[W]^{DP} \subseteq \mathcal{R}$. (The most important example is $W = R_{d-1}(f)$. If $\operatorname{ann}_R(f)_1 \neq 0$, then $W \subsetneq \mathcal{R}_1$ and $\mathcal{S} \subsetneq \mathcal{R}$.) We note that $D(g) \in \mathcal{S}$ for all $g \in \mathcal{S}$ and $D \in R$. Thus for any subset $S \subseteq R$, the action of R on $\mathcal{R}$ restricts to an action of S on $\mathcal{S}$. We usually want a polynomial ring $S = k[V]$ with $V \subseteq R_1$ acting as the dual of $\mathcal{S}$ (i.e. $S \cong \mathcal{S}^*$).

To ensure that the choice of $V \subseteq R_1$ implies $S \cong \mathcal{S}^*$, we need $V \cong W^*$. Note that $R_1 \cong W^\perp \oplus W^*$. Thus choosing $S = k[V] \subseteq R$ such that $S \cong \mathcal{S}^*$ with the action induced by R, is equivalent to choosing $V \subseteq R_1$ such that $R_1 = W^\perp \oplus V$. Note that $\mathcal{S} \subseteq \mathcal{R}$ determines the ideal $\operatorname{ann}_R \mathcal{S} = \{D \in R \mid Dg = 0 \text{ for all } g \in \mathcal{S}\}$, which equals $(W^\perp)$, the ideal in R generated by $W^\perp$. Since $R = (W^\perp) \oplus S$ as graded k-vector spaces, $\mathcal{S}$ determines S only as a direct summand.

Remark 3.11: Note that $E \in \mathrm{Mat}_k(r,r)$ is idempotent if and only if E acts as the identity on its image and $k^r = \operatorname{im} E \oplus \ker E$. Hence specifying E is equivalent to choosing subspaces $\operatorname{im} E, \ker E \subseteq k^r$ such that $k^r = \operatorname{im} E \oplus \ker E$.

A pair $(W \subseteq \mathcal{R}_1, V \subseteq R_1)$ satisfying $W \oplus V^\perp = \mathcal{R}_1$ determines an idempotent $E \in \mathrm{Mat}_k(r,r)$ by the equations

$$\operatorname{im} E = \{v \in k^r \mid v^\mathsf{T} x \in W\} \quad \text{and} \quad \ker E = \{v \in k^r \mid v^\mathsf{T} x \in V^\perp\}.$$

Note that by remark 3.10, a pair (W, V) satisfying $W \oplus V^\perp = \mathcal{R}_1$ is equivalent to a pair $(\mathcal{S} = k[W]^{DP} \subseteq \mathcal{R}, S = k[V] \subseteq R)$ satisfying $S = \mathcal{S}^*$.

Conversely, an idempotent $E \in \mathrm{Mat}_k(r,r)$ determines $\mathcal{S} = k[W]^{DP} \subseteq \mathcal{R}$ and $S = k[V] \subseteq R$ by the equations

$$\begin{aligned} W &= \{v^\mathsf{T} x \mid v \in \operatorname{im} E\} \ = \{x^\mathsf{T} E u \mid u \in k^r\} \subseteq \mathcal{R}_1, \\ V &= \{v^\mathsf{T} \partial \mid v \in \operatorname{im} E^\mathsf{T}\} = \{u^\mathsf{T} E \partial \mid u \in k^r\} \subseteq R_1. \end{aligned}$$

We note that

$$V^\perp = \{v^\mathsf{T} x \mid (u^\mathsf{T} E \partial)(v^\mathsf{T} x) = u^\mathsf{T} E v = 0 \,\forall\, u \in k^r\} = \{v^\mathsf{T} x \mid v \in \ker E\}.$$

Since E is idempotent, we know that $k^r = \operatorname{im} E \oplus \ker E$. This implies that $W \oplus V^\perp = \mathcal{R}_1$ and $W^\perp \oplus V = R_1$, and therefore $S \cong \mathcal{S}^*$.

Let $s = \operatorname{rank} E = \dim_k V = \dim_k W$. Choose a basis $\{v_1, \dots, v_s\}$ for $\operatorname{im} E$, and a basis $\{v_{s+1}, \dots, v_r\}$ for $\ker E$. Since $\operatorname{im} E \oplus \ker E = k^r$, it follows that the matrix $P = [v_1, \dots, v_r]$ is invertible. Furthermore,

$$P^{-1} E P = P^{-1}[v_1, \dots, v_s, 0, \dots, 0] = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix},$$

where I is the $s \times s$ identity matrix, cf. remark 3.6. The similarity transformation $E \mapsto P^{-1} E P$ corresponds to a change of variables in $\mathcal{R}$ and R, transforming $\mathcal{S}$ into $\mathcal{S} = k[x_1, \dots, x_s]^{DP}$ and S into $S = k[\partial_1, \dots, \partial_s]$, cf. proposition 3.15.

It is usually more convenient to specify an idempotent E instead of specifying a pair $(\mathcal{S} = k[W]^{DP}, S = k[V])$ such that $R_1 = W^\perp \oplus V$. Therefore, we will formulate most of the results in this section using idempotents, and we will define and use $\mathcal{S}$ and S only when we have to. If $f \in \mathcal{R}$ and $\mathcal{S} = k[W]^{DP} \subseteq \mathcal{R}$, then we will often need to know when $f \in \mathcal{S}$. Since $f \in k[R_{d-1}(f)]^{DP}$, this is equivalent to $R_{d-1}(f) \subseteq W$. The next lemma allows us to express this in terms of the idempotent E.

Lemma 3.12: *Assume $d > 0$ and $f \in \mathcal{R}_d$. Let $E \in \operatorname{Mat}_k(r,r)$ be idempotent, and define $W = \{v^{\mathsf{T}}x \mid v \in \operatorname{im} E\}$. Then*

(a) *$R_{d-1}(f) \subseteq W$ if and only if $E\partial f = \partial f$,*

(b) *$R_{d-1}(f) = W$ if and only if $E\partial f = \partial f$ and $\operatorname{rank} E = \dim_k R_{d-1}(f)$.*

Proof: Clearly, $R_{d-1}(f) = \{Df \mid D \in R_{d-1}\} \subseteq W$ if and only if $\{\partial Df \mid D \in R_{d-1}\} \subseteq \{\partial h \mid h \in W\} = \operatorname{im} E$. Since E is idempotent, this is equivalent to $E\partial Df = \partial Df$ for all $D \in R_{d-1}$, i.e. $E\partial f = \partial f$. This proves (a). (b) follows immediately, since $\operatorname{rank} E = \dim_k W$. Note that $E\partial f = \partial f$ implies that $\operatorname{rank} E \geq \dim_k R_{d-1}(f)$ by (a), thus (b) is the case of minimal rank. □

When $f \in \mathcal{S} \subsetneq \mathcal{R}$, the definition of M_f is ambiguous in the following way.

Remark 3.13: Let $\mathcal{S} = k[x_1, \dots, x_s]^{DP}$ and $S = k[\partial_1, \dots, \partial_s]$. Assume $s < r$, so that $\mathcal{S} \subsetneq \mathcal{R}$ and $S \subsetneq R$. Let $\partial' = [\partial_1, \dots, \partial_s]^{\mathsf{T}}$. There are two ways to interpret definition 2.14 when $f \in \mathcal{S}$. We may consider f to be an element of $\mathcal{R}$, giving $M_f = \{A \in \operatorname{Mat}_k(r,r) \mid I_2(\partial A \partial)_2 \subseteq \operatorname{ann}_R f\}$. Or we may think of f as an element of $\mathcal{S}$, in which case $M'_f = \{A \in \operatorname{Mat}_k(s,s) \mid I_2(\partial' A \partial')_2 \subseteq \operatorname{ann}_S f\}$.

Notice that we choose to write $I_2(\partial A \partial)_2$. This is the degree two part of the ideal $I_2(\partial A \partial)$ and generates the ideal. The reason for doing this is that $I_2(\partial' A \partial')$ is ambiguous; is it an ideal in R or an ideal in S? But its degree two piece is the same in both cases; $I_2(\partial' A \partial')_2$ is simply the k-vector space spanned by the 2×2 minors of $(\partial' A \partial')$. The ideals in R and S generated by these minors are therefore equal to $I_2(\partial' A \partial')_2 R$ and $I_2(\partial' A \partial')_2 S$, respectively.

Since $\mathcal{R}$ is our default ring, M_f will always mean what definition 2.14 says, i.e. $M_f = \{A \in \operatorname{Mat}_k(r,r) \mid I_2(\partial A \partial)_2 \subseteq \operatorname{ann}_R f\}$. It is not immediately clear what the analogue of M'_f should be for a more general subring $\mathcal{S} \subseteq \mathcal{R}$. We will in proposition 3.15 prove that the following definition gives us what we want.

Definition 3.14: Assume $f \in \mathcal{R}_d$. Let $E \in M_f$ be idempotent. Define

$$M_f^E = M_f \cap E \operatorname{Mat}_k(r,r) E.$$

Of course, $M_f^I = M_f$. Note that $E \operatorname{Mat}_k(r,r)E$ is closed under multiplication. Hence M_f^E is a k-algebra if M_f is closed under matrix multiplication. In any case, we note that $E \in M_f^E$, and that E acts as the identity on M_f^E.

We want to show that if $E\partial f = \partial f$ then M_f^E reduces to M'_f (cf. remark 3.13) when we perform a suitable base change and forget about extra variables. In remark 3.13 we used both $\operatorname{ann}_R f$ and $\operatorname{ann}_S f$. In general, if $f \in \mathcal{S} \subseteq \mathcal{R}$ and $S \cong \mathcal{S}^*$, then by definition $\operatorname{ann}_S f = \{D \in S \mid Df = 0\}$. Hence

$$\operatorname{ann}_S f = S \cap \operatorname{ann}_R f$$

is always true. Recall that, if $P \in \mathrm{GL}_r$, then $\phi_P : \mathcal{R} \to \mathcal{R}$ is the k-algebra homomorphism induced by $x \mapsto P^\mathsf{T} x$, and $\phi_P : R \to R$ is induced by $\partial \mapsto P^{-1}\partial$.

***Proposition* 3.15:** *Let $f \in \mathcal{R}_d$, $d > 0$. Suppose $E \in M_f$ is idempotent and satisfies $E\partial f = \partial f$. Let $s = \operatorname{rank} E$, $W = \{v^\mathsf{T} x \mid v \in \operatorname{im} E\}$ and $V = \{v^\mathsf{T}\partial \mid v \in \operatorname{im} E^\mathsf{T}\}$. Define $\mathcal{S} = k[W]^{DP} \subseteq \mathcal{R}$ and $S = k[V] \subseteq R$. Choose $P \in \mathrm{GL}_r$ such that*

$$E' = PEP^{-1} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $\mathcal{S}' = \phi_P(\mathcal{S})$, $S' = \phi_P(S)$ and $f' = \phi_P(f)$. Then $\mathcal{S}' = k[x_1, \ldots, x_s]^{DP}$, $S' = k[\partial_1, \ldots, \partial_s]$ and $f' \in \mathcal{S}'$. Let $\partial' = [\partial_1, \ldots, \partial_s]^\mathsf{T}$. Then

$$M_f^E \cong M'_{f'} = \bigl\{A \in \operatorname{Mat}_k(s,s) \bigm| I_2(\partial' \, A\partial')_2 \subseteq \operatorname{ann}_{S'}(f')\bigr\}.$$

Proof: We start by proving that $\phi_P(\mathcal{S}) = k[x_1, \ldots, x_s]^{DP}$. We know that $W = \{x^\mathsf{T} Eu \mid u \in k^r\}$. Since $\phi_P(x^\mathsf{T} Eu) = (P^\mathsf{T} x)^\mathsf{T} Eu = x^\mathsf{T} PEu = x^\mathsf{T} E'Pu$, it follows that $\phi_P(W) = \{x^\mathsf{T} E'v \mid v \in k^r\} = \langle x_1, \ldots, x_s\rangle$. Thus $\phi_P(\mathcal{S}) = k[x_1, \ldots, x_s]^{DP}$. In a similar fashion we get $\phi_P(V) = \{v^\mathsf{T} E'\partial \mid v \in k^r\} = \langle \partial_1, \ldots, \partial_s\rangle$, implying $\phi_P(S) = k[\partial_1, \ldots, \partial_s]$. Furthermore, $E\partial f = \partial f$ implies $R_{d-1}(f) \subseteq W$ by lemma 3.12. Thus $f \in k[R_{d-1}(f)]^{DP} \subseteq \mathcal{S}$, and therefore, $f' = \phi_P(f) \in \mathcal{S}'$.

In order to show that $M_f^E \cong M'_{f'}$, we first prove that

$$M_f^E = \bigl\{A \in E \operatorname{Mat}_k(r,r)E \bigm| I_2(E\partial \, A\partial)_2 \subseteq \operatorname{ann}_S f\bigr\}. \tag{3.3}$$

Assume that $A \in E \operatorname{Mat}_k(r,r)E$. Since $A = AE$ and$(E\partial)_i \in S$ for all i, it follows that $I_2(E\partial \, A\partial)_2 \subseteq S$ automatically. Hence $I_2(E\partial \, A\partial)_2 \subseteq \operatorname{ann}_S f$ if and only if

$I_2(E\partial\ A\partial)_2 \subseteq \operatorname{ann}_R f$. By lemma 2.13 this latter statement holds if and only if $(A\partial)(E\partial)^{\mathsf{T}}(f)$ is symmetric, which is equivalent to $A\partial\partial^{\mathsf{T}}f$ being symmetric, since $E\partial f = \partial f$. And $A\partial\partial^{\mathsf{T}}f$ is symmetric if and only if $A \in M_f$. Hence, if $A \in E\operatorname{Mat}_k(r,r)E$, then $I_2(E\partial\ A\partial)_2 \subseteq \operatorname{ann}_S f \Leftrightarrow A \in M_f$, which proves equation (3.3).

Now let $M = E\operatorname{Mat}_k(r,r)E$ and

$$M' = PMP^{-1} = E'\operatorname{Mat}_k(r,r)E' = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \,\middle|\, A \in \operatorname{Mat}_k(s,s) \right\}.$$

Applying equation (3.3) to f' and E', we see that

$$M_{f'}^{E'} = \big\{ A \in M' \,\big|\, I_2(E'\partial\ A\partial)_2 \subseteq \operatorname{ann}_{S'}(f') \big\}.$$

Clearly, $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$ defines an isomorphism $M'_{f'} \to M_{f'}^{E'}$. Thus to finish the proof, it is enough to show that $M_{f'}^{E'} = PM_f^E P^{-1}$.

Let $A \in M'$. Then $A \in M_{f'}^{E'}$ if and only if $A\partial\partial^{\mathsf{T}}(f')$ is symmetric. Note that $\phi_{P^{-1}}(\partial\partial^{\mathsf{T}}(f')) = (P\partial)(P\partial)^{\mathsf{T}}(f) = P\partial\partial^{\mathsf{T}}(f)P^{\mathsf{T}}$. Hence $A\partial\partial^{\mathsf{T}}(f')$ is symmetric if and only if $\phi_{P^{-1}}(P^{-1}A\partial\partial^{\mathsf{T}}(f')(P^{-1})^{\mathsf{T}}) = P^{-1}AP\partial\partial^{\mathsf{T}}f$ is symmetric, which is equivalent to $P^{-1}AP \in M_f^E$. Thus $M_{f'}^{E'} = PM_f^E P^{-1} \cong M_f^E$, and we are done. □

Before we go on to theorem 3.18, we need two more lemmas.

***Lemma* 3.16:** *Suppose $d \geq 2$ and $f \in \mathcal{R}_d$. Let $E \in M_f$ be idempotent. Then $M_f^E = EM_fE$. If $E\partial f = \partial f$ and $\operatorname{rank} E = \dim_k R_{d-1}(f)$, then $M_f^E = M_fE$ and $M_f = M_f^E \oplus \ker \gamma_f$.*

Proof: $E\partial\partial^{\mathsf{T}}f$ is symmetric since $E \in M_f$. If $A \in M_f$, then $A\partial\partial^{\mathsf{T}}f$ is symmetric, hence $EA\partial\partial^{\mathsf{T}}(f)E^{\mathsf{T}} = EAE\partial\partial^{\mathsf{T}}f$ is also symmetric. This proves $EAE \in M_f$, and therefore $EAE \in M_f^E$. Hence $A \mapsto EAE$ defines a k-linear map $M_f \to M_f^E$. It is clearly surjective. Indeed, if $EAE \in M_f^E \subseteq M_f$, then $EAE \mapsto E^2AE^2 = EAE$. Thus $M_f^E = EM_fE$.

If $E\partial f = \partial f$, then $EA\partial\partial^{\mathsf{T}}f = E\partial\partial^{\mathsf{T}}A^{\mathsf{T}}(f) = \partial\partial^{\mathsf{T}}A^{\mathsf{T}}(f) = A\partial\partial^{\mathsf{T}}f$, hence $EA\partial f = A\partial f$ because $d \geq 2$. Since $R_{d-1}(f) = \{v^{\mathsf{T}}x \,|\, v \in \operatorname{im} E\}$ by lemma 3.12, we have $\{\partial Df \,|\, D \in R_{d-1}\} = \operatorname{im} E$. It follows that

$$\begin{aligned} EA\partial f = A\partial f &\Leftrightarrow EA\partial Df = A\partial Df \;\forall\, D \in R_{d-1} \\ &\Leftrightarrow EAEv = AEv \;\forall\, v \in k^r \Leftrightarrow EAE = AE. \end{aligned}$$

Similarly, $A\partial f = 0$ if and only if $AE = 0$. Hence the map $M_f \to M_f^E$ above is also given by $A \mapsto AE$. This proves that $M_f^E = M_f E$. Furthermore, the kernel of this map is obviously $\{A \in M_f \mid AE = 0\} = \{A \in M_f \mid A\partial f = 0\} = \ker \gamma_f$. Finally, the composition $M_f^E \subseteq M_f \to M_f^E$ is the identity, implying $M_f = M_f^E \oplus \ker \gamma_f$. □

Lemma 3.17: *Suppose $d \geq 2$ and $f \in \mathcal{R}_d$. Let $E \in M_f$ be idempotent and $g = \gamma_f(E)$. Then $M_g^E = M_f^E$. If $d \geq 3$, then even $M_g E = M_f E$.*

Proof: Since $\partial g = E\partial f$, we get $A\partial\partial^\mathsf{T} g = AE\partial\partial^\mathsf{T} f$. It follows that

$$M_g = \{A \in \mathrm{Mat}_k(r,r) \mid AE \in M_f\}. \tag{3.4}$$

Indeed, $A \in M_g$ if and only if $A\partial\partial^\mathsf{T} g$ is symmetric. But $A\partial\partial^\mathsf{T} g = AE\partial\partial^\mathsf{T} f$, and $AE\partial\partial^\mathsf{T} f$ is symmetric if and only if $AE \in M_f$. This proves equation (3.4).

Let $A \in M_g$. Then $AE \in M_f$, and therefore $AE = (AE)E \in M_f E$. Thus $M_g E \subseteq M_f E$. This implies that $M_g^E = EM_gE \subseteq EM_fE = M_f^E$. Conversely, let $A \in EM_fE \subseteq M_f$. Since $AE = A$, we have $AE \in M_f$, and therefore $A \in M_g$. Hence $A = EAE \in EM_gE$. This proves that $M_g^E = M_f^E$.

Assume $d \geq 3$, and let $A \in M_f$. Since $E \in M_f$ and M_f is closed under multiplication, it follows that $AE \in M_f$, which implies $A \in M_g$. This shows that $M_f \subseteq M_g$. Thus $M_f E \subseteq M_g E \subseteq M_f E$, and we are done. □

We are now in a position to prove a generalization of theorem 3.7. This time we do not assume $\mathrm{ann}_R(f)_1 = 0$. More importantly, however, is that we are able to show how M_f and the M_{g_i}'s are related. Recall that E acts as the identity on M_f^E. Therefore $\{E_1, \dots, E_n\}$ is a complete set of idempotents in M_f^E if and only if $\sum_{i=1}^n E_i = E$ and $E_iE_j = 0$ for all $i \neq j$.

Theorem 3.18:
Let $d \geq 2$ and $f \in \mathcal{R}_d$. Choose a matrix $E \in M_f$ such that $E\partial f = \partial f$ and $\operatorname{rank} E = \dim_k R_{d-1}(f)$. Let

$$\mathrm{Coid}\left(M_f^E\right) = \left\{ \{E_i\}_{i=1}^n \;\middle|\; 0 \neq E_i \in M_f^E, \sum_{i=1}^n E_i = E \text{ and } E_iE_j = 0 \,\forall\, i \neq j \right\},$$

$$\mathrm{Reg}(f) = \left\{ \{g_1, \dots, g_n\} \;\middle|\; f = g_1 + \dots + g_n \text{ is a regular splitting of } f \right\}.$$

The map $\{E_i\}_{i=1}^n \mapsto \{g_i = \gamma_f(E_i)\}_{i=1}^n$ *defines a bijection*

$$\mathrm{Coid}\left(M_f^E\right) \to \mathrm{Reg}(f).$$

Assume $d \geq 3$. *Then* M_f^E *is a commutative* k*-algebra, and there exists a unique maximal regular splitting of* f. *Let* $\{E_1, \ldots, E_n\}$ *be a complete set of orthogonal idempotents in* M_f^E, *and let* $g_i = \gamma_f(E_i)$. *Then*

$$M_{g_i}^{E_i} = M_f E_i = M_f^E E_i \quad \text{for all } i, \quad \text{and} \quad M_f^E = \bigoplus_{i=1}^{n} M_{g_i}^{E_i}.$$

Proof: We know that every regular splitting happens inside $\mathcal{S} = k[R_{d-1}(f)]^{DP}$ by corollary 2.10. Using the isomorphism of proposition 3.15, the first statements of the theorem are equivalent to the corresponding statements about $M'_{f'}$ and $\mathrm{Reg}(f')$, and follows from theorem 3.7.

Let $d \geq 3$. It follows from proposition 2.21 and lemma 3.16 that M_f^E is a commutative k-algebra. (Or by the isomorphism with $M'_{f'}$.) The existence of the unique maximal regular splitting of f then follows by proposition 3.2b.

It remains only to prove the last two statements. Let $\{E_1, \ldots, E_n\}$ be a complete set of orthogonal idempotents in M_f^E, and let $g_i = \gamma_f(E_i)$. Note that $\{\partial Df \,|\, D \in R_{d-1}\} = \operatorname{im} E$ by lemma 3.12, and recall that E is the identity in M_f^E. Since $\partial g_i = E_i \partial f$, it follows that $\{\partial D g_i \,|\, D \in R_{d-1}\} = \operatorname{im}(E_i E) = \operatorname{im} E_i$ and $M_{g_i}^{E_i} = M_{g_i} E_i$, cf. the proof of lemma 3.16. Moreover, $M_{g_i} E_i = M_f E_i$ by lemma 3.17, and $M_f E_i = M_f E E_i = M_f^E E_i$ by lemma 3.16. It follows that $M_{g_i}^{E_i} = M_f E_i = M_f^E E_i$ for all i, and $M_f^E = \oplus_{i=1}^n M_f^E E_i = \oplus_{i=1}^n M_{g_i}^{E_i}$. □

Remark 3.19: Note that an idempotent E as in theorem 3.18 always exists. Given $f \in \mathcal{R}_d$, let $W = R_{d-1}(f)$, and choose $W' \subseteq \mathcal{R}_1$ such that $W \oplus W' = \mathcal{R}_1$. Let $E \in \mathrm{Mat}_k(r, r)$ be the idempotent determined by

$$\operatorname{im} E = \{v \in k^r \,|\, v^{\mathsf{T}} x \in W\} \quad \text{and} \quad \ker E = \{v \in k^r \,|\, v^{\mathsf{T}} x \in W'\},$$

cf. remark 3.11. Then $E\partial f = \partial f$ and $\operatorname{rank} E = \dim_k R_{d-1}(f)$ by lemma 3.12. Moreover, $E\partial f = \partial f$ implies $E \in M_f$. Also note that this E is not unique since we have the choice of $W' \in \mathcal{R}_1$.

Remark 3.20: One goal of this paper is to find out what the algebra M_f can tell us about f. Assume that $\operatorname{ann}_R(f)_1 = 0$. The idempotent E in theorem 3.18

must then be the identity matrix I, and therefore $M_f^E = M_f$. Then the first part of theorem 3.18 reduces to theorem 3.7, and tells us that the idempotents in M_f determines the regular splittings of f, and how this happens.

Assume $d \geq 3$. The last two statements of theorem 3.18 have no counter part in theorem 3.7. They say that if $A \in M_f$, then $A_i = AE_i \in M_{g_i}^{E_i}$ and $A = \sum_{i=1}^n A_i$. Thus any "information" about f contained in M_f is passed on as "information" about g_i contained in $M_{g_i}^{E_i}$. For example, M_f contains a nilpotent matrix if and only if (at least) one of the $M_{g_i}^{E_i}$ contains a nilpotent matrix.

In other words, in order to figure out what M_f can tell us about f, it should be enough to find out what $M_{g_i}^{E_i}$ can tell us about g_i for all i. (Proposition 3.24 can be used for similar purposes.) Hence we may assume that M_f does not contain any non-trivial idempotents. If k contains every eigenvalue of each $A \in M_f$, then this implies that $M_f = \langle I \rangle \oplus M_f^{\text{nil}}$ by proposition 3.5. And if $k = \bar{k}$, then it is always so, hence modulo theorem 3.18 it is enough to study all $f \in \mathcal{R}_d$ such that $M_f = \langle I \rangle \oplus M_f^{\text{nil}}$. It is this situation we study in chapter 4.

Theorem 3.18 is formulated using a non-unique idempotent E. We will now give an intrinsic reformulation of that theorem when $d \geq 3$. For that purpose, we define the following k-algebra.

Definition 3.21: Assume $d \geq 3$ and $f \in \mathcal{R}_d$. Define $G_f = \gamma_f(M_f)$, and let

$$\star : G_f \times G_f \to G_f$$

be the map induced by multiplication in M_f.

Of course, we could define G_f also for smaller d, but then we would not get an induced multiplication. The induced map is clearly the following. For any $g, h \in G_f$, we may choose $A, B \in M_f$ such that $g = \gamma_f(A)$ and $h = \gamma_f(B)$, and define $g \star h = \gamma_f(AB)$. We can prove that this is well defined, and that $\star$ is a bilinear, associative and commutative multiplication on G_f, like we do in proposition 5.8. But here we choose a different approach.

The idempotent $E \in M_f$ in theorem 3.18 satisfies $E\partial f = \partial f$ and $\operatorname{rank} E = \dim_k R_{d-1}(f)$. Hence $M_f = M_f^E \oplus \ker \gamma_f$ by lemma 3.16. Therefore,

$$G_f = \gamma_f(M_f) = \gamma_f(M_f^E) \cong M_f^E.$$

The map $\star$ is clearly induced by the multiplication in M_f^E, proving that $\star$ is well defined and giving G_f the structure of a commutative k-algebra. Note that $\star$ is independent of E, by its definition 3.21.

Note that f is the identity element of $(G_f, \star)$ since $f = \gamma_f(I)$. We have the following immediate consequence of theorem 3.18.

Corollary 3.22: *Let $d \geq 3$ and $f \in \mathcal{R}_d$. Then $f = \sum_{i=1}^n g_i$ is a regular splitting of f if and only if $\{g_1, \dots, g_n\}$ is a complete set of orthogonal idempotents in G_f. In particular, there is a unique maximal regular splitting. If $f = \sum_{i=1}^n g_i$ is any regular splitting, then $G_{g_i} = G_f \star g_i$ for all i, and $G_f = \oplus_{i=1}^n G_{g_i}$.*

Example 3.23: Let $r = d = 3$ and $f = x_1 x_2^{(2)} + x_2 x_3^{(2)} + x_3^{(3)}$. Then

$$\partial f = \begin{pmatrix} x_2^{(2)} \\ x_1 x_2 + x_3^{(2)} \\ x_2 x_3 + x_3^{(2)} \end{pmatrix} \quad \text{and} \quad \partial \partial^{\mathsf{T}} f = \begin{pmatrix} 0 & x_2 & 0 \\ x_2 & x_1 & x_3 \\ 0 & x_3 & x_2 + x_3 \end{pmatrix}.$$

It follows that $\operatorname{ann}_R(f)_1 = 0$ and $\operatorname{ann}_R(f)_2 = \langle \partial_1^2, \partial_1\partial_3, \partial_1\partial_2 + \partial_2\partial_3 - \partial_3^2 \rangle$. Thus

$$I_2 \begin{pmatrix} \partial_1 & \partial_2 & \partial_3 \\ 0 & \partial_3 & \partial_1 + \partial_3 \end{pmatrix} \subseteq \operatorname{ann}_R f.$$

It follows that

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \in M_f.$$

We note that $\det(\lambda I - A) = \lambda^2(\lambda - 1)$. Since A has both 0 and 1 as eigenvalues, A is neither invertible nor nilpotent. Hence there must exists a non-trivial idempotent in M_f! Indeed, we know that

$$A^2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \in M_f,$$

and we see that $A^3 = A^2$. Thus $E = A^2$ is such an idempotent.

So far we have shown that $M_f \supseteq k[A] = \langle I, A, A^2 \rangle$. To prove equality, we show that $\operatorname{ann}_R f$ has exactly two generators of degree 3. Since $R/\operatorname{ann}_R f$ is Gorenstein of codimension 3, the structure theorem of Buchsbaum-Eisenbud [BE77] applies. Because we already know that $\operatorname{ann}_R f$ has three generators of degree 2 and at least two generators of degree 3, it follows easily that it cannot have more generators. Hence

$$\operatorname{ann}_R f = \left(\partial_1^2, \partial_1\partial_3, \partial_1\partial_2 + \partial_2\partial_3 - \partial_3^2, \partial_2^3, \partial_2^2\partial_3\right),$$

which are the five Pfaffians of

$$\begin{pmatrix} 0 & 0 & \partial_1 & \partial_2 & \partial_3 \\ 0 & 0 & 0 & \partial_3 & \partial_1 + \partial_3 \\ -\partial_1 & 0 & 0 & 0 & \partial_2^2 \\ -\partial_2 & -\partial_3 & 0 & 0 & 0 \\ -\partial_3 & -\partial_1 - \partial_3 & -\partial_2^2 & 0 & 0 \end{pmatrix}.$$

Thus $M_f = \langle I, A, A^2 \rangle$, and $E = A^2$ is an idempotent of rank 1. We note that

$$M_f \cdot E = \langle E \rangle \quad \text{and} \quad M_f \cdot (I - E) = \langle I - E, A - A^2 \rangle.$$

Since $A - A^2$ obviously is nilpotent, M_f cannot contain another idempotent (in addition to I, E and $I - E$). Let g be the additive component of f satisfying $\partial g = E\partial f$. Since

$$E\partial f = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2^{(2)} \\ x_1x_2 + x_3^{(2)} \\ x_2x_3 + x_3^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ (x_2 + x_3)^{(2)} \\ (x_2 + x_3)^{(2)} \end{pmatrix},$$

it follows that

$$g = (x_2 + x_3)^{(3)} \in k[x_2 + x_3]^{DP}.$$

The other additive component is therefore

$$h = f - g = (x_1 - x_3)x_2^{(2)} - x_2^{(3)} \in k[x_1 - x_3, x_2]^{DP}.$$

This verifies that $f = g + h$ is a regular splitting of f, as promised by theorem 3.7. Furthermore, $M_g^E = M_f E$ and $M_h^{I-E} = M_f(I - E)$. Since M_h^{I-E} contains a nilpotent matrix, we will in chapter 4 see that h has a degenerate splitting.

We also see that $G_f = \langle f, g, x_2^{(3)} \rangle = \langle (x_1 - x_3)x_2^{(2)}, x_2^{(3)}, (x_2 + x_3)^{(3)} \rangle$. And we note that $f \sim x_1x_2^{(2)} + x_2^{(3)} + x_3^{(3)}$, and $f \sim x_1x_2^{(2)} + x_3^{(3)}$ as long as $\operatorname{char} k \neq 3$.

In remark 2.37 we claimed that results concerning M_f often corresponds to results about $I(M)$. In this section we have seen how idempotents in M_f are related to regular splittings of f. We end this section with a result showing how $I(M)$ and $X(M)$ "splits" if M contains a complete set of orthogonal idempotents. Recall that

$$I(M) = \sum_{A\in M} I_2(\partial\, A\partial) \quad \text{and} \quad X(M) = \big\{f \in \mathcal{R}\,\big|\, \operatorname{ann}_R f \supseteq I(M)\big\}.$$

Proposition 3.24: *Let $M \subseteq \operatorname{Mat}_k(r,r)$ be a commutative subalgebra containing the identity matrix I. Let $\{E_1,\dots,E_n\}$ be a complete set of orthogonal idempotents in M. For every i, let $M_i = ME_i$, $V_i = \{v^{\mathsf{T}}\partial \,|\, v \in \operatorname{im} E_i^{\mathsf{T}}\} \subseteq R_1$, $S_i = k[V_i]$ and $\mathcal{S}_i = k[\{v^{\mathsf{T}}x \,|\, v \in \operatorname{im} E_i\}]^{DP} \cong S_i^*$. Define $I_{S_i}(M) = S_i \cap I(M)$ and $X_{\mathcal{S}_i}(M) = \mathcal{S}_i \cap X(M)$. Then*

(a) $I_R(M) = \big(\sum_{i<j} RV_iV_j\big) \oplus \big(\oplus_{i=1}^n I_{S_i}(M_i)\big)$,

(b) $\big(R/I_R(M)\big)_d = \oplus_{i=1}^n \big(S_i/I_{S_i}(M_i)\big)_d$ *for all* $d > 0$, *and*

(c) $X_{\mathcal{R}}(M)_d = \oplus_{i=1}^n X_{\mathcal{S}_i}(M_i)_d$ *for all* $d > 0$.

Proof: Note that $R_1 = \oplus_{i=1}^n V_i$ by proposition 3.5d. This implies

$$R_d = \Big(\sum_{i<j} R_{d-2}V_iV_j\Big) \oplus \Big(\mathop{\oplus}_{i=1}^n V_i^d\Big)$$

for all $d \geq 1$. Since $V_i^d = (S_i)_d$, the degree d part of S_i, we get

$$\Big(R\Big/\sum_{i<j} RV_iV_j\Big)_d = \mathop{\oplus}_{i=1}^n (S_i)_d$$

for all $d > 0$. Thus (b) follows immediately from (a).

Since $M = \oplus_{i=1}^n M_i$ (proposition 3.5c), it follows by definition that

$$I_R(M) = I(M) = \sum_{A\in M} I_2(\partial\, A\partial) = \sum_{i=1}^n \sum_{A\in M_i} I_2(\partial\, A\partial). \tag{3.5}$$

Fix i, and let $A \in M_i$. Putting $(A,B) = (A,E_i)$ into equation (2.2) proves that

$$I_2(\partial\, A\partial) \subseteq I_2(\partial\, E_i\partial) + I_2(E_i\partial\, A\partial),$$

and putting $(A,B)=(E_i,A)$ gives

$$I_2(E_i\partial\, A\partial)\subseteq I_2(\partial\, A\partial).$$

Since $E_i\in M_i$, this shows that

$$\sum_{A\in M_i} I_2(\partial\, A\partial)=I_2(\partial\, E_i\partial)+\sum_{A\in M_i} I_2(E_i\partial\, A\partial). \tag{3.6}$$

Note that $(E_i\partial)_k\in V_i$ and $((I-E_i)\partial)_k\in\sum_{j\neq i}V_j$ for all k. Hence the minors of $(\partial\, E_i\partial)$ satisfy

$$\begin{vmatrix}\partial_k & (E_i\partial)_k\\ \partial_l & (E_i\partial)_l\end{vmatrix}=\begin{vmatrix}((I-E_i)\partial)_k & (E_i\partial)_k\\ ((I-E_i)\partial)_l & (E_i\partial)_l\end{vmatrix}\in\sum_{j\neq i}V_iV_j. \tag{3.7}$$

For all $u,v\in k^r$ and $j\neq i$ we have (cf. equation (2.1))

$$\sum_{k,l=1}^{r}(E_j^{\mathsf T}u)_k v_l\begin{vmatrix}\partial_k & (E_i\partial)_k\\ \partial_l & (E_i\partial)_l\end{vmatrix}=\begin{vmatrix}u^{\mathsf T}E_j\partial & u^{\mathsf T}E_jE_i\partial\\ v^{\mathsf T}\partial & v^{\mathsf T}E_i\partial\end{vmatrix}=(u^{\mathsf T}E_j\partial)\cdot(v^{\mathsf T}E_i\partial)$$

because $E_jE_i=0$. Since $\{v^{\mathsf T}E_i\partial\mid v\in k^r\}=V_i$, this means that $I_2(\partial E_i\partial)$ contains every product V_iV_j, $j\neq i$. Hence $I_2(\partial\, E_i\partial)=\sum_{j\neq i}RV_iV_j$ for all i by equation (3.7). Therefore,

$$\sum_{i=1}^{n} I_2(\partial\, E_i\partial)=\sum_{i<j}RV_iV_j. \tag{3.8}$$

Combining equations (3.5), (3.6) and (3.8), we have proven so far that

$$I(M)=\sum_{i<j}RV_iV_j+\sum_{i=1}^{n}\sum_{A\in M_i}I_2(E_i\partial\, A\partial).$$

If $A\in M_i$, then $A\partial=AE_i\partial$, and therefore $I_2(E_i\partial\, A\partial)_2\subseteq V_i^2\subseteq S_i$. Hence

$$I(M)=\Big(\sum_{i<j}RV_iV_j\Big)\oplus\Big(\bigoplus_{i=1}^{n}\sum_{A\in M_i}I_2(E_i\partial\, A\partial)_2S_i\Big), \tag{3.9}$$

a direct sum of graded k-vector spaces. What we have proven also shows that

$$I(M_i)=\sum_{A\in M_i}I_2(\partial\, A\partial)=\Big(\sum_{j\neq i}RV_iV_j\Big)\oplus\Big(\sum_{A\in M_i}I_2(E_i\partial\, A\partial)_2S_i\Big) \tag{3.10}$$

for all i. It follows that

$$I_{S_i}(M_i) = S_i \cap I(M_i) = S_i \cap I(M) = \sum_{A \in M_i} I_2(E_i\partial\, A\partial)_2 S_i. \tag{3.11}$$

With equation (3.9) this proves (a).

To prove (c), note for any i and $f \in \mathcal{S}_i$ that $\operatorname{ann}_R f = (\sum_{j\neq i} V_j) \oplus \operatorname{ann}_{S_i} f$. It follows from equations (3.9), (3.10) and (3.11) that

$$\begin{aligned} X_{\mathcal{S}_i}(M_i) &= \bigl\{f \in \mathcal{S}_i \bigm| \operatorname{ann}_R f \supseteq I(M_i)\bigr\} \\ &= \bigl\{f \in \mathcal{S}_i \bigm| \operatorname{ann}_{S_i} f \supseteq I_{S_i}(M_i)\bigr\} = X_{\mathcal{S}_i}(M) \subseteq X(M). \end{aligned}$$

Since $\mathcal{S}_i \cap \mathcal{S}_j = k$ for $i \neq j$, it follows that $\oplus_{i=1}^n X_{\mathcal{S}_i}(M_i)_d \subseteq X(M)_d$ for all $d > 0$. To prove equality it is enough to show that their dimensions are equal. And this follows from (b), since $X_{\mathcal{S}_i}(M_i)_d = \{f \in (\mathcal{S}_i)_d \,|\, Df = 0 \;\forall\, D \in I_{S_i}(M_i)_d\}$ (by lemma 2.36d) implies $\dim_k X_{\mathcal{S}_i}(M_i)_d = \dim_k\bigl(S_i/I_{S_i}(M_i)\bigr)_d$. □

Remark 3.25: We can give a direct proof of the other inclusion in part (c). By definition, $f \in X(M)$ if and only if $M \subseteq M_f$. Let $f \in X(M)_d$. Since $\{E_i\} \subseteq M \subseteq M_f$, there exists $g_i \in \mathcal{S}_i$ such that $f = \sum_{i=1}^n g_i$ is a regular splitting by theorem 3.7 ($d = 1$ is trivial). Let $D \in I_{S_i}(M_i)$. Then $D(g_j) = 0$ for all $j \neq i$ since $D \in (V_i)$, and $D(f) = 0$ since $D \in I(M)$. Hence $D(g_i) = 0$. This proves that $I_{S_i}(M_i) \subseteq \operatorname{ann}_{S_i} g_i$, i.e. $g_i \in X_{\mathcal{S}_i}(M_i)_d$ for all i.

3.3 Minimal resolutions

Now that we know how to find all regular splittings of a form $f \in \mathcal{R}_d$, we turn to consequences for the graded Artinian Gorenstein quotient $R/\operatorname{ann}_R f$. In this section we obtain a minimal free resolution of $R/\operatorname{ann}_R f$ when f splits regularly. This allows us to compute the (shifted) graded Betti numbers of $R/\operatorname{ann}_R f$.

Fix $n \geq 1$, and let $W_1, \dots, W_n \subseteq \mathcal{R}_1$ satisfy $\mathcal{R}_1 = \oplus_{i=1}^n W_i$. For all i define $\mathcal{S}^i = k[W_i]^{DP}$. Note that $\mathcal{R}_1 = \oplus_{i=1}^n W_i$ implies $\mathcal{R} = \mathcal{S}^1 \otimes_k \cdots \otimes_k \mathcal{S}^n$. For each i, let $V_i = (\sum_{j\neq i} W_i)^\perp \subseteq R_1$ and $S^i = k[V_i] \cong (\mathcal{S}^i)^*$. Then $R_1 = \oplus_{i=1}^n V_i$, and therefore $R = S^1 \otimes_k \cdots \otimes_k S^n$.

Remark 3.26: Let $s_i = \dim_k W_i = \dim_k V_i$, and note that $\sum_{i=1}^n s_i = r$. Let

$$\mathcal{J}_i = \Big\{ j \in \mathbb{Z} \,\Big|\, \sum_{k<i} s_k < j \leq \sum_{k\leq i} s_k \Big\}.$$

for all i. There is a base change (that is, a homogeneous change of variables) of $\mathcal{R}$ such that $\mathcal{S}^i = k[\{x_j \,|\, j \in \mathcal{J}_i\}]^{DP}$ for all i (cf. remark 2.6). This implies for all i that $S^i = k[\{\partial_k \,|\, j \in \mathcal{J}_i\}]$. Note that the subspaces $\{W_i\}_{i=1}^r$, or equivalently $\{V_i\}_{i=1}^r$, determine and is determined by a unique set of orthogonal idempotents $\{E_i\}_{i=1}^n \subseteq \operatorname{Mat}_k(r,r)$, cf. remark 3.11. Thus the "rectifying" base change above corresponds to a simultaneous diagonalization of $\{E_i\}_{i=1}^r$ as in remark 3.6. We will not assume that this base change has been made when we state and prove our results, but some claims may be easier to understand with this in mind.

Let $f = \sum_{i=1}^r g_i$ be a regular splitting with $g_i \in \mathcal{S}_d^i$, $g_i \neq 0$, $d > 0$. The following result is fundamental to this section, comparing the ideals $\operatorname{ann}_R(f)$, $\operatorname{ann}_R(g_i)$ and $\operatorname{ann}_{S^i}(g_i)$.

Lemma 3.27: *With the notation above, the following statements are true.*

(a) *For every i we have $\operatorname{ann}_{S^i}(g_i) = S^i \cap \operatorname{ann}_R(g_i)$ and*

  (i) $\operatorname{ann}_R(g_i) = \big(\sum_{j\neq i} S_1^j\big) \oplus \operatorname{ann}_{S^i}(g_i)$ *as graded k-vector spaces,*

  (ii) $\operatorname{ann}_R(g_i) = \big(\sum_{j\neq i} S_1^j\big) + R \operatorname{ann}_{S^i}(g_i)$ *as ideals in R, and*

  (iii) $R/\operatorname{ann}_R(g_i) \cong S^i/\operatorname{ann}_{S^i}(g_i)$.

(b) *There exist nonzero $D_i \in S_d^i$, $i = 1, \ldots, n$, such that*

$$\operatorname{ann}_R(f) = \bigcap_{i=1}^n \operatorname{ann}_R(g_i) + (D_2 - D_1, \ldots, D_n - D_1).$$

(c) *We may express $\cap_{i=1}^n \operatorname{ann}_R(g_i)$ as a direct sum of graded k-vector spaces;*

$$\bigcap_{i=1}^n \operatorname{ann}_R(g_i) = \Big(\sum_{i<j} RS_1^i S_1^j\Big) \oplus \Big(\bigoplus_{i=1}^n \operatorname{ann}_{S^i}(g_i)\Big),$$

(d) *or as a sum of ideals in R;*

$$\bigcap_{i=1}^n \operatorname{ann}_R(g_i) = \sum_{i<j} RS_1^i S_1^j + \sum_{i=1}^n R \operatorname{ann}_{S^i}(g_i).$$

(e) *The Hilbert function H of $R/\operatorname{ann}_R(f)$ satisfies*

$$H\big(R/\operatorname{ann}_R(f)\big) = \sum_{i=1}^{n} H\big(S^i/\operatorname{ann}_{S^i}(g_i)\big) - (n-1)\big(\delta_0 + \delta_d\big),$$

where δ_e is 1 in degree e and zero elsewhere.

Proof: By definition, $\operatorname{ann}_{S^i}(g_i) = \{D \in S^i \mid D(g_i) = 0\}$, which clearly equals $S^i \cap \operatorname{ann}_R(g_i)$. By construction, $D(g_i) = 0$ for all $D \in S_1^j$, $j \neq i$. Hence $\big(\sum_{j\neq i} S_1^j\big) \subseteq \operatorname{ann}_R(g_i)$. Since $R/\big(\sum_{j\neq i} S_1^j\big) = S^i$, we get

$$\operatorname{ann}_R(g_i) = \Big(\sum_{j\neq i} S_1^j\Big) \oplus \operatorname{ann}_{S^i}(g_i)$$

as graded k-vector subspaces of R. The rest of (a) follows immediately.

Consider the regular splitting $f = \sum_{i=1}^n g_i$. By lemma 2.9 we have

$$\operatorname{ann}_R(f)_e = \bigcap_{i=1}^{n} \operatorname{ann}_R(g_i)_e \text{ for all } e < d.$$

Thus the ideals $\operatorname{ann}_R(f)$ and $\cap_{i=1}^n \operatorname{ann}_R(g_i)$ are equal in every degree $e \neq d$. In degree d the right-hand side has codimension n (since the g_i are linearly independent), hence $\operatorname{ann}_R(f)$ must have $n-1$ extra generators of degree d. If we choose $D_i \in S_d^i$ such that $D_1(g_1) = \cdots = D_n(g_n) \neq 0$, then clearly

$$\operatorname{ann}_R(f) = \bigcap_{i=1}^{n} \operatorname{ann}_R(g_i) + (D_2 - D_1, \ldots, D_n - D_1).$$

By (a) we have $\sum_{i<j} RS_1^i S_1^j \subseteq \operatorname{ann}_R(g_k)$ for all k. Note that

$$R_e = \Big(\sum_{i<j} R_{e-2} S_1^i S_1^j\Big) \oplus \Big(\bigoplus_{i=1}^{n} S_e^i\Big) \text{ for all } e > 0.$$

Because $\big(\cap_{i=1}^n \operatorname{ann}_R(g_i)\big) \cap S^j = \operatorname{ann}_{S^j}(g_j)$, this implies both (c) and (d). Combining (b) and (c), it follows that $(R/\operatorname{ann}_R f)_e = \oplus_{i=1}^n (S^i/\operatorname{ann}_{S^i} g_i)_e$ for all $e \neq 0, d$, proving (e). □

Most of the time in this section we will assume $n = 2$. This makes it easier to state and prove our results. Let $\mathcal{S} = \mathcal{S}^1$ and $\mathcal{T} = \mathcal{S}^2$. (Of course, we may think of $\mathcal{T}$ as $\mathcal{T} = \mathcal{S}^2 \otimes_k \cdots \otimes_k \mathcal{S}^n$, reaching $n > 2$ by induction.) Similarly, let $S = S^1$

and $T = S^2$, and $s = s_1$ and $t = s_2 = r - s$. Hence $\mathcal{R} = \mathcal{S} \otimes_k \mathcal{T}$ and $R = S \otimes_k T$. We will often compare ideals of R, S and T, and some words are in order.

Given a homogeneous ideal $I \subseteq S$, the inclusion $S \subseteq R$ makes I into a graded k-vector subspace of R. If $J \subseteq T$ is another homogeneous ideal, then IJ is the k-vector subspace of R spanned by all products ij with $i \in I$ and $j \in J$. Since IJ automatically is closed under multiplication from R, it is equal to the ideal in R generated by all products ij. In particular, IT is simply the ideal in R generated by I. There are many ways to think of and write this ideal, including

$$(I) = R \cdot I = I \otimes_S R = I \otimes_S (S \otimes_k T) = I \otimes_k T = IT.$$

Similarly, $IT \cdot SJ = (I \otimes_S R) \otimes_R (R \otimes_T J) = I \otimes_k J = IJ = (IJ)$. We have used here a property of tensor products often called *base change*, cf. [Eis95, proposition A2.1]. Note that $IT \cap SJ = IT \cdot SJ = IJ$. It follows that

$$I_1 J_1 \cap I_2 J_2 = (I_1 \cap I_2)(J_1 \cap J_2) \tag{3.12}$$

for all homogeneous ideals $I_1, I_2 \subseteq S$ and $J_1, J_2 \subseteq T$.

Fix $d \geq 1$, and let $g \in \mathcal{S}_d$ and $h \in \mathcal{T}_d$. We want to point out what lemma 3.27 says in this simpler situation. Note that the ideal $\operatorname{ann}_S(g)$ in S generates the ideal $T \operatorname{ann}_S(g)$ in R. Let

$$\mathsf{m}_S = (S_1) \subseteq S \quad \text{and} \quad \mathsf{m}_T = (T_1) \subseteq T$$

be the maximal homogeneous ideals in S and T, respectively. Since $T = \mathsf{m}_T \oplus k$, we get $R = S\mathsf{m}_T \oplus S$. Lemma 3.27 tells us that $\operatorname{ann}_R(g) = S\mathsf{m}_T \oplus \operatorname{ann}_S(g)$ and $\operatorname{ann}_R(g) = S\mathsf{m}_T + T \operatorname{ann}_S(g)$. Furthermore,

$$\operatorname{ann}_R(g) \cap \operatorname{ann}_R(h) = \mathsf{m}_S \mathsf{m}_T + T \operatorname{ann}_S(g) + S \operatorname{ann}_T(h) \tag{3.13}$$

as ideals in R, and there exist $D \in S_d$ and $E \in T_d$ such that

$$\operatorname{ann}_R(f) = \operatorname{ann}_R(g) \cap \operatorname{ann}_R(h) + (D - E). \tag{3.14}$$

We will use these equations to calculate the minimal resolution of $R/\operatorname{ann}_R(f)$. They involve products of ideals, and we start with the following lemma.

Lemma 3.28: *Given homogeneous ideals $I \subseteq S$ and $J \subseteq T$, let $\mathcal{F}$ and $\mathcal{G}$ be their resolutions*

$$\mathcal{F} : 0 \to F_s \xrightarrow{\varphi_s} \dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} I \to 0,$$
$$\mathcal{G} : 0 \to G_t \xrightarrow{\psi_t} \dots \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} J \to 0,$$

where the F_i's are free S-modules and the G_i's are free T-modules. Then the tensor complex

$$\mathcal{F} \otimes_k \mathcal{G} : 0 \to H_{s+t-1} \xrightarrow{\eta_{s+t-1}} \dots \xrightarrow{\eta_2} H_1 \xrightarrow{\eta_1} IJ \to 0$$

is exact, hence a free resolution of IJ in $R = S \otimes_k T$, and minimal if both $\mathcal{F}$ and $\mathcal{G}$ are minimal.

The definition of the tensor complex can be found in [Eis95, section 17.3]. Its construction gives $H_i = \oplus_{j=1}^{i} F_j \otimes_k G_{i+1-j}$ for all $i \geq 1$. Note that this is a free R-module. The maps $\eta_i : H_i \to H_{i-1}$ for $i > 1$ are given by

$$\begin{array}{ccccccccc}
F_1 \otimes_k G_i & \oplus & F_2 \otimes_k G_{i-1} & \oplus & \cdots & \oplus & F_{i-1} \otimes_k G_2 & \oplus & F_i \otimes_k G_1 \\
\mathrm{id}_{F_1} \otimes \psi_i \searrow & & \swarrow \varphi_2 \otimes \mathrm{id}_{G_{i-1}} & & & & (-1)^i \, \mathrm{id}_{F_{i-1}} \otimes \psi_2 \searrow & & \swarrow \varphi_i \otimes \mathrm{id}_{G_1} \\
& F_1 \otimes_k G_{i-1} & \oplus & & \cdots & & \oplus & F_{i-1} \otimes_k G_1 &
\end{array}$$

that is, $\eta_i = \oplus_{j=1}^{i-1} \left(\varphi_{j+1} \otimes \mathrm{id}_{G_{i-j}} - (-1)^j \, \mathrm{id}_{F_j} \otimes \psi_{i-j+1} \right)$, and $\eta_1 = \varphi_1 \otimes \psi_1$.

Proof of lemma 3.28: The complex is exact since we get it by tensoring over k, and I and J are free over k, hence flat. It is trivially minimal when $\mathcal{F}$ and $\mathcal{G}$ are minimal by looking at the maps η_i. □

Note that $\mathcal{F} \otimes_S R = \mathcal{F} \otimes_k T$ is a resolution of $I \otimes_S R = IT$, the ideal in R generated by I. Similarly, $R \otimes_T \mathcal{G}$ is a resolution of SJ. Furthermore, $(\mathcal{F} \otimes_S R) \otimes_R (R \otimes_T \mathcal{G}) = \mathcal{F} \otimes_k \mathcal{G}$.

Example 3.29: Let

$$\mathcal{M} : 0 \to M_s \to \cdots \to M_1 \to \mathsf{m}_S \to 0,$$
$$\mathcal{N} : 0 \to N_t \to \cdots \to N_1 \to \mathsf{m}_T \to 0$$

be the Koszul resolutions of $\mathsf{m}_S \subseteq S$ and $\mathsf{m}_T \subseteq T$, respectively. We know that $M_k = \binom{s}{k} S(-k)$ and $N_k = \binom{t}{k} T(-k)$ for all k. If we apply lemma 3.28 to $I = \mathsf{m}_S$ and $J = \mathsf{m}_T$, we get a graded minimal free resolution

$$\mathcal{MN} = \mathcal{M} \otimes_k \mathcal{N} : 0 \to MN_{s+t-1} \to \cdots \to MN_1 \to \mathsf{m}_S \mathsf{m}_T \to 0$$

of $\mathsf{m}_S \mathsf{m}_T \subseteq R = S \otimes_k T$. Here $MN_k = \oplus_{i=1}^k M_i \otimes_k N_{k+1-i}$ for all $k > 0$. Hence $MN_k = \nu_k R(-k-1)$ where

$$\nu_k = \sum_{i=1}^{k} \binom{s}{i} \binom{t}{k+1-i} = \binom{s+t}{k+1} - \binom{s}{k+1} - \binom{t}{k+1}.$$

This agrees with the Eagon-Northcott resolution of

$$I_2 \begin{pmatrix} \partial_1 & \dots & \partial_s & 0 & \dots & 0 \\ 0 & \dots & 0 & \partial_{s+1} & \dots & \partial_{s+t} \end{pmatrix}.$$

We chose to write lemma 3.28 in terms of ideals $I \subseteq S$ and $J \subseteq T$ because this is how we will use it most of the time. Of course, the result is true more generally. Indeed, if $\mathcal{F}$ and $\mathcal{G}$ are resolutions of an S-module M and a T-module N, respectively, then the tensor complex $\mathcal{F} \otimes_k \mathcal{G}$ is a resolution of $M \otimes_k N$, with the same proof. We will use this is in the next lemma.

Lemma 3.30: *Let $I \subseteq S$ be a homogeneous ideal, and let $I' = S\mathsf{m}_T + IT \subseteq R$. Denote the shifted graded Betti numbers of S/I and R/I' by $\hat{\beta}^I_{ij}$ and $\hat{\beta}^{I'}_{ij}$, respectively. Then for all $j, k \geq 0$, we have*

$$\hat{\beta}^{I'}_{kj} = \sum_{i=0}^{k} \binom{t}{k-i} \hat{\beta}^I_{ij}.$$

Proof: The proof rests upon the following observation. If $I \subseteq S$ and $J \subseteq T$ are ideals, then $S/I \otimes_k T/J \cong R/(IT + SJ)$. Indeed,

$$\begin{aligned} S/I \otimes_k T/J &= S/I \otimes_S (S \otimes_k T/J) = S/I \otimes_k R/SJ \\ &= (S/I \otimes_S R) \otimes_R R/SJ = R/IT \otimes_R R/SJ = R/(IT + SJ). \end{aligned}$$

It follows that we may compute a resolution of $R/(IT+SJ)$ as the tensor complex of the resolutions of S/I and T/J. We do this with $J = \mathsf{m}_T$.

Let $\mathcal{F}$ and $\mathcal{N}$ be the graded minimal free resolutions of S/I and T/m_T, respectively, cf. example 3.29. That is,

$$\mathcal{F} : 0 \to F_s \to \cdots \to F_1 \to F_0 \to S/I \to 0,$$
$$\mathcal{N} : 0 \to N_t \to \cdots \to N_1 \to N_0 \to T/\mathsf{m}_T \to 0,$$

with $F_i = \oplus_{j \geq 0} \hat{\beta}^I_{ij} S(-i-j)$ and $N_i = \binom{t}{i} T(-i)$ for all $i \geq 0$.

The tensor complex $\mathcal{F} \otimes_k \mathcal{N}$ gives a graded minimal free resolution

$$\mathcal{H} : 0 \to H_{s+t} \to \cdots \to H_1 \to H_0$$

of $R/(S\mathsf{m}_T + IT) = R/I'$, where for all $k \geq 0$ we have

$$H_k = \bigoplus_{i=0}^{k} F_i \otimes_k N_{k-i} = \bigoplus_{i=0}^{k} \bigoplus_{j \geq 0} \binom{t}{k-i} \hat{\beta}^I_{ij} R(-k-j).$$

The result follows by reading off the Betti numbers from this equation. □

Since $\operatorname{ann}_R(g) = S\mathsf{m}_T + T \operatorname{ann}_S(g)$, we may use this lemma to compare the (shifted) graded Betti numbers of $R/\operatorname{ann}_R g$ and $S/\operatorname{ann}_S g$. In the next two results we use the short exact sequence

$$0 \to I \cap J \to I \oplus J \to I + J \to 0$$

and the mapping cone construction (cf. [Eis95, appendix A3.12]) several times.

Proposition 3.31: *Let $I \subseteq S$ and $J \subseteq T$ be homogeneous ideals, and let m_S and m_T be the maximal homogeneous ideals in S and T, respectively. Assume that $I_1 = J_1 = 0$. Let $\mathcal{F}$ and $\mathcal{G}$ be graded minimal free resolutions*

$$\mathcal{F} : 0 \to F_s \xrightarrow{\varphi_s} \dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} I \to 0,$$
$$\mathcal{G} : 0 \to G_t \xrightarrow{\psi_t} \dots \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} J \to 0.$$

Denote the shifted graded Betti numbers of S/I and T/J by $\hat{\beta}^I_{ij}$ and $\hat{\beta}^J_{ij}$. Then $\mathsf{m}_S\mathsf{m}_T + IT + SJ \subseteq R = S \otimes_k T$ has a graded minimal free resolution

$$\mathcal{H} : 0 \to H_r \to \cdots \to H_1 \to \mathsf{m}_S\mathsf{m}_T + IT + SJ \to 0$$

where $r = s + t$ and

$$H_k = \nu_k R(-k-1) \oplus \left(\bigoplus_{j \geq 0} \sum_{i=1}^{k} \left(\binom{t}{k-i} \hat{\beta}^I_{ij} + \binom{s}{k-i} \hat{\beta}^J_{ij} \right) R(-k-j) \right)$$

for all $k > 0$. Here $\nu_k = \binom{r}{k+1} - \binom{s}{k+1} - \binom{t}{k+1}$.

Proof: Remember, by definition of the shifted graded Betti numbers, we have

$$F_i = \bigoplus_{j\geq 0} \hat\beta^I_{ij}\, S(-i-j) \quad \text{and} \quad G_i = \bigoplus_{j\geq 0} \hat\beta^J_{ij}\, T(-i-j)$$

for every i. We will construct the minimal resolution in two similar steps.
Step 1. Note that $IT \cap \mathsf{m}_S\mathsf{m}_T = (I\cap \mathsf{m}_S)(T \cap \mathsf{m}_T) = I\mathsf{m}_T$ by equation (3.12). This gives us a short exact sequence

$$0 \to I\mathsf{m}_T \to IT \oplus \mathsf{m}_S\mathsf{m}_T \to \mathsf{m}_S\mathsf{m}_T + IT \to 0. \tag{3.15}$$

Let $\mathcal{M}$ and $\mathcal{N}$ be the Koszul resolutions of $\mathsf{m}_S \subseteq S$ and $\mathsf{m}_T \subseteq T$, respectively, as in example 3.29. By lemma 3.28 we have four minimal resolutions;

$$\begin{aligned}
\mathcal{F}' = \mathcal{F}\otimes_k T &: 0 \to F'_s \xrightarrow{\varphi_s} \dots \xrightarrow{\varphi_2} F'_1 \xrightarrow{\varphi_1} IT \to 0,\\
\mathcal{G}' = S\otimes_k \mathcal{G} &: 0 \to G'_t \xrightarrow{\psi_t} \dots \xrightarrow{\psi_2} G'_1 \xrightarrow{\psi_1} SJ \to 0,\\
\mathcal{F}'' = \mathcal{F}\otimes_k \mathcal{N} &: 0 \to F''_{s+t-1} \xrightarrow{\zeta_{s+t-1}} \dots \xrightarrow{\zeta_2} F''_1 \xrightarrow{\zeta_1} I\mathsf{m}_T \to 0,\\
\mathcal{G}'' = \mathcal{M}\otimes_k \mathcal{G} &: 0 \to G''_{s+t-1} \xrightarrow{\xi_{s+t-1}} \dots \xrightarrow{\xi_2} G''_1 \xrightarrow{\xi_1} \mathsf{m}_S J \to 0.
\end{aligned}$$

The free modules in the first resolution are $F'_i = F_i \otimes_k T = \oplus_{j\geq 0} \hat\beta^I_{ij} R(-i-j)$, and we identify the map $\varphi_i \otimes \mathrm{id}_T$ with φ_i since they are given by the same matrix. Similarly, for the second resolution, we have $G'_i = S\otimes_k G_i = \oplus_{j\geq 0} \hat\beta^J_{ij} R(-i-j)$. The modules in the third and fourth resolution satisfy

$$\begin{aligned}
F''_{k-1} &= \bigoplus_{i=1}^{k-1} F_i \otimes_k N_{k-i}\\
&= \bigoplus_{i=1}^{k-1} \left(\Big(\bigoplus_{j\geq 0} \hat\beta^I_{ij}\, S(-i-j)\Big) \otimes_k \binom{t}{k-i} T(-k+i)\right)\\
&= \bigoplus_{j\geq 0} \left(\sum_{i=1}^{k-1} \binom{t}{k-i} \hat\beta^I_{ij}\right) R(-k-j),
\end{aligned}$$

and similarly, $G''_{k-1} = \oplus_{j\geq 0} \left(\sum_{i=1}^{k-1} \binom{s}{k-i} \hat\beta^J_{ij}\right) R(-k-j)$.

By tensoring the exact sequence $0 \to \mathsf{m}_T \to T \to T/\mathsf{m}_T \to 0$ with I, we get a short exact sequence

$$0 \to I\mathsf{m}_T \to IT \to I \otimes_k T/\mathsf{m}_T \to 0.$$

We need to lift the inclusion $I\mathsf{m}_T \subseteq IT$ to a map of complexes $\mathcal{F}'' \to \mathcal{F}'$. This is easily achieved by defining the map $F''_i \to F'_i = F_i \otimes_k T$ to be $\mathrm{id}_{F_i} \otimes \psi_1$ on the summand $F_i \otimes_k N_1$, and zero on all other direct summands of F''_i. The mapping cone construction now gives a resolution $\cdots \to F'_3 \oplus F''_2 \to F'_2 \oplus F''_1 \to F'_1$ of $I \otimes_k T/\mathsf{m}_T$ that actually equals the tensor complex associated to $I \otimes_k T/\mathsf{m}_T$ (similar to lemma 3.28). It is obviously minimal by looking at the maps.

Next we lift the inclusion $I\mathsf{m}_T \subseteq \mathsf{m}_S\mathsf{m}_T$ to a map of complexes $\mathcal{F}'' \to \mathcal{MN}$. By looking at the degrees of these maps, we see that they must be minimal when $I_1 = 0$, that is, when I has no linear generators. Indeed, one such lift is

$$\bar{\pi}_i = \bigoplus_{j=1}^{i} \pi_j \otimes \mathrm{id} : \bigoplus_{j=1}^{i} F_j \otimes_k N_{i+1-j} \to \bigoplus_{j=1}^{i} M_j \otimes_k N_{i+1-j},$$

where π is a lift of $I \subseteq \mathsf{m}_S$ to a map of complexes $\mathcal{F} \to \mathcal{M}$.

Thus we can lift the map $I\mathsf{m}_T \hookrightarrow IT \oplus \mathsf{m}_S\mathsf{m}_T$, $z \mapsto (z, -z)$, in the exact sequence (3.15) to a map $(\mathrm{id} \otimes \psi_1) \oplus (-\bar{\pi})$ of complexes $\mathcal{F}'' \to \mathcal{F}' \oplus \mathcal{MN}$. The mapping cone construction now gives a minimal free resolution

$$\mathcal{H}' : 0 \to H'_{s+t} \to \cdots \to H'_1$$

of $\mathsf{m}_S\mathsf{m}_T + IT$, where

$$H'_k = MN_k \oplus F'_k \oplus F''_{k-1} = \nu_k\, R(-k-1) \oplus \left(\bigoplus_{j \geq 0} \sum_{i=1}^{k} \binom{t}{k-i} \hat{\beta}^I_{ij}\, R(-k-j) \right)$$

for all $k \geq 1$. This concludes the first step.

Step 2. We notice that $\mathsf{m}_S\mathsf{m}_T + IT \subseteq \mathsf{m}_S T$, and therefore

$$\mathsf{m}_S J \subseteq (\mathsf{m}_S\mathsf{m}_T + IT) \cap SJ \subseteq \mathsf{m}_S T \cap SJ = \mathsf{m}_S J.$$

Hence $(\mathsf{m}_S\mathsf{m}_T + IT) \cap SJ = \mathsf{m}_S J$, and we have a short exact sequence

$$0 \to \mathsf{m}_S J \to (\mathsf{m}_S\mathsf{m}_T + IT) \oplus SJ \to \mathsf{m}_S\mathsf{m}_T + IT + SJ \to 0. \qquad (3.16)$$

We now proceed as in the first step, getting a lift of the inclusion $\mathsf{m}_S J \subseteq SJ$ to a map of complexes $\mathcal{G}'' \to \mathcal{G}'$. To lift the inclusion $\mathsf{m}_S J \subseteq \mathsf{m}_S\mathsf{m}_T + TI$ to a map of complexes $\mathcal{G}'' \to \mathcal{H}'$, we take the lift of $\mathsf{m}_S J \subseteq \mathsf{m}_S\mathsf{m}_T$ to $\mathcal{G}'' \to \mathcal{MN}$, as in step one, and extend it by zero, since $H'_k = MN_k \oplus F'_k \oplus F''_{k-1}$ for all $k \geq 1$. And then the mapping cone construction produces a free resolution

$$\mathcal{H} : 0 \to H_r \to \cdots \to H_1 \to \mathsf{m}_S\mathsf{m}_T + IT + SJ \to 0,$$

which is minimal since all maps are minimal. Here $H_k = H'_k \oplus G'_k \oplus G''_{k-1}$ is for all $k > 0$ equal to

$$H_k = \nu_k \, R(-k-1) \oplus \left(\bigoplus_{j\geq 0} \sum_{i=1}^{k} \left(\binom{t}{k-i} \hat{\beta}^I_{ij} + \binom{s}{k-i} \hat{\beta}^J_{ij} \right) R(-k-j) \right). \quad \square$$

Remark 3.32: Because $\operatorname{ann}_R(g) \cap \operatorname{ann}_R(h) = \mathsf{m}_S \mathsf{m}_T + T \operatorname{ann}_S(g) + S \operatorname{ann}_T(h)$, we will use proposition 3.31 with $I = \operatorname{ann}_S(g)$ and $J = \operatorname{ann}_T(h)$ when we calculate the resolution of $\operatorname{ann}_R(f) = \operatorname{ann}_R(g) \cap \operatorname{ann}_R(h) + (D - E)$. There is another way to find the resolution of $\operatorname{ann}_R(g) \cap \operatorname{ann}_R(h)$, using the sequence

$$0 \to \operatorname{ann}_R(g) \cap \operatorname{ann}_R(h) \to \operatorname{ann}_R(g) \oplus \operatorname{ann}_R(h) \to \mathsf{m}_R \to 0.$$

This is a short exact sequence, and we know the minimal resolutions of the middle and right-hand side modules. Since the quotients are Artinian, these resolutions all have the "right" length. Hence we may dualize the sequence, use the mapping cone to construct a resolution of $\operatorname{Ext}^{r-1}_R\big(\operatorname{ann}_R(g) \cap \operatorname{ann}_R(h), R\big)$, and dualize back. Compared to the proof of proposition 3.31, this is done in one step, but the resulting resolution is not minimal. Thus more work is needed to find the cancelations, and in the end the result is obviously the same.

We are now ready to find the minimal resolution of $R/\operatorname{ann}_R f$. Note that we here use the convention that $\binom{a}{b} = 0$ for all $b < 0$ and all $b > a$.

Theorem 3.33:
Let $g \in \mathcal{S}_d$ and $h \in \mathcal{T}_d$ for some $d \geq 2$. Let $f = g + h \in \mathcal{R}_d$, and assume that $\operatorname{ann}_S(g)_1 = \operatorname{ann}_T(h)_1 = 0$. Let $\mathcal{F}$ and $\mathcal{G}$ be graded minimal free resolutions of $\operatorname{ann}_S g \subseteq S$ and $\operatorname{ann}_T h \subseteq T$,

$$\mathcal{F} : 0 \to F_s \xrightarrow{\varphi_s} \dots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} \operatorname{ann}_S g \to 0,$$
$$\mathcal{G} : 0 \to G_t \xrightarrow{\psi_t} \dots \xrightarrow{\psi_2} G_1 \xrightarrow{\psi_1} \operatorname{ann}_T h \to 0.$$

Denote the shifted graded Betti numbers of $S/\operatorname{ann}_S g$ and $T/\operatorname{ann}_T h$ by $\hat{\beta}^g_{ij}$ and $\hat{\beta}^h_{ij}$, respectively. That is,

$$F_i = \bigoplus_{j=0}^{d} \hat{\beta}^g_{ij} \, S(-i-j) \quad \text{and} \quad G_i = \bigoplus_{j=0}^{d} \hat{\beta}^h_{ij} \, T(-i-j)$$

for every i. Then $\operatorname{ann}_R f \subseteq R = S \otimes_k T$ has a graded minimal free resolution

$$\mathcal{H} : 0 \to H_r \to \cdots \to H_1 \to \operatorname{ann}_R f \to 0$$

with $H_r = R(-r-d)$ and

$$\begin{aligned} H_k = \nu_k\, R(-k-1) &\oplus \nu_{r-k}\, R(-d-k+1) \\ &\oplus \left(\bigoplus_{j=1}^{d-1} \left(\sum_{i=1}^{s-1} \binom{r-s}{k-i} \hat{\beta}^g_{ij} + \sum_{i=1}^{t-1} \binom{r-t}{k-i} \hat{\beta}^h_{ij} \right) R(-k-j) \right) \end{aligned}$$

for all $0 < k < r$. Here $r = s + t$ and $\nu_k = \binom{r}{k+1} - \binom{s}{k+1} - \binom{t}{k+1}$.

Proof: Since $\operatorname{ann}_R g \cap \operatorname{ann}_R h = \mathsf{m}_S \mathsf{m}_T + T \operatorname{ann}_S g + S \operatorname{ann}_T h$ by equation (3.13) (or lemma 3.27d), we may apply proposition 3.31. This gives us a graded minimal free resolution

$$\mathcal{H}' : 0 \to H'_r \to \cdots \to H'_1 \to \operatorname{ann}_R g \cap \operatorname{ann}_R h \to 0$$

with

$$H'_k = \nu_k\, R(-k-1) \oplus \left(\bigoplus_{j=0}^{d} \sum_{i=1}^{k} \left(\binom{t}{k-i} \hat{\beta}^g_{ij} + \binom{s}{k-i} \hat{\beta}^h_{ij} \right) R(-k-j) \right).$$

By lemma 3.27b, we may choose $D \in S_d$ and $E \in T_d$ such that

$$\operatorname{ann}_R f = (\operatorname{ann}_R g \cap \operatorname{ann}_R h) + (D - E).$$

Since $(\operatorname{ann}_R g \cap \operatorname{ann}_R h) \cap (D - E) = (D - E)\mathsf{m}_R$, we have a short exact sequence

$$0 \to (D-E)\mathsf{m}_R \to (\operatorname{ann}_R g \cap \operatorname{ann}_R h) \oplus (D-E) \to \operatorname{ann}_R f \to 0. \tag{3.17}$$

Evidently, $(D-E)\mathsf{m}_R$ has a Koszul type resolution with k^{th} free module $M_k = \binom{r}{k} R(-d-k)$. Thus by the mapping cone construction we have a resolution

$$\mathcal{H}'' : 0 \to H''_{r+1} \to \cdots \to H''_1$$

of $\operatorname{ann}_R f$, with

$$\begin{aligned} H''_k = M_{k-1} \oplus H'_k = \binom{r}{k-1} &R(-d-k+1) \oplus \nu_k\, R(-k-1) \\ \oplus \left(\bigoplus_{j=0}^{d} \sum_{i=1}^{k} \left(\binom{t}{k-i} \hat{\beta}^g_{ij} + \binom{s}{k-i} \hat{\beta}^h_{ij} \right) R(-k-j) \right)&. \end{aligned} \tag{3.18}$$

Since $R/\operatorname{ann}_R f$ is Gorenstein, its minimal resolution is self-dual. We now use this to find terms in $\mathcal{H}''$ that must be canceled. When we dualize $\mathcal{H}''$ (using $M^\vee = \operatorname{Hom}_R(M,R)$), we get a resolution whose k^{th} term is

$$\begin{aligned}(H''_{r-k})^\vee \otimes_k k(-d-r) &= \nu_{r-k}\, R(-d-k+1) \oplus \binom{r}{k+1} R(-k-1)\\ &\oplus \left(\bigoplus_{j=0}^{d} \left(\sum_{i=k-t}^{s-1} \binom{t}{k-i} \hat\beta^g_{ij} + \sum_{i=k-s}^{t-1} \binom{s}{k-i} \hat\beta^h_{ij} \right) R(-k-j) \right).\end{aligned} \tag{3.19}$$

Here we have used $\hat\beta^g_{s-i,d-j} = \hat\beta^g_{ij}$ and $\hat\beta^h_{t-i,d-j} = \hat\beta^h_{ij}$, which follow from the symmetry of the resolutions $\mathcal{F}$ and $\mathcal{G}$.

Since $\operatorname{ann}_S(g)_1 = 0$, we know that $\hat\beta^g_{sd} = \hat\beta^g_{00} = 1$, but otherwise the "rim" of the Betti diagram is zero, i.e. $\hat\beta^g_{ij} = 0$ for $i=0$, $j \neq 0$, for $j=0$, $i \neq 0$, for $i = s$, $j \neq d$, and for $j = d$, $i \neq s$. Similar statements hold for $\hat\beta^h_{ij}$. Putting this into equations (3.18) and (3.19), we see that the first has no terms with twist $(-k)$, whereas the second has $[\binom{t}{k} + \binom{s}{k}]R(-k)$. Thus we see that at least a summand

$$\rho = \left[\tbinom{t}{k-s} + \tbinom{s}{k-t}\right] R(-d-k)$$

must be canceled from every H''_k. By looking at the expression for H''_k, we see that its summand with twist equal to $(-d-k)$, is exactly ρ.

By the construction, the only part of the map $H''_{k+1} \to H''_k$ that can possibly be non-minimal, is the map from the direct summand $M_k = \binom{r}{k}R(-d-k)$ of H''_{k+1} to the summand ρ of H''_k. By the previous paragraph, all of ρ must cancel. But ρ is mapped into H''_{k-1} by a map that we know is minimal, hence it must cancel against M_k. When we have done so for all k, every resulting map is minimal. So we are left with a graded free resolution that must be minimal. Since $\binom{r}{k} - \binom{t}{k-s} - \binom{s}{k-t} = \nu_{r-k-1}$, we see that this resolution is $\mathcal{H} : 0 \to H_r \to \cdots \to H_1 \to \operatorname{ann}_R f \to 0$ with $H_r = R(-d-r)$ and

$$\begin{aligned} H_k &= \nu_k\, R(-k-1) \oplus \nu_{r-k}\, R(-d-k+1)\\ &\oplus \left(\bigoplus_{j=1}^{d-1} \left(\sum_{i=1}^{s-1} \binom{r-s}{k-i} \hat\beta^g_{ij} + \sum_{i=1}^{t-1} \binom{r-t}{k-i} \hat\beta^h_{ij} \right) R(-k-j) \right)\end{aligned}$$

for all $0 < k < r$. □

Remark 3.34: If we compare theorem 3.33 in the case $(s,t) = (3,1)$ with the resolution obtained by Iarrobino and Srinivasan in [IS, theorem 3.9], we see that they agree. Our methods are, however, very different.

As a consequence we can compute the graded Betti numbers of $R/\operatorname{ann}_R f$.

THEOREM 3.35:
Let $d \geq 2$ and $f, g_1, \dots, g_n \in \mathcal{R}_d$. Suppose $f = g_1 + \cdots + g_n$ is a regular splitting of f. Let $s_i = \dim_k R_{d-1}(g_i)$ for every i. Let $s = \sum_{i=1}^n s_i$, and define

$$\nu_{nk} = (n-1)\binom{r}{k+1} + \binom{r-s}{k+1} - \sum_{i=1}^{n} \binom{r-s_i}{k+1}.$$

Denote by $\hat{\beta}^f_{kj}$ and $\hat{\beta}^{g_i}_{kj}$ the shifted graded Betti numbers of $R/\operatorname{ann}_R(f)$ and $R/\operatorname{ann}_R(g_i)$, respectively. Then

$$\hat{\beta}^f_{kj} = \sum_{i=1}^{n} \hat{\beta}^{g_i}_{kj} + \nu_{nk}\delta_{1j} + \nu_{n,r-k}\delta_{d-1,j} \tag{3.20}$$

for all $0 < j < d$ and all $k \in \mathbb{Z}$. Here the symbol δ_{ij} is defined by $\delta_{ii} = 1$ for all i, and $\delta_{ij} = 0$ for all $i \neq j$.

Proof: Since $\hat{\beta}^f_{kj} = v_{nk} = 0$ for all $k \geq r$ and all $k \leq 0$, it is enough to prove (3.20) for $0 < k < r$. Let $\mathcal{S} = k[R_{d-1}(f)]^{DP}$ and $\mathcal{S}_i = k[R_{d-1}(g_i)]^{DP}$. Recall that $f \in \mathcal{S}$ and $g_i \in \mathcal{S}_i$. It follows from the definition of a regular splitting that $R_{d-1}(f) = \oplus_{i=1}^n R_{d-1}(g_i)$, and therefore $\mathcal{S} = \mathcal{S}_1 \otimes_k \cdots \otimes_k \mathcal{S}_n \subseteq \mathcal{R}$, cf. remark 2.11. In particular, $s = \sum_{i=1}^n s_i = \dim_k R_{d-1}(f) \leq r$.

Choose $V \subseteq R_1$ such that $R_1 = R_{d-1}(f)^\perp \oplus V$, and let $S = k[V]$. Then $S \cong \mathcal{S}^*$, cf. remark 3.10. Denote the shifted graded Betti numbers of $S/\operatorname{ann}_S(f)$ by $\hat{\beta}^{S/f}_{kj}$. It follows from lemma 3.30 that

$$\hat{\beta}^f_{kj} = \sum_{i=1}^{s-1} \binom{r-s}{k-i}\hat{\beta}^{S/f}_{ij} + \binom{r-s}{k}\delta_{0j} + \binom{r-s}{k-s}\delta_{dj} \tag{3.21}$$

for all $j, k \geq 0$. Note that $\operatorname{ann}_S(f)_1 = 0$.

For every i let $V_i = \big(\sum_{j \neq i} R_{d-1}(g_j)\big)^\perp \cap V \subseteq R_1$ and $S_i = k[V_i]$. Then $V = \oplus_{i=1}^n V_i$, and therefore $S = S_1 \otimes_k \cdots \otimes_k S_n \subseteq R$. Furthermore, $S_i \cong \mathcal{S}_i^*$ for all i, and $\operatorname{ann}_S(f)_1 = \oplus_{i=1}^n \operatorname{ann}_{S_i}(g_i)_1$ by lemma 3.27. Thus $\operatorname{ann}_R(f)_1 = 0$ is equivalent to $\operatorname{ann}_{S_i}(g_i)_1 = 0$ for all i.

Denote the shifted graded Betti numbers of $S_i/\operatorname{ann}_{S_i}(g_i)$ by $\hat{\beta}_{kj}^{S_i/g_i}$. If we apply equation (3.21) to g_i, we get

$$\hat{\beta}_{kj}^{g_i} = \sum_{l=1}^{s_i-1} \binom{r-s_i}{k-l} \hat{\beta}_{lj}^{S_i/g_i} \tag{3.22}$$

for all $k \geq 0$ and all $0 < j < d$. To prove the theorem we first show that

$$\hat{\beta}_{kj}^{S/f} = \sum_{i=1}^{n} \sum_{l=1}^{s_i-1} \binom{s-s_i}{k-l} \hat{\beta}_{lj}^{S_i/g_i} + \nu_{nk}\delta_{1j} + \nu_{n,s-k}\delta_{d-1,j}. \tag{3.23}$$

for all $0 < j < d$ and $0 < k < r$.

Note that $\nu_{1k} = 0$ for all k, since $n = 1$ implies $s = s_1$. Thus equation (3.23) is trivially fulfilled for $n = 1$. We proceed by induction on n.

Assume (3.23) holds for $h = g_1 + \cdots + g_{n-1}$. Let $T = S_1 \otimes_k \cdots \otimes_k S_{n-1}$, which is a polynomial ring in $t = \sum_{i=1}^{n-1} s_i$ variables. Since $f = h + g_n$ and $\operatorname{ann}_T(h)_1 = \operatorname{ann}_{S_n}(g_n)_1 = 0$, we may use theorem 3.33 to find the minimal resolution of $S/\operatorname{ann}_S f$. We see that its graded Betti numbers are given by

$$\hat{\beta}_{kj}^{S/f} = \sum_{c=1}^{t-1} \binom{s-t}{k-c} \hat{\beta}_{cj}^{T/h} + \sum_{l=1}^{s_n-1} \binom{s-s_n}{k-l} \hat{\beta}_{lj}^{S_n/g_n} + \nu_{2k}\delta_{1j} + \nu_{2,s-k}\delta_{d-1,j}$$

for all $0 < k < s$ and $0 < j < d$. Since by induction

$$\hat{\beta}_{cj}^{T/h} = \sum_{i=1}^{n-1} \sum_{l=1}^{s_i-1} \binom{t-s_i}{c-l} \hat{\beta}_{lj}^{S_i/g_i} + \nu_{n-1,c}\delta_{1j} + \nu_{n-1,t-c}\delta_{d-1,j},$$

the proof of equation (3.23) reduces to the following three binomial identities.

(1) $\displaystyle\sum_{c=1}^{t-1} \binom{s-t}{k-c} \binom{t-s_i}{c-l} = \binom{s-s_i}{k-l}$

(2) $\displaystyle\sum_{c=1}^{t-1} \binom{s-t}{k-c} \nu_{n-1,c} + \nu_{2k} = \nu_{nk}$

(3) $\displaystyle\sum_{c=1}^{t-1} \binom{s-t}{k-c} \nu_{n-1,t-c} + \nu_{2,s-k} = \nu_{n,s-k}$

They all follow from the well known formula $\sum_{i\in\mathbb{Z}} \binom{a}{i}\binom{b}{k-i} = \binom{a+b}{k}$.

The first follows immediately since we may extend the summation to $c \in \mathbb{Z}$ because $1 \le l < s_i$. In the second we note that

$$\nu_{n-1,c} = (n-2)\binom{t}{c+1} + \binom{0}{c+1} - \sum_{i=1}^{n-1} \binom{t-s_i}{c+1}.$$

Note that $\nu_{n-1,c} = 0$ for all $c \ge t$ and all $c \le 0$, even $c = -1$ since $\binom{0}{0} = 1$. Hence we can extend the summation in equation (2) to all $c \in \mathbb{Z}$, implying

$$\sum_{c=1}^{t-1} \binom{s-t}{k-c} \nu_{n-1,c} = (n-2)\binom{s}{k+1} + \binom{s-t}{k+1} - \sum_{i=1}^{n-1} \binom{s-s_i}{k+1}.$$

Since

$$\nu_{2k} = \binom{s}{k+1} + \binom{0}{k+1} - \binom{s-t}{k+1} - \binom{s-s_n}{k+1},$$

equation (2) follows easily. Finally, the third equation equals the second by letting $(c,k) \mapsto (t-c, s-k)$, finishing the proof of equation (3.23).

The theorem now follows by combining equations (3.21), (3.22) and (3.23). Also here the proof reduces to three binomial identities, and their proofs are similar to equation (1) above. □

Remark 3.36: We may express $\hat{\beta}^f_{kj}$ in terms of $\hat{\beta}^{S_i/g_i}_{lj}$, the shifted graded Betti numbers of $S_i/\operatorname{ann}_{S_i}(g_i)$. From the proof of theorem 3.35, we see that

$$\hat{\beta}^f_{kj} = \sum_{i=1}^{n} \sum_{l=1}^{s_i-1} \binom{r-s_i}{k-l} \hat{\beta}^{S_i/g_i}_{lj} + \nu_{nk}\delta_{1j} + \nu_{n,r-k}\delta_{d-1,j}.$$

Remark 3.37: For any $f \in \mathcal{R}_d$ we may arrange the shifted graded Betti numbers $\hat{\beta}_{ij}$ of $R/\operatorname{ann}_R f$ into the following $(d+1) \times (r+1)$ box.

1	$\hat{\beta}_{r-1,d}$	$\cdots$	$\hat{\beta}_{1d}$	0
0	$\hat{\beta}_{r-1,d-1}$	$\cdots$	$\hat{\beta}_{1,d-1}$	0
$\vdots$	$\vdots$		$\vdots$	$\vdots$
0	$\hat{\beta}_{r-1,1}$	$\cdots$	$\hat{\beta}_{11}$	0
0	$\hat{\beta}_{r-1,0}$	$\cdots$	$\hat{\beta}_{10}$	1

We call this the Betti diagram of $R/\operatorname{ann}_R f$. The Betti numbers are all zero outside this box, i.e. $\hat{\beta}_{ij} = 0$ for $i < 0$, for $j < 0$, for $i > r$, and for $j > d$. Thus the socle degree d is equal to the Castelnuovo-Mumford regularity of $R/\operatorname{ann}_R f$. In addition, $\hat{\beta}_{ij}$ will always be zero for $i = 0$, $j > 0$ and for $i = r$, $j < d$, and $\hat{\beta}_{00} = \hat{\beta}_{rd} = 1$, as indicated.

The values of $\hat{\beta}_{ij}$ when $j = 0$ or $j = d$ are easily determined by equation (3.21). Since $\operatorname{ann}_S(f)_1 = 0$, it follows that

$$\hat{\beta}_{i0} = \binom{r-s}{i} \quad \text{and} \quad \hat{\beta}_{id} = \binom{r-s}{i-s}$$

for all i. In particular, if $\operatorname{ann}_R(f)_1 = 0$, then they are all zero (except $\hat{\beta}_{00} = \hat{\beta}_{rd} = 1$).

The "inner" rectangle of the Betti diagram, that is, $\hat{\beta}_{ij}$ with $0 < i < r$ and $0 < j < d$, is determined by theorem 3.35. We note that it is simply the sum of the "inner" rectangles of the Betti diagrams of $R/\operatorname{ann}_R(g_i)$, except an addition to the rows with $j = 1$ and $j = d - 1$.

3.4 The parameter space

The closed points of the quasi-affine scheme $\mathbf{Gor}(r, H)$ parameterize every $f \in \mathcal{R}_d$ such that the Hilbert function of $R/\operatorname{ann}_R f$ equals H. We will in this section define some "splitting subfamilies" of $\mathbf{Gor}(r, H)$, and discuss some of their properties. We assume here that k is an algebraically closed field. We start by defining $\mathbf{Gor}(r, H)$, cf. [IK99, definition 1.10].

Let

$$A = \left\{ \alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{Z}^r \;\middle|\; \alpha_i \geq 0 \text{ for all } i \text{ and } \sum_{i=1}^{r} \alpha_i = d \right\}, \tag{3.24}$$

and note that $|A| = \binom{r+d-1}{d} = \dim_k \mathcal{R}_d$. We consider $\mathcal{A} = k[\{z_\alpha \mid \alpha \in A\}]$, which is a polynomial ring in $|A|$ variables, to be the coordinate ring of $\mathbb{A}(\mathcal{R}_d)$. We think of

$$F = \sum_{\alpha \in A} z_\alpha x^{(\alpha)} \in \mathcal{A} \otimes_k \mathcal{R}_d$$

as the generic element of $\mathcal{R}_d$. The action of R on $\mathcal{R}$ extend by $\mathcal{A}$-linearity to an action on $\mathcal{A} \otimes_k \mathcal{R}$. In particular, if $D \in R_d$, then $D(F) = \sum_{\alpha \in A} z_\alpha D(x^{(\alpha)})$ is an element of $\mathcal{A}_1$.

For any $0 \le e \le d$, fix bases $\mathcal{D} = \{D_1, \dots, D_M\}$ and $\mathcal{E} = \{E_1, \dots, E_N\}$ for R_{d-e} and R_e, respectively. Let $D = [D_1, \dots, D_M]^\mathsf{T}$ and $E = [E_1, \dots, E_N]^\mathsf{T}$, and define $\mathrm{Cat}_e^d = DE^\mathsf{T}$. It is customary to require that $\mathcal{D}$ and $\mathcal{E}$ are the standard bases $\{\partial^\alpha\}$ ordered lexicographically, and to call Cat_e^d the "catalecticant" matrix of this size. Note that the $(i,j)^{\text{th}}$ entry of $\mathrm{Cat}_e^d(F)$ is

$$\big(\mathrm{Cat}_e^d(F)\big)_{ij} = D_i E_j(F) = \sum_{\alpha \in A} z_\alpha D_i E_j\big(x^{(\alpha)}\big) \in \mathcal{A}_1.$$

If $f \in \mathcal{R}_d$, then $\mathrm{Cat}_e^d(f)$ is a matrix representation of the map $R_e \to \mathcal{R}_{d-e}$ given by $D \mapsto D(f)$. Hence

$$\dim_k(R/\operatorname{ann} f)_e = \operatorname{rank} \mathrm{Cat}_e^d(f) = \dim_k(R/\operatorname{ann} f)_{d-e}$$

by lemma 1.2. Therefore the $k \times k$ minors of $\mathrm{Cat}_e^d(F)$ cut out the subset

$$\big\{f \in \mathcal{R}_d \,\big|\, \dim_k(R/\operatorname{ann} f)_e < k\big\} \subseteq \mathbb{A}(\mathcal{R}_d).$$

Definition 3.38: Let $H = (h_0, \dots, h_d)$ be a symmetric sequence of positive integers (i.e. $h_{d-i} = h_i$ for all i) such that $h_0 = 1$ and $h_1 \le r$. We define $\mathbf{Gor}_{\le}(r,H)$ to be the affine subscheme of $\mathbb{A}(\mathcal{R}_d)$ defined by the ideal

$$I_H = \sum_{e=1}^{d-1} I_{h_e+1}\big(\mathrm{Cat}_e^d(F)\big).$$

We let $\mathbf{Gor}(r,H)$ be the open subscheme of $\mathbf{Gor}_{\le}(r,H)$ where some $h_e \times h_e$ minor is nonzero for each e. We denote by $\mathrm{Gor}(r,H)$ the corresponding reduced scheme, which is then the quasi-affine algebraic set parameterizing all $f \in \mathcal{R}_d$ such that $H(R/\operatorname{ann} f) = H$. Furthermore, let $\mathbf{PGor}(r,H)$ and $\mathrm{PGor}(r,H)$ be the projectivizations of $\mathbf{Gor}(r,H)$ and $\mathrm{Gor}(r,H)$, respectively. By virtue of the Macaulay duality (cf. lemma 1.3), $\mathrm{PGor}(r,H)$ parameterizes the graded Artinian Gorenstein quotients R/I with Hilbert function H.

We are now ready to define a set of $f \in \mathrm{Gor}(r,H)$ that split. This subset will depend on the Hilbert function of every additive component of f. Recall that if $f = \sum_{i=1}^n g_i$ is a regular splitting of f, then by lemma 3.27 (a and e)

$$H\big(R/\operatorname{ann}_R f\big) = \sum_{i=1}^n H\big(R/\operatorname{ann}_R g_i\big) - (n-1)\big(\delta_0 + \delta_d\big).$$

Definition 3.39: Let $r \geq 1$, $d \geq 2$ and $n \geq 1$. For each $i = 1, \dots, n$, suppose $H_i = (h_{i0}, \dots, h_{id})$ is a symmetric sequence of positive integers such that $h_{i0} = 1$ and $\sum_{i=1}^n h_{i1} \leq r$. Let $\underline{H} = (H_1, \dots, H_n)$ and $H = \sum_{i=1}^n H_i - (n-1)(\delta_0 + \delta_d)$, i.e $H = (h_0, \dots, h_d)$ where $h_0 = h_d = 1$ and $h_j = \sum_{i=1}^n h_{ij}$ for all $0 < j < d$. Define

$$\operatorname{Split}(r, \underline{H}) = \operatorname{Split}(r, d, n, \underline{H}) \subseteq \operatorname{Gor}(r, H)$$

to be the subset parameterizing all $f \in \mathcal{R}_d$ with the following property: There exist a regular splitting $f = \sum_{i=1}^n g_i$ such that $H(R/\operatorname{ann}_R g_i) = H_i$ for all i. Let $\operatorname{PSplit}(r, \underline{H}) \subseteq \operatorname{PGor}(r, H)$ be the projectivization of $\operatorname{Split}(r, \underline{H})$.

Obviously, $\operatorname{Split}(r, \underline{H})$ reduces to $\operatorname{Gor}(r, H)$ if $n = 1$. $\operatorname{Split}(r, \underline{H})$ is always a constructible subset of $\operatorname{Gor}(r, H)$, since it is the image of the morphism ρ, see lemma 3.40. Note that every linear map $k^s \to k^r$, that is, every matrix $C \in \operatorname{Mat}_k(r, s)$, induces a homomorphism of k-algebras $k[x_1, \dots, x_s]^{DP} \to \mathcal{R}$, determined by $[x_1, \dots, x_s] \mapsto [x_1, \dots, x_r]C$, that we denote ϕ_C.

Lemma 3.40: *Let $\underline{H} = (H_1, \dots, H_n)$ be an n-tuple of symmetric h-vectors $H_i = (h_{i0}, \dots, h_{id})$ such that $h_{i0} = 1$ for all i, and $\sum_{i=1}^n h_{i1} \leq r$. Let $s_i = h_{i1}$, $\underline{s} = (s_1, \dots, s_n)$ and $H = \sum_{i=1}^n H_i - (n-1)(\delta_0 + \delta_d)$, where δ_e is 1 in degree e and zero elsewhere. Define*

$$\Phi_{\underline{s}} = \left\{ (\phi_{C_1}, \dots, \phi_{C_n}) \;\middle|\; C_i \in \operatorname{Mat}_k(r, s_i) \text{ and } \dim_k \sum_{i=1}^n \operatorname{im} C_i = \sum_{i=1}^n s_i \right\}.$$

Then $\operatorname{Split}(r, \underline{H})$ is the image of the morphism

$$\rho : \Phi_{\underline{s}} \times \prod_{i=1}^n \operatorname{Gor}(s_i, H_i) \to \operatorname{Gor}(r, H),$$

$$\big((\phi_{C_1}, \dots, \phi_{C_n}), (g_1, \dots, g_n)\big) \mapsto \sum_{i=1}^n \phi_{C_i}(g_i).$$

Furthermore, the fiber over any closed point has dimension $\sum_{i=1}^n s_i^2$.

Proof: The first part is clear from definition 3.39. Note that the condition $\dim_k \sum_{i=1}^n \operatorname{im} C_i = \sum_{i=1}^n s_i$ in the definition of $\Phi_{\underline{s}}$ is equivalent to $\operatorname{rank} C_i = s_i$ and $\operatorname{im} C_i \cap \sum_{j \neq i} \operatorname{im} C_j = 0$ for all i.

To find the dimension of the fibers, we will start by describing a group that acts on $\Phi_{\underline{s}} \times \Pi_{i=1}^n \operatorname{Gor}(s_i, H_i)$ in such a way that the morphism ρ is constant on the orbits of the group action.

First, let the group $\Pi_{i=1}^n \operatorname{GL}_{s_i}$ act on $\Phi_{\underline{s}} \times \Pi_{i=1}^n \operatorname{Gor}(s_i, H_i)$ by

$$(P_i)_{i=1}^n \times \Big((\phi_{C_i})_{i=1}^n, (g_i)_{i=1}^n \Big) \mapsto \Big((\phi_{C_i P_i^{-1}})_{i=1}^n, (\phi_{P_i}(g_i))_{i=1}^n \Big).$$

Obviously, $\phi_{C_i P_i^{-1}} = \phi_{C_i} \circ \phi_{P_i^{-1}}$, and therefore, $\big(\phi_{C_i P_i^{-1}}\big)(\phi_{P_i} g_i) = \phi_{C_i}(g_i)$.

Second, let Σ_n denote the symmetric group on n symbols. A permutation $\sigma \in \Sigma_n$ acts on the n-tuple $\underline{H} = (H_1, \ldots, H_n)$ by permuting its coordinates, i.e., $\sigma(\underline{H}) = \big(H_{\sigma^{-1}(1)}, \ldots, H_{\sigma^{-1}(n)}\big)$. Let $G_{\underline{H}}$ be the subgroup of Σ_n defined by

$$G_{\underline{H}} = \{\sigma \in \Sigma_n \mid \sigma(\underline{H}) = \underline{H}\}.$$

Note that $G_{\underline{H}}$ is a product of symmetric groups. Indeed, let k be the number of distinct elements of $\{H_1, \ldots, H_n\}$. Call these elements $H'_1, \ldots, H'_k$, and let $n_i \geq 1$ be the number of j such that $H_j = H'_i$. Then $\sum_{i=1}^k n_i = n$, and

$$G_{\underline{H}} \cong \Sigma_{n_1} \times \cdots \times \Sigma_{n_k}.$$

The group $G_{\underline{H}}$ acts on $\Phi_{\underline{s}} \times \Pi_{i=1}^n \operatorname{Gor}(s_i, H_i)$ by

$$\sigma \times \Big((\phi_{C_i})_{i=1}^n, (g_i)_{i=1}^n \Big) \mapsto \Big((\phi_{C_{\sigma^{-1}(i)}})_{i=1}^n, (g_{\sigma^{-1}(i)})_{i=1}^n \Big).$$

Indeed, since any $\sigma \in G_{\underline{H}}$ fixes $\underline{H}$, we have $H_{\sigma^{-1}(i)} = H_i$, and in particular $s_{\sigma^{-1}(i)} = s_i$ since $s_i = h_{i1}$. Thus $C_{\sigma^{-1}(i)} \in \operatorname{Mat}_k(r, s_i)$ and $g_{\sigma^{-1}(i)} \in \operatorname{Gor}(s_i, H_i)$. Clearly, $\sum_{i=1}^n \phi_{C_{\sigma^{-1}(i)}}\big(g_{\sigma^{-1}(i)}\big) = \sum_{i=1}^n \phi_{C_i}(g_i)$. Thus the morphism ρ is constant on the orbits of also this group action.

Suppose $f \in \operatorname{im} \rho$. By theorem 3.18 f has a unique maximal regular splitting $f = \sum_{i=1}^m f'_i$, and every other regular splitting is obtained by grouping some of the summands. Evidently, since $f \in \operatorname{Split}(r, d, n, \underline{H})$, there is at least one way to group the summands such that $f = \sum_{i=1}^n f_i$ is a regular splitting and $H\big(R/\operatorname{ann}_R(f_i)\big) = H_i$ for all i, and there are only finitely many such "groupings". If $f = \sum_{i=1}^n f_i$ is any such expression, then clearly there exists $\big((\phi_{C_i})_{i=1}^n, (g_i)_{i=1}^n\big) \in \Phi_{\underline{s}} \times \Pi_{i=1}^n \operatorname{Gor}(s_i, H_i)$ such that $f_i = \phi_{C_i}(g_i)$ for all i.

Now, if $\big((\phi_{C_i})_{i=1}^n, (g_i)_{i=1}^n\big) \in \rho^{-1}(f)$ is any element of the fiber over f, then the expression $f = \sum_{i=1}^n \phi_{C_i}(g_i)$ is one of those finitely many groupings. Assume $\big((\phi_{C_i'})_{i=1}^n, (g_i')_{i=1}^n\big)$ is another element of the fiber such that the expression $f = \sum_{i=1}^n \phi_{C_i'}(g_i')$ corresponds to the same grouping. Since

$$H\big(R/\operatorname{ann}_R(\phi_{C_i}(g_i))\big) = H_i = H\big(R/\operatorname{ann}_R(\phi_{C_i'}(g_i'))\big),$$

there exists $\sigma \in G_{\underline{H}}$ such that $\phi_{C_i'}(g_i') = \phi_{C_{\sigma^{-1}(i)}}\big(g_{\sigma^{-1}(i)}\big)$ for all i. By composing with σ, we may assume $\phi_{C_i'}(g_i') = \phi_{C_i}(g_i)$ for all i. Note that $\partial(\phi_{C_i}(g_i)) = C_i\phi_{C_i}(\partial g)$ and $R_{d-1}(\partial g) = k^{s_i}$. It follows that $R_{d-1}\partial(\phi_{C_i}(g_i)) = \operatorname{im} C_i$, and therefore $\operatorname{im} C_i' = \operatorname{im} C_i$. Thus there exists $P_i \in \mathrm{GL}_{s_i}$ such that $C_i' = C_i P_i^{-1}$ for all i. Moreover, $\phi_{C_i}(g_i) = \phi_{C_i'}(g_i') = \phi_{C_i}(\phi_{P_i^{-1}} g_i')$ implies $g_i' = \phi_{P_i}(g_i)$ since ϕ_{C_i} is injective. This proves that $\big((\phi_{C_i'})_{i=1}^n, (g_i')_{i=1}^n\big)$ and $\big((\phi_{C_i})_{i=1}^n, (g_i)_{i=1}^n\big)$ are in the same orbit.

We have shown that the fiber $\rho^{-1}(f)$ over f is of a finite union of $(G_{\underline{H}} \times \Pi_{i=1}^n \mathrm{GL}_{s_i})$-orbits; one orbit for each grouping $f = \sum_{i=1}^n f_i$ of the maximal splitting of f such that $H\big(R/\operatorname{ann}_R(f_i)\big) = H_i$. By considering how the group acts on $\Phi_{\underline{s}}$, we see that different group elements give different elements in the orbit. It follows that the dimension of any fiber equals $\dim(\Pi_{i=1}^n \mathrm{GL}_{s_i}) = \sum_{i=1}^n s_i^2$. $\square$

Example 3.41: Let $n = 2$. The fiber over $f = x_1^{(d)} + x_2^{(d)} \in \mathrm{Split}(r, d, 2, \underline{H})$ is a single orbit. However, the fiber over $f = x_1^{(d)} + x_2^{(d)} + x_3^{(d)} \in \mathrm{Split}(r, d, 2, \underline{H})$ consists of three orbits, one for each of the expressions $f = x_i^{(d)} + \sum_{j \neq i} x_j^{(d)}$.

Remark 3.42: We have seen that ρ is constant on the orbits of the action of $G_{\underline{H}} \times \Pi_{i=1}^n \mathrm{GL}_{s_i}$. If the geometric quotient exists, we get an induced map

$$\Big(\Phi_{\underline{s}} \times \prod_{i=1}^n \mathrm{Gor}(s_i, H_i)\Big) \Big/ \Big(G_{\underline{H}} \times \prod_{i=1}^n \mathrm{GL}_{s_i}\Big) \to \mathrm{Gor}(r, H).$$

Let $U_i \subseteq \mathrm{Gor}(s_i, H_i)$ parameterize all $g \in k[x_1, \dots, x_{s_i}]^{DP}$ that do not have any non-trivial regular splitting, and $U \subseteq \mathrm{Split}(r, n, \underline{H})$ be those $f \in \mathcal{R}_d$ where $f = \sum_{i=1}^n g_i$ is a maximal splitting. The morphism above restricts to a map $\big(\Phi_{\underline{s}} \times \Pi_{i=1}^n U_i\big)/\big(G_{\underline{H}} \times \Pi_{i=1}^n \mathrm{GL}_{s_i}\big) \to U$. By the proof of lemma 3.40, this is a bijection.

Remark 3.43: We would like to identify $\mathfrak{W}_{\underline{s}} = \Phi_{\underline{s}} / \prod_{i=1}^n \mathrm{GL}_{s_i}$. Let $\mathrm{Grass}(s_i, r)$ be the Grassmannian that parameterizes s_i-dimensional k-vector subspaces of $\mathcal{R}_1 \cong k^r$. We may think of $\mathrm{Grass}(s_i, r)$ as the set of equivalence classes of injective, linear maps $k^{s_i} \hookrightarrow \mathcal{R}_1$, two maps being equivalent if they have the same image. It follows that $\mathfrak{W}_{\underline{s}}$ is the open subscheme of $\prod_{i=1}^n \mathrm{Grass}(s_i, r)$ parameterizing all n-tuples $W = (W_1, \dots, W_n)$ of subspaces $W_i \subseteq \mathcal{R}_1$ such that $\dim_k W_i = s_i$ and $W_i \cap \sum_{j \neq i} W_j = 0$ for all i.

Remark 3.44: For completeness, we want to describe the corresponding map of structure sheafs, $\rho^{\#} : \mathcal{O}_{\mathrm{Gor}(r,H)} \to \rho_* \mathcal{O}_{\Phi_{\underline{s}} \times \prod_{i=1}^n \mathrm{Gor}(s_i, H_i)}$.

For each i, let (c_{ijk}) be the entries of $C_i \in \mathrm{Mat}_k(r, s_i)$, i.e.

$$C_i = \begin{pmatrix} c_{i11} & \dots & c_{i1s_i} \\ \vdots & & \vdots \\ c_{ir1} & \dots & c_{irs_i} \end{pmatrix}.$$

Since $\sum_{i=1}^n \operatorname{im} C_i = \operatorname{im}[C_1, \dots, C_n]$, it follows that $\Phi_{\underline{s}}$ is isomorphic to the set of $r \times (\sum_i s_i)$-matrices of maximal rank. Let Y be the coordinate ring of $\mathrm{Mat}_k(r, \sum_i s_i)$. We choose to write Y as

$$Y = \bigotimes_{i=1}^{n}{}_k \, k\big[\{y_{ijk} \,\big|\, 1 \le j \le r \text{ and } 1 \le k \le s_i\}\big].$$

Let $\mathcal{S}^i = k[x_1, \dots, x_{s_i}]^{DP}$ and $S^i = k[\partial_1, \dots, \partial_{s_i}]$. By definition, $\mathrm{Gor}(s_i, H_i)$ parametrizes all $g_i \in \mathcal{S}^i_d$ such that the Hilbert function of $S^i / \operatorname{ann}_{S^i}(g_i)$ is H_i. The coordinate ring of $\mathbb{A}(\mathcal{S}^i_d)$ is $\mathcal{A}_i = k[\{z_{i\gamma} \,|\, \gamma \in A_i\}]$, where

$$A_i = \left\{ \gamma = (\gamma_1, \dots, \gamma_{s_i}) \in \mathbb{Z}^{s_i} \,\middle|\, \gamma_k \ge 0 \text{ for all } k \text{ and } \sum_{k=1}^{s_i} \gamma_k = d \right\}.$$

$\mathbf{Gor}_{\le}(s_i, H_i)$ is the affine subscheme of $\mathbb{A}(\mathcal{S}^i_d)$ whose coordinate ring is $\mathcal{A}_i / I_{H_i}$, cf. definition 3.38. Any $g_i \in \mathcal{S}^i_d$ can be written as

$$g_i = \sum_{\gamma \in A_i} a_{i\gamma} \prod_{k=1}^{s_i} x_k^{(\gamma_k)}.$$

It follows that

$$\sum_{i=1}^n \phi_{C_i}(g_i) = \sum_{i=1}^n \sum_{\gamma \in A_i} a_{i\gamma} \prod_{k=1}^{s_i} \Big(\sum_{j=1}^r c_{ijk} x_j \Big)^{(\gamma_k)}.$$

When we expand this, we see that for any $\alpha = (\alpha_1, \dots, \alpha_r) \in A$ (cf. equation (3.24)) the coefficient in front of $x^{(\alpha)} = \Pi_{j=1}^{r} x_j^{(\alpha_j)}$ is

$$\sum_{i=1}^{n} \sum_{\gamma \in A_i} a_{i\gamma} \cdot \sum_{\substack{\{\beta_{jk} \geq 0\} \\ \sum_{j=1}^{r} \beta_{jk} = \gamma_k \\ \sum_{k=1}^{s_i} \beta_{jk} = \alpha_j}} \prod_{j=1}^{r} \left[\binom{\alpha_j}{\beta_{j1}, \dots, \beta_{js_i}} \prod_{k=1}^{s_i} c_{ijk}^{\beta_{jk}} \right].$$

The multinomial

$$\binom{\alpha_j}{\beta_{j1}, \dots, \beta_{js_i}} = \frac{\alpha_j!}{\beta_{j1}! \cdots \beta_{js_i}!}$$

appears as a result of how the multiplication in $\mathcal{R}$ is defined.

The coordinate ring of $\mathbb{A}(\mathcal{R}_d)$ is $\mathcal{A} = k[\{z_\alpha \,|\, \alpha \in A\}]$. Let

$$\mathcal{A} \to Y \otimes_k \mathcal{A}_1 \otimes_k \cdots \otimes_k \mathcal{A}_n$$

be the k-algebra homomorphism induced by

$$z_\alpha \mapsto \sum_{i=1}^{n} \sum_{\gamma \in A_i} z_{i\gamma} \cdot \sum_{\substack{\{\beta_{jk} \geq 0\} \\ \sum_{j=1}^{r} \beta_{jk} = \gamma_k \\ \sum_{k=1}^{s_i} \beta_{jk} = \alpha_j}} \prod_{j=1}^{r} \left[\binom{\alpha_j}{\beta_{j1}, \dots, \beta_{js_i}} \prod_{k=1}^{s_i} y_{ijk}^{\beta_{jk}} \right]$$

for all $\alpha \in A$. This implies that $F = \sum_{\alpha \in A} z_\alpha x^{(\alpha)} \in \mathcal{A} \otimes_k \mathcal{R}_d$ is mapped to $\sum_{i=1}^{n} \phi_i(F_i)$, where $F_i = \sum_{\gamma \in A_i} z_{i\gamma} x^{(\gamma)} \in \mathcal{A}_i \otimes_k \mathcal{S}_d^i$ and

$$\phi_i : \begin{pmatrix} x_1 \\ \vdots \\ x_{s_i} \end{pmatrix} \mapsto \begin{pmatrix} y_{i11} & \dots & y_{ir1} \\ \vdots & & \vdots \\ y_{i1s_i} & \dots & y_{irs_i} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_r \end{pmatrix}.$$

Hence $\operatorname{Cat}_e^d(F) \mapsto \sum_{i=1}^{n} \operatorname{Cat}_e^d(\phi_i(F_i)) = \sum_{i=1}^{n} P_i \operatorname{Cat}_e^d(F_i) P_i'$ for suitable matrices P_i and P_i' with entries in Y. Since every $(h_{ie}+1) \times (h_{ie}+1)$-minor of $\operatorname{Cat}_e^d(F_i)$ is zero in $\mathcal{A}_i / I_{H_i}$, it follows that every $(h_e+1) \times (h_e+1)$-minor of $\operatorname{Cat}_e^d(F)$ maps to zero in $Y \otimes_k \mathcal{A}_1 / I_{H_1} \otimes_k \cdots \otimes_k \mathcal{A}_n / I_{H_n}$. This induces a map

$$\mathcal{A}/I_H \to Y \otimes_k \mathcal{A}_1 / I_{H_1} \otimes_k \cdots \otimes_k \mathcal{A}_n / I_{H_n}.$$

This ringhomomorphism is equivalent to a morphism of affine schemes;

$$\psi : \operatorname{Mat}_k(r, \textstyle\sum_i s_i) \times \displaystyle\prod_{i=1}^{n} \mathbf{Gor}_{\leq}(s_i, H_i) \to \mathbf{Gor}_{\leq}(r, H).$$

Let $f = \sum_{i=1}^n \phi_{C_i}(g_i) \in \operatorname{im}\psi \cap \operatorname{Gor}(r,H)$. Since $R_{d-1}\partial(\phi_{C_i}(g_i)) = \operatorname{im} C_i$, it follows that

$$\operatorname{im}\operatorname{Cat}_{d-1}^d\Big(\sum_{i=1}^n \phi_{C_i}(g_i)\Big) = R_{d-1}\partial\Big(\sum_{i=1}^n \phi_{C_i}(g_i)\Big) \subseteq \sum_{i=1}^n \operatorname{im} C_i.$$

Hence $\operatorname{rank}\operatorname{Cat}_{d-1}^d(f) = h_1 = \sum_{i=1}^n s_i$ implies that $\dim_k \sum_{i=1}^n \operatorname{im} C_i = \sum_{i=1}^n s_i$. Thus

$$\psi^{-1}\big(\operatorname{Gor}(r,H)\big) = \Phi_{\underline{s}} \times \prod_{i=1}^n \operatorname{Gor}(s_i, H_i).$$

Since $\mathbf{Gor}(r,H)$ is an open subscheme of $\mathbf{Gor}_{\le}(r,H)$, it follows that $(\psi,\psi^{\#})$ restricts to $(\rho,\rho^{\#})$.

The next lemma rewrites the definition of $\operatorname{Split}(r,\underline{H})$ so that it gives conditions on the ideal $I = \operatorname{ann}_R f$ instead of conditions on f directly.

Lemma 3.45: $\operatorname{PSplit}(r,\underline{H})$ *parameterizes all* $R/I \in \operatorname{PGor}(r,H)$ *that have the following properties: There exist subspaces* $V_1,\dots,V_n \subseteq R_1$ *with* $\dim_k V_i = h_{i1}$ *such that* $R_1 = I_1 \oplus \big(\oplus_{i=1}^n V_i\big)$ *and* $V_iV_j \subseteq I_2$ *for all* $i \neq j$. *Furthermore,* $S^i/I\cap S^i \in \operatorname{PGor}(h_{i1},H_i)$ *for all* i, *where* $S^i = k[V_i] \subseteq R$.

Proof: Pick $f \in \operatorname{Split}(r,\underline{H})$ such that $I = \operatorname{ann}_R f$. By definition 3.39 there exists a regular splitting $f = \sum_{i=1}^n g_i$ such that $H(R/\operatorname{ann}_R g_i) = H_i$ for all i, and $g_i \in \mathcal{S} = k[R_{d-1}(f)]^{DP}$ by corollary 2.10. Choose $V \subseteq R_1$ such that $R_1 = I_1 \oplus V$, and let $S = k[V] \cong \mathcal{S}^*$. By lemma 3.27(ai) we get $\operatorname{ann}_R f = (I_1) \oplus \operatorname{ann}_S f$. For all i let $W_i = R_{d-1}(g_i) \subseteq \mathcal{R}_1$ and define $V_i = (\sum_{j\neq i} W_j)^{\perp} \cap S \subseteq V$.

Note that $\dim_k W_i = \dim_k(R/\operatorname{ann}_R g_i)_1 = h_{i1}$. Since $\mathcal{S}_1 = \oplus_{i=1}^n W_i$, it follows that $S_1 = V = \oplus_{i=1}^n V_i$. Therefore $V_i \cong W_i^*$, and $\dim_k V_i = h_{i1}$. Let $S^i = k[V_i]$. By lemma 3.27 (b and c) there exist nonzero $D_i \in S^i_d$ such that

$$\operatorname{ann}_S f = \Big(\sum_{i<j} SV_iV_j\Big) \oplus \Big(\bigoplus_{i=1}^n \operatorname{ann}_{S^i}(g_i)\Big) + (D_2 - D_1,\dots,D_n - D_1).$$

It follows that $\operatorname{ann}_{S^i}(g_i) = \operatorname{ann}_S(f) \cap S^i = I \cap S^i$. Therefore,

$$I = (I_1) \oplus \Big(\sum_{i<j} SV_iV_j\Big) \oplus \Big(\bigoplus_{i=1}^n (I\cap S^i)\Big) + (D_2 - D_1,\dots,D_n - D_1). \tag{3.25}$$

In particular, $V_iV_j \subseteq I_2$ for all $i \neq j$. This proves all the properties listed in lemma 3.45. The opposite implication follows from equation (3.25). □

Remark 3.46: Note that the existence of the D_i's in equation (3.25) implies that the map $I \mapsto (I_1, \{V_i\}, \{I \cap S^i\})$ is not 1-to-1. This is easily understood if we translate to polynomials. Since annihilator ideals determine polynomial only up to a nonzero scalar, it follows that the fiber over $\{I \cap S^i = \operatorname{ann}_{S^i}(g_i)\}$ are all $I = \operatorname{ann}_R(f)$ such that $f = \sum_{i=1}^n c_i g_i$ and $c_i \neq 0$ for all i.

If $R/I \in \mathbf{PGor}(r, H)$, we denote by $\mathcal{T}_{R/I}$ the tangent space to $\mathbf{Gor}(r, H)$ (the affine cone over $\mathbf{PGor}(r, H)$) at a point corresponding to R/I. Recall that $\operatorname{PSplit}(r, \underline{H})$ parametrizes all $R/\operatorname{ann}_R f$ such that $f \in \mathcal{R}_d$ and there exist a regular splitting $f = \sum_{i=1}^n g_i$ such that $H(R/\operatorname{ann}_R g_i) = H_i$ for all i, cf. definition 3.39.

Theorem 3.47:
Assume $k = \bar{k}$. Let $r \geq 1$, $d \geq 4$ and $n \geq 1$. Let $\underline{H} = (H_1, \dots, H_n)$ be an n-tuple of symmetric h-vectors $H_i = (h_{i0}, \dots, h_{id})$ such that $\sum_{i=1}^n h_{i1} \leq r$ and $h_{i0} = 1$ for all i. Let $s_i = h_{i1} \geq 1$ and $H = \sum_{i=1}^n H_i - (n-1)(\delta_0 + \delta_d)$ where δ_e is 1 in degree e and zero elsewhere.

(a) *The dimension of* $\operatorname{PSplit}(r, \underline{H}) \subseteq \operatorname{PGor}(r, H) \subseteq \mathbb{P}(\mathcal{R}_d)$ *is*

$$\dim \operatorname{PSplit}(r, \underline{H}) = n - 1 + \sum_{i=1}^n \dim \operatorname{PGor}(s_i, H_i) + \sum_{i=1}^n s_i(r - s_i).$$

(b) $\operatorname{PSplit}(r, \underline{H})$ *is irreducible if* $\operatorname{PGor}(s_i, H_i)$ *is irreducible for all i.*

Let $R/I \in \operatorname{PSplit}(r, \underline{H})$. Choose $V_1, \dots V_n \subseteq R_1$ such that $\dim_k V_i = s_i$ for all i, $R_1 = I_1 \oplus (\oplus_{i=1}^n V_i)$ and $V_i V_j \subseteq I_2$ for all $i \neq j$, cf. lemma 3.45. Let $S^i = k[V_i]$ and $J_i = I \cap S^i \in \operatorname{PGor}(s_i, H_i)$. For each i, let β_{1j}^i be the minimal number of generators of degree j of J_i (as an ideal in S^i).

(c) *The dimension of the tangent space to the affine cone over* $\mathbf{PGor}(r, H)$ *at a point corresponding to R/I is*

$$\dim_k \mathcal{T}_{R/I} = \sum_{i=1}^n \dim_k \mathcal{T}_{S^i/J_i} + \sum_{i=1}^n s_i(r - s_i) + \sum_{i=1}^n \sum_{j \neq i} s_j \, \beta_{1,d-1}^i.$$

(d) *Assume in addition for all i that S^i/J_i is a smooth point of* $\mathbf{PGor}(s_i, H_i)$ *and $\beta_{1,d-1}^i = 0$. Then R/I is a smooth point of* $\mathbf{PGor}(r, H)$*. Moreover, R/I is contained in a unique irreducible component of the closure* $\overline{\operatorname{PSplit}(r, \underline{H})}$*. This component is also an irreducible component of* $\mathbf{PGor}(r, H)$*.*

In particular, if $\mathbf{PGor}(s_i, H_i)$ *is irreducible and generically smooth for all* i*, and* $\beta_{1,d-1}(J_i) = 0$ *for general* $S^i/J_i \in \mathbf{PGor}(s_i, H_i)$*, then the closure* $\overline{\mathrm{PSplit}(r, \underline{H})}$ *is an irreducible component of* $\mathbf{PGor}(r, H)$*, and* $\mathbf{PGor}(r, H)$ *is smooth in some non-empty open subset of* $\mathrm{PSplit}(r, \underline{H})$*.*

This is a generalization of [IS, theorem 3.11].

Proof: (a) follows from lemma 3.40, since the lemma implies that

$$\dim \mathrm{Split}(r, \underline{H}) = \sum_{i=1}^{n} \dim \mathrm{Gor}(s_i, H_i) + \sum_{i=1}^{n} r s_i - \sum_{i=1}^{n} s_i^2.$$

Alternatively, we can count dimensions using equation (3.25), just note that the V_i's are determined only modulo I_1. Let $s = \dim_k (R/I)_1 = \sum_{i=1}^{n} s_i$. Then we get $s(r-s)$ for the choice of $I_1 \subseteq R_1$, $s_i(s - s_i)$ for the choice on V_i (modulo I_1), $\dim \mathrm{PGor}(s_i, H_i)$ for the choice of $I \cap S^i \subseteq S^i$, and finally $n-1$ for the choice of $D_2 - D_1, \dots, D_n - D_1 \in R_d$. Adding these together proves (a).

(b) follows immediately from lemma 3.40.

To prove (c), we use theorem 3.9 in [IK99] (see also remarks 3.10 and 4.3 in the same book), which tells us that $\dim_k \mathcal{T}_{R/I} = \dim_k (R/I^2)_d$. Note that $H(S^i/J_i) = H_i$ for all i by definition of $\mathrm{PSplit}(r, \underline{H})$.

Assume first that $I_1 = 0$. Note that this implies $R_1 = \oplus_{i=1}^{n} V_i$, and therefore $R = S^1 \otimes_k \cdots \otimes_k S^n$ and $r = \sum_{i=1}^{n} s_i$. By equation (3.25) we have

$$I_e = \Big(\sum_{i<j} R_{e-2} S_1^i S_1^j \Big) \oplus \Big(\bigoplus_{i=1}^{n} J_{i,e} \Big)$$

as a direct sum of k-vector subspaces of R_e for all degrees $e < d$. In particular, $I_1 = 0$ is equivalent to $J_{i,1} =$ for all i.

Let $S = S^1 \otimes_k \cdots \otimes_k S^{n-1}$, $J_S = I \cap S$ and $s = \sum_{i=1}^{n-1} s_i$, and let $T = S^n$, $J_T = I \cap T$ and $t = s_n$. Then $I_e = R_{e-2} S_1 T_1 \oplus J_{S,e} \oplus J_{T,e}$ for all $e < d$. It follows for all $2 \le e \le d-2$ that

$$\begin{aligned} I_e \cdot I_{d-e} = {} & R_{d-4} S_2 T_2 \oplus J_{S,e} \cdot J_{S,d-e} \oplus J_{T,e} \cdot J_{T,d-e} \\ & \oplus T_1 (S_{d-e-1} J_{S,e} + S_{e-1} J_{S,d-e}) \oplus S_1 (T_{d-e-1} J_{T,e} + T_{e-1} J_{T,d-e}). \end{aligned}$$

Since $I_1 = 0$ implies $J_{S,1} = J_{T,1} = 0$, and $\sum_{e=2}^{d-2} S_{d-e-1} J_{S,e} = S_1 J_{S,d-2}$, we get

$$(I^2)_d = \sum_{e=2}^{d-2} I_e \cdot I_{d-e} = R_{d-4} S_2 T_2 \oplus (J_S^2)_d \oplus (J_T^2)_d \oplus S_1 T_1 J_{S,d-2} \oplus S_1 T_1 J_{T,d-2}.$$

Because $R_d = S_d \oplus T_1 S_{d-1} \oplus R_{d-4} S_2 T_2 \oplus S_1 T_{d-1} \oplus T_d$, it follows that

$$\big(R/I^2\big)_d = \big(S/J_S^2\big)_d \oplus \big(T/J_T^2\big)_d \oplus T_1 \big(S_{d-1}/S_1 J_{S,d-2}\big) \oplus S_1 \big(T_{d-1}/T_1 J_{T,d-2}\big).$$

To find the dimension of $(R/I^2)_d$, we need the dimension of $S_{d-1}/S_1 J_{S,d-2}$. We note that $S_{d-1}/S_1 J_{S,d-2} \cong S_{d-1}/J_{S,d-1} \oplus J_{S,d-1}/S_1 J_{S,d-2}$ as k-vector spaces. And furthermore, $\dim_k S_{d-1}/J_{S,d-1} = \dim_k (S/J_S)_{d-1} = \dim_k (S/J_S)_1 = s$ and $\dim_k (J_{S,d-1}/S_1 J_{S,d-2}) = \beta_{1,d-1}^{J_S}$. Thus

$$\dim_k T_1 \big(S_{d-1}/S_1 J_{S,d-2}\big) = t\big(s + \beta_{1,d-1}^{J_S}\big),$$

and similarly $\dim_k S_1 \big(T_{d-1}/T_1 J_{T,d-2}\big) = s\big(t + \beta_{1,d-1}^{J_T}\big)$. Therefore,

$$\dim_k \big(R/I^2\big)_d = \dim_k \big(S/J_S^2\big)_d + \dim_k \big(T/J_T^2\big)_d + 2st + t\beta_{1,d-1}^{J_S} + s\beta_{1,d-1}^{J_T}.$$

Note that $\beta_{1,d-1}^{J_S} = \sum_{i=1}^{n-1} \beta_{1,d-1}^{i}$ since $d \geq 4$. Induction on n now gives

$$\dim_k \big(R/I^2\big)_d = \sum_{i=1}^{n} \dim_k \big(S^i/J_{S^i}^2\big)_d + \sum_{i=1}^{n} s_i(r - s_i) + \sum_{i=1}^{n} (r - s_i)\beta_{1,d-1}^{i}. \qquad (*)$$

Next we no longer assume $I_1 = 0$. Let $V = \oplus_{i=1}^n V_i$, $S = k[V]$, $J = I \cap S$ and $s = \sum_{i=1}^n s_i \leq r$. Let $T = k[I_1]$ so that $R = S \otimes_k T$. Since $I_e = R_{e-1} T_1 \oplus J_e$ for all e, it follows that $(I^2)_d = R_{d-2} T_2 \oplus T_1 J_{d-1} \oplus (J^2)_d$. This implies that $\dim_k (R/I^2)_d = \dim_k (S/J^2)_d + s(r - s)$. Since $J_1 = 0$, we can find $\dim_k (S/J^2)_d$ by using $(*)$ (with r replaced by s). Doing this proves (c).

To prove (d), we use the morphism $\rho : \Phi_{\underline{s}} \times \Pi_{i=1}^n \operatorname{Gor}(s_i, H_i) \to \operatorname{Gor}(r, H)$ from lemma 3.40. For each i let X_i be the unique irreducible component of $\operatorname{Gor}(s_i, H_i)$ containing S^i/J_i. It is indeed unique since S^i/J_i is a smooth point on $\operatorname{PGor}(s_i, H_i)$. Let $\rho' : \Phi_{\underline{s}} \times \Pi_{i=1}^n X_i \to \operatorname{Gor}(r, H)$ be the restriction of ρ, and let $\overline{\operatorname{im} \rho'}$ be the closure of $\operatorname{im} \rho'$ in $\operatorname{Gor}(r, H)$. Note that $\overline{\operatorname{im} \rho'}$ is irreducible. It is well known that the fiber $(\rho')^{-1}(R/I)$ must have dimension

$$\geq \dim \Big(\Phi_{\underline{s}} \times \prod_{i=1}^{n} X_i\Big) - \dim \overline{\operatorname{im} \rho'}.$$

Furthermore, $\dim(\rho')^{-1}(R/I) \leq \dim \rho^{-1}(R/I) = \sum_{i=1}^n s_i^2$ by lemma 3.40. Note that $\dim X_i = \dim_k \mathcal{T}_{S^i/J_i}$ since S^i/J_i is a smooth point on $\mathrm{PGor}(s_i, H_i)$. Since $\beta^i_{1,d-1} = 0$, it follows from (c) that the dimension of $\mathrm{Gor}(r, H)$ at R/I is

$$\begin{aligned}\dim_{R/I} \mathrm{Gor}(r,H) &\geq \dim \overline{\operatorname{im} \rho'} \\ &\geq \dim\Big(\Phi_{\underline{s}} \times \prod_{i=1}^n X_i\Big) - \sum_{i=1}^n s_i^2 \\ &= \sum_{i=1}^n \dim_k \mathcal{T}_{S^i/J_i} + \sum_{i=1}^n s_i(r - s_i) \\ &= \dim_k \mathcal{T}_{R/I} \geq \dim_{R/I} \mathrm{Gor}(r,H)\end{aligned}$$

Hence $\dim_k \mathcal{T}_{R/I} = \dim_{R/I} \mathrm{Gor}(r,H) = \dim \overline{\operatorname{im} \rho'}$. Thus R/I is a smooth point on $\mathrm{PGor}(r, H)$, and is therefore contained in a unique irreducible component X of $\mathrm{PGor}(r, H)$. Since $\dim X = \dim_{R/I} \mathrm{Gor}(r,H) = \dim \overline{\operatorname{im} \rho'}$, it follows that only one component of $\overline{\mathrm{Split}(r, \underline{H})}$ contains R/I, namely $\overline{\operatorname{im} \rho'}$.

The final statement follows easily. □

Remark 3.48: We assume in this remark that $d = 3$. We see from the proof of theorem 3.47 that the dimension formula in (a) is valid also in this case. But the formula in (b) is no longer true in general. We need an additional correction term on the right-hand side. It is not difficult to show that this correction term is $\sum_{i<j<k} s_i s_j s_k$. Note that if $d = 3$ then $\beta^i_{1,d-1} = \binom{s_i}{2}$ for all i. It follows that the tangent space dimension when $d = 3$ is

$$\dim_k \mathcal{T}_{R/I} = \sum_{i=1}^n \dim_k \mathcal{T}_{S^i/J_i} + \sum_{i=1}^n s_i(r - s_i) + \binom{s}{3} - \sum_{i=1}^n \binom{s_i}{3}.$$

Thus $\dim_k \mathcal{T}_{R/I} > \dim \mathrm{PSplit}(r, \underline{H})$ when $n \geq 2$, except $n = 2$ and $s_1 = s_2 = 1$.

Remark 3.49: Let $\hat{\beta}_{ij}$ be the shifted graded Betti numbers of $R/\operatorname{ann}_R f$. The Hilbert function of $R/\operatorname{ann}_R f$ for a general $f \in \mathcal{R}_d$ is equal to

$$H_{d,r}(e) = \min(\dim_k R_e, \dim_k R_{d-e})$$

by [IK99, Proposition 3.12]. This is equivalent to $\operatorname{ann}_R(f)_e = 0$ for all $e \leq d/2$, that is, $\hat{\beta}_{1j} = 0$ for all $j \leq d/2 - 1$. It follows that $\hat{\beta}_{ij} = 0$ for all $i > 0$ and

$j \le d/2 - 1$. Recall that $\hat{\beta}_{ij} = \hat{\beta}_{r-i,d-j}$ since the minimal resolution of $R/\operatorname{ann}_R f$ is symmetric, hence $\hat{\beta}_{ij} = 0$ for all $i < r$ and $j \ge d - (d/2 - 1) = d/2 + 1$. This shows that, if $d = 2m$, then $\hat{\beta}_{ij} = 0$ for all $j \ne m$, and if $d = 2m+1$, then $\hat{\beta}_{ij} = 0$ for all $j \ne m, m+1$, except $\hat{\beta}_{00} = \hat{\beta}_{rd} = 1$. Therefore, when $d \ge 6$, it follows that $\beta_{1,d-1} = \hat{\beta}_{1,d-2} = 0$ for a general $f \in \mathcal{R}_d$.

It is known that $\operatorname{PGor}(r, H)$ is smooth and irreducible for $r \le 3$. (For $r = 3$ see [Die96] and [Kle98].) It is also known to be generically smooth in some cases with $r > 3$, see [IK99]. Hence we can use theorem 3.47 to produce irreducible, generically smooth components of $\operatorname{PGor}(r, H)$ for suitable H when $d \ge 6$.

Chapter 4

Degenerate splittings

In chapter 3 we proved that if $A \in M_f$ is idempotent, then the polynomial g satisfying $\partial g = A\partial f$ is an additive component of f. In this chapter we will study what happens when A is nilpotent. The idea is to "deform" the situation so that $f, g \in \mathcal{R}_d$ becomes $f_t, g_t \in \mathcal{R}_d[t_1, \ldots, t_n]$ and A becomes an idempotent $A_t \in \mathrm{Mat}_{k[t_1,\ldots,t_n]}(r, r)$, preserving the relation $\partial g_t = A_t \partial f_t$.

Our investigations in this chapter were guided by the following question.

Question 4.1: *Given $f \in \mathcal{R}_d$, $d \geq 3$, is it possible to find $f_t \in \mathcal{R}_d[t_1, \ldots, t_n]$ such that $f_0 = f$ and f_t splits regularly $\dim_k M_f - 1$ times over $k(t_1, \ldots, t_n)$?*

Sections 4.1 and 4.2 deal with cases where we can give a positive answer to this question, and cases in which we can produce counter examples, respectively. The motivation behind the question is that $\dim_k M_f - 1$ is an upper bound for the number of times that f_t can split when we require $f_0 = f$, see lemma 4.2 below. There is also a flatness condition we would like f_t to satisfy, but we will ignore that in this paper, cf. remark 4.4.

Note that $\dim_k M_f - 1 = r\beta_{11} + \beta_{1d}$ by lemma 2.17. Since f_t can split at most $r - 1$ times (that is, have at most r additive components), we see that question 4.1 automatically has a negative answer if $\beta_{11} > 0$, i.e. if $\mathrm{ann}_R(f)_1 \neq 0$.

Recall that by corollary 2.10 the "regular splitting properties" of f does not change if we add dummy variables since any regular splitting must happen inside the subring $k[R_{d-1}(f)]^{DP} \subseteq \mathcal{R}$. It is not so for degenerate splittings, as seen in example 4.3 below. For this reason most f we consider in this chapter will satisfy

$\operatorname{ann}_R(f)_1 = 0$. Note that this implies that $\dim_k M_f - 1 = \beta_{1d}$.

We will now prove that the number $\dim_k M_f - 1$ in question 4.1 is an upper bound. Recall that by theorem 3.18 the regular splittings of f_t inside $\mathcal{R}_d \otimes_k k(t_1, \ldots, t_n) = \mathcal{R}_d(t_1, \ldots, t_n)$ are determined by the idempotents in

$$M_{f_t} = \{A \in \operatorname{Mat}_{k(t_1,\ldots,t_n)}(r, r) \mid I_2(\partial\, A\partial) \subseteq \operatorname{ann}_{R(t_1,\ldots,t_n)} f_t\}.$$

Lemma 4.2: *Let $f_t \in \mathcal{R}_d[t_1, \ldots, t_n]$. Then $\dim_{k(t_1,\ldots,t_n)} M_{f_t} \leq \dim_k M_{f_0}$. In particular, if f_t splits regularly m times, then $m \leq \dim_k M_{f_0} - 1$.*

Proof: First assume that $n = 1$. Then $f_t = \sum_{k\geq 0} t^k f_k$ for some $f_k \in \mathcal{R}_d$. Let $A_1, \ldots, A_m \in \operatorname{Mat}_{k(t)}(r, r)$ form a basis for M_{f_t} as a $k(t)$-vector space. We may multiply by denominators and assume $A_i \in \operatorname{Mat}_{k[t]}(r, r)$ for all i. Write $A_i = \sum_{k=0}^{a_i} t^k A_{ik}$ with $A_{ik} \in \operatorname{Mat}_k(r, r)$. Assume that $A_{10}, \ldots, A_{m0}$ are linearly dependent, say $\sum_{i=0}^{m} c_i A_{i0} = 0$ where $c_i \in k$, not all zero. Choose j such that $a_j = \max\{a_i \mid c_i \neq 0\}$, and replace A_j with $(c_j t)^{-1} \sum_{i=0}^{m} c_i A_i$. The new A_i's still form a $k(t)$-basis for M_{f_t}, and the degree of A_j as a polynomial in t has decreased. Continuing this process, we arrive at a basis $\{A_i\}$ such that $A_{10}, \ldots, A_{m0}$ are linearly independent.

For every i, since $A_i \in M_{f_t}$, there exists a polynomial $g_i \in \mathcal{R}_d(t)$ such that $\partial g_i = A_i \partial f_t$. And because $A_i \in \operatorname{Mat}_{k[t]}(r, r)$ it follows that $g_i \in \mathcal{R}_d[t]$. Thus $g_i = \sum_{k\geq 0} t^k g_{ik}$ for suitable $g_{ik} \in \mathcal{R}_d$. It follows that

$$\sum_{k\geq 0} t^k \partial g_{ik} = \partial g_i = A_i \partial f_t = \sum_{j,k\geq 0} t^{j+k} A_{ij} \partial f_k.$$

In particular, $\partial g_{i0} = A_{i0} \partial f_0$, implying $A_{i0} \in M_{f_0}$ for all i. Since $\{A_{i0}\}$ are linearly independent, it follows that $\dim_k M_{f_0} \geq \dim_{k(t)} M_{f_t}$.

For general $n \geq 1$, let $k' = k(t_1, \ldots, t_{n-1})$. There exist $f'_k \in \mathcal{R}_d[t_1, \ldots, t_{n-1}]$ such that $f_t = \sum_{k\geq 0} t_n^k f'_k$, and the above argument shows that $\dim_{k'} M_{f'_0} \geq \dim_{k'(t_n)} M_{f_t}$. Induction on n proves that $\dim_k M_{f_0} \geq \dim_{k(t_1,\ldots,t_n)} M_{f_t}$.

If f_t splits regularly m times, then M_{f_t} contains $m+1$ orthogonal idempotents, hence $\dim_k M_{f_0} \geq \dim_{k(t_1,\ldots,t_n)} M_{f_t} \geq m + 1$. □

Example 4.3: Let $d \geq 4$ and $f = x_1^{(d-2)} x_2^{(2)} \in \mathcal{R} = k[x_1, x_2]^{DP}$. With $R = k[\partial_1, \partial_2]$ we get $\operatorname{ann}_R f = (\partial_2^3, \partial_1^{d-1})$ and $M_f = \langle I \rangle$, hence f cannot be a specialization of an $f_t \in \mathcal{R}_d[t_1, \ldots, t_n]$ that splits. But it is easy to find $f_t \in k[t][x_1, x_2, x_3]^{DP}$

such that $f_0 = f$ and f_t splits! Indeed, one such choice is

$$f_t = t^{-3}[t(x_1 + tx_2 + t^3x_3)^{(d)} - (x_1 + t^2x_2)^{(d)} + (1-t)x_1^{(d)}] \equiv f \bmod (t).$$

Note that even this is in concordance with lemma 4.2.

Remark 4.4: Let $f \in \mathcal{R}_d$. When we look for $f_t \in \mathcal{R}_d[t_1, \ldots, t_n]$ such that $f_0 = f$, there are several properties we would like f_t to have. Our main concern in this chapter is that we want f_t to split regularly $\dim_k M_f - 1$ times over $k(t_1, \ldots, t_n)$, giving a positive answer to question 4.1. But in addition, we would like $R(t_1, \ldots, t_n)(f_t) \cong R(t_1, \ldots, t_n)/\operatorname{ann}_{R(t_1,\ldots,t_n)}(f_t)$ and $R(f) \cong R/\operatorname{ann}_R f$ to have equal Hilbert functions, for the following reason.

Let $k_t = k[t_1, \ldots, t_n]$, $\mathcal{R}_t = \mathcal{R} \otimes_k k_t$ and $R_t = R \otimes_k k_t$. An $f_t \in \mathcal{R}_d \otimes_k k_t$ determines a family $k_t \to \mathcal{R}_t/R_t(f_t)$. Let $C_t = R_t(f_t) = R(f_t) \otimes_k k_t \subseteq \mathcal{R}_t$. It is easy to show that $\mathcal{R}/C_0 = \mathcal{R}_t/C_t \otimes_{k_t} k_t/(t_1, \ldots, t_n) = \mathcal{R}/R(f_0)$, thus $R(f_0)$ is a specialization of the family. We would like this family to be flat, at least in an open neighbourhood of the origin. This simply means that the generic fiber $R(t_1, \ldots, t_n)(f_t)$ has the same Hilbert function as $R(f_0)$. (The condition that f_t should have a regular splitting of length $\dim_k M_f$ inside $\mathcal{R}_d(t_1, \ldots, t_n)$, is also a statement about the generic fiber.)

Note that, although the family $k_t \to R_t/J_t$ where $J_t = \operatorname{ann}_{R_t}(f_t)$ is maybe more natural to consider, it is also more problematic, since $f_t \mapsto R_t/J_t \mapsto R/J_0$ does not generally commute with med $f_t \mapsto f_0 \mapsto R/\operatorname{ann}_R(f_0)$. In general we only have an inclusion $J_0 \subseteq \operatorname{ann}_R(f_0)$. If $f \neq 0$, then $(J_0)_d = \operatorname{ann}_R(f_0)_d$, and since $\operatorname{ann}_R(f_0)$ is determined by its degree d piece by lemma 1.2a, it follows that $\operatorname{ann}_R(f_0) = \operatorname{sat}_{\le d} J_0 = \oplus_{e=0}^d \{D \in R_e \mid R_{d-e} \cdot D \subseteq J_0\} + (R_{d+1})$.

Of course we would like $R(f)$ to be a specialization of a flat, splitting family, but in this chapter we study question 4.1 without the additional flatness requirement. Note that we do not know of any example in which question 4.1 has a positive answer, but would have had a negative answer if we had required $H(R(t_1, \ldots, t_n)(f_t)) = H(R(f))$.

4.1 Positive results

In this section we consider some cases where we are able to prove that question 4.1 has a positive answer. We start with a result that effectively "deforms" a relation $\partial g = A\partial f$ with A nilpotent to a relation $\partial g_t = A_t \partial f_t$ with A_t idempotent. The proof is an explicit construction of f_t using the nilpotent matrix $A \in M_f$ as input data. This will later allow us to answer question 4.1 positively when $r \leq 4$.

Suppose A is nilpotent, i.e. $A^k = 0$ for $k \gg 0$. The *index* of A is defined by

$$\operatorname{index}(A) = \min\{k \geq 1 \mid A^k = 0\}.$$

Let A be a nilpotent matrix of index $n+1$, i.e., $A^{n+1} = 0$ and $A^n \neq 0$. Then $A^0 = I, A, A^2, \dots, A^n$ are linearly independent. To see why, assume there is a non-zero relation $\sum_{k=0}^{n} c_k A^k = 0$, and let $i = \min\{k \mid c_k \neq 0\} \leq n$. Multiplying the relation by A^{n-i} implies that $c_i A^n = 0$, which is a contradiction.

Theorem 4.5:
Let $d \geq 3$ and $f \in \mathcal{R}_d$. Assume that M_f contains a non-zero nilpotent matrix $A \in \operatorname{Mat}_k(r,r)$, and let $n = \operatorname{index}(A) - 1 \geq 1$. Then f is a specialization of some $f_t \in \mathcal{R}_d[t_1, \dots, t_n]$ that splits regularly n times inside $\mathcal{R}_d(t_1, \dots, t_n)$.

Proof: Since M_f is closed under multiplication by proposition 2.21, it contains $k[A] = \langle I, A, \dots, A^n \rangle$, the k-algebra generated by A.

Choose an idempotent $E \in \operatorname{Mat}_k(r,r)$ such that $\ker E = \ker A^n$. (I.e. let $U = \ker A$ and choose W such that and $U \cap W = 0$ and $U + W = k^r$. Then let E represent the linear map that acts as the identity on W and takes U to 0.) This implies that $A^n E = A^n$ and that there exists a matrix $Q \in \operatorname{Mat}_k(r,r)$ such that $E = QA^n$. Note that $EA = 0$. Define

$$A_t = A + tE.$$

Then $A_t^n = A^n + tA^{n-1}E + \cdots + t^n E$, and

$$A_t^{n+1} = A^{n+1} + tA^n E + \cdots + t^{n+1} E = tA_t^n.$$

It follows that $(A_t^n)^2 = t^n A_t^n$, hence $t^{-n} A_t^n$ is idempotent. Now define

$$P = I + \sum_{k=1}^{n} t^k A^{n-k} Q.$$

P is chosen so that $A_t^n = PA^n$. Since $\det P \equiv 1 \pmod{t}$, P is an invertible element of $\mathrm{Mat}_{k(t)}(r,r)$. Let ϕ_P be the homomorphism defined by $x \mapsto P^{\mathsf{T}}x$ on $\mathcal{R}$ and by $\partial \mapsto P^{-1}\partial$ on R, as usual. Recall that for all $g \in \mathcal{R}$ and $D \in R$ we have $\phi_P(Dg) = \phi_P(D)\phi_P(g)$. Also note that $(PA^n)^2 = t^n PA^n$ implies $A^n PA^n = t^n A^n$.

Since $A^n \in M_f$, there exists a polynomial $g \in \mathcal{R}_d$ such that $\partial g = A^n \partial f$. Let $g_t = \phi_P(g) = \sum_{k \geq 0} t^k g_k \in \mathcal{R}_d[t]$, and define

$$f_t = f + t^{-n}\left(g_t - \sum_{k=0}^{n} t^k g_k\right) = f + \sum_{k>0} t^k g_{n+k} \in \mathcal{R}_d[t].$$

We want to prove that $A_t \in M_{f_t}$. We start by calculating ∂g_t.

$$\partial g_t = \partial \phi_P(g) = P\phi_P(\partial g) = P\phi_P(A^n \partial f) = A_t^n \phi_P(\partial f) \tag{4.1}$$

Multiplying (4.1) by A^n, and using $A^n PA^n = t^n A^n$, gives $A^n \partial g_t = t^n \phi_P(\partial g)$. Since the entries of ∂g and $\phi_P(\partial g)$ are in $\mathcal{R}[t]$, this implies that $A^n \partial g_i = 0$ for all $i < n$, and $A^n \partial g_n = \partial g = A^n \partial f$. In particular, $E\partial g_n = QA^n \partial g_n = E\partial f$.

When we multiply (4.1) by A_t, the result is $A_t \partial g_t = t\partial g_t$. As polynomials in t this equals $(A + tE)(\sum_{i\geq 0} t^i \partial g_i) = t(\sum_{i \geq 0} t^i \partial g_i)$, and implies that

$$A\partial g_i + E\partial g_{i-1} = \partial g_{i-1} \text{ for all } i \geq 0.$$

(Actually, this implies that $A\partial g_i = \partial g_{i-1}$ for all $0 \leq i \leq n$, since $E = QA^n$ and we have already proven that $A^n \partial g_{i-1} = 0$ for $i \leq n$.) Also, since $A \in M_f$, there exists $h \in \mathcal{R}_d$ such that $\partial h = A\partial f$.

Putting all this together, we get

$$\begin{aligned} A_t \partial f_t &= \big(A + tE\big)\left(\partial f + \sum_{k>0} t^k \partial g_{n+k}\right) \\ &= A\partial f + tE\partial f + \sum_{k>0} t^k A\partial g_{n+k} + \sum_{k>0} t^{k+1} E\partial g_{n+k} \\ &= \partial h + \sum_{k>0} t^k \big(A\partial g_{n+k} + E\partial g_{n+k-1}\big) \\ &= \partial h + \sum_{k>0} t^k \partial g_{n-1+k} = \partial\big(h + tg_n + t^2 g_{n+1} + \dots\big). \end{aligned}$$

This proves that $A_t \in M_{f_t}$. And since M_{f_t} is closed under multiplication, it follows that $k[A_t] = \langle I, A_t, \dots, A_t^n \rangle \subseteq M_{f_t}$.

Since $E' = I - t^{-n}A_t^n$ is idempotent, we may apply theorem 3.18. It tells us that f_t has a regular splitting with two additive components, $t^{-n}g_t$ and $f' = t^{-n}(t^n f - g_0 - tg_1 - \cdots - t^n g_n)$, and furthermore that

$$k[A_t] \cdot E' = \langle E', A_t E', \dots, A_t^{n-1}E' \rangle \subseteq M_{f'}^{E'}.$$

Hence we may repeat our procedure on f'. By induction on n, we arrive at some $f_{\underline{t}} \in \mathcal{R}_d[t_1, \dots, t_n]$ such that $f_0 = f$ and $f_{\underline{t}}$ splits regularly n times. □

Remark 4.6: The choice of E in the proof of theorem 4.5 boils down to choosing $Q \in \mathrm{Mat}_k(r,r)$ such that $A^n Q A^n = A^n$, and then letting $E = QA^n$. This then implies $\ker E = \ker A^n$ and that E is idempotent. We note that Q is certainly not unique. If A^n is in Jordan normal form, then we may let $Q = A^{\mathsf{T}}$. This is what we will do in most explicit cases.

***Corollary* 4.7:** *Suppose* $k = \bar{k}$ *and* $d \geq 3$. *Let* $f \in \mathcal{R}_d$. *Assume that* $\mathrm{ann}_R(f)_1 = 0$, *and let* β_{1j} *be the minimal number of generators of* $\mathrm{ann}_R f$ *of degree* j. *Then* f *has a regular or degenerate splitting if and only if* $\beta_{1d} > 0$.

Proof: Since $\beta_{11} = 0$, we have $\dim_k M_f - 1 = \beta_{1d}$. Thus $\beta_{1d} > 0$ if and only if M_f contains a matrix $A \notin \langle I \rangle$. Since $k = \bar{k}$, we may assume that A is either idempotent or nilpotent. It follows from theorem 3.18 that M_f contains a non-trivial idempotent if and only if f splits regularly. By theorem 4.5, if $A \in M_f$ is non-zero and nilpotent, then f has a degenerate splitting. Finally, if f has a degenerate splitting, then $\dim_k M_f - 1 \geq 1$ by lemma 4.2. □

Let $f \in \mathcal{R}_d$ with $d \geq 3$. If M_f is generated by one matrix, then theorem 4.5 answers question 4.1 affirmatively, that is, we can find $f_t \in \mathcal{R}_d[t_1, \dots, t_n]$ such that $f_0 = f$ and f_t splits regularly $\dim_k M_f - 1$ times over $k(t_1, \dots, t_n)$. This is the best we can hope for by lemma 4.2, and our next theorem proves that this is always possible when $r \leq 4$. But first we need some facts about matrices.

***Lemma* 4.8:** *Given matrices* $A, B \in \mathrm{Mat}_k(r,r)$ *the following are true.*

(a) $\operatorname{rank} A + \operatorname{rank} B - r \leq \operatorname{rank}(AB) \leq \min(\operatorname{rank} A, \operatorname{rank} B)$.

(b) *If* $AB = BA$, $A \neq 0$ *and* B *is nilpotent, then* $\operatorname{rank}(AB) < \operatorname{rank} A$.

(c) *If* $AB = BA$, $\operatorname{rank} A = r - 1$ *and* A *is nilpotent, then* $A^{r-1} \neq 0$ *and*

$$B \in k[A] = \langle I, A, \dots, A^{r-1} \rangle.$$

Proof: (a) The right inequality follows from the inclusions $\ker(AB) \supseteq \ker B$ and $\operatorname{im}(AB) \subseteq \operatorname{im} A$. To prove the left inequality, let β be the restriction of the map $B : k^r \to k^r$ to $\ker(AB)$. Obviously, $\ker \beta = \{v \in \ker(AB) \mid Bv = 0\} = \ker B$, and $\operatorname{im} \beta \subseteq \ker A$. Hence

$$\dim_k \ker(AB) = \dim_k \ker \beta + \dim_k \operatorname{im} \beta \leq \dim_k \ker B + \dim_k \ker A,$$

which is equivalent to $\operatorname{rank}(AB) \geq \operatorname{rank} A + \operatorname{rank} B - r$.

(b) Assume that $\operatorname{rank}(AB) = \operatorname{rank} A$. We know that $\operatorname{im}(AB) \subseteq \operatorname{im} A$, hence equal ranks implies $\operatorname{im}(AB) = \operatorname{im} A$. It follows that $\operatorname{im}(AB^k) = \operatorname{im} A$ for all k by induction on k. Indeed, since $AB = BA$, we have

$$\operatorname{im} AB^{k+1} = \operatorname{im} BAB^k = B(\operatorname{im} AB^k) = B(\operatorname{im} A) = \operatorname{im} BA = \operatorname{im} AB = \operatorname{im} A.$$

But B is nilpotent, implying $\operatorname{im} A = \operatorname{im} AB^r = \operatorname{im} 0 = 0$. Hence $A = 0$. Therefore, when $A \neq 0$, it follows that $\operatorname{rank} AB < \operatorname{rank} A$.

(c) Let $A^0 = I$. Part (a) implies for all $k \geq 0$ that

$$\operatorname{rank} A^{k+1} \geq \operatorname{rank} A^k + \operatorname{rank} A - r = \operatorname{rank} A^k - 1.$$

Since A is nilpotent, we know that $A^r = 0$. Therefore,

$$0 = \operatorname{rank} A^r \geq \operatorname{rank} A^{r-1} - 1 \geq \operatorname{rank} A^{r-2} - 2 \geq \cdots \geq \operatorname{rank} A - (r-1) = 0.$$

It follows that all inequalities must be equalities, that is, $\operatorname{rank} A^k = r - k$ for all $0 \leq k \leq r$. In particular, $A^{r-1} \neq 0$. Moreover, the quotient $\ker A^k / \ker A^{k-1}$ has dimension 1 for all $1 \leq k \leq r$. Consider the filtration

$$0 = \ker I \subsetneq \ker A \subsetneq \ker A^2 \subsetneq \cdots \subsetneq \ker A^{r-1} \subsetneq \ker A^r = k^r.$$

Choose $v_1 \notin \ker A^{r-1}$, and let $v_k = A^{k-1} v_1$ for $k = 2, \dots, r$. Then $\{v_1, \dots, v_r\}$ is a basis for k^r. To prove this, note that $v_k \notin \ker A^{r-k}$ because $A^{r-1} v_1 \neq 0$, but $v_k \in \ker A^{r-k+1}$ since $A^r = 0$. Assume that $v_1, \dots, v_r$ are linearly dependent. Then there exist $c_1, \dots, c_r \in k$, not all zero, such that $\sum_{i=1}^r c_i v_i = 0$. If we let

$k = \min\{i \mid c_i \neq 0\}$, then $v_k = c_k^{-1}(\sum_{i=k+1}^r c_i v_i)$. But $v_i \in \ker A^{r-k}$ for all $i > k$, implying $v_k \in \ker A^{r-k}$, a contradiction.

There exist $c_1, \dots, c_r \in k$ such that $Bv_1 = \sum_{i=1}^r c_i v_i = \sum_{i=1}^r c_i A^{i-1} v_1$ since $\{v_1, \dots, v_r\}$ is a basis for k^r. Since $AB = BA$ it follows for all k that

$$\begin{aligned} Bv_k &= BA^{k-1}v_1 = A^{k-1}Bv_1 \\ &= A^{k-1}\sum_{i=1}^r c_i A^{i-1} v_1 = \sum_{i=1}^r c_i A^{i-1} A^{k-1} v_1 = \sum_{i=1}^r c_i A^{i-1} v_k. \end{aligned}$$

Since $\{v_i\}$ is a basis, it follows that $B = \sum_{i=1}^r c_i A^{i-1}$, that is, $B \in k[A]$. □

The following theorem gives a positive answer to question 4.1 when $r \leq 4$.

Theorem 4.9:
Assume that $r \leq 4$ and $\bar{k} = k$. Let $f \in \mathcal{R}_d$, $d \geq 3$, satisfy $\operatorname{ann}_R(f)_1 = 0$. Then for some $n \geq 1$ there exists $f_t \in \mathcal{R}_d[t_1, \dots, t_n]$ such that $f_0 = f$ and f_t splits regularly $\dim_k M_f - 1$ times over $k(t_1, \dots, t_n)$.

Proof: We may assume that M_f does not contain any non-trivial idempotent, because if it does, we apply theorem 3.18 first, and then the following proof on each additive component. Since $\bar{k} = k$, it follows by proposition 3.5 that $M_f = \langle I \rangle \oplus M_f^{\text{nil}}$ where $M_f^{\text{nil}} = \{A \in M_f \mid A \text{ is nilpotent}\}$.

The conclusion follows from theorem 4.5 if M_f is generated by a single matrix. And if M_f^{nil} contains a matrix A of rank $r-1$, then $M_f = k[A]$ by lemma 4.8. Therefore, we now assume that M_f is *not* generated by a single matrix, and in particular, that all matrices in M_f^{nil} have rank $\leq r-2$.

If $r = 1$, then $f = cx_1^{(d)}$ and $M_f = \langle I \rangle$, thus there is nothing to prove. If $r = 2$, then M_f must be generated by a single matrix, and we are done.

If $r = 3$, then M_f^{nil} may only contain matrices of rank 1. Since M_f cannot be generated by a single matrix, M_f^{nil} must contain two matrices $A \nparallel B$ of rank 1. We may write $A = u_1 v_1^{\mathsf{T}}$ and $B = u_2 v_2^{\mathsf{T}}$ for suitable vectors $u_i, v_j \in k^r$. Note that $A^2 = B^2 = AB = BA = 0$ since their ranks are < 1 by lemma 4.8b. Thus $u_i^{\mathsf{T}} v_j = 0$ for all $i, j = 1, 2$. If $u_1 \nparallel u_2$, then this implies $v_1 \parallel v_2$ since $r = 3$. Similarly, $v_1 \nparallel v_2$ implies $u_1 \parallel u_2$. However, both cases are impossible, since each imply $\operatorname{ann}_R(f)_1 \neq 0$ by corollary 2.29. (These are essentially the two cases in example 2.30.)

Suppose $r = 4$ and that M_f^{nil} only contains matrices of rank ≤ 2. We will break down the proof of this case into four subcases.

Case 1. Assume M_f^{nil} contains two matrices $A \nparallel B$ of rank 1, i.e. $A = u_1 v_1^{\mathsf{T}}$ and $B = u_2 v_2^{\mathsf{T}}$. Then $u_i v_j^{\mathsf{T}} = 0$ for all $i, j = 1, 2$ as above. Again, both $u_1 \parallel u_2$ and $v_1 \parallel v_2$ lead to contradictions by corollary 2.29. Thus we may up to a base change assume $u_1 = [1000]^{\mathsf{T}}$ and $u_2 = [0100]^{\mathsf{T}}$. Hence $v_i = [00{*}{*}]^{\mathsf{T}}$, and after another change of basis, $v_1 = [0010]^{\mathsf{T}}$ and $v_2 = [0001]^{\mathsf{T}}$. In other words,

$$A = \left(\begin{smallmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{smallmatrix}\right) \qquad \text{and} \qquad B = \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \end{smallmatrix}\right).$$

Since $I_2(\partial\, A\partial\, B\partial) \subseteq \operatorname{ann} f$, this already implies that there exist $c_1, c_2 \in k$ and $g \in k[x_1, x_2]^{DP}$ such that $f = c_1 x_3 x_1^{(d-1)} + c_2 x_4 x_2^{(d-1)} + g$. Note that $c_1, c_2 \neq 0$ since $\operatorname{ann}(f)_1 = 0$, and we may assume $c_1 = c_2 = 1$.

Suppose that M_f^{nil} contains a matrix C in addition to A and B. Then $CA = AC = CB = BC = 0$ because their ranks are < 1. This implies that

$$C = \left(\begin{smallmatrix} 0 & \star \\ 0 & 0 \end{smallmatrix}\right) \text{ as a } 2 \times 2 \text{ block matrix using } 2 \times 2 \text{ blocks,}$$

and modulo A and B we may assume that $\star = \left(\begin{smallmatrix} 0 & a \\ b & 0 \end{smallmatrix}\right)$. It follows that

$$I_2(\partial\, C\partial) = (b\partial_1\partial_3 - a\partial_2\partial_4) \subseteq \operatorname{ann}_R f.$$

Hence $0 = (b\partial_1\partial_3 - a\partial_2\partial_4)(f) = bx_1^{(d-2)} - ax_2^{(d-2)}$. This implies $a = b = 0$ since $d \geq 3$. Thus we have proven that $M_f = \langle I, A, B \rangle$. Let

$$f_t = \tfrac{1}{t}\Big((x_1 + tx_3)^{(d)} - x_1^{(d)} + (x_2 + tx_4)^{(d)} - x_2^{(d)}\Big) + g.$$

Then $f_0 = f$, and $f_t \sim x_3^{(d)} + x_4^{(d)} - \big(x_1^{(d)} + x_2^{(d)} - tg\big)$ obviously splits twice.

Case 2. Suppose M_f does not contain any matrix of rank 1. If $A, B \in M_f^{\mathrm{nil}}$, then both have rank 2 and $A^2 = B^2 = AB = BA = 0$. We may assume that $A = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$, which implies that $B = \left(\begin{smallmatrix} 0 & B' \\ 0 & 0 \end{smallmatrix}\right)$. But then $B - \lambda A$ has rank 1 when λ is an eigenvalue for B', a contradiction. Therefore, for the rest of the proof we may assume that M_f contains exactly one matrix of rank 1.

Case 3. Assume M_f does not contain any A of rank 2 satisfying $A^2 = 0$. Then M_f must contain an A such that $\operatorname{rank} A = 2$ and $A^2 \neq 0$. Note that $\operatorname{rank} A^2 = 1$.

Because $M_f \neq k[A]$, there exists $B \in M_f$, $B \notin k[A]$. Then $\operatorname{rank} B = 2$ since M_f cannot contain several matrices of rank 1. Thus $B^2 \neq 0$, and therefore $B^2 = bA^2$, $b \neq 0$. Also $\operatorname{rank} AB \leq 1$, hence $AB = BA = aA^2$. Let t be a root of $t^2 + 2at + b$. Since $\operatorname{rank}(tA + B) \leq 1$ implies $B \in k[A]$, we get $\operatorname{rank}(tA + B) = 2$. But $(tA + B)^2 = (t^2 + 2at + b)A^2 = 0$, contradicting our assumption.

Case 4. Hence M_f contains a matrix A of rank 2 satisfying $A^2 = 0$ and a matrix B of rank 1. We may assume that $A = \left(\begin{smallmatrix} 0 & I \\ 0 & 0 \end{smallmatrix}\right)$. From $AB = BA$ it follows that $B = \left(\begin{smallmatrix} B_1 & B_2 \\ 0 & B_1 \end{smallmatrix}\right)$, and $B_1 = 0$ since $\operatorname{rank} B = 1$. Modulo a similarity transformation $B \mapsto PBP^{-1}$ with $P = \left(\begin{smallmatrix} Q & 0 \\ 0 & Q \end{smallmatrix}\right)$ we may assume that

$$B_2 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{or} \qquad B_2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix},$$

and modulo A this becomes $B_2 \in \left\{ \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \right\}$. Since B is the only matrix in M_f of rank 1 (up to a scalar), the first must be disregarded. (It reduces to case 1 above.) Hence $B_2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. It follows that

$$f = x_4 x_1^{(d-1)} + x_3 x_2 x_1^{(d-2)} + g \quad \text{where} \quad g \in k[x_1, x_2]^{DP},$$

up to a base change. Define $f_t \in \mathcal{R}_d[t]$ by

$$f_t = \tfrac{1}{st}\Big((x_1 + sx_2 + tx_3 + stx_4)^{(d)} - (x_1 + sx_2)^{(d)} - (x_1 + tx_3)^{(d)} + x_1^{(d)} \Big) + g.$$

Then $f_0 = f$, and $f_t \cong$ splits twice. If $M_f = \langle I, A, B \rangle$, then we are done.

Thus assume that M_f^{nil} contains a matrix $C \notin \langle A, B \rangle$. Because $CA = AC$ and $CB = BC$, we have

$$C = \begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ 0 & c_1 & c_5 & c_6 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & 0 & c_1 \end{pmatrix}.$$

Clearly, $c_1 = 0$ since C is nilpotent. If $c_2 = 0$, then $\operatorname{rank}(C - c_3 A - c_4 B) \leq 1$, thus $C \in \langle A, B \rangle$ since B is the only matrix in M_f of rank 1. This contradiction allows us to assume that $c_2 = 1$. It also implies that M_f^{nil} cannot contain yet another matrix, since we then would have to get another one of rank 1. Therefore, $M_f = \langle I, A, B, C \rangle$. Now, $\operatorname{rank} C < 3$ implies $c_5 = 0$, and modulo B we may assume $c_4 = 0$. If $\operatorname{char} k \neq 2$, we may also assume $c_3 = c_6 = 0$. This follows from the similarity transformation $C \mapsto PCP^{-1}$ where $P = \left(\begin{smallmatrix} I & Q \\ 0 & I \end{smallmatrix}\right)$ with $Q = \left(\begin{smallmatrix} 0 & 0 \\ q & 0 \end{smallmatrix}\right)$ and $q = \frac{1}{2}(c_3 - c_6)$. It follows that

$$f = x_4 x_1^{(d-1)} + x_3 x_2 x_1^{(d-2)} + c x_1^{(d)}$$

up to a base change. (We may even assume $c = 0$ if $\operatorname{char} k \nmid d$.) Let

$$f_t = \tfrac{1}{st}\Big((x_1 + sx_2 + tx_3 + stx_4)^{(d)} - (x_1 + sx_2)^{(d)} - (x_1 + tx_3)^{(d)} + x_1^{(d)}\Big) + cx_1^{(d)}.$$

Then $f_0 = f$, and $f_t \sim x_1^{(d)} + x_2^{(d)} + x_3^{(d)} + x_4^{(d)}$ splits regularly three times.

If $\operatorname{char} k = 2$, then the case $(c_3, c_6) = (0, 1)$ is not in the $\mathrm{GL}_k(4)$ orbit of $(c_3, c_6) = (0, 0)$. A base change shows that this additional case is isomorphic to $M_f = \langle I, A, B, A^2\rangle$ where $A^2 = B^2$ and

$$A = \begin{pmatrix} 0&1&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \qquad B = \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}.$$

This implies that $f = x_4 x_1^{(d-1)} + x_3^{(2)} x_1^{(d-2)} + x_2^{(2)} x_1^{(d-2)} + cx_1^{(d)}$. Let

$$\begin{aligned} f_t = t^{-3}\Big(&t(x_1 + tx_2 + t^2x_4)^{(d)} + t(x_1 + tx_3)^{(d)} \\ &- (x_1 + t^2x_2 + t^2x_3)^{(d)} + (1 - 2t + ct^3)x_1^{(d)}\Big). \end{aligned}$$

Again, $f_0 = f$, and $f_t \sim x_1^{(d)} + x_2^{(d)} + x_3^{(d)} + x_4^{(d)}$ splits regularly three times. Hence in each case we have found an $f_t \in \mathcal{R}_d[t_1, \dots, t_n]$ such that $f_0 = f$ and f_t splits regularly $\dim_k M_f - 1$ times over $k(t_1, \dots, t_n)$, and we are done. □

Remark 4.10: Note that the last case of the proof says the following. Suppose M_f contains two matrices of rank 2 that are non-proportional. If $\operatorname{char} k \neq 2$, then M_f contains exactly two of rank 2 such that $A^2 = 0$. If $\operatorname{char} k = 2$, then there are two possibilities. Either every matrix in M_f of rank 2 satisfies $A^2 = 0$, or only one matrix is of this type, and the rest satisfy $A^2 \neq 0$.

We will end this section with a generalization of theorem 4.5.

Theorem 4.11:
Suppose $d \geq 3$ and $f \in \mathcal{R}_d$. Let $A_1, \dots, A_m \in \mathrm{Mat}_k(r, r)$ be nonzero and nilpotent, and assume there exist orthogonal idempotents $E_1, \dots, E_m$ such that $E_iA_i = A_iE_i = A_i$ for all i. Let $n_i = \operatorname{index} A_i$ and $1 \leq a_i < n_i$. Assume that $A_i^k \in M_f$ for all $k \geq a_i$ and all $i = 1, \dots, m$. Let $n = \sum_{i=1}^m (n_i - a_i)$. Then f is a specialization of some $f_t \in \mathcal{R}_d[t_1, \dots, t_n]$ that splits regularly n times over $k(t_1, \dots, t_n)$.

Proof: The proof uses the same ideas as the proof of theorem 4.5, with some modifications. Fix one $i \in \{1, \dots, m\}$, and choose $Q \in E_i \mathrm{Mat}_k(r, r) E_i$ such that $A_i^{n_i-1}QA_i^{n_i-1} = A_i^{n_i-1}$. Define matrices $P = I + \sum_{k=1}^{n_i-1} t^k A_i^{n_i-1-k}Q$ and $A_{it} = A_i + tQA_i^{n_i-1}$. It follows that $A_{it}^{n_i-1} = PA_i^{n_i-1}$ and $A_{it}^{n_i} = tA_{it}^{n_i-1}$. Because $A_i^{n_i-1} \in M_f$, there exists $g \in \mathcal{R}_d$ such that $\partial g = A_i^{n_i-1}\partial f$. Define

$$g_t = \phi_P(g) = \sum_{k\geq 0} t^k g_k \quad \text{and} \quad f_t = f + \sum_{k\geq 1} t^k g_{n_i-1+k}.$$

For all $i \neq j$, it follows from $E_iE_j = 0$ that $A_iE_j = E_jA_i = A_iA_j = 0$. Thus $A_jA_{it} = 0$. Since $\partial g_t = P\phi_P(\partial g) = A_{it}^{n_i-1}\phi_P(\partial f)$, it follows that $A_j\partial g_t = 0$, and therefore, $A_j\partial g_k = 0$ for all $k \geq 0$. Hence $A_j^k \in M_{f_t}$ for all $j \neq i$ and $k \geq a_j$.

We will now prove that $A_{it}^k\partial f_t = A_i^k\partial f + \sum_{j\geq 1} t^j \partial g_{n_i-1-k+j}$ for all $k \geq 0$. Assume it is true for some $k \geq 0$. The arguments following equation (4.1) in the proof of theorem 4.5 apply here and show that $A_i^{n_i-1}\partial g_{n_i-j} = 0$ for all $j > 1$, $A_i^{n_i-1}\partial f = A_i^{n_i-1}\partial g_{n_i-1}$ and $A_i\partial g_{n_i-1+j} + QA_i^{n_i-1}\partial g_{n_i-2+j} = \partial g_{n_i-2+j}$ for all j. It follows that

$$\begin{aligned} A_{it}^{k+1}\partial f_t &= \Big(A_i + tQA_i^{n_i-1}\Big)\Big(A_i^k\partial f + \sum_{j\geq 1} t^j \partial g_{n_i-1-k+j}\Big) \\ &= A_i^{k+1}\partial f + \sum_{j\geq 1} t^j\Big(A_i\partial g_{n_i-1-k+j} + QA_i^{n_i-1}\partial g_{n_i-2-k+j}\Big) \\ &= A_i^{k+1}\partial f + \sum_{j\geq 1} t^j \partial g_{n_i-2-k+j}. \end{aligned}$$

Since $A_i^k \in M_f$ for all $k \geq a_i$ it follows that $A_{it}^k \in M_{f_t}$ for all $k \geq a_i$. In particular, $E' = I - (t^{-1}A_{it})^{n_i-1} \in M_{f_t}$.

Since E' is idempotent, we may apply theorem 3.18. It tells us that f_t has a regular splitting with the following two additive components, $t^{-n_i+1}g_t$ and

$$f' = t^{-n_i+1}\big(t^{n_i-1}f - g_0 - tg_1 - \cdots - t^{n_i-1}g_{n_i-1}\big),$$

and furthermore that $(A_{it}E')^k = A_{it}^kE' \in M_{f'}^{E'}$ for all $k \geq a_i$. Hence we may repeat our procedure on f'. By induction on n_i and i, we arrive at some $f_{\underline{t}} \in \mathcal{R}_d[t_1, \ldots, t_n]$ such that $f_0 = f$ and $f_{\underline{t}}$ splits regularly n times. □

Remark 4.12: We assume in theorem 4.11 that $A_i^k \in M_f$ for all $k \geq a_i$. It is in fact enough to assume $A_i^{a_i}B_i, A_i^{a_i+1}C_i \in M_f$ for some invertible $B_i, C_i \in k[A_i]$. Indeed, apply proposition 2.26 with $A = A_iB_i^{-1}C_i$, $B = I$ and $C = A_i^{a_i}B_i$.

It follows that $A^kC = A_i^{a_i+k}B_i^{1-k}C_i^k \in M_f$ for all $k \geq 0$. In particular, with $k = n_i - a_i - 1$, we get $A_i^{n_i-1}P \in M_f$ where $P \in k[A_i]$ is invertible. This implies $A_i^{n_i-1} \in M_f$ since $A_i^{n_i} = 0$. Now letting $k = n_i - a_i - 2$ implies $A_i^{n_i-2} \in M_f$. By descending induction on k we get $A_i^k \in M_f$ for all $k \geq a_i$.

4.2 Counter examples

In this section we will produce examples of $f \in \mathcal{R}_d$ in which we cannot find an $f_t \in \mathcal{R}_d[t_1, \ldots, t_n]$ such that $f_0 = f$ and f_t splits regularly $\dim_k M_f - 1$ times over $k(t_1, \ldots, t_n)$. Thus question 4.1 has a negative answer for these f. There exist many such examples due to purely numerical reasons, and the following theorem enables us to find some.

Theorem 4.13:
Let $d \geq 3$, $s \leq r$ and $\mathcal{S} = k[x_1, \ldots, x_s]^{DP} \subseteq \mathcal{R}$. Suppose $h \in \mathcal{S}_d$ does not split regularly. Let $f = h + x_{s+1}^{(d)} + \cdots + x_r^{(d)} \in \mathcal{R}_d$. Assume that there exists an $f_t \in \mathcal{R}_d[t_1, \ldots, t_n]$ such that $f_0 = f$ and f_t splits regularly $m - 1$ times over $k(t_1, \ldots, t_n)$. Suppose $m > r - s + 1$. Then M_h must contain a non-zero nilpotent matrix of rank $\leq s/(m - r + s)$.

Proof: Clearly, $m \leq r$. Note that if $\operatorname{ann}_R(f)_1 \neq 0$, then $\operatorname{ann}_S(h)_1 \neq 0$. In this case M_h will contain nilpotent matrices of rank 1, and we are done. Therefore, we may assume $\operatorname{ann}_R(f)_1 = 0$. This implies $\operatorname{ann}_{R(t_1,\ldots,t_n)}(f_t)_1 = 0$. It also implies that $f \neq 0$ since $s > r - m + 1 \geq 1$.

For each $k = 1, \ldots, r - s$, define $E_k \in \operatorname{Mat}_k(r, r)$ by

$$(E_k)_{ij} = \begin{cases} 1 & \text{if } i = j = k + s, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, E_k is a diagonal idempotent of rank 1. Furthermore, $\partial\big(x_{s+k}^{(d)}\big) = E_k \partial f$, thus $E_k \in M_f$. Let $E_0 = I - \sum_{k=1}^{r-s} E_k \in M_f$. It follows by theorem 3.18 that $M_f = M_0 \oplus M_1 \oplus \ldots \oplus M_{r-s}$ where $M_k = M_f E_k = \langle E_k \rangle$ for $k = 1, \ldots, r - s$, and $M_0 = M_f E_0 \cong M_h$. To be precise, $M_0 = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix} \right) \mid A \in M_h \right\}$.

Choose a multiplicative (monomial) order on $k' = k[t_1, \ldots, t_n]$ with 1 as the smallest element. If V is any k-vector space and $v \in V' = V \otimes_k k[t_1, \ldots, t_n]$,

$v \neq 0$, denote by $\operatorname{lc}(v) \in V$ the leading coefficient of v, which to us the coefficient of the *smallest* non-zero term of v in the ordering. Note that if $\varphi : U \times V \to W$ is a k-bilinear map, it induces a k'-bilinear map $\varphi' : U' \times V' \to W'$. Then $\operatorname{lc}\big(\varphi'(u,v)\big) = \varphi\big(\operatorname{lc}(u), \operatorname{lc}(v)\big)$ as long as $\varphi\big(\operatorname{lc}(u), \operatorname{lc}(v)\big) \neq 0$.

There exist orthogonal idempotents $A_1, \dots, A_m \in M_{f_t}$ and non-zero polynomials $g_1, \dots, g_m \in \mathcal{R}_d(t_1, \dots, t_n)$ such that $\sum_{i=1}^m A_i = I$ and $\partial g_i = A_i \partial f_t$. Let the common denominator of the entries of A_i be $\lambda_i \in k[t_1, \dots, t_n]$. We may scale λ_i such that $\operatorname{lc}(\lambda_i) = 1$. Replace A_i by $\lambda_i A_i$. Then $A_i \in \operatorname{Mat}_{k[t_1,\dots,t_n]}(r,r)$ and $A_i^2 = \lambda_i A_i$. Moreover, replace g_i by $\lambda_i g_i$ to preserve the relation $\partial g_i = A_i \partial f_t$. This implies that $g_i \in \mathcal{R}_d[t_1, \dots, t_n]$.

Let $A_{i0} = \operatorname{lc}(A_i) \neq 0$. Note that $\operatorname{lc}(f_t) = f$, and $A_{i0}\partial f \neq 0$ because $\operatorname{ann}_R(f)_1 = 0$. It follows that

$$\partial \operatorname{lc}(g_i) = \operatorname{lc}(\partial g_i) = \operatorname{lc}(A_i \partial f_t) = \operatorname{lc}(A_i) \partial \operatorname{lc}(f_t) = A_{i0} \partial f.$$

Hence $A_{i0} \in M_f$. If $A_{i0}^2 \neq 0$, then $A_{i0}^2 = \operatorname{lc}(A_i^2) = \operatorname{lc}(\lambda_i A_i) = A_{i0}$. Thus $A_{i0}^2 = 0$ or $A_{i0}^2 = A_{i0}$ for all i. Furthermore, $A_{i0}A_{j0} = 0$ for all $i \neq j$, because $A_i A_j = 0$. In addition, $\operatorname{rank} A_{i0} \leq \operatorname{rank} A_i$. (If some minor of A_i is zero, then the corresponding minor of A_{i0} must also be zero.)

Since h does not split regularly, M_h does not contain any non-trivial idempotents. Hence $\{E_i\}$ is the unique maximal coid in M_f, and any idempotent in M_f is a sum of some of the E_i's. Assume A_{i0} is idempotent. We want to prove that $A_{i0} \in \langle E_1, \dots, E_{r-s} \rangle$. If it is not, then $A_{i0}E_0 = E_0$. For all $j \neq i$, we have $A_{j0}A_{i0} = 0$, and therefore $A_{j0}E_0 = 0$ and $A_{j0} \neq A_{i0}$. This implies $A_{j0} \in \oplus_{i=1}^{r-s} M_i = \langle E_1, \dots, E_{r-s} \rangle$, and it follows that $A_{j0}^2 \neq 0$. Hence A_{j0} must be an idempotent! Therefore $\{A_{j0}\}_{j=1}^m$ is a set of orthogonal idempotents, but $\{E_j\}_{j=0}^{r-s}$ is maximal, hence $m \leq r - s + 1$, a contradiction.

Let $J = \{i \,|\, A_{i0}^2 = A_{i0}\}$ and $k = \sum_{i \in J} \operatorname{rank} A_{i0} \geq |J|$. By the last paragraph, $k \leq r - s$. Clearly, the number of nilpotents among $\{A_{i0}\}_{i=1}^m$ is

$$m - |J| \geq m - k \geq m - r + s \geq 2.$$

Now suppose that M_h does not contain any non-zero nilpotent matrix of rank

$\leq s/(m-r+s)$. Then $\operatorname{rank} A_{i0} > s/(m-r+s)$ for all $i \notin J$. It follows that

$$\begin{aligned} r = \sum_{i=1}^{m} \operatorname{rank} A_i &\geq \sum_{i=1}^{m} \operatorname{rank} A_{i0} > k + (m-k)\frac{s}{m-r+s} \\ &= \frac{ms-(r-m)k}{m-r+s} \geq \frac{ms-(r-m)(r-s)}{m-r+s} = r, \end{aligned}$$

which is the contradiction we sought. □

Remark 4.14: It is not correct that if M_{f_t} contains m idempotents of rank $\leq k$, then M_{f_0} must contain m idempotents or nilpotents of rank $\leq k$. A simple example is $f = x_2 x_1^{(d-1)}$, $r = 2$. Then $M_f = \langle I, A\rangle$ where $A_1 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Let $f_t = t^{-1}[(x_1 + tx_2)^{(d)} - x_1^{(d)}]$, so that $f_0 = f$. Then $M_{f_t} = \langle A_t, B_t\rangle$ where $A_t = \left(\begin{smallmatrix} -t & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B_t = \left(\begin{smallmatrix} 0 & 1 \\ 0 & t \end{smallmatrix}\right)$. Thus both $A_0 = B_0 = A$. We see that M_{f_t} can contain two idempotents of rank 1 even though $\dim_k\{A \in M_f \mid \operatorname{rank} A \leq 1\} = 1$.

Now that we have theorem 4.13 at our disposal, we are ready to give the first example in which question 4.1 has a negative answer.

Example 4.15: Suppose $r = 5$ and $a, b \geq 2$. Let

$$f = x_1^{(a-1)} x_2^{(b+1)} x_3 + x_1^{(a)} x_2^{(b)} x_4 + x_1^{(a+1)} x_2^{(b-1)} x_5.$$

Then $f \in \mathcal{R}_d$ where $d = a+b+1 \geq 5$. The annihilator ideal is

$$\begin{aligned} \operatorname{ann}_R(f) = (\partial_3,\partial_4,\partial_5)^2 &+ (\partial_1\partial_4 - \partial_2\partial_3, \partial_1\partial_5 - \partial_2\partial_4) \\ &+ (\partial_1^a\partial_3, \partial_2^b\partial_5, \partial_1^{a+2}, \partial_2^{b+2}) + (\partial_1^{a+1}\partial_2^b, \partial_1^a\partial_2^{b+1}). \end{aligned}$$

It is easy to check that $\operatorname{ann}_R f$ contains the right-hand side. For the converse, assume that $D \in \operatorname{ann}_R(f)_e$. Modulo $(\partial_3,\partial_4,\partial_5)^2$ there exist $D_i \in k[\partial_1,\partial_2]$ such that $D = \partial_3 D_1 + \partial_4 D_2 + \partial_5 D_3 + D_4$, and modulo $(\partial_1\partial_4 - \partial_2\partial_3, \partial_1\partial_5 - \partial_2\partial_4)$ we may assume that $D_2 = 0$ and $D_3 = c_1\partial_1\partial_2^{e-2} + c_2\partial_2^{e-1}$. Computing Df, we see that $Df = 0$ is equivalent to $D_1(x_1^{(a-1)}x_2^{(b+1)}) + D_3(x_1^{(a+1)}x_2^{(b-1)}) = D_4(f) = 0$. This implies that $D_1 \in (\partial_1^a, \partial_2^{b+2})$, $D_3 \in (\partial_2^b)$ and $D_4 \in (\partial_1^{a+2}, \partial_2^{b+2}, \partial_1^{a+1}\partial_2^b, \partial_1^a\partial_2^{b+1})$, and proves that D is contained in the right-hand side.

Since $a, b \geq 2$, we see that $\operatorname{ann}_R f$ has two generators of degree d. Thus $\dim_k M_f = 3$. Let $g_1 = x_1^{(a)}x_2^{(b+1)}$, $g_2 = x_1^{(a+1)}x_2^{(b)}$ and

$$A_1 = \begin{pmatrix} 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0&0&0&1&0 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{pmatrix}.$$

A simple calculation shows that $\partial g_1 = A_1 \partial f$ and $\partial g_2 = A_2 \partial f$. This implies that $A_1, A_2 \in M_f$, and it follows that $M_f = \langle I, A_1, A_2 \rangle$. (Note that $g_1 = \partial_1 h$ and $g_2 = \partial_2 h$ where $h = x_1^{(a+1)} x_2^{(b+1)}$.)

Since M_f does not contain any non-zero nilpotent matrix of rank 1, theorem 4.13 implies that there does not exist an $f_t \in \mathcal{R}_d[t_1, \dots, t_n]$ such that $f_0 = f$ and f_t splits regularly $\dim_k M_f - 1$ times over $k(t_1, \dots, t_n)$. Moreover, by adding terms $x_i^{(d)}$ with $i > 5$, we have produced such examples for all $r \geq 5$ and $d \geq 5$.

Example 4.16: Let us consider the following two polynomials.

(a) $$f_1 = x_4\bigl(x_2 x_3^{(2)}\bigr) + x_5\bigl(x_1 x_3^{(2)} + x_2^{(2)} x_3\bigr) + x_6\bigl(x_1 x_2 x_3 + x_2^{(3)}\bigr) + x_7\bigl(x_1^{(2)} x_3 + x_1 x_2^{(2)}\bigr) \in \mathcal{R}_4,\ r = 7.$$

(b) $$f_2 = x_5\bigl(x_3 x_4\bigr) + x_6\bigl(x_2 x_4 + x_3^{(2)}\bigr) + x_7\bigl(x_1 x_4 + x_2 x_3\bigr) + x_8\bigl(x_1 x_3 + x_2^{(2)}\bigr) + x_9\bigl(x_1 x_2\bigr) \in \mathcal{R}_3,\ r = 9.$$

Tedious but simple computations show that the annihilators are:

$$\begin{aligned}
\operatorname{ann}_R(f_1) = {} & (\partial_4, \partial_5, \partial_6, \partial_7)^2 + (\partial_1\partial_4, \partial_2\partial_4 - \partial_1\partial_5, \partial_3\partial_4 - \partial_2\partial_5, \partial_2\partial_5 - \partial_1\partial_6, \\
& \partial_3\partial_5 - \partial_2\partial_6, \partial_2\partial_6 - \partial_1\partial_7, \partial_3\partial_6 - \partial_2\partial_7) + (\partial_1\partial_3 - \partial_2^2) \\
& + (\partial_2\partial_3\partial_7, \partial_3^2\partial_7) + (\partial_1^3, \partial_1^2\partial_2, \partial_3^3) + (\partial_2^4, \partial_2^3\partial_3) \\
\operatorname{ann}_R(f_2) = {} & (\partial_5, \dots, \partial_9)^2 + (\partial_1\partial_5, \partial_2\partial_5, \partial_1\partial_6, \partial_3\partial_5 - \partial_2\partial_6, \partial_4\partial_5 - \partial_3\partial_6, \\
& \partial_2\partial_6 - \partial_1\partial_7, \partial_3\partial_6 - \partial_2\partial_7, \partial_4\partial_6 - \partial_3\partial_7, \partial_2\partial_7 - \partial_1\partial_8, \partial_3\partial_7 - \partial_2\partial_8, \\
& \partial_4\partial_7 - \partial_3\partial_8, \partial_2\partial_8 - \partial_1\partial_9, \partial_3\partial_8 - \partial_2\partial_9, \partial_4\partial_8, \partial_3\partial_9, \partial_4\partial_9) \\
& + (\partial_1^2, \partial_2^2 - \partial_1\partial_3, \partial_2\partial_3 - \partial_1\partial_4, \partial_3^2 - \partial_2\partial_4, \partial_4^2) + (\partial_2^2\partial_3, \partial_2\partial_3^2)
\end{aligned}$$

In both cases, $\dim_k M_{f_i} = 3$. It is easy to check that the two nilpotent matrices in M_{f_1} are of rank 3, and of rank 4 in M_{f_2}. By theorem 4.13, there does not exist an $f_t \in \mathcal{R}_d[t_1, \dots, t_n]$ such that $f_0 = f_i$ and f_t splits regularly $\dim_k M_{f_i} - 1$ times over $k(t_1, \dots, t_n)$. Again, we may add terms $x_i^{(d)}$ to produce such examples for all $r \geq 7$ when $d = 4$ and all $r \geq 9$ when $d = 3$.

The next proposition allows us to construct f such that M_f does not contain nilpotent matrices of small rank. The previous examples are special cases of this proposition.

Proposition 4.17: *Suppose $d \geq 3$, $s \geq 2$, $q \geq 1$ and $r = 2s + q$. Let $\mathcal{S} = k[x_1, \ldots, x_s]^{DP} \subseteq \mathcal{R} = k[x_1, \ldots, x_r]^{DP}$. Let $g_1, \ldots, g_{s+q} \in \mathcal{S}_{d-1}$ satisfy $\partial_{i+1} g_j = \partial_i g_{j+1} = h_{i+j-2} \in \mathcal{S}_{d-2}$ for all $1 \leq i < s$ and $1 \leq j < s + q$. Define $f = \sum_{i=1}^{s+q} x_{s+i} g_i \in \mathcal{R}_d$. Assume that $h_i = 0$ for all $i < s-1$, and that $h_{s-1}, \ldots, h_{s+q+1}$ are linearly independent. Then $M_f = \langle I, B_0, \ldots, B_q \rangle$ where, for each $k = 0 \ldots, q$,*

$$(B_k)_{ij} = \begin{cases} 1, & \text{if } i \leq s \text{ and } j = s + k + i, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: For each k we note that B_k is block matrix of the form $\left(\begin{smallmatrix} 0 & B'_k \\ 0 & 0 \end{smallmatrix}\right)$, where $B'_k \in \mathrm{Mat}_k(s, s+q)$ is a "displaced" identity matrix. That is, B'_k is a block matrix of the form $\begin{pmatrix} O_1 & I & O_2 \end{pmatrix}$, where O_1 is an $s \times k$ zero matrix, I is an $s \times s$ identity matrix, and O_2 is an $s \times (q - k)$ zero matrix. In particular, $\operatorname{rank} B_k = s$.

By computing $\partial \partial^{\mathsf{T}} f$, we see that it has a block decomposition,

$$\partial \partial^{\mathsf{T}} f = \begin{pmatrix} X_1 & X_2 \\ X_3 & 0 \end{pmatrix},$$

where $X_1 \in \mathrm{Mat}_k(s, s)$ and $X_2 \in \mathrm{Mat}_k(s, s + q)$. X_2 is a Hankel matrix in the sense that $(X_2)_{ij} = \partial_i g_j = h_{i+j-1}$ for all $1 \leq i \leq s$ and $1 \leq j \leq s + q$, i.e

$$X_2 = X_3^{\mathsf{T}} = \begin{pmatrix} h_1 & \ldots & h_{s+q} \\ \vdots & & \vdots \\ h_s & \ldots & h_{r-1} \end{pmatrix}.$$

We note that the columns and rows of X_2 are linearly independent over k. This implies that $\operatorname{ann}_R(f)_1 = 0$.

By lemma 2.13, $A = \left(\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix}\right) \in M_f$ if and only if

$$A \partial \partial^{\mathsf{T}} f = \begin{pmatrix} A_1 X_1 + A_2 X_3 & A_1 X_2 \\ A_3 X_1 + A_4 X_3 & A_3 X_2 \end{pmatrix}$$

is symmetric. Since the entries of X_1 and $X_2 = X_3^{\mathsf{T}}$ are linearly independent, this is equivalent to both

$$\begin{pmatrix} A_1 X_1 & 0 \\ A_3 X_1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} A_2 X_3 & A_1 X_2 \\ A_4 X_3 & A_3 X_2 \end{pmatrix}$$

being symmetric. In particular, it implies that $A_3X_1 = 0$. Let a^T be a row in A_3, and define $\delta = \sum_{i=1}^{s} a_i\partial_i$. Then $0 = a^\mathsf{T}X_1 = [\delta\partial_1 f, \dots, \delta\partial_s f]$, i.e. $0 = \partial_i\delta f = \sum_{j=1}^{s+q} x_{s+j}\partial_i\delta g_j$ for all $i \le s$. This implies $\delta g_j = 0$ for all j, and therefore, $\delta f = 0$. Since $\operatorname{ann}_R(f)_1 = 0$, it follows that $A_3 = 0$.

Next we investigate $A_4X_3 = (A_1X_2)^\mathsf{T}$. We will use induction to prove that both A_1 and A_4 are identity matrices, up to a scalar. Let $a_{ij} = (A_4)_{ij}$ for all $1 \le i, j \le s+q$ and $b_{ij} = (A_1)_{ij}$ for all $1 \le i, j \le s$. Then

$$(A_4X_3)_{ij} = \sum_{k=1}^{s+q}(A_4)_{ik}(X_3)_{kj} = \sum_{k=1}^{s+q} a_{ik}h_{j+k-1},$$

and similarly, $(A_1X_2)_{ji} = \sum_{k=1}^{s} b_{jk}h_{i+k-1}$. Thus $A_4X_3 = (A_1X_2)^\mathsf{T}$ is equivalent to the following set of equations;

$$\sum_{k=1}^{s+q} a_{ik}h_{j+k-1} = \sum_{k=1}^{s} b_{jk}h_{i+k-1} \text{ for all } 1 \le i \le s+q \text{ and } 1 \le j \le s. \qquad (*_1)$$

Let $c = a_{11}$. Consider first the equation $\sum_{k=1}^{s+q} a_{1k}h_k = \sum_{k=1}^{s} b_{1k}h_k$, which we get from $(*_1)$ by letting $i = j = 1$. Since the non-zero h_k's involved are linearly independent, it follows that $a_{1k} = 0$ for all $k > s$. Next put $i = 1$ into $(*_1)$ to get $\sum_{k=1}^{s+q} a_{1k}h_{j+k-1} = \sum_{k=1}^{s} b_{jk}h_k$. If $a_{1k} = 0$ for all $k \ge s-j+3$, then this equation implies $a_{s-j+2} = 0$. By induction on j, $a_{1k} = 0$ for all $k > 1$. Hence $(*_1)$ with $j = 1$ reduces to $a_{11}h_j = \sum_{k=1}^{s} b_{jk}h_k = b_{j,s-1}h_{s-1} + b_{js}h_s$ for all j. This implies that $b_{jk} = c\delta_{jk}$ for $k = s-1$ and $k = s$. The symbol δ_{jk} is defined by $\delta_{jj} = 1$ for all j, and $\delta_{jk} = 0$ for all $j \ne k$.

Now assume for some $2 \le i \le s+q$, that $b_{jk} = c\delta_{jk}$ for all $1 \le j \le s$ and $k > s-i$. Consider the right-hand side of $(*_1)$. If $k < s-i$, then $h_{i+k-1} = 0$. When $k > s-i$, all b_{jk} are zero by the induction hypothesis, except $b_{jj} = c$. Thus $\sum_{k=1}^{s} b_{jk}h_{i+k-1}$ consists of at most two terms, $b_{j,s-i}h_{s-1}$ ($k = s-i$, requires $i < s$) and ch_{i+j-1} ($k = j$, requires $s-i < j \le s$). Hence if $j = 1$ and $i \ge s$, then $(*_1)$ becomes $\sum_{k=1}^{s+q} a_{ik}h_k = ch_i$. Since $h_{s-1}, \dots, h_{s+q}$ are linearly independent, it follows that $a_{ik} = c\delta_{ik}$ for all $k \ge s$ and $b_{1,s-i} = a_{i,s-1}$.

Assume for some $2 \le j \le s$ that we know $a_{ik} = c\delta_{ik}$ for all $k > s-j+1$. Then the left-hand side of $(*_1)$ consist of at most three terms, corresponding to

$k = s - j$, $k = s - j + 1$ and $k = i > s - j + 1$. Hence $(*_1)$ reduces to

$$\underset{(j<s)}{a_{i,s-j}h_{s-1}} + a_{i,s-j+1}h_s + \underset{(i>s-j+1)}{ch_{i+j-1}} = \underset{(i<s)}{b_{j,s-i}h_{s-1}} + \underset{(i>s-j)}{ch_{i+j-1}}.$$

We have written under each term what it requires. The two terms ch_{i+j-1} cancel each other, except when $i = s - j + 1$. It follows that $a_{i,s-j+1} = c\delta_{i,s-j+1}$ and $b_{j,s-i} = a_{i,s-j}$. By induction on j, $a_{ik} = c\delta_{ik}$ for all $k \geq 1$, and $b_{j,s-i} = a_{i,s-j} = c\delta_{j,s-i}$ for all $j \geq 1$. By induction on i, $b_{jk} = c\delta_{jk}$ for all $1 \leq j, k \leq s$, and $a_{ik} = c\delta_{ik}$ for all $1 \leq i, k \leq s + q$. This means that $A_1 = cI$ and $A_4 = cI$.

Finally, to finish the proof, we need to show that A_2X_3 is symmetric if and only if $A_2 \in \langle B'_0, \ldots, B'_q \rangle$. Let $a_{ij} = (A_2)_{ij}$ for all $1 \leq i \leq s$ and $1 \leq j \leq s + q$, and let $a_{ij} = 0$ for $j \leq 0$. A_2X_3 is symmetric if and only if

$$\sum_{k=1}^{s+q} a_{ik}h_{j+k-1} = \sum_{k=1}^{s+q} a_{jk}h_{i+k-1} \text{ for all } 1 \leq j < i \leq s. \tag{$*_2$}$$

Assume for some $2 \leq i \leq s$ that $a_{1k} = 0$ for all $k > s + q + 2 - i$. Equation $(*_2)$ with $j = 1$ says that $\sum_{k=1}^{s+q} a_{ik}h_k = \sum_{k=1}^{s+q} a_{1k}h_{i+k-1}$. Since $h_k = 0$ for $k < s - 1$ and $h_{s-1}, \ldots, h_{s+q+1}$ are linearly independent, it follows that $a_{1,s+q+2-i} = 0$ and $a_{ik} = a_{1,k-i+1}$ for all $k = s - 1, \ldots, s + q$. By induction on i, $a_{1k} = 0$ for all $k \geq q + 2$ and $a_{ik} = a_{1,k-i+1}$ for all $(i, k) \in \{2, \ldots, s\} \times \{s - 1, \ldots, s + q\}$.

Assume for some $2 \leq \alpha < s$ that

$$a_{ij} = a_{1,j-i+1} \text{ for all pairs } \big\{(i, j) \,\big|\, i < \alpha \text{ or } j > s - \alpha\big\}. \tag{$*_3$}$$

This is true for $\alpha = 2$. For some $\alpha < \beta \leq s$ assume in addition that

$$\begin{gathered} a_{ij} = a_{1,j-i+1} \text{ for all pairs} \\ \big\{(i, j) \,\big|\, (i \leq \beta - 2 \text{ and } j = s - \alpha) \text{ or } (i = \alpha \text{ and } j \geq s - \beta + 2)\big\}, \end{gathered} \tag{$*_4$}$$

and also that

$$a_{\beta-1,s-\alpha} = a_{\alpha,s-\beta+1}. \tag{$*_5$}$$

These assumptions hold for $\beta = \alpha + 1$. For all $k \geq s - \beta + 2$ it follows in particular that $a_{\beta,k-\alpha+\beta} = a_{1,k-\alpha+1} = a_{\alpha k}$ by putting $(i, j) = (\beta, k - \alpha + \beta)$ in $(*_3)$ and $(i, j) = (\alpha, k)$ in $(*_4)$. Therefore, any term on the left-hand side of $\sum_{k=1}^{s+q} a_{\alpha k} h_{\beta+k-1} = \sum_{k=1}^{s+q} a_{\beta k} h_{\alpha+k-1}$ with $s-\beta+2 \leq k \leq s+q$ cancel the corresponding term on the right-hand side. In addition, we already know that any term on the right-hand side with $k \geq q+2$ are zero. Hence the equation reduces to

$$a_{\alpha,s-\beta} h_{s-1} + a_{\alpha,s-\beta+1} h_s = a_{\beta,s-\alpha} h_{s-1} + a_{\beta,s-\alpha+1} h_s.$$

This implies that $a_{\beta,s-\alpha} = a_{\alpha,s-\beta}$ and $a_{\alpha,s-\beta+1} = a_{\beta,s-\alpha+1}$. And because $a_{\beta-1,s-\alpha} = a_{\alpha,s-\beta+1}$ by $(*_5)$ and $a_{\beta,s-\alpha+1} = a_{1,s-\alpha-\beta+2}$ by $(*_3)$, it follows that $a_{\beta-1,s-\alpha} = a_{1,s-\alpha-\beta+2}$. These equations are exactly what we need to proceed with induction on β. This induction ends after $\beta = s$, proving $(*_4)$ and $(*_5)$ with $\beta = s+1$. In order to continue with induction on α, we need $(*_3)$ with $\alpha \mapsto \alpha+1$. Now $(*_4)$ with $\beta = s+1$ contains all these equations, except $a_{s,s-\alpha} = a_{1,1-\alpha}$. But $a_{s,s-\alpha} = a_{\alpha 0}$ by $(*_5)$ with $\beta = s+1$, implying $a_{s,s-\alpha} = a_{\alpha 0} = 0 = a_{1,1-\alpha}$. Hence we may do induction on α, finally proving $(*_3)$ with $\alpha = s$. Since $\alpha_{1k} = 0$ for all $k \leq 0$ and all $k \geq q+2$, this gives us exactly what we wanted, namely $A_2 = \sum_{k=0}^{q} a_{1,k+1} B'_k$.

The converse statement, that $A_2 \in \langle B'_0, \ldots, B'_q \rangle$ implies that $A_2 X_3$ is symmetric, follows easily from equation $(*_2)$. This completes the proof. □

Remark 4.18: Proposition 4.17 involves polynomials $g_1, \ldots, g_{s+q} \in \mathcal{S}_{d-1}$ that satisfy $\partial_{i+1} g_j = \partial_i g_{j+1}$ for all $1 \leq i < s$ and $1 \leq j < s+q$. Using the $\{g_i\}$ we defined $h_1, \ldots, h_{r-1} \in \mathcal{S}_{d-2}$ by $h_{i+j-1} = \partial_i g_j$. This actually implies that $\partial_{i+1} h_j = \partial_i h_{j+1}$ for all $1 \leq i < s$ and $1 \leq j < r-1$. Indeed, if $i < s$ and $j < r-1$, then we may choose $k < s+q$ such that $h_j = \partial_{j-k+1} g_k$. Hence

$$\partial_{i+1} h_j = \partial_{i+1} \partial_{j-k+1} g_k = \partial_i \partial_{j-k+1} g_{k+1} = \partial_i h_{j+1}.$$

Assume conversely that we have polynomials $h_1, \ldots, h_{r-1} \in \mathcal{S}_{d-2}$ satisfying $\partial_{i+1} h_j = \partial_i h_{j+1}$ for all $1 \leq i < s$ and $1 \leq j < r-1$. For some $k \in \{1, \ldots, s+q\}$, consider $\{h_k, \ldots, h_{k+r-1}\}$. Since this set satisfies $\partial_i h_{k-1+j} = \partial_j h_{k-1+i}$ for all $1 \leq i, j \leq r$, it follows that there exists g_k such that $\partial_i g_k = h_{k-1+i}$ for all $1 \leq i \leq r$. This defines $g_1, \ldots, g_{s+q} \in \mathcal{S}_{d-1}$, and $\partial_{i+1} g_j = h_{i+j} = \partial_i g_{j+1}$.

Remark 4.19: Let $f_{ij} = (A_2 X_3)_{ij} = \sum_{k=1}^{s+q} a_{ik} h_{j+k-1}$ for $1 \leq i, j \leq s$. $A_2 X_3$ is symmetric if and only if it is a Hankel matrix, i.e. $f_{i+1,j} = f_{i,j+1}$ for all

$1 \leq i, j < s$. One implication is obvious. To prove the other, assume that A_2X_3 is symmetric. Note that $\partial_{i+1}h_j = \partial_i h_{j+1}$ by remark 4.18. Therefore, $\partial_{k+1}f_{ij} = \partial_k f_{i,j+1}$ for all $1 \leq i \leq s$ and all $1 \leq j, k < s$. Assume for some $2 \leq k \leq 2s-2$ that $f_{i+1,j} = f_{i,j+1}$ for all $1 \leq i, j < s$ such that $i + j = k$. The following now follows for all $1 \leq i < s$ and $1 < j < s$ such that $i + j = k + 1$.

If $l < s$, then $\partial_l f_{i+1,j} = \partial_{l+1} f_{i+1,j-1} = \partial_{l+1} f_{ij} = \partial_l f_{i,j+1}$. Similarly, if $l > 1$, then $\partial_l f_{i+1,j} = \partial_{l-1} f_{i+1,j+1} = \partial_{l-1} f_{j+1,i+1} = \partial_l f_{j+1,i} = \partial_l f_{i,j+1}$. Here we also used that A_2X_3 is symmetric. Together this shows that $\partial_l f_{i+1,j} = \partial_l f_{i,j+1}$ for all l, and therefore $f_{i+1,j} = f_{i,j+1}$. We have assumed $j > 1$ here, thus we still need to prove that $f_{k+1,1} = f_{k,2}$ when $k < s$. But this follows by the symmetry of A_2X_3, which implies $f_{k+1,1} = f_{1,k+1}$. By induction on k, A_2X_3 is Hankel.

Remark 4.20: The assumption in proposition 4.17 that $\partial_{i+1}g_j = \partial_i g_{j+1}$ for all $1 \leq i < s$ and $1 \leq j < s+q$ ensures that $B_k \in M_f$ for all $k = 0, \ldots, q$. The extra restrictions on the h_i's guarantee that $M_f = \langle I, B_0, \ldots, B_q \rangle$. There are other restrictions we could impose on $\{h_i\}$ to achieve the same ends, but at least $q+3$ of the h_i's must be linearly independent. To prove this, let $\nu = \dim_k \langle h_1, \ldots, h_{r-1} \rangle$. Let us count the number of linearly independent equations that the symmetry of A_2X_3 imposes on the entries of A_2. Let $f_{ij} = (A_2X_3)_{ij}$. By remark 4.19 we may use the equivalent statement that A_2X_3 is a Hankel matrix.

For every $i = 1, \ldots, s-1$, the equation $f_{i2} = f_{i+1,1}$ reduces to at most ν equations over k. For every $j = 3, \ldots, s$, the equation $f_{ij} = f_{i+1,j-1}$ gives at most one more equation, namely $\partial_s^{d-2} f_{ij} = \partial_s^{d-2} f_{i+1,j-1}$. All others are covered by $f_{i,j-1} = f_{i+1,j-2}$ since $\partial_k f_{ij} = \partial_{k+1} f_{i,j-1}$ for all $k < s$. Thus we get at most $(s-1)(\nu+s-2)$ linearly independent equations. In order to make $\dim_k M_f = q+2$, we need to reduce the $s(s+q)$ entries of A_2 to $q+1$. We can only hope to achieve this if

$$(s-1)(\nu + s - 2) \geq s(s+q) - (q+1) = (s-1)(s+q+1).$$

Since $s \geq 2$, this is equivalent to $\nu \geq q + 3$.

When using proposition 4.17, we need to construct the g_i's involved. By remark 4.18, the condition on the g_i's is equivalent to the corresponding condition on the h_i's. Since the h_i's have extra restrictions, it is easier to work directly with them. The next lemma tells us how the $\{h_i\}$ can and must be chosen.

Lemma 4.21: *Let $f \in \mathcal{R}_d$. Define a homogeneous ideal $J \subseteq R$ by*

$$J = I_2\begin{pmatrix} \partial_1 & \dots & \partial_{r-1} \\ \partial_2 & \dots & \partial_r \end{pmatrix} = \Big(\Big\{\partial_i\partial_{j+1} - \partial_{i+1}\partial_j \,\Big|\, i,j = 1,\dots,r-1\Big\}\Big).$$

Then the following statements are equivalent.

(a) $J \subseteq \operatorname{ann}_R f$.

(b) *There exists $g \in \mathcal{R}_d$ such that $\partial_i g = \partial_{i+1} f$ for all $i = 1,\dots,r-1$. This g is unique modulo $\langle x_r^{(d)}\rangle$.*

(c) *There exists $h \in \mathcal{R}_d$ such that $\partial_i h = \partial_{i-1} f$ for all $i = 2,\dots,r$. This h is unique modulo $\langle x_1^{(d)}\rangle$.*

(d) *f is a linear combination of the terms in $(x_1 + tx_2 + \dots + t^{r-1}x_r)^{(d)}$.*

(e) *f is a linear combination of the terms in $(x_r + tx_{r-1} + \dots + t^{r-1}x_1)^{(d)}$.*

Furthermore, if $n \geq 2$, then $f_1,\dots,f_n \in \mathcal{R}_d$ satisfy $\partial_i f_{j+1} = \partial_{i+1} f_j$ for all $1 \leq i < s$ and $1 \leq j < n$ if and only if $f_1,\dots,f_n$ are n consecutive terms in $c_t(x_r + tx_{r-1} + \dots + t^{r-1}x_1)^{(d)}$ for some $c_t \in k[t]$.

Remark 4.22: For any $\alpha = (\alpha_1,\dots,\alpha_r) \in \mathbb{N}_0^r$ define $\sigma(\alpha) = \sum_{i=1}^r (r-i)\alpha_i$. Let $|\alpha| = \sum_{i=1}^r \alpha_i$ and $m = \max\{\sigma(\alpha) \mid \sum_{i=1}^r \alpha_i = d\} = (r-1)d$, and define

$$g_{dk} = \sum_{\substack{|\alpha|=d \\ \sigma(\alpha)=k}} x^{(\alpha)} \in \mathcal{R}_d$$

for all $0 \leq k \leq m$. Clearly, $g_{d0},\dots,g_{dm}$ are linearly independent, and

$$\big(x_r + tx_{r-1} + \dots + t^{r-1}x_1\big)^{(d)} = \sum_{k=0}^m t^k g_{dk}.$$

Thus $\{g_{dk}\}$ are the terms we speak of in lemma 4.21e. The lemma implies that $J_d^\perp = \{f \in \mathcal{R}_d \mid J \subseteq \operatorname{ann}_R f\} = \langle g_{d0},\dots,g_{dm}\rangle$, hence $\dim_k (R/J)_d = m+1$ for all $d \geq 0$.

Proof of lemma 4.21: The implications (b) $\Rightarrow$ (a), (c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (a) are all obvious. Furthermore, (d) $\Leftrightarrow$ (e), because the two expansions have the same terms, just in opposite order, since

$$\big(x_r + tx_{r-1} + \dots + t^{r-1}x_1\big)^{(d)} = t^{(r-1)d}\big(x_1 + \tfrac{1}{t}x_2 + \dots + (\tfrac{1}{t})^{r-1}x_r\big)^{(d)}.$$

To prove (a) $\Rightarrow$ (b), assume that $J \subseteq \operatorname{ann}_R f$. For any $i = 1, \dots, r$ let $e_i \in k^r$ be the i^{th} unit vector, i.e. $(e_i)_j = 1$ if $j = i$, and $(e_i)_j = 0$ otherwise. In particular, $\alpha = (\alpha_1, \dots, \alpha_r) = \sum_{i=1}^r \alpha_i e_i$. For any α such that $|\alpha| = d$, let

$$g_\alpha = \begin{cases} \partial^{\alpha - e_i + e_{i+1}}(f), & \text{if } \alpha_i > 0 \text{ for some } i < r, \\ 0, & \text{if } \alpha_r = d. \end{cases}$$

This is well defined since $J \subseteq \operatorname{ann}_R f$. Note that g_α is an element of k. Define a polynomial $g \in \mathcal{R}_d$ by $g = \sum_{|\alpha|=d} g_\alpha x^{(\alpha)}$. It follows that $\partial_i g = \partial_{i+1} f$ for all $i < r$. Indeed, for all $|\alpha| = d - 1$ we get $\partial^\alpha \partial_i g = g_{\alpha + e_i} = \partial^{\alpha + e_{i+1}} f = \partial^\alpha \partial_{i+1} f$. Obviously, if both g and g' satisfy (b), then $\partial_i g' = \partial_{i+1} f = \partial_i g$ for all $i < r$, hence $g' - g \in \langle x_r^{(d)} \rangle$. This proves (a) $\Rightarrow$ (b). Moreover, we obtain a proof of (a) $\Rightarrow$ (c) by renaming the variables $(x_1, \dots, x_r) \mapsto (x_r, \dots, x_1)$.

Note that (a) $\Rightarrow$ (e) follows from (a) $\Rightarrow$ (b) and the last statement. Thus we are done when we prove the last statement. One implication is obvious. To prove the other, let $n \geq 2$ and assume that $f_1, \dots, f_n \in \mathcal{R}_d$ satisfy $\partial_i f_{j+1} = \partial_{i+1} f_j$ for all $1 \leq i < s$ and $1 \leq j < n$. In particular, $J \subseteq \operatorname{ann}_R(f_i)$ for all i. From what we have already proven, we may for $k > n$ inductively choose $f_k \in \mathcal{R}_d$ such that $\partial_i f_{j+1} = \partial_{i+1} f_j$ for all $i < r$ and $\partial_r^d(f_k) = 0$, and similarly for $k \leq 0$, except then $\partial_1^d(f_k) = 0$. For all $\alpha = (\alpha_1, \dots \alpha_r)$, $\alpha_i \geq 0$, let $\sigma(\alpha) = \sum_{i=1}^r (r - i)\alpha_i$. Since $\partial_i(f_k) = \partial_r(f_{k-(r-i)})$, it follows that $\partial^\alpha(f_k) = \partial_r^d(f_{k-\sigma(\alpha)})$ for all k. Obviously, $\max\{\sigma(\alpha) \mid \sum_{i=1}^r \alpha_i = N\} = (r-1) \cdot N$. If $k > n + (r-1)N$, then for all $|\alpha| \geq N$ we have $\partial^\alpha(f_k) = \partial_r^d(f_{k-\sigma(\alpha)}) = 0$, hence $f_k = 0$. Similarly, $f_k = 0$ for all $k \ll 0$.

Pick $a, b \geq 0$ such that $f_{-a}, f_b \neq 0$ and $f_{-a-1} = f_{b+1} = 0$. (In fact, $f_{-a} = c_1 x_r^{(d)}$ and $f_b = c_2 x_1^{(d)}$.) Define $f_t = \sum_{k=0}^{a+b} t^k f_{k-a} \in \mathcal{R}_d[t]$. It follows for all $i < r$ that

$$\begin{aligned} (\partial_i - t\partial_{i+1})(f_t) = \partial_i f_t - t\partial_{i+1} f_t &= \sum_{k=0}^{a+b} t^k \partial_i f_{k-a} - t \sum_{k=0}^{a+b} t^k \partial_{i+1} f_{k-a} \\ &= \sum_{k \in \mathbb{Z}} t^k \partial_i f_{k-a} - \sum_{k \in \mathbb{Z}} t^{k+1} \partial_i f_{k-a+1} = 0. \end{aligned}$$

Thus $\operatorname{ann}_{R(t)}(f_t) \supseteq (\partial_1 - t\partial_2, \dots, \partial_{r-1} - t\partial_r, \partial_r^{d+1})$. Note that

$$\operatorname{ann}_{R(t)}\big((x_r + \cdots + t^{r-1} x_1)^{(d)}\big) = \big(\partial_1 - t\partial_2, \dots, \partial_{r-1} - t\partial_r, \partial_r^{d+1}\big).$$

By lemma 1.4 there exists $c_t \in k(t)$ such that $f_t = c_t (x_r + \cdots + t^{r-1} x_1)^{(d)}$. Since $f_t \in \mathcal{R}_d[t]$, it follows that $c_t = \partial_r^d f_t \in k[t]$, finishing the proof. $\square$

Remark 4.23: By remark 4.18 and lemma 4.21, the polynomials $h_1, \dots, h_{r-1}$ in proposition 4.17 must be $r-1$ consecutive terms in $c_t\big(\sum_{k=0}^{s-1} t^k x_{s-k}\big)^{(d-2)}$ for some $c_t \in k[t]$. We also need $h_i = 0$ for all $i < s-1$ and $h_{s-1}, \dots, h_{s+q+1}$ linearly independent. Since there are $(d-2)(s-1)+1$ linearly independent terms in $\big(\sum_{k=0}^{s-1} t^k x_{s-k}\big)^{(d-2)}$, those conditions can be met if and only if

$$q + 2 \leq (d-2)(s-1).$$

In particular, it is possible to construct such examples with $q = 1$ as long as $(d-2)(s-1) \geq 3$, i.e. $s \geq 4$ when $d = 3$, $s \geq 3$ when $d = 4$, and $s \geq 2$ when $d \geq 5$. This is what we did in examples 4.15 and 4.16. We may now also construct examples having $q > 1$.

Remark 4.24: We started this chapter with the following question 4.1. Given a polynomial $f \in \mathcal{R}_d$, $d \geq 3$, is it possible to find $f_t \in \mathcal{R}_d[t_1, \dots, t_n]$ such that $f_0 = f$ and f_t splits regularly $\dim_k M_f - 1$ times over $k(t_1, \dots, t_n)$? When $r \leq 4$ we proved in theorem 4.9 that this is always possible. When $r \geq 5$ and $d \geq 5$, or $r \geq 7$ and $d = 4$, or $r \geq 9$ and $d = 3$, we have found examples that this is not always possible. This leaves only the six pairs

$$(r, d) \in \big\{(5,3), (6,3), (7,3), (8,3), (5,4), (6,4)\big\}.$$

We end this chapter with the following example. It is basically the first degenerate splitting example we ever considered, and theorem 4.5 was formulated and proven with this example as a model.

Example 4.25: Let $A \in \operatorname{Mat}_k(r,r)$ be the fundamental Jordan block, i.e.

$$A_{ij} = \begin{cases} 1, & \text{if } j = i+1, \\ 0, & \text{otherwise.} \end{cases}$$

Let the ideal $J \subseteq R$ be defined as in lemma 4.21, and let

$$I = I_2(\partial A \partial) = I_2\left(\begin{smallmatrix} \partial_1 & \partial_2 & \dots & \partial_{r-1} & \partial_r \\ \partial_2 & \partial_3 & \dots & \partial_r & 0 \end{smallmatrix}\right) = J + \partial_r \cdot (\partial_2, \dots \partial_r).$$

For all $d \geq 0$ and $k = 0, \dots, (r-1)d$, define $h_{dk} \in \mathcal{R}_d$ by

$$\big(x_1 + t x_2 + \dots + t^{r-1} x_r\big)^{(d)} = \sum_{k=0}^{(r-1)d} t^k h_{dk}. \tag{4.2}$$

If we let $\tau(\alpha) = \sum_{i=1}^{r}(i-1)\alpha_i$, then this simply means that

$$h_{dk} = \sum_{\substack{|\alpha|=d \\ \tau(\alpha)=k}} x^{(\alpha)}.$$

Note that $\partial_i h_{dk} = h_{d-1,k-i+1}$ for all $i = 1,\dots,r$. Let $f \in \mathcal{R}_d$. It follows from lemma 4.21 that $I \subseteq \operatorname{ann}_R f$ if and only if $f \in \langle h_{d0},\dots,h_{d,r-1}\rangle$. This implies that

$$I_d^{\perp} = \{f \in \mathcal{R}_d \mid I \subseteq \operatorname{ann}_R f\} = \langle h_{d0},\dots,h_{d,r-1}\rangle,$$

and therefore $\dim_k(R/I)_d = r$ for all $d > 0$. Note that $\partial_r(h_{dk}) = 0$ for all $k < r-1$, thus $\operatorname{ann}_R(f)_1 \neq 0$ if $f \in \langle h_{d0},\dots,h_{d,r-2}\rangle$.

Let $d \geq 3$ and $f = h_{d,r-1}$. Clearly $\operatorname{ann}_R(f)_1 = 0$, hence proposition 2.21 implies that M_f is a commutative k-algebra. Since $A \in M_f$, it follows by lemma 4.8c that $M_f = k[A]$. Let us prove that

$$\operatorname{ann}_R f = I + \partial_1^{d-1} \cdot (\partial_1,\dots,\partial_{r-1}). \tag{4.3}$$

Since $\partial_i h_{dk} = h_{d-1,k-i+1}$, it follows that $\partial_1^{d-2}\partial_i f = h_{1,r-i} = x_{r+1-i}$ for all $i = 1,\dots,r$. These are linearly independent, and it follows that $\{\partial_1^k \partial_i f\}_{i=1}^r$ are linearly independent for all $0 \leq k \leq d-2$. Hence for all $0 < e < d$ we get $\dim_k(R/\operatorname{ann}_R f)_e \geq r = \dim_k(R/I)_e$. Since $I \subseteq \operatorname{ann}_R f$, it follows that $\operatorname{ann}_R(f)_e = I_e$ for all $e < d$ and $H(R/\operatorname{ann}_R f) = (1,r,r,\dots,r,1)$. In degree d $\operatorname{ann}_R f$ needs $r-1$ extra generators. Since $\partial_1^{d-1}\partial_i f = 0$ for all $i < r$, equation (4.3) follows. Note that $\operatorname{ann}_R f$ is generated in degree two and d only.

Equation (4.2) can be used to define a degenerate splitting of length r of f. Indeed, substituting $k+1$ for r, the equation may be rewritten as

$$h_{dk} + \sum_{i>k} t^{i-k} h_{di} = t^{-k}\Big(\big(x_1 + tx_2 + \cdots + t^k x_{k+1}\big)^{(d)} - \sum_{i<k} t^i h_{di}\Big).$$

Since $h_{di} \in k[x_1,\dots,x_k]^{DP}$ for all $i < k$, we may proceed carefully by induction and prove that there exists a polynomial $h'_t \in k[t_1,\dots,t_k][x_1,\dots,x_{k+1}]^{DP}$ such that $h'_0 = h_{dk}$ and h'_t splits k times inside $k(t_1,\dots,t_k)[x_1,\dots,x_{k+1}]^{DP}$. In particular, there exists $f_t \in \mathcal{R}_d[t_1,\dots,t_{r-1}]$ such that $f_0 = f$ and f_t splits $r-1$ times over $k(t_1,\dots,t_{r-1})$, which is also what theorem 4.5 guarantees. In fact, the degenerate splitting f_t we get from equation (4.2) is essentially the same as the one theorem 4.5 gives us, since $A^k \partial f = \partial h_{d,r-k-1}$ for all k.

Note that $f_t \sim x_1^{(d)} + \cdots + x_r^{(d)}$, thus this example is an extremal case. Other examples of $f \in \mathcal{R}_d$ such that $M_f = k[A]$ and A is in Jordan normal form can be constructed from this one.

Chapter 5

Generalizations

A central object in this paper has been M_f, the matrix algebra that we have associated to any $f \in \mathcal{R}_d$. In this chapter we consider how to generalize the construction of M_f and some of the results in section 2.2. In fact, we will define two different generalizations of M_f, and both give is new algebras. Indeed, we show that both $\widehat{M}^f = \big(\oplus_{e=0}^{d-3} M_e^f\big) \oplus \big(\oplus_{e \geq d-2} \operatorname{Mat}_{R_e}(r,r)\big)$, where M_e^f is defined below, and $M_{f,D} = \{A \in \operatorname{Mat}_k(N,N) \,|\, I_2(D\,AD) \subseteq \operatorname{ann}_R f\}$ are (non-commutative) k-algebras, see propositions 5.5 and 5.11.

We start by defining a k-vector space M_e^f that generalizes M_f in the sense that $M_0^f = M_f$.

Definition 5.1: Let $d \geq 0$ and $f \in \mathcal{R}_d$. For all $e \geq 0$ define M_e^f by

$$M_e^f = \{A \in \operatorname{Mat}_{R_e}(r,r) \,|\, I_2(\partial\, A\partial) \subseteq \operatorname{ann}_R f\}.$$

Lemmas 2.12 and 2.13 were important tools in the study of M_f. They provided a connection between M_f and polynomials $g \in \mathcal{R}_d$ that we later used to find regular and degenerate splittings of f. Lemma 5.2 updates both lemmas, connecting M_e^f to polynomials $g \in \mathcal{R}_{d-e}$ that are related to f.

Lemma 5.2: *Suppose $d \geq e \geq 0$ and $f \in \mathcal{R}_d$.*

(a) *Let $A \in \operatorname{Mat}_{R_e}(r,r)$. The following are equivalent.*

  (i) *$I_2(\partial\, A\partial) \subseteq \operatorname{ann}_R f$.*

  (ii) *$A\partial\partial^{\mathsf{T}} f$ is a symmetric matrix.*

(iii) *There exists* $g \in \mathcal{R}_{d-e}$ *such that* $\partial g = A\partial f$.

Furthermore, this g is unique if $e < d$.

(b) *Let* $g \in \mathcal{R}_{d-e}$. *The following are equivalent.*

(i) *There exists* $A \in \mathrm{Mat}_{R_e}(r,r)$ *such that* $\partial g = A\partial f$.

(ii) $R_1(g) \subseteq R_{e+1}(f)$.

(iii) $\mathrm{ann}_R(f)_{d-e-1} \subseteq \mathrm{ann}_R(g)_{d-e-1}$.

Proof: The proof of the equivalences in (a) is an exact copy of the proof of lemma 2.13, and the uniqueness of g is obvious. To prove (b), the existence of an A such that $\partial g = A\partial f$ simply means that $R_1(g) \subseteq R_{e+1}(f)$. By duality this is equivalent to $\mathrm{ann}_R(g)_{d-e-1} = R_1(g)^\perp \supseteq R_{e+1}(f)^\perp = \mathrm{ann}_R(f)_{d-e-1}$. □

Definition 5.3: If $d > e \geq 0$ and $f \in \mathcal{R}_d$, let

$$\gamma_e^f : M_e^f \to \mathcal{R}_{d-e}$$

be the k-linear map defined by sending a matrix $A \in M_e^f$ to the unique polynomial $g \in \mathcal{R}_{d-e}$ satisfying $\partial g = A\partial f$, cf. lemma 5.2a.

γ_e^f is indeed a map of k-vector spaces since $\partial g = A\partial f$ is k-linear in both A and g. In chapters 3 and 4 we used elements in the image of $\gamma_f = \gamma_0^f$ to produce regular and degenerate splittings of f. Even though we do not find such an explicit use of the polynomials in $\mathrm{im}\,\gamma_e^f$ when $e > 0$, we are still interested in its image. We start by calculating the kernel and image of γ_e^f.

Lemma 5.4: *Suppose* $d > e \geq 0$ *and* $f \in \mathcal{R}_d$. *Then*

$$\begin{aligned} \mathrm{im}\,\gamma_e^f &= (\mathsf{m}_R \,\mathrm{ann}_R f)_{d-e}^\perp, \\ \ker \gamma_e^f &= \{A \in \mathrm{Mat}_{R_e}(r,r) \mid A\partial f = 0\}. \end{aligned}$$

Moreover, if we let β_{1j} *be the minimal number of generators of* $\mathrm{ann}_R(f)$ *of degree* j, *then*

$$\begin{aligned} \dim_k \mathrm{im}\,\gamma_e^f &= \dim_k (R/\mathrm{ann}\, f)_{d-e} + \beta_{1,d-e}, \\ \dim_k \ker \gamma_e^f &= re \cdot \tbinom{r-1+e}{e+1} + r \cdot \dim_k \mathrm{ann}(f)_{e+1}. \end{aligned}$$

Proof: By lemma 5.2b, $\operatorname{im}\gamma_e^f = \{g \in \mathcal{R}_{d-e} \mid \operatorname{ann}_R(f)_{d-e-1} \subseteq \operatorname{ann}_R(g)_{d-e-1}\}$. Since $\operatorname{ann}_R g$ is determined by its degree $d-e$ piece by lemma 1.2a, it follows that $\operatorname{im}\gamma_e^f = (R_1 \cdot \operatorname{ann}_R(f)_{d-e-1})^\perp = (\mathfrak{m}_R \operatorname{ann}_R f)_{d-e}^\perp$. Evidently, $R_1 \operatorname{ann}_R f_{d-e-1}$ is a k-vector subspace of $\operatorname{ann}_R(f)_{d-e}$ of codimension $\beta_{1,d-e}$. Hence

$$\dim_k \operatorname{im}\gamma_e^f = \operatorname{codim}_k(R_1 \cdot \operatorname{ann}_R(f)_{d-e-1}) = \dim_k(R/\operatorname{ann} f)_{d-e} + \beta_{1,d-e}.$$

Since $\partial\gamma_e^f(A) = A\partial f$, we get $\ker\gamma_e^f = \{A \in \operatorname{Mat}_{R_e}(r,r) \mid A\partial f = 0\}$. If we let $V_e = \{D = [D_1 \dots D_r]^\mathsf{T} \in R_e^r \mid \sum_i D_i\partial_i \in \operatorname{ann}(f)_{e+1}\}$, we see that $\dim_k \ker\gamma_e^f = r \cdot \dim_k V_e$. We note that V_e is the kernel of the map $R_e^r \to \mathcal{R}_{d-e-1}$ given by $D \mapsto \sum_i D_i\partial_i(f)$. This map is the composition $R_e^r \to R_{e+1} \to \mathcal{R}_{d-e-1}$, and its image is $R_{e+1}(f)$ since $R_e^r \to R_{e+1}$ is surjective. It follows that

$$\dim_k V_e = r \cdot \binom{r-1+e}{e} - \dim_k R_{e+1}(f) = e \cdot \binom{r-1+e}{e+1} + \dim_k \operatorname{ann}(f)_{e+1}. \qquad \square$$

The first significant property that M_f possesses is that it is closed under matrix multiplication when $d \geq 3$. Our definition of M_e^f allows us to transfer this to $M^f = \oplus_{e\geq 0} M_e^f$, with a similar restriction. The following proposition should therefore come as no surprise.

***Proposition* 5.5:** *Suppose $a + b \leq d - 3$. Matrix multiplication defines a map*

$$M_a^f \times M_b^f \to M_{a+b}^f,$$

and all commutators belong to $\ker\gamma_{a+b}^f$. In particular, the augmentation

$$\widehat{M}^f = \left(\bigoplus_{e=0}^{d-3} M_e^f\right) \oplus \left(\bigoplus_{e\geq d-2} \operatorname{Mat}_{R_e}(r,r)\right)$$

is a (non-commutative) graded k-algebra with unity.

Proof: The proof of proposition 2.21 generalizes immediately. $\square$

Since $M_e^f = \operatorname{Mat}_{R_e}(r,r)$ for all $e \geq d-1$, we see that $\widehat{M}^f$ differs from M^f only in degree $d-2$. It is interesting that the image of the multiplication map $M_a^f \times M_b^f \to \operatorname{Mat}_{R_{a+b}}(r,r)$ is generally not contained in M_{a+b}^f if $a+b = d-2$. An easy example is $r = 2$ and $f = x_1^{(2)} + x_2^{(2)} \in \mathcal{R}_2$. Then $\partial\partial^\mathsf{T} f = I$, thus M_0^f consists of all symmetric matrices. But the product of two symmetric matrices is not symmetric, unless they commute.

We now want to study $\operatorname{im}\gamma_e^f$ in more detail. To help us do that we define the following graded R-modules.

Definition 5.6: If $f \in \mathcal{R}_d$, let $F^f = \oplus_e F_e^f$ and $G^f = \oplus_e G_e^f$ where

$$F_e^f = \{g \in \mathcal{R}_{d-e} \mid \operatorname{ann}(f)_k \subseteq \operatorname{ann}(g)_k \,\forall\, k \le d-e\},$$
$$G_e^f = \{g \in \mathcal{R}_{d-e} \mid \operatorname{ann}(f)_k \subseteq \operatorname{ann}(g)_k \,\forall\, k < d-e\}.$$

In the following we will often drop the superscripts $({}^f)$. Obviously, $G_d = k$ and $G_e = F_e = 0$ for all $e > d$. Note that $G_e = \{g \in \mathcal{R}_{d-e} \mid \operatorname{ann}_R(f)_{d-e-1} \subseteq \operatorname{ann}_R(g)_{d-e-1}\}$ for all e by lemma 1.2a. In particular, lemma 5.2b implies that

$$G_e = \operatorname{im}\gamma_e^f \quad \text{for all} \quad 0 \le e < d.$$

The next lemma summarizes some nice properties of F and G.

Lemma 5.7: *Suppose $f \in \mathcal{R}_d$. Then the following are true.*

(a) $G = \{g \in \mathcal{R} \mid \partial_i g \in F \,\forall\, i\} \supseteq F = R(f)$,

(b) $\dim_k(G/F)_e = \beta_{1,d-e}$ *for all e, and*

(c) $G \cong \operatorname{Hom}_k(R/\mathsf{m}_R \operatorname{ann}_R f, k)$.

In particular, G is a graded canonical module for $R/\mathsf{m}_R \operatorname{ann}_R f$, and we can get a free resolution of G (as a graded R-module) by computing one for $R/\mathsf{m}_R \operatorname{ann}_R f$ and dualizing.

Proof: Recall that $R_e(f)^\perp = \operatorname{ann}_R(f)_{d-e}$ by lemma 1.2b. Dualizing this equation gives $R_e(f) = \{g \in \mathcal{R}_{d-e} \mid Dg = 0 \,\forall\, D \in \operatorname{ann}_R(f)_{d-e}\}$, which equals F_e by lemma 1.2a. Combining this with lemma 5.2b, we get $G_e = \{g \in \mathcal{R}_{d-e} \mid R_1(g) \subseteq R_{e+1}(f) = F_{e+1}\}$. This proves (a).

(b) follows from lemma 5.4 if $0 \le e < d$, and it is trivial otherwise.

Before we prove (c), we want to say something about dualizing F. Note that $\mathcal{R}_e = \operatorname{Hom}_k(R_e, k)$ since $\mathcal{R}$ by definition is the graded dual of R. This implies $R_e = \operatorname{Hom}_k(\mathcal{R}_e, k)$. Since $F_{d-e} \subseteq \mathcal{R}_e$, the map $R_e \to \operatorname{Hom}_k(F_{d-e}, k)$ is clearly surjective, and its kernel is $\{D \in R_e \mid D(g) = 0 \,\forall\, g \in F_{d-e}\} = F_{d-e}^\perp = \operatorname{ann}_R(f)_e$. Thus $\operatorname{Hom}_k(F_{d-e}, k) \cong (R/\operatorname{ann}_R f)_e$, and therefore $\operatorname{Hom}_k(F, k) \cong R/\operatorname{ann}_R f$. This explains why $F^* \cong F$, which is the Gorenstein property of F.

Turning to G, the map $R_e \to \operatorname{Hom}_k(G_{d-e}, k)$ is surjective as above. Its kernel is $\{D \in R_e \mid D(g) = 0 \,\forall\, g \in G_{d-e}\} = G_{d-e}^{\perp}$, and $G_{d-e}^{\perp} = (\mathsf{m}_R \operatorname{ann}_R f)_e$ by lemma 5.4. This shows that $\operatorname{Hom}_k(G,k) \cong R/\mathsf{m}_R \operatorname{ann}_R f$, proving (c). The last statements follow since $R/\mathsf{m}_R \operatorname{ann}_R f$ is Artinian. □

Since $F = R(f)$, multiplication in R induces a ring structure on F given by $D(f) \star E(f) = DE(f)$. For all a, b such that $a + b \neq d$, we can extend $\star$ to a bilinear map $F_a \times G_b \to G_{a+b}$ by $D(f) \star g = D(g)$. This is well defined because $a \neq d - b$ implies $\operatorname{ann}_R(f)_a \subseteq \operatorname{ann}_R(g)_a$. The equation $D(f) \star g = D(g)$ is not well defined when $a = d - b$ and $g \in G_b \setminus F_b$, thus G is not quite an F-module.

In order to extend the multiplication to all of G, we need an even larger restriction on the degrees, as seen in the following proposition. Note that M^f contains $R \cdot I = \{D \cdot I \mid D \in R\}$, the subalgebra consisting of all multiples of the identity matrix. Clearly, if $D \in R_e$, then $\gamma_e^f(D \cdot I) = D(f)$. Thus $\gamma_e^f : M_e^f \to G_e$ maps $R_e \cdot I$ onto F_e.

***Proposition* 5.8:** *$\gamma = \oplus_e \gamma_e$ induces a multiplication $\star : G_a \times G_b \to G_{a+b}$ for $a + b \leq d - 3$ that is associative, commutative and k-bilinear. $f \in G_0$ acts as the identity. Furthermore, $D(f) \star h = D(h)$ for all $D \in R_a$ and $h \in G_b$.*

Proof: Given $g \in G_a$ and $h \in G_b$, we can find $A \in M_a$ and $B \in M_b$ such that $g = \gamma_a(A)$ and $h = \gamma_b(B)$ since $G_e = \operatorname{im} \gamma_e$. Since $a + b \leq d - 3$ it follows from proposition 5.5 that $AB \in M_{a+b}$ and $BA\partial f = AB\partial f$. We define $g \star h$ to be

$$g \star h = \gamma_{a+b}(AB) \in G_{a+b}.$$

First we prove that this is well defined. Assume that $\gamma_a(A') = \gamma_a(A)$ and $\gamma_b(B') = \gamma_b(B)$. Then $A'\partial f = A\partial f$ and $B'\partial f = B\partial f$, and therefore

$$\begin{aligned} \partial\big(\gamma_{a+b}(A'B')\big) &= A'B'\partial f = A'B\partial f \\ &= BA'\partial f = BA\partial f = AB\partial f = \partial\big(\gamma_{a+b}(AB)\big). \end{aligned}$$

Hence $\gamma_{a+b}(A'B') = \gamma_{a+b}(AB)$.

Now, $AB\partial f = BA\partial f$ is equivalent to $\gamma_{a+b}(AB) = \gamma_{a+b}(BA)$, which implies $g \star h = h \star g$. Associativity follows from associativity of matrix multiplication, and the bilinearity is obvious. Furthermore, from $f = \gamma_0(I)$ it follows that $f \star g = g$

for all $g \in G_a$, $a \leq d-3$. Finally, if $D \in R_a$, then $D(f) = \gamma_a(D \cdot I)$. Hence $D(f) \star h = \gamma_a(D \cdot I) \star \gamma_b(B) = \gamma_{a+b}(D \cdot B) = D(h)$. □

The last statement, $D(f) \star h = D(h)$, says that $\star$ restricts to the "module" action $F_a \times G_b \to G_{a+b}$, but with the stronger requirement $a + b \leq d - 3$. Let us extend the multiplication $\star : G_a \times G_b \to G_{a+b}$ by zero if $a+b \geq d-2$. We do this to get an algebra, but note that $\star$ no longer restricts to $D(f) \star E(f) = DE(f)$ on F when $a + b \geq d - 2$.

***Corollary* 5.9:** *The truncation $\widetilde{G} = \oplus_{e=0}^{d-3} G_e$ is a commutative k-algebra.*

Proof: This is immediate from proposition 5.8. □

Remark 5.10: Proposition 5.8 implies in particular that G_e is a module over G_0 for all $e \leq d-3$. We first discovered this the following way. Let $N = \binom{r+e}{e+1}$, and fix a basis $\{D_1, \ldots, D_N\}$ be for R_{e+1}. Define $D = [D_1, \cdots, D_N]^{\mathsf{T}}$ and $M'_e = \{A \in \mathrm{Mat}_k(r, N) \mid I_2(\partial\, AD) \subseteq \operatorname{ann} f\}$. Just slightly modifying ideas in this chapter, it is easy to see that there is a surjective map $M'_e \to G_e$, and that matrix multiplication $M'_0 \times M'_e \to M'_e$ induces the same module action $G_0 \times G_e \to G_e$ as above.

There are other ways, in addition to M^f, to generalize the construction of M_f. We feel the following is worth mentioning. Fix some $e \geq 1$, and let $N = \dim_k R_e = \binom{r-1+e}{e}$. Choose a basis $\mathcal{D} = \{D_1, \ldots, D_N\}$ for R_e, and let $D = [D_1 \ldots D_N]^{\mathsf{T}}$. For any $d \geq 0$ and $f \in \mathcal{R}_d$, we define

$$M_{f,D} = \{A \in \mathrm{Mat}_k(N, N) \mid I_2(D\; AD) \subseteq \operatorname{ann}_R f\}.$$

$M_{f,D}$ is clearly a k-vector space containing the identity matrix. We note that $M_{f,\partial} = M_f$, thus this is another generalization of M_f. However, one of the basic lemmas we used to study M_f, lemma 2.13, does not generalize to $M_{f,D}$ when $e \geq 2$. That is, $I_2(D\; AD) \subseteq \operatorname{ann} f$ does not imply that there exists $g \in \mathcal{R}_d$ such that $Dg = ADf$. The converse implication is obviously still true. On the other hand, lemma 2.12 generalizes, i.e. $\operatorname{ann}(f)_{d-e} \subseteq \operatorname{ann}(g)_{d-e}$ if and only if there exists $A \in \mathrm{Mat}_k(N, N)$ such that $Dg = ADf$. But the reason for including $M_{f,D}$ here, is that proposition 2.21 generalizes.

Proposition 5.11: *Suppose $e \geq 1$ and $d \geq 3e$. Let $f \in \mathcal{R}_d$. Then $M_{f,D}$ is closed under matrix multiplication. If furthermore* $\operatorname{ann}(f)_e = 0$, *then $M_{f,D}$ is a commutative k-algebra.*

Proof: Pick $A, B \in M_{f,D}$. Note that for all i, j, k the 3×3 minor

$$\begin{vmatrix} D_i & (AD)_i & (BD)_i \\ D_j & (AD)_j & (BD)_j \\ D_k & (AD)_k & (BD)_k \end{vmatrix}$$

belongs to $\operatorname{ann}(f)_{3e}$ by expansion along the third column. Expanding along the third row proves that

$$D_k \cdot \begin{vmatrix} (AD)_i & (BD)_i \\ (AD)_j & (BD)_j \end{vmatrix} \in \operatorname{ann}(f)_{3e}$$

for all i, j and k. Since $d \geq 3e$ it follows that $I_2(AD\ BD) \subseteq \operatorname{ann} f$. Hence $(AD)(BD)^{\mathsf{T}}(f) = ADD^{\mathsf{T}}(f)B^{\mathsf{T}}$ is symmetric, and therefore

$$ABDD^{\mathsf{T}}(f) = ADD^{\mathsf{T}}(f)B^{\mathsf{T}} = BDD^{\mathsf{T}}(f)A^{\mathsf{T}} = DD^{\mathsf{T}}(f)B^{\mathsf{T}}A^{\mathsf{T}} = DD^{\mathsf{T}}(f)(AB)^{\mathsf{T}}.$$

This means that $AB \in M_{f,D}$. Moreover,

$$ABDD^{\mathsf{T}}(f) = DD^{\mathsf{T}}(f)B^{\mathsf{T}}A^{\mathsf{T}} = BDD^{\mathsf{T}}(f)A^{\mathsf{T}} = BADD^{\mathsf{T}}(f),$$

which implies that $(AB - BA)Df = 0$. If $\operatorname{ann}(f)_e = 0$, then $AB = BA$. □

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