Topics in elementary tropical geometry

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Introduction

The importance of a scientific theory can often be measured by its relations to other research areas. In view of this, there is no doubt that the emerging field of tropical geometry very much deserves its recent popularity. Tropical geometry appears in the intersection of a varied bouquet of mathematical areas, both pure and applied. It has provided new insight in old problems, and it stands out as a natural working environment for new research in fields such as enumerative geometry ([13], [7], [8], [16]), real geometry ([22]), secant varieties ([4]), statistics ([17]), mirror symmetry ([2], [9]) and phylogenetics ([19], [18]).

The first appearance of tropical varieties came in 1971, under the name of "logarithmic limit-sets" of complex algebraic varieties ([1]). The modern formulation of this uses the concept of *amoebas*, introduced by Gelfand, Kapranov and Zelevinsky in their book [2]. The amoeba of an algebraic variety in complex n-space is the image of the variety when taking the logarithms of the absolute value of each coordinate. In particular, the amoeba is a subset of real n-space. Letting the base of the logarithm tend to infinity, the amoeba shrinks to its "spine", a polyhedral complex which we now call a tropical variety.

To avoid limits in the above construction, one can replace the ground field $\mathbb{C}$ by an algebraically closed field with a non-Archimedean valuation, for example the field of Puiseux series with complex coefficients. The topological closure of the valuation of an algebraic variety defined over such a field is a *non-Archimedean amoeba*, commonly called the *tropicalization* of the original variety. This algebraic approach to tropical varieties is used by many authors. It turns out that a tropicalized variety can be interpreted as the *Gröbner fan* of a homogeneous polynomial ideal. Thus many computations lend themselves to well-developed algebraic techniques.

An ongoing project of Mikhalkin ([12]) takes a different view on tropical varieties. He aims to build a theory of tropical algebraic geometry completely from scratch, in parallel to algebraic geometry.

The underlying idea motivating much of the existing tropical geometry is the following: Given a problem involving algebraic geometry, tropicalization might lead to an easier problem, due to the piecewise linear nature of tropical geometry. Of course, this simplification does not come for free: Upon solving the tropical problem, one must then show that the result allows a "lifting" back to the algebraic setting. Understanding such liftings is an active research area (see e.g. [10] and [20]), and there are many open problems.

The prime example of how this idea works - in fact, the example that really sparked off the interest in tropical geometry - is Mikhalkin's Correspondence Theorem ([13]). It states that the number $N_{g,d}$ of plane complex curves of genus g and degree d through $3d + g - 1$ points in general position, equals the number of tropical plane curves with similar properties. The latter can be counted through combinatorial methods, hence the Correspondence Theorem gives a new formula for $N_{g,d}$, as an alternative to the algorithmic solution found by Caporaso and Harris in 1998 ([3]).

While Mikhalkin's Correspondence Theorem gave important new insight, it did not solve any open problems of classical geometry. However, several similar correspondence theorems have since followed, doing exactly that. We mention here the computation of Zeuthen numbers in enumerative geometry (see [14, Remark 7.2]), and the Welschinger invariants in real geometry ([22]), for which there were no known formulas prior to tropical geometry.

A well known tropical geometer once described the technique of tropicalization as "a train going from the world of algebraic geometry, to the world of tropical geometry". Continuing this metaphor, we suggest that there are two ways of enjoying the trip. The first is to travel as tourists: You bring everything you need from home, but take full advantage of the friendly environment. (Alas, the train going back has broken down, so you have to walk all the way home, carrying your souvenirs. This, of course, refers to the "lifting" process.)

Alternatively, you can travel light, and let yourself be swept away to explore the new world. You might lose track from time to time, and even forget the purpose of the whole trip. But hopefully, when eventually returning home, it is with more fundamental knowledge than that obtained by the tourist. This is the philosophy followed in this dissertation. In other words, instead of viewing the objects of tropical geometry solely as the result of tropicalization, they are studied for their own sake, without concern of liftings.

This point of view has important consequences for the mathematical methods employed. When seeing tropical varieties as tropicalizations of algebraic varieties, the main focus lies on algebraic techniques. While this certainly has proved to be effective in many cases, it is in some sense unsatisfactory. Whatever definition one uses of tropical varieties, they turn out as polyhedral complexes in real Euclidean space. When trying to learn their secrets, it is of great interest to see how far one can get using their own language, which is based on real convex geometry, polytopes and combinatorics.

The starting point for much of the material in this dissertation came in 2000, when Kapranov showed that non-Archimedean amoebas of algebraic hypersurfaces has a simple description in terms of what we now call *tropical polynomials*, highlighting the geometric aspects of tropical geometry. A tropical polynomial is a real Laurent polynomial where the operation of addition is exchanged with taking the maximum, and multiplication is exchanged with addition. It can be regarded as a real, convex, piecewise-linear function with integer slopes. Kapranov showed that a subset of real n-space is the non-Archimedean amoeba of a hypersurface if and only if it is the non-linear locus of a tropical polynomial (see [11], and also [5]). This latter description is what we will take as our definition of tropical hypersurfaces. It is not hard to see

that a tropical hypersurface is a polyhedral complex of codimension one, and many interesting geometric properties follow.

A fundamental concept of tropical geometry is the duality between the cells of the tropical hypersurface associated to a tropical polynomial, and the elements of a certain subdivision of the Newton polytope of the polynomial. Because of this, many aspects of tropical hypersurfaces are best studied through the theory of convex lattice polytopes and their subdivisions. A tropical hypersurface is smooth if the dual subdivision is elementary (unimodular).

A sometimes confusing issue at this early stage of tropical geometry, is the lack of uniform terminology in the existing literature, and differing definitions of basic concepts. An illustrating example is the definition of the degree of a tropical hypersurface. Let us denote by Γ_d^n the n-dimensional simplex in real n-space spanned by the origin and the standard basis vectors scaled by the factor d. According to different authors, a tropical hypersurface[1] has degree d if i) the support set of the defining polynomial is exactly the set of lattice points in Γ_d^n (used e.g. in [21]), ii) the convex hull of the support set is Γ_d^n (used e.g. in [6] and [13]), iii) the convex hull of the support set fits inside Γ_d^n, but not inside Γ_{d+1}^n (equivalent to the definition in [15]). The definitions according to i), ii) and iii) are increasingly inclusive. Hence in this case, and several others, the preference of either definition is mostly a matter of scope, rather than choosing among conflicting schools. Also for the papers in this dissertation, some definitions differ slightly in generality. In particular, the varying of definition of degree is essential for the stated results.

As a real traveler finds comfort in things and places reminding of home, it is an unavoidable impulse for a tropical geometer to look for analogies between tropical geometry and classical geometry. And there is indeed a lot to be found. Tropical varieties, though different in looks, have properties which are remarkable similar to those of algebraic varieties. Even more fascinating is the fact that these similarities often come with a "twist". For example, through two general points in real n-space there is a unique tropical line. But for special pairs of points, there are infinitely many tropical lines containing them. In these cases we say that the tropical lines form a *two-point family*.

Another example of an analogy with a twist is given by Bezout's Theorem. This holds for tropical plane curves - but only if the definition of degree is well chosen (see Section 4 of the first paper of this dissertation).

The main theme of this dissertation is to find tropical analogues of well known results of classical geometry, and to prove these analogues by elementary methods. Our arguments are geometric and combinatorial, and rarely require heavy theory. The dissertation consists of five papers. In each of these, we study a specific subject of tropical geometry with clear analogies to complex algebraic geometry.

The subject of the first paper is tropical elliptic plane curves. These are smooth tropical plane curves of degree 3 and genus 1. Here we use alternative iii) above (the least restrictive). Mimicking the setup in classical algebraic geometry, we define the

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[1]In the following cited articles, only the case $n = 2$ (i.e., tropical plane curves) is considered. However, the generalization to higher dimensions is immediate.

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Jacobian as an abelian group associated to a tropical curve. Unlike in the classical case, the Jacobian of a tropical elliptic curve is not equal as a set to the curve itself, but to a smaller part of it, namely the curve's unique cycle. We show that the induced group structure on the cycle is isomorphic to the circle group. Moreover, the group operation has a geometric interpretation similar to the classical picture.

Moving up one step dimension-wise, papers two and three study tropical surfaces in real three-space, and the tropical lines contained in these. The definition of degree used here is alternative ii) above. In the first of these two papers, the emphasis is on tropical surfaces with infinitely many tropical lines. In analogy to the classical fact that any smooth quadric surface has two rulings of lines, we prove that for any point on the unique compact 2-cell of a smooth tropical quadric surface, the surface contains two tropical lines passing through the point. We also show that there exist smooth tropical surfaces of arbitrary degree containing infinitely many tropical lines. However, this can only happen when the lines form two-point families.

The third paper is a self-contained continuation of the previous. Here, we explain a specific method for counting the number of tropical lines on smooth tropical surfaces of degree at least three. We obtain an upper bound for the number of tropical lines on a general tropical surface with a given subdivision, by counting certain subcomplexes of the subdivision. (The concept of generality here refers to the Euclidean topology on the parameter cone of tropical surfaces associated to a given subdivision.) If the general surface has infinitely many tropical lines, this information can also be extracted from the subdivision. As a concrete example, we offer a subdivision for which the associated tropical surfaces are smooth cubics with exactly 27 tropical lines in the general case, and always at least 27 tropical lines. We also give examples of smooth tropical surfaces of arbitrary degree greater than three containing no tropical lines.

In the fourth paper we study transversal intersections of tropical hypersurfaces in arbitrary dimension. If such an intersection is one-dimensional, we call it a tropical complete intersection curve. We calculate the number of vertices (counting multiplicities) of such a curve, as a function of the degrees of the intersecting tropical hypersurfaces. If the curve is smooth and connected, this allows us to compute the curve's genus. The obtained formula coincides with the genus formula for complete intersection curves in complex projective space.

The fifth paper is a short note, containing a single, surprising theorem. The starting point is the classical Fano's axiom of plane geometry, which states that there are no quadrangles with the property that its three diagonal points (i.e., the intersection points of opposite sides) are collinear. For example, this is known to hold in projective planes over any field of characteristic different from two. In the tropical plane, however, the situation is completely opposite: For every plane tropical quadrangle, its three diagonal points are tropically collinear.

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The group law on a tropical elliptic curve

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Abstract

In analogy with the classical group law on a plane cubic curve, we define a group law on a smooth plane tropical cubic curve. We show that the resulting group is isomorphic to S^1.

1 Introduction

Tropical geometry is a recent, but rapidly growing field of research in mathematics, in which one seeks to establish connections between complex algebraic geometry and the combinatorics of certain piecewise linear objects, called tropical varieties. Such connections has led to new insight in various areas, like enumerative geometry [3], mirror symmetry [1] and statistics [5].

A favorite subject among many tropical geometers is the study of plane tropical curves, and their many fascinating similarities with classical plane algebraic curves. The purpose of this paper is to give a contribution to the list of such analogies by - in a way resembling the classical case - defining a group law on a smooth plane tropical cubic curve.

We define the *Jacobian* as an abelian group associated to a tropical curve. Unlike the classical situation, the Jacobian of a tropical elliptic curve C is not equal as a set to the curve itself, but to a smaller part of it, namely the curve's unique cycle $\bar{C}$. For $P, Q \in \bar{C}$ we define $d_C(P, Q)$ to be the displacement from P to Q with respect to the $\mathbb{Z}$-metric on $\bar{C}$ (and a chosen orientation of $\bar{C}$). This plays a crucial role in the main results, which can be summarized as follows:

Theorem 1.1. *Let C be a tropical elliptic curve, and let $\bar{C}$ be its unique cycle. Let $\mathcal{O}$ be a point on $\bar{C}$.*

   a) We have a bijection of sets $\bar{C} \longrightarrow \mathrm{Jac}(C)$, given by $P \longmapsto P - \mathcal{O}$.

   b) The induced group law on $\bar{C}$ satisfies the relation

$$d_C(\mathcal{O}, P + Q) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q).$$

   c) As a group, $\bar{C}$ is isomorphic to the circle group S^1.

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2   Preliminaries

Let $\mathbb{R}_{tr} := (\mathbb{R}, \oplus, \odot)$ be the *tropical semiring*, where the binary operations are defined by $a \oplus b := \max\{a, b\}$ and $a \odot b := a + b$. The multiplicative identity element of $\mathbb{R}_{tr}$ is 0, while there is no additive identity (since $-\infty$ is not included as an element in $\mathbb{R}_{tr}$).

Remark 2.1. The operations $\oplus$ and $\odot$ can be extended to $\mathbb{R}^n$ as follows:

$$(a_1, \ldots, a_n) \oplus (b_1, \ldots, b_n) := (\max\{a_1, b_1\}, \ldots, \max\{a_n, b_n\}), \quad \text{and}$$
$$\lambda \odot (a_1, \ldots, a_n) := (\lambda + a_1, \ldots, \lambda + a_n), \quad \text{for } \lambda \in \mathbb{R}.$$

Moreover, we can define *tropical projective n-space* by setting $\mathbb{P}_{tr}^n := \mathbb{R}^{n+1} / \sim$, where $\boldsymbol{x} \sim \boldsymbol{y} \iff \boldsymbol{x} = \lambda \odot \boldsymbol{y}$ for some $\lambda \in \mathbb{R}$. Note that unlike the classical situation, $\mathbb{P}_{tr}^n$ does not have more points than $\mathbb{R}^n$. For example, every equivalence class in $\mathbb{P}_{tr}^n$ has a representative in $\mathbb{R}^{n+1}$ with 0 as the last coordinate.

Let $\mathcal{A} \subseteq \mathbb{Z}^n$ be a finite set of vectors $\boldsymbol{a} = (a_1, \ldots, a_n)$. A tropical (Laurent) polynomial in indeterminates $x_1, \ldots, x_n$, with support $\mathcal{A}$, is an expression of the form

$$f = \bigoplus_{\boldsymbol{a} \in \mathcal{A}} \lambda_{\boldsymbol{a}} \odot x_1^{a_1} \odot \cdots \odot x_n^{a_n} = \max_{\boldsymbol{a} \in \mathcal{A}} \{\ldots, \lambda_{\boldsymbol{a}} + \sum_{i=1}^{n} a_i x_i, \ldots\},$$

where each $\lambda_{\boldsymbol{a}} \in \mathbb{R}_{tr}$. The convex hull of $\mathcal{A}$ is called the *Newton polytope of f* and is denoted by Δ. When in danger of ambiguity, we use indices to indicate the polynomial, as in $\mathcal{A}_f$ and Δ_f.

Notice that as a function $\mathbb{R}^n \to \mathbb{R}$, f is convex and piecewise linear.

Definition 2.2. The *tropical hypersurface* $V(f)$ defined by f is the set of points in $\mathbb{R}^n$ where the function $f \colon \mathbb{R}^n \to \mathbb{R}$ is not linear.

Remark 2.3. Note that if f consists of a single monomial, $V(f)$ is the empty set.

Remark 2.4. Different tropical polynomials can define the same tropical variety. In particular, it is easy to see that if $g = f \odot m$, where $m = x^a y^b$ is a tropical monomial, then $V(g) = V(f)$. Note that in this case $\mathcal{A}_g$ (resp. Δ_g) is a translation of $\mathcal{A}_f$ (resp. Δ_f) by the vector (a, b).

3   Tropical curves

We now focus our attention to tropical hypersurfaces in $\mathbb{R}^2$:

Definition 3.1. Let $f(x, y)$ be a tropical polynomial in two indeterminates. The tropical hypersurface $V(f) \subseteq \mathbb{R}^2$ is called a *tropical curve* in $\mathbb{R}^2$.

We recall some basic properties of tropical curves. For proofs and more details, see [6, Section 3], or [3, Sections 1-3] for a more exhaustive approach.

Given a tropical polynomial f, we can associate a lattice subdivision of the Newton polygon Δ of f in the following way: Let $\widehat{\Delta}$ be the convex hull of the set $\{(a, b, \lambda_{abc})\} \subseteq \mathbb{R}^2 \times \mathbb{R}$, where (a, b) runs through $\mathcal{A}$. Then define Subdiv_f to be the image under the projection to $\mathbb{R}^2$ of the top facets of $\widehat{\Delta}$, i.e., the facets whose outer normal unit vector has positive last coordinate.

The subdivision Subdiv_f is in a natural way dual to the tropical variety $V(f)$. In particular, each edge of $V(f)$ corresponds to an edge of Subdiv_f, and corresponding edges are perpendicular to each other. The unbounded rays in $V(f)$ correspond to the edges of $\partial \Delta$. (Cf. [6, Proposition 3.5] and [3, Proposition 3.11].)

Let E be an edge of a tropical curve $C = V(f)$, and let $E^\vee$ be the corresponding edge in Subdiv_f. We define the *weight* of E to be the lattice length of $E^\vee$, i.e. $1 + \sharp\{$interior lattice points of $E^\vee\}$.

Lemma 3.2. *For any node V of a tropical curve, the following balancing condition holds: Let $E_1, \ldots, E_n$ be the edges adjacent to V. For each $i = 1, \ldots, n$ let m_i be the weight of E_i, and $\boldsymbol{v}_i$ the primitive integer vector pointing into E_i from V. Then*

$$(1) \qquad\qquad m_1 \boldsymbol{v}_1 + \cdots + m_n \boldsymbol{v}_n = \boldsymbol{0},$$

where $\boldsymbol{0} = (0, 0) \in \mathbb{R}^2$.

The balancing condition characterizes tropical curves: Assume C is a 1-dimensional polyhedral complex in $\mathbb{R}^2$, consisting of rays and line segments with rational slopes, each assigned some positive integral weight. Then $C = V(f)$ for some tropical polynomial f if and only if (1) is satisfied at every vertex of C.

Next we define the degree of a tropical curve. For each $d \in \mathbb{N}_0$, let Γ_d be the triangle with vertices $(0, 0), (d, 0), (0, d)$. (When $d = 0$ we get the degenerated triangle $\Gamma_0 = \{(0, 0)\}$.)

Definition 3.3. Let $C = V(f)$ be a tropical curve in $\mathbb{R}^2$, and let Δ be the Newton polygon of f. If Δ fits inside Γ_d, but not inside Γ_{d-1}, then C has *degree d*. If $\Delta = \Gamma_d$, we say that C has degree d *with full support*.

Remark 3.4. There seems to be no clear consensus in the literature on how to define the degree of a tropical curve. Definition 3.3 differs slightly from the ones in [6] and [3], but serves the purpose of this paper better. In particular, as we will see in the next section, Definition 3.3 gives room for an extended version of the tropical Bezout's theorem compared to that in [6].

Example 3.5. A *tropical line* is a tropical curve of degree 1 with full support. For instance, if $f = ax \oplus by \oplus c$, then the tropical line $L = V(f)$ consists of three unbounded rays, emanating from the "center" $(c - a, c - b)$ in the directions $(-1, 0), (0, -1)$ and $(1, 1)$ respectively.

Example 3.6. If f is any monomial, then Δ consists of a single point. Hence $V(f)$ has degree 0. This is appropriate since $V(f)$ is an empty set. (Cf. Remark 2.3.)
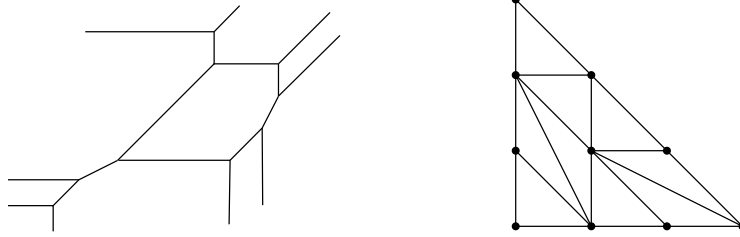
Figure 1: A tropical curve and its associated subdivision. The subdivision shows that the curve is smooth of degree 3 and genus 1.

A vertex V of a tropical curve is called *3-valent* if V has exactly 3 adjacent edges. Furthermore, if these edges have weights m_1, m_2, m_3 and primitive integer direction vectors $\boldsymbol{u} = (u_0, u_1), \boldsymbol{v} = (v_0, v_1), \boldsymbol{w} = (w_0, w_1)$ respectively, we define the *multiplicity* of V to be the absolute value of the number

$$m_1 m_2 \begin{vmatrix} u_0 & u_1 \\ v_0 & v_1 \end{vmatrix} = m_2 m_3 \begin{vmatrix} v_0 & v_1 \\ w_0 & w_1 \end{vmatrix} = m_1 m_3 \begin{vmatrix} w_0 & w_1 \\ u_0 & u_1 \end{vmatrix}.$$

Definition 3.7. A tropical curve is called *smooth* if every vertex is 3-valent and has multiplicity 1.

Notice that in a smooth tropical curve, every edge has weight 1.

Definition 3.8. The *genus* of a smooth tropical curve $C = V(f)$ is the number of vertices of Subdiv_f in the interior of the Newton polygon Δ_f.

Figure 1 shows a smooth curve of degree 3 and genus 1, and its associated subdivision.

3.1 The $\mathbb{Z}$-metric

Let $C \subseteq \mathbb{R}^2$ be a smooth tropical curve. If E is any edge of C, we define a metric on E called the $\mathbb{Z}$-*metric*, in the following way. For any two points $\boldsymbol{x}, \boldsymbol{y} \in E$, we set their distance in the $\mathbb{Z}$-metric to be the number $\frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{\|\boldsymbol{v}\|}$, where $\|\cdot\|$ denotes the Euclidean norm, and $\boldsymbol{v}$ is a primitive integral direction vector of E. In particular, if E is a bounded edge, we define its *lattice length*, $l(E)$, to be the distance (in the $\mathbb{Z}$-metric) between its endpoints. Note that if both endpoints of E have integral coordinates, then $l(E) = 1 + \sharp\{\text{interior lattice points on } E\}$.

Remark 3.9. By identifying each edge E of C with the real interval $[0, l(E)]$ (or $[0, \infty)$ if E is unbounded), C can be thought of as a "metric graph with possibly unbounded edges". This is equivalent to giving C a $\mathbb{Z}$-*affine structure*, or *tropical structure* as described e.g. in [4].

4   Intersections of tropical curves

We say that two tropical curves C and D intersect *transversally* if no vertex of C lies on D and vice versa. In a transversal intersection we define intersection multiplicities as follows: Let P be an intersection point of C and D, where the two edges meeting have weights m_1 and m_2, and primitive direction vectors (v_0, v_1) and (w_0, w_1) respectively. The *intersection multiplicity* $\mathrm{mult}_P(C \cap D)$ is then the absolute value of

$$
m_1 m_2 \begin{vmatrix} v_0 & v_1 \\ w_0 & w_1 \end{vmatrix}.
$$

Non-transversal intersections are dealt with in the following way: For *any* intersecting tropical curves C and D, let C_ε and D_ε be nearby translations of C and D such that C_ε and D_ε intersect transversally. We then have ([6, Theorem 4.3]):

Theorem-Definition 4.1. *Let the* stable intersection *of C and D, denoted $C \cap_{st} D$, be defined by*

$$
C \cap_{st} D = \lim_{\varepsilon \to 0} (C_\varepsilon \cap D_\varepsilon).
$$

This limit is independent of the choice of perturbations, and is a well-defined subset of points with multiplicities in $C \cap D$.

Theorem 4.2 (Tropical Bezout). *Assume C and D are tropical curves of degrees c and d respectively. If both curves have full support, then their stable intersection consists of cd points, counting multiplicities.*

Proof. See [6, Theorem 4.2 and Corollary 4.4]. The idea is to show that the number of (stable) intersection points is invariant under translations of the curves. Thus we can arrange the two curves such that for each of them, the intersection points lie on the unbounded rays in one of the three coordinate directions. It is then trivial to check that $\sharp(C \cap_{st} D) = cd$. □

There is also a tropical version of Bernstein's Theorem: Recall that the *mixed area* of two convex polygons R and S is defined as the number $\mathrm{Area}(R+S) - \mathrm{Area}(R) - \mathrm{Area}(S)$, where $R + S$ is the *Minkowski sum* of R and S.

Theorem 4.3 (Tropical Bernstein). *Let $C = V(f)$ and $D = V(g)$ be any tropical curves intersecting transversally, with Newton polygons Δ_f and Δ_g respectively. Then the number of intersection points, counting multiplicities, equals the mixed area of Δ_f and Δ_g.*

Proof. See [7, Theorem 9.5]. □

Although perhaps not as enlightening as the homotopy argument given in [6], one can prove Theorem 4.2 as a special case of Theorem 4.3. In fact, we can get a stronger result:

Theorem 4.4 (Strong version of Tropical Bezout). *Assume C and D are tropical curves of degrees c and d respectively. If at least one of the curves have full support, then their stable intersection consists of cd points, counting multiplicities.*

Proof. Because of Theorem-Definition 4.1 we can assume that the intersection is transversal. Note that for any positive integers c and d, we have the Minkowski sum $\Gamma_c + \Gamma_d = \Gamma_{c+d}$. Hence the mixed area of $\Gamma_c + \Gamma_d$ equals $\frac{1}{2}(c+d)^2 - \frac{1}{2}c^2 - \frac{1}{2}d^2 = cd$. This proves Theorem 4.2.

Suppose now C has full support, i.e. $\Delta_f = \Gamma_c$, and that Δ_g is a convex polygon of the form $\Gamma_d \smallsetminus Q$, where $Q \subseteq \Gamma_d$ is a lattice polygon containing exactly one of the corners of Γ_d, say $(d, 0)$. Then $\text{Area}(\Delta_f + \Delta_g) = \text{Area}(\Gamma_c + (\Gamma_d \smallsetminus Q)) = \text{Area}(\Gamma_c + \Gamma_d) - \text{Area}(Q)$. Thus the mixed area of Δ_f and Δ_g is

$$\text{Area}(\Delta_f + \Delta_g) - \text{Area}(\Delta_f) - \text{Area}(\Delta_g) = \\ (\text{Area}(\Gamma_c + \Gamma_d)) - \text{Area}(Q) - \text{Area}(\Gamma_d) - (\text{Area}(\Gamma_d) - \text{Area}(Q)) = cd.$$

The same argument shows that we can do the same at the other corners, without changing the mixed area. In this way we can form any Newton polygon Δ_g associated to a tropical curve of degree d. Hence $\sharp(C \cap_{st} D) = cd$ for any tropical curve D of degree d. $\square$

Remark 4.5. If neither of the two curves have full support, the theorem will not hold in general. For example, if C and D are the quadric curves given by $C = V(x^2 \oplus y)$ and $D = V(x \oplus y^2)$, then $C \cap D$ consists of a single point with multiplicity 3. Another example is given by the non-intersecting lines $V(0 \oplus x)$ and $V(1 \oplus x)$.

An important special case of Theorem 4.4 is the following corollary:

Corollary 4.6. *Let D be any tropical curve of degree d. Then any tropical line meets D stably in exactly d points, counting multiplicities.*

5 Divisors on smooth tropical curves

Let C be a smooth tropical curve in $\mathbb{R}^2$.

Definition 5.1. We define the *group of divisors* on C, $\text{Div}(C)$, to be the free abelian group generated by the points on C. A *divisor* D on C is an element of $\text{Div}(C)$, i.e. a finite formal sum of the form $D = \sum \mu_P P$.

The number $\sum \mu_P$ is as usual called the *degree* of D. Observe that the elements of degree 0 in $\text{Div}(C)$ form a group, denoted by $\text{Div}^0(C)$.

To define principle divisors, we must first define rational functions. By a *tropical rational function* $h \colon \mathbb{R}^2 \to \mathbb{R}$ we mean a function of the form $h = f - g$, where f and g are tropical polynomials with equal Newton polygons.

Definition 5.2. Given a tropical polynomial f, we define the divisor $\text{div}(f) \in \text{Div}(C)$ as the formal sum of points in $C \cap_{st} V(f)$, counted with their respective intersection multiplicities. Furthermore, if $h = f - g$ is a tropical rational function on $\mathbb{R}^2$, we set $\text{div}(h) := \text{div}(f) - \text{div}(g)$. A divisor $D \in \text{Div}(C)$ is called a *principal divisor* if $D = \text{div}(h)$ for some tropical rational function h.

It follows from Theorem 4.3 that any principal divisor on C has degree 0.

Remark 5.3. Suppose the Newton polygons of f and g differ by a translation. Then we would still have $\text{div}(f) - \text{div}(g) \in \text{Div}^0(C)$, because of Theorem 4.3. In fact, $\text{div}(f) - \text{div}(g)$ is a principle divisor. Indeed, if (a, b) is the translation vector from Δ_f to Δ_g, let $m = x^a y^b$ be the corresponding tropical monomial. Since $V(f) = V(f \odot m)$ (by Remark 2.4) it follows that $\text{div}(f) - \text{div}(g) = \text{div}(h)$, where h is the tropical rational function $(f \odot m) - g$.

Definition 5.4. Two divisors D_1 and D_2 are *linearly equivalent*, denoted as $D_1 \sim D_2$, if $D_1 - D_2$ is principal.

Linear equivalence is an equivalence relation, and as in the classical case one can show that it restricts to an equivalence relation on the subgroup $\text{Div}^0(C)$. Hence we can make the following definition:

Definition 5.5. The group $\text{Div}^0(C)/\sim$ is called the *Jacobian* of C, $\text{Jac}(C)$.

5.1 A formula for the divisor of a tropical rational function

The purpose of this section is to develop a formula for the divisor of a tropical rational function $h\colon \mathbb{R}^2 \to \mathbb{R}$, using only the properties of h restricted to C. We begin with some easy observations:

Lemma 5.6. *Let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be a tropical rational function, and $C \subseteq \mathbb{R}^2$ a tropical curve. The restriction of h to C is then*

  a) *continuous on C,*

  b) *piecewise linear on each edge of C, with integer slopes (with respect to the $\mathbb{Z}$-metric on the edge),*

  c) *eventually constant on each unbounded ray of C.*

Proof. a) Note that h is the difference of tropical polynomials, which are continuous.

b) It is enough to prove this for tropical polynomials. Let E be an edge of C, with primitive integer direction vector $\boldsymbol{v}$, and let $f(\boldsymbol{x}) = \max_{\boldsymbol{a} \in \mathcal{A}}\{\alpha_{\boldsymbol{a}} + \boldsymbol{a} \cdot \boldsymbol{x}\}$ be a tropical polynomial function. Clearly, $f|_E$ is piecewise linear on E. Furthermore, consider any point $P \in E$ such that f is linear in an open interval $I \subseteq E$ containing P. Then $f(\boldsymbol{x}) = \alpha_{\boldsymbol{a}} + \boldsymbol{a} \cdot \boldsymbol{x}$ for all $\boldsymbol{x} \in I$, for some $\boldsymbol{a} \in \mathcal{A}$, and the slope of f at P in the direction of $\boldsymbol{v}$ (w.r.t the $\mathbb{Z}$-metric on E), is $f(P + \boldsymbol{v}) - f(P) = \boldsymbol{a} \cdot \boldsymbol{v} \in \mathbb{Z}$.

c) Suppose $h(\boldsymbol{x}) = f(\boldsymbol{x}) - g(\boldsymbol{x})$, where $f(\boldsymbol{x}) = \max_{\boldsymbol{a} \in \mathcal{A}_f}\{\alpha_{\boldsymbol{a}} + \boldsymbol{a} \cdot \boldsymbol{x}\}$ and $g(\boldsymbol{x}) = \max_{\boldsymbol{b} \in \mathcal{A}_g}\{\beta_{\boldsymbol{b}} + \boldsymbol{b} \cdot \boldsymbol{x}\}$ are tropical polynomials with $\Delta_f = \Delta_g$. Let $\ell = \{V + t\boldsymbol{u} \mid t \geq 0\}$ be an unbounded ray of C, starting at the vertex V and with primitive direction vector $\boldsymbol{u}$. Then $f(V + t\boldsymbol{u}) = \max_{\boldsymbol{a} \in \mathcal{A}_f}\{\alpha_{\boldsymbol{a}} + \boldsymbol{a} \cdot V + t(\boldsymbol{a} \cdot \boldsymbol{u})\}$. For all $t >> 0$ this maximum is achieved for some $\boldsymbol{a} = \boldsymbol{a}_\ell$ with the property that $\boldsymbol{a}_\ell \cdot \boldsymbol{u} \geq \boldsymbol{a} \cdot \boldsymbol{u}$ for all $\boldsymbol{a} \in \mathcal{A}_f$. In particular this implies that $\boldsymbol{a}_\ell \in \partial \Delta_f$. Similarly, when $t >> 0$, we have $g(V + t\boldsymbol{u}) = \beta_{\boldsymbol{b}_\ell} + \boldsymbol{b}_\ell \cdot V + t(\boldsymbol{b}_\ell \cdot \boldsymbol{u})$, for some $\boldsymbol{b}_\ell \in \partial \Delta_g$ such that $\boldsymbol{b}_\ell \cdot \boldsymbol{u} \geq \boldsymbol{b} \cdot \boldsymbol{u}$ for all $\boldsymbol{b} \in \mathcal{A}_g$. Since $\Delta_f = \Delta_g$ this implies that $\boldsymbol{a}_\ell \cdot \boldsymbol{u} = \boldsymbol{b}_\ell \cdot \boldsymbol{u}$, and we conclude that for $t >> 0$ we have $h(V + t\boldsymbol{u}) = \alpha_{\boldsymbol{a}_\ell} - \beta_{\boldsymbol{b}_\ell} + (\boldsymbol{a}_\ell - \boldsymbol{b}_\ell) \cdot V$, which is constant. $\square$

For any function $r\colon C \to \mathbb{R}$ satisfying a) and b) above, we associate to each point $P \in C$ an integer $\operatorname{ord}_P(r)$ as follows: If P is a vertex of C, then $\operatorname{ord}_P(r)$ is the sum of the outgoing slopes of r along the edges adjacent to P. If $P \in C$ is not a vertex, we use the same definition, after first having inserted a (2-valent) vertex at P (but otherwise keeping C unchanged).

We say that r is locally linear at $P \in C$ if there exists an open neighborhood $U \subseteq \mathbb{R}^2$ containing P, and an affine-linear function $s\colon \mathbb{R}^2 \to \mathbb{R}$ such that $r|_{C \cap U} = s|_{C \cap U}$. It is easy to see that $\operatorname{ord}_P(r) = 0$ if r is locally linear at P. Note however, that the converse is not true if P is a vertex of C.

We are now ready to prove the following:

Lemma 5.7. *For any tropical rational function $h\colon \mathbb{R}^2 \to \mathbb{R}$ we have*

$$\operatorname{div}(h) = \sum_{P \in C} \operatorname{ord}_P(h|_C)\, P \,.$$

Proof. Let $f = \max_{\boldsymbol{a} \in \mathcal{A}_f}\{\alpha_{\boldsymbol{a}} + \boldsymbol{a} \cdot \boldsymbol{x}\}$ be a tropical polynomial, and let $\overline{f} := f|_C$. It is clear that $\overline{f}$ is locally linear at any $P \in C \smallsetminus (C \cap_{st} V(f))$, and therefore $\operatorname{ord}_P(\overline{f}) = 0$ for such P. We show below that for each $P \in C \cap_{st} V(f)$, the intersection multiplicity at P equals $\operatorname{ord}_P(\overline{f})$. This implies $\operatorname{div}(f) = \sum_{P \in C} \operatorname{ord}_P(\overline{f})\, P$. By the obvious extension from tropical polynomials to tropical rational functions, the lemma follows from this.

Consider first the case where P is a transversal intersection point, between an edge E_C of C and an edge E_f of $V(f)$. We can choose primitive direction vectors $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{u} = (u_1, u_2)$ of E_C and E_f respectively, such that if m is the weight of E_f, the intersection multiplicity is $\operatorname{mult}_P(C \cap V(f)) = m(v_1 u_2 - v_2 u_1)$.

To find $\operatorname{ord}_P(\overline{f})$, suppose $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}_f$ are such that f equals $\alpha_{\boldsymbol{a}} + \boldsymbol{a} \cdot \boldsymbol{x}$ on one side of P, and $\alpha_{\boldsymbol{b}} + \boldsymbol{b} \cdot \boldsymbol{x}$ on the other side. Then $\boldsymbol{b} - \boldsymbol{a}$ is orthogonal to E_f. Moreover, by the definition of weight, we have (possibly after swapping $\boldsymbol{a}$ and $\boldsymbol{b}$) that $\boldsymbol{b} - \boldsymbol{a} = m(u_2, -u_1)$. This implies that $\operatorname{ord}_P(\overline{f}) = \boldsymbol{v} \cdot \boldsymbol{b} + (-\boldsymbol{v}) \cdot \boldsymbol{a} = \boldsymbol{v} \cdot (\boldsymbol{b} - \boldsymbol{a}) = m(v_1 u_2 - v_2 u_1)$, which equals the intersection multiplicity found above.

Next, suppose $P \in C \cap_{st} V(f)$ is a non-transversal intersection point, i.e., that P is a vertex of either C or $V(f)$. In either case, consider $f_\varepsilon = \max_{\boldsymbol{a} \in \mathcal{A}_f}\{\alpha_{\boldsymbol{a}}^\varepsilon + \boldsymbol{a} \cdot \boldsymbol{x}\}$ such that $V(f_\varepsilon)$ is a small translation of $V(f)$ intersecting C transversally, and $P \notin V(f_\varepsilon)$. Let $P_1, \ldots, P_k \in C \cap V(f_\varepsilon)$ be the intersection points close to P (i.e. the points tending to P when $f_\varepsilon \to f$). Then we have

$$(2) \qquad \operatorname{mult}_P(C \cap_{st} V(f)) = \sum_{i=1}^{k} \operatorname{mult}_{P_i}(C \cap V(f_\varepsilon)).$$

We proceed to show that $\operatorname{ord}_P(\overline{f})$ shows a similar, stable behavior. Let $\ell_1, \ldots, \ell_s$ be the edges of C emanating from P, with primitive direction vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s$. (If P is not a vertex of C, we insert a vertex at P, making $s = 2$, and $v_1 = -v_2$.) Furthermore, let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_s \in \mathcal{A}_f$ be such that for $\boldsymbol{x} \in \ell_i$ close to P, we have $f(\boldsymbol{x}) = \lambda_{\boldsymbol{a}_i} + \boldsymbol{a}_i \cdot \boldsymbol{x}$. In particular, with this notation, we have $\operatorname{ord}_P(\overline{f}) = \sum_{i=1}^{s} \boldsymbol{a}_i \cdot \boldsymbol{v}_i$.

Because $P \notin V(f_\varepsilon)$, f_ε is locally linear at P, and we can assume w.l.o.g. that $f_\varepsilon(\boldsymbol{x}) = \alpha_{\boldsymbol{a}_1}^\varepsilon + \boldsymbol{a}_1 \cdot \boldsymbol{x}$ in a neighborhood of P. For $j = 1, \ldots, s$, let $B_j \subseteq \{P_1, \ldots, P_k\}$

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be the subset whose elements lies on ℓ_j. It is not hard to see that if $B_j \neq \emptyset$, then $\sum_{Q \in B_j} \mathrm{ord}_Q(\overline{f}_\varepsilon) = \boldsymbol{a}_j \cdot \boldsymbol{v}_j - \boldsymbol{a}_1 \cdot \boldsymbol{v}_j$. Hence,

$$(3) \quad \sum_{i=1}^{k} \mathrm{ord}_{P_i}(\overline{f}_\varepsilon) = \sum_{B_j \neq \emptyset}(\boldsymbol{a}_j \cdot \boldsymbol{v}_j - \boldsymbol{a}_1 \cdot \boldsymbol{v}_j) = \sum_{B_j \neq \emptyset} \boldsymbol{a}_j \cdot \boldsymbol{v}_j + \sum_{B_j = \emptyset} \boldsymbol{a}_1 \cdot \boldsymbol{v}_j = \mathrm{ord}_P(\overline{f}),$$

where in the second to last transition we used the balancing condition, and in the final transition the easily proved observation that if $B_j = \emptyset$ then $\boldsymbol{a}_j = \boldsymbol{a}_1$.

From (2) and (3) we deduce that $\mathrm{mult}_P(C \cap_{st} V(f) = \mathrm{ord}_P(\overline{f})$ also when P is non-transversal, and hence that $\mathrm{div}(f) = \sum \mathrm{ord}_P(\overline{f})P$. This proves the lemma. $\quad\square$

Remark 5.8. A consequence of the above two lemmas is that our definitions of tropical rational functions and their divisors are in agreement with those used by Gathmann and Kerber in [2]. The set $\mathcal{R} = \{h|_C \mid h \text{ is a tropical rational function}\}$ is a subset of what they call *rational functions on C*, i.e. functions $r : C \to \mathbb{R}$ which satisfy parts a) and b) of Lemma 5.6. Moreover, Lemma 5.7 implies that for any function in $\mathcal{R}$, the definition of its associated divisor given in [2] is equivalent to our Definition 5.2. In particular, the endpoints at infinity of unbounded rays (these are included as part of the curve in [2]) are avoided because of Lemma 5.6c).

6   Tropical elliptic curves

In the remainder of the paper C will denote a *tropical elliptic curve*, by which we mean a smooth tropical curve of degree 3 and genus 1. We assume that $C = V(f)$, where $f(x, y)$ has Newton polygon $\Delta_f \subseteq \Gamma_3$. Since $(1, 1)$ is the only lattice point in the interior of Γ_3, the definition of genus requires that $(1, 1)$ is a vertex of Subdiv_f lying in the interior of Δ_f. Hence C contains a unique cycle, which we will denote by $\overline{C}$. Finally, each connected component of $C \smallsetminus \overline{C}$ is called a *tentacle* of C.

6.1   An explicit homeomorphism $\overline{C} \to S^1$

Obviously, as a topological space, $\overline{C}$ is homeomorphic to the circle group S^1. We will now construct one such homeomorphism, based on the $\mathbb{Z}$-metric on the edges of C.

Choose any fixed point $\mathcal{O} \in \overline{C}$. Let $V_1, \ldots, V_n$ be the vertices of $\overline{C}$ in counterclockwise direction, such that if $\mathcal{O}$ is a vertex then $V_1 = \mathcal{O}$, otherwise $\mathcal{O}$ lies between V_1 and V_n. Let $E_1, \ldots, E_n$ be the edges of $\overline{C}$, such that $E_1 = V_1 V_2$ and so on. Recall that for each i, $l(E_i)$ denotes the length of E_i in the $\mathbb{Z}$-metric on E_i. Let l be the *cycle length* of $\overline{C}$, i.e., $l = l(E_1) + \cdots + l(E_n)$.

We now define a homeomorphism $\mu \colon \overline{C} \longrightarrow \mathbb{R}/l\mathbb{Z} \approx S^1$, linear in the Euclidean metric of each edge E_i. It is then enough to specify the images in $\mathbb{R}/l\mathbb{Z}$ of the points $\mathcal{O}, V_1, \ldots, V_n$, which we do recursively:

$$(4) \quad \begin{aligned} \mu(\mathcal{O}) &= 0 \\ \mu(V_1) &= l(\mathcal{O}V_1) \\ \mu(V_{i+1}) &= \mu(V_i) + l(E_i), \quad i = 1, \ldots, n-1. \end{aligned}$$
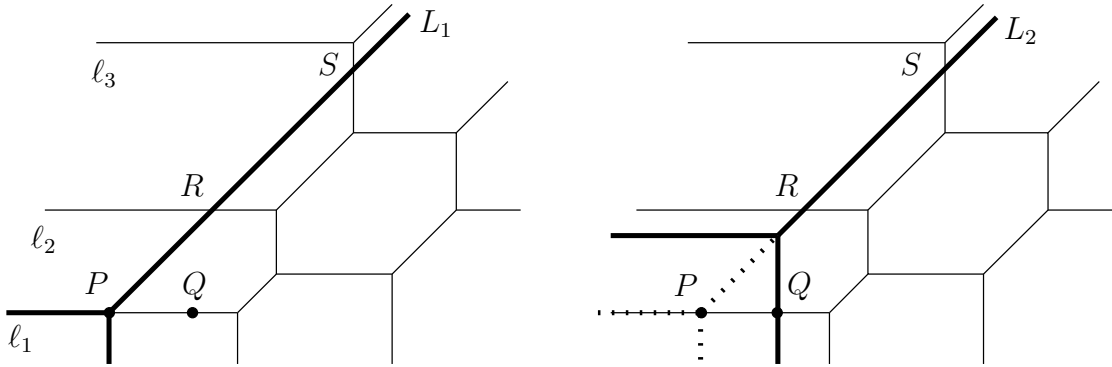
Figure 2: Sufficiently close points P and Q on ℓ_1 are linearly equivalent.

Finally, identifying $\mathbb{R}/l\mathbb{Z}$ with the interval $[0, l)$, we define the (signed) displacement function $d_C \colon \bar{C} \times \bar{C} \to \mathbb{R}$ by the formula

$$d_C(P, Q) = \mu(Q) - \mu(P).$$ (5)

Note that $d_C(Q, P) = -d_C(P, Q)$ for any $P, Q \in \bar{C}$. Moreover, for any three points $P, Q, R \in \bar{C}$ we have $d_C(P, Q) + d_C(Q, R) = d_C(P, R)$.

6.2 When are two points on C linearly equivalent?

In this section we give two propositions, which together give a complete answer to the question in the title. Namely, we prove that any two points on the same tentacle are linearly equivalent, while two distinct points on $\bar{C}$ are never linearly equivalent.

Proposition 6.1. *Let P and Q be points on the same tentacle of C. Then $P \sim Q$.*

Proof. We begin by showing that the points on any unbounded ray are equivalent. By symmetry, it is enough to prove this for the rays that are unbounded in, say, the x-coordinate. Figure 2 shows a typical situation with three such rays, ℓ_1, ℓ_2 and ℓ_3.

The following argument shows that any two sufficiently close points P and Q on ℓ_1 are equivalent: Assume P is further away from $\bar{C}$ than Q. Let $h = f_1 - f_2$ be the tropical rational function where f_1 and f_2 are tropical linear polynomials such that $L_1 = V(f_1)$ is the tropical line with center in P, and $L_2 = V(f_2)$ is the line passing through Q and with center on the ray of L_1 with direction vector $(1, 1)$. Denote this ray by ρ. Then $\mathrm{div}(f_1) = P + R + S$, where R and S lies on ρ, and $\mathrm{div}(f_2) = Q + R + S$ (as long as P and Q are close enough). It follows that $\mathrm{div}(h) = P - Q$, in other words $P \sim Q$.

To show that *any* two points P and Q on ℓ_1 are equivalent, we can choose a finite sequence of points $P = P_1, P_2, \ldots, P_m = Q$ on ℓ_1 such that each pair (P_i, P_{i+1}) is close enough for the above technique to work. Then $P = P_1 \sim \cdots \sim P_m = Q$.

A similar argument shows that the points on ℓ_2 are equivalent. The idea is sketched in Figure 3. To show that P and Q are equivalent, take the tropical line L_1 with center
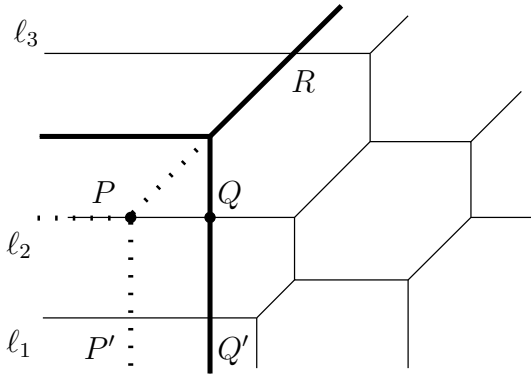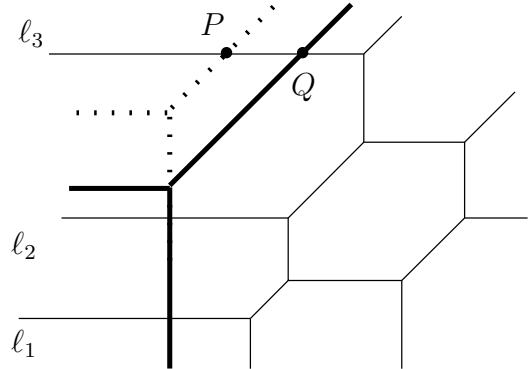
Figure 3: $P \sim Q$ on ℓ_2.

Figure 4: $P \sim Q$ on ℓ_3.

in P and slide it along the ray with direction $(1,1)$ (i.e. keeping R as intersection point with C) until it passes through Q. With the notation on Figure 3, we see that $P + P' + R \sim Q + Q' + R$. But $P' \sim Q'$, since they are on ℓ_1, thus $P \sim Q$.

The same technique works for ℓ_3 (see Figure 4) and also for the bounded line segments of the tentacles. Any tentacle of a tropical elliptic curve can be handled in this way. $\square$

Proposition 6.2. *If $P, Q \in \overline{C}$ and $P \sim Q$, then $P = Q$.*

Proof. Suppose otherwise that $P \neq Q$, and that there exists a tropical rational function h such that $\mathrm{div}(h) = P - Q$. We will apply Lemma 5.7 to show that this leads to a contradiction.

Let $\overline{h} = h|_C$. As a first observation, note that $\overline{h}$ is constant on each tentacle of C. Indeed, this follows from Lemma 5.6c) and the fact that $\mathrm{ord}_R(\overline{h}) = 0$ for all points $R \in C \smallsetminus \overline{C}$. (Note in particular that if $\overline{h}$ is constant on two edges adjacent to a 3-valent vertex V, then $\mathrm{ord}_V(\overline{h}) = 0$ implies that $\overline{h}$ is constant on the third edge as well.)

Now, let c_1 and c_2 be the two directed polygonal arcs of $\overline{C}$ from P to Q. We claim that for each $i = 1, 2$, $\overline{h}$ has constant slope along c_i, w.r.t. the $\mathbb{Z}$-metric. To see this, observe that $\overline{h}$ is clearly linear along any edge of c_i. Furthermore, suppose two edges of c_i intersect in a vertex $V \in \overline{C}$, and that the slopes of $\overline{h}$ along these edges (directed from P to Q) are s_1 and s_2. Because $\overline{h}$ is constant on the tentacle adjacent to V, $\mathrm{ord}_V(\overline{h})$ is of the form $\pm(s_1 - s_2 + 0)$. This equals 0, hence $s_1 = s_2$. This proves the claim.
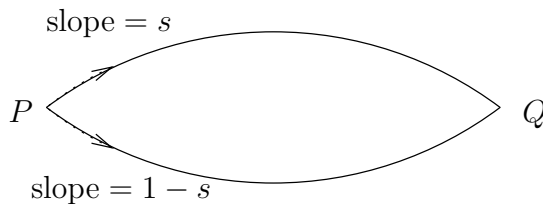

Figure 5: Slope properties of a function h satisfying $\mathrm{div}(h) = P - Q$.

Since $\operatorname{ord}_P(\overline{h}) = 1$, the slopes of $\overline{h}$ along the paths c_1 and c_2 must be s and $1 - s$ for some $s \in \mathbb{Z}$. But this contradicts the assumption of continuity of $\overline{h}$ at Q, since for any choice of s, one of the numbers s and $1 - s$ is positive, while the other is non-positive. (See Figure 5.) $\qquad\square$

6.3 The group law

In this final section we will show that the Jacobian $\operatorname{Jac}(C)$ is set-theoretically equal to $\overline{C}$, and describe the resulting group structure on $\overline{C}$. A crucial step towards this goal is to determine when divisors of the form $P + Q$ are linearly equivalent. When trying to imitate the techniques from the classical case, we stumble across the following problem: Given two points P and Q on $\overline{C}$, we cannot always find a tropical line L that intersects C stably in P and Q. (Recall that a stable intersection is defined as a limit of transversal intersections.) If there exists such a tropical line, we call (P, Q) a *good pair*.

We fix the notation $\boldsymbol{p}_1 = (-1, 0)$, $\boldsymbol{p}_2 = (0, -1)$ and $\boldsymbol{p}_3 = (1, 1)$ for the primitive integer direction vectors of a tropical line.

Lemma 6.3. *Let P, Q, P', Q' be any points on $\overline{C}$. Then*

$$P + Q \sim P' + Q' \quad \Longleftrightarrow \quad d_C(P, P') = -d_C(Q, Q').$$

Proof. We proceed in two steps. First, we prove the result when (P, Q) and (P', Q') are good pairs. Using this, we then generalize to any pairs.

• *Step 1.* Assume (P, Q) and (P', Q') are good pairs, and that $P + Q \sim P' + Q'$. Then there exists (unique) tropical lines L and L', and a point $R \in \overline{C}$ such that $L \cap_{st} C = P + Q + R$ and $L' \cap_{st} C = P' + Q' + R$. (Note that the existence of R follows from Proposition 6.2.) Consider a homotopy L_t of lines containing R such that $L_0 = L$ and $L_1 = L'$. It is enough to consider the case where P and P' are on the same edge, Q and Q' are on the same edge, and L' is a parallel displacement of L along one of the axes. Indeed, in more complex cases, the homotopy can be broken down into parts with the above properties.

Let $\boldsymbol{v}_P$ and $\boldsymbol{v}_Q$ be primitive integer direction vectors of the edges of $\overline{C}$ containing P, P' and Q, Q' respectively, and assume that L' equals the shifting of L δ units in the direction of, say, $\boldsymbol{p}_1$ (see Figure 6). Then from the general formula for (non-orthogonal) vector projection (Figure 7), we find the displacements of P and Q:

$$
\text{(6)} \qquad
\begin{aligned}
PP' &= \frac{\|\boldsymbol{p}_2 \times \delta\boldsymbol{p}_1\|}{\|\boldsymbol{p}_2 \times \boldsymbol{v}_P\|}\, \boldsymbol{v}_P = \delta \boldsymbol{v}_P \quad \Longrightarrow \quad |d_C(P, P')| = \frac{\|\delta\boldsymbol{v}_P\|}{\|\boldsymbol{v}_P\|} = \delta, \\
QQ' &= \frac{\|\boldsymbol{p}_3 \times \delta\boldsymbol{p}_1\|}{\|\boldsymbol{p}_3 \times \boldsymbol{v}_Q\|}\, \boldsymbol{v}_Q = \delta \boldsymbol{v}_Q \quad \Longrightarrow \quad |d_C(Q, Q')| = \frac{\|\delta\boldsymbol{v}_Q\|}{\|\boldsymbol{v}_Q\|} = \delta.
\end{aligned}
$$

(Notice that both the denominators above equals 1, since the intersections at hand have multiplicity 1.) According to the orientation of $\overline{C}$, P and Q are moved in opposite direction. Hence $d_C(P, P') = -d_C(Q, Q')$ as claimed.
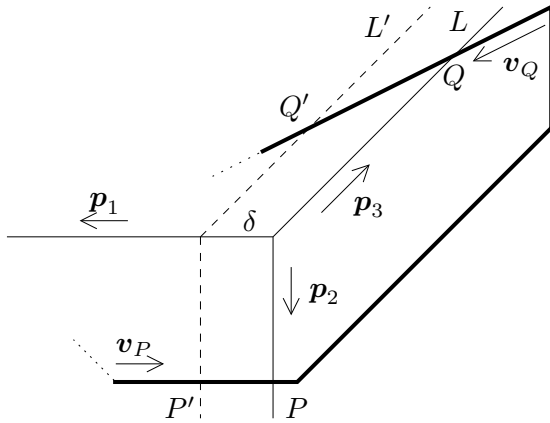
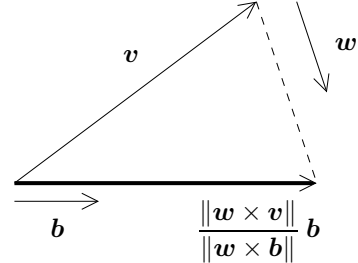Figure 6: Illustrating Step 1.

Figure 7: Non-orthogonal projection

The implication $\Leftarrow$ follows by a similar argument.

• *Step 2.* Now assume (P, Q) is not a good pair. Let L_1 and L_2 be tropical lines through P and Q respectively, and let R_1, S_1, R_2, S_2 be the other intersection points. The idea is to move L_1 and L_2 into new lines L_1' and L_2' in such a way that R_1, S_1, R_2, S_2 are preserved as intersection points. P and Q will not be preserved; they will move to new points P' and Q'. (See Figure 8.) By construction, these points satisfy $P' + Q' \sim P + Q$. Using our results in Step 1 on each of the lines L_1 and L_2, it follows that $d_C(P, P') = -d_C(Q, Q')$. Conversely, it is not hard to see that in this way one can reach any nearby pair (P', Q') satisfying $d_C(P, P') = -d_C(Q, Q')$.

Finally, by choosing L_1 and L_2 in the right way, (P', Q') will form a good pair. Since we proved in Step 1 that the lemma is true for good pairs, it then follows that the lemma holds for any pairs (P, Q) and (P', Q'). $\qquad\square$
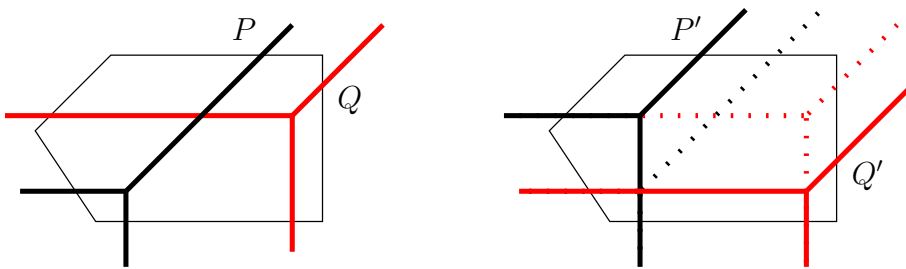

Figure 8: Moving a bad pair (P, Q).

Proposition 6.4. *For any fixed point $\mathcal{O} \in \bar{C}$, the map $\tau_{\mathcal{O}} \colon \bar{C} \to \mathrm{Jac}(C)$ given by $P \mapsto P - \mathcal{O}$ is a bijection of sets.*

Proof. Injectivity follows immediately from Lemma 6.3, since

$$P - \mathcal{O} \sim Q - \mathcal{O} \implies P + \mathcal{O} \sim Q + \mathcal{O} \implies d_C(P, Q) = 0 \implies P = Q.$$

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To prove surjectivity, let D be any divisor of degree 0 . We must show that there exists $P \in \bar{C}$ such that $D \sim P - \mathcal{O}$. Assume first that $D = P_1 - Q_1$, where $P_1, Q_1 \in \bar{C}$. Choose P such that $d_C(P, P_1) = d_C(\mathcal{O}, Q_1)$, then Lemma 6.3 gives $P + Q_1 \sim P_1 + \mathcal{O}$. Thus we have $D = P_1 - Q_1 \sim P - \mathcal{O}$.

Now assume $D = D_1 - D_2$, where $D_1 = P_1 + \cdots + P_n$ and $D_2 = Q_1 \cdots + Q_n$ are any effective divisors of degree $n > 1$. Let P_{12} and Q_{12} be points such that $P_1 + P_2 \sim \mathcal{O} + P_{12}$ and $Q_1 + Q_2 \sim \mathcal{O} + Q_{12}$. Then

$$D \sim \mathcal{O} + P_{12} + \cdots + P_n - (\mathcal{O} + Q_{12} + \cdots + Q_n) = P_{12} + \cdots + P_n - (Q_{12} + \cdots + Q_n).$$

Hence $D \sim D_1' - D_2'$, where D_1 and D_2 are effective of degree $n - 1$. This way we can reduce to the case $n = 1$, which we already proved. $\square$

Because of Proposition 6.4, $\bar{C}$ has a natural group structure:

Definition 6.5. Define $(\bar{C}, \mathcal{O})$ to be the group consisting of points on $\bar{C}$, with the group structure induced from $\mathrm{Jac}(C)$ such that the bijection $\tau_{\mathcal{O}}$ is an isomorphism of groups.

The next theorem and its corollary are the main results of this paper.

Theorem 6.6. *Let P and Q be any points on $\bar{C}$, and let $+$ denote addition in the group $(\bar{C}, \mathcal{O})$. Then the point $P + Q$ satisfies the relation*

$$d_C(\mathcal{O}, P + Q) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q).$$

Proof. Because $\tau_{\mathcal{O}}$ is a group isomorphism, the following equalities hold in $\mathrm{Jac}(C)$:

$$(P + Q) - \mathcal{O} = \tau_{\mathcal{O}}(P + Q) = \tau_{\mathcal{O}}(P) + \tau_{\mathcal{O}}(Q) = P - \mathcal{O} + Q - \mathcal{O}.$$

Thus in $\mathrm{Jac}(C)$ we have $(P + Q) + \mathcal{O} = P + Q$. This means that the divisors $(P + Q) + \mathcal{O}$ and $P + Q$ are equivalent, which by Lemma 6.3 implies the relation

$$d_C(P, P + Q) = d_C(\mathcal{O}, Q).$$

Adding $d_C(\mathcal{O}, P)$ on each side then gives $d_C(\mathcal{O}, P + Q) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q)$ as wanted. $\square$

Remark 6.7. We can describe the group law geometrically just as in the classical case of elliptic curves: To add P and Q we do the following. If (P, Q) is a good pair, consider the tropical line L through P and Q, and let R be the third intersection point of L and $\bar{C}$. Now if $(R, \mathcal{O})$ is a good pair, let L' be the through R and $\mathcal{O}$. Then $P + Q$ is the third intersection point of L' and $\bar{C}$. (See Figure 9 for an example.)

If any of the pairs (P, Q) and $(R, \mathcal{O})$ fail to be good, then move the two points involved equally far (in the $\mathbb{Z}$-metric) in opposite directions until they form a good pair, and use this new pair as described above.

Corollary 6.8. *The map $\mu \colon (\bar{C}, \mathcal{O}) \longrightarrow \mathbb{R}/l\mathbb{Z} \approx S^1$ defined in (4) is a group isomorphism.*
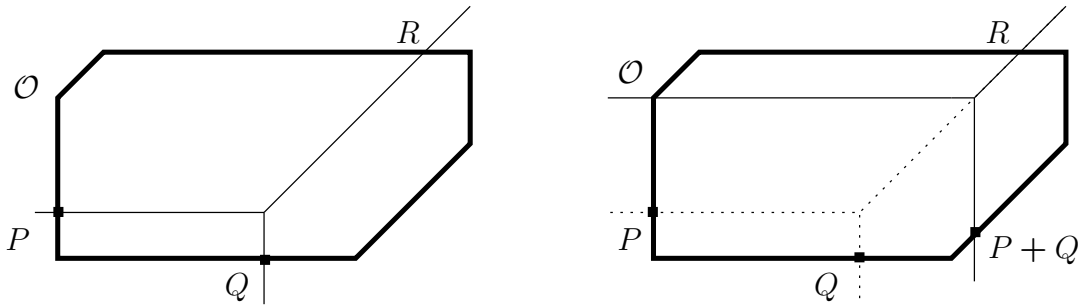
Figure 9: Adding points on a tropical elliptic curve.

Proof. It follows from the relation (5) that for any P we have $\mu(P) = d_C(\mathcal{O}, P)$. Thus

$$\mu(P + Q) = d_C(\mathcal{O}, P + Q) = d_C(\mathcal{O}, P) + d_C(\mathcal{O}, Q) = \mu(P) + \mu(Q).$$

$\square$

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Smooth tropical surfaces with infinitely many tropical lines

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Abstract

We study the tropical lines contained in smooth tropical surfaces in $\mathbb{R}^3$. On smooth tropical quadric surfaces we find two one-dimensional families of tropical lines, like in classical algebraic geometry. Unlike the classical case, however, there exist smooth tropical surfaces of any degree with infinitely many tropical lines.

1 Introduction

Tropical geometry has during the last few years become an increasingly popular field of mathematics. This is not least due to the many fascinating similarities with classical geometry. In this paper we examine tropical analogues of the following well-known results in classical algebraic geometry:

(I) Any smooth quadric surface has two rulings of lines,

(II) Any smooth surface of degree greater than two, has at most finitely many lines.

While a lot of work has been done lately on tropical plane curves, comparatively little is known in higher dimensions. The usual way of defining a tropical variety is as the *tropicalization* of an algebraic variety defined over an algebraically closed field with a non-Archimedean valuation (see e.g. [3]). In the case of hypersurfaces, however, a more inviting, geometric definition is possible. For example, a tropical surface in $\mathbb{R}^3$ is precisely the non-linear locus of a continuous convex piecewise linear function $f : \mathbb{R}^3 \to \mathbb{R}$ with rational slopes. It is an unbounded two-dimensional polyhedral complex, with *zero tension* at each 1-cell. Furthermore, it is dual to a regular subdivision of the Newton polytope of f (when f is regarded as a *tropical polynomial*). The tropical surface is *smooth* if this subdivision is an elementary (unimodular) triangulation.

Tropical varieties of higher codimension are in general more difficult to grasp. However, the only such varieties we are interested in here, are tropical lines in $\mathbb{R}^3$. These were given an explicit geometric description in [3], on which we base our definition. As an analogue of (I) above, we prove that:

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Theorem. *Any smooth tropical quadric surface X has a unique compact 2-cell $\overline{X}$. For any point $p \in \overline{X}$, there exist two tropical lines on X containing p.*

While in classical geometry, any two distinct points in $\mathbb{R}^3$ lie on a unique line, this is only true generically for tropical lines. In fact, for special choices of $p, q \in \mathbb{R}^3$ there are infinitely many tropical lines containing p and q. We show that such families of tropical lines can also exist on a smooth tropical surface. As a consequence, we get the following result, in contrast to (II) above:

Theorem. *There exist tropical surfaces of any degree, with infinitely many tropical lines.*

The paper is organized as follows: In sections 2 and 3 we give some necessary background on convex geometry and tropical geometry, respectively. In particular, the concept of a *two-point family* of tropical lines in $\mathbb{R}^3$ is defined in Section 3.3. Then follows two technical sections, 4 and 5. The former of these deals with constructions of regular elementary triangulations, while the latter contains an analysis of certain lattice polytopes. In Section 6 we explore the general properties of tropical lines contained in smooth tropical surfaces, and in Section 7 we use these to study tropical lines on quadric surfaces. Section 8 concerns two-point families of tropical lines on smooth tropical surfaces. Finally, Section 9 contains our results for tropical surfaces of higher degrees.

2 Lattice polytopes and subdivisions

2.1 Convex polyhedra and polytopes

A *convex polyhedron* in $\mathbb{R}^n$ is the intersection of finitely many closed halfspaces. A *cone* is a convex polyhedron, all of whose defining hyperplanes contain the origin. A *convex polytope* is a bounded convex polyhedron. Equivalently, a convex polytope can be defined as the convex hull of a finite set of points in $\mathbb{R}^n$. Throughout this paper, all polyhedra and polytopes will be assumed to be convex unless explicitly stated otherwise.

For any polyhedron $\Delta \subseteq \mathbb{R}^n$ we denote its affine hull by $\mathrm{Aff}(\Delta)$, and its relative interior (as a subset of $\mathrm{Aff}(\Delta)$) by $\mathrm{int}(\Delta)$. The dimension of Δ is defined as $\dim \mathrm{Aff}(\Delta)$. By convention, $\dim \emptyset = -1$. A *face* of Δ is a polyhedron of the form $\Delta \cap H$, where H is a hyperplane such that Δ is entirely contained in one of the closed halfspaces defined by H. In particular, the empty set is considered a face of Δ. Faces of dimensions 0, 1 and $n-1$ are called *vertices*, *edges* and *facets* of Δ, respectively. If Δ is a polytope, then the vertices of Δ form the minimal set $\mathcal{A}$ such that $\Delta = \mathrm{conv}(\mathcal{A})$.

Let F be a facet of a polyhedron $\Delta \subseteq \mathbb{R}^n$, where $\dim \Delta \leq n$. A vector v is *pointing inwards* (resp. *pointing outwards*) from F relative to Δ if, for some positive constant t, the vector tv (resp. $-tv$) starts in F and ends in $\Delta \setminus F$. If in addition v is orthogonal to F, v is an *inward normal vector* (resp. *outward normal vector*) of F relative to Δ. Using the notation $\langle\,,\,\rangle$ for the Euclidean inner product, a straightforward consequence of these definitions is:

Lemma 2.1. *A vector v is an inward (resp. outward) normal vector of F relative to Δ, if and only if $\langle u, v \rangle > 0$ (resp. $\langle u, v \rangle < 0$) for all vectors u pointing inwards from F relative to Δ.*

If all the vertices of Δ are contained in $\mathbb{Z}^n$, we call Δ a *lattice polyhedron*, or *lattice polytope* if it is bounded. A lattice polytope in $\mathbb{R}^n$ is *primitive* if it contains no lattice points other than its vertices. It is *elementary* (or *unimodular*) if it is n-dimensional and its volume is $\frac{1}{n!}$. Obviously, every elementary polytope is also primitive, while the other implication is not true in general. For instance, the unit square in $\mathbb{R}^2$ is primitive, but not elementary.

Most of the polytopes we are interested in will be simplices, i.e., the convex hull of $n+1$ affinely independent points. In $\mathbb{R}^2$, the primitive simplices are precisely the elementary ones, namely the lattice triangles of area $\frac{1}{2}$. (This is an immediate consequence of Pick's theorem.) In higher dimensions, the situation is very different: There is no upper limit for the volume of an primitive simplex in $\mathbb{R}^n$, when $n \geq 3$. The standard example of this is the following: Let $p, q \in \mathbb{N}$ be relatively prime, with $p < q$, and let $T_{p,q}$ be the tetrahedron with vertices in $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,p,q)$. Then $T_{p,q}$ is an primitive simplex of volume $\frac{q}{6}$.

2.2   Polyhedral complexes and subdivisions

A (finite) *polyhedral complex* in $\mathbb{R}^n$ is a finite collection X of convex polyhedra, called *cells*, such that

- if $C \in X$, then all faces of C are in X, and

- if $C, C' \in X$, then $C \cap C'$ is a face of both C and C'.

The d-dimensional elements of X are called the *d-cells* of X. The dimension of X itself is defined as $\max\{\dim C \mid C \in X\}$. Furthermore, if all the maximal cells (w.r.t. inclusion) have the same dimension, we say that X is of *pure dimension*.

A polyhedral complex, all of whose cells are cones, is a *fan*.

A *subdivision* of a polytope Δ is a polyhedral complex $\mathcal{S}$ such that $|\mathcal{S}| = \Delta$, where $|\mathcal{S}|$ denotes the union of all the elements of $\mathcal{S}$. It follows that $\mathcal{S}$ is of pure dimension $\dim \Delta$. If all the maximal elements of $\mathcal{S}$ are simplices, we call $\mathcal{S}$ a *triangulation*. If $\mathcal{S}$ and $\mathcal{S}'$ are subdivisions of the same polytope, we say that $\mathcal{S}'$ is a *refinement* of $\mathcal{S}$ if for all $C' \in \mathcal{S}'$ there is a $C \in \mathcal{S}$ such that $C' \subseteq C$.

If Δ is a lattice polytope, we can consider *lattice subdivisions* of Δ, i.e., subdivisions in which every element is a lattice polytope. In particular, a lattice subdivision is *primitive* (resp. *elementary*) if all its maximal elements are primitive (resp. elementary). We write down some noteworthy properties of these subdivisions:

- Every elementary subdivision is necessarily a triangulation, and also primitive.

- In a primitive subdivision, all elements (not only the maximal) are primitive.

- For any lattice polytope, its lattice subdivisions with no non-trivial refinements are precisely its primitive triangulations.

2.3   Regular subdivisions and the secondary fan

Let $\Delta = \mathrm{conv}(\mathcal{A})$ where $\mathcal{A}$ is a finite set of points in $\mathbb{R}^n$. Any function $\alpha\colon \mathcal{A} \to \mathbb{R}$ will induce a lattice subdivision of Δ in the following way. Consider the polytope

$$\mathrm{conv}(\{(v, \alpha(v)) \mid v \in \mathcal{A}\}) \in \mathbb{R}^{n+1}.$$

Projecting the top faces of this polytope to $\mathbb{R}^n$, forgetting the last coordinate, gives a collection of subpolytopes of Δ. They form a subdivision $\mathcal{S}_\alpha$ of Δ. The function α is called a *lifting function* associated to $\mathcal{S}_\alpha$.

Definition 2.2. A lattice subdivision $\mathcal{S}$ of $\mathrm{conv}(\mathcal{A})$ is *regular* if $\mathcal{S} = \mathcal{S}_\alpha$ for some $\alpha\colon \mathcal{A} \to \mathbb{R}$.

The set of regular subdivisions of $\mathrm{conv}(\mathcal{A})$ has an interesting geometric structure, as observed by Gelfand, Kapranov and Zelevinsky in [2]. Suppose $\mathcal{A} \subseteq \mathbb{R}^n$ consists of k points. For a fixed ordering of the points in $\mathcal{A}$, the space $\mathbb{R}^{\mathcal{A}} \simeq \mathbb{R}^k$ is a parameter space for all functions $\alpha\colon \mathcal{A} \to \mathbb{R}$. For a given given regular subdivision $\mathcal{S}$ of $\mathrm{conv}(\mathcal{A})$, let $K(\mathcal{S})$ be the set of all functions $\alpha \in \mathbb{R}^{\mathcal{A}}$ which induce $\mathcal{S}$. The following is proved in [2, Chapter 7]:

Proposition 2.3. *Let $\mathcal{S}$ and $\mathcal{S}'$ be any regular subdivisions of $\mathrm{conv}(\mathcal{A})$. Then:*

a) *$K(\mathcal{S})$ is a cone in $\mathbb{R}^{\mathcal{A}}$.*

b) *$\mathcal{S}'$ is a refinement of $\mathcal{S}$ if and only if $K(\mathcal{S})$ is a face of $K(\mathcal{S}')$.*

c) *The cones $\{K(\mathcal{S}) \mid \mathcal{S}$ is a regular subdivision of $\mathrm{conv}(\mathcal{A})\}$ form a fan, $\Phi(\mathcal{A})$, in $\mathbb{R}^{\mathcal{A}}$.*

The fan $\Phi(\mathcal{A})$ is called the *secondary fan* of $\mathcal{A}$. Proposition 2.3b) shows that a subdivision corresponding to a maximal cone of $\Phi(\mathcal{A})$ can have no refinements. Hence the maximal cones correspond precisely to the primitive regular lattice triangulations of $\mathrm{conv}(\mathcal{A})$.

3   Basic tropical geometry

3.1   Tropical hypersurfaces

The purpose of this section is to recall the basics about tropical hypersurfaces and their dual subdivisions. Good references for proofs and details are [3], [4], and [1].

We work over the *tropical semiring* $\mathbb{R}_{tr} := (\mathbb{R}, \max, +)$. Note that some authors use min instead of max in the definition of the tropical semiring. This gives a semiring isomorphic to $\mathbb{R}_{tr}$. Most statements of tropical geometry are independent of this choice, but sometimes care has to be taken (cf. Lemma 3.3).

To simplify the reading of tropical expressions, we adopt the following convention: If a expression is written in quotation marks, all arithmetic operations should

be interpreted as tropical. Hence, if $x, y \in \mathbb{R}$ and $k \in \mathbb{N}_0$ we have for example "$x + y$" $= \max\{x, y\}$, "xy" $= x + y$ and "x^k" $= kx$.

A *tropical monomial* in n variables is an expression of the form "$x_1^{a_1} \cdots x_n^{a_n}$", or in vector notation, "x^a", where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $a = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$. Note that "$x^a$" $= \langle a, x \rangle$, the Euclidean inner product of a and x in $\mathbb{R}^n$. A *tropical polynomial* is a tropical linear combination of tropical monomials, i.e.

$$(1) \qquad f(x) = \text{“}\sum_{a \in \mathcal{A}} \lambda_a x^a\text{”} = \max_{a \in \mathcal{A}}\{\lambda_a + \langle a, x \rangle\},$$

where $\mathcal{A}$ is a finite subset of $\mathbb{N}_0^n$, and $\lambda_a \in \mathbb{R}$ for each $a \in \mathcal{A}$. From the rightmost expression in (1) we see that as a function $\mathbb{R}^n \to \mathbb{R}$, f is concave and piecewise linear. The *tropical hypersurface* $V_{tr}(f) \subseteq \mathbb{R}^n$ is defined to be the non-linear locus of $f \colon \mathbb{R}^n \to \mathbb{R}$. Equivalently, it is the set of points $x \in \mathbb{R}^n$ where the maximum in (1) is attained at least twice.

It is well known (see e.g. [3] and [4]) that $V_{tr}(f)$ is a connected polyhedral complex of pure dimension $n - 1$. As a subset of $\mathbb{R}^n$, $V_{tr}(f)$ is unbounded, although some of its cells may be bounded.

We next describe the very useful duality between a tropical hypersurface $V_{tr}(f)$ and a certain lattice subdivision. With f as in (1), we define the *Newton polytope* of f to be the convex hull of the exponent vectors, i.e., the lattice polytope $\text{conv}(\mathcal{A}) \subseteq \mathbb{R}^n$. As explained in Section 2.3, the map $a \mapsto \lambda_a$ induces a regular subdivision of the Newton polytope $\text{conv}(\mathcal{A})$; we denote this subdivision by $\text{Subdiv}(f)$.

Any element $\Delta \in \text{Subdiv}(f)$ of dimension at least 1, corresponds in a natural way to a subset $\Delta^\vee \subseteq V_{tr}(f)$. Namely, if the vertices of Δ are $a_1, \ldots, a_r$, then $\Delta^\vee$ is the solution set of the equalities and inequalities

$$(2) \qquad \lambda_{a_1} + a_1 x = \cdots = \lambda_{a_r} + \langle a_r, x \rangle \geq \lambda_b + \langle b, x \rangle, \quad \text{for all } b \in \mathcal{A} \smallsetminus \{a_1, \ldots, a_r\}.$$

That $\Delta^\vee \subseteq V_{tr}(f)$ follows immediately from the definition of $V_{tr}(f)$, once we observe that $r \geq 2$ (this is implied by the assumption $\dim \Delta \geq 1$). In fact, $\Delta^\vee$ is a closed cell of $V_{tr}(f)$. Moreover, we have the following theorem (see [4]):

Theorem 3.1. *The association $\Delta \mapsto \Delta^\vee$ gives a one-to-one correspondence between the k-cells of $\text{Subdiv}(f)$ and the $(n - k)$-cells of $V_{tr}(f)$, for each $k = 1, 2, \ldots, n$. Furthermore, for any cells $\Delta, \Lambda \in \text{Subdiv}(f)$ of dimensions at least 1, we have that*

   i) If Δ is a face of Λ, then $\Lambda^\vee$ is a face of $\Delta^\vee$ in $V_{tr}(f)$.

  ii) The affine-linear subspaces $\text{Aff}(\Delta)$ and $\text{Aff}(\Delta^\vee)$ are orthogonal in $\mathbb{R}^n$.

 iii) $\Delta^\vee$ is an unbounded cell of $V_{tr}(f)$ if and only if Δ is contained in a facet of the Newton polytope of f.

If C is a cell of $V_{tr}(f)$, we denote its corresponding cell in $\text{Subdiv}(f)$ by $C^\vee$. The cells C and $C^\vee$ are said to be *dual* to each other.

Theorem 3.1 is independent of the choice of max or min as the tropical addition. However, the following lemma is not (cf. Remark 3.3 below). For lack of reference, we include a proof.

Lemma 3.2. *a) Let $X \subseteq \mathbb{R}^2$ be a tropical curve, and $E \in X$ a vertex. If C is an edge of X adjacent to E, then the outgoing direction vector of C from E is an outward normal vector of $C^\vee$ relative to $E^\vee$.*

b) Let X be a tropical hypersurface in $\mathbb{R}^n$, where $n \geq 2$, and let $C \subseteq X$ be a $(n-1)$-cell adjacent to a $(n-2)$-cell E. If v is an inward normal vector of E relative to C, then v is an outward normal vector of $C^\vee$ relative to $E^\vee$.

Proof. a) Let X be defined by the polynomial $f = "\sum_{a \in \mathcal{A}} \lambda_a x^a" = \max_{a \in \mathcal{A}}\{\lambda_a + \langle a, x \rangle\}$, where $\mathcal{A} \subseteq \mathbb{Z}^2$ is finite. Let E be a vertex of X, and C an edge of X adjacent to E. We consider first the case where C is bounded. Then C has a second endpoint F, and $\overrightarrow{EF}$ is a direction vector of C pointing away from E. Dually, $C^\vee$ is the common edge of the polygons $E^\vee$ and $F^\vee$. Since we already know (by Theorem 3.1) that $\overrightarrow{EF}$ is orthogonal to $C^\vee$, Lemma 2.1 implies that all we have to do is to show that $\langle u, \overrightarrow{EF} \rangle < 0$ for some vector u pointing inwards from $C^\vee$ relative to $E^\vee$.

Let $\mathcal{V}(E^\vee) = \{a_1, a_2, \ldots, a_r\}$ be the vertices of $E^\vee$, named such that $C^\vee = \overline{a_1 a_2}$. Then $u = \overrightarrow{a_2 a_3}$ points inwards from $C^\vee$ relative to $E^\vee$. We claim that $\langle \overrightarrow{a_2 a_3}, \overrightarrow{EF} \rangle < 0$. To prove this, observe that the vertex E satisfies the system of (in)equalities

(3) $\qquad \lambda_{a_1} + \langle a_1, E \rangle = \lambda_{a_2} + \langle a_2, E \rangle = \cdots = \lambda_{a_r} + \langle a_r, E \rangle > \lambda_b + \langle b, E \rangle,$

for all $b \in \mathcal{A} \smallsetminus \mathcal{V}(E^\vee)$. Similarly, F satisfies the relations

(4) $\qquad \lambda_{a_1} + \langle a_1, F \rangle = \lambda_{a_2} + \langle a_2, F \rangle = \cdots = \lambda_c + \langle c, F \rangle = \cdots > \lambda_d + \langle d, F \rangle,$

for all $c \in \mathcal{V}(F^\vee)$ and $d \in \mathcal{A} \smallsetminus \mathcal{V}(F^\vee)$. Now, in particular, (3) gives $\langle a_2, E \rangle - \langle a_3, E \rangle = \lambda_{a_3} - \lambda_{a_2}$, while (4) implies (setting $d = a_3$) that $\langle a_2, F \rangle - \langle a_3, F \rangle > \lambda_{a_3} - \lambda_{a_2}$. Combining this, we find:

$$\langle \overrightarrow{a_2 a_3}, \overrightarrow{EF} \rangle = \langle a_3 - a_2, F - E \rangle = \langle a_3, F \rangle - \langle a_2, F \rangle + \langle a_2, E \rangle - \langle a_3, E \rangle$$
$$< \lambda_{a_2} - \lambda_{a_3} + \lambda_{a_3} - \lambda_{a_2} = 0.$$

This proves the claim, and therefore that $\overrightarrow{EF}$ is an outer normal vector of $C^\vee$ relative to $E^\vee$.

Finally we consider the case when C is unbounded. If C is unbounded, then $C^\vee \subseteq \partial(\Delta_f)$, where Δ_f is the Newton polytope of f. Let $f' = "f + \lambda_b x^b"$, where the exponent vector $b \in \mathbb{Z}^2$ is chosen outside of Δ_f in such a way that $C^\vee$ is not in the boundary of $\Delta_{f'}$. If the coefficient λ_b is set low enough, all elements of Subdiv(f) will remain unchanged in Subdiv(f'). Furthermore, all vertices of X, and all direction vectors of the edges of X, remain unchanged in $V_{tr}(f')$. In particular, E is a vertex of $V_{tr}(f')$, and its adjacent edge whose dual is $C^\vee$, has the same direction vector as C. Since $C^\vee$ is not in the boundary, we have reduced the problem to the bounded case above. This proves the lemma.

b) Let π be the orthogonal projection of $\mathbb{R}^n$ from Aff(E) to Aff$(E^\vee) \simeq \mathbb{R}^2$. If $C_1, \ldots, C_r$ are the $(n-1)$-cells adjacent to E, then $\pi(C_1), \ldots, \pi(C_r)$ are mapped to rays or line segments in Aff$(E^\vee)$, with $\pi(E)$ as their common endpoint. Furthermore, if v is an inward normal vector of E relative to C_i, then v is a direction vector of $\pi(C_i)$ pointing away from $\pi(E)$. The lemma now follows from the argument in a). $\square$

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Remark 3.3. For readers used to working over the semiring $(\mathbb{R}, \min, +)$ instead of $(\mathbb{R}, \max, +)$, note that when using min, the result of Lemma 3.2 changes: The vector v is then an *inward* normal vector of $C^\vee$ relative to $E^\vee$.

3.2   Tropical surfaces in $\mathbb{R}^3$

A tropical hypersurfaces in $\mathbb{R}^3$ will be called simply a *tropical surface*. We will usually restrict our attention to those covered by the following definition:

Definition 3.4. Let $X = V_{tr}(f)$ be a tropical surface, and let $\delta \in \mathbb{N}$. We say that the *degree* of X is δ if the Newton polytope of f is the simplex

$$\Gamma_\delta := \text{conv}(\{(0,0,0), (\delta,0,0), (0,\delta,0), (0,0,\delta)\}).$$

If $\text{Subdiv}(f)$ is an elementary (unimodular) triangulation of Γ_δ, then X is *smooth*.

Remark 3.5. We will frequently talk about a tropical surface X of degree δ without referring to any defining tropical polynomial. It is then to be understood that $X = V_{tr}(f)$ for some f with Newton polytope Γ_δ. In this setting, the notation Subdiv_X refers to $\text{Subdiv}(f)$.

Let us note some immediate consequences of Definition 3.4. For example, since any elementary triangulation of Γ_δ has δ^3 maximal elements, X must have δ^3 vertices. Furthermore, any 1-cell $E \subseteq X$ has exactly 3 adjacent 2-cells, namely those dual to the sides of the triangle $E^\vee$. This last property makes it particularly easy to state and prove the so-called *balancing property*, or *zero-tension property* for smooth tropical surfaces. (A generalization of this holds for any tropical hypersurface. However, this involves assigning an integral *weight* to each maximal cell of X, a concept we will not need here.)

Lemma 3.6 (Balancing property for smooth tropical surfaces). *For any 1-cell E of a smooth tropical surface X, consider the 2-cells C_1, C_2, C_3 adjacent to E. Choosing an orientation around E, each C_i has a unique primitive normal vector v_i compatible with this orientation. Then $v_1 + v_2 + v_3 = 0$.*

Proof. As explained above, $C_1^\vee$, $C_2^\vee$ and $C_3^\vee$ are the sides of the triangle $E^\vee$. Theorem 3.1 implies that $C_i^\vee$ is parallel to v_i for each $i = 1, 2, 3$. In fact, since $C_i^\vee$ is primitive, it must also have the same length as (the primitive) vector v_i. The vectors forming the sides of any polygon (following a given orientation), sum to zero, thus the lemma is proved. $\square$

Note that when $\dim E = 1$, Theorem 3.1 guarantees that $\dim E^\vee = 2$; in particular $E^\vee$ is non-degenerate. This implies that no two of the vectors v_1, v_2, v_3 in Lemma 3.6 are parallel. Thus:

Lemma 3.7. *Let C_1, C_2, C_3 be the adjacent 2-cells to a 1-cell of a smooth tropical surface. Then C_1, C_2, C_3 span different planes in $\mathbb{R}^3$.*

We conclude these introductory remarks on tropical surfaces with a description of some important group actions. Let S_4 be the group of permutations of four elements, so that S_4 is the symmetry group of the simplex Γ_δ. In the obvious way this gives an action of S_4 on the set of subdivisions of Γ_δ.

We can also define an action of S_4 on the set of tropical surfaces of degree δ. Let $X = V_{tr}(f)$, where $f(x_1, x_2, x_3) = $ "$\sum_{a \in \Gamma_\delta} \lambda_a x_1^{a_1} x_2^{a_2} x_3^{a_3}$". For a given permutation $\sigma \in S_4$, we define $\sigma(X)$ as follows. First, homogenize f, giving a polynomial in four variables:

$$f^{hom}(x_1, x_2, x_3, x_4) = \text{``} \sum_{a \in \Gamma_\delta} \lambda_a x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{\delta - a_1 - a_2 - a_3} \text{''}.$$

Now σ acts on f^{hom} in the obvious way by permuting the variables, giving a new tropical polynomial $\sigma(f^{hom})$. Dehomogenizing again, we set $\sigma(f) := \sigma(f^{hom})(x_1, x_2, x_3, 0)$. (Note that 0 is the multiplicative identity element of $\mathbb{R}_{tr}$.) Finally, we define $\sigma(X)$ to be the surface $V_{tr}(\sigma(f))$. Clearly, $\sigma(X)$ is still of degree δ. The resulting action is compatible with the action of S_4 on the subdivisions of Γ_δ. In other words, $\mathrm{Subdiv}_{\sigma(X)} = \sigma(\mathrm{Subdiv}_X)$.

3.3   Tropical lines in $\mathbb{R}^3$

Let L be an unrooted tree with five edges, and six vertices, two of which are 3-valent and the rest 1-valent. We define a *tropical line* in $\mathbb{R}^3$ to be any realization of L in $\mathbb{R}^3$ such that

- the realization is a polyhedral complex, with four unbounded rays (the 1-valent vertices of L are pushed to infinity),

- the unbounded rays have direction vectors $-e_1, -e_2, -e_3, e_1 + e_2 + e_3$,

- The realization is balanced at each vertex, i.e., the primitive integer vectors in the directions of all outgoing edges adjacent to a given vertex, sum to zero.

If the bounded edge has length zero, the tropical line is called *degenerate*. For non-degenerate tropical lines, there are three combinatorial types, shown in Figure 1. From left to right we denote these combinatorial types by $(12)(34)$, $(13)(24)$ and $(14)(23)$, respectively, so that each pair of digits indicate the directions of two adjacent rays. Likewise, the combinatorial type of a degenerate tropical line is written (1234).
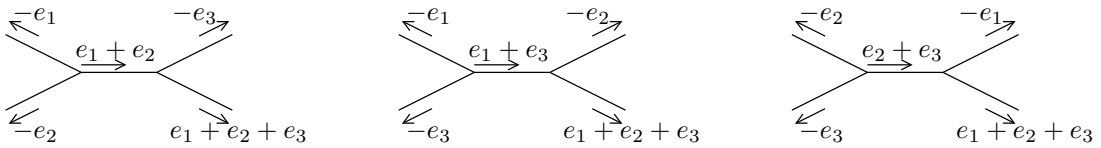

Figure 1: The combinatorial types of tropical lines in $\mathbb{R}^3$.

Remark 3.8. This definition is equivalent to the more standard algebraic definition of tropical lines in $\mathbb{R}^3$. See [3, Examples 2.8 and 3.8].

The *Tropical Grassmannian*, $G_{tr}(1,3)$, is the space of all tropical lines in $\mathbb{R}^3$. It is a polyhedral fan in $\mathbb{R}^4$ consisting of three 4-dimensional cones, one for each combinatorial type. These cones are glued along their common lineality space of dimension 3 (corresponding to rigid translations in $\mathbb{R}^3$).

Remark 3.9. One can define tropical lines in $\mathbb{R}^n$ and their Grassmannians for any $n \geq 2$. A detailed description of these spaces are given in [5].

In classical geometry, any two distinct points lie on a unique line. When we turn to tropical lines, this is true only for generic points. In fact, for special choices of points P and Q there are infinitely many tropical lines passing through P and Q. The precise statement is as follows:

Lemma 3.10. *Let* $P, Q \in \mathbb{R}^3$. *There exist infinitely many tropical lines containing* P *and* Q *if and only if the coordinate vector* $Q - P$ *contains either a zero, or two equal coordinates. In all other cases,* P *and* Q *lie on a unique tropical line.*

An infinite collection of tropical lines in $\mathbb{R}^3$, is called a *two-point family* if there exist two points lying on all tropical lines in the collection. Using Lemma 3.10 it is not hard to see that the tropical lines of any two-point family have in fact a one-dimensional common intersection.

4   Constructing regular elementary triangulations

The aim of this section is to prove a precise version of the following: If Δ is a sufficiently nice polytope contained in Γ_δ, and Δ admits a regular, elementary triangulation (or *RE-triangulation* for short), then this can be extended to a RE-triangulation of Γ_δ. This fact and the lemmas building up to its proof are useful for proving existence of smooth tropical surfaces with particular properties.

We start with an easy observation, which we state in some generality for later convenience:

Lemma 4.1. *Suppose* $\Delta \subseteq \mathbb{R}^n$ *is a* n-*dimensional lattice polytope,* $F_1, F_2 \subseteq \Delta$ *are disjoint closed faces of* Δ, *and* $\alpha_j \colon F_j \to \mathbb{R}$ *is a lifting function for each* $j = 1, 2$, *such that the following properties are fulfilled:*

   i) $\Delta = \mathrm{conv}(F_1 \cup F_2)$,

  ii) $\dim(F_1) + \dim(F_2) = n - 1$,

 iii) Δ *contains no lattice points outside* F_1 *and* F_2,

 iv) α_j *induces an primitive triangulation of* F_j, *with* N_j *maximal elements.*

Then $\alpha\colon \Delta \cap \mathbb{Z}^n \to \mathbb{R}$, *defined by* $\alpha(v) := \alpha_j(v)$ *if* $v \in F_j$, *induces an primitive triangulation of* Δ. *This triangulation has* $N_1 \cdot N_2$ *maximal elements, each of which is of the form* $\mathrm{conv}(\Lambda_1 \cup \Lambda_2)$, *where* $\Lambda_i \subseteq F_j$ *is a maximal element of the triangulation induced by* α_j.

Proof. For each $j = 1, 2$, pick an arbitrary maximal element $\Lambda_j \subseteq F_j$ of the triangulation induced by α_j, and let $\Omega = \mathrm{conv}(\Lambda_1 \cup \Lambda_2)$. Then Ω is the convex hull of $\dim(F_1) + 1 + \dim(F_2) + 1 = n + 1$ lattice points, and is an primitive simplex contained in Δ. All we have to prove is that Ω is in the subdivision induced by α. To show this, it is enough to check that

(5) $$\alpha(v) < \mathrm{Aff}_{\alpha,\Omega}(v),$$

for all $v \in (\Delta \cap \mathbb{Z}^n) \smallsetminus \Omega$, where $\mathrm{Aff}_{\alpha,\Omega}$ is the unique affine function extending $\alpha|_\Omega$ to all of $\mathbb{R}^n$. Suppose $v \in F_j$. Then v lies in the affine hull of Λ_j, which implies $\mathrm{Aff}_{\alpha,\Omega}(v) = \mathrm{Aff}_{\alpha_j,\Lambda_j}(v)$. Hence (5) is equivalent to $\alpha_j(v) < \mathrm{Aff}_{\alpha_j,\Lambda_j}(v)$. But this follows from the fact that Λ_j is an element of the subdivision induced by α_j. $\square$

Lemma 4.2. *Let* Δ_1 *and* Δ_2 *be lattice polytopes such that* $\Delta_1 \cup \Delta_2$ *is convex, and* $F := \Delta_1 \cap \Delta_2$ *is a facet of both. Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be regular subdivisions of* Δ_1 *and* Δ_2 *respectively, such that the induced subdivisions on* F *are equal. Suppose furthermore that* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *have associated lifting functions* α_1 *and* α_2 *that are equal on* F. *Then* $\mathcal{S}_1 \cup \mathcal{S}_2$ *is a regular subdivision of* $\Delta_1 \cup \Delta_2$.

Proof. Let $L(x) = 0$ be the equation of the affine hyperplane spanned by F, and consider the lifting function $\alpha\colon \Delta_1 \cup \Delta_2 \to \mathbb{R}$ defined for any $\lambda \in \mathbb{R}$ by

$$\alpha(v) := \begin{cases} \alpha_1(v) & \text{if } v \in \Delta_1, \\ \alpha_2(v) - \lambda L(v) & \text{if } v \in \Delta_2. \end{cases}$$

For λ large enough, α is convex at every point of F, and the induced subdivisions on Δ_1 and Δ_2 will be $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. $\square$

Zooming in to $\mathbb{R}^3$, we now prove an auxiliary result:

Lemma 4.3. *Let* $d > e$ *be natural numbers, and define the triangles* $T_0, T_1 \subseteq \mathbb{R}^3$ *by*

$$T_0 = \mathrm{conv}(\{(0,0,0), (d,0,0), (0,d,0)\}),$$
$$T_1 = \mathrm{conv}(\{(0,0,1), (e,0,1), (0,e,1)\}).$$

Let $\mathcal{T}_i$ *be any RE-triangulation of* T_i, $i = 0, 1$. *Then there exists a RE-triangulation* $\mathcal{T}$ *of the polytope* $\Delta = \mathrm{conv}(T_0 \cup T_1)$ *such that* $\mathcal{T}|_{T_i} = \mathcal{T}_i$ *for* $i = 0, 1$.

Proof. The strategy is as follows: We decompose Δ into three tetrahedra, find RE-triangulations of each of them, and show that these fit together to form a RE-triangulation of Δ. For $i = 0, 1$, let $\alpha_i\colon T_i \to \mathbb{R}$ be a lifting function associated to $\mathcal{T}_i$, and let $\alpha\colon \Delta \to \mathbb{R}$ be defined by $\alpha(v) = \alpha_i(v)$ if $v \in T_i$.

The decomposition of a triangular prism into three tetrahedra is well known: Let

$$\Delta_0 = \text{conv}(T_0 \cup \{(0,0,1)\}),$$
$$\Delta_1 = \text{conv}(T_1 \cup \{(d,0,0)\}),$$
$$\Delta_2 = \Delta \smallsetminus (\Delta_0 \cup \Delta_1) = \text{conv}(\{(d,0,0),(0,d,0),(0,0,1),(0,e,1)\}).$$

Now we apply Lemma 4.1 three times: On Δ_0 (with $F_1 = T_0$ and $F_2 = (0,0,1)$), on Δ_1 (with $F_1 = T_1$ and $F_2 = (d,0,0)$),) and finally on Δ_2 (with $F_1 = [(d,0,0),(0,d,0)]$ and $F_2 = [(0,0,1),(0,e,1)]$). In each case it follows that α restricted to Δ_i induces an primitive triangulation $\mathcal{T}_i$ on Δ_i. $\mathcal{T}_0$ and $\mathcal{T}_1$ are obviously elementary: Their maximal elements are tetrahedra with base area $\frac{1}{2}$ and height 1. To see that $\mathcal{T}_2$ is elementary, note that $\mathcal{T}_2$ has de maximal elements, since the faces $[(d,0,0),(0,d,0)]$ and $[(0,0,1),(0,e,1)]$ are triangulated into d and e pieces respectively (cf. condition iv) of Lemma 4.1). On the other hand, $\text{vol}(\Delta_2) = \frac{1}{6}de$, so $\mathcal{T}_2$ must be elementary.

Now use Lemma 4.2 twice: First let $\Delta' = \Delta_0 \cup \Delta_2$. Obviously, since $\mathcal{T}_0$ and $\mathcal{T}_2$ come from the same lifting function, they induce the same triangulation on $\Delta_0 \cap \Delta_2$. Thus, the lemma guarantees that $\mathcal{T}_0 \cup \mathcal{T}_2$ is a RE-triangulation on Δ'. Also, as seen in the proof of the lemma, we can find an associated lifting function which is equal to α on Δ_2. But then we can use Lemma 4.2 again, on $\Delta = \Delta' \cup \Delta_1$. We conclude that $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \mathcal{T}_2$ is a RE-triangulation of Δ. □

Corollary 4.4. *Let $\Gamma \subseteq \mathbb{R}^3$ be a lattice polytope congruent to Γ_δ for some δ. Then any RE-triangulation of one of it's facets can be extended to a RE-triangulation of Γ.*

Proof. After translating and rotating, we can assume that $\Gamma = \Gamma_\delta$, and that the triangulated facet is the one at the bottom, i.e., T_0 in the above lemma. Now choose any RE-triangulation of each triangle $T_k := \text{conv}\{(0,0,k),(\delta-k,0,k),(0,\delta-k,k)\}$, $k = 1,\ldots,\delta$. The lemma then implies that each layer (of height 1) $\text{conv}\{T_{k-1},T_k\}$ has a RE-triangulation extending these. Finally we can glue these together one by one, as in Lemma 4.2. □

To simplify the statement of the main result in this section, we introduce the following notion: We say that a lattice polytope $\Delta \subseteq \Gamma_\delta$ is a *truncated* version of Γ_δ, if Δ results from chopping off one or several corners of Γ_δ such that i) each chopped off piece is congruent to Γ_s for some $s < \delta$, and ii) any two chopped off pieces have disjoint interiors.

Proposition 4.5. *Let Δ be a truncated version of Γ_δ for some $\delta \in \mathbb{N}$. If $\mathcal{T}$ is a RE-triangulation of Δ, then $\mathcal{T}$ can be extended to a RE-triangulation of Γ_δ.*

Proof. Each "missing piece" is a tetrahedron congruent to Γ_s for some integer $s < \delta$, with a RE-triangulation (induced by $\mathcal{T}$) on one of its facets. Hence, by Corollary 4.4, each missing piece has a RE-triangulation that fits. By Lemma 4.2, we can glue these triangulations onto $\mathcal{T}$ one by one, making a RE-triangulation of Γ_δ. □

5 Polytopes with exits in Γ_δ

Let $\omega_1, \omega_2, \omega_3, \omega_4$ be the vectors $-e_1, -e_2, -e_3$ and $e_1 + e_2 + e_3$, respectively. For any $\delta \in \mathbb{N}$, and each $i = 1, 2, 3, 4$, let F_i be the facet of Γ_δ with ω_i as an outwards normal vector. For any $p \in \mathbb{R}^n$, let $\ell_{p,i}$ be the unbounded ray emanating from p in the direction of ω_i. Hence any tropical line in $\mathbb{R}^3$ with vertices v_1 and v_2, contain the rays $\ell_{v_1,i_1}, \ell_{v_1,i_2}, \ell_{v_2,i_3}, \ell_{v_2,i_4}$ for some permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$. The central theme of this paper is to examine under what conditions a tropical line can be contained in a tropical surface. A simple, but crucial observation is the following:

Lemma 5.1. *Let C be a (closed) 2-cell of a tropical surface. Then,*

$$\ell_{p,i} \subseteq C \text{ for any point } p \in C \quad \Longleftrightarrow \quad C^\vee \text{ is contained in } F_i.$$

Motivated by this lemma, we make the following definition:

Definition 5.2. Let Δ be a lattice polytope contained in Γ_δ. We say that Δ has an *exit in the direction of ω_i* if $\dim(\Delta \cap F_i) \geq 1$. If Δ has exits in the directions of k of the ω_i's, we say that Δ has *k exits.*

It is a fun task to establish how many exits different types of subpolytopes of Γ_δ can have. We leave the proof of this lemma to the reader:

Lemma 5.3. *If $\delta \geq 2$, then a primitive triangle in Γ_δ can have at most 3 exits.*

The case of tetrahedra with 4 exits in Γ_δ is an interesting one, which will be important for us towards the end of the paper. Let $\mathcal{T}_\delta$ be the set of all such tetrahedra. We proceed to give a classification of the elements of $\mathcal{T}_\delta$, and analyze under what conditions they can be elementary.

For any lattice tetrahedron $\Omega \subseteq \Gamma_\delta$ we define its *facet distribution* $\mathrm{Fac}(\Omega)$ to be the unordered collection of four (possibly empty) subsets of $[4] := \{1, 2, 3, 4\}$ obtained in the following way: For each vertex of Ω take the set of indices i of the facets F_i containing that vertex. For example, if $\Omega' \subseteq \Gamma_2$ has vertices $(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1)$, then $\mathrm{Fac}(\Omega') = \{\{1, 2, 3\}, \{1, 2\}, \{3, 4\}, \{1, 4\}\}$.

A collection of four subsets of $[4]$ is called a *four-exit distribution* (FED) if each $i \in [4]$ appears in exactly two of the subsets. Clearly, Ω has four exits if and only if $\mathrm{Fac}(\Omega)$ contains a FED. (A collection $\{J_1, J_2, J_3, J_4\}$ is *contained* in another collection $\{J_1', J_2', J_3', J_4'\}$ if (possibly after renumerating) $J_i \subseteq J_i'$, for all $i = 1, \ldots, 4$.) For example, with Ω' as above, $\mathrm{Fac}(\Omega')$ contains two FEDs: $\{\{1, 2, 3\}, \{1, 2\}, \{3, 4\}, \{4\}\}$ and $\{\{2, 3\}, \{1, 2\}, \{3, 4\}, \{1, 4\}\}$.

Let $\mathcal{F}$ be the set of all FEDs, and consider the incidence relation

$$\mathcal{Q} \subseteq \mathcal{T}_\delta \times \mathcal{F}, \qquad \mathcal{Q} := \{(\Omega, c) \mid c \text{ is contained in } \mathrm{Fac}(\Omega)\}.$$

Let π_1 and π_2 be the projections from $\mathcal{Q}$ to $\mathcal{T}_\delta$ and $\mathcal{F}$ respectively. Then π_1 is obviously surjective, but not injective (for example, the last paragraph shows that $\pi_1^{-1}(\Omega')$ consists of two elements). Note that the group S_4 acts on $\mathcal{T}_\delta$ (induced by the symmetry

action on Γ_δ), on $\mathcal{F}$ (in the obvious way), and on $\mathcal{Q}$ (letting $\sigma(\Omega, c) = (\sigma(\Omega), \sigma(c))$). Hence we can consider the quotient incidence

$$\tilde{\mathcal{Q}} := \mathcal{Q}/S_4 \subseteq \mathcal{T}_\delta/S_4 \times \mathcal{F}/S_4,$$

with the projections $\tilde{\pi}_1$ and $\tilde{\pi}_2$. We claim that the image of $\tilde{\mathcal{Q}}$ under $\tilde{\pi}_2$ has exactly six elements, namely the equivalence classes of the following FEDs:

$$
\begin{aligned}
& c_1 = \{\{1,2,3\}, \{1,2,4\}, \{3\}, \{4\}\}, && c_4 = \{\{1,2,3\}, \{1,2\}, \{3,4\}, \{4\}\}, \\
(6) \quad & c_2 = \{\{1,2,3\}, \{1,2,4\}, \{3,4\}, \{\,\}\}, && c_5 = \{\{1,2,3\}, \{1,4\}, \{2,4\}, \{3\}\}, \\
& c_3 = \{\{1,2\}, \{1,2\}, \{3,4\}, \{3,4\}\}, && c_6 = \{\{1,2\}, \{1,3\}, \{2,4\}, \{3,4\}\}.
\end{aligned}
$$

The proof of this claim is a matter of simple case checking: One finds that the set $\mathcal{F}/S_4$ has 11 elements. In addition to the six given in (6) there are four elements represented by FEDs of the form $\{\{1,2,3,4\}, \{..\}, \{..\}, \{..\}\}$. These cannot be in the image of $\tilde{\pi}_2$, since no vertex lies on all four facets. Finally there is the equivalence class of $\{1,2,3\}, \{1,2,3\}, \{4\}, \{4\}$, which corresponds to a degenerate tetrahedron.

Now, for $\delta \in \mathbb{N}$, and each $j = 1 \ldots, 6$, we define the following subsets of $\mathcal{T}_\delta$:

$$
\begin{aligned}
(7) \qquad \mathcal{G}_\delta^j &:= \{\Omega \in \mathcal{T}_\delta \mid \tilde{\Omega} \in \tilde{\pi}_1(\tilde{\pi}_2^{-1}(c_j))\} \\
\mathcal{E}_\delta^j &:= \{\Omega \in \mathcal{G}_\delta^j \mid \Omega \text{ is elementary}\}.
\end{aligned}
$$

(Here, $\tilde{\Omega}$ denotes the image of Ω in $\mathcal{T}_\delta/S_4$.) Note that for a fixed δ, the subsets $\mathcal{G}_\delta^j$ cover $\mathcal{T}_\delta$, but may overlap. For instance, our running example Ω' lies in $\mathcal{G}_2^4 \cap \mathcal{G}_2^6$.

In the particular case $\delta = 1$, we have trivially that for all $j = 1, \ldots, 6$, the sets $\mathcal{G}_1^j$ and $\mathcal{E}_1^j$ both consist of the single tetrahedron Γ_1. For higher values of δ, we have the following results for the subsets $\mathcal{E}_\delta^j$:

Proposition 5.4. *Let $\delta \geq 2$ be a natural number. Then*

a) $\mathcal{E}_\delta^1 = \mathcal{E}_\delta^2 = \mathcal{E}_\delta^3 = \emptyset$.

b) $\mathcal{E}_\delta^4 \cap \mathcal{E}_\delta^5 \neq \emptyset$.

c) $\mathcal{E}_\delta^5 \setminus (\mathcal{E}_\delta^4 \cup \mathcal{E}_\delta^6) = \emptyset$.

d) $\mathcal{E}_\delta^6 \setminus (\mathcal{E}_\delta^4 \cup \mathcal{E}_\delta^5) = \emptyset \iff$ either $\delta = 3$, or δ is even and contained in a certain sequence, starting with $2, 4, 6, 8, 14, 16, 18, 20, 26, 30, 56, 76, \ldots$.

Proof. a) Any tetrahedron Ω in $\mathcal{G}_\delta^1$ or $\mathcal{G}_\delta^2$ contains a complete edge of Γ_δ. Such an edge is not primitive when $\delta > 1$, hence Ω cannot be elementary.

If $\Omega \in \mathcal{G}_\delta^3$, then (modulo the action of S_4) the vertices of Ω are of the form $(0, 0, a), (0, 0, b), (c, \delta - c, 0), (d, \delta - d, 0)$. Its volume is

$$\left| \frac{1}{6} \begin{vmatrix} 0 & 0 & a & 1 \\ 0 & 0 & b & 1 \\ c & \delta - c & 0 & 1 \\ d & \delta - d & 0 & 1 \end{vmatrix} \right| = \left| \frac{1}{6} \delta(a - b)(c - d) \right|,$$

which is either equal to 0 or $\geq \frac{\delta}{6}$. Hence Ω cannot be elementary when $\delta > 1$.

b) Given any natural number δ, let Ω be the convex hull of $(0,0,0), (1,0,0), (\delta-1,0,1)$ and $(0,1,\delta-1)$. Then $\Omega \in \mathcal{G}_\delta^4 \cap \mathcal{G}_\delta^5$. Also, $\text{vol}(\Omega) = \frac{1}{6}$, so Ω is elementary.

c) Any $\Omega \in \mathcal{G}_\delta^5$ has (modulo S_4) vertices with coordinates $(0,0,0), (\delta-a,0,a), (0,b,\delta-b)$, and $(c,d,0)$, where a,b,c,d are natural numbers such that $0 \leq a,b,c,d \leq \delta$ and $c+d \leq \delta$. Furthermore, if $\Omega \notin \mathcal{G}_\delta^j$ for all $j \neq 5$, then all these inequalities are strict. If Ω is elementary, we must have $\text{vol}(\Omega) = \frac{1}{6}$, which is implies that

$$(8) \qquad 6\,\text{vol}(\Omega) = \left\| \begin{matrix} \delta-a & 0 & a \\ 0 & b & \delta-b \\ c & d & 0 \end{matrix} \right\| = |abc + (\delta-a)(\delta-b)d|$$

is equal to 1. This is impossible when $\delta \geq 2$, as shown in Lemma 5.5 below.

d) The vertices of $\Omega \in \mathcal{G}_\delta^6 \smallsetminus (\mathcal{G}_\delta^1 \cup \mathcal{G}_\delta^2 \cup \mathcal{G}_\delta^3 \cup \mathcal{G}_\delta^4 \cup \mathcal{G}_\delta^5)$ are (modulo S_4) of the form $(a,0,0)$, $(0,b,0), (0,c,\delta-c)$, and $(d,0,\delta-d)$, where $1 \leq a,b,c,d \leq \delta-1$. We find

$$6\,\text{vol}(\Omega) = |ac(\delta-b-d) - bd(\delta-a-c)| =: f(\delta,a,b,c,d).$$

When $\delta = 3$, it is straightforward to check by hand that the equation $f(\delta,a,b,c,d) = 1$ has no solutions in the required domain. However, if $\delta = 2n+1$ for any $n \geq 2$, then $(a,b,c,d) = (n-1,n,n+1,n)$ is a solution, since $f(2n+1,n-1,n,n,n+1) = |(n-1)(n+1) - n^2| = 1$.

When δ is even we do not have any general results. A computer search shows that the equation $f(\delta,a,b,c,d) = 1$ has solutions (in the allowable domain) for all δ less than 1000 except for $\delta \in \{2,4,6,8,14,16,18,20,26,30,56,76\}$. It would be interesting to know whether more exceptions exist. $\qquad\square$

Lemma 5.5. *The equation*

$$abc + (\delta-a)(\delta-b)d = \pm 1$$

has no integer solutions in the domain $1 \leq a,b \leq \delta-1, \ c,d \neq 0$.

Proof. Keep $c,d \in \mathbb{Z} \smallsetminus \{0\}$ and $\delta \in \mathbb{N}$ fixed, and let ϵ be either 1 or -1. Then the equation $cxy + d(\delta-x)(\delta-y) = \epsilon$ describes a hyperbola C intersecting the x-axis in $x^* = (\delta - \frac{\epsilon}{d\delta}, 0)$ and the y-axis in $y^* = (0, \delta - \frac{\epsilon}{d\delta})$. Observe that $\delta - \frac{\epsilon}{d\delta}$ is strictly bigger than $\delta - 1$, and furthermore that the slope $y'(x) = \frac{d(\delta-y)-cy}{cx-d(\delta-x)}$ is positive at both x^* and y^*. It follows that C never meets the square $1 \leq x,y \leq \delta-1$. This proves the lemma. $\qquad\square$

6 Properties of tropical lines on tropical surfaces

From now on, unless explicitly stated otherwise, X will always be a smooth tropical surface of degree δ in $\mathbb{R}^3$, and L a tropical line in $\mathbb{R}^3$. We fix the notation $\ell_1,\ldots,\ell_4$ for the unbounded rays of L in the directions $-e_1, -e_2, -e_3$ and $e_1+e_2+e_3$, respectively, and ℓ_5 the bounded line segment.

Any tropical surface X induces a map c_X from the underlying point set of X to the set of cells of X, mapping a point on X to the minimal cell (w.r.t. inclusion) on

X containing it. In particular we introduce the following notion: If v is a vertex of $L \subseteq X$, and $\dim c_X(v) = k$, we say that v is a *k-vertex* of L (on X).

An important concept for us is the possibility of a line segment on X to pass from one cell to another. When X is smooth, it turns out that this can only happen in one specific way, making life a lot simpler for us. We prove this after giving a precise definition:

Definition 6.1. Let X be a tropical surface (not necessarily smooth), and let $\ell \subseteq X$ be a ray or line segment. Let $\mathcal{C}_X(\ell)$ be the set

$$\mathcal{C}_X(\ell) := \{c_X(p) \mid p \in \ell, \text{ and } c_X(q) = c_X(p) \text{ for all } q \in \ell \text{ sufficiently close to } p.\}.$$

If $|\mathcal{C}_X(\ell)| \geq 2$, then we say that ℓ is *trespassing* on X.

Note that $\mathcal{C}_X(\ell)$ consists of the cells $C \subseteq X$ which satisfy $\dim(\text{int}(C) \cap \ell) \geq 1$. Thus Definition 6.1 corresponds well to the intuitive concept of "passing from one cell to another".

Lemma 6.2. *Suppose X is smooth, $\ell \subseteq X$ a trespassing line segment, and $C, C' \subseteq X$ cells such that*

$$\mathcal{C}_X(\ell) = \{C, C'\}.$$

Then C and C' are maximal cells of X whose intersection is a vertex of X.

Proof. Let $E = C \cap C'$, and let v be a direction vector of ℓ. Clearly, $\dim E$ is either 1 or 0. If E is a 1-cell, then C and C' are 2-cells adjacent to E. But since X is smooth, Lemma 3.7 implies that ℓ cannot intersect the interiors of both C and C', contradicting that $\mathcal{C}_X(\ell) = \{C, C'\}$.

Hence $\dim E = 0$, i.e., E is a vertex of X. Since X is smooth, $E^\vee$ is a tetrahedron in Subdiv_X. Now, if $\dim C = \dim C' = 1$, then both C and C' are parallel to v, implying that $E^\vee$ has two parallel facets ($C^\vee$ and $C'^\vee$). This contradicts that $E^\vee$ is a tetrahedron. The case where $\dim C = 1$ and $\dim C' = 2$ (or vice versa) is also impossible. Here, $C^\vee$ and $C'^\vee$ would be, respectively, a facet and an edge of $E^\vee$, where v is the normal vector of $C^\vee$ and v also is normal to $C'^\vee$ (since $C'^\vee$ is normal to C' which contains ℓ). This would lead to $E^\vee$ being degenerate. The only possibility left is that $\dim C = \dim C' = 2$, in other words that C and C' are both maximal. This proves the lemma. $\square$

In the following, we will call a tropical line L trespassing on X, if $L \subseteq X$, and at least one of the edges of L is trespassing. Obviously, Lemma 6.2 implies that:

Corollary 6.3. *Any trespassing tropical line on X contains a vertex of X.*

Proof. By definition, a trespassing tropical line on X has a trespassing edge (either a ray or a line segment). Then we can find a line segment ℓ contained in this edge, such that $|\mathcal{C}_X(\ell)| = 2$. By Lemma 6.2, ℓ contains a vertex of X. $\square$

Lemma 6.4. *Suppose $L \subseteq X$ is non-degenerate, and that L has a 1-vertex v on X. Let $E = c_X(v)$. Then we have:*

a) *E contains no other points of L.*

b) *The edges of the triangle $E^\vee \subseteq \mathrm{Subdiv}_X$ are orthogonal to the vectors ω_i, ω_j and $\omega_i + \omega_j$ (in some order), where ω_i and ω_j are the directions of the unbounded edges of L adjacent to v.*

Proof. a) Since L is non-degenerate, v has exactly three adjacent edges. Let m_1, m_2, m_3 be the intersections of these with a neighborhood of v, small enough so that each m_i is contained in a closed cell of X. It is sufficient to prove that none of these segments are contained in E. Assume otherwise that $m_1 \subseteq E$. Since $v \in \mathrm{int}(E)$, the only other cells of X meeting v are the three (since X is smooth) 2-cells adjacent to E. Hence $m_2 \subseteq C$ and $m_3 \subseteq C'$, where C and C' are 2-cells adjacent to E. We must have $C \neq C'$, otherwise L cannot be balanced at v. But then, since X is smooth, C and C' span different planes in $\mathbb{R}^3$ (see Lemma 3.7). This again contradicts the balancing property of L at v. Indeed, balance at v immediately implies that the plane spanned by m_1 and m_2 equals the plane spanned by m_1 and m_3.
b) Follows from a) and Lemma 3.7. $\square$

Corollary 6.5. *Let v_1 and v_2 be the (possibly coinciding) vertices of $L \subseteq X$, and let $V_i = c_X(v_i)$ for $i = 1, 2$. Then L is degenerate if and only if $V_1 = V_2$.*

Proof. One implication is true by definition. For the other implication, suppose $V_1 = V_2 =: V$. If $\dim V = 0$, then L is clearly degenerate. If $\dim V = 1$, then we must have $v_1 = v_2$ (indeed, $v_1 \neq v_2$ would contradict Lemma 6.4a)), thus L is degenerate. Finally, $\dim V$ cannot be 2, as this would imply the absurdity that V spans $\mathbb{R}^3$. $\square$

We are now ready to prove the following proposition:

Proposition 6.6. *If $\deg X \geq 3$, then any tropical line $L \subseteq X$ passes through at least one vertex of X.*

Proof. Suppose $L \cap X^0 = \emptyset$. By Corollary 6.3, L must be non-trespassing. Also, L cannot be degenerate. Indeed, if it were, let v be its vertex. Then $c_X(v)^\vee$ would have to be a primitive triangle in Γ_δ with four exits, contradicting Lemma 5.3. For non-degenerate tropical lines, it is easy to rule out all cases except for one, namely when both of L's vertices are 1-vertices (necessarily on different edges on X), as suggested to the left in Figure 2. We can assume w.l.o.g. that the combinatorial type L is $((1, 2), (3, 4))$. Applying Lemma 6.4b), it is clear that Subdiv_X contains two triangles with a common edge, with exits as shown to the right in Figure 2. The points A, B, C, D lie on $F_{14}, F_{23}, F_{12}, F_{34}$ respectively, and the middle edge AB is orthogonal to $e_1 + e_2$. It follows that the points are situated as in Figure 3, with coordinates of the form $A = (a, 0, 0)$, $B = (0, a, \delta - a)$, $C = (0, 0, c)$ and $D = (d, \delta - d, 0)$. Since X is smooth, the triangles ABC and ABD must be facets of some elementary tetrahedra $ABCP$ and $ABDQ$. Setting $P = (p_1, p_2, p_3)$ and $Q = (q_1, q_2, q_3)$ we find that

$$6\,\mathrm{vol}(ABCP) = \left\| \begin{vmatrix} a & 0 & 0 & 1 \\ 0 & a & \delta - a & 1 \\ 0 & 0 & c & 1 \\ p_1 & p_2 & p_3 & 1 \end{vmatrix} \right\| = |a(ac + \delta p_2 - ap_2 - ap_3 - c_2 - cp_1)|,$$
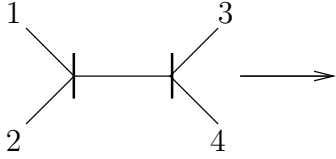
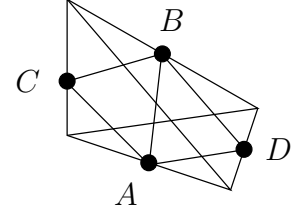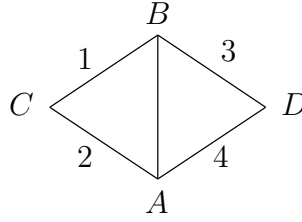Figure 2: A tropical line not containing any vertices of X.

Figure 3: Positions of $A, B, C, D \in \Gamma_\delta$.

implying that $a = 1$, and similarly that

$$6\,\mathrm{vol}(ABDQ) = \left\| \begin{matrix} a & 0 & 0 & 1 \\ 0 & a & \delta - a & 1 \\ d & \delta - d & 0 & 1 \\ q_1 & q_2 & q_3 & 1 \end{matrix} \right\| = |(\delta - a)(da - \delta a + aq_2 + aq_3 + \delta q_1 - dq_2 - dq_1)|,$$

necessitating $\delta - a = 1$. Hence we conclude that $\delta = 2$, as claimed. □

7   Tropical lines on smooth tropical quadric surfaces

The aim of this section is to prove a tropical analogue of the following famous theorem in classical geometry: A smooth algebraic surface of degree two has two rulings of lines.

We begin by describing the compact maximal cells of a smooth tropical quadric. It turns out that there is always exactly one such cell:

Proposition 7.1. *A smooth tropical quadric surface has a unique compact 2-cell. This cell has a normal vector of the form $-e_i + e_j + e_k$, for some permutation (i, j, k) of the numbers $(1, 2, 3)$.*

Proof. Let X be the smooth quadric. A compact 2-cell of X corresponds to a 1-cell in Subdiv_X in the interior of the Newton polytope Γ_2. Such 1-cells will in the following be called *diagonals*.

The only possible diagonals in Γ_2 are the line segments (see Figure 4)

$$(9)\ \ PP' = \overline{(1,0,0),(0,1,1)}, \quad QQ' = \overline{(1,0,1),(0,1,0)} \quad \text{and} \quad RR' = \overline{(0,0,1),(1,1,0)}.$$
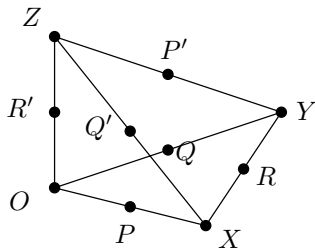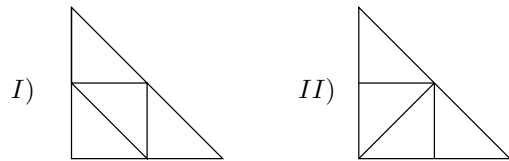

Figure 4: The lattice points in Γ_2.

Figure 5: The two unique elementary triangulations of a lattice triangle with side length 2.
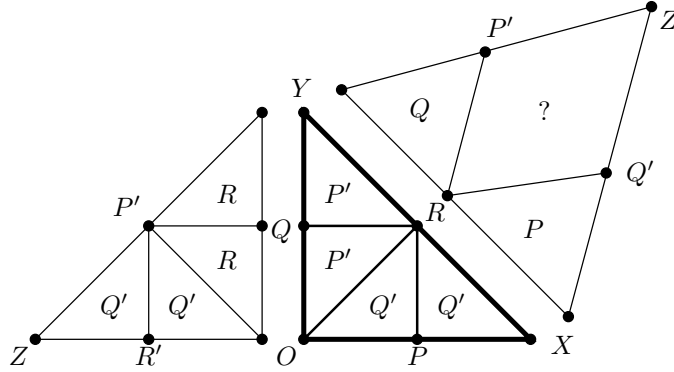
Figure 6: Induced subdivisions on three facets of Γ_2. A letter inside a triangle indicates the fourth point in the corresponding tetrahedron. The points X, Y, Z, O are $(2, 0, 0), (0, 2, 0), (0, 0, 2), (0, 0, 0)$ respectively.

Note that all these intersect in $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \notin \mathbb{Z}^3$, so at most one of them can be in Subdiv_X. This proves uniqueness. To complete the proof we must show that Subdiv_X contains at least one diagonal. (The final statement in the proposition follows trivially from the direction vectors of the diagonals in (9).)

Since X is smooth, Subdiv_X is an elementary triangulation of Γ_2. In particular, the induced subdivisions of the four facets of Γ_2 are also elementary triangulations. Up to symmetry, there are only two possibilities for these triangulations, shown as I and II in Figure 5. Suppose the triangulation of the bottom facet is of type I. Then, in particular, it contains the triangle $\triangle PQR$ as an element. Let $T \in \mathrm{Subdiv}_X$ be the (unique) elementary tetrahedron having this triangle as a facet. For T to have volume $\frac{1}{6}$, its height must be 1, so the fourth vertex is either P', Q' or R'. In either case, T contains one of the diagonals (9) as an edge.

The same argument can be used on the three other facets of Γ_2, so we are left with the case where all the subdivisions induced on the facets are of type II (cf. Figure 5). Suppose this is the case, and that Subdiv_X contains no diagonals. We will show that this leads to a contradiction.

Figure 6 shows three of the facets of Γ_2 folded out. Starting from the bottom facet OXY (drawn in bold lines in Figure 6), we can assume (after a rotation if necessary) that its induced subdivision is as in Figure 6. Now, since Subdiv_X contains neither PP', QQ' nor RR', the tetrahedron containing OPR as a facet, must have Q' as its fourth vertex. Similarly, the other three tetrahedra on the bottom of Subdiv_X are uniquely determined. This in turn determines the subdivision of the facet OYZ, and the corresponding closest tetrahedra (see Figure 6). In particular, it follows that $P'Q' \in \mathrm{Subdiv}_X$. But turning to the facet XYZ, we see that this is impossible. Indeed, we already know that $P'R$ and $Q'R$ are in Subdiv_X. Together with $P'Q'$, this implies that the induced subdivision of XYZ is of type I, violating the assumption. $\square$

Let $\overline{X}$ denote the compact 2-cell of X found in Proposition 7.1. Our main result about tropical lines on tropical quadrics is the following:
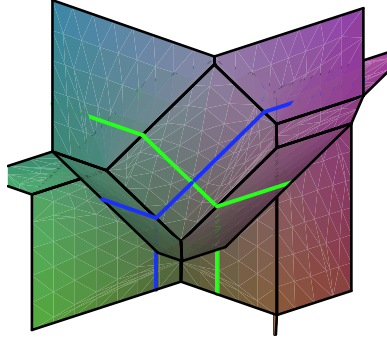
Figure 7: A smooth tropical quadric surface X, with two tropical lines passing through a point in $\overline{X}$.

Theorem 7.2. *For each point $p \in \overline{X}$ there exist two distinct tropical lines on X passing through p.*

Proof. We can assume (using if necessary the action of S_4) that $\overline{X}$ has a normal vector $-e_1 + e_2 + e_3$, i.e., that the edge in Subdiv_X corresponding to $\overline{X}$ is PP' (see Figure 4). Let p be any point on $\overline{X}$, and consider the line given by $p + t(e_1 + e_2)$, $t \in \mathbb{R}$. Let L^- and L^+ be the rays where $t \leq 0$ and $t \geq 0$ respectively, and let p^-, p^+ be the points on the boundary of $\overline{X}$ where L^- and L^+ leave $\overline{X}$. We will show that the tropical line L_p with vertices p^- and p^+, lie on X.

Let $E^- := c_X(p^-)$ and $E^+ := c_X(p^+)$. If E^- (resp. E^+) is a vertex, redefine it to be any adjacent edge (of $\overline{X}$) not parallel to v. To prove that $L_p \subseteq X$, it is enough (by Lemma 5.1) to show that the triangle $(E^-)^\vee \in \mathrm{Subdiv}_X$ has exits in the directions ω_1, ω_2, and that $(E^+)^\vee$ has exits in the directions ω_3, ω_4.

The boundary of $\overline{X}$ is made up precisely by the 1-cells of X whose dual triangles in Subdiv_X has PP' as one edge. In particular there are lattice points $A, B \in \Gamma_2$ such that $(E^-)^\vee = \triangle APP'$ and $(E^+)^\vee = \triangle BPP'$. We claim that

(10) $\qquad\qquad\qquad A$ and B lies on the edges F_{12} and F_{34} respectively.

If this claim is true, it follows immediately that the triangles $\triangle APP'$ and $\triangle BPP'$ have the required exits, and therefore that $L_p \subseteq X$. To prove the claim, we utilize Lemma 7.3 below. By the construction of E^-, it is clear that the vector $e_1 + e_2$ points inwards from E^- into $\overline{X}$. The lemma then implies that $\langle e_1 + e_2, u \rangle < 0$ for all vectors u pointing inwards from PP' into $\triangle APP'$. In particular, choosing u as the vector from P to $A = (a_1, a_2, a_3)$, this gives $a_1 + a_2 < 1$. The only lattice points in Γ_2 satisfying this are those on F_{12}, so $A \in F_{12}$. That $B \in F_{34}$ follows similarly. This proves the claim, and we conclude that $L_p \subseteq X$.

Next, consider the affine line $p + t(e_1 + e_3)$, $t \in \mathbb{R}$. The points where this line leaves $\overline{X}$ are again the vertices of a tropical line, L'_p, which we claim is contained in X. Indeed, this follows after swapping the coordinates e_2 and e_3 (i.e., letting the transposition $\sigma = (23) \in S_4$ act on X), and repeating the above proof word for word. Figure 7 shows L_p and L'_p in a typical situation. $\qquad\square$
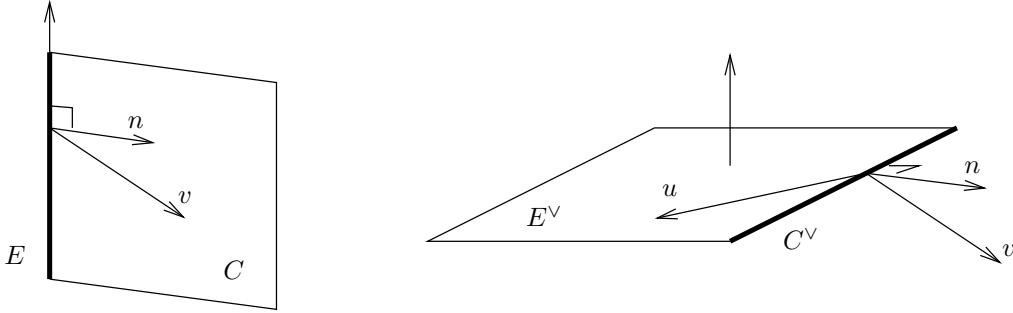
41

Figure 8: Illustration of Lemma 7.3.

Lemma 7.3. *Let E be an edge of a 2-cell C on a tropical surface. For any vector v pointing inwards from E into C, and any vector u pointing inwards from $C^\vee$ into $E^\vee$, we have*

$$\langle v, u \rangle < 0.$$

Proof. Let n be the unit inwards normal vector of E relative to C. By Lemma 3.2, n is an outwards normal vector of $C^\vee$ relative to $E^\vee$. In particular, we have $\langle v, n \rangle > 0$ and $\langle u, n \rangle < 0$. (See Figure 8.)

For $v = n$, the lemma is clearly true, so assume $v \neq n$. The vector product $v \times n$ is then a normal vector of C, and therefore a direction vector of $C^\vee$. Hence $u \times (v \times n)$ is a normal vector of $E^\vee$, i.e., it is a direction vector of E. But since n is a normal vector of E, this implies that $\langle u \times (v \times n), n \rangle = 0$. Expanding this, using the familiar formula $a \times (b \times c) = \langle a, c \rangle b - \langle a, b \rangle c$, we find that

$$\langle u, n \rangle \langle v, n \rangle = \langle u, v \rangle \langle n, n \rangle = \langle u, v \rangle.$$

(In the last step we used that $|n| = 1$.) The lemma follows from this, since $\langle u, n \rangle < 0$ and $\langle v, n \rangle > 0$. $\square$

8 Two-point families on X

To any $L \subseteq X$, with edges $\ell_1, \ldots, \ell_5$, we can associate a set of data, $\mathcal{D}_X(L) = \{V_1, V_2, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_5, \kappa\}$, where,

- $V_i = c_X(v_i)$, where v_1, v_2 are the (possibly coinciding) vertices of L.

- $\mathcal{C}_i$ is the set $\mathcal{C}_X(\ell_i)$ (cf. Definition 6.1).

- κ is the combinatorial type of L.

Recall in particular that ℓ_i is trespassing on X if and only if $|\mathcal{C}_i| \geq 2$.

One might wonder if different tropical lines on X can have the same set of data. It is not hard to imagine an example giving an affirmative answer, e.g. as in Figure 9.

In this Figure one of the vertices of the tropical line can be moved along the middle segment, creating infinitely many tropical lines with the same set of data. Clearly, the collection of all these tropical lines is a two-point family. As we will show in the remainder of this section, this is not a coincidence.
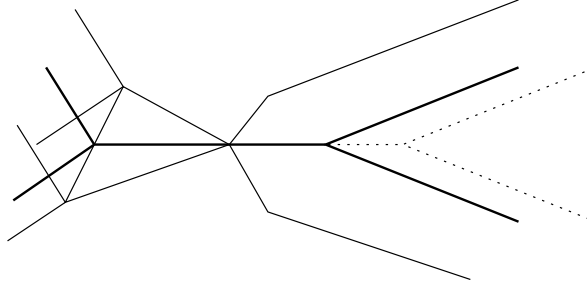

Figure 9: A two-point family of tropical lines on a tropical surface.

By a perturbation of a point $p \in \mathbb{R}^3$ we mean a continuous map $\mu\colon [0,1) \to \mathbb{R}^3$, possibly constant, such that $\mu(0) = p$.

Definition 8.1. A tropical line $L \subseteq X$ can be *perturbed on X* if there exist perturbations μ_1 and μ_2 - not both constant - of the vertices of L such that for all $t \in [0,1)$, $\mu_1(t)$ and $\mu_2(t)$ are the vertices of a tropical line $L_t \subseteq X$. In this case, we call the map $[0,1) \to G_{tr}(1,3)$ given by $t \mapsto L_t$ a *perturbation* of L on X.

If L is degenerate, we think of L as having two coinciding vertices. Thus Definition 8.1 allows perturbations of L where the vertices are separated, creating non-degenerate tropical lines.

By a *two-point family of tropical lines on X*, or simply a two-point family on X, we mean a two-point family of tropical lines, all of which are contained in X. A two-point family on X is *maximal* (on X) if it not contained in any strictly larger two-point family on X. A tropical line on X is *isolated* if it does not belong to any two-point family on X.

Special perturbations, as the one in Figure 9, give rise to two-point families on X. We state a straightforward generalization of this example in the following lemma, for later reference. Note that if μ is a perturbation of L on X, we say that the vertex v_i is *perturbed along an edge* of L, if $\mathrm{im}(\mu_i) \subseteq \mathrm{Aff}(\ell)$ for some edge $\ell \subseteq L$ (cf. the notation in Definition 8.1).

Lemma 8.2. *If a non-degenerate $L \subseteq X$ has a perturbation on X where at least one of the vertices is perturbed along an edge of L, then L belongs to a two-point family on X.*

Proposition 8.3. *Let L be a tropical line on a smooth tropical surface X, where $\deg X \geq 3$. If L is isolated, then L is uniquely determined by $\mathcal{D}_X(L)$.*

Proof. Let $\mathcal{D} = \mathcal{D}_X(L) = \{V_1, V_2, \mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_5, \kappa\}$ be a given set of data. We will identify all situations where L is not uniquely determined by $\mathcal{D}$, and show that Lemma 8.2 applies in each of these cases.

We first consider the case where $\kappa \neq (1234)$, meaning that L is non-degenerate. The following observations will be used frequently:

A) L is determined by (the positions of) its two vertices.

B) The direction vector of the bounded segment ℓ_5 is determined by κ.

C) If $|\mathcal{C}_j| \geq 2$, then $\mathrm{Aff}(\ell_j)$ is determined by the elements of C_j (and the index j).

D) If $\dim V_i = 1$, and $\mathrm{Aff}(\ell_j)$ is known for any edge ℓ_j adjacent to v_i, then v_i is determined.

Of these, A) and B) are clear, C) is a consequence of Lemma 6.2, and D) follows from Lemma 6.4a).

Now, assume that V_1 and V_2 are ordered so that $\dim V_1 \leq \dim V_2$. Under this assumption, we examine the uniqueness of L for different sets of data, according to the pair $(\dim V_1, \dim V_2)$:

 • $(\dim V_1, \dim V_2) = (0,0)$: Obviously, by A), L is determined.
 • $(\dim V_1, \dim V_2) = (0,1)$: In this case $\mathrm{Aff}(\ell_5)$ is determined by V_1 and κ (cf. B)). Hence v_2 is determined (by D)). Since $v_1 = V_1$, it follows that L is determined.
 • $(\dim V_1, \dim V_2) = (0,2)$: Again, v_1 and $\mathrm{Aff}(\ell_5)$ are determined by V_1 and κ. Write $\kappa = ((a,b),(c,d))$, and consider first the case where either $|\mathcal{C}_c| \geq 2$ or $|\mathcal{C}_d| \geq 2$. We can assume the former. Then $\mathrm{Aff}(\ell_c)$ is determined, which again determines $v_2 = \mathrm{Aff}(\ell_5) \cap \mathrm{Aff}(\ell_c)$. Thus, in this case L is determined.

Otherwise, we have $\mathcal{C}_c = \mathcal{C}_d = V_2$. In this situation L is not uniquely determined by $\mathcal{D}$, as v_2 can be perturbed to anywhere in the intersection of $\mathrm{Aff}(\ell_5)$ and V_2 without changing $\mathcal{D}$.

 • $(\dim V_1, \dim V_2) = (1,1)$: Observe first that we must have $|\mathcal{C}_i| \geq 2$ for some i. (Otherwise L is not trespassing, and since none of its vertices are vertices of X, this would contradict Proposition 6.6.) Hence $\mathrm{Aff}(\ell_i)$ is determined for some i. If $i = 5$, then (by D)) both v_1 and v_2 are determined by this. If $i \neq 5$, then in the first place only the endpoint of ℓ_i is determined. But this together with κ determines $\mathrm{Aff}(\ell_5)$, and thus both vertices. Hence, in any case, L is determined.

 • $(\dim V_1, \dim V_2) = (1,2)$: Let $\kappa = ((a,b),(c,d))$. We consider five cases:

i) $|\mathcal{C}_j| \geq 2$ for both $j = c, d$. Then $\mathrm{Aff}(\ell_c)$ and $\mathrm{Aff}(\ell_d)$ are determined, and therefore also $v_2 = \mathrm{Aff}(\ell_c) \cap \mathrm{Aff}(\ell_d)$. This and κ determines $\mathrm{Aff}(\ell_5)$, which in turn (by D)) determines v_1. Hence L is determined.

ii) $|\mathcal{C}_j| \geq 2$ for exactly one index $j \in \{c, d\}$ (assume d), and also for at least one index $j \in \{a, b, 5\}$. This last condition determines $\mathrm{Aff}(\ell_5)$, either directly (if $j = 5$) or via v_1 and κ. Thus $v_2 = \mathrm{Aff}(\ell_d) \cap \mathrm{Aff}(\ell_5)$ is determined, and therefore L as well.

iii) $|\mathcal{C}_j| \geq 2$ for exactly one index $j \in \{c, d\}$ (assume d), and for no other indices j. In this case v_2 can be perturbed along ℓ_d without changing $\mathcal{D}$, so L is not determined by $\mathcal{D}$. (The perturbation of v_1 (along V_1) will be determined by the perturbation of v_2.)

iv) $|\mathcal{C}_j| \geq 2$ for no $j \in \{c, d\}$, but at least one $j \in \{a, b, 5\}$. As in ii) above, the last condition determines $\mathrm{Aff}(\ell_5)$ and therefore v_1. The vertex v_2 can be perturbed along ℓ_5, so L is not determined.

v) $|\mathcal{C}_j| = 1$ for all $j \in \{1, 2, 3, 4, 5\}$. This is not possible when $\deg X \geq 3$. In fact, it follows from Lemma 5.3 that $\deg X = 1$. Indeed, since no edge of L is trespassing, the triangle $V_1^\vee$ must have four exits in $\Gamma_{\deg X}$.

• $(\dim V_1, \dim V_2) = (2, 2)$: Note first that $V_1 \neq V_2$, since L spans $\mathbb{R}^3$. Hence $|\mathcal{C}_5| \geq 2$, determining $\mathrm{Aff}(\ell_5)$. Now, for both $i = 1, 2$ we have: If any adjacent unbounded edge of v_i is trespassing, then v_i is determined. If not, v_i can be perturbed along ℓ_5 keeping $\mathcal{D}$ unchanged.

Going through the above list, we see that in each case where L is not uniquely determined by $\mathcal{D}$, L has a perturbation where a vertex is perturbed along an edge of X. Hence, by Lemma 8.2, L belongs to a two-point family on X.

Finally, suppose $\kappa = (1234)$, so L is degenerate. We show that in this case, L is determined by $\mathcal{D}$. Corollary 6.5 (and its proof) tells us that $V_1 = V_2 := V$ where $\dim V$ is either 0 or 1. In the first case, L is obviously uniquely determined. If $\dim V = 1$ then $|\mathcal{C}_j| \geq 2$ for some $j \in \{1, 2, 3, 4\}$, otherwise L would contain no vertex of X, contradicting Proposition 6.6. Hence $\mathrm{Aff}(\ell_j)$ is determined. We claim that $V_1 \not\subseteq \mathrm{Aff}(\ell_j)$. Note that this would determine $v_1 = v_2 = \mathrm{Aff}(\ell_j) \cap V_1$, and therefore also L. To prove the claim, note that if $V_1 \subseteq \mathrm{Aff}(\ell_j)$, then $V_1 \in \mathcal{C}_j$. This is impossible, since any element of $\mathcal{C}_j$ must be of dimension 2 (cf. Lemma 6.2). This concludes the proof of the proposition. □

9   Tropical lines on higher degree tropical surfaces

In this section we present our main results about tropical lines on smooth tropical surfaces of degree greater than two. The proofs rest heavily on what we have done so far. The first is indeed a corollary of Proposition 8.3:

Corollary 9.1. *Let X be a smooth tropical surface where $\deg X \geq 3$. Then X contains at most finitely many isolated tropical lines. Furthermore, X contains at most finitely many maximal two-point families.*

Proof. The first statement is immediate from Proposition 8.3, since there are only finitely many possible sets of data $\mathcal{D}_X(L)$. For the last statement, observe that any two-point family contains a non-degenerate tropical line. Going through the proof of Proposition 8.3, we see that if $\mathcal{D}$ is the data set of is a non-degenerate tropical line, then there can be at most one maximal two-point family containing tropical lines with data set $\mathcal{D}$. Hence there are at most finitely many maximal two-point families on X. □

The next theorem show that two-point families exist on smooth tropical surfaces of any degree.

Theorem 9.2. *For any integer δ, there exists a full dimensional cone in $\Phi(\Gamma_\delta)$ in which each point corresponds to a smooth tropical surface containing a two-point family of tropical lines. In particular, there exist smooth tropical surfaces of degree δ with infinitely many tropical lines.*

Proof. According to Proposition 5.4b) there exists an elementary tetrahedron with four exits Γ_δ. An example of such a tetrahedron is (see Figure 10)

$$\Omega := \mathrm{conv}(\{(0,0,0),(0,0,1),(\delta-1,1,0),(1,0,\delta-1)\}).$$

Assume for the moment that there exists a smooth tropical surface X such that Subdiv_X contains Ω. Then Lemma 5.1 implies the vertex $v := \Omega^\vee \in X$ is the center of degenerate tropical line $L \subseteq X$. We claim that L belongs to a two-point family on X. Indeed, this also follows from Lemma 5.1: Let $C \subseteq X$ be the cell dual to the line segment in Subdiv_X with vertices $(0,0,0)$ and $(0,0,1)$. Then for any point $p(t) = v + t(-e_1 - e_2)$, where $t > 0$, the line segment with endpoints v and $p(t)$ is contained in C. Let L_t be the tropical line with vertices v and $p(t)$. Lemma 5.1 guarantees that the rays starting in $p(t)$ in the directions $-e_1$ and $-e_2$ are contained in C. Hence $L_t \subseteq X$. Clearly, the lines L_t form a two-point family on X, thus the claim is true. (See Figure 11.)

What remains to prove is the existence of a RE-triangulation of Γ_δ containing Ω. Using the techniques in Section 4, it is not hard to construct such a triangulation. For example, consider the polytope

$$\Delta = \mathrm{conv}(\{(0,0,0),(\delta,0,0),(\delta-1,1,0),(0,1,0),(0,1,\delta-1),(0,0,\delta)\}).$$

Then Δ is a truncated version of Γ_δ, so by Proposition 4.5 it is enough to construct a RE-triangulation of Δ which contains Ω. Write $\Delta = \Omega \cup \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$, where

$$\Delta_1 = \mathrm{conv}(\{(0,0,0),(\delta,0,0),(\delta-1,1,0),(1,0,\delta-1)\})$$
$$\Delta_2 = \mathrm{conv}(\{(0,0,1),(\delta-1,1,0),(1,0,\delta-1),(0,0,\delta)\})$$
$$\Delta_3 = \mathrm{conv}(\{(0,0,0),(\delta-1,1,0),(0,1,0),(0,0,\delta)\})$$
$$\Delta_4 = \mathrm{conv}(\{(\delta-1,1,0),(0,1,0),(0,1,\delta-1),(0,0,\delta)\})$$

Repeated use of Lemma 4.1 gives a RE-triangulation of each of these (for Δ_1 and Δ_4 choose any RE-triangulation of the facets $\mathrm{conv}(\{(0,0,0),(\delta,0,0),(1,0,\delta-1)\})$ and $\mathrm{conv}(\{(\delta-1,1,0),(0,1,0),(0,1,\delta-1)\})$ respectively). Finally, it is easy to check that these triangulations patch together to a RE-triangulation of Δ, using Lemma 4.2. □

In light of the above theorem, one might ask whether there exist tropical surfaces of high degree containing an *isolated* degenerate tropical line L. If we add the requirement that L is non-trespassing on X, we can give the following partial answer:

Proposition 9.3. *Let $\delta \in \mathbb{N}$. There exists a smooth tropical surface X of degree δ containing an isolated, non-trespassing, degenerate tropical line, if and only if δ is*

- *an odd number greater than 3, or*

- *an even number except 2, 4, 6, 8, 14, 16, 18, 20, 26, 30, 56, 76,...*

Proof. We know that the vertex of such a line must be a vertex of X, corresponding to an elementary tetrahedron $\Omega \in \mathrm{Subdiv}_X$ with four exits. Furthermore, no edge of Ω
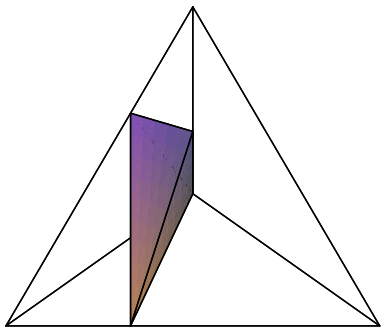
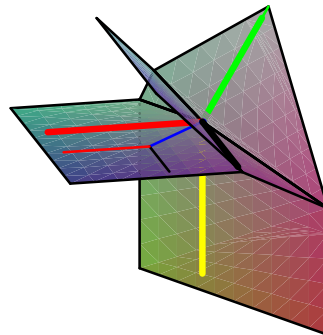Figure 10: A tetrahedron with four exits in Γ_δ.

Figure 11: The degenerated tropical line corresponding to the tetrahedron in Figure 10 belongs to a two-point family

can have more than one exit. Indeed, an edge with exits ω_i and ω_j will be orthogonal to the vector $\omega_i + \omega_j$, implying (as in the proof of Theorem 9.2) that L belongs to a two-point family.

From the classification in (6) of tetrahedra with four exits in Γ_δ, we observe the following: A tetrahedron with four exits, in which no edge has more than one exit, must belong either exclusively to the subset $\mathcal{G}_\delta^5$, or exclusively to the subset $\mathcal{G}_\delta^6$. The result then follows from Proposition 5.4c) and d). As we remarked in that proposition, we do not know how (or if) the list of even degrees continues. $\qquad\square$

Both Theorem 9.2 and Proposition 9.3 show that there exist plenty of tropical surfaces of arbitrarily high degree containing tropical lines. It is natural to wonder whether there also exist smooth tropical surfaces containing *no* tropical lines, isolated or not. This is indeed true in all degrees greater than three, as we prove in [6]. In that paper we present a classification of tropical lines on general smooth tropical surfaces, and propose a method for counting the isolated tropical lines on such surfaces.

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Tropical lines on smooth tropical surfaces

Magnus Dehli Vigeland*

Abstract

Given a tropical line L on a tropical surface X, we define its combinatorial position on X to be a certain decorated graph, showing the relative positions of vertices of X on L, and how the vertices of L are positioned on X. We classify all possible combinatorial positions of a tropical line on general smooth tropical surfaces of any degree. This classification allows one to give an upper bound for the number of tropical lines on a general smooth tropical surface of degree ≥ 3 with a given subdivision. As a concrete example, we offer a subdivision for which the associated tropical surfaces are smooth cubics with exactly 27 tropical lines in the general case, and always at least 27 tropical lines. We also give examples of smooth tropical surfaces of arbitrary degree > 3 containing no tropical lines.

1 Introduction

A celebrated theorem in classical geometry states that any smooth algebraic cubic surface in complex projective three-space contains exactly 27 distinct lines. This was first established in 1849 in a correspondence between Arthur Cayley and George Salmon.

Since the appearance of tropical geometry a few years ago, it has been a recurring question whether there is a tropical analogue of this result. It is a common opinion among tropical geometers that this is indeed the case. Explicit examples of smooth tropical surfaces with 27 distinct tropical lines have been found by Mikhalkin and by Gross [5].

However, nothing on the subject has been published as yet. Furthermore, it is far from obvious what the correct formulation of the tropical analogue should be. For example, in [7] we showed that there exist smooth tropical cubic surfaces containing infinitely many tropical lines.

The purpose of this paper is to give a systematic approach to the subject of tropical lines on smooth tropical surfaces of arbitrary degree. As it turns out, this allows us to give a partial answer to the above questions.

Tropical surfaces in $\mathbb{R}^3$ are unbounded polyhedral cell complexes of dimension 2 with certain properties. Most importantly, each tropical surface is dual to a regular lattice subdivision of a lattice polytope in $\mathbb{R}^3$. We say that the tropical surface is smooth of

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*Department of Mathematics, University of Oslo, Norway. *Email*: magnusv@math.uio.no
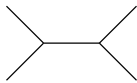
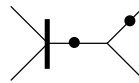Figure 1: The underlying graph of a tropical line in $\mathbb{R}^3$.

Figure 2: Example of the combinatorial position of a tropical line $L \subseteq X$.

degree δ, if the dual subdivision is an elementary (unimodular) triangulation of the polytope $\Gamma_\delta = \mathrm{conv}(\{(0,0,0),(\delta,0,0),(0,\delta,0),(0,0,\delta)\})$.

A large portion of our results hold only for *general* smooth tropical surfaces. The concept of generality used here should be noted: The parameter spaces of tropical surfaces are cones (or, more generally, fans) in some large Euclidean space. When we speak about general tropical surfaces with a given dual subdivision, we mean the surfaces corresponding to points in some open dense subset (in the Euclidean topology) of the parameter cone.

One can show that the general intersection of two tropical planes (i.e., tropical surfaces of degree 1), is an unbounded one-dimensional polyhedral cell complex, called a *tropical line*. Its underlying topological space is homeomorphic to the graph in Figure 1, with its 1-valent vertices removed.

The main core of this paper is an analysis of the different ways in which a tropical line L can lie on a smooth tropical surface X. A crucial concept in our arguments is the notion of the *combinatorial position* of L on X. This is a decoration of the underlying graph of L, displaying the relative positions of vertices of X on L, and vertices of L on X (see Figure 2 for a typical example). We are able to show that for a general smooth X, only 17 such combinatorial positions are possible. Moreover, only nine of these can occur if X has degree greater than two.

Let X be a tropical surface, and $\mathcal{S}$ its associated subdivision. The elements of $\mathcal{S}$ dual to the cells of X intersecting L, form a subcomplex called a *line subcomplex*. In most cases, the cell structure of this subcomplex is determined by the combinatorial position of L. Hence, by counting subcomplexes of $\mathcal{S}$, we obtain an upper bound for the number of tropical lines on X.

As an application of the above technique we provide examples of general smooth tropical surfaces of arbitrary degree greater than 3, containing no tropical lines (see Proposition 6.4). This complements a result in [7], where we found general smooth tropical surfaces of arbitrary degree containing infinitely many tropical lines.

In the final section of this paper we consider smooth tropical cubic surfaces. For the subdivision $\mathcal{S}_{\alpha,3}$, shown in Figure 18, we prove:

Theorem 7.1.

    a) A general tropical surface with subdivision $\mathcal{S}_{\alpha,3}$ contains exactly 27 tropical lines.

    b) Any tropical surface with subdivision $\mathcal{S}_{\alpha,3}$ contains at least 27 tropical lines.

    c) There exist tropical surfaces with subdivision $\mathcal{S}_{\alpha,3}$ containing infinitely many tropical lines.

2 Preliminaries

2.1 Convex polyhedra and polytopes

A *convex polyhedron* in $\mathbb{R}^n$ is the intersection of finitely many closed halfspaces. A *cone* is a convex polyhedron, all of whose defining hyperplanes contain the origin. A *convex polytope* is a bounded convex polyhedron. Equivalently, a convex polytope can be defined as the convex hull of a finite set of points in $\mathbb{R}^n$. Throughout this paper, all polyhedra and polytopes will be assumed to be convex unless explicitly stated otherwise.

For any polyhedron $\Delta \subseteq \mathbb{R}^n$ we denote its affine hull by $\mathrm{Aff}(\Delta)$, and its relative interior (as a subset of $\mathrm{Aff}(\Delta)$) by $\mathrm{int}(\Delta)$. The dimension of Δ is defined as $\dim \mathrm{Aff}(\Delta)$. By convention, $\dim \emptyset = -1$. A *face* of Δ is a polyhedron of the form $\Delta \cap H$, where H is a hyperplane such that Δ is entirely contained in one of the closed halfspaces defined by H. In particular, the empty set is considered a face of Δ. Faces of dimensions 0, 1 and $n-1$ are called *vertices*, *edges* and *facets* of Δ, respectively. If Δ is a polytope, then the vertices of Δ form the minimal set $\mathcal{A}$ such that $\Delta = \mathrm{conv}(\mathcal{A})$.

A *lattice polytope* in $\mathbb{R}^n$ is a polytope of the form $\Delta = \mathrm{conv}(\mathcal{A})$, where $\mathcal{A}$ is a finite subset of $\mathbb{Z}^n$. We say that Δ is *elementary*, or *unimodular*, if it is n-dimensional and its volume is $\frac{1}{n!}$. It is easy to see that a necessary condition for Δ to be elementary is that it is a *simplex*, that is, the convex hull of $n+1$ affinely independent points.

2.2 Polyhedral complexes and subdivisions

A (finite) *polyhedral complex* in $\mathbb{R}^n$ is a finite collection X of convex polyhedra, called *cells*, such that

- if $C \in X$, then all faces of C are in X, and

- if $C, C' \in X$, then $C \cap C'$ is a face of both C and C'.

The d-dimensional elements of X are called the *d-cells* of X. The dimension of X itself is defined as $\max\{\dim C \mid C \in X\}$. Furthermore, if all the maximal cells (w.r.t. inclusion) have the same dimension, we say that X is of *pure dimension*.

A polyhedral complex, all of whose cells are cones, is a *fan*.

A *subdivision* of a polytope Δ is a polyhedral complex $\mathcal{S}$ such that $|\mathcal{S}| = \Delta$, where $|\mathcal{S}|$ denotes the union of all the elements of $\mathcal{S}$. It follows that $\mathcal{S}$ is of pure dimension $\dim \Delta$. If all the maximal elements of $\mathcal{S}$ are simplices, we call $\mathcal{S}$ a *triangulation*.

If Δ is a lattice polytope, we can consider *lattice subdivisions* of Δ, i.e., subdivisions in which every element is a lattice polytope. In particular, a lattice subdivision is an *elementary triangulation* if all its maximal elements are elementary simplices.

2.3 Regular subdivisions and their secondary cones

Let $\mathcal{A} \subseteq \mathbb{R}^n$ be a finite set of points, and let $\Delta = \mathrm{conv}(\mathcal{A})$. For any function $\alpha \colon \mathcal{A} \to \mathbb{R}$ we consider the *lifted polytope*

$$\widetilde{\Delta} = \mathrm{conv}(\{(v, \alpha(v)) \mid v \in \mathcal{A}\}) \subseteq \mathbb{R}^n \times \mathbb{R} \cong \mathbb{R}^{n+1}.$$

Projecting the top faces of $\widetilde{\Delta}$ to $\mathbb{R}^n$ by forgetting the last coordinate, gives a collection of sub-polytopes of Δ. They form a subdivision $\mathcal{S}_\alpha$ of Δ, called the *regular (or coherent) subdivision* induced by α. The function α is called a *lifting function* associated to $\mathcal{S}_\alpha$. Note that if $\mathcal{A} \subseteq \mathbb{Z}^n$, then $\mathcal{S}_\alpha$ is a lattice subdivision of Δ. Most of the subdivisions we will encounter in this paper, are regular elementary triangulations, or RE-triangulations for short.

Fixing an order of the elements of $\mathcal{A}$, there is a natural 1-1 correspondence between the set of functions $\alpha\colon \mathcal{A} \to \mathbb{R}$ and $\mathbb{R}^N$, where $N = |\mathcal{A}|$. Hence, for any regular subdivision $\mathcal{S}$ of $\mathrm{conv}(\mathcal{A})$, we can regard the set

$$K(\mathcal{S}) := \{\alpha\colon \mathcal{A} \to \mathbb{R} \mid \mathcal{S}_\alpha = \mathcal{S}\}$$

as a subset of $\mathbb{R}^N$. The following was observed in [2, Chapter 7]:

Proposition 2.1. $K(\mathcal{S})$ *is an open cone in* $\mathbb{R}^N$. *If* $\mathcal{S}$ *is an RE-triangulation, then* $\dim K(\mathcal{S}) = N$.

The cone $K(\mathcal{S})$ is called the *secondary cone* associated to $\mathcal{S}$.

2.3.1 Example

For $\delta \in \mathbb{N}$, let $\mathcal{A}_\delta$ to be the set of lattice points contained in the simplex

$$\Gamma_\delta := \mathrm{conv}(\{(0,0,0), (\delta,0,0), (0,\delta,0), (0,0,\delta)\}).$$

The number of points in $\mathcal{A}_\delta$ is $\binom{\delta+3}{3}$, the $(\delta-1)$'th tetrahedral number. Let $\alpha\colon \mathbb{R}^3 \to \mathbb{R}$ be the polynomial function given by

(1) $$\alpha(x,y,z) = -2x^2 - 2y^2 - 2z^2 - xy - 2xz - 2yz.$$

For any given δ, the restriction of α to $\mathcal{A}_\delta$ induces - as explained above - a regular subdivision of Γ_δ. We denote this subdivision by $\mathcal{S}_{\alpha,\delta}$.

Proposition 2.2. *For any* $\delta \in \mathbb{N}$, $\mathcal{S}_{\alpha,\delta}$ *is an RE-triangulation of* Γ_δ.

Proof. We introduce the following six families of elementary tetrahedra in $\mathbb{R}^3$: For each lattice point $P = (p,q,r) \in \mathbb{Z}^3$, let

$$\Delta_P^1 = \mathrm{conv}\{(p,q,r), (p+1,q,r), (p,q+1,r), (p,q,r+1)\}$$
$$\Delta_P^2 = \mathrm{conv}\{(p+1,q+1,r), (p+1,q,r), (p,q+1,r), (p,q,r+1)\}$$
$$\Delta_P^3 = \mathrm{conv}\{(p+1,q,r), (p+1,q+1,r), (p,q,r+1), (p+1,q,r+1)\}$$
$$\Delta_P^4 = \mathrm{conv}\{(p,q+1,r), (p+1,q+1,r), (p,q,r+1), (p,q+1,r+1)\}$$
$$\Delta_P^5 = \mathrm{conv}\{(p+1,q+1,r), (p+1,q,r+1), (p,q+1,r+1), (p,q,r+1)\}$$
$$\Delta_P^6 = \mathrm{conv}\{(p+1,q+1,r), (p+1,q,r+1), (p,q+1,r+1), (p+1,q+1,r+1)\}$$

The tetrahedra $\Delta_P^1, \ldots, \Delta_P^6$ have disjoint interiors, and they form a subdivision of the unit cube with diagonal $(p,q,r)(p+1,q+1,r+1)$, shown in Figure 3. In particular, the set $\{\Delta_P^i\}_{i=1,\ldots,6;P\in\mathbb{Z}^3}$ is a covering of $\mathbb{R}^3$.
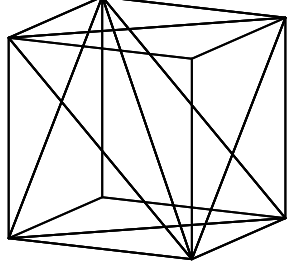
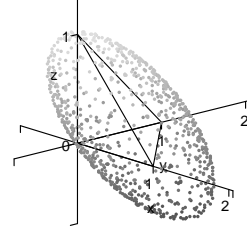Figure 3: The subdivision of the unit cube induced by α.

Figure 4: The ellipsoid Q_1.

To prove the proposition, it is enough to show that for any δ, each maximal element of $\mathcal{S}_{\alpha,\delta}$ equals Δ_P^i for some $1 \le i \le 6$ and $P \in \mathbb{N}_0^3$. Equivalently, it suffices to prove that any tetrahedron of the form Δ_P^i is a maximal element of $\mathcal{S}_{\alpha,\delta}$, for all δ such that $\Delta_P^i \subseteq \Gamma_\delta$.

Let $P = (p, q, r)$ be any point in $\mathbb{N}_0^3$, and consider the lifting (by α) of the tetrahedron Δ_P^1. The affine hull of the resulting polytope $\widetilde{T}_P^1$ is an affine hyperplane in $\mathbb{R}^4$, namely the graph of the function

$$\beta(x, y, z) = 2p + 2q + 2r - \alpha(p, q, r) - (4p + q + 2r + 2)x - (p + 4q + 2r + 2)y - (2p + 2q + 4r + 2)z.$$

(Proof: It is easy to check that the functions β and α are equal on the vertices of Δ_P^1.)

We claim that the difference $\gamma := \beta - \alpha$ is strictly positive at all lattice points $(x, y, z) \in \mathbb{Z}^3 \smallsetminus \Delta_P^1$. Note that correctness of this claim implies that $\widetilde{T}_P^1$ is a top facet of $\widetilde{\mathcal{A}}_\delta$, and therefore that $\Delta_P^1 \in \mathcal{S}_{\alpha,\delta}$ (assuming δ big enough). To prove the claim, observe that the subset $\{(x, y, z) \mid \gamma(x, y, z) \le 0\}$ is a solid ellipsoid circumscribing Δ_P^1. Translating such that $P \mapsto (0, 0, 0)$, we get the ellipsoid Q_1 with equation $\gamma(x + p, y + q, z + r) = 0$, or

$$2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 2x - 2y - 2z = 0.$$

The ellipsoid Q_1 is shown in Figure 4. It is clear that is contains no lattice points other than the vertices of $\Delta_{(0,0,0)}^1$. This proves the claim.

The five remaining cases are treated similarly. More precisely, for each $i = 2, \ldots, 6$, the problem of proving that $\Delta_P^i \in \mathcal{S}_{\alpha,\delta}$ reduces to that of showing that a certain ellipsoid, Q_i, contains no lattice points outside $\Delta_{(0,0,0)}^i$. This is a trivial task, once one calculates the equations of these ellipsoids:

$$\begin{aligned}
Q_2: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 3x - 3y - 3z + 1 = 0 \\
Q_3: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 4x - 3y - 4z + 2 = 0 \\
Q_4: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 3x - 4y - 4z + 2 = 0 \\
Q_5: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 4x - 4y - 5z + 3 = 0 \\
Q_6: \quad & 2x^2 + 2y^2 + 2z^2 + xy + 2xz + 2yz - 5x - 6z - 5y + 5 = 0
\end{aligned}$$

We conclude that any tetrahedron of the form Δ_P^i contained in Γ_δ, is a maximal element of $\mathcal{S}_{\alpha,\delta}$. Since the Δ_P^i's are elementary, have disjoint interiors and cover $\mathbb{R}^3$, this proves that $\mathcal{S}_{\alpha,\delta}$ is elementary. $\qquad\square$

3 Tropical surfaces and tropical lines in $\mathbb{R}^3$

We begin by going through the basic definitions and our notation concerning tropical surfaces in $\mathbb{R}^3$. Note that these concepts can be immediately generalized to hypersurfaces in $\mathbb{R}^n$. (See [3], [4], and [1]).

We work over the *tropical semiring* $\mathbb{R}_{tr} := (\mathbb{R}, \max, +)$. To simplify the reading of tropical expressions, we adopt the following convention: If an expression is written in quotation marks, all arithmetic operations should be interpreted as tropical. Hence, if $x, y \in \mathbb{R}$ and $k \in \mathbb{N}_0$ we have for example "$x + y$" $= \max(x, y)$, "xy" $= x + y$ and "x^k" $= kx$.

A *tropical polynomial* in indeterminates x_1, x_2, x_3 is an expression of the form

$$(2) \qquad f(x_1, x_2, x_3) = \text{``} \sum_{(a_1, a_2, a_3) \in \mathcal{A}} \lambda_{a_1 a_2 a_3} x_1^{a_1} x_2^{a_2} x_3^{a_3} \text{''}$$

where the *support* $\mathcal{A}$ is a finite subset of $\mathbb{Z}^3$, and the coefficients $\lambda_{a_1 a_2 a_3}$ are real numbers. We can write the expression for f more compactly using vector notation, with $x = (x_1, x_2, x_3)$ and $a = (a_1, a_2, a_3)$, as $f(x) = \text{``} \sum_{a \in \mathcal{A}} \lambda_a x^a \text{''}$. Translating to classical arithmetic, we see that f is the maximum of a finite number of affine-linear expressions with integral coefficients (except for the constant terms). Hence, $f \colon \mathbb{R}^3 \to \mathbb{R}$ is a convex, piecewise linear function with rational slopes. The non-linear locus of f, denoted $V_{tr}(f)$, is called the *tropical surface* associated to f. It is well known (see e.g. [3] and [4]) that $V_{tr}(f)$ is a connected polyhedral complex of pure dimension 2, some of whose cells are unbounded in $\mathbb{R}^3$.

Definition 3.1. Let $\delta \in \mathbb{N}$. A *tropical surface of degree δ* is a subset of $\mathbb{R}^3$ of the form $V_{tr}(f)$, where f is a tropical polynomial whose support is the set $\mathcal{A}_\delta$ defined in Section 2.3.1

3.1 Duality

Many of the techniques used in this paper rest on the duality - detailed below - between cells in a tropical surface of degree δ and in its subdivision of Γ_δ.

Let X be a tropical surface of degree δ. Writing $\mathcal{A}_\delta := \Gamma_\delta \cap \mathbb{Z}^3$, this means (by Definition 3.1) that X is of the form $X = V_{tr}(f)$, for some tropical polynomial $f(x) = \text{``} \sum_{a \in \mathcal{A}_\delta} \lambda_a x^a \text{''}$. As explained in Section 2.3, the function $a \mapsto \lambda_a$ induces a regular lattice subdivision of Γ_δ. We denote this by $\mathrm{Subdiv}(f)$. Any element $\Delta \in \mathrm{Subdiv}(f)$ of dimension at least 1, corresponds in a natural way to a cell $\Delta^\vee \subseteq V_{tr}(f)$. Namely, if the vertices of Δ are $a_1, \ldots, a_r$, then $\Delta^\vee$ is the solution set of the equalities and inequalities

$$(3) \quad \lambda_{a_1} + \langle a_1, x \rangle = \cdots = \lambda_{a_r} + \langle a_r, x \rangle \geq \lambda_b + \langle b, x \rangle, \quad \text{for all } b \in \mathcal{A}_\delta \smallsetminus \{a_1, \ldots, a_r\}.$$

(Here, $\langle \ , \ \rangle$ denotes the Euclidean inner product on $\mathbb{R}^3$.) Moreover, we have the following theorem (see [4, Proposition 3.11]):

Theorem 3.2. *The association* $\Delta \mapsto \Delta^\vee$ *gives a one-to-one correspondence between the* k-*cells of* $\mathrm{Subdiv}(f)$ *and the* $(n-k)$-*cells of* $V_{tr}(f)$, *for each* $k = 1, 2, 3$. *Furthermore, for any cells* $\Delta, \Lambda \in \mathrm{Subdiv}(f)$ *of dimensions at least 1, we have that*

i) *If* Δ *is a face of* Λ, *then* $\Lambda^\vee$ *is a face of* $\Delta^\vee$ *in* $V_{tr}(f)$.

ii) *The affine-linear subspaces* $\mathrm{Aff}(\Delta)$ *and* $\mathrm{Aff}(\Delta^\vee)$ *are orthogonal in* $\mathbb{R}^3$.

iii) $\Delta \subseteq \partial(\Gamma_\delta)$ *if and only if* $\Delta^\vee$ *is an unbounded cell of* $V_{tr}(f)$.

If C is a cell of $V_{tr}(f)$, we denote its corresponding cell in $\mathrm{Subdiv}(f)$ by $C^\vee$. The cells C and $C^\vee$ are said to be *dual* to each other.

Definition 3.3. We say that $V_{tr}(f)$ is a *smooth* tropical surface if $\mathrm{Subdiv}(f)$ is an elementary (unimodular) triangulation.

For example, let $f_\delta(x) = \text{``}\sum_{a \in \mathcal{A}_\delta} \alpha(a) x^a\text{''}$, where α is the lifting function defined in (1). Then according to Definitions 3.1 and 3.3, the tropical surface $V_{tr}(f_\delta)$ is smooth of degree δ.

3.2   Tropical lines in $\mathbb{R}^3$

Let L be an unrooted tree with five edges, and six vertices, two of which are 3-valent and the rest 1-valent. We define a *tropical line in* $\mathbb{R}^3$ to be any realization of L in $\mathbb{R}^3$ such that

- the realization is a polyhedral complex, with four unbounded rays (the 1-valent vertices of L are pushed to infinity),

- the unbounded rays have direction vectors $-e_1, -e_2, -e_3, e_1 + e_2 + e_3$,

- The realization is balanced at each vertex, i.e., the primitive integer vectors in the directions of all outgoing edges adjacent to a given vertex, sum to zero.

If the bounded edge has length zero, the tropical line is called *degenerate*. For non-degenerate tropical lines, there are three combinatorial types of tropical lines in $\mathbb{R}^3$, as shown in Figure 5. The combinatorial types of the lines in Figure 5, from left to right, are denoted by $((12)(34))$, $((13)(24))$ and $((14)(23))$. Each innermost pair of digits indicate the directions of two adjacent rays.

Remark 3.4. This definition is equivalent to the more standard algebraic definition of tropical lines in $\mathbb{R}^3$. See [3, Examples 2.8 and 3.8].

In classical geometry, any two distinct points lie on a unique line. When we turn to tropical lines, this is true only for generic points. In fact, for special choices of points P and Q there are infinitely many tropical lines passing through P and Q. The precise statement is as follows:
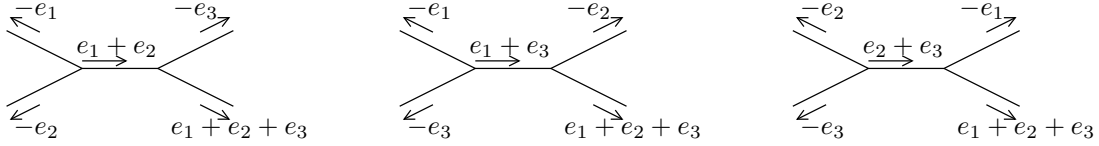
Figure 5: The combinatorial types of tropical lines in $\mathbb{R}^3$.

Lemma 3.5. *Let $P, Q \in \mathbb{R}^3$. There exist infinitely many tropical lines containing P and Q if and only if one of the coordinates of the vector $Q - P$ is zero, or two of them coincide. In all other cases, P and Q lie on a unique line.*

Definition 3.6. An infinite collection of tropical lines in $\mathbb{R}^3$, is called a *two-point family* if there exist two points lying on all lines in the collection.

3.3 Group actions of S_4

The group of permutations of four elements, S_4, acts naturally on many of the spaces involved with tropical surfaces. Firstly, observe that S_4 is the symmetry group of the simplex $\Gamma_\delta \subseteq \mathbb{R}^\vee$. This induces an action of the set of lattice points $\mathcal{A}_\delta = \Gamma_\delta \cap \mathbb{Z}^3$ (in fact on all of $\mathbb{Z}^3$), described explicitly as follows: Let $\sigma \in S_4$ be a permutation of four elements. For any $a = (a_1, a_2, a_3) \in \mathcal{A}_\delta$, let $a^{\mathrm{hom}} := (a_1, a_2, a_3, \delta - a_1 - a_2 - a_3)$. We define $\sigma(a)$ to be the point in $\mathcal{A}$ whose coordinates are the first three coordinates of $\sigma(a^{\mathrm{hom}})$. Obviously, this action of S_4 on $\mathcal{A}_\delta$ also induce an action of S_4 on the set of subdivisions of Γ_δ.

Secondly, S_4 acts on the set of tropical surfaces of degree δ. Let $X = V_{tr}(f)$, where $f(x) = \text{``}\sum_{a \in \mathcal{A}_\delta} \lambda_a x^a\text{''}$. For a given $\sigma \in S_4$, we define $\sigma(X)$ to be the surface $V_{tr}(\sigma(f))$, where $\sigma(f) = \text{``}\sum_{a \in \mathcal{A}_\delta} \lambda_a x^{\sigma(a)}\text{''}$ Clearly, $\sigma(X)$ is still of degree δ, and the resulting action is compatible with the action of S_4 on the subdivisions of Γ_δ. In other words, $\mathrm{Subdiv}_{\sigma(X)} = \sigma(\mathrm{Subdiv}_X)$.

4 Properties of tropical lines on tropical surfaces

4.1 Notation

The topic of this paper is to study tropical lines contained in tropical surfaces. It is important to note that 'containment' here is meant purely set-theoretically. For notational convenience, we fix the following: The symbols X and L will always refer to the underlying point set in $\mathbb{R}^3$ of a tropical surface of degree δ, and a tropical line, respectively. The associated polyhedral cell complexes are denoted by $\mathrm{Complex}(X)$ and $\mathrm{Complex}(L)$ respectively. Hence in particular, the statement $L \subseteq X$ means that L is contained in X as subsets of $\mathbb{R}^3$. There is a natural map $c_X \colon X \to \mathrm{Complex}(X)$, taking a point $p \in X$ to the minimal cell of X containing p.

Furthermore, for the remaining part of the paper we fix the following notation:

- For $\delta \in \mathbb{N}$, Γ_δ is the simplex with vertices $(0,0,0), (\delta,0,0), (0,\delta,0), (0,0,\delta)$, and $\mathcal{A}_\delta := \Gamma_\delta \cap \mathbb{Z}^3$.

- The vectors $-e_1, -e_2, -e_3$ and $e_1+e_2+e_3$ are denoted $\omega_1, \ldots, \omega_4$. The coordinate variables in $\mathbb{R}^3$ are x_1, x_2, x_3, and we set $x_4 := \delta - x_1 - x_2 - x_3$.

- If ℓ is the equation of a plane in $\mathbb{R}^3$, then $\mathcal{P}_l$ denotes this plane.

- For a fixed $\delta \in \mathbb{N}$, F_i is the facet of Γ_δ with outer normal vector ω_i, $i = 1, \ldots, 4$. (Note that F_i is contained in the plane $\mathcal{P}_{x_i=0}$.) Moreover, for distinct $i,j \in \{1, \ldots, 4\}$ we set $F_{ij} := F_i \cap F_j$.

- If X is a tropical surface of degree δ, then we set $\mathrm{Subdiv}_X := \mathrm{Subdiv}(f)$, where f is any tropical polynomial with support $\mathcal{A}_\delta$, such that $X = V_{tr}(f)$. (It is easy to see that $\mathrm{Subdiv}(f)$ is the same for all such f, so Subdiv_X is well defined.)

- If $\alpha \in K(\mathcal{S})$, where $\mathcal{S}$ is a regular subdivision of Γ_δ, then X_α is the associated tropical surface. More precisely, $X_\alpha = V_{tr}(f)$, where

$$f(x) = \text{``} \sum_{a \in \mathcal{A}_\delta} \alpha(a) x^a \text{''}.$$

4.2   Trespassing line segments on X

In [7] we introduced the notion of *trespassing* line segments on X. If $\ell \subseteq X$ is any ray or line segment, we say that ℓ is trespassing on X if there exist distinct cells $C, C' \subseteq X$ such that

$$(4) \qquad \dim(\mathrm{int}(C) \cap \ell) = \dim(\mathrm{int}(C') \cap \ell) = 1.$$

Alternatively, ℓ is trespassing on X if it is not contained in the closure of any single cell of X. For smooth X, trespassing can happen essentially in one way only, as shown in [7, Lemma 6.2]:

Lemma 4.1. *Let $\ell \subseteq X$, where ℓ is a line segment and X is smooth. If $C, C' \subseteq X$ are any cells satisfying (4) and such that $\ell \subseteq C \cup C'$, then C and C' are maximal cells of X whose intersection is a vertex V of X.*

An immediate consequence of this is that $C^\vee$ and $(C')^\vee$ are opposite edges of the tetrahedron $V^\vee$. The following converse to Lemma 4.1 is straightforward:

Lemma 4.2. *Let Λ and Λ' be opposite edges of a tetrahedron $\Delta \in \mathrm{Subdiv}_X$. Then there is a trespassing line segment on X passing through the vertex $\Delta^\vee \in X$, and which is orthogonal to both Λ and Λ'.*

We recall one more result from [7], again valid for smooth X (cf. [7, Lemma 6.4]):

Lemma 4.3. *Suppose $L \subseteq X$ is non-degenerate, and that the vertex v of L lies in the interior of a 1-cell E of X. Then $L \cap E = \{v\}$, and the three edges of L adjacent to v start off in different 2-cells of X adjacent to E.*

The last statement of Lemma 4.3 can be reformulated "dually" as follows: Suppose ω_i and ω_j are the direction vectors of the unbounded edges of L emanating from v. Then the edges of the triangle $E^\vee$ are orthogonal to ω_i, ω_j and $\omega_i + \omega_j$ respectively.
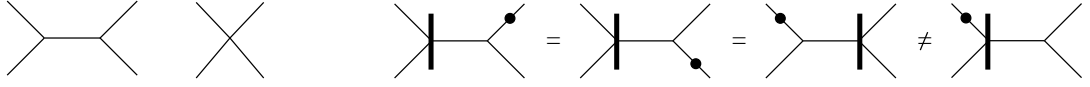
Figure 6: Underlying graphs of tropical lines in $\mathbb{R}^3$.

Figure 7: Four combinatorial positions, three of which are equal.

4.3 The combinatorial position of $L \subseteq X$

We now describe a way of displaying the essential information of how a tropical line lies on a tropical surfaces.

For any tropical line $L \subseteq \mathbb{R}^3$, the underlying graph of L is one of the two shown in Figure 6, dependent on whether L is degenerate or not. A *decoration* of either of these consists of a finite number of dots (possibly none) on each edge, and at each vertex either a dot, a vertical line segment, or nothing. Note that we consider the graphs without metrics, so moving an edge-dot along its edge does not change the decoration. Also, two decorations $\mathcal{C}$ and $\mathcal{C}'$ (of the same graph) are said to be equal if there is an automorphism of the graph taking $\mathcal{C}$ to $\mathcal{C}'$. See Figure 7 for examples.

Definition 4.4. Let X be a tropical surface. The *combinatorial position* of L on X is the following decoration of the underlying graph of L:

- If an edge of L passes through k vertices of X, the corresponding edge of the underlying graph has k dots.

- For each vertex v of L, the corresponding vertex of the graph has a dot if $\dim c_X(v) = 0$, and a vertical line segment if $\dim c_X(v) = 1$.

Remark 4.5. There is nothing special about the graphs of Figure 6 in this context. Thus, using the same definition, one can speak of the combinatorial position of any connected one-dimensional polyhedral complex contained pointwise in X.

4.4 Line subcomplexes of Subdiv_X

Recall from Section 4.1 the map $c_X \colon X \to \mathrm{Complex}(X)$, taking a point $p \in X$ to the minimal cell of X containing p. Combining c_X with dualization, we get the map

(5)
$$\begin{aligned} c_X^\vee \colon X &\longrightarrow \mathrm{Subdiv}_X \\ p &\longmapsto c_X(p)^\vee. \end{aligned}$$

If $Y \subseteq X$ is any subset, we set

$$c_X^\vee(Y) := \bigcup_{y \in Y} c_X^\vee(y).$$

Note that if Y is connected, then $c_X^\vee(Y)$ is a connected subcomplex of Subdiv_X.
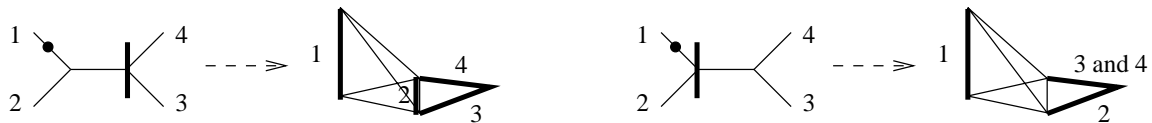
Figure 8: Two combinatorial positions giving the same line subcomplex structure, but with different sets of required exits (indicated by bold lines). The exits are determined using Lemmas 4.1 and 4.3.

Definition 4.6. Let $\mathcal{S}$ be a regular subdivision of Γ_δ. A subcomplex $\mathcal{R} \subseteq \mathcal{S}$ is called a *line subcomplex* if there exists a tropical surface X and a tropical line $L \subseteq X$ such that $\mathrm{Subdiv}_X = \mathcal{S}$, and $c_X^\vee(L) = \mathcal{R}$.

Conversely, suppose $\mathcal{R} \subseteq \mathcal{S}$ is a line subcomplex. Then if X' is any tropical surface with $\mathrm{Subdiv}_{X'} = \mathcal{S}$, we say that $\mathcal{R}$ is *realized on X'* if there is a tropical line $L \subseteq X'$ such that $c_{X'}^\vee(L) = \mathcal{R}$.

Because tropical lines in $\mathbb{R}^3$ are unbounded, any line subcomplex in Subdiv_X contain cells dual to unbounded cells of X. Recall from Theorem 3.2c) that such cells of Subdiv_X lie in the boundary of the Newton polytope Γ_δ. This motivates the concept of subpolytopes with *exits* in Γ_δ, introduced in [7]:

Definition 4.7. Let Δ be a lattice polytope (of dimension 1,2 or 3) contained in Γ_δ. We say that Δ has an *exit in the direction of ω_i* if at least one edge of Δ lies in F_i. If Δ has exits in the directions of k of the ω_i's, we say that Δ has *k exits*.

The relevance of this definition should be clear from the following observation: Let C be any cell of X, and let $p \in C$ be an arbitrary point. Then C contains the ray with starting point p and direction ω_i, if and only if $C^\vee$ has an exit in direction ω_i.

When X is smooth, the cell structure of a line subcomplex $c_X^\vee(L)$ is in many cases uniquely determined by the combinatorial position of L on X. Moreover, using Lemmas 4.1 and 4.3, we can often describe explicitly the exits required of the edges of $c_X^\vee(L)$. For example, the two rightmost combinatorial positions in Figure 7 imply the same cell structure of $c_X^\vee(L)$, but with different exit properties (see Figure 8).

Remark 4.8. A line subcomplex often has more exits than those required by the combinatorial position. Hence it is usually more difficult to reverse the process described in the last paragraph, i.e., to determine the combinatorial position of L on X, given a line subcomplex $c_X^\vee(L) \subseteq \mathrm{Subdiv}_X$.

We conclude this section by mentioning one case where the cell structure of $c_X^\vee(L)$ is not determined by the combinatorial position of L on X. Namely, when both vertices of L are vertices of X, and the middle edge of L is not trespassing. In this case, the middle edge of L may or may not be an edge of X, giving different structures of $c_X^\vee(L)$. (See Figure 9.) The two tetrahedra $P^\vee$ and $Q^\vee$ have a common facet if PQ is an edge of X (case i), but only an edge in common otherwise (i.e., if PQ goes across a 2-cell of X). Note that if the middle edge were trespassing, there would be no ambiguity: By Lemma 4.1, no point of PQ could then be in the interior of a 1-cell of X.
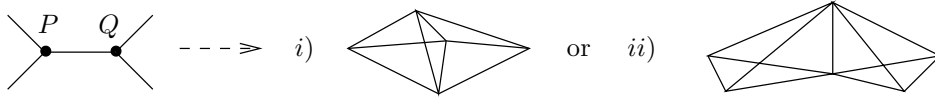
Figure 9: A combinatorial position giving two possible line subcomplexes, depending on whether PQ is a 1-cell of X (case i) of not (case i).

4.5 Deformations and specializations

Let $\mathcal{S}$ be a given subdivision of Γ_δ, and let $K = K(\mathcal{S})$ be the corresponding secondary cone. We define the incidence $\mathfrak{X}_\mathcal{S} \subseteq K \times \mathbb{R}^3$ by

$$\mathfrak{X}_\mathcal{S} := \{(\alpha, x) \mid x \in X_\alpha)\} \subseteq K \times \mathbb{R}^3.$$

Using the Euclidean metrics on K and $\mathbb{R}^3$, we give $\mathfrak{X}_\mathcal{S}$ the topology induced by the product topology on $K \times \mathbb{R}^3$. This makes the projections on K and $\mathbb{R}^3$, denoted by p_1 and p_2 respectively, continuous. Note that for any $\alpha \in K$, $p_2(p_1^{-1}(\alpha))$ is the tropical surface X_α.

Definition 4.9. A *family of tropical lines associated to $\mathcal{S}$* is a subset $\mathfrak{L} \subseteq \mathfrak{X}_\mathcal{S}$ satisfying the following conditions:

- For any $\alpha \in K$, $p_2(p_1^{-1}(\alpha) \cap \mathfrak{L})$ is a tropical line $L_\alpha \subseteq X_\alpha$.

- The projections from $\mathfrak{L}$ to K and $\mathbb{R}^3$ are continuous.

Definition 4.10. A *deformation* of $L \subseteq X_\alpha$ is a family $\mathfrak{L}$ of tropical lines associated to $\mathcal{S}$, such that

- $p_1(\mathfrak{L})$ contains α, and is homeomorphic to an interval,

- for any two points $\beta \neq \gamma$ in $p_1(\mathfrak{L})$, we have $X_\beta \neq X_\gamma$.

Note that a deformation of $L \subseteq X$ can be thought of as a map $t \mapsto (L_t, X_t)$, where t runs through some interval $I \subseteq \mathbb{R}$ containing 0, and where $(L, X) = (L_0, X_0)$. In particular, 0 can be an endpoint of I, as in $I = [0, 1)$.

Definition 4.11. Let L be a tropical line with combinatorial position $\mathcal{C}$ on X. We say that L *deforms into combinatorial position $\mathcal{C}'$*, if there exist a deformation $t \mapsto (L_t, X_t)$ of $L \subseteq X$ such that for all $t \in I \smallsetminus \{0\}$, the combinatorial position of L_t on X_t is $\mathcal{C}'$.

The following lemma gives a simple property of deformations, namely that one cannot deform a tropical line away from a vertex through which it is trespassing.

Lemma 4.12. *Suppose $L \subseteq X$ has a trespassing edge ℓ, passing through the vertex $\Delta^\vee \in X$, where Δ is a tetrahedron in Subdiv_X. If $t \mapsto (L_t, X_t)$, $t \in I$, is any deformation of $L \subseteq X$, let ℓ_t be the edge of L_t parallel to ℓ. Then for small t, ℓ_t is trespassing through $\Delta^\vee \in X_t$.*

Proof. Since ℓ is trespassing through $\Delta^\vee \in X$, Lemma 4.1 gives that $\dim(\ell \cap \mathrm{int}(\Lambda_1^\vee)) = \dim(\ell \cap \mathrm{int}(\Lambda_2^\vee)) = 1$, for some pair of opposite edges Λ_1, Λ_2 of Δ. By continuity of the deformation, this implies that $\dim(\ell_t \cap \mathrm{int}(\Lambda_1^\vee)) = \dim(\ell_t \cap \mathrm{int}(\Lambda_2^\vee)) = 1$ for small enough t. Hence ℓ_t is trespassing through $\Delta^\vee$. $\qquad\square$

Remark 4.13. Note that the proof of Lemma 4.12 rests on Lemma 4.1, which requires X to be smooth. In fact, it is not hard to produce examples of non-smooth X where one *can* deform away from trespassed vertices.

Related to deformations is the concept of specialization:

Definition 4.14. Let $t \mapsto (L_t, X_t)$ be a deformation of $L_0 \subseteq X_0$, where $t \in [0,1]$. We say that $L_0 \subseteq X_0$ *specializes* to $L_1 \subseteq X_1$ if the combinatorial position of $L_t \subseteq X_t$ is constant for all $t \in [0,1)$ but differs for $t = 1$.

5   Classification of combinatorial positions

Let $\delta \in \mathbb{N}$ be fixed, and let $\mathcal{S}$ be an RE-triangulation of Γ_δ. By Proposition 2.1, the secondary cone $K(\mathcal{S})$ has dimension N in $\mathbb{R}^N$, where $N = |\mathcal{A}_\delta| = \binom{\delta+3}{3}$. Recall that each $\alpha \in K(\mathcal{S})$ corresponds to a smooth tropical surface X_α with subdivision $\mathcal{S}$.

Definition 5.1. We say that a property Π holds for *general tropical surfaces with subdivision $\mathcal{S}$* if Π holds for X_α for every α in some open, dense subset of $K(\mathcal{S})$.

More generally, Π holds for *general smooth tropical surfaces of degree δ* if Π holds for general tropical surface with subdivision $\mathcal{S}$, for all RE-triangulations $\mathcal{S}$ of Γ_δ.

Finally, Π holds for *general smooth tropical surfaces* if Π holds for general smooth tropical surfaces of degree δ, for all $\delta \in \mathbb{N}$.

The next lemma gives an important example of a property held by general smooth tropical surfaces (in all degrees).

Lemma 5.2. *A general smooth X contains no doubly trespassing line segments.*

Proof. Let $X = V_{tr}(f)$ be a tropical surface, given by a tropical polynomial $f = $ "$\sum \lambda_a x^a$", and suppose $\ell \subseteq X$ is a line segment containing two vertices of X, say P and Q, in its relative interior. We will show that this implies a linear relation between the coefficients λ_a.

The situation is shown in Figure 10. From Lemma 4.1, it follows that Subdiv_X contains three 1-cells AB, CD, EF such that $\ell \subseteq (AB)^\vee \cup (CD)^\vee \cup (EF)^\vee$, and such that $P^\vee = ABCD$ and $Q^\vee = CDEF$. Obviously, a necessary condition for this to happen is that ℓ is parallel to both vectors products $\vec{AB} \times \vec{CD}$ and $\vec{CD} \times \vec{EF}$.

Since ℓ is contained in each of the planes spanned by $(AB)^\vee$, $(CD)^\vee$ and $(EF)^\vee$, any point $p = (p_1, p_2, p_3) \in \ell$ satisfies the defining equations of these planes:

$$
\begin{aligned}
\lambda_A + \langle A, x \rangle &= \lambda_B + \langle B, x \rangle \\
\lambda_C + \langle C, x \rangle &= \lambda_D + \langle D, x \rangle \\
\lambda_E + \langle E, x \rangle &= \lambda_F + \langle B, x \rangle
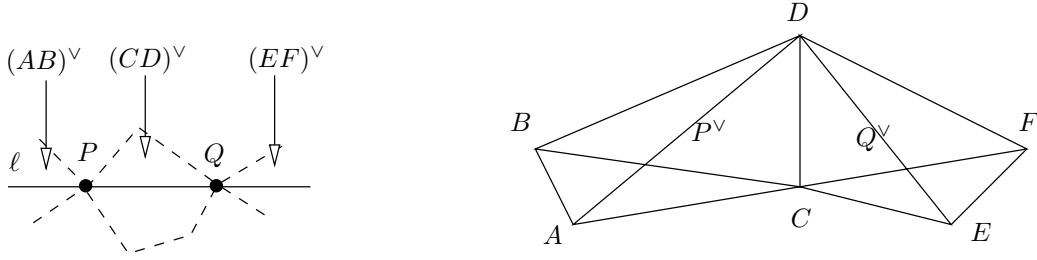\end{aligned}
$$

(6)

Figure 10: A doubly trespassing line, and the dual configuration.

Substituting p for x, this amounts to the matrix equation

$$(7) \qquad M\begin{pmatrix} p \\ 1 \end{pmatrix} = 0, \quad \text{where} \quad M = \begin{pmatrix} \vec{AB} & \lambda_B - \lambda_A \\ \vec{CD} & \lambda_D - \lambda_C \\ \vec{EF} & \lambda_F - \lambda_E \end{pmatrix} = \begin{pmatrix} B_1 - A_1 & B_2 - A_2 & B_3 - A_3 & \lambda_B - \lambda_A \\ D_1 - C_1 & D_2 - C_2 & D_3 - C_3 & \lambda_D - \lambda_C \\ F_1 - E_1 & F_2 - E_2 & F_3 - E_3 & \lambda_F - \lambda_E \end{pmatrix}.$$

Since (7) holds for all $p \in \ell$, the nullity of M must be at least 2, i.e., rank $M \leq 2$. In fact, rank $M = 2$, since the vectors $\vec{AB}$, $\vec{CD}$, $\vec{EF}$ are not parallel. Therefore there is a linear relation between $\lambda_A, \ldots, \lambda_F$. Note that this relation gives a hyperplane section of the secondary cone $K(\mathrm{Subdiv}_X)$ containing the point corresponding to X.

To prove the lemma, let $\mathcal{S}$ be any RE-triangulation of Γ_δ, for any δ. In $\mathcal{S}$, we look for all pairs of tetrahedra with a common edge, and such that (with the notation of Figure 10) $\vec{AB} \times \vec{CD} \parallel \vec{CD} \times \vec{EF}$. Clearly there are at most finitely many such pairs. As seen above, each such pair gives rise to a hyperplane section of $K(\mathcal{S})$, and any surface containing a doubly trespassing line, corresponds to a point on one of these hyperplanes. Since the complement of the union of the hyperplanes is open and dense, the lemma follows. $\qquad \square$

The above lemma greatly limits the number of ways in which a tropical line can lie on a general smooth tropical surface. In particular, the lemma says that for general X, each of the five edges of $L \subseteq X$ contains at most one vertex of X in its relative interior. An immediate implication of this is the following interesting result: There exists a *finite* list of combinatorial positions, such that for a general smooth X (of any degree), the combinatorial position of any tropical line on X is in the list.

However, there are some combinatorial positions of L that do not occur on general X, but which are not excluded by Lemma 5.2. Many of these can be identified using the lemma to follow.

By a *3-star on X* we mean the union of 3 line segments on X, no two parallel, with a common endpoint called the *center* of the 3-star. If Y is a 3-star on X with center v, we say that Y is *special* if the number of trespassing edges of Y is exactly $\dim c_X(v) + 1$. Obviously, there are three possible combinatorial positions of a special 3-star on X (cf. Remark 4.5), as shown in Figure 11.

Lemma 5.3. *A general smooth X contains no special 3-stars.*

Proof. Let $X = V_{tr}(f)$ be smooth of some arbitrary degree δ, where $f = $ "$\sum \lambda_a x^a$". For any 3-star $Y \subseteq X$, we consider the *3-star subcomplex $c_X^\vee(Y)$* in Subdiv_X. (Cf. the
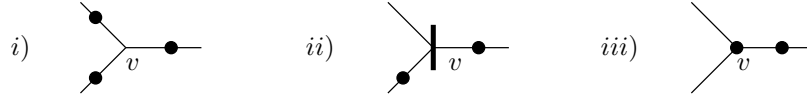
Figure 11: Special 3-stars on X, where $\dim c_X(v)$ equals i) 2, ii) 1, and iii) 0.

definition of line subcomplexes in Section 4.4.) For the three special 3-stars in Figure 11, the structures of the corresponding 3-star subcomplexes are depicted in Figure 12. We claim that given any special 3-star subcomplex $\mathcal{R} \subseteq \mathrm{Subdiv}_X$, it is realized as a 3-star on X only if the coefficients λ_a satisfy a linear condition, dependent of $\mathcal{R}$. To show this, the idea is to find in each case the equations v must satisfy and arrange them in a matrix form, similar to (7). For example, in case i), we see that v lies on each of the planes spanned by $(AB)^\vee$, $(CD)^\vee$, $(EF)^\vee$ and $(GH)^\vee$. Writing out the corresponding equations, we obtain

$$(8) \qquad \begin{pmatrix} \vec{AB} & \lambda_B - \lambda_A \\ \vec{CD} & \lambda_D - \lambda_C \\ \vec{EF} & \lambda_F - \lambda_E \\ \vec{GH} & \lambda_H - \lambda_G \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} = 0.$$

Observe that the leftmost matrix in (8) is a 4×4-matrix; let us call it M. Since the null-space of M is non-trivial (it contains $(v, 1)^T$), we must have $\det M = 0$, giving a linear relation in the λ's. Note that this would reduce to the trivial condition $0 = 0$ if $\mathrm{rank}\, M \leq 2$. However, it is easy to see that in our case $\vec{AB}, \vec{CD}, \vec{EF}, \vec{GH}$ span all of $\mathbb{R}^3$, so $\mathrm{rank}\, M = 3$. This proves the claim in case i).

The cases ii) and iii) are done in the same way, but with the matrix M exchanged with

$$ii)\ M' = \begin{pmatrix} \vec{AB} & \lambda_B - \lambda_A \\ \vec{AC} & \lambda_C - \lambda_A \\ \vec{DE} & \lambda_E - \lambda_D \\ \vec{FG} & \lambda_G - \lambda_F \end{pmatrix}, \qquad iii)\ M'' = \begin{pmatrix} \vec{AB} & \lambda_B - \lambda_A \\ \vec{AC} & \lambda_C - \lambda_A \\ \vec{AD} & \lambda_D - \lambda_A \\ \vec{EF} & \lambda_F - \lambda_E \end{pmatrix}.$$

It is now straightforward to show that a general smooth X contains no special 3-stars. Indeed, let $\mathcal{S}$ be any elementary subdivision of Γ_δ, and α a point in the parameter cone $K(\mathcal{S})$. Then as we have seen, any 3-star subcomplex in $\mathcal{S}$ like those in Figure 12 can be realized on X_α only if α lies on a certain hyperplane. Moreover, $\mathcal{S}$ contains at most finitely many of the 3-star subcomplexes in Figure 12. Hence for any α in the complement of a finite union of hyperplanes, X_α contains no special 3-stars. $\qquad \square$
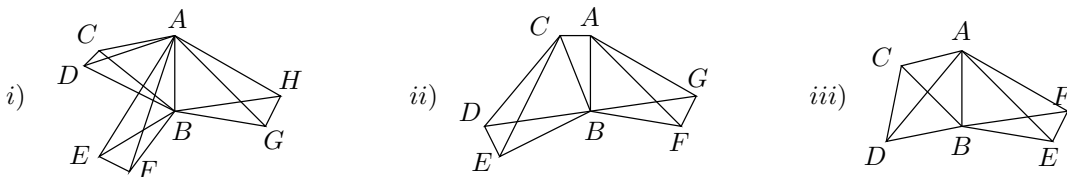

Figure 12: Configurations in Subdiv_X implied by the 3-stars in Figure 11.
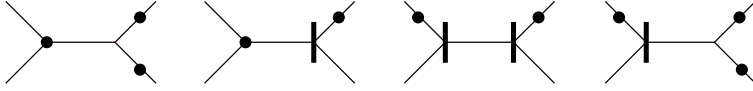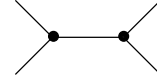
63

Figure 13: Lemma 5.5a).   Figure 14: Lemma 5.5b).

Corollary 5.4. *On a general smooth tropical surface X, any vertex v of a tropical line $L \subseteq X$ satisfies*

$$(9) \qquad \sharp\{\text{trespassing edges of } L \text{ adjacent to } v\} \ \leq \ \dim c_X(v).$$

Proof. It is easy to see that if v is any vertex of a tropical line $L \subseteq X$, for which (9) is *not* satisfied, then v is the center vertex of a special 3-star. The result therefore follows from Lemma 5.3. □

By the help of Corollary 5.4 it is straightforward to construct a list containing all possible combinatorial positions of a non-degenerate tropical line on a general smooth tropical surface. The result is shown in Table 1.

Note that we do not claim that all the entries of Table 1 actually occur on general smooth surfaces. For starters, the following combinatorial positions are clearly impossible on any tropical surface:
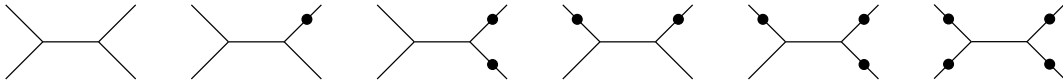
In each case, the middle segment of L is contained in a 2-cell of X. But the part of L contained in this cell spans $\mathbb{R}^3$, which is a contradiction.

Furthermore, we have the following lemma:

Lemma 5.5.

 a) *A general smooth X has no tropical lines with the combinatorial positions shown in Figure 13.*

 b) *A general smooth X has no tropical line such that i) its combinatorial position is the one in Figure 14 and ii) its middle segment goes across a 2-cell of X.*

Proof. a) The idea is basically the same in in the proofs of Lemma 5.2 and Lemma 5.3. Each case implies some linear relation between the coefficients λ_a of the polynomial defining X. We sketch the argument for the leftmost combinatorial position: Observe that if L has this combinatorial position, then the line subcomplex of L is homeomorphic to case *i*) of Figure 12. Let v_1 be the vertex of L which is also a vertex of X; assume this is dual to the tetrahedron $ABGH$. Then, by the definition of duality, v_1 is uniquely determined by $\lambda_A, \lambda_B, \lambda_G$ and λ_H. (In fact, the coordinates of v_1 are linear forms in these λ's.) Similarly, the other vertex of L, v_2, is determined by $\lambda_A, \lambda_B, \lambda_C, \lambda_D, \lambda_E$ and λ_F (this corresponds to solving the equation (8), but with the last row removed.) Finally, since $v_1 v_2$ is the middle segment of a tropical line, it

Table 1: Combinatorial positions of tropical lines on a general smooth X. Gray=nonexistent or non-general; dash=non-compatible 3-stars.

has a prescribed direction. This forces a linear relation between the coefficients. The remaining three cases of the above claim are done similarly.

b) This combinatorial position was discussed in the last paragraph of Section 4.4. If the middle segment of L goes across a 2-cell of X, the line subcomplex is homeomorphic to case ii) of Figure 9. In this case the argument sketched in a) applies again: Each vertex of L is determined by the λ's, and the direction vector of the middle segment implies a linear relation between these.

Note that this argument does not apply if the middle segment of L is a 1-cell of X: In this case the direction of the middle segment is encoded in the line subcomplex as a normal vector of the common facet of the two tetrahedra (cf. case i) of Figure 9). $\square$

5.1   The classification theorem

In the last section we identified 10 entries of Table 1 that were either impossible on any tropical surface X, or *non-general*, meaning that they do not occur on general smooth X. In this section we analyze the remaining 17 combinatorial positions. The main result is the following classification:

Theorem 5.6. *For a general smooth tropical surface X, the combinatorial position of any non-degenerate tropical line on X is one of the 17 listed in Table 2. Moreover, we have:*

a) *14 of the combinatorial positions occur only of surfaces of a particular degree:*

- *1A and 1B occur only on surfaces of degree 1.*
- *2A, 2B, 2C, 2D, 2E and 2F occur only on surfaces of degree 2.*
- *3A, 3B, 3C, 3D, 3E and 3F occur only on surfaces of degree 3.*

b) *3 of the combinatorial positions occur on surfaces of arbitrary degree. More precisely:*

- *3G and 3H occur on surfaces of any degree $\delta \geq 2$.*
- *3I occur on surfaces of any degree $\delta \geq 1$.*

The 17 combinatorial positions mentioned in Theorem 5.6 are called *general combinatorial positions*.

Proof. The proof of Theorem 5.6 will occupy most of this section. To avoid repeating ourselves too much, we start by giving some auxiliary observations about *tropical half lines* on X, which will apply frequently. A tropical half line in $\mathbb{R}^3$ is the remaining part of a non-degenerate tropical line, after removing two adjacent rays. Figure 15 shows tropical half lines on X in different positions.

For a tropical half line H, let H^b be its bounded segment. Note that if $H^b \subseteq X$ is non-trespassing, then there is a unique cell of X, denoted C^b, containing H^b. The following lemma gives information on the position of the dual cell $(C^b)^\vee \subseteq \mathrm{Subdiv}_X$. As always, X is assumed to be smooth of some fixed degree δ.

	1A	1B	
Only deg $X = 1$			
Only deg $X = 2$	2A	2B	2C
	2D	2E	2F
Only deg $X = 3$	3A	3B	3C
	3D	3E	3F
Any deg $X \geq 2$	3G	3H	
Any deg $X \geq 1$	3I		

Table 2: The 17 general combinatorial positions.

Lemma 5.7. *Let $H \subseteq X$ be a tropical half line with unbounded rays in the directions ω_i and ω_j, such that H^b is non-trespassing and contained in a cell C^b of X. Then $(C^b)^\vee$ is contained in a plane with equation $x_i + x_j = K$ for some $K \in \mathbb{N}_0$.*

Proof. Recall that any vector contained in C^b is orthogonal to $(C^b)^\vee$. In the case where $i, j \neq 4$, this immediately proves the assertion, since by assumption C^b contains the vector $-\omega_i - \omega_j = e_i + e_j$. For the remaining case, suppose $j = 4$, and let (i, i', i'') be any permutation of $(1, 2, 3)$. Then C^b contains the vector $\omega_i + \omega_4 = e_{i'} + e_{i''}$, so $(C^b)^\vee$ lies in a plane with equation $x_{i'} + x_{i''} = $ constant. This is equivalent to the statement in the lemma, since $x_i + x_4 = K \iff x_{i'} + x_{i''} = \delta - K$. $\square$

Lemma 5.8. *Let H be as in Lemma 5.7, and suppose in addition that $\dim c_X(v) \geq k$, where v is the vertex of H, and k is the number of unbounded rays of H which are trespassing on X. If either*

- $\dim c_X(v) > 0$, *or*

- $\dim c_X(v) = 0$ *and* $\dim C^b = 1$,

then $(C^b)^\vee$ lies in the plane with equation $x_i + x_j = \dim C^b - \dim c_X(v) + k$.
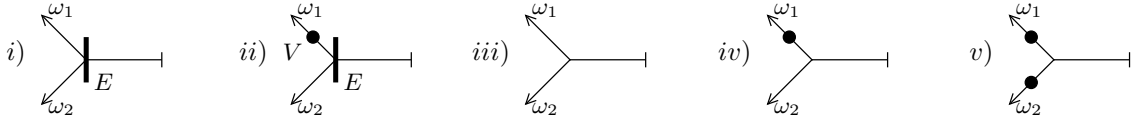
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Figure 15: Tropical half lines on X, with varying values of $\dim c_X(v)$ and numbers of trespassing rays.

Remark 5.9. Lemma 5.8 implies in particular that in the cases mentioned, the integer K in Lemma 5.7 is dependent only of $\dim C^b$, $\dim c_X(v)$ and k. As we will see, this does no longer hold if $\dim c_X(v) = 0$ and $\dim C^b = 2$ (the only case not covered by the lemma), and this fact is what allows the positions $3G$ and $3H$ to occur on surfaces of arbitrarily high degree.

Proof. By symmetry we can assume that $i = 1$ and $j = 2$, so by Lemma 5.7, $(C^b)^\vee$ lies in a plane given by $x_1 + x_2 = K$ for some K. Suppose first $\dim c_X(v) > 0$. It is easy to see that it suffices to consider the five cases shown in Figure 15. Note that in all these cases, $\dim C^b = 2$.

In case i), $\dim c_X(v) = 1$ and $k = 0$, so we must show that $K = 2 + 0 - 1 = 1$. We see that the triangle $E^\vee$ has one edge on F_1, another edge on F_2, while its last edge is $(C^b)^\vee$. Hence the vertices of $E^\vee$ are of the form $(0, 0, a)$, $(0, K, b)$ and $(K, 0, c)$, where $a, b, c \in \mathbb{N}_0$. Let $P = (p, q, r)$ be the fourth vertex of a tetrahedron in Subdiv_X having $E^\vee$ as a facet. A standard calculation shows that the volume of this tetrahedron is divisible by K. Hence unimodularity of Subdiv_X implies $K = 1$, as wanted.

In ii), we must show that $K = 2$. Here, $E^\vee$ has one edge in F_1, another in the plane $x_2 = 1$, and the third is $(C^b)^\vee$. Thus the vertices of $E^\vee$ are $(0, 1, a), (0, K, b), (K-1, 1, c)$ for some $a, b, c \in \mathbb{N}_0$. A volume calculation shows that any integral tetrahedron having $E^\vee$ as facet, has a volume divisible by $K - 1$. Thus $K = 2$.

In iii) we must show $K = 0$. It is clear that $(C^b)^\vee$ lies in both facets F_1 and F_2, and therefore in the edge F_{12} of Γ_δ. Since F_{12} is contained in the plane $x_1 + x_2 = 0$, we are done.

In iv) we find similarly that $(C^b)^\vee$ lies in the intersection of F_1 (where $x_1 = 0$) and the plane given by $x_2 = 1$. In particular, $(C^b)^\vee$ lies in the plane where $x_1 + x_2 = 1$, as claimed in the lemma.

Finally, in case v) $(C^b)^\vee$ lies in the intersection of the planes with equations $x_1 = 1$ and $x_2 = 1$. In particular, this means $x_1 + x_2 = 2$, which is again what we needed to prove.

It remains to treat the case where $\dim c_X(v) = 0$ and $\dim C^b = 1$. In other words, v is a vertex of X and H^b is contained in an edge C^b of X. Dually, $(C^b)^\vee$ is a triangle $ABC \in \mathrm{Subdiv}_X$, and $v^\vee$ is a tetrahedron having $(C^b)^\vee$ as a facet, i.e. $v^\vee = ABCD$ for some integral point D. By Lemma 5.7, ABC lies in a plane with equation $x_1 + x_2 = K$, and by obvious volume considerations, D must then lie in a plane given by $x_1 + x_2 = K \pm 1$. We have to prove that $K = 1$.

Observe that $ABCD$ has exits in both directions ω_1 and ω_2, since H has no trespassing rays (recall the assumption $k \leq \dim c_X(v)$). Now, if the triangle ABC has exits in neither of the two directions, then we must have $D \in F_{12}$, implying $K = 1$.

If ABC has an exit in exactly one of the two directions, say ω_1, then we must have (possibly after renaming) that $AB \subseteq F_1$ and $CD \subseteq F_2$. Writing out what this means for the coordinates of A, B, C and D, a volume calculation of $ABCD$ shows that unimodularity again implies $K = 1$. Note that ABC (being a non-degenerate triangle contained in the plane of the form $x_1 + x_2 = K$) cannot have exits in both directions ω_1 and ω_2. Hence we have covered all cases.

This concludes the proof of the lemma. □

After this preparatory work, we turn to the proof of part a) of Theorem 5.6:

Proof of Theorem 5.6a). We examine the combinatorial positions individually:

• Position $1A$: Suppose $L \subseteq X$, where X has degree δ, and L has combinatorial position $3A$ on X. We can assume w.l.o.g. that L has combinatorial type $((12)(34))$, so the situation is as follows:
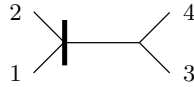
Regard L as the union of two tropical half lines on X, sharing the same bounded segment. Let C^b be the 2-cell of X containing this bounded segment. Applying Lemma 5.8 to the half line on the left (i.e. the one with rays 1 and 2) it follows that $(C^b)^\vee$ lies in the plane with equation $x_1 + x_2 = 1$. On the other hand, the same lemma applied to the other half line implies that $(C^b)^\vee$ lies in the plane with equation $x_3 + x_4 = 0$, i.e. $x_1 + x_2 = \delta$. We conclude that $\delta = 1$.

• An analogue argument works in all cases mentioned in part a) where the middle segment of L is not trespassing on X, i.e. positions $2A$, $2B$, $2D$, $2F$, $3A$, $3D$ and $3E$. Note in particular that in case $2F$, we have $\dim C^b = 1$ by Lemma 5.5b), so Lemma 5.8 applies.

• Position $1B$: Again we assume that L has combinatorial type $((12)(34))$:
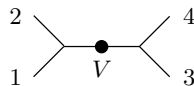
This time we regard L as the union of two tropical half lines on X, intersecting in the point V only. Let C_1^b be the 2-cell containing the bounded segment of the half line with rays 1 and 2, and similarly C_2^b the 2-cell containing the bounded segment of the other half line. Now we apply Lemma 5.8 twice, to find that $(C_1^b)^\vee$ and $(C_2^b)^\vee$ lie in the planes with equations $x_1 + x_2 = 0$, and $x_1 + x_2 = \delta$ respectively. But $(C_1^b)^\vee$ and $(C_2^b)^\vee$ are opposite edges of the unimodular tetrahedron $V^\vee$. This is only possibly if $\delta = 1$.

• An analogue argument works in all cases mentioned in part a) where the middle segment is trespassing, i.e. the positions $2C$, $2E$, $3B$, $3C$ and $3F$. □

In Table 3 we have summarized the cell structures of the line subcomplexes associated to the combinatorial positions $3A, \ldots, 3I$, including the additional information provided by Lemma 5.8.

We make the following definition:

Combinatorial position of $L \subseteq X$	Structure of line subcomplex $c_X^\vee(L) \subseteq \mathrm{Subdiv}_X$	Necessary conditions
3A		Exits: $AB \subseteq F_i$, $\quad BD \subseteq F_j$, $\qquad AC \subseteq F_k$, $\quad EF \subseteq F_l$. $AD \subseteq \mathcal{P}_{x_i+x_j=1}$, $\quad CD \subseteq \mathcal{P}_{x_l=1}$.
3B		Exits: $AB \subseteq F_i$, $\quad AC \subseteq F_j$, $\qquad DF \subseteq F_k$, $\quad EF \subseteq F_l$. $BC \subseteq \mathcal{P}_{x_i+x_j=1}$, $\quad ED \subseteq \mathcal{P}_{x_i+x_j=2}$.
3C		Exits: $AB \subseteq F_i$, $\quad AC \subseteq F_j$, $\qquad DE \subseteq F_k$, $\quad FG \subseteq F_l$. $AD \subseteq \mathcal{P}_{x_i+x_j=1}$, $DE \subseteq \mathcal{P}_{x_i+x_j=2} \cap F_k$.
3D		Exits: $CE \subseteq F_i$, $\quad AB \subseteq F_j$, $\qquad DE \subseteq F_k$, $\quad FG \subseteq F_l$. $CD \subseteq \mathcal{P}_{x_j=1}$, $DE \subseteq \mathcal{P}_{x_i+x_j=2} \cap F_k$.
3E		Exits: $AB \subseteq F_i$, $\quad AC \subseteq F_j$, $\qquad DE \subseteq F_k$, $\quad FG \subseteq F_l$. $BC \subseteq \mathcal{P}_{x_k=1} \cap \mathcal{P}_{x_l=1}$.
3F		Exits: $CD \subseteq F_i$, $\quad AB \subseteq F_j$, $\qquad EF \subseteq F_k$, $\quad GH \subseteq F_l$. $CD \subseteq \mathcal{P}_{x_i+x_j=1} \cap F_i$, $EF \subseteq \mathcal{P}_{x_i+x_j=2} \cap F_k$.
3G		Exits: $CD \subseteq F_k$, $\quad EF \subseteq F_l$, $ABCD$ has edges also in F_i and F_j. $CD \subseteq \mathcal{P}_{x_l=1} \cap F_k$.
3H		Exits: $CE \subseteq F_k$, $\quad DE \subseteq F_l$, $ABCD$ has edges also in F_i and F_j. $CD \subseteq \mathcal{P}_{x_k+x_l=1}$.
3I		Exits: $CD \subseteq F_k \cap F_l$, $ABCD$ has edges also in F_i and F_j. (In particular, $ABCD$ has four exits.)

Table 3: Cell structures of all line subcomplexes in Subdiv_X for general smooth X of degree $\delta \geq 3$. For positions $3A, \ldots, 3F$ we have $\delta = 3$. The bold lines indicate edges with required exits.
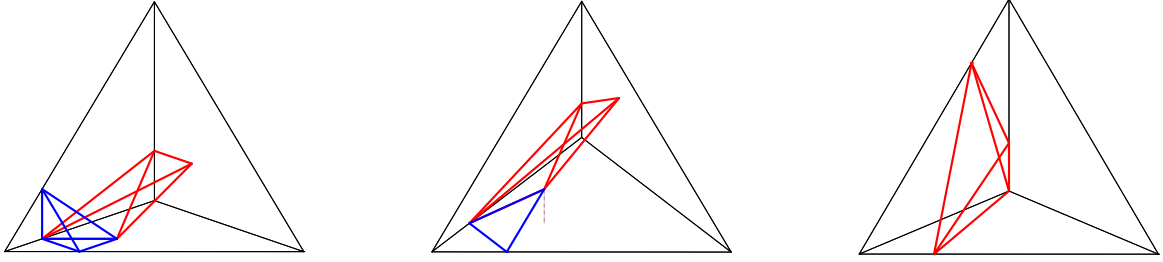
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Figure 16: Line subcomplexes in Γ_4 associated to tropical lines in position $3G$, $3H$ and $3I$, respectively.

Definition 5.10. A lattice complex contained in Γ_δ is said to be of *type* $3A$ (resp. $3B, \dots$) if it has the same cell structure as the line subcomplex given in Table 3 for position $3A$ (resp. $3B, \dots$), and it meets the associated conditions given in Table 3.

It should be noted that the conditions given in Table 3 are only necessary conditions: If $\mathcal{S}$ is an RE-triangulation of Γ_δ, then a subcomplex of $\mathcal{S}$ of a given type is not necessarily a line subcomplex in $\mathcal{S}$.

We apply this terminology in the proof of the remaining part of Theorem 5.6:

Proof of Theorem 5.6b). As usual, let δ be the degree of X.
• Position $3G$: Assume $\delta \geq 2$ (this is clearly necessary for $L \subseteq X$ to have combinatorial position $3G$), and consider the complex $\mathcal{R}_G$ with maximal elements $\Delta_1 = ABCD$ and $\Delta_2 = CDEF$, where $A = (0,0,1), B = (0,1,1), C = (\delta-1,0,0), D = (\delta-2,1,0), E = (\delta-1,1,0), F = (\delta-1,0,1)$. The leftmost picture of Figure 16 shows $\mathcal{R}_G$ for $\delta = 4$; it is clear that $\mathcal{R}_G$ is of type $3G$.

We claim that if Subdiv_X contains $\mathcal{R}_G$, then X contains a line with combinatorial position $3G$. Indeed, this can be checked directly by examining the shape of the 2-cell of X dual to the edge CD. Furthermore, by applying the techniques described in [7, Section 4], one can construct RE-triangulations of Γ_δ containing $\mathcal{R}_G$, for all $\delta \geq 2$. This proves the assertion in Theorem 5.6b) concerning position $3G$.

• Position $3H$: Let $\mathcal{R}_H$ be the complex with maximal elements the tetrahedron $ABCD$ and the triangle CDE, where

$$A = (0,0,1), \quad B = (0,1,2), \quad C = (\delta-1,0,0), \quad D = (\delta-2,1,1), \quad E = (\delta-1,1,0).$$

(See Figure 16, middle picture.) Then $\mathcal{R}_H$ is of type $3H$. Suppose $\mathcal{S}$ is an RE-triangulation of Γ_δ containing $\mathcal{R}_H$. By examining the shape of the 2-cell $(CD)^\vee$, one can see that $\mathcal{R}_H$ is a line subcomplex in $\mathcal{S}$, realizable on X_α for all α in some open, full-dimensional cone in $K(\mathcal{S})$. Finally, as above, one can construct RE-triangulations of Γ_δ containing $\mathcal{R}_H$, for all $\delta \geq 2$.

• Position $3I$: Consider the tetrahedron Δ with vertices $(0,0,0)$, $(0,0,1)$, $(1,0,\delta-1)$ and $(\delta-1,1,0)$, shown to the left in Figure 16 for $\delta = 4$. Clearly, Δ is a complex of type $3I$. In the proof of [7, Theorem 9.2] we showed that if $\Delta \in \mathrm{Subdiv}_X$, then X has

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a tropical line with combinatorial position $3I$. Moreover, we showed that for all $\delta \in \mathbb{N}$ there exists an RE-triangulation of Γ_δ which contains Δ. $\qquad\square$

This concludes the proof of Theorem 5.6. $\qquad\square$

We conclude this section with a result important for the next section.

Proposition 5.11. *Let X be a smooth tropical surface of degree at least 3.*

  a) If X contains a tropical line with combinatorial position $3I$, then X contains a two-point family of tropical lines.

  b) Suppose $L \subseteq X$ has general combinatorial position other than $3I$. If $L' \subseteq X$ is any tropical line, we have

$$c_X^\vee(L') = c_X^\vee(L) \implies L' = L.$$

Alternatively, b) can be formulated as follows: Let $\mathcal{R}$ be a subcomplex of Subdiv_X of type either $3A, 3B, 3C, 3D, 3E, 3F, 3G$ or $3H$. Then there is either none or exactly one tropical line on X with line subcomplex $\mathcal{R}$.

Proof. a) This follows from the convexity of the cells of X: Suppose $L \subseteq X$ has combinatorial position $3I$, with vertices v_1 and v_2, where $C := c_X(v_2)$ has dimension 2. If $\vec{u}$ is the vector from v_1 to v_2, then convexity of C implies that the tropical line L_t with vertices v_1 and $v_t := v_1 + t\vec{u}$ lies on X for all $t \geq 0$. Let V is the common vertex of L and X, and let C be the cell of X holding the other vertex of L.

b) This is a consequence of [7, Proposition 8.3], which states that if $\deg X \geq 3$, then any $L \subseteq X$ not belonging to a two-point family on X is uniquely determined by its *set of data*, $\mathcal{D}_X(L)$, introduced in [7]. The main difference between $c_X^\vee(L)$ and $\mathcal{D}_X(L)$ is that the latter includes the combinatorial type of L. However, one can check that if $L \subseteq X$ has any general combinatorial position (and $\deg X \geq 3$), then its combinatorial type - and thus $\mathcal{D}_X(L)$ - is uniquely determined by $c_X^\vee(L)$ and X.

To prove the lemma, we argue as follows: Let L be as in the statement, and suppose $c_X^\vee(L') = c_X^\vee(L)$ for some $L' \subseteq X$. This implies that $\mathcal{D}_X(L) = \mathcal{D}_X(L')$, and thus, by [7, Proposition 8.3], that either $L = L'$, or that L belong to a two-point family on X. It is straightforward to check that the latter possibility cannot happen (see the proof [7, Proposition 8.3] for details). $\qquad\square$

6 Counting tropical lines on tropical surfaces

The classification in the previous section can be used to count tropical lines on smooth tropical surfaces. More precisely, let X be any smooth tropical surface of degree ≥ 3. First, check whether Subdiv_X contains a subcomplex of type $3I$, i.e., a tetrahedron with four exits. If it does, then by Proposition 5.11a) X contains infinitely many tropical lines.

Suppose Subdiv_X contains no subcomplexes of type $3I$. Then Proposition 5.11b) implies that for any general combinatorial position $\mathfrak{P}$, there is an injection of sets

$$\{\text{tropical lines on } X \text{ with combinatorial position } \mathfrak{P}\}$$
$$\downarrow c_X^\vee$$
$$\{\text{subcomplexes of } \mathrm{Subdiv}_X \text{ of type } \mathfrak{P}\}.$$

Since on a general smooth X, every tropical line has general combinatorial position (Theorem 5.6), we have:

Proposition 6.1. *Let $\mathcal{S}$ be an RE-triangulation of Γ_δ without subcomplexes of type $3I$, where $\delta \geq 3$. If X is a general smooth tropical surface with subdivision $\mathcal{S}$, then*

$$\sharp\{\text{tropical lines on } X\} \leq \sharp\{\text{subcomplexes of } \mathcal{S} \text{ of general type}\}.$$

Remark 6.2. Proposition 6.1 gives an computationally accessible upper bound for the number of tropical lines on a general tropical surface with given subdivision. Namely, if $\mathcal{S}$ is a subdivision of Γ_δ, its subcomplexes of general type can be found in the following easily programmable way: For each type, identify all subcomplexes in $\mathcal{S}$ with the cell structure associated to that type, as given in Table 3. Thereafter, check which of these satisfy the associated conditions (in the rightmost column of Table 3).

The upper bound given in Proposition 6.1 is not sharp in general. However, in concrete examples, it is often fairly easy to improve the inequality, or even find the exact number of tropical lines. We give a detailed example of this in Section 7, where we analyze the subdivision $\mathcal{S}_{\alpha,3}$. However, we first look at tropical surfaces without any tropical lines.

6.1   Tropical surfaces with no tropical lines

In classical geometry, it is well known that a general smooth algebraic surface of degree higher than 3 in projective three-space contains no lines. (See [6, p. 28] for an early reference.)

As shown in [7], this statement fails to hold for tropical surfaces. To restate the result precisely, recall our notion of generality for smooth tropical surfaces of degree δ: For a general such surface to have a certain property, we require that for each RE-triangulation $\mathcal{S}$ of Γ_δ there is an open dense subset $U \subseteq K(\mathcal{S})$ such that for all $\alpha \in U$, X_α has the property. In [7, Theorem 9.2] we showed that

Theorem 6.3. *For any $\delta \in \mathbb{N}$ there exists an RE-triangulation $\mathcal{S}$ of Γ_δ such that X_α contains infinitely many tropical lines for all $\alpha \in K(\mathcal{S})$.*

We will now prove a theorem to the converse effect: There exist RE-triangulations of Γ_δ for arbitrary $\delta \geq 4$, for which a general surface contains no tropical lines.

Recall the RE-triangulation $\mathcal{S}_{\alpha,\delta}$ of Γ_δ, defined in Section 2.3.1 as the subdivision induced by the lifting function $\alpha(a,b,c) = -2a^2 - 2b^2 - 2c^2 - ab - 2ac - 2bc$. (Figure 17 shows $\mathcal{S}_{\alpha,4}$ and one of its associated tropical surfaces.)
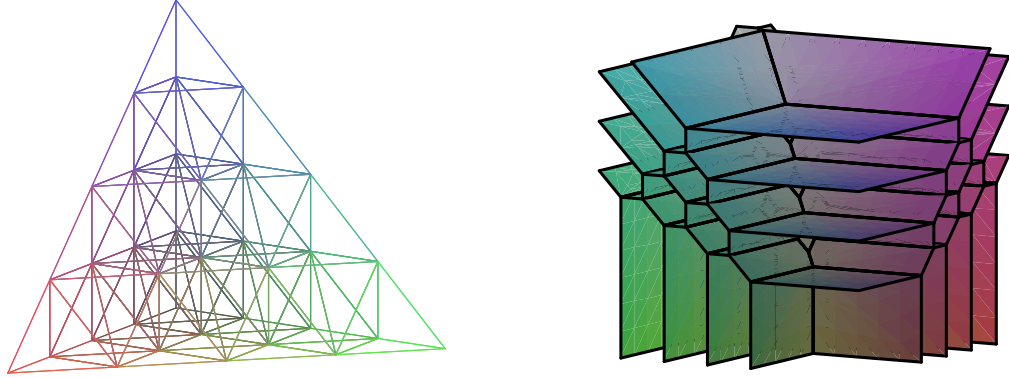
Figure 17: A quartic subdivision (left) whose general tropical surface (right) has no tropical lines.

Proposition 6.4. *Let* $\delta \geq 4$. *A general tropical surface with subdivision* $\mathcal{S}_{\alpha,\delta}$ *contains no tropical lines.*

Proof. Let X be a general tropical surface with subdivision $\mathcal{S}_{\alpha,\delta}$. Since we assume $\deg X \geq 4$, Theorem 5.6 guarantees that any tropical line on X has combinatorial position either $3G, 3H$ or $3I$. Hence to prove the proposition it is enough to show that $\mathcal{S}_{\alpha,\delta}$ has no subcomplexes of types $3G, 3H, 3I$. Combining the description of the maximal elements of $\mathcal{S}_{\alpha,\delta}$ (Section 2.3.1) with the information given in Table 3, this is a simple exercise. $\square$

7 Tropical lines on smooth tropical cubic surfaces.

We start this section by giving two conjectures concerning tropical lines on smooth tropical cubic (i.e. degree 3) surfaces. Subsequently, we examine one specific subdivision, and show that the first conjecture hold for the tropical cubics associated to this subdivision.

Conjecture 1. *A general smooth tropical cubic surface contains exactly 27 tropical lines.*

Extending to all smooth tropical cubics, we conjecture the following:

Conjecture 2. *For a smooth tropical surface* X, *let* f *be the number of two-point families on* X, *and* i *the number of tropical lines on* X *not part of any two-point family on* X. *Then we have*

$$f + i = 27.$$

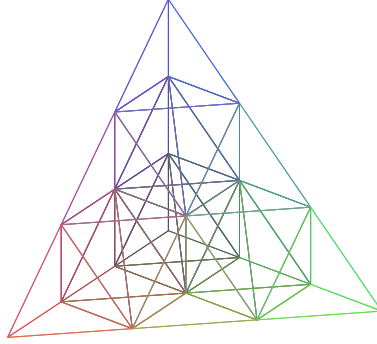*In particular,* X *contains either 27 or infinitely many tropical lines.*

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Figure 18: The subdivision $\mathcal{S}_{\alpha,3}$, invariant under involutions (12), (34) and (13)(24).

7.1   An example

In this section we will analyze the RE-triangulation $\mathcal{S}_{\alpha,3}$ (shown in Figure 18), which was defined in Section 2.3.1. The aim is to prove the following theorem:

Theorem 7.1.

   a) A general tropical surface with subdivision $\mathcal{S}_{\alpha,3}$ contains exactly 27 tropical lines.

   b) Any tropical surface with subdivision $\mathcal{S}_{\alpha,3}$ contains at least 27 tropical lines.

   c) There exist tropical surfaces with subdivision $\mathcal{S}_{\alpha,3}$ containing infinitely many tropical lines.

   We will show this through a series of lemmas, looking at how many lines X has in the different general combinatorial positions. We will frequently use that $\mathcal{S}_{\alpha,3}$ is invariant under the subgroup $G \subseteq S_4$ generated by the three involutions (12), (34) and (13)(24). In particular, $|G| = 8$.

   Some local notation used in this section: The elements of $\mathcal{A}_3$ will be denoted $A_{000}, A_{100}, \ldots, A_{003}$, where the indices indicates the coordinates of the lattice points. Furthermore, X is assumed to be a tropical surface with subdivision $\mathcal{S}_{\alpha,3}$. Thus X corresponds to a point $(\lambda_{000}, \lambda_{100}, \ldots, \lambda_{003})$ in the secondary cone $K(\mathcal{S}_{\alpha,3})$, where the ordering is chosen such that λ_{ijk} is the lifting value of A_{ijk}. In other words, $X = V_{tr}(f)$, where

$$f(x_1, x_2, x_3) = \text{``} \sum_{(i,j,k) \in \mathcal{A}_3} \lambda_{ijk} x_1^i x_2^j x_3^k \text{''}.$$

Lemma 7.2. *X has no tropical lines in either of the combinatorial positions 3C, 3G, 3H or 3I.*

Proof. It is enough to observe that $\mathcal{S}_{\alpha,3}$ has no subcomplexes of types 3C, 3G, 3H or 3I. This is a straight-forward (although somewhat tedious if done by hand) check, using the cell structures given in Table 3. $\qquad\square$

Lemma 7.3.

   a) A general X has exactly 12 tropical lines with combinatorial position 3A or 3D.

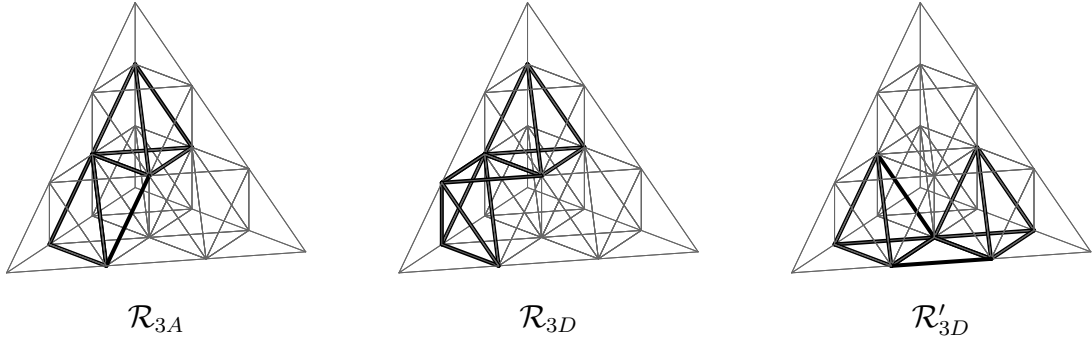$$\mathcal{R}_{3A} \qquad\qquad \mathcal{R}_{3D} \qquad\qquad \mathcal{R}'_{3D}$$

Figure 19: Subcomplexes in $\mathcal{S}_{\alpha,3}$ of types $3A$ and $3D$

b) *Exactly 4 of the tropical lines in a) specialize to a two-point family.*

c) *Any X has exactly 12 tropical lines which deforms into combinatorial position $3A$ or $3D$. Neither of these deforms into any other general combinatorial position.*

Proof. a) Consider the three subcomplexes $\mathcal{R}_{3A}, \mathcal{R}_{3D}, \mathcal{R}'_{3D} \subseteq \mathcal{S}_{\alpha,3}$ shown in Figure 19. In $\mathcal{S}_{\alpha,3}$ we find a total of eight subcomplexes of type $3A$; these are all equivalent modulo G to $\mathcal{R}_{3A}$. Furthermore, there are 12 subcomplexes of type $3D$. Of these, eight are equivalent to $\mathcal{R}_{3D}$, while the remaining four are equivalent to $\mathcal{R}'_{3D}$.

Let $h_1 = \lambda_{210} + \lambda_{002} - \lambda_{201} - \lambda_{011}$, and $h_2 = 2\lambda_{210} - 2\lambda_{120} + \lambda_{020} - \lambda_{200}$. We claim that:

i) $h_1 > 0 \iff \mathcal{R}_{3A}$ is uniquely realized on X as a tropical line with comb. pos. $3A$,

ii) $h_1 < 0 \iff \mathcal{R}_{3D}$ is uniquely realized on X as a tropical line with comb. pos. $3D$,

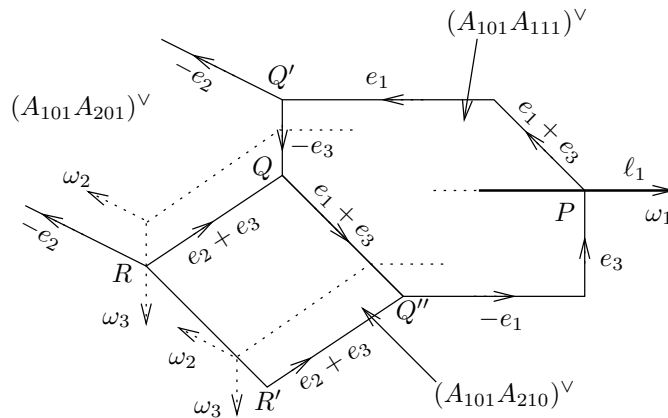iii) $h_2 \neq 0 \iff \mathcal{R}'_{3D}$ is uniquely realized on X as a tropical line with comb. pos. $3D$.

Figure 20: If the ray ℓ_1 meets the interior of the segment QQ'' (resp. the interior of QQ'), it can be extended uniquely to a tropical line on X with combinatorial position $3A$ (resp. $3D$).
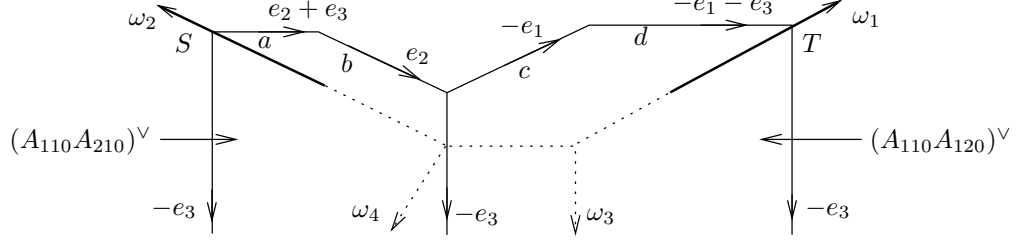
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Figure 21: A tropical line whose line subcomplex is $\mathcal{R}'_{3D}$.

To prove claims i) and ii) we sketch the 2-cells of X dual to the three edges $A_{101}A_{111}$, $A_{101}A_{201}$ and $A_{101}A_{210}$ (see Figure 20). In the figure, P and Q are the points dual to the tetrahedra $A_{002}A_{101}A_{111}A_{011}$ and $A_{101}A_{111}A_{201}A_{210}$ respectively. By Lemma 4.2, X contains a line segment trespassing through P, with direction vector ω_1. This segment can be extended uniquely to a ray $\ell_1 \subseteq X$, starting somewhere on the polygonal arc $Q'QQ''$.

A calculation shows that the coordinates of P and Q are

$$P = (\lambda_{011} - \lambda_{111}, \lambda_{101} - \lambda_{111}, \lambda_{101} - \lambda_{111} + \lambda_{011} - \lambda_{002})$$
$$Q = (\lambda_{101} - \lambda_{201}, \lambda_{101} - \lambda_{111}, \lambda_{101} - \lambda_{201} + \lambda_{210} - \lambda_{111}).$$

In particular, $Q_3 - P_3 = h_1$. Suppose first that $h_1 > 0$. Then $Q_3 > P_3$, so ℓ_1 starts in the interior of QQ''. Observe that $(QQ'')^\vee = A_{101}A_{111}A_{210}$ has an exit in the direction ω_4, and that $(RR')^\vee = A_{101}A_{200}A_{210}$ has exits in both directions ω_2 and ω_3. Hence it is evident from Figure 20 that if $h_1 > 0$, then ℓ_1 can be extended uniquely to a tropical line $L \subseteq X$ of combinatorial type $((23)(14))$, with one vertex in each of $\mathrm{int}(QQ'')$ and $\mathrm{int}(RR')$. Clearly, L has combinatorial position $3A$, and $c_X^\vee(L) = \mathcal{R}_{3A}$, so claim i) is proved.

Similarly, if $h_1 < 0$, then ℓ_1 starts in $\mathrm{int}(QQ')$. From the facts that $(QQ')^\vee = A_{101}A_{111}A_{201}$ has an exit in the direction ω_4, and that the vertex R allows a trespassing ray with direction ω_3 (cf. Lemma 4.2), we see that ℓ_1 can be extended uniquely to a tropical line on X of combinatorial type $((23)(14))$, with one vertex in $\mathrm{int}(QQ')$ and the other in the interior of the 2-cell $(A_{101}A_{201})^\vee$. The combinatorial position of this line is $3D$, and the associated line subcomplex in Subdiv_X is precisely $\mathcal{R}_{3D}$. Thus claim ii) is proved.

For claim iii) we refer to Figure 21, showing the 2-cells dual to the edges $A_{110}A_{210}$ and $A_{110}A_{120}$. If the side lengths $a + b \neq c + d$ then X contains a unique tropical line L containing the vertices $S, T \in X$: If $a + b < c + d$, as in Figure 21, then L has combinatorial type $((13)(24))$ and one vertex in $\mathrm{int}((A_{110}A_{120})^\vee)$. If $a + b > c + d$ then L has combinatorial type $((14)(23))$ and one vertex in $\mathrm{int}((A_{110}A_{210})^\vee)$. In both cases, L has one vertex on the edge $(A_{110}A_{210}A_{120})^\vee)$ joining the two 2-cells. Clearly, L has combinatorial position $3D$, and $c_X^\vee(L) = \mathcal{R}'_{3D}$. Furthermore, calculating the vertex coordinates, one finds that $a + b - c - d = h_2$. This proves claim iii).

Observe that claim i) remains valid if we exchange h_1 and $\mathcal{R}_{3A}$ by $\sigma(h_1)$ and $\sigma(\mathcal{R}_{3A})$, where σ is any element of $G \subseteq S_4$, and similarly for the claims ii) and iii). From this we conclude two things. Firstly, if α lies away from the hyperplanes given by $\sigma(h_1) = 0$, for

all $\sigma \in G$, then the 16 subcomplexes in the orbits of $\mathcal{R}_{3A}$ and $\mathcal{R}_{3D}$ give rise to exactly 8 tropical lines on X_α. Secondly, if α lies away from the hyperplanes $\sigma(h_2) = 0$, for all $\sigma \in G$, then the four subcomplexes in the orbit if $\mathcal{R}'_{3D}$ give rise to exactly four tropical lines on X_α. Hence, a general X with subdivision $\mathcal{S}_{\alpha,3}$ has exactly $8 + 4 = 12$ tropical lines with combinatorial position either $3A$ or $3D$.

b) Let us first analyze the cases $h_1 = 0$ and $h_2 = 0$. If $h_1 = 0$, then X contains the tropical line with vertices Q and R (see Figure 20). It has non-general combinatorial position, and it does not belong to any two-point family on X.

Next, suppose $h_2 = 0$. In this case $a + b = c + d$ (cf. Figure 21), and the lines through S and T with direction vectors e_2 and e_1 respectively, meet in the point $v := S + (0, a + b, 0) = T + (c + d, 0, 0)$ on the 1-cell dual to the triangle $A_{210}A_{120}A_{110}$. Since this triangle has exits in both directions ω_3 and ω_4, it follows that X contains the degenerate tropical line with vertex v. In fact, it is easy to see that for all $t \geq 0$, the tropical line with vertices v and $v + t(e_1 + e_2)$ lies on X. Hence X contains the complete two-point family of tropical lines passing through S and T.

Now for the specializations. As seen in a) the 12 tropical lines in question come in two groups, 8 associated to $\mathcal{R}_{3A}$ or $\mathcal{R}_{3D}$, and 4 associated to $\mathcal{R}'_{3D}$. Suppose X is general, and that $L \subseteq X$ is in the first group. We can assume that $c_X^\vee(L)$ equals either $\mathcal{R}_{3A}$ or $\mathcal{R}_{3D}$. Any perturbation of X which keeps $h_1 \neq 0$, induces a deformation of $L \subseteq X$ that preserves the combinatorial position of L. Hence to obtain a specialization of L, we must let $h_1 \to 0$. As observed above, this results in a specialization of L to an isolated tropical line.

Next, let (on a general X) L be in the last group, i.e., we can assume L to be the realization of $\mathcal{R}'_{3D}$. Choose any perturbation of X such that $h_2 \to 0$. As shown above, this will induce a specialization of L to a degenerate tropical line which belongs to a two-point family on X.

c) On general X, the 12 tropical lines are of course those found in a); these clearly satisfy the requirements. If X is non-general, then either $\sigma(h_1) = 0$ or $\sigma(h_2) = 0$ for some $\sigma \in G$. Suppose the former. It is enough to consider the case $h_1 = 0$, in which X contains the tropical line L_0 with vertices Q and R. As seen in b), L_0 is the unique specialization of any realization of $\mathcal{R}_{3A}$ or $\mathcal{R}_{3D}$. In particular, it deforms into combinatorial positions $3A$ and $3D$.

We claim that L_0 cannot be deformed into any combinatorial position other than $3A$ and $3D$. Let $X_0 := X$, and consider any deformation $t \mapsto (L_t, X_t)$ of $L_0 \subseteq X_0$ into some general combinatorial position $\mathcal{C}$. For each t, let $P_t \in X_t$ be the vertex corresponding to $P \in X_0$. Then we know, by Lemma 4.12, that the ω_1-ray of L_t is trespassing through P_t for each t. But this, together with the assumption that the combinatorial position $\mathcal{C}$ of L_t is general, implies that $\mathcal{C}$ equals either $3A$ or $3D$. (This follows from our discussion in a), in particular Figure 20.)

Finally, suppose $\sigma(h_2) = 0$; as before it is enough to consider the case $h_2 = 0$. Then X contains the two-point family of tropical lines passing through S and T (cf. Figure 21). Let L_{deg} be the degenerate member of this two-point family. Clearly, L_{deg} deforms into combinatorial position $3D$ (it is the unique specialization of any realization of $\mathcal{R}'_{3D}$), and it is easy to see that it does not deform into any other general combinatorial
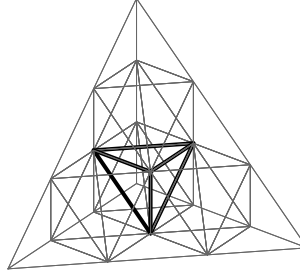
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Figure 22: The tetrahedron T common to all subcomplexes of type $3B$.

position. As for the non-degenerate tropical lines in the two-point family, none of them have general combinatorial position, nor can any of them be deformed into any general combinatorial position.

We conclude that the 12 tropical lines found in a) all have unique (and distinct) specializations, which satisfy the requirements given in the lemma. This completes the proof. $\square$

Lemma 7.4.

  a) *A general X has exactly 3 tropical lines with combinatorial position $3B$.*

  b) *Each of the tropical lines in a) specialize to a two-point family.*

  c) *Any X has exactly 3 tropical lines which deforms into combinatorial position $3B$. Neither of these deforms into any other general combinatorial position.*

Proof. a) There are 12 subcomplexes of type $3B$ in $\mathcal{S}_{\alpha,3}$. All of these contain the tetrahedron T, shown in Figure 22, with vertices A_{110}, A_{101}, A_{011} and A_{111}. Using Lemma 4.2 we see that the dual vertex $T^\vee \in X$ allows trespassing line segments in three directions simultaneously: $e_1 + e_2$, $e_1 + e_3$ and $e_2 + e_3$. Drawing the shapes of the 2-cells adjacent to $T^\vee$, one sees immediately that each of these trespassing line segments can be extended to a tropical line on X. For general X, each extension is unique on X, and the three resulting tropical lines all have combinatorial position $3B$.

b) Non-generality in this case means that at least one of the three trespassing line segments in a) meets a second vertex of X. One can check that this always allows for a second trespassing, resulting in a two-point family on X. It is even possible for the lines segment to meet a third vertex of X, giving rise to a 2-dimensional two-point family on X. Any of the tropical lines in a) specializes to both a 1-dimensional and 2-dimensional family obtained in this way.

c) For any of the two-point families described in b), none of its members has general combinatorial position. Using arguments similar to those in the proof of Lemma 7.3, it is not hard to show that there is exactly one tropical line in the family that can be deformed into some general combinatorial position, which must be $3B$. The truth of the statement follows from this. $\square$
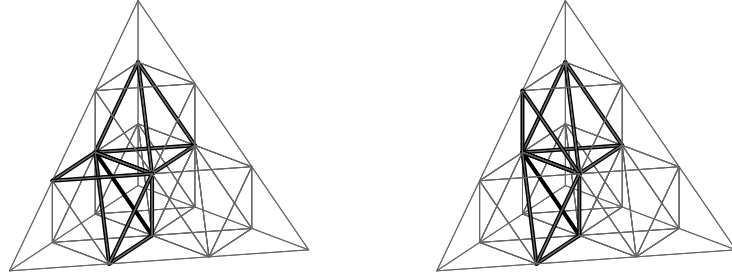
Figure 23: Subcomplexes $\mathcal{R}_{3E}$ (to the left) and $\mathcal{R}'_{3E}$ (to the right).

Lemma 7.5.

    a) A general X has exactly 4 tropical lines with combinatorial position $3E$.

    b) Each of the tropical lines in b) specialize to a two-point family.

    c) Any X has exactly 4 tropical lines which deforms into combinatorial position $3E$. Neither of these deforms into any other general combinatorial position.

Proof. There are 8 subcomplexes of type $3E$ in $\mathcal{S}_{\alpha,3}$, all equivalent modulo G. These 8 can be divided into 4 pairs, such that the subcomplexes in each pair contain the same tetrahedra. One of these pairs, $\mathcal{R}_{3E}$ and $\mathcal{R}'_{3E}$, is shown in Figure 23.

We claim that a general X contains exactly one tropical line L with line subcomplex either $\mathcal{R}_{3E}$ or $\mathcal{R}'_{3E}$. To prove this, we refer to Figure 24, which shows the 2-cell dual to $A_{111}A_{101}$. The cell is a parallel hexagon whose edge directions are given in the figure. The vertices P and Q are the duals of the tetrahedra $A_{111}A_{101}A_{110}A_{210}$ and $A_{111}A_{101}A_{011}A_{001}$, allowing (by Lemma 4.2) trespassing in directions ω_3 and ω_1 respectively. Let L be the tropical line with vertices $v_1 = P + (a, 0, 0)$ and $v_2 = v_1 + (\min(b,c), 0, \min(b,c))$. Observe that v_2 lies either on the edge RR' (if $c \leq b$) or on the edge RR'' (if $b \leq c$). Hence, since both $(RR')^\vee = A_{111}A_{101}A_{201}$ and $(RR'')^\vee = A_{111}A_{101}A_{102}$ has exits in directions ω_2 and ω_4, this ensures that $L \subseteq X$. For a general X, we can assume $b \neq c$. If $c < b$ (as shown in Figure 24), we have $v_2 \in \text{int}(RR')$, giving $c_X^\vee(L) = \mathcal{R}_{3E}$. If $b < c$, then $v_2 \in \text{int}(RR'')$, and $c_X^\vee(L) = \mathcal{R}'_{3E}$. In either case it is clear that L is the only tropical line on X passing through P and Q. This proves the claim.

Finally, the same argument applies to the three other pairs of subcomplexes, giving a total of 4 lines with combinatorial position $3E$.

Parts b) and c) are proved in a similar fashion as in the corresponding parts of Lemma 7.3. $\qquad\square$

Lemma 7.6. *Any X has exactly 8 tropical lines with combinatorial position $3F$. Neither of these specialize into any other combinatorial position.*

Proof. Modulo G, the only subcomplexes of type $3F$ in $\mathcal{S}_{\alpha,3}$ are $\mathcal{R}_{3F}$ and $\mathcal{R}'_{3F}$, shown in Figure 25. Both have orbits of length 4 under the action of G.

It is not hard to see that any X contains exactly one tropical line with line subcomplex $\mathcal{R}_{3F}$. Indeed, Figure 26 shows how to construct such a tropical line. For
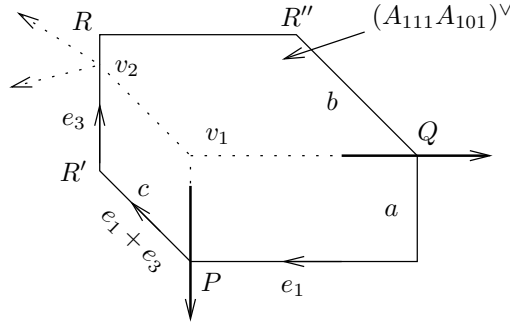
Figure 24: A tropical line on X with combinatorial position $3E$.

uniqueness we can e.g. apply Lemma 3.5: Denoting the side lengths by a, b, c, d, as indicated, we find that $\vec{PR} = (0, a, a) + (0, b, 0) + (-c, 0, 0) + (0, d, d) = (-c, a+b+d, a+d)$. Since a, b, c, d are strictly positive, the lemma implies that there is a unique tropical line through P and R, and, *a fortiori*, that there is a unique line on X with associated line subcomplex $\mathcal{R}_{3F}$.

The same argument applies to the subcomplexes in the orbit of $\mathcal{R}_{3F}$. Similarly, by studying the 2-cells dual to $A_{111}A_{102}$ and $A_{101}A_{201}$, one can show that X always contains exactly one tropical line with $\mathcal{R}'_{3F}$ as its line subcomplex. Hence we have a total of 8 tropical lines with combinatorial position $3F$. □

The lemmas 7.2 through 7.6 provide everything needed to prove Theorem 7.1.

Proof of Theorem 7.1. a) To sum up, we have on a general X, 12 tropical lines with combinatorial position $3A$ or $3D$, 3 with $3B$, 4 with $3E$, 8 with $3F$ and none with $3C$, $3G$, $3H$ or $3I$. No tropical line can have more than one combinatorial position on X, hence the total number of lines is exactly $12 + 3 + 4 + 8 = 27$.

b) Part c) of the lemmas 7.3 through 7.5, and Lemma 7.6 identifies, on any X, four sets of tropical lines. Moreover, it follows from the same results that these four sets are mutually disjoint, and contains altogether 27 tropical lines.

c) As shown in part b) of the lemmas 7.3 through 7.5 there exist tropical surfaces X with subdivision $\mathcal{S}_{\alpha,3}$, which contains one or more two-point families of tropical lines. In particular, such X has infinitely many tropical lines. □

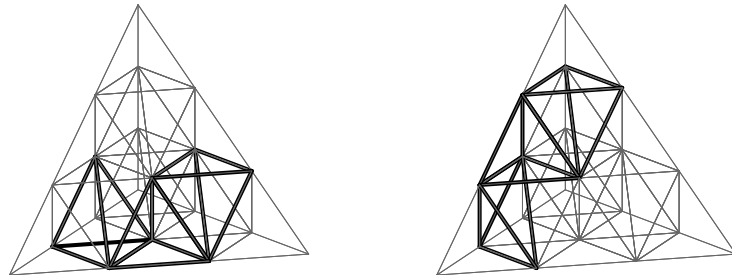Figure 25: Subcomplexes $\mathcal{R}_{3F}$ (to the left) and $\mathcal{R}'_{3F}$ (to the right).
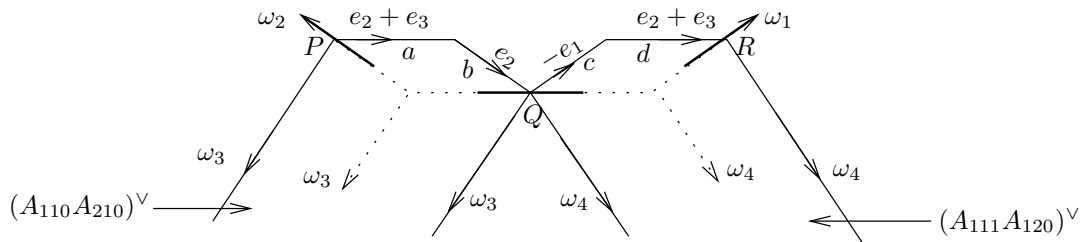
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Figure 26: A tropical line on X with combinatorial position $3F$.

7.2 Comments

In principle, Conjecture 1 could be proved by subjecting every RE-subdivision of Γ_3 (up to the action of S_4) to an analysis similar to that in the proof of Theorem 7.1. It is not known to the author how many such subdivisions exist. Using computer-randomized lifting functions we generated over 5000 RE-subdivisions of Γ_3, but the actual number is presumably a lot larger. For each subdivision, we calculated the total number of subcomplexes of types $3A, \ldots, 3I$. The resulting numbers ranged between 27 and 110. (For the subdivision $\mathcal{S}_{\alpha,3}$ examined in Section 7.1, the corresponding number is 48.)

Going through the proofs of part b) of Lemmas 7.3 - 7.6, we see a clear pattern: Let L be any of the 27 tropical lines on a general X with subdivision $\mathcal{S}_{\alpha,3}$. When we pass to a non-general surface, then L specializes to either a unique tropical line, or a unique two-point family. This is almost enough to prove Conjecture 2 for the subdivision $\mathcal{S}_{\alpha,3}$. To complete the proof, one has to show in addition that it is impossible for any X to have an isolated tropical line which is not the specialization of any of the 27 general lines. It seems probable that this could be tackled by a case study of combinatorial positions, but we leave this for future research.

Finally we are tempted to pose the following question, on whose answer we dare not speculate:

Question. *Does there exist an RE-triangulation of Γ_3, for which any associated tropical surface contains exactly 27 tropical lines?*

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Tropical complete intersection curves

Magnus Dehli Vigeland[*]

Abstract

A tropical complete intersection curve $C \subseteq \mathbb{R}^{n+1}$ is a transversal intersection of n smooth tropical hypersurfaces. We give a formula for the number of vertices of C given by the degrees of the tropical hypersurfaces. We also compute the genus of C (defined as the number of independent cycles of C) when C is smooth and connected.

1 Notation and definitions

We work over the tropical semifield $\mathbb{R}_{tr} = (\mathbb{R}, \oplus, \odot) = (\mathbb{R}, \max, +)$. A *tropical (Laurent) polynomial* in variables $x_1, \ldots, x_m$ is an expression of the form

$$(1) \qquad f = \bigoplus_{\boldsymbol{a}=(a_1,\ldots,a_m)\in\mathcal{A}} \lambda_{\boldsymbol{a}}\, x_1^{a_1} \cdots x_m^{a_m} = \max_{\boldsymbol{a}\in\mathcal{A}}\{\lambda_{\boldsymbol{a}} + a_1 x_1 + \cdots + a_m x_m\},$$

where the coefficients $\lambda_{\boldsymbol{a}}$ are real numbers, and the *support set* $\mathcal{A}$ is a finite subset of $\mathbb{Z}^m$. (In the middle expression of (1), all products and powers are tropical.) The convex hull of $\mathcal{A}$ in $\mathbb{R}^m$ is called the *Newton polytope* of f, denoted Δ_f.

Any tropical polynomial f induces a regular lattice subdivision of Δ_f in the following way: With f as in (1), let the *lifted Newton polytope* $\tilde{\Delta}_f$ be the polyhedron defined as

$$\tilde{\Delta}_f := \mathrm{conv}(\{(\boldsymbol{a}, t) \mid \boldsymbol{a} \in \mathcal{A}, t \leq \lambda_{\boldsymbol{a}}\}) \subseteq \Delta_f \times \mathbb{R} \subseteq \mathbb{R}^m \times \mathbb{R}$$

Furthermore, we define the *top complex* $\mathcal{T}_f$ to be the complex whose maximal cells are the bounded facets of $\tilde{\Delta}_f$. Projecting the cells of $\mathcal{T}_f$ to $\mathbb{R}^m$ by deleting the last coordinate gives a collection of lattice polytopes contained in Δ_f, forming a regular subdivision of Δ_f. We denote this subdivision by $\mathrm{Subdiv}(f)$.

1.1 Tropical hypersurfaces

Note that any tropical polynomial $f(x_1, \ldots, x_m)$ is a convex, piecewise linear function $f : \mathbb{R}^m \to \mathbb{R}$.

Definition 1.1. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a tropical polynomial. The *tropical hypersurface* $V_{tr}(f)$ associated to f is the non-linear locus of f.

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It is well known that for any tropical polynomial f, $V_{tr}(f)$ is a finite connected polyhedral cell complex in $\mathbb{R}^m$ of pure dimension $m-1$, some of whose cells are unbounded. Furthermore, $V_{tr}(f)$ is in a certain sense dual to $\mathrm{Subdiv}(f)$: There is a one-one correspondence between the k-cells of $V_{tr}(f)$ and the $(m-k)$-cells of $\mathrm{Subdiv}(f)$. A cell C of $V_{tr}(f)$ is unbounded if and only if its dual $C^\vee \in \mathrm{Subdiv}(f)$ is contained in the boundary of Δ_f.

Let $m \in \mathbb{N}$, and let $e_1, \ldots, e_m$ denote the standard basis of $\mathbb{R}^m$. For any $d \in \mathbb{N}_0$, we define the simplex $\Gamma_d^m := \mathrm{conv}\{0, de_1, \ldots, de_m\} \subseteq \mathbb{R}^m$, where 0 denotes the origin of $\mathbb{R}^m$. For example, Γ_3^2 is the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(3,0)$ and $(0,3)$. Note that the volume of Γ_d^m is given by

$$(2) \qquad \mathrm{vol}(\Gamma_d^m) = \frac{1}{m!}d^m.$$

Definition 1.2. A tropical hypersurface $X = V_{tr}(f) \subseteq \mathbb{R}^m$ is *smooth* if every maximal cell of $\mathrm{Subdiv}(f)$ is a simplex of volume $\frac{1}{m!}$. If in addition we have $\Delta_f = \Gamma_d^m$ for some $d \in \mathbb{N}$, we say that X is smooth of *degree* d.

1.2 Minkowski sums and mixed subdivisions

The set $\mathcal{K}^m$ of all convex sets in $\mathbb{R}^m$ has a natural structure of a semiring, as follows: If K_1 and K_2 are convex sets, we define binary operators $\oplus$ and $\odot$ by

$$(3) \qquad K_1 \oplus K_2 := \mathrm{conv}(K_1 \cup K_2)$$

$$(4) \qquad K_1 \odot K_2 := K_1 + K_2.$$

The operator $+$ in (4) is the *Minkowski sum*, defined for any two subsets $A, B \subseteq \mathbb{R}^m$ by $A + B := \{a + b \mid a \in A, b \in B\}$. The Minkowski sum of two convex sets are again convex, so (4) is well defined. Furthermore, it is easy to see that $\odot$ distributes over $\oplus$, and it follows that $\mathcal{K}^m$ is indeed a semiring.

Lemma 1.3. *Let $\mathbb{R}_{tr}[x_1, \ldots, x_m]$ be the semiring of tropical polynomials in n variables. The map $\mathbb{R}_{tr}[x_1, \ldots, x_m] \to \mathcal{K}^{m+1}$ defined by $f \mapsto \tilde{\Delta}_f$, is a homomorphism of semirings.*

Proof. This is a straightforward exercise. The key ingredients are the identities

$$\mathrm{conv}(A \cup B) = \mathrm{conv}(\mathrm{conv}(A) \cup \mathrm{conv}(B)) \quad \text{and}$$
$$\mathrm{conv}(A + B) = \mathrm{conv}(A) + \mathrm{conv}(B),$$

which hold for any (not necessarily convex) subsets $A, B \subseteq \mathbb{R}^m$. $\qquad \square$

Let $f_1, \ldots, f_n$ be tropical polynomials, and set $f := f_1 \odot \cdots \odot f_n$. As a consequence of Lemma 1.3, we find that $\mathrm{Subdiv}(f)$ is the subdivision of $\Delta_f = \Delta_{f_1} + \cdots + \Delta_{f_n}$ obtained by projecting the top complex of $\tilde{\Delta}_f = \tilde{\Delta}_{f_1} + \cdots + \tilde{\Delta}_{f_n} \subseteq \mathbb{R}^m \times \mathbb{R}$ to $\mathbb{R}^m$ by deleting the last coordinate.

For any cell $\Lambda \in \mathrm{Subdiv}(f)$, the lifted cell $\tilde{\Lambda} \in \mathcal{T}_f$ can be written uniquely as a Minkowski sum $\tilde{\Lambda} = \tilde{\Lambda}_1 + \cdots + \tilde{\Lambda}_n$, where $\tilde{\Lambda}_i \in \mathcal{T}_{f_i}$ for each i. Projecting each term to $\mathbb{R}^m$ gives a representation of Λ as a Minkowski sum $\Lambda = \Lambda_1 + \cdots + \Lambda_n$. The subdivision $\mathrm{Subdiv}(f)$, together with the associated Minkowski sum representation of each cell, is called the regular *mixed subdivision* of Δ_f induced by $f_1, \ldots, f_n$.

Remark 1.4. Note that the representation of Λ as a Minkowski sum of cells of the Subdiv(f_i)'s is not unique in general. Following [1], we call the representation obtained from the lifted Newton polytopes as described above, the *privileged representation* of Λ.

Definition 1.5. The *mixed cells* of the mixed subdivision are the cells with privileged representation $\Lambda = \Lambda_1 + \cdots + \Lambda_n$, where $\dim \Lambda_i \geq 1$ for all $i = 1, \ldots, n$.

2 Intersections of tropical hypersurfaces

In this section we go through some basic properties and definitions regarding unions and intersections of tropical hypersurfaces. Most of the material here also appear in the recent article [1].

We begin by observing that any union of tropical hypersurfaces is itself a tropical hypersurface. This follows by inductive use of the following lemma:

Lemma 2.1. *If X and Y are tropical hypersurfaces in $\mathbb{R}^m$, and f, g are tropical polynomials such that $X = V_{tr}(f)$ and $Y = V_{tr}(g)$, then $X \cup Y = V_{tr}(f \odot g)$.*

Proof. By definition, $V_{tr}(f \odot g)$ is the non-linear locus of the function $f \odot g = f + g$. Since f and g are both convex and piecewise linear, this is exactly the union of the non-linear loci of f and g respectively. $\square$

Remark 2.2. Let $U = X_1 \cup \cdots \cup X_n$, where $X_i = V_{tr}(g_i) \subseteq \mathbb{R}^m$ is a tropical hypersurface for each i. We denote by Subdiv$_U$ the mixed subdivision of $\Delta_{g_1} + \cdots + \Delta_{g_n}$ induced by $g_1, \ldots, g_n$. It follows from Lemma 2.1 and the discussion in Section 1.2 that Subdiv$_U$ is dual to U in the sense explained in Section 1.1.

Moving on to intersections, we will only consider smooth hypersurfaces. Let I be the intersection of smooth tropical hypersurfaces $X_1, \ldots, X_n \subseteq \mathbb{R}^m$, where $n \leq m$. As a first observation, notice that I is a polyhedral complex, since the X_i's are. The intersection is *proper* if $\dim(I) = m - n$.

Let C be a non-empty cell of I. Then C can be written uniquely as $C = C_1 \cap \cdots \cap C_n$, where for each i, C_i is a cell of X_i containing C in its relative interior. (The relative interior of a point must here be taken to be the point itself.)

Regarding C as a cell of the union $U = X_1 \cup \cdots \cup X_n$, we consider the dual cell $C^\vee \in$ Subdiv$_U$ (cf. Remark 2.2). From Section 1.2, we know that $C^\vee$ has a privileged representation as a Minkowski sum of cells of the subdivisions dual to the X_i's. It is not hard to see that this representation is precisely $C^\vee = C_1^\vee + \cdots + C_n^\vee$. In particular, since $\dim C_i \leq m - 1$, and therefore $\dim C_i^\vee \geq 1$, for each i, $C^\vee$ is a mixed cell of Subdiv$_U$.

Definition 2.3. With the notation as above, the intersection $X_1 \cap \cdots \cap X_n$ is *transversal along C* if

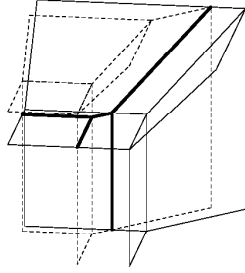$$(5) \qquad \dim C^\vee = \dim C_1^\vee + \cdots + \dim C_n^\vee.$$

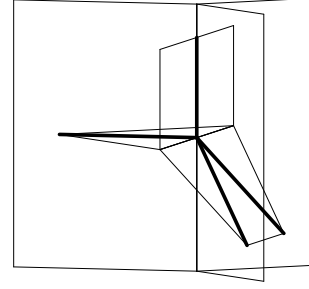Figure 1: Tropical planes intersecting in a tropical line.

Figure 2: A proper intersection which is not transversal.

More generally, the intersection $X_1 \cap \cdots \cap X_n$ is said to be *transversal* if for any subset $J \subseteq \{1, \ldots, n\}$ (of size at least two), the intersection $\bigcap_{i \in J} X_i$ is proper and transversal along each cell.

Remark 2.4. Definition 2.3 implies that if smooth tropical hypersurfaces $X_1, \ldots, X_n$ intersect transversely, then Subdiv_U is a *tight coherent mixed subdivision* (see e.g. [5]).

Recall from standard theory that the *k-skeleton* $X^{(k)}$ of a polyhedral complex X, is the subcomplex consisting of all cells of dimension less or equal to k. It is not hard to see from Definition 2.3 that if X and Y are tropical hypersurfaces intersecting transversely in $\mathbb{R}^n$, then

$$(6) \qquad\qquad X^{(j)} \cap Y^{(k)} = \emptyset$$

for all nonnegative integers j, k such that $j + k < n$. More generally, we find that:

Lemma 2.5. *Suppose $X_1, \ldots, X_n$ intersect transversally, and let $I_J = \bigcap_{i \in J} X_i$, where J is a subset of $\{1, 2, \ldots, n\}$. For each $s \notin J$ we have*

$$I_J^{(j)} \cap X_s^{(k)} = \emptyset,$$

for all j, k such that $j + k < n$.

Example 2.6. Figure 1 shows a *tropical line* in $\mathbb{R}^3$ as the transversal intersection of two tropical planes (i.e., tropical hypersurfaces of degree 1).

Example 2.7. Figure 2 shows an intersection in $\mathbb{R}^3$ which is proper, but not transversal. The surfaces are $X = V_{tr}(0x \oplus 0y \oplus 0)$ and $Y = V_{tr}(0xy \oplus 0z \oplus 0xyz)$. (Since the "spines" meet in a point, the intersection is not transversal.)

2.1 Intersection multiplicities

Let $X_1, \ldots, X_n \subseteq \mathbb{R}^m$ be smooth tropical hypersurfaces such that the intersection $I = X_1 \cap \cdots \cap X_n$ is transversal. Let $U = X_1 \cup \cdots \cup X_n$ and denote by Subdiv_U the mixed subdivision associated to U. In [1, Definition 4.3], a general formula is given for the *intersection multiplicity* at each cell of I. For our purposes, two special cases suffice. If $P \in I^{(0)}$, let $P^\vee$ be the associated dual cell in Subdiv_U.

Definition 2.8. Suppose $n = m$, so I consists of finitely many points. The intersection multiplicity at a point $P \in I$ is defined by $m_P = \mathrm{vol}(P^\vee)$.

Remark 2.9. This generalizes the standard definition of intersection multiplicities of tropical plane curves.

Definition 2.10. Suppose $n = m - 1$, so I is one-dimensional. The intersection multiplicity at a vertex $P \in I$ is defined by $m_P = 2\,\mathrm{vol}(P^\vee)$.

Remark 2.11. It follows from the definition of transversality that $P^\vee$ has a privileged representation of the form $P^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta$, where each Λ_i is a primitive lattice interval, and Δ is a primitive lattice triangle. It follows from this that $\mathrm{vol}(P^\vee)$ is always a positive multiple of $\frac{1}{2}$.

2.2   Tropical versions of Bernstein's Theorem and Bezout's Theorem

Given polytopes $\Delta_1, \ldots, \Delta_m$ in $\mathbb{R}^m$, we consider the map $\gamma : (\mathbb{R}_{\geq 0})^m \to \mathbb{R}$ defined by $(\lambda_1, \ldots, \lambda_m) \mapsto \mathrm{vol}(\lambda_1 \Delta_1 + \cdots + \lambda_m \Delta_m)$. One can show that γ is given by a homogeneous polynomial in $\lambda_1, \ldots, \lambda_m$ of degree m. We define the *mixed volume* of $\Delta_1, \ldots, \Delta_m$ to be the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_m$ in the polynomial expression for γ. The following tropical version of Bernstein's Theorem is proved in [1, Corollary 4.7]:

Theorem 2.12. *Suppose tropical hypersurfaces $X_1, \ldots, X_m \subseteq \mathbb{R}^m$ with Newton polytopes $\Delta_1, \ldots, \Delta_m$ intersect in finitely many points. Then the total number of intersection points counted with multiplicities is equal to the mixed volume of $\Delta_1, \ldots, \Delta_m$.*

As a special case of this we get a tropical version of Bezout's Theorem:

Corollary 2.13. *Suppose the tropical hypersurfaces $X_1, \ldots, X_m \subseteq \mathbb{R}^m$ have degrees $d_1, \ldots, d_m$, and intersect in finitely many points. Then the number of intersection points counting multiplicities is $d_1 \cdots d_m$.*

Proof. By Theorem 2.12, the number of intersection points, counting multiplicities, is the coefficient of $\lambda_1 \lambda_2 \cdots \lambda_m$ in

$$\mathrm{vol}(\lambda_1 \Gamma_{d_1}^m + \cdots + \lambda_m \Gamma_{d_m}^m) = \mathrm{vol}(\Gamma_{\lambda_1 d_1 + \cdots + \lambda_m d_m}^m) = \frac{1}{m!}(\lambda_1 d_1 + \cdots + \lambda_m d_m)^m.$$

By the multinomial theorem, the wanted coefficient is $d_1 \cdots d_m$, as claimed. $\qquad \square$

3   Tropical complete intersection curves

A *tropical complete intersection curve* $\mathcal{C}$ is a transversal intersection of n smooth tropical hypersurfaces $X_1, \ldots, X_n \subseteq \mathbb{R}^{n+1}$, for some $n \geq 2$. It is a one-dimensional polyhedral complex, some of whose edges are unbounded. We say that $\mathcal{C}$ is *smooth* if the intersection multiplicity is 1 at each vertex (cf. Definition 2.10).

Recall that any cell C of $\mathcal{C}$ is also a cell of the tropical hypersurface $U = X_1 \cup \cdots \cup X_n$. In particular, the notation $C^\vee$ always refers to the cell of Subdiv_U dual to $C \subseteq U$.

Lemma 3.1. *Each vertex of $\mathcal{C}$ has valence 3.*

Proof. If P is a vertex of $\mathcal{C}$, then by Remark 2.11, $P^\vee$ has a privileged representation $P^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta$, where each Λ_i is a primitive interval, and Δ is a primitive lattice triangle. If E is any edge of $\mathcal{C}$ adjacent to P, then $E^\vee$ must be a mixed cell of Subdiv_U which is also a facet of $P^\vee$. This means that $E^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta'$, where Δ' is a side of Δ. Hence there are exactly 3 such adjacent edges - one for each side of Δ. $\qquad\square$

Our first goal is to calculate the number of vertices of $\mathcal{C}$. Before stating the general formula, let us discuss the easiest case as a warm up example:

3.1   Example: Complete intersections in $\mathbb{R}^3$

Let $\mathcal{C} = X \cap Y \subseteq \mathbb{R}^3$ be a tropical complete intersection curve, where $X = V_{tr}(f)$ and $Y = V_{tr}(g)$ are smooth tropical surfaces of degrees d and e respectively.

Theorem 3.2. *The number of vertices of $\mathcal{C}$, counting multiplicities, is $de(d+e)$.*

Proof. The idea is to look at all the vertices of the union $X \cup Y$, and their dual polytopes in the subdivision corresponding to $X \cup Y$. Since the intersection of X and Y is transversal, we can write the set of vertices of $X \cup Y$ as a disjoint union,

$$(7) \qquad (X \cup Y)^{(0)} = X^{(0)} \sqcup Y^{(0)} \sqcup (X \cap Y)^{(0)}.$$

Now, any element $P \in (X \cup Y)^{(0)}$ corresponds to a maximal cell $P^\vee$ in $\mathrm{Subdiv}(f \odot g)$. The privileged representation of $P^\vee$ is of one of the following forms:

- $P^\vee = (\text{3-cell of } \mathrm{Subdiv}(f)) + (\text{0-cell of } \mathrm{Subdiv}(g)) \implies P \in X^{(0)}$.

- $P^\vee = (\text{0-cell of } \mathrm{Subdiv}(f)) + (\text{3-cell of } \mathrm{Subdiv}(g)) \implies P \in Y^{(0)}$.

- $P^\vee = (\text{2-cell of } \mathrm{Subdiv}(f)) + (\text{1-cell of } \mathrm{Subdiv}(g))$ or
  $P^\vee = (\text{1-cell of } \mathrm{Subdiv}(f)) + (\text{2-cell of } \mathrm{Subdiv}(g)) \implies P \in (X \cap Y)^{(0)}$.

Hence, dualizing (7) and taking volumes, we get the relation

$$(8) \qquad \sum_{P \in (X \cup Y)^{(0)}} \mathrm{vol}(P^\vee) = \sum_{P \in X^{(0)}} \mathrm{vol}(P^\vee) + \sum_{P \in Y^{(0)}} \mathrm{vol}(P^\vee) + \sum_{P \in (X \cap Y)^{(0)}} \mathrm{vol}(P^\vee).$$

Now, if $P \in (X \cap Y)^{(0)}$, the volume of $P^\vee$ is $\frac{1}{2}m_P$ (by definition of intersection multiplicity). Hence, (8) gives

$$\mathrm{vol}(\Delta_{f \odot g}) = \mathrm{vol}(\Delta_f) + \mathrm{vol}(\Delta_g) + \sum_{P \in (X \cap Y)^{(0)}} \frac{1}{2}m_P.$$

Since $\Delta_f = \Gamma_d^3$, $\Delta_g = \Gamma_e^3$, and $\Delta_{f \odot g} = \Gamma_d^3 + \Gamma_e^3 = \Gamma_{d+e}^3$, we find that

$$\sum_{P \in (X \cap Y)^{(0)}} m_P = 2\left[\frac{(d+e)^3}{6} - \frac{d^3}{6} - \frac{e^3}{6}\right] = de(d+e).$$

$\qquad\square$

3.2   The number of vertices in the general case

In this section we prove the following generalization of Theorem 3.2:

Theorem 3.3. *Let $\mathcal{C} = X_1 \cap \cdots \cap X_n$ be a tropical complete intersection curve in $\mathbb{R}^{n+1}$, where $X_1, \ldots, X_n$ are smooth of degrees $d_1, \ldots, d_n$. The number of vertices of $\mathcal{C}$, counting multiplicities, is*

$$\sum_{P \in \mathcal{C}^{(0)}} m_P = d_1 d_2 \cdots d_n (d_1 + d_2 + \cdots + d_n).$$

To prove Theorem 3.3, we will use the same setup as in the previous section. Note that in the proof of the case $n = 3$, the relation (7) is the key giving us control over $(X \cap Y)^{(0)}$. So as an auxiliary lemma, we first state and prove a generalization of this.

To simplify the writing, we introduce the following notation: Let $[n] = \{1, 2, \ldots, n\}$. For any nonempty subset $J = \{j_1, \ldots, j_k\} \subseteq [n]$, we put

$$
(9) \qquad
\begin{aligned}
U_J &:= X_{j_i} \cup \cdots \cup X_{j_k}, \\
I_J &:= X_{j_i} \cap \cdots \cap X_{j_k}.
\end{aligned}
$$

In the special case $J = [n]$, we simply write U and I, i.e. $U := U_{[n]}$ and $I = \mathcal{C} = I_{[n]}$.

By the assumption of transversality, we have $I_J^{(0)} \cap I_K^{(0)} = \emptyset$ whenever $J, K \subseteq [n]$ are distinct nonempty subsets. Thus we can split the 0-cells of $U = X_1 \cup \cdots \cup X_n$ into a disjoint union:

$$U^{(0)} = \bigsqcup_{J \subseteq [n]} I_J^{(0)}.$$

Similarly, for any nonempty subset $J \subseteq [n]$, we get

$$(10) \qquad U_J^{(0)} = \bigsqcup_{J' \subseteq J} I_{J'}^{(0)}.$$

Lemma 3.4. *For a transversal intersection of tropical hypersurfaces $X_1, \ldots, X_n$, we have:*

$$(11) \qquad I^{(0)} \sqcup \bigsqcup_{|J|=n-1} U_J^{(0)} \sqcup \bigsqcup_{|J|=n-3} U_J^{(0)} \sqcup \cdots = U^{(0)} \sqcup \bigsqcup_{|J|=n-2} U_J^{(0)} \sqcup \bigsqcup_{|J|=n-4} U_J^{(0)} \sqcup \cdots.$$

Proof. By applying (10) to every set $U_J^{(0)}$ in (11), we see that the following expression is equivalent to (11):

$$(12) \qquad I^{(0)} \sqcup \bigsqcup_{\substack{|J|=n-1 \\ J' \subseteq J}} I_{J'}^{(0)} \sqcup \bigsqcup_{\substack{|J|=n-3 \\ J' \subseteq J}} I_{J'}^{(0)} \sqcup \cdots = \bigsqcup_{J' \subseteq [n]} I_{J'}^{(0)} \sqcup \bigsqcup_{\substack{|J|=n-2 \\ J' \subseteq J}} I_{J'}^{(0)} \sqcup \bigsqcup_{\substack{|J|=n-4 \\ J' \subseteq J}} I_{J'}^{(0)} \sqcup \cdots.$$

We claim that for each fixed subset $J' \subseteq [n]$, the set $I_{J'}^{(0)}$ appears equally many times on each side of (12). By inspection, this is true for $J' = [n]$. Assume now $|J'| = k < n$. Then for any integer s with $k \leq s \leq n$, there are exactly $\binom{n-k}{s-k}$ sets $J \subseteq [n]$ containing

J' such that $|J| = s$. Hence, the number of times $I_{J'}^{(0)}$ appears on the left side of (12) is $\binom{n-k}{n-1-k} + \binom{n-k}{n-3-k} + \cdots = \binom{n-k}{1} + \binom{n-k}{3} + \cdots = 2^{n-k-1}$, while the number of appearances on the right side is $\binom{n-k}{n-k} + \binom{n-k}{n-2-k} + \cdots = \binom{n-k}{0} + \binom{n-k}{2} + \cdots = 2^{n-k-1}$. This proves the claim, and the lemma follows. $\qquad\square$

Proof of Theorem 3.3. Suppose $\mathcal{C}$ and $X_1, \ldots, X_n$ are as in the statement of the theorem. We assume that for each i, X_i has degree d_i, so the associated Newton polytope is the simplex $\Gamma_{d_i}^{n+1}$. Let U denote the union $X_1 \cup \cdots \cup X_n$, and Subdiv_U the associated subdivision of $\Gamma_{d_1 + \cdots + d_n}^{n+1}$.

For each nonempty $J = \{j_1, \ldots, j_k\} \subseteq [n]$, let U_J and I_J be as in (9). In particular, U_J is a tropical hypersurface (set-theoretically contained in U) with an associated subdivision Subdiv_{U_J} of the simplex $\Delta_J := \Gamma_{d_{j_1} + \cdots + d_{j_k}}^{n+1}$.

Each vertex of U_J is also a vertex of U, and therefore corresponds to a maximal cell of Subdiv_U. Let S_J be the set of maximal cells of Subdiv_U corresponding to the vertices of U_J. By transversality, the elements of S_J are simply translations of the maximal cells of Subdiv_{U_J}. Hence the total volume of the cells of S_J, denoted $\text{vol}(S_J)$, is

$$\text{vol}(S_J) = \sum_{P \in U_J^{(0)}} \text{vol}(P^\vee) = \text{vol}(\Delta_J) = \frac{1}{(n+1)!}(d_{j_1} + \cdots + d_{j_k})^{n+1}.$$

Now we turn to Lemma 3.4. Dualizing (11), we find that

$$(13) \qquad \sum_{P \in I^{(0)}} \text{vol}(P^\vee) + \sum_{|J|=n-1} \text{vol}(S_J) + \cdots = \text{vol}(S) + \sum_{|J|=n-2} \text{vol}(S_J) + \cdots$$

By the definition of intersection multiplicity, the dual $P^\vee \in \text{Subdiv}_U$ of a vertex $P \in I^{(0)}$ has volume $\frac{1}{2}m_P$. It follows that

$$\sum_{P \in I^{(0)}} \frac{1}{2}m_P = \frac{1}{(n+1)!} \sum_{\{j_i, \ldots, j_k\} \subseteq [n]} (-1)^{n-k}(d_{j_1} + \cdots + d_{j_k})^{n+1},$$

which after some elementary manipulation reduces to

$$\sum_{P \in I^{(0)}} m_P = d_1 d_2 \cdots d_n (d_1 + d_2 + \cdots + d_n).$$

$\qquad\square$

3.3 The genus of tropical complete intersection curves

Definition 3.5. The *genus* $g = g(\mathcal{C})$ of a tropical complete intersection curve $\mathcal{C}$ is the first Betti number of $\mathcal{C}$, i.e., the number of independent cycles of $\mathcal{C}$.

Lemma 3.6. *For a connected tropical complete intersection curve $\mathcal{C}$, we have*

$$2g(\mathcal{C}) - 2 = v - x,$$

where v is the number of vertices, and x the numbers unbounded edges of $\mathcal{C}$.

For the proof, recall that a graph is called *3-valent* if every vertex has 3 adjacent edges. Furthermore, we apply the following terminology: A one-dimensional polyhedral complex in $\mathbb{R}^m$ with unbounded edges is regarded as a graph, where the 1-valent vertices have been removed. For example, a tropical line in $\mathbb{R}^3$ is considered a 3-valent graph with 2 vertices and 5 edges.

Proof. By Lemma 3.1, $\mathcal{C}$ is 3-valent. Since $\mathcal{C}$ is connected, it has a spanning tree T, such that $\mathcal{C} \setminus T$ consists of g edges. While T is not 3-valent, we can construct a 3-valent tree T' from T by adding unbounded edges wherever necessary. Clearly, we must add exactly $2g$ such edges. Thus if $\mathcal{C}$ has v vertices and e edges, T' has v vertices and $e + g$ edges. Since T' is 3-valent, it is easy to see (for example by induction) that the number of edges is one more that twice the number of vertices, i.e.,

$$(14) \qquad\qquad\qquad\qquad e + g - 1 = 2v.$$

On the other hand, since $\mathcal{C}$ is 3-valent, we must have $e = \frac{1}{2}(3v + x)$. Combining this with (14) gives the wanted result. $\qquad\square$

Lemma 3.7. *Let $\mathcal{C}$ be the transversal intersection of $X_1, \ldots, X_n \subseteq \mathbb{R}^{n+1}$, where each $X_i = V_{tr}(f_i)$ is a smooth tropical hypersurface of degree d_i. If $\mathcal{C}$ is smooth, the number of unbounded edges of $\mathcal{C}$ is $x = (n + 2)d_1 \cdots d_n$.*

Proof. Let $U = X_1 \cup \cdots \cup X_n$, and let Subdiv_U be the associated subdivision of the simplex $\Gamma := \Gamma_{d_1 + \cdots + d_n}^{n+1}$. The unbounded edges of $\mathcal{C}$ are then in one-one correspondence with the mixed n-cells of Subdiv_U contained in the boundary of Γ. To prove the lemma, it therefore suffices to show that there are exactly $d_1 \cdots d_n$ mixed n-cells in each of the $n + 2$ facets of Γ. We do this below for the facet Γ' with $e_1 = (1, 0, \ldots, 0)$ as an inner normal vector; the others follow similarly.

For each $i = 1, \ldots, n$ let $\mathcal{S}_i$ be the subdivision induced by $\mathrm{Subdiv}(f_i)$ on the facet of $\Gamma_{d_i}^{n+1}$ with e_1 as an inner normal vector. We can then regard $\mathcal{S}_i$ as the subdivision associated to the tropical hypersurface $X_i' := V_{tr}(f_i') \subseteq \mathbb{R}^n$, where f_i' is the tropical polynomial obtained from f_i by removing all terms containing x_1. Furthermore, X_i' is homeomorphic to the intersection $X_i \cap H$, where H is any (classical) hyperplane with equation $x_1 = k$ and $k << 0$. Note that $\deg X_i' = \deg X_i = d_i$.

Let $\mathcal{S}$ be the subdivision of Γ' induced by Subdiv_U. As above, we regard $\mathcal{S}$ as the subdivision associated to the union $X_1' \cup \cdots \cup X_n' \subseteq \mathbb{R}^n$. Thus, the (finitely many) points in the intersection $I := X_1' \cap \cdots \cap X_n'$ are precisely the duals of the mixed n-cells of $\mathcal{S}$. We know from Theorem 2.13 that the number of points in I is $d_1 \cdots d_n$ when counting with intersection multiplicities; in other words (by Definition 2.8) we have $\sum_{Q \in I} \mathrm{vol}(Q^\vee) = d_1 \cdots d_n$.

All that remains is to show that if $Q \in I$, then $\mathrm{vol}(Q^\vee) = 1$. This is where smoothness of $\mathcal{C}$ comes in: Let P be the vertex of $\mathcal{C}$ such that $Q^\vee$ is a facet of $P^\vee \in \mathrm{Subdiv}_U$. Writing (as in Remark 2.11) $P^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta$, where the Λ_i's are primitive intervals and Δ a primitive triangle, we must have $Q^\vee = \Lambda_1 + \cdots + \Lambda_{n-1} + \Delta'$, where Δ' is a side in Δ. Since $\mathrm{vol}(P^\vee) = \frac{1}{2}$ (by smoothness), it follows from this that $\mathrm{vol}(Q^\vee) = 1$. $\qquad\square$
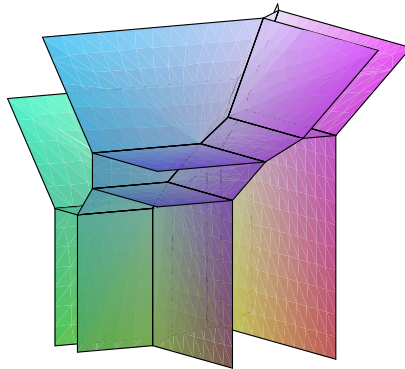
Figure 3: A tropical quadric surface in $\mathbb{R}^3$.

Theorem 3.8. *Let $\mathcal{C}$ be the transversal intersection of n smooth tropical hypersurfaces in $\mathbb{R}^{n+1}$ of degrees $d_1, \ldots, d_n$. If $\mathcal{C}$ is smooth and connected, the genus g of $\mathcal{C}$ is given by*

$$(15) \qquad 2g - 2 = d_1 \cdots d_n(d_1 + \cdots + d_n - (n+2)).$$

Proof. Since $\mathcal{C}$ is smooth, it has exactly $v = d_1 d_2 \cdots d_n(d_1 + d_2 + \cdots + d_n)$ vertices (by Theorem 3.3) and $x = (n+2)d_1 \cdots d_n$ unbounded edges (by Lemma 3.7). Combined with Lemma 3.6, this proves the theorem. $\square$

Remark 3.9. In complex projective space it is well known that any complete intersection curve is connected. This follows from standard cohomological arguments (see also [4, Section 3.4.6] for a direct geometric argument due to Serre). In the tropical setting, it is known that any transversal intersection of tropical hyperplanes is a *tropical variety*, i.e., the tropicalization of an algebraic variety defined over the field of Puiseux series ([2, Section 3, and Lemma 1.2 for the relation to Puiseux series]). Furthermore, if a tropical variety is the tropicalization of an irreducible variety, then it is connected ([3, Theorem 2.2.7]). This suggests that - at least in the general case - a tropical complete intersection curve is connected. However, to the author's knowledge, this has not been proved.

Remark 3.10. The formula (15) coincides with the genus formula for a smooth complete intersection in $\mathbb{P}^{n+1}_{\mathbb{C}}$ of n hypersurfaces of degrees $d_1, \ldots, d_n$.

4 Example: Tropical elliptic curves in $\mathbb{R}^3$

By a *tropical quadric surface* in $\mathbb{R}^3$, we mean a smooth tropical hypersurface of degree 2. In this section we take a closer look at intersections of *tropical quadric surfaces* in $\mathbb{R}^3$, i.e., smooth tropical hypersurfaces in $\mathbb{R}^3$ of degree 2. Figure 4 shows a typical tropical quadric surface.
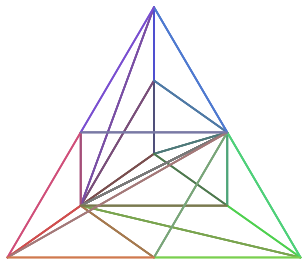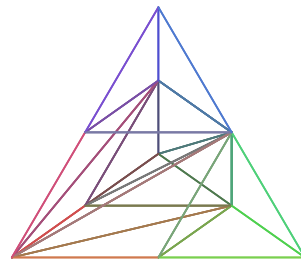
Figure 4: The subdivision Subdiv(f).

Figure 5: The subdivision Subdiv(g).

Let $\mathcal{C}$ be a smooth, connected complete intersection curve of two tropical quadric surfaces in $\mathbb{R}^3$. We call $\mathcal{C}$ a *tropical elliptic curve.* The name is justified by Theorem 3.8, which tells us the the genus g of $\mathcal{C}$ satisfies $2g - 2 = 2 \cdot 2 \cdot (2 + 2 - 4)$, that is, $g = 1$. In particular, $\mathcal{C}$ contains a unique cycle.

Since $\mathcal{C}$ is smooth, it has exactly $2 \cdot 2 \cdot (2 + 2) = 16$ vertices, by Theorem 3.3. We divide these into two categories: Those on the cycle (called *internal vertices*), and the rest (*external vertices*). Clearly, $\mathcal{C}$ has at least 3 internal vertices. But what is the maximum number of internal vertices? As the following example shows, all 16 vertices can be internal:

Example 4.1. Let $Q_1 = V_{tr}(f)$ and $Q_2 = V_{tr}(g)$, where

$$f(x, y, z) = (-6) \oplus 13x \oplus (-3)y \oplus (-4)z \oplus 10x^2 \oplus 2xy \oplus 4xz \oplus (-9)y^2 \oplus 5yz \oplus (-9)z^2,$$

and

$$g(x, y, z) = (-15) \oplus (-10)x \oplus (-4)y \oplus 2z \oplus (-7)x^2 \oplus (-2)xy$$
$$\oplus 0xz \oplus 2y^2 \oplus 15yz \oplus (-1)z^2.$$

Figures 4 and 5 show the subdivisions of Γ_2^3 induced by f and g respectively.

The intersection curve $\mathcal{C} = Q_1 \cap Q_2$ has genus 1 and 16 internal vertices. Figure 6 shows the the two quadrics intersecting. In Figure 7 we see the intersection curve alone from a different angle, clearly showing the cycle with all its 16 vertices.

Remark 4.2. A computer search shows that for every integer m, with $3 \leq m \leq 16$, there exist two tropical quadric surfaces in $\mathbb{R}^3$ intersecting transversally in a tropical elliptic curve with m internal vertices.

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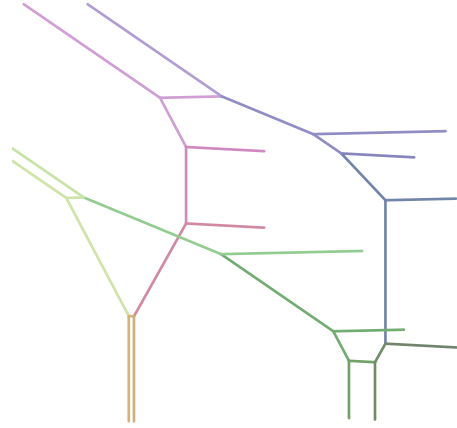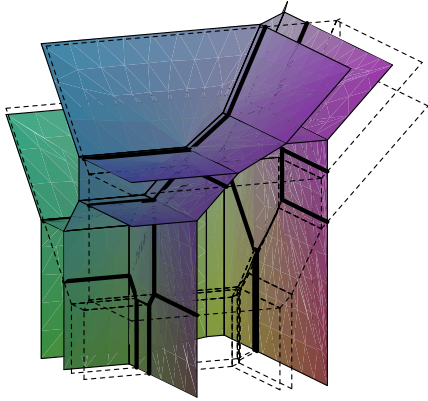
Figure 6: The intersection $\mathcal{C} = Q_1 \cap Q_2$.

Figure 7: $\mathcal{C}$ has 16 internal vertices.

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A Fano theorem in tropical geometry

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Abstract

Given a quadrangle in a projective plane over a field of characteristic other than 2, it is well known that the three diagonal points, i.e., the intersection points of opposite sides, are never collinear. We show that in the tropical plane the complete opposite is true: For any four points in the tropical plane, the three diagonal points are tropically collinear.

1   Introduction

We define a *Fano quadrangle* to be any set of four points a, b, c, d, no three on a line, such that the diagonal points $p := ab \cap cd$, $q := ac \cap bd$ and $r := ad \cap bc$ are well defined and collinear.

Fano's axiom. *There exists no Fano quadrangles.*

If K is any field of characteristic different from 2, then Fano's axiom holds in the projective plane $\mathbb{P}_K^2$. In fact, it is commonly included as an axiom for projective geometry (see e.g. [1, p. 231]).

Fano's axiom does not hold in $\mathbb{P}_K^2$ if K has characteristic 2. For example, if K is the finite field of two elements, the resulting projective plane is commonly called the *Fano plane*. It is straightforward to show that any set of four points in the Fano plane, no three on a line, is a Fano quadrangle.

In this note, we show that Fano's axiom does not hold in the tropical plane. In fact, every quadrangle in the tropical plane is a (tropical equivalent of a) Fano quadrangle.

2   The tropical projective plane

We work over the tropical semiring $(\mathbb{R}, \oplus, \odot)$, where the binary operations are given by $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$. The *tropical projective plane* $\mathbb{TP}^2$ is defined as $\mathbb{R}^3 / \sim$, where $(a, b, c) \sim (a', b', c')$ if and only if there exists $k \in \mathbb{R}$ such that $(a', b', c') = (a \odot k, b \odot k, c \odot k)$.

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Remark 2.1. Note that $\mathbb{TP}^2$ is *not* a compactification of $\mathbb{R}^2$. In fact, we do not get any new points compared to $\mathbb{R}^2$, since any point in $\mathbb{TP}^2$ has a unique representative of the form $(a, b, 0)$. However, working projectively gives a nice symmetrization of the variables, well suited to the presentation of the material in this note.

A *tropical linear form* in x, y, z is an expression of the form $f = a \odot x \oplus b \odot y \oplus c \odot z = \max\{x+a, y+b, z+c\}$, where $a, b, c \in \mathbb{TP}^2$. The *tropical line* $V_{tr}(f) \subseteq \mathbb{TP}^2$ is defined as the set of points $(x, y, z) \in \mathbb{TP}^2$ where the maximum in the expression for f is attained at least twice. In other words, we have

$$(1) \quad V_{tr}(f) = \{ (x, y, z) \in \mathbb{TP}^2 \mid x + a = y + b \geq z + c \text{ or}$$
$$y + b = z + c \geq x + a \text{ or } x + a = z + c \geq y + b\}.$$

Note that there is a duality between points and tropical lines in $\mathbb{TP}^2$, such that a point $(a, b, c) \in \mathbb{TP}^2$ corresponds to the tropical line $V_{tr}(a \odot x \oplus b \odot y \oplus c \odot z)$, and vice versa. It is easy to see (e.g. using (1)) that this is well defined.

2.1 Stable joins and intersections

Definition 2.2. The *tropical determinant* of a $n \times n$ matrix $M = (a_{ij})$ with coefficients in $\mathbb{R}$, is defined by the formula

$$|M|_t := \bigoplus_{\sigma \in S_n} a_{1\sigma(1)} \odot \cdots \odot a_{n\sigma(n)} = \max_{\sigma \in S_n}\{a_{1\sigma(1)} + \cdots + a_{n\sigma(n)}\}.$$

Here, S_n denotes the symmetric group of permutations of n elements. We say that M is *tropically singular* if, in the expression for $|M|_t$, the maximum is attained at least twice.

Given any two (not necessarily distinct) points $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ in $\mathbb{TP}^2$, we define $u \in \mathbb{TP}^2$ by

$$(2) \qquad u = (u_1, u_2, u_3) = \left(\left| \begin{smallmatrix} p_2 & p_3 \\ q_2 & q_3 \end{smallmatrix} \right|_t, \left| \begin{smallmatrix} p_1 & p_3 \\ q_1 & q_3 \end{smallmatrix} \right|_t, \left| \begin{smallmatrix} p_1 & p_2 \\ q_1 & q_2 \end{smallmatrix} \right|_t \right).$$

Definition 2.3. The *stable join* $p \vee q$ of p and q is the tropical line associated to the tropical linear form with coefficients (u_1, u_2, u_3).

For general points $p, q \in \mathbb{TP}^2$, the stable join $p \vee q$ is the unique tropical line containing them. For special choices of p and q however, there are infinitely many tropical lines passing through p and q. Among these, the stable join $p \vee q$ is the unique one such that for any generic perturbations $p_s \to p$ and $q_s \to q$ of p and q we have $p_s \vee q_s \to p \vee q$.

We define similarly the *stable intersection* of two tropical lines: If L_1 and L_2 are the tropical lines associated to the tropical linear forms with coefficients (p_1, p_2, p_3) and (q_1, q_2, q_3) respectively, their stable intersection $L_1 \wedge L_2$ is the point u defined as in (2).

Definition 2.4. Three points $a, b, c \in \mathbb{TP}^2$ are said to be *tropically collinear* if there exists a tropical line containing them.

Lemma 2.5. *Three points $a, b, c \in \mathbb{TP}^2$ are collinear if and only if the matrix with a, b, c as row vectors is tropically singular.*

Proof. See [2, Lemma 5.1] for a proof involving lifting the entries of the matrix to the field of Puiseux-series. Alternatively, one can prove Lemma 2.5 directly by simple case study. $\qquad\square$

3   A tropical Fano theorem

A collection of four points $a, b, c, d \in \mathbb{TP}^2$, not necessarily distinct, is called a *tropical Fano quadrangle* if the points

(3)
$$
\begin{aligned}
p &:= (a \vee b) \wedge (c \vee d), \\
q &:= (a \vee c) \wedge (b \vee d), \\
r &:= (a \vee d) \wedge (b \vee c)
\end{aligned}
$$

are tropically collinear.

Theorem 3.1. *Any collection of four points in $\mathbb{TP}^2$ is a tropical Fano quadrangle.*

Proof. Let $a, b, c, d \in \mathbb{TP}^2$ be arbitrary points, not necessarily distinct, and let $p = (p_1, p_2, p_3)$, $q = (q_1, q_2, q_3)$ and $r = (r_1, r_2, r_3)$ be as in (3). We must prove that the matrix

(4)
$$
M = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ r_1 & r_2 & r_3 \end{pmatrix}
$$

is tropically singular. Note that each entry of M is the maximum of 8 linear forms, each consisting of 4 terms. For instance,

$$
p_1 = \left|\begin{smallmatrix} a_1 & a_3 \\ b_1 & b_3 \end{smallmatrix}\right|_t \odot \left|\begin{smallmatrix} c_1 & c_2 \\ d_1 & d_2 \end{smallmatrix}\right|_t \oplus \left|\begin{smallmatrix} a_1 & a_2 \\ b_1 & b_2 \end{smallmatrix}\right|_t \odot \left|\begin{smallmatrix} c_1 & c_3 \\ d_1 & d_3 \end{smallmatrix}\right|_t = \max_{(i,j,k,l)\in A} \{a_i + b_j + c_k + d_l\},
$$

where A is the set $\big\{(i, j, k, l) \mid \{\{i, j\}, \{k, l\}\} = \{\{1, 2\}, \{1, 3\}\}\big\}$.

Let S_3 be the group of permutations of three letters. For each permutation $\sigma \in S_3$, let M_σ be the set of all linear forms appearing in the formal expression for the product $p_{\sigma(1)} \odot q_{\sigma(2)} \odot r_{\sigma(3)}$, after expanding as much as possible. Each set M_σ has at most $8 \cdot 8 \cdot 8 = 512$ elements, all linear forms corresponding to tropical monomials of degree 12. By construction, the value of $p_{\sigma(1)} \odot q_{\sigma(2)} \odot r_{\sigma(3)}$ is $\max(M_\sigma)$.

Now, we claim that for any element $\tau \in S_3$, we have the inclusion

(5)
$$
M_\tau \subseteq \bigcup_{\sigma \in S_3 \smallsetminus \tau} M_\sigma.
$$

The verification of this claim is straightforward, but tedious to do by hand. We used computer software to compute the sets M_σ explicitly and check (5) in each case.

The theorem now follows easily. Indeed, if the matrix M were tropically non-singular, there would be some element $\tau \in S_3$ such that

$$p_{\tau(1)} \odot q_{\tau(2)} \odot r_{\tau(3)} > p_{\sigma(1)} \odot q_{\sigma(2)} \odot r_{\sigma(3)}, \quad \forall \sigma \in S_3 \smallsetminus \tau.$$

This is equivalent to the statement

$$(6) \qquad \max(M_\tau) > \max(M_\sigma), \quad \forall \sigma \in S_3 \smallsetminus \tau.$$

But because of (5), every monomial in M_τ also appears in M_σ for some $\sigma \neq \tau$. Hence, (6) cannot be true. $\qquad\square$

Remark 3.2. L. F. Tabera found an elegant proof of Theorem 3.1, without need of computers. His idea is to lift the configuration of points to a plane over a power series field of characteristic 2. See [3] for details.

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