Apolar varieties of Schubert cycles

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Thanksgiving

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Chapter 1

Introduction

Grassmannians and their Schubert cycles are varieties which occur in many different areas in mathematics. They are fundamental objects in algebraic geometry. When studying rational varieties, Schubert cycles are frequently encountered, for instance, every rational scroll is a linear intersection of a Schubert cycle.

In this thesis, we will look for varieties which contain a given Schubert cycle. More precisely, given a Schubert cycle σ, we are looking for varieties Y such that $\sigma \subset Y$. Such a variety Y is said to be *apolar* to σ. We want the variety Y to have certain nice properties, in particular, we want σ to have codimension one in Y and we want Y to be arithmetically Cohen-Macaulay.

Every projective variety is defined by an ideal. If $R = k[x_0, \ldots, x_n]$ is a polynomial ring over an algebraically closed field k, let $Z(f)$ denote the zero locus of a homogeneous polynomial f in R. A variety V in $\mathbb{P}^n$ is defined as

$$V = Z(f_1, \ldots, f_r)$$

where $f_i \in R$ for all i, and all f_i are homogeneous. The ideal which defines a variety V will be denoted I_V. There is an order reversing correspondence between projective varieties and their ideals: If a variety V is contained in a variety W, the ideal I_V contains the ideal I_W.

When a variety V is a complete intersection, finding nice apolar varieties is not so hard: Let I_V be the ideal $(f_1, \ldots, f_{r-1}, f_r)$, and let I_W be the ideal $(f_1, \ldots, f_{r-1})$. Assume at least one of the f_i-s are non-linear, for example, let f_r have degree $d > 1$. Then W contains V, and W is arithmetically Cohen-Macaulay since any complete intersection is. Furthermore, V has codimension one in W, and the degree of W is less than the degree of V.

Now, let V be a variety defined by the ideal $I_V = (f_1, \ldots, f_r)$. Assume all f_i are irreducible. Then V is contained in all the hypersurfaces $S_i = Z(f_i)$, and these hypersurfaces are all arithmetically Cohen-Macaulay. Of course, the codimension of V in S_i is generally big, and often there exist varieties S_i' such that $V \subset S_i' \subset S_i$.

Let V be a variety of degree $d > 1$ in $\mathbb{P}^n$ and let p be a point in $\mathbb{P}^n$. Then the union

$$\bigcup_{q \in V} \langle q, p \rangle$$

is the union of all lines joining p to points on V. This union is called *the cone over V with vertex in p*, and we denote it C_pV. If the point p is not on V, the cone is a projective variety of dimension $\dim V + 1$ and degree d. If p is a smooth point on V, the cone still has dimension $\dim V + 1$, but the degree is $d - 1$. If p is a point on V with multiplicity m, and if the projection map $\pi_p : V \to \mathbb{P}^{n-1}$ is birational onto its image, the degree of C_pV is $d - m$.

We can define the cone C_LV for any linear space $L = \mathbb{P}^r$. It is done by iteration of the preceding construction; The cone C_LV is given by

$$C_LV = C_{p_{r+1}}(C_{p_r}(\cdots(C_{p_1}V)\cdots))$$

where $p_1, \ldots, p_{r+1}$ spans L.

Any cone over V will contain V, and cones are thus apolar varieties for V. As we have seen, choosing the vertex point wisely, the cones even has lower degree than Y.

In the case when the variety is a curve C of degree δ, Ciliberto and Harris have proved ([3]) that if C is a canonical curve of genus $g \geq 23$, general in moduli, then any irreducible surface of degree d, where $\delta - 1 \leq d \leq \delta$, must be a cone over C.

Following is an outline of this thesis. In the **first chapter**, we fix the notation and state some necessary results from algebra and basic results about Grassmannians. We define what we mean by a *Schubert cycle*, and we state the formulas of Pieri and Giambelli which enables us to do intersection theory on these cycles. In this thesis, every time we deal with a cycle, it is a Schubert cycle.

The **second chapter** is about the tangent spaces of the Grassmannians. If σ is a cycle on a Grassmannian G, let Y_σ be the union of the tangent spaces along σ. We define a map τ which takes a cycle σ and maps it to the cycle $Y_\sigma \cap G$. The linear span of $\tau(\sigma)$ equals the linear span of Y_σ. We give explicit formulas for the dimension of the linear span of Y_σ in the cases when σ satisfy some given conditions (propositions 2.10 and 2.11). In any given example, the method can be used to find the linear span of Y_σ for any σ.

The varieties Y_σ contains σ, and are therefore apolar varieties for Schubert cycles. In the case of Grassmannians of lines, we find the ideal of Y_σ, the dimension of Y_σ, and the codimension of σ in Y_σ. This codimension is almost always greater than one.

A detailed description of the natural habitat of the Grassmannians of lines, can be found in the **third chapter**. The Grassmannian of lines in projective n-space is a variety in the space of skew symmetric $(n+1) \times (n+1)$ matrices, and they are defined by the 4-Pfaffians of such a matrix. The matrices have properties which are dependent of whether n is odd or even, and we treat the two cases separately. We find an explicit description of the dual variety of Grassmannians of lines (theorems 3.4 and 3.5).

After establishing the language of 2-forms and skew matrices, we investigate the intersection of the Grassmannian with tangent hyperplanes, all tangent at the same point. We write the intersection as a union of Grassmannians of lines in projective 3-space. Then the intersection locus is contained in the union Y of the linear spans of the smaller Grassmannians, and Y is therefore an apolar variety for the intersection locus. Furthermore, the codimension of the intersection locus in Y is one. It turns out that all the Grassmannians in the union have a point p in common, and that these apolar varieties are cones over the intersection locus with vertex in p. A Y constructed in this way, corresponds to a cycle on the Grassmannian of 3-spaces in n-space. This is the cycle of all 3-spaces which contain the line which corresponds to the common point p.

Inspired by the promising properties of these apolar varieties, we generalize. Starting with a cycle σ on the Grassmannian of 3-spaces in a fixed n-space, let $\upsilon(\sigma)$ be the union of lines in the 3-spaces in σ. The map $\sigma \mapsto \upsilon(\sigma)$ takes Schubert cycles on the Grassmannian of 3-spaces in n-space to Schubert cycles on the Grassmannian of lines in n-space. Let Y_σ be the union of the linear spans of the Grassmannians of lines in 3-spaces in σ. Then Y_σ is an apolar variety of $\upsilon(\sigma)$. The cycle $\upsilon(\sigma)$ is defined by some linear forms and some forms of degree two, and we find explicitly the quadrics which are also in the ideal of Y_σ (proposition 3.10). When σ is a linear space, we are able to fully describe the ideal of Y_σ (theorem 3.11), and we also find the degree of Y_σ in this case (theorem 3.13). The degree is strictly less than the degree of $\upsilon(\sigma)$. Also in the case when σ is isomorphic to a Grassmannian of lines in $(n-2)$-space, we find the ideal of Y_σ explicitly (theorem 3.14). Finally, we see that all the apolar varieties constructed in this way, are cones and that they are minimal in the sense that there can be no variety Y_0 such that $\upsilon(\sigma) \subset Y_0 \subset Y_\sigma$ (propositions 3.15 and 3.16).

The thesis ends in the **fourth chapter** with a treatment of the powersum problem and Grassmannians of lines. Starting with a Grassmannian of lines, we go via a theorem by Macaulay (theorem 4.3) and get a homogeneous polynomial of a certain degree. We find this degree (proposition 4.4), and investigate how to write this form as a sum of powers of linear forms. It reduces to the following problem: How many tangent hyperplanes, all tangent at the same point, contain a fixed linear space of dimension one less than the codimension of the Grassmannian? In three examples we do specific calculations with Chern classes (examples 4.4, 4.5 and 4.6). For general n we isolate precisely what we need to answer the problem.

This thesis will mostly deal with Grassmannians of lines. Most definitions are included when needed, and repeated throughout. However, the necessary algebra is included in the following section only.

1.1 Preparatory algebra

In this section, we will include some preliminary algebra.

Definition 1.1 (Local ring, from [16]). *A ring R is a* local ring *if it is commutative and has a unique maximal ideal.*

Definition 1.2 (Height of a prime ideal and dimension of a ring, from [5],[11], [19]). *In a ring R, the* height *of a prime ideal $\mathfrak{p}$ is the supremum of all integers n such that there exists a chain*

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$$

of distinct prime ideals. The dimension *of the ring R is the supremum of the heights of all prime ideals in R. If I is a proper ideal in R, we define the* height of *I to be the minimum of the heights of the prime ideals containing I:*

$$height(I) = inf\{height(\mathfrak{p}) \mid \mathfrak{p} \supset I\}$$

The codimension *of a prime $\mathfrak{p}$ is defined to be the dimension of the local ring $R_{\mathfrak{p}}$.*

Definition 1.3 (Regular sequence and depth, from [11]). *Let R be a ring, and let M be an R-module. A sequence $r_1, \ldots, r_k$ of elements in R is called a* regular sequence *for M if r_1 is not a zero divisor in M, and for $i = 2, \ldots, k$, r_i is not a zero divisor in $M/(r_1, \ldots, r_{i-1})M$. If R is a local ring with maximal ideal $\mathfrak{m}$, then the* depth *of M is the maximum length of a regular sequence $r_1, \ldots, r_k$ for M with all $r_i \in \mathfrak{m}$. The* depth of an ideal *I in R is the maximal length of a regular sequence in I.*

Recall that a ring R is *Noetherian* if every ideal of R is finitely generated, and that this is equivalent to the *ascending chain condition on ideals in R*, which says that every strictly ascending chain of ideals must terminate (see [5] page 27). In particular, any field is Noetherian (the only ideals are 0 and the whole field). A ring R is *Artinian* if every descending chain of ideals must stabilize. That is, if

$$R \supset I_1 \supset I_2 \supset \cdots$$

is a descending chain of ideals in R, there exists an integer N such that $I_n = I_{n+1}$ for all $n \geq N$.

Theorem 1.1. *Let R be a commutative Noetherian ring, and let $S = R[x_1, \ldots, x_n]$ be a commutative ring, finitely generated over R. Then S is Noetherian. Furthermore, any homomorphic image of a Noetherian ring is Noetherian.*

Proof. The first statement is a corollary of the Hilbert basis theorem on page 186-187 in [16], and the second statement is corollary 1.3 on page 28 in [5]. □

Since the ring R itself is an R-module, the definition of regular sequence and depth applies to R. We are ready for the important definition of a Cohen-Macaulay ring:

Definition 1.4 (Cohen-Macaulay ring/variety, from [2], [5], [11]). *A local Noetherian ring R is* Cohen-Macaulay *if the depth of R equals the dimension of R. A ring R such that depth($\mathfrak{m}$) = codim($\mathfrak{m}$) for every maximal ideal $\mathfrak{m}$ of R is called a* Cohen-Macaulay *ring. By an* arithmetically Cohen-Macaulay *(abbreviated ACM) projective variety, we will mean a projective variety whose homogeneous coordinate ring is Cohen-Macaulay.*

If $(R, \mathfrak{m})$ is a local ring, and M is an R-module, the *socle* of M is defined as the annihilator in M of the maximal ideal $\mathfrak{m}$.

Definition 1.5 (Gorenstein, from [5]). *A zero-dimensional local ring $(R, \mathfrak{m})$ is* Gorenstein *if the socle of R is isomorphic to $R/\mathfrak{m}$. A local Cohen-Macaulay ring R is* Gorenstein *if there exists a non-zerodivisor $r \in R$ such that $R/(r)$ is Gorenstein. A positively graded Cohen-Macaulay ring $R = k \oplus R_1 \oplus \cdots$ is* Gorenstein *if there exists a non-zerodivisor $r \in R$ such that $R/(r)$ is Gorenstein.*

In this thesis, the ring will often be a quotient of a polynomial ring, so we treat this case in particular.

Lemma 1.2 ([19], 16D). *Let R be a Cohen-Macaulay ring. Then the polynomial ring $R[x_0, \ldots, x_n]$ is also Cohen-Macaulay. In particular, a polynomial ring over a field is Cohen-Macaulay.*

Lemma 1.3 ([19], 16F). *Let R be a Cohen-Macaulay ring and let $I = (r_1, \ldots, r_k)$ be an ideal of height k. Then R/I^n is Cohen-Macaulay for every $n > 0$.*

Let T be a polynomial ring over a field (the field will usually be the complex numbers), and let A be a quotient ring of T. If T_d denotes the d-th graded piece of T, the *socle* of A is given by

$$\operatorname{soc}(A) = (0 : T_1) = \{\overline{a} \in A \mid t \cdot \overline{a} = 0 \ \forall t \in T_1\}$$

Definition 1.6 (Gorenstein ring). *A graded Cohen-Macaulay quotient ring of a polynomial ring over a field is* Gorenstein *if its socle is generated in only one degree. By an* arithmetically Gorenstein *(abbreviated AG) projective variety, we will mean a projective variety whose coordinate ring is Gorenstein.*

Definition 1.7 (Injective object, injective resolution, from [11]). *Let A be an object in an abelian category $\mathfrak{A}$. The object A is* injective *if the functor Hom$(\cdot, A)$ is exact. An* injective resolution *of A is a complex $I^\bullet$, defined in degrees $i \geq 0$, together* *with a morphism* $\epsilon : A \to I^0$, *such that* I^i *is an injective object in* $\mathfrak{A}$ *for each* $i \geq 0$, *and such that the sequence*

$$0 \to A \xrightarrow{\epsilon} I^0 \xrightarrow{d^0} I^1 \to \cdots \to I^i \xrightarrow{d^i} I^{i+1} \to \cdots$$

is exact.

Lemma 1.4 ([5], Corollary A3.11). *If* R *is any ring, and* M *is any* R*-module, then* M *has a unique minimal injective resolution.*

Now, let M be an R-module, and let $I^\bullet$ be the minimal injective resolution. If F is a covariant left exact functor from the category of modules to another abelian category, applying F to $0 \to M \to I^\bullet$ gives a complex

$$0 \to F(M) \to F(I^0) \to F(I^1) \to F(I^2) \to \cdots$$

where only $0 \to F(M) \to F(I^0) \to F(I^1)$ is exact. The *i-th cohomology object* $h^i(I^\bullet)$ of the complex $I^\bullet$ is defined to be $\ker(d^i)/\mathrm{im}(d^{i-1})$. The *right derived functors* R^iF, $i \geq 0$, is defined to be $R^iF(M) = h^i(F(I^\bullet))$. From the definition of right derived functors, it follows that $R^0F \simeq F$.

If X is a topological space, let $\Gamma(X, \cdot)$ be the global section functor. The *cohomology functors* $H^i(X, \cdot)$ are the right derived functors of $\Gamma(X, \cdot)$. For any sheaf $\mathscr{F}$, the groups $H^i(X, \mathscr{F})$ are the *cohomology groups* of $\mathscr{F}$.

Now, let X be a variety of dimension m in $\mathbb{P}^n$ and let $\mathscr{I}_X$ be its sheaf of ideals. Let H be the hyperplane defined by a linear form h, and assume H does not contain any component of X. For a fixed natural number n_0, we can define a map

$$\phi : \mathscr{I}_X(n_0 + 1) \to \mathscr{I}_{X \cap H}(n_0 + 1)$$

Notice that the kernel of ϕ are all elements $h \cdot f$ such that f is in $\mathscr{I}_X(n_0)$. Thus we have an exact sequence

$$0 \to \mathscr{I}_X(n_0) \to \mathscr{I}_X(n_0 + 1) \to \mathscr{I}_{X \cap H}(n_0 + 1) \to 0$$

where the first map is simply multiplication by h. Applying the functor $\Gamma(X, \cdot)$ to this exact sequence, we get a long exact sequence in cohomology (see page 637-639 in [5]):

$$\begin{aligned} 0 \to &H^0(X, \mathscr{I}_X(n_0)) \to H^0(X, \mathscr{I}_X(n_0 + 1)) \to H^0(X, \mathscr{I}_{X \cap H}(n_0 + 1)) \\ \to &H^1(X, \mathscr{I}_X(n_0)) \to H^1(X, \mathscr{I}_X(n_0 + 1)) \to H^1(X, \mathscr{I}_{X \cap H}(n_0 + 1)) \to \\ &H^2(X, \mathscr{I}_X(n_0)) \to H^2(X, \mathscr{I}_X(n_0 + 1)) \to H^2(X, \mathscr{I}_{X \cap H}(n_0 + 1)) \to \cdots \end{aligned}$$

Now, consider the following lemma:

Lemma 1.5. *A projective variety X of dimension $m \geq 1$ is arithmetically Cohen-Macaulay if and only if $H^i(X, \mathscr{I}_X(j)) = 0$ for all $1 \leq i \leq m$ and for all j.*

Using this, we get the exact sequence

$$0 \to H^0(X, \mathscr{I}_X(n_0)) \to H^0(X, \mathscr{I}_X(n_0+1)) \to H^0(X, \mathscr{I}_{X\cap H}(n_0+1)) \to 0$$

which implies the isomorphism

$$H^0(X, \mathscr{I}_{X\cap H}(n_0+1)) \simeq H^0(X, \mathscr{I}_X(n_0+1))/\{h \cdot f | f \in H^0(X, \mathscr{I}_X(n_0))\}$$

Thus if I_X is generated by the elements $g_0, \ldots, g_r$, then $I_{X\cap H}$ is generated by $\overline{g_0}, \ldots, \overline{g_r}$ where $\overline{g_s} = g_s$ modulo h.

Let X and Y be two arithmetically Cohen-Macaulay varieties in $\mathbb{P}^n$, and assume that $I_X \subset I_Y$. Let H be a hyperplane in $\mathbb{P}^n$. Then it follows from the above discussion that $I_{X\cap H} \subset I_{Y\cap H}$, and by repeating this, we get that $I_{X\cap L} \subset I_{Y\cap L}$ for any general linear space $L \subset \mathbb{P}^n$.

The long exact sequence in cohomology above gives that general linear intersections of ACM varieties are ACM. In fact, if H is a hyperplane that does not contain any component of X, the dimension of $X \cap H$ is less than the dimension of X. Since all $H^i(X, \mathscr{I}_X(j))$ in the sequence is zero, $H^i(X, \mathscr{I}_{X\cap H}(j))$ are zero for all $1 \leq i \leq \dim X - 1$. But then lemma 1.5 implies that $X \cap H$ is ACM.

Theorem 1.6 ([12], theorem 3.1 and corollary 3.2). *All Grassmann varieties have homogeneous coordinate rings which are Gorenstein. Any Schubert subvariety of a Grassmannian has homogeneous coordinate ring which is Cohen-Macaulay.*

To find the tangent space to an affine variety $X \subset \mathbb{A}^n$ at a point p, we take all f in the ideal of X, expand around p and take their linear parts. The tangent space T_pX is the zero locus of these homogeneous linear forms. The *tangent cone* to X at p is obtained in the following way ([10], lecture 20): Take all $f \in I_X$, expand around p, and take their *leading terms.* The tangent cone TC_pX is defined by these leading terms. As we have defined it, the tangent cone at p is contained in the tangent space at p, since the linear parts are among the leading terms in the expansions. Note that the polynomials that cut out the tangent cone to X at p, does not necessarily generate the ideal of TC_pX. Also not that the dimension of the tangent cone at p is always the local dimension of X at p. Since the tangent cone is defined by homogeneous polynomials, there is a projective variety assigned to it, called the *projectivized tangent cone.* From now on, when dealing with tangent cones, we will mean the projectivized ones. Following [10], the tangent cone of X at p is simply the intersection $\tilde{X} \cap E$ of the strict transform of X with the exceptional divisor of the blow up of $\mathbb{A}^n$ at p.

Proposition 1.7. *Let X be an ACM projective variety in $\mathbb{P}^n$, and let p be a point on X. Let C_pX denote the cone over X with vertex in p. If the ideal of X is generated by quadrics, and the ideal of the tangent cone TC_pX is generated by the leading terms discussed above, and if the tangent cone TC_pX over X in p is ACM, then C_pX is ACM.*

Proof. Since X is ACM, the length of the resolution of the ideal I_X is equal to the codimension of X in $\mathbb{P}^n$. That is, the projective dimension $pd(X)$ is equal to $\operatorname{codim}(X)$. Furthermore, the ideal of C_pX is contained in the ideal of X, and therefore $pd(C_pX)$ is less than or equal to $pd(X)$. Let R be the homogeneous polynomial ring of $\mathbb{P}^n$, and let $\mathfrak{m} = R_1 \oplus R_2 \oplus \cdots$ be the homogeneous maximal ideal. The Auslander-Buchsbaum formula ([5], exercise 19.8) gives that $depth(\mathfrak{m}, I_{C_pX}) \geq depth(\mathfrak{m}, I_X)$. Thus there exist elements $f_1, f_2, \ldots, f_{depth(X)} \in \mathfrak{m}$ such that the dimension of $C_pX \cap Z(f_1, \ldots, f_{depth(X)})$ is one. We may assume that all f_i are linear. This implies that the cone C_pX is ACM if and only if the one dimensional $C_pX \cap Z(f_1, \ldots, f_{depth(X)})$ is ACM. Thus we may reduce to the case when X is a curve. Since the tangent cone TC_pX is ACM, the tanget cone over the curve is also generated by the leading terms.

From now on, let X be a curve. Let X_p denote the image of X after projecting from p. Blow up $\mathbb{P}^n$ in p, and let $\tilde{X}$ be the strict transform of X on $\tilde{\mathbb{P}}^n$. Let $H = \mathcal{O}_{\tilde{X}}(1)$ and let E be the exceptional divisor on $\tilde{\mathbb{P}}^n$. Then $H^i(X, \mathcal{I}_X(j)) = H^i(\tilde{X}, \mathcal{I}_{\tilde{X}}(jH))$ and $H^i(X_p, \mathcal{I}_{X_p}(j)) = H^i(\tilde{X}, \mathcal{I}_{\tilde{X}}(j(H-E)))$. Now, C_pX is a cone over X_p with vertex in a point outside the span of X_p, and they therefore have the same homogeneous ideal. Lemma 1.5 implies that it is enough to prove that $H^1(\tilde{X}, \mathcal{I}_{\tilde{X}}(j(H-E))) = 0$ for all j. Consider the exact sequence

$$0 \to \mathcal{I}_{\tilde{X}}(j(H-E)) \xrightarrow{E} \mathcal{I}_{\tilde{X}}(jH-(j-1)E) \to \mathcal{I}_{\tilde{X}\cap E}((j-1)h) \to 0$$

where h is $-E$ restricted to $\tilde{X}$. This exact sequence gives a long exact sequence in cohomology

$$\begin{aligned} 0 \to H^0(\tilde{X}, \mathcal{I}_{\tilde{X}}(j(H-E))) &\to H^0(\tilde{X}, \mathcal{I}_{\tilde{X}}(jH-(j-1)E)) \\ &\xrightarrow{\theta} H^0(\tilde{X}, \mathcal{I}_{\tilde{X}\cap E}((j-1)h)) \to H^1(\tilde{X}, \mathcal{I}_{\tilde{X}}(j(H-E))) \\ &\to H^1(\tilde{X}, \mathcal{I}_{\tilde{X}}(jH-(j-1)E)) \to H^1(\tilde{X}, \mathcal{I}_{\tilde{X}\cap E}((j-1)h)) \to \cdots \end{aligned}$$

We want to show that $H^1(\tilde{X}, \mathcal{I}_{\tilde{X}}(j(H-E))) = 0$ by showing that $H^1(\tilde{X}, \mathcal{I}_{\tilde{X}}(jH-(j-1)E)) = 0$ and that θ is surjective. We do this via the exact sequence

$$0 \to \mathcal{I}_{\tilde{X}}(\alpha H - \beta E) \xrightarrow{E} \mathcal{I}_{\tilde{X}}(\alpha H - (\beta-1)E) \to \mathcal{I}_{\tilde{X}\cap E}((\beta-1)h) \to 0$$

where $\beta \leq \alpha$. To ease the notation, let the cohomology groups be denoted $H^i(\alpha, \beta)$, $H^1(\alpha, \beta-1)$ and $H^i(\beta-1)$ respectively. We know that $H^1(\alpha, 0) = 0$ for all α. When $\alpha = \beta = 1$ the sequence becomes

$$0 \to \mathcal{I}_{\tilde{X}}(H-E) \xrightarrow{E} \mathcal{I}_{\tilde{X}}(H) \to \mathcal{I}_{\tilde{X}\cap E} \to 0$$

Since p is a point on X, $H^0(\tilde{X}, \mathscr{I}_{\tilde{X}\cap E}) = 0$, and we already know that $H^1(1,0) = 0$. Thus $H^1(1,1)$ must be zero. When $\alpha = 2$ and $\beta = 1$ the sequence becomes

$$0 \to \mathscr{I}_{\tilde{X}}(2H - E) \xrightarrow{E} \mathscr{I}_{\tilde{X}}(2H) \to \mathscr{I}_{\tilde{X}\cap E} \to 0$$

As above $H^0(\tilde{X}, \mathscr{I}_{\tilde{X}\cap E}) = 0$, and $H^1(2,0) = 0$, which forces $H^1(2,1)$ to be zero. In the case $\alpha = \beta = 2$, the sequence is

$$0 \to \mathscr{I}_{\tilde{X}}(2H - 2E) \xrightarrow{E} \mathscr{I}_{\tilde{X}}(2H - E) \to \mathscr{I}_{\tilde{X}\cap E}(h) \to 0$$

Since the ideal of the tangent cone is generated by the leading terms in the expansions around p of elements in the ideal of X, and the ideal of X is generated by quadrics, the map $\theta : H^0(2,1) \to H^0(1)$ is surjective. Furthermore, the previous case implies $H^1(2,1) = 0$. This forces $H^1(2,2)$ to be zero. Moving on, consider the case $\alpha = 3$ and $\beta = 1$. The sequence becomes

$$0 \to \mathscr{I}_{\tilde{X}}(3H - E) \xrightarrow{E} \mathscr{I}_{\tilde{X}}(3H) \to \mathscr{I}_{\tilde{X}\cap E} \to 0$$

Again, $H^0(\tilde{X}, \mathscr{I}_{\tilde{X}\cap E}) = 0$, and $H^1(3,0) = 0$, which forces $H^1(3,1)$ to be zero. When $\alpha = 3$ and $\beta = 2$, the sequence is

$$0 \to \mathscr{I}_{\tilde{X}}(3H - 2E) \xrightarrow{E} \mathscr{I}_{\tilde{X}}(3H - E) \to \mathscr{I}_{\tilde{X}\cap E}(h) \to 0$$

The ideal of X is generated by quadrics, so any cubic in the ideal is a quadric multiplied by a linear form. Thus the map $\theta : H^0(3,1) \to H^0(1)$ is surjective. Furthermore, $H^1(3,1) = 0$ from the previous case, and this forces $H^1(3,2)$ to be zero. In the case $\alpha = \beta = 3$, the sequence becomes

$$0 \to \mathscr{I}_{\tilde{X}}(3H - 3E) \xrightarrow{E} \mathscr{I}_{\tilde{X}}(3H - 2E) \to \mathscr{I}_{\tilde{X}\cap E}(2h) \to 0$$

Any quadric in the ideal of the tangent cone comes from a quadric in the ideal of X singular at p, and therefore the map $\theta : H^0(3,2) \to H^0(2)$ is surjective. Moreover, $H^1(3,2) = 0$ from the previous case, and thus $H^1(3,3) = 0$. Continuing in this way, we get that $H^1(\alpha, \alpha) = 0$ for all $\alpha \geq 0$.

Assume now that α is negative, and consider the exact sequence

$$0 \to \mathscr{I}_{\tilde{X}}(\alpha(H - E)) \xrightarrow{H-E} \mathscr{I}_{\tilde{X}}((\alpha + 1)(H - E)) \to \mathscr{I}_{\tilde{X}\cap(H-E)}((\alpha + 1)h) \to 0$$

For any negative α, $H^0(\tilde{X}, \mathscr{I}_{\tilde{X}\cap(H-E)}((\alpha+1)h)) = 0$. When $\alpha = -1$, $H^1(\tilde{X}, \mathscr{I}_{\tilde{X}}((\alpha + 1)(H - E))) = 0$, and this forces $H^1(\tilde{X}, \mathscr{I}_{\tilde{X}}(\alpha(H - E)))$ to be zero. When $\alpha = -2$, the sequence becomes

$$0 \to \mathscr{I}_{\tilde{X}}(-2(H - E)) \xrightarrow{H-E} \mathscr{I}_{\tilde{X}}(-(H - E)) \to \mathscr{I}_{\tilde{X}\cap(H-E)}(-h) \to 0$$

Since $H^1(\tilde{X}, \mathscr{I}_{\tilde{X}}(-(H - E))) = 0$ by the previous case, $H^1(\tilde{X}, \mathscr{I}_{\tilde{X}}(-2(H - E))) = 0$. Continuing in this way, we see that $H^1(\tilde{X}, \mathscr{I}_{\tilde{X}}(\alpha(H - E))) = 0$ for any negative α, too. $\square$

1.2 Fixing the notation

There are a lot of different notations for the Grassmannian, and the one used here is just one of many. The Grassmannian of $(k+1)$-dimensional subvector spaces of an $(n+1)$-dimensional vector space V is denoted

$$G(k+1, n+1) \qquad \text{or} \qquad G(k+1, V^{n+1}).$$

The vector space V will always be $\mathbb{C}^{n+1}$ for some n. This Grassmannian can be identified with the Grassmannian of k-dimensional linear subspaces of the projective n-space $\mathbb{P}(V)$. Once and for all, fix the basis $\{e_i\}_{i=1}^{n+1}$ for V, i.e. $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is in position number i. Vectors in V are denoted $\mathbf{v}$.

If $\mathbf{v}_1, \dots, \mathbf{v}_{k+1}$ are independent vectors in V, the $(k+1)$-space spanned by them is denoted $\langle \mathbf{v}_1, \dots, \mathbf{v}_{k+1} \rangle$, and it is mapped to the point $\mathbf{v}_1 \wedge \dots \wedge \mathbf{v}_{k+1}$ in $\mathbb{P}(\wedge^{k+1} V)$. This map is an embedding and is called the Plücker embedding. The image of the Plücker embedding is the Grassmannian $G(k+1, n+1)$. We may view the Plücker embedding in the following way:

If $\mathbf{v}_1, \dots, \mathbf{v}_{k+1}$ are independent vectors in V, the subspace $U^{k+1} = \langle \mathbf{v}_1, \dots, \mathbf{v}_{k+1} \rangle$ is completely described by the $(k+1) \times (n+1)$ matrix

$$M = \begin{pmatrix} v_{11} & v_{12} & v_{13} & \cdots & v_{1,n+1} \\ v_{21} & v_{22} & v_{23} & \cdots & v_{2,n+1} \\ \vdots & \vdots & \vdots & & \vdots \\ v_{k+1,1} & v_{k+1,2} & v_{k+1,3} & \cdots & v_{k+1,n+1} \end{pmatrix}$$

For a sequence $I : 1 \leq i_1 < \cdots < i_{k+1} \leq n+1$, the determinant of the maximal minor corresponding to columns in I is called the Plücker coordinate P_I. Since M has maximal rank, at least one $(k+1) \times (k+1)$-minor is non-zero. Moreover, changing the basis of U^{k+1}, we must multiply M on the left by an invertible matrix M', and P_I is multiplied with $\det(M')$.

There are $\binom{n+1}{k+1}$ such maximal minors, and it makes sense to define a map

$$G(k+1, n+1) \to \mathbb{P}^{\binom{n+1}{k+1}-1}$$

by sending U^{k+1} to the set of Plücker coordinates

$$U^{k+1} \mapsto (P_{12\dots k+1}, \dots, P_I, \dots)$$

This presupposes an ordering of the indices I, of course.

A point on $G(k+1, n+1) \subset \mathbb{P}(\wedge^{k+1} V) = \mathbb{P}^{N-1}$ where $N = \binom{n+1}{k+1}$ is often called just p. The k-dimensional linear subspace of $\mathbb{P}(V) = \mathbb{P}^n$ corresponding to the point p is called $\mathbb{P}^k_p$. If we are dealing with a particular Grassmannian, we often denote this by a G. If there is more than one Grassmannian involved, we usually write them as above.

The *dimension* of the Grassmannian $G(k+1, n+1)$ is given by the formula

$$\dim G(k+1, n+1) = (k+1)(n-k)$$

Notice that this is exactly the number of squares in a grid system of width $(k+1)$ and height $(n-k)$, as in figure 1.1. We will use this image of the Grassmannian repeatedly throughout this thesis. Following [6] this is a picture of the Chow ring of the Grassmannian, and we will sometimes refer to this grid system as the Chow ring.

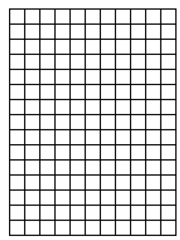


Figure 1.1: The Grassmannian $G(k+1, n+1)$ presented as a grid system

It is a well known fact (see for example [10], lecture 6) that all Grassmannians are defined as the common zero locus of some forms of degree two. Thus they are projective varieties. If $p_{i_1 \dots i_{k+1}}$ denotes the $(k+1)$-minor obtained by deleting all columns except columns number i_1 to i_{k+1}, the forms defining the Grassmannian $G(k+1, n+1)$ is given by ([15])

$$\sum_{t=1}^{k+1} (-1)^t p_{i_1 \dots i_k j_t} p_{j_1 \dots \hat{j}_t \dots j_{k+1}}$$

where $\hat{j}_t$ means that this element is deleted from the sequence.

The Grassmannians are not complete intersections, and the degree is given by the following formula ([6], page 274)

$$\deg(G(k+1, n+1)) = \frac{1!2! \cdots k!(\dim(G))!}{(n-k)!(n-k+1)! \cdots n!} \tag{1.1}$$

Being varieties, the Grassmannians have subvarieties, some so important they have their own name. They are called *Schubert cycles*.

1.2.1 Schubert cycles

We will now define what we mean by a *Schubert cycle*. We will give two formulas which enables us to do intersection theory on these cycles. Moreover, we give formulas for the degree and dimension of a cycle. A flag in V^{n+1} is a chain of inclusions

$$F = (V_1 \subset V_2 \subset \cdots \subset V_n \subset V_{n+1} = V^{n+1})$$

where V_i is a subvector space of V^{n+1} of dimension i. For any non-increasing sequence of numbers

$$n - k \geq c_1 \geq c_2 \geq \cdots \geq c_{k+1} \geq 0 \tag{1.2}$$

let

$$\begin{aligned} \sigma_{c_1 \ldots c_{k+1}} &= \{\Lambda \in G(k+1, n+1) \mid \dim(\Lambda \cap V_{n-k+i-c_i}) \geq i \text{ for } i = 1, \ldots, k+1\} \\ &= \{\mathbb{P}^k \subset \mathbb{P}^n \mid \dim(\mathbb{P}^k \cap \mathbb{P}^{n-k+i-c_i-1}) \geq i-1 \text{ for } i = 1, \ldots, k+1\} \end{aligned} \tag{1.3}$$

Then $\sigma_{c_1 \ldots c_{k+1}}$ is a subvariety of $G(k+1, n+1)$, and all subvarieties defined this way are called *Schubert cycles* of the Grassmannian.

The notation for a Schubert cycle will vary a little bit. Sometimes they are written projectively and sometimes not. Also, sometimes indices which are zero will be omitted, sometimes not. However, it is important that a Schubert cycle on $G(k+1, n+1)$ is defined by $k+1$ indices, and if a cycle on $G(k+1, n+1)$ is written with fewer indices, it will imply that the omitted indices are all zero.

Example 1.1 ($G(2,5)$). The Grassmannian of two dimensional subvector spaces of a five dimensional vector space V^5 is denoted $G(2,5)$. For a given flag F, consider the cycle

$$\begin{aligned} \sigma_{10}(F) &= \{\Lambda \in G(2,5) \mid \dim(\Lambda \cap V_{5-2+i-c_i}) \geq i \text{ for } i = 1, 2\} \\ &= \{\Lambda \in G(2,5) \mid \dim(\Lambda \cap V_3) \geq 1 \text{ and } \dim(\Lambda \cap V_5) \geq 2\} \end{aligned}$$

Notice that the condition $\dim(\Lambda \cap V_5) \geq 2$ is automatically fulfilled. Alternatively, if we think of $G(2,5)$ as the set of lines l in $\mathbb{P}^4$ and fix the projective flag $\mathbb{P}(F)$ consisting of a point, a line, a plane and a hyperplane in $\mathbb{P}^4$, the cycle σ_{10} is the set of lines in $\mathbb{P}^4$ which intersect the fixed plane in at least a point.

A slightly more complicated cycle is

$$\sigma_{21}(F) = \{\Lambda \in G(2,5) \mid \dim(\Lambda \cap V_2) \geq 1 \text{ and } \dim(\Lambda \cap V_4) \geq 2\}$$

Projectively, it is the set of lines in $\mathbb{P}^4$ which intersect a fixed line in at least a point and is contained in a fixed $\mathbb{P}^3$. $\triangle$

The sequence consisting of zeros only is perfectly legal according to equation 1.2. The corresponding cycle is

$$\sigma_{00\cdots0}(F) = \{\Lambda \in G(k+1, n+1) \mid \dim(\Lambda \cap V_{n-k+i}) \geq i \text{ for } i = 1, \ldots, k+1\}$$

Projectively,

$$\sigma_{000\ldots0}(F) = \{\mathbb{P}^k \in \mathbb{P}^n \mid \mathbb{P}^k \cap \mathbb{P}^{n-k+i-1} \text{ is at least a } \mathbb{P}^{i-1}\}$$

But this is automatically fulfilled, so the cycle $\sigma_{\mathbf{0}}$ is the Grassmannian itself. Recall the picture of the Grassmannian as a $(k+1) \times (n-k)$ grid system. We have a similar image for any Schubert cycle: Think of the grid system as $k+1$ columns of height $n-k$. Shade a subcolumn of height c_1 in the first column, a subcolumn of height c_2 in the second column and so on. The result is an image of the cycle $\sigma_{c_1c_2\ldots c_{k+1}}$. The image sits inside the original grid system, as it should, since the cycle is a subvariety of the Grassmannian. The codimension of the cycle $\sigma_{c_1c_2\ldots c_{k+1}}$ is

$$\operatorname{codim}(\sigma_{c_1c_2\cdots_{k+1}}) = \sum_{j=1}^{k+1} c_j$$

i.e. the number of shaded squares in the grid system. Equivalently, the dimension of any Schubert cycle is equal to the number of unshaded squares:

$$\dim(\sigma) = (k+1)(n-k) - \sum_{j=1}^{k+1} c_j$$

For more on this, see [9], pages 193-211.

Example 1.2 ($G(2,5)$ continued). The cycle $\sigma_{10}(F) \subset G(2,5)$ has codimension one in $G(2,5)$, and the cycle $\sigma_{21}(F)$ has codimension three. Their pictures are shown in figure 1.2. △

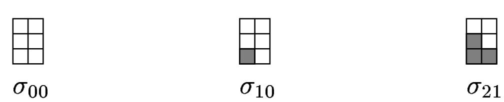


Figure 1.2: The cycles $\sigma_{00} = G(2,5)$, σ_{10} and σ_{21} on $G(2,5)$.

The cycles on the form $\sigma_{c_10\ldots0}$ are called *special Schubert cycles*. It would be nice to know how the Schubert cycles intersect inside the Grassmannian, and in the case when one is special, we have the following simple lemma:

Lemma 1.8 (Pieri's formula, [9], page 203). *If* $\mathbf{a} = a_1 0 \ldots 0$, *then for any* $\mathbf{b}$,

$$\sigma_{\mathbf{a}} \cdot \sigma_{\mathbf{b}} = \sum \sigma_{\mathbf{c}}$$

where $\mathbf{c}$ *is such that* $b_i \leq c_i \leq b_{i-1}$ *and* $\sum c_i = a_1 + \sum b_i$.

Notice that Pieri's formula implies that when we intersect a special Schubert cycle with another cycle, the codimension of the intersection is the sum of the codimensions. The intersection is generally a union of different cycles. Pieri's formula is so important, it deserves an example:

Example 1.3. Which lines in $\mathbb{P}^4$ intersect two given planes? This is exactly the intersection of two σ_{10}-s, where the planes are parts of two different flags F and F' in $\mathbb{P}^4$. Now, Pieri's formula says that

$$\sigma_{10}(F) \cdot \sigma_{10}(F') = \sigma_{11} + \sigma_{20}$$

Figure 1.3 illustrates this intersection. △

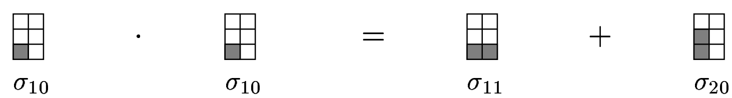


Figure 1.3: The intersection $(\sigma_{10})^2 = \sigma_{11} + \sigma_{20}$.

Pieri's formula tells us how to intersect two cycles when one of them is special. This is all we need to know, thanks to the following lemma:

Lemma 1.9 (Giambelli's formula, [9], page 205). *Every Schubert cycle can be written as a polynomial in special Schubert cycles. In fact,*

$$\sigma_{c_1 c_2 \ldots c_{k+1}} = \begin{vmatrix} \sigma_{c_1} & \sigma_{c_1+1} & \sigma_{c_1+2} & \cdots & \sigma_{c_1+d-1} \\ \sigma_{c_2-1} & \sigma_{c_2} & \sigma_{c_2+1} & \cdots & \sigma_{c_2+d-2} \\ \sigma_{c_3-2} & \sigma_{c_3-1} & \sigma_{c_3} & \cdots & \sigma_{c_3+d-3} \\ \vdots & & & \ddots & \vdots \\ \sigma_{c_d-d+1} & & \cdots & & \sigma_{c_d} \end{vmatrix}$$

where d *is the number of non-zero indices.*

Example 1.4. The non-special cycle σ_{43} on $G(2,7)$ equals the following polynomial in special Schubert cycles:

$$\sigma_{43} = \begin{vmatrix} \sigma_4 & \sigma_5 \\ \sigma_2 & \sigma_3 \end{vmatrix}$$
$$= \sigma_4 \cdot \sigma_3 - \sigma_5 \cdot \sigma_2$$

△

Let σ be a Schubert cycle on $G(k+1, n+1)$. We know that σ is determined by a non-increasing sequence of $k+1$ numbers, $0 \leq c_i \leq n-k$. To the cycle

$$\sigma = \sigma_{c_1 c_2 c_3 \cdots c_{k+1}}$$

where $n-k \geq c_1 \geq c_2 \geq \cdots \geq c_{k+1} \geq 0$ we can assign a non-decreasing sequence

$$(a_1, \ldots, a_{k+1})$$

where

$$a_i = n - k + (i-1) - c_i$$

The degree of the Schubert cycle σ is given by the formula (see [6] page 274)

$$\deg(\sigma) = \frac{(\dim(\sigma))!}{a_1! \cdots a_{k+1}!} \prod_{i<j} (a_j - a_i) \tag{1.4}$$

For any Grassmannian $G(k+1, n+1)$, the cycle $\sigma_{10\cdots0}$ has codimension one in G, and its assigned sequence of numbers is

$$(n-k-1, n-k+1, n-k+2, \ldots, n)$$

Thus the degree of the special Schubert cycle σ_{10} is

$$\begin{aligned}
\deg(\sigma_{10}) &= \frac{(\dim G - 1)!}{(n-k-1)!(n-k+1)! \cdots n!} \cdot 2 \cdot \frac{3!}{2} \cdot \frac{4!}{3} \cdots \frac{k!}{k-1} \cdot \frac{(k+1)!}{k} \\
&= \frac{(\dim G)!}{(k+1)(n-k)(n-k-1)!(n-k+1)! \cdots n!} \cdot \\
&\qquad\qquad 2 \cdot \frac{3!}{2} \cdots \frac{k!}{k-1} \cdot \frac{(k+1)!}{k} \\
&= \frac{2 \cdot 3! \cdots k! (\dim G)!}{(n-k)(n-k-1)!(n-k+1)! \cdots n!} \\
&= \frac{2 \cdot 3! \cdots k! (\dim G)!}{(n-k)!(n-k+1)! \cdots n!}
\end{aligned}$$

Comparing with equation 1.1 gives that the degree of the cycle $\sigma_{10\cdots0}$ is equal to the degree of the Grassmannian itself. It is thus a hyperplane section on G.

Example 1.5 (The degree of $G(2,4)$). In this case, the Schubert cycle σ_{10} is the set of lines l in $\mathbb{P}^3$ that intersects a fixed line L in at least a point. To find the degree of $G(2,4)$, we must intersect with a hyperplane $\dim G(2,4)$ times, i.e. we must calculate $(\sigma_{10})^{(2\cdot2)} = (\sigma_{10})^4$:

$$\begin{aligned}
\sigma_{10}^4 =& (\sigma_{10})^2 \cdot (\sigma_{11} + \sigma_{20}) \\
=& \sigma_{10} \cdot (\sigma_{21} + \sigma_{21}) \\
=& 2\sigma_{22}
\end{aligned}$$

The cycle σ_{22} has dimension zero in $G(2,4)$, thus it is a point. This calculation shows that the intersection of $G(2,4)$ with four hyperplanes is two points, which implies that the degree of $G(2,4)$ is two. $\triangle$

Example 1.6 (The degree of $G(2,5)$). The dimension of $G(2,5)$ is 6, and

$$\begin{aligned}\sigma_{10}^6 =& \sigma_{10}^4 \cdot (\sigma_{11} + \sigma_{20})\\ =& \sigma_{10}^3 \cdot (\sigma_{21} + \sigma_{21} + \sigma_{30})\\ =& \sigma_{10}^2 \cdot (2\sigma_{22} + 2\sigma_{31} + \sigma_{31})\\ =& \sigma_{10} \cdot (2\sigma_{32} + 2\sigma_{32} + \sigma_{32})\\ =& 5\sigma_{33}\end{aligned}$$

The cycle σ_{33} is a point on $G(2,5)$, and this implies that the degree of $G(2,5)$ is 5. $\triangle$

In general, substituting $k = 1$ into the formula for the degree of $G(k+1, n+1)$, gives

$$\deg(G(2, n+1)) = \frac{1}{n}\binom{2n-2}{n-1}$$

If the sum of the codimensions of two Schubert cycles is bigger than the dimension of the Grassmannian, the two Schubert cycles will not intersect. In fact, the following is true (see [9]):

$$\sigma_{\mathbf{a}} \cdot \sigma_{\mathbf{b}} = \emptyset \quad \text{unless } a_i + b_{k-i+2} \leq n-k \text{ for all } i$$

Example 1.7 ($G(2,5)$). The formula says that the Schubert cycles $\sigma_{a_1a_2}$ and $\sigma_{b_1b_2}$ has non-empty intersection only if

$$a_1 + b_2 \leq 3 \text{ and } a_2 + b_1 \leq 3$$

$\triangle$

1.3 Some general theory on Grassmannians of lines

The Grassmannian $G(2, n+1)$ is the Grassmannian of two dimensional subvector spaces L of a vector space V^{n+1} of dimension $n+1$. We can think of the vector space L as a line in the n-dimensional projective space $\mathbb{P}(V^{n+1})$, and $G(2, n+1)$ as the Grassmannian of lines in $\mathbb{P}(V^{n+1})$. The Plücker embedding is now defined as

$$\begin{array}{ccc} \mathbb{P}^n & \to & \mathbb{P}^{\binom{n+1}{2}-1} \\ L = \langle v_1, v_2\rangle & \mapsto & v_1 \wedge v_2 \end{array} \tag{1.5}$$

Now let L be a fixed line in $\mathbb{P}^n$ spanned be the two points $(p_{10},\ldots,p_{1n})$ and $(p_{20},\ldots,p_{2n})$. We can form a $2\times(n+1)$ matrix in the following way:

$$A=\begin{pmatrix} p_{10} & p_{11} & \cdots & p_{1n} \\ p_{20} & p_{21} & \cdots & p_{2n}\end{pmatrix}$$

The Plücker coordinates are the (2×2)-minors of this matrix.

Example 1.8 (**$G(2,4)$**)**.** The matrix A now becomes

$$\begin{pmatrix} p_{10} & p_{11} & p_{12} & p_{13} \\ p_{20} & p_{21} & p_{22} & p_{23}\end{pmatrix}$$

and the minors are

$$q_{01}=\begin{vmatrix} p_{10} & p_{11} \\ p_{20} & p_{21}\end{vmatrix} \quad q_{02}=\begin{vmatrix} p_{10} & p_{12} \\ p_{20} & p_{22}\end{vmatrix} \quad q_{03}=\begin{vmatrix} p_{10} & p_{13} \\ p_{20} & p_{23}\end{vmatrix}$$

$$q_{12}=\begin{vmatrix} p_{11} & p_{12} \\ p_{21} & p_{22}\end{vmatrix} \quad q_{13}=\begin{vmatrix} p_{11} & p_{13} \\ p_{21} & p_{23}\end{vmatrix} \quad q_{23}=\begin{vmatrix} p_{12} & p_{13} \\ p_{22} & p_{23}\end{vmatrix}$$

$\triangle$

By construction of the Plücker coordinates it is easy to see that

$$\begin{aligned} q_{ij} &= -\,q_{ji} \\ q_{ii} &= 0 \end{aligned} \tag{1.6}$$

1.3.1 Quadratic relations

The Plücker coordinates of points on $G(2,n+1)$ satisfy the following quadratic relations, see [15]:

$$\sum_{\lambda=0}^{2}(-1)^{\lambda}q_{j_0k_\lambda}q_{k_0\ldots\hat{k_\lambda}\ldots k_2}=0 \tag{1.7}$$

where $\hat{k_\lambda}$ means that k_λ is removed from the sequence. Here, j_0 is any number between 0 and n and $k_0k_1k_2$ is any sequence of numbers between 0 and n. Written out, the sum in equation 1.7 becomes

$$q_{j_0k_0}q_{k_1k_2}-q_{j_0k_1}q_{k_0k_2}+q_{j_0k_2}q_{k_0k_1}=0 \tag{1.8}$$

So, the natural question to ask is "How many quadratic relations does the Plücker coordinates satisfy?", that is, "How many quadratic relations define the Grassmannian of lines as a subvariety of the big projective space?" We know that j_0 is any number between 0 and n, and the same is true for the k_i-s. Therefore, we must pick 4 numbers out of $n+1$ possibilities. It is not given a priori that all the numbers must be different, but by inspection and equation 1.6, we see that the equations are trivially satisfied if two numbers are equal. Thus, the only sequences that give us independent quadratic relations are those where none of the numbers are equal, and therefore we have $\binom{n+1}{4}$ independent relations. We have just proved the following lemma:

Lemma 1.10. *The Grassmannian of lines in* $\mathbb{P}^n$ *is defined by* $\binom{n+1}{4}$ *quadratic relations as a subvariety of* $\mathbb{P}^{\binom{n+1}{2}-1}$.

Example 1.9 ($G(2,4)$ continued). From lemma 1.10 it follows that there is $\binom{4}{4} = 1$ equation defining $G(2,4)$. From the proof of the lemma it follows that the equation is defined by setting $j_0 = 0$ and $k_0, k_1, k_2 = 1, 2, 3$. Thus, $G(2,4)$ is the zero locus of $Q = q_{01}q_{23} - q_{02}q_{13} + q_{03}q_{12}$. $\triangle$

1.3.2 The points on the Grassmannian of lines as skew symmetric matrices

In this section, we will explain why we may consider the points in the Plücker space of the Grassmannians of lines as skew symmetric matrices. We will also prove that the intersection of the linear span of a Schubert cycle and $G(k+1, n+1)$ is the cycle itself.

We know that $G(2, n+1) \subset \mathbb{P}(\wedge^2\mathbb{C}^{n+1})$. If $e_1, \ldots, e_{n+1}$ is a basis for $\mathbb{C}^{n+1}$, then

$$\{e_i \wedge e_j | i < j\}$$

if a basis for $\wedge^2\mathbb{C}^{n+1}$. We know that $e_i \wedge e_j = -e_j \wedge e_i$ for all i, j, and $e_i \wedge e_i = 0$ for all i. It is useful to list these vectors in the following way:

$$\begin{array}{ccccc}
e_1 \wedge e_2 & e_1 \wedge e_3 & e_1 \wedge e_4 & \ldots & e_1 \wedge e_{n+1} \\
 & e_2 \wedge e_3 & e_2 \wedge e_4 & \ldots & e_2 \wedge e_{n+1} \\
 & & e_3 \wedge e_4 & \ldots & e_3 \wedge e_{n+1} \\
 & & & \ddots & \vdots \\
 & & & & e_n \wedge e_{n+1}
\end{array}$$

If we now define $x_{ij} := e_i \wedge e_j$, these basis vectors form the matrix

$$\begin{pmatrix}
0 & x_{12} & x_{13} & x_{14} & \cdots & x_{1,n+1} \\
0 & 0 & x_{23} & x_{24} & \cdots & x_{2,n+1} \\
0 & 0 & 0 & x_{34} & \cdots & x_{3,n+1} \\
\vdots & \vdots & \vdots & \ddots & & \vdots \\
0 & 0 & 0 & & & x_{n,n+1} \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}$$

Since $x_{ij} = -x_{ji}$, this matrix can be expanded to a skew symmetric matrix

$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} & \cdots & x_{1,n+1} \\ -x_{12} & 0 & x_{23} & x_{24} & \cdots & x_{2,n+1} \\ -x_{13} & -x_{23} & 0 & x_{34} & \cdots & x_{3,n+1} \\ -x_{14} & -x_{24} & -x_{34} & 0 & & \\ \vdots & \vdots & \vdots & & & \vdots \\ & & & & & x_{n,n+1} \\ -x_{1,n+1} & -x_{2,n+1} & -x_{3,n+1} & & \cdots & 0 \end{pmatrix}$$

Every point p in $\wedge^2\mathbb{C}^{n+1}$ can be written in the form

$$p = \sum_{i<j} a_{ij} e_i \wedge e_j$$

and may thus be considered as an $(n+1) \times (n+1)$ skew symmetric matrix.

Let $A = [a_{ij}]$ be a $(r \times r)$ skew symmetric matrix. Suppose that r is even. We then define a polynomial $\mathrm{Pf}(A)$ in the a_{ij}-s by induction on r:

$$\begin{aligned} r = 2: &\qquad \mathrm{Pf}\begin{pmatrix} 0 & a_{12} \\ -a_{12} & 0 \end{pmatrix} = a_{12} \\ r \text{ is even, } r \geq 4: &\qquad \mathrm{Pf}(A) = \textstyle\sum_{j=2}^{r} (-1)^j a_{1j} \mathrm{Pf}(A_{1j}) \end{aligned} \tag{1.9}$$

where A_{1j} is the skew symmetric sub-matrix we get be deleting the first and j-th row and column in A.

It is well known that a skew symmetric matrix of even dimension generally has maximal rank. The determinant of such a matrix is the square of a polynomial, and this polynomial is exactly the maximal Pfaffian. When the dimension is odd, the Pfaffian is always zero, and thus the matrix has rank at most one less than the dimension (See [7], appendix D).

Example 1.10. Let A be the matrix

$$\begin{pmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{pmatrix}$$

Then

$$\begin{aligned} \mathrm{Pf}(A) =& x_{12}\mathrm{Pf}(A_{12}) - x_{13}\mathrm{Pf}(A_{13}) + x_{14}\mathrm{Pf}(A_{14}) \\ =& x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} \end{aligned}$$

△

If r is odd, we modify the sum in equation 1.9 to Pfaffians of the matrix A_j, where A_j is the even-dimensional skew symmetric matrix we get by deleting the j-th row and column of the matrix A.

The $(4\times4)-$Pfaffians of a skew symmetric matrix is obtained by taking the Pfaffians of all the (4×4) skew symmetric submatrices. Doing this, we see that the (4×4)-Pfaffians give us quadratic relations in the entries, and in one relation each entry appears at most once. By inspection, we see that these are exactly the same relations as the Plücker relations, and thus we may conclude that the Grassmannian of lines in $\mathbb{P}^n$ is determined by the (4×4)-Pfaffians of the skew symmetric matrix of dimension $n+1$, where the entries are the basis vectors of $\wedge^2 V^{n+1}$ relative to a fixed basis for V^{n+1}.

There are $\binom{n+1}{4}$ Pfaffians which defines the ideal of $G(2,n+1)$. They are obtained by choosing four rows and columns in the matrix of basisvectors for $\wedge^2 V^{n+1}$. Let Q_{ijkl} the 4-Pfaffian involving rows and columns number i,j,k and l. A basis for $\wedge^4 V^{n+1}$ is

$$\{e_i\wedge e_j\wedge e_k\wedge e_l \mid 1\leq i<j<k<l\leq n+1\}$$

This gives an isomorphism

$$\begin{array}{ccc}\wedge^4 V^{n+1} & \to & I_{G(2,n+1)(2)}\\ e_i\wedge e_j\wedge e_k\wedge e_l & \mapsto & Q_{ijkl}\end{array}$$

In this thesis, the quadrics Q_{ijkl} will always be the basis for the ideal of $G(2,n+1)$. Every quadric in the basis of the ideal of $G(2,n+1)$ is a (4×4)-Pfaffian. By itself, such a Pfaffian defines a $G(2,4)$ in $G(2,n+1)$. Thus every quadric in the basis of the ideal of $G(2,n+1)$ corresponds to a $G(2,W^4)$, where W^4 is an element in $G(4,n+1)$ and thus a point in $\wedge^4 V^{n+1}$.

When all the (4×4)-Pfaffians are zero, the matrix has rank 2. Thus, the points on the Grassmannian is represented by skew symmetric matrices of rank 2.

Now, let σ be a Schubert cycle in $G(2,n+1)$. The ideal of $G(2,n+1)$ is generated by the 4-Pfaffians of an $(n+1)\times(n+1)$ matrix, i.e of $\binom{n+1}{4}$ quadrics. The ideal of σ contains the ideal of $G(2,n+1)$:

$$I_\sigma \supset I_{G(2,n+1)} = (q_1,\ldots,q_{\binom{n+1}{4}})$$

The cycle σ is defined by two indices:

$$\sigma_{c_1c_2} = \{\mathbb{P}^1\subset\mathbb{P}^n \mid \mathbb{P}^1\cap\mathbb{P}_F^{n-c_1-1}\supset\mathbb{P}^0 \text{ and } \mathbb{P}^1\subset\mathbb{P}^{n-c_2}\}$$

for some fixed partial flag $(\mathbb{P}^{n-c_1-1}\subset\mathbb{P}^{n-c_2})$. Choosing coordinates wisely, the $\mathbb{P}^{n-c_1-1}$ can be represented by a matrix of the form

$$\mathbb{P}^{n-c_1-1} = (I_{n-c_1} \mid 0)$$

where I_j is the identity matrix of size j. Furthermore, the $\mathbb{P}^{n-c_2}$ can be represented by a matrix of the form

$$\mathbb{P}^{n-c_2} = (I_{n-c_2+1} \mid 0)$$

Therefore, an element in $\sigma_{c_1c_2}$ can be represented by a $2 \times (n+1)$ matrix of the form

$$\begin{array}{cc} & \text{columns number } 1, \dots, n-c_1 \\ & \downarrow \;\cdots\; \downarrow \\ \text{element in } \sigma_{c_1c_2}: & \begin{pmatrix} * & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \end{pmatrix} \\ & \uparrow \;\cdots\cdots\; \uparrow \\ & \text{columns number } 1, \dots, n-c_2+1 \end{array}$$

The 2-minors

$$x_{ij} \text{ where } j \geq n-c_2+2$$

are all zero, and so are the minors

$$x_{ij} \text{ where } n-c_1+1 \leq i < j \leq n-c_2+1$$

and therefore, an element in $\sigma_{c_1c_2}$ can be represented by a $(n+1) \times (n+1)$ matrix M where these entries are zero. The ideal of $\sigma_{c_1c_2}$ is generated by the 4-Pfaffians of the matrix M. They only involve the non-zero minors, and therefore, the linear span of $\sigma_{c_1c_2}$ is exactly the intersection of the hyperplanes $Z(x_{ij})$ where $Z(f)$ denotes the zero locus of the polynomial f, and x_{ij} is as specified above.

Notice that this is also true for general k: If σ is a cycle on $G = G(k+1, n+1)$, the ideal of σ contains the ideal of G. The ideal of G is generated by quadrics. Moreover, the cycle σ is generated by some pieces of a flag, and a point in σ can be represented by a $(k+1) \times (k+1)$ matrix M of maximal rank. The linear span of σ is precisely the zero locus of the Plücker coordinates corresponding to $(k+1)$-minors of M which are zero. The ideal of σ is defined by the Plücker relations and these coordinates. Thus $\langle\sigma\rangle \cap G(k+1, n+1)$ is σ itself. This proves

Lemma 1.11. *For any Schubert cycle σ on $G(k+1, n+1)$, let $\langle\sigma\rangle$ denote the linear span of σ. Then*

$$\langle\sigma\rangle \cap G(k+1, n+1) = \sigma$$

Chapter 2

Tangent spaces of Grassmannians

As the title says, this chapter is about the tangent spaces of Grassmannians. Given a linear space L in $G = G(k+1, n+1)$, we will investigate the union of the tangent spaces along L. We then generalize to arbitrary cycles. Furthermore, we define a map τ from the Chow ring of G to itself. The image of a cycle under τ is a new cycle, and the linear span of $\tau(\sigma)$ is equal to the linear span of the union of tangent spaces along σ.

2.1 Intersections of tangent spaces

In this section we will describe the tangent space of $G(k+1, n+1)$ at a fixed point p. In particular, we find the cycle on $G(k+1, n+1)$ which spans T_pG.

Let p be a point on $G = G(k+1, V^{n+1})$. After a suitable choice of coordinates, we may assume that p corresponds to the subspace $\langle e_1, \ldots, e_{k+1}\rangle \subset V^{n+1}$. Assume q is another point on G, chosen in such a way that $\mathbb{P}^k_q$ intersects $\mathbb{P}^k_p$ in codimension one, i.e $\mathbb{P}^k_p \cap \mathbb{P}^k_q = \mathbb{P}^{k-1}_{p\cap q}$. The union of these two spaces will span a $(k+1)$-dimensional space in $\mathbb{P}^n$, and we call this $\mathbb{P}^{k+1}_{p\cup q}$. Inside $\mathbb{P}^{k+1}_{p\cup q}$ there is a one dimensional family of $\mathbb{P}^k$-s which contain the $\mathbb{P}^{k-1}_{p\cap q}$. This family forms a line through p on G. By choosing a different point q', we get a different $\mathbb{P}^{k-1}$, and thus a different line through p on G. All these lines must lie in the tangent space T_pG of G at the point p.

The $\mathbb{P}^k_q$-s span the tangent space

The discussion above gives that the cycle

$$\{\mathbb{P}^k \in \mathbb{P}^n \mid \mathbb{P}^k \cap \mathbb{P}^k_p \subset \mathbb{P}^{k-1}\} = \sigma_{n-k-1,\ldots,n-k-1,0}$$

is contained in the tangent space T_pG. Now, $\mathbb{P}^k_p$ can be represented by a $(k+1)\times(n+1)$ matrix

$$\mathbb{P}^k_p = (I_{k+1} \mid 0)$$

where I_j is the identity matrix of size j. Thus any element in $\sigma_{n-k-1,\dots,n-k-1,0}$ can be represented by a matrix of the form

$$\mathbb{P}^k \in \sigma_{n-k-1,\dots,n-k-1,0} : \qquad \begin{pmatrix} * & * & \cdots & * & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ * & * & \cdots & * & 0 & \cdots & 0 \\ * & * & \cdots & * & * & \cdots & * \end{pmatrix}$$

$$\uparrow \quad \cdots \quad \uparrow$$

columns number $1, \dots, k+1$

A $(k+1) \times (k+1)$ minor of this matrix is non-zero if and only if it involves at most one of the $n-k$ last columns. Thus the number of non-zero Plücker coordinates is

$$1 + \binom{k+1}{k}(n-k) = 1 + (k+1)(n-k)$$

Thus the linear span of this cycle of a projective space of dimension $(k+1)(n-k)$. Since the dimension of T_pG is precisely $(k+1)(n-k)$, the linear span of the cycle must be equal to T_pG. We have proved the following proposition:

Proposition 2.1. *Let p be a point on the Grassmannian $G(k+1, V^{n+1})$. The tangent space T_pG of $G(k+1, n+1)$ at p is spanned by all $\mathbb{P}^k \subset \mathbb{P}(V)$ which intersect $\mathbb{P}^k_p$ in codimension one.*

Corollary 2.1. *The cycle $\sigma_{10\cdots0}$ is a tangent hyperplane section on the Grassmannian $G(k+1, n+1)$.*

Proof. Proposition 2.1 says that the tangent space T_pG at the point p is spanned by

$$\{\Lambda \in G(k+1, n+1) \mid \dim(\Lambda \cap V_{k+1}) \geq k\}$$

where $\mathbb{P}^k_p = \mathbb{P}(V_{k+1})$. This is recognized as the cycle $\sigma_{n-k-1,\dots,n-k-1,0}$, and this cycle is contained in $\sigma_{10\cdots0}(V_{k+1})$. Thus the linear span of the small cycle (i.e the tangent space) is contained in the linear span of the big cycle (i.e the hyperplane $\langle\sigma_{10\cdots0}\rangle$), and we are done. □

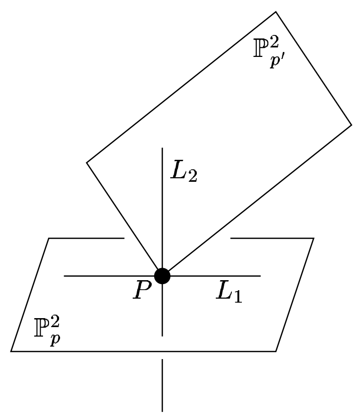
We have seen that the tangent space at the point $\mathbb{P}^k_p = \mathbb{P}(V_{k+1})$ is spanned by the cycle $\sigma_{n-k-1,\dots,n-k-1,0}(V_{k+1})$. Lemma 1.11 gives that the intersection of the tangent space and the Grassmannian is exactly this cycle, i.e

$$T_pG \cap G(k+1, n+1) = \langle\sigma_{n-k-1,\dots,n-k-1,0}\rangle \cap G(k+1, n+1) = \sigma_{n-k-1,\dots,n-k-1,0} \quad (2.1)$$

2.1.1 The intersection of two tangent spaces

We will now investigate how two tangent spaces may intersect. To illustrate the general idea, look at the following example:

Example 2.1 ($G(3, n+1)$)**.** Let p be a fixed point on $G(3, n+1)$, corresponding to the plane $\mathbb{P}^2_p$ in projective n-space. According to proposition 2.1, the tangent space of G at the point p is spanned by all $\mathbb{P}^2$-s which intersect $\mathbb{P}^2_p$ in a line. If p' is another point on G, the tangent space of G in p' is spanned by all $\mathbb{P}^2$-s which intersect $\mathbb{P}^2_{p'}$ in a line. Equation 2.1 gives that a point in the triple intersection $T_pG \cap T_{p'}G \cap G$ thus corresponds to a $\mathbb{P}^2$ which intersect $\mathbb{P}^2_p$ in a line L_1 *and* $\mathbb{P}^2_{p'}$ in a (generally different) line L_2. Now, the lines L_1 and L_2 are two lines in a $\mathbb{P}^2$, and thus they must intersect in at least a point. It follows that $\mathbb{P}^2_p$ intersects $\mathbb{P}^2_{p'}$ in at least a point P.



Assume the two planes intersect only in the point P. Every time we choose a line in $\mathbb{P}^2_p$ through P and a line in $\mathbb{P}^2_{p'}$ through P, these lines span a plane, and thus determine a unique point in $T_pG \cap T_{p'}G \cap G$. There is a $\mathbb{P}^1$ of lines in $\mathbb{P}^2_p$ through P and the same is true in $\mathbb{P}^2_{p'}$. Thus, the triple intersection $T_pG \cap T_{p'}G \cap G$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. $\triangle$

In the case when $G = G(k+1, n+1)$, proposition 2.1 and equation 2.1 says that a point in the triple intersection $T_pG \cap T_{p'}G \cap G$ corresponds to a $\mathbb{P}^k$ which intersects both $\mathbb{P}^k_p$ and $\mathbb{P}^k_{p'}$ in codimension one. These two $\mathbb{P}^{k-1}$-s are both subsets of the $\mathbb{P}^k$, so they must intersect in a least a $\mathbb{P}^{k-2}$. This shows that $\mathbb{P}^k_p$ and $\mathbb{P}^k_{p'}$ must intersect in at least a $\mathbb{P}^{k-2}$, and this is non-empty whenever $k \geq 2$.

Assume now that $\mathbb{P}^k_p \cap \mathbb{P}^k_{p'} = \mathbb{P}^{k-2}_{p\cap p'}$. Every time we choose a $\mathbb{P}^{k-1}$ in $\mathbb{P}^k_p$ which contains $\mathbb{P}^{k-2}_{p\cap p'}$ and a $\mathbb{P}^{k-1}$ in $\mathbb{P}^k_{p'}$ which contains $\mathbb{P}^{k-2}_{p\cap p'}$, their union will span a $\mathbb{P}^k$ and this $\mathbb{P}^k$ corresponds to a point in $T_pG \cap T_{p'}G \cap G$. There is a $\mathbb{P}^1$ of such $\mathbb{P}^{k-1}$ in $\mathbb{P}^k_p$ and the same is true for $\mathbb{P}^k_{p'}$. Thus the triple intersection $T_pG \cap T_{p'}G \cap G$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

In the case when $G = G(2, n+1)$, the points in $T_pG \cap T_{p'}G \cap G$ corresponds to lines which intersect both $\mathbb{P}^1_p$ and $\mathbb{P}^1_{p'}$ in a point. Fix a point q on $\mathbb{P}^1_p$. There is a $\mathbb{P}^1$ of lines though q and a point on $\mathbb{P}^1_{p'}$. There is also a $\mathbb{P}^1$ of points on $\mathbb{P}^1_p$, so again we get that the triple intersection $T_pG \cap T_{p'}G \cap G$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This proves

Proposition 2.2. *Let $G = G(k+1, n+1)$ be the Grassmannian of k-dimensional linear subspaces of a fixed n-dimensional projective space where $k \geq 2$. Let p and p' be points on G. If $\mathbb{P}^k_p$ intersects $\mathbb{P}^k_{p'}$ in dimension less than $k-2$, the triple intersection $T_pG \cap T_{p'}G \cap G$ is empty. If $\mathbb{P}^k_p$ intersects $\mathbb{P}^k_{p'}$ in at least dimension $k-2$, the triple intersection $T_pG \cap T_{p'}G \cap G$ is at least a $\mathbb{P}^1 \times \mathbb{P}^1$. When $k = 1$, the triple intersection is a $\mathbb{P}^1 \times \mathbb{P}^1$ as long as $\mathbb{P}^1_p$ does not intersect $\mathbb{P}^1_{p'}$. If the two lines intersect, the triple intersection consists of all lines in the plane they span union all lines in $\mathbb{P}^n$ through the point of intersection. Thus the triple intersection is a plane union a $\mathbb{P}^{n-1}$ in this case.*

Let $G = G(k+1, n+1)$ and let p and p' be points on G. Assume that $T_pG \cap T_{p'}G \cap G$ is non-empty and minimal. As we saw above, this means that $\mathbb{P}^k_p$ and $\mathbb{P}^k_{p'}$ intersect in a $\mathbb{P}^{k-2}_{p\cap p'}$. The linear span of the union $\mathbb{P}^k_p \cup \mathbb{P}^k_{p'}$ is a $\mathbb{P}^{k+2}_{p\cup p'}$. Now look at the set

$$\{\mathbb{P}^k \subset \mathbb{P}^n | \mathbb{P}^{k-2}_{p\cap p'} \subset \mathbb{P}^k \text{ and } \mathbb{P}^k \subset \mathbb{P}^{k+2}_{p\cup p'}\}$$

Notice that $T_pG \cap T_{p'}G \cap G$ is contained in this set. We can identify this set as the Schubert cycle

$$\{\Lambda \in G(k+1, n+1) | \dim(\Lambda \cap V_{k-1}) \geq k-1 \text{ and } \dim(\Lambda \cap V_{k+3}) \geq k+1\}$$

By inspecting the indices, we see that $n+1-(k+1)+(k-1)-c_{k-1} = k-1$, which implies that $c_{k-1} = n-k$. We also get that $n+1-(k+1)+(k+1)-c_{k+1} = k+3$, which implies that $c_{k+1} = n-k-2$. Thus this is the Schubert cycle $\sigma_{n-k,\ldots,n-k,n-k-2,n-k-2}$. It has dimension $(n-k)(k+1)-(n-k)(k-1)-2(n-k-2) = 4$, and is isomorphic to the Grassmannian $G(2,4)$. Every time we fix a flag $\mathbb{P}^{k-2} \subset \mathbb{P}^{k+2}$ in $\mathbb{P}^n$, we get such a four dimensional quadric Q on $G(k+1, n+1)$. A picture of the quadric in G is shown in figure 2.1.

2.2 Tangent spaces along linear subspaces of $G(k+1, n+1)$

In this section, we describe the linear subspaces of $G(k+1, n+1)$. We find that there are two distinct types. Furthermore, we give formulas for the dimension of the linear span of the union of tangent spaces along linear subspaces.

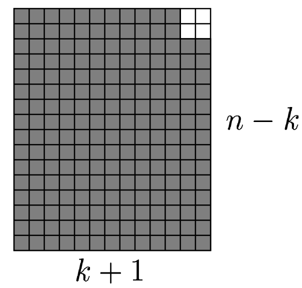


Figure 2.1: The quadric Q on G with respect to a fixed flag $\mathbb{P}^{k-2} \subset \mathbb{P}^{k+2}$

Let σ be a Schubert cycle on G. We know that σ is determined by a non-increasing sequence of $k+1$ numbers, $0 \leq c_i \leq n-k$. Recall that to the cycle

$$\sigma = \sigma_{c_1 c_2 c_3 \ldots c_{k+1}}$$

where $n-k \geq c_1 \geq c_2 \geq \cdots \geq c_{k+1} \geq 0$, we can assign a non-decreasing sequence

$$(a_1, \ldots, a_{k+1})$$

where

$$a_i = n - k + (i-1) - c_i$$

The degree of the Schubert cycle σ is given by the formula (see equation 1.4)

$$\deg(\sigma) = \frac{(\dim(\sigma))!}{a_1! \ldots a_{k+1}!} \prod_{i<j} (a_j - a_i)$$

and the dimension of σ is

$$\dim(\sigma) = (k+1)(n-k) - \sum_{i=1}^{k+1} c_i$$

Example 2.2. Let σ be a cycle of dimension one. Then

$$\sigma = \sigma_{n-k, n-k, \ldots, n-k, n-k-1} \tag{2.2}$$

The assigned sequence is

$$(0, 1, 2, 3, \ldots, k-1, k+1)$$

and the degree is

$$\begin{aligned} \deg(\sigma) &= \frac{1}{1!2!3! \cdot \cdots \cdot (k-1)!(k+1)!} \cdot (2!3!4! \ldots (k-1)!(k+1) \cdot k \cdot \cdots \cdot 2) \\ &= 1 \end{aligned}$$

Thus a Schubert cycle of dimension one on G has degree one. A picture of such a cycle is given in figure 2.2. Since only the upper right corner of the Chow ring is of importance, the picture only shows this corner. The shading fills the entire rectangle except the upper right corner box. We draw it this way to save some space, and to focus the attention on the important part.

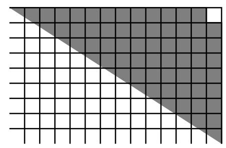


Figure 2.2: A Schubert cycle of dimension one on G. It has degree one

All lines on G are Schubert cycles of the type given in equation 2.2. For a fixed flag $F = \{V_j\} = \{\mathbb{P}_F^{j-1}\}$, $j = 1, \ldots, n+1$ in $\mathbb{P}^n$, this cycle is defined as

$$\begin{aligned}\sigma_{n-k,n-k,\ldots,n-k,n-k-1} &= \{\Lambda \in G(k+1, n+1) \,|\, \dim(\Lambda \cap V_k) \geq k \text{ and} \\ &\qquad\qquad \dim(\Lambda \cap V_{k+2}) \geq k+1\} \\ &= \{\mathbb{P}^k \subset \mathbb{P}^n | \mathbb{P}_F^{k-1} \subset \mathbb{P}^k \subset \mathbb{P}_F^{k+1}\}\end{aligned}$$

Proposition 2.1 says that the tangent spaces along this line are spanned by the $\mathbb{P}^k$-s which intersect some $\mathbb{P}^k$ in the cycle in codimension one. Thus a $\mathbb{P}_t^k$ which corresponds to a point t on G which lies in a tangent space along the line, satisfies the following conditions:

$$\begin{aligned}\mathbb{P}_t^k \cap \mathbb{P}_F^{k-1} &= \mathbb{P}^{k-2} \\ \mathbb{P}_t^k \cap \mathbb{P}_F^{k+1} &= \mathbb{P}^{k-1}\end{aligned} \tag{2.3}$$

We will prove later (lemma 2.6) that the opposite is also true, i.e a $\mathbb{P}^k$ corresponds to a point on G in some tangent space along the line given by the cycle above if and only if it satisfies the conditions in equation 2.3. After a suitable choice of coordinates, we may assume that $\mathbb{P}_F^{k-1}$ is represented by a matrix of the form

$$\mathbb{P}_F^{k-1} = (I_k | \; 0)$$

where I_k is the $(k \times k)$ identity matrix. Furthermore, $\mathbb{P}_F^{k+1}$ is represented by a matrix of the form

$$\mathbb{P}_F^{k+1} = (I_{k+2} | \; 0)$$

The conditions in equation 2.3 implies that $\mathbb{P}^k$ must be represented by a matrix of the form

$$\mathbb{P}^k = \left(\begin{array}{ccccc|cccccc} * & * & \dots & \dots & * & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & \dots & * & 0 & 0 & 0 & \dots & 0 & 0 \\ * & * & \dots & \dots & * & * & * & 0 & \dots & 0 & 0 \\ \hline * & * & \dots & \dots & * & * & * & * & & \dots & * \end{array} \right)^{(k+1)\times(n+1)}$$

$$\uparrow \; \uparrow$$

columns number $(k+1)$ and $(k+2)$

where the boxed submatrix has dimension $k \times (n+1-k)$, and a star means we can fill in whichever complex number we want. Notice that when $k = 1$, the first condition of equation 2.3 har no meaning. Thus, only the two *bottom* rows of the matrix above is left.

A $(k+1) \times (k+1)$-minor is non-zero if and only if it involves at least $k-1$ of the first k columns and at most one of the last $n-1-k$. The total number of minors not identically zero is

$$S_1(k,n) = \begin{cases} (n+1-k) + \binom{k}{k-1}(1+2(n-k-1)) & \text{if } k \neq 1 \\ \binom{n+1}{2} - \binom{n-2}{2} = 3(n-1) & \text{if } k = 1 \end{cases}$$

Thus the linear span of the union of the tangent spaces along the line has dimension $S_1(k,n) - 1$. $\triangle$

This proves

Proposition 2.3. *The dimension of the linear span of the union of tangent spaces along a line on $G(k+1, n+1)$ is given by*

$$S_1(k,n) - 1 = \begin{cases} (n+1-k) + \binom{k}{k-1}(1+2(n-k-1)) - 1 & \text{if } k \neq 1 \\ \binom{n+1}{2} - \binom{n-2}{2} - 1 = 3(n-1) - 1 & \text{if } k = 1 \end{cases}$$

Example 2.3. Let σ be a Schubert cycle of dimension 2 on G. There are two possibilities:

$$\sigma = \sigma_{n-k,n-k,\dots,n-k,n-k-2}$$

or

$$\sigma = \sigma_{n-k,n-k,\dots,n-k,n-k-1,n-k-1}$$

Their pictures are given in figure 2.3. Their assigned sequences are

$$(0, 1, 2, \dots, k-1, k+2)$$

Figure 2.3: Schubert cycles of dimension 2 on G

and

$$(0, 1, 2, \ldots, k-2, k, k+1)$$

and the degrees are one in both cases. Thus Schubert cycles of dimension 2 always have degree one.

We have seen that there are two types of Schubert planes on the Grassmannian G. Keeping the fixed flag F from above, the plane of type 1 is given by

$$\begin{aligned}\sigma_{n-k,\ldots,n-k,n-k-2} &= \{\Lambda \in G(k+1, n+1) | \dim(\Lambda \cap V_k) \geq k \\ &\qquad\qquad\qquad \text{and } \dim(\Lambda \cap V_{k+3}) \geq k+1\} \\ &= \{\mathbb{P}^k \subset \mathbb{P}^n | \mathbb{P}_F^{k-1} \subset \mathbb{P}^k \subset \mathbb{P}_F^{k+2}\}\end{aligned}$$

Proposition 2.1 says that the union of the tangent spaces along this plane is spanned by all $\mathbb{P}_t^k$-s which intersect some $\mathbb{P}^k$ in the cycle in codimension one. Such $\mathbb{P}_t^k$-s satisfy the conditions

$$\begin{aligned}\mathbb{P}_t^k \cap \mathbb{P}_F^{k-1} &= \mathbb{P}^{k-2} \\ \mathbb{P}_t^k \cap \mathbb{P}_F^{k+2} &= \mathbb{P}^{k-1}\end{aligned} \tag{2.4}$$

We will show (lemma 2.6) that the opposite is true, also. Thus, we have that a $\mathbb{P}^k$ corresponds to a point on G in some tangent space along the plane given by the cycle if and only if it satisfies the conditions in equation 2.4.

After a suitable choice of coordinates, we may assume that $\mathbb{P}_F^{k-1}$ is represented by a matrix of the form

$$\mathbb{P}_F^{k-1} = (I_k | \ 0)$$

and that $\mathbb{P}_F^{k+2}$ is represented by a matrix of the form

$$\mathbb{P}_F^{k+2} = (I_{k+3} | \ 0)$$

and $\mathbb{P}^k$ must be represented by a matrix of the form

$$\mathbb{P}^k = \left(\begin{array}{ccccc|ccccccc} * & * & \dots & \dots & * & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & \dots & * & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ * & * & \dots & \dots & * & * & * & * & 0 & \dots & 0 & 0 \\ \hline * & * & \dots & \dots & * & * & * & * & * & & \dots & * \end{array}\right)^{(k+1)\times(n+1)}$$

$$\uparrow \; \uparrow \; \uparrow$$

columns number $(k+1),(k+2)$ and $(k+3)$

where the boxed submatrix has dimension $k \times (n+1-k)$. When $k=1$, the first condition in equation 2.4 has no meaning, and thus lines which span the union of the tangent spaces along the plane is represented by matrices of the form given by the two *bottom* rows of the matrix above.

A $(k+1)\times(k+1)$-minor is non-zero if and only if it involves at least $k-1$ of the first k columns and at most one of the last $n-2-k$. The total number of minors not identically zero is

$$S_2^1(k,n) = \begin{cases} (n-k+1) + \binom{k}{k-1}\left(\binom{3}{2} + 3(n-k-2)\right) & \text{if } k \neq 1 \\ \binom{n+1}{2} - \binom{n-3}{2} = 4n-6 & \text{if } k = 1 \end{cases}$$

Thus the linear span of the union of the tangent spaces along the plane has dimension $S_2^1(k,n)-1$.

The plane of type 2 is given by

$$\begin{aligned} \sigma_{n-k,\dots,n-k,n-k-1,n-k-1} &= \{\Lambda \in G(k+1,n+1) \mid \dim(\Lambda \cap V_{k-1}) \geq k-1 \\ &\qquad \text{and } \dim(\Lambda \cap V_{k+2}) \geq k+1\} \\ &= \{\mathbb{P}^k \subset \mathbb{P}^n \mid \mathbb{P}_F^{k-2} \subset \mathbb{P}^k \subset \mathbb{P}_F^{k+1}\} \end{aligned}$$

Proposition 2.1 says that the union of the tangent spaces along the plane is spanned by $\mathbb{P}_t^k$-s which satisfy the following conditions:

$$\begin{aligned} \mathbb{P}_t^k \cap \mathbb{P}_F^{k-2} &= \mathbb{P}^{k-3} \\ \mathbb{P}_t^k \cap \mathbb{P}_F^{k+1} &= \mathbb{P}^{k-1} \end{aligned} \tag{2.5}$$

We will show later (lemma 2.6) that the opposite is also true, i.e. that any $\mathbb{P}^k$ which satisfies the conditions in equation 2.5 is in some tangent space along the plane.

After a suitable choice of coordinates, we may assume that $\mathbb{P}_F^{k-2}$ is represented by a matrix of the form

$$\mathbb{P}^{k-2} = (I_{k-1} \mid 0)$$

and that $\mathbb{P}_F^{k+1}$ is represented by a matrix of the form

$$\mathbb{P}_F^{k+1} = (I_{k+2}|\ 0)$$

This gives that every $\mathbb{P}^k$ which satisfies the conditions in equation 2.5 must be represented by a matrix of the form

$$\mathbb{P}^k = \left(\begin{array}{ccccc|ccccccc} * & * & \dots & \dots & * & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ * & * & \dots & \dots & * & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ * & * & \dots & \dots & * & * & * & * & 0 & \dots & 0 & 0 \\ * & * & \dots & \dots & * & * & * & * & 0 & \dots & 0 & 0 \\ \hline * & * & \dots & \dots & * & * & * & * & * & & \dots & * \end{array}\right)^{(k+1)\times(n+1)}$$

$\uparrow\ \uparrow\ \uparrow$
columns number $k, (k+1)$ and $(k+2)$

where the boxed submatrix has dimension $k \times (n+2-k)$.

A $(k+1)\times(k+1)$-minor is non-zero if and only if it involves at least $k-2$ of the first $k-1$ columns, and at most one of the last $n-1-k$. The total number of minors not identically zero is

$$S_2^2(k,n) = \begin{cases} \binom{3}{2} + 3(n-1-k) + \binom{k-1}{k-2}\left(1+\binom{3}{2}(n-1-k)\right) & \text{if } k \geq 3 \\ \binom{k+2}{k+1} + \binom{k+2}{k}(n-1-k) & \text{if } k = 1 \text{ or } 2 \end{cases}$$

Thus the linear span of the union of the tangent spaces along the plane has dimension $S_2^2(k,n) - 1$. △

To summarize:

Proposition 2.4. *Let $\sigma = \sigma_{n-k,\dots,n-k,n-k-2}$ be a plane on $G(k+1,n+1)$ isomorphic to $G(2,3)$. The dimension of the linear span of the union of tangent spaces along σ is*

$$S_2^1(k,n) - 1 = \begin{cases} (n-k+1) + \binom{k}{k-1}\left(\binom{3}{2} + 3(n-k-2)\right) - 1 & \textit{if } k \neq 1 \\ \binom{n+1}{2} - \binom{n-3}{2} - 1 = 4n-7 & \textit{if } k = 1 \end{cases}$$

If $\sigma = \sigma_{n-k,\dots,n-k,n-k-1,n-k-1}$ is a plane isomorphic to $G(1,3)$, the dimension of the linear span of the union of tangent spaces along σ is

$$S_2^2(k,n) - 1 = \begin{cases} \binom{3}{2} + 3(n-1-k) + \binom{k-1}{k-2}\left(1+\binom{3}{2}(n-1-k)\right) - 1 & \textit{if } k \geq 3 \\ \binom{k+2}{k+1} + \binom{k+2}{k}(n-1-k) - 1 & \textit{if } k = 1 \textit{ or } 2 \end{cases}$$

Example 2.4. There are three possible Schubert cycles of dimension three:

$$\sigma = \sigma_{n-k,\ldots,n-k,n-k-3}$$
$$\sigma = \sigma_{n-k,\ldots,n-k,n-k-1,n-k-1,n-k-1}$$
$$\sigma = \sigma_{n-k,\ldots,n-k,n-k-1,n-k-2}$$

Their assigned sequences are

$$(0, 1, \ldots, k-1, k+3)$$
$$(0, 1, \ldots, k-3, k-1, k, k+1)$$
$$(0, 1, \ldots, k-2, k, k+2)$$

and their pictures are shown in figure 2.4. It is easy to see that the degree of the first two cycles are both 1, while the degree of the third cycle is 2. △

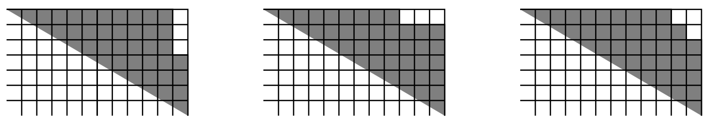


Figure 2.4: Schubert cycles of dimension 3 on G. The first two have degree 1, and the last one has degree 2

Proposition 2.5. *Any linear space $\mathbb{P}^r$ lying on the Grassmannian $G(k+1, n+1)$ is a subgrassmannian of the form $G(1, r+1)$ or $G(r, r+1)$*

Proof. To prove this, we will use the formula for the degree of a Schubert cycle given above. We have already seen that the proposition is true when $r = 1, 2$ and 3: Figure 2.2 shows the Chow ring of a line on G. The open spaces is the Chow ring of the Grassmannian $G(1, 2)$, and this proves the proposition when $r = 1$. Figure 2.3 shows the Chow ring of the two types of planes on G. The open spaces form the Chow rings of $G(1, 3)$ and $G(2, 3)$, and this proves the proposition when $r = 2$. Figure 2.4 shows the Chow rings of the three possible types of three dimensional cycles on G. We have seen that only the two first cycles have degree 1, and the open spaces are the Chow rings of $G(1, 3)$ and $G(3, 4)$. Thus the proposition is true when $r = 3$.

There are five possibilities for a cycle of dimension four:

$$\sigma_{n-k,\ldots,n-k,n-k-4}$$
$$\sigma_{n-k,\ldots,n-k,n-k-1,n-k-1,n-k-1,n-k-1}$$
$$\sigma_{n-k,\ldots,n-k,n-k-1,n-k-1,n-k-2}$$
$$\sigma_{n-k,\ldots,n-k,n-k-1,n-k-3}$$
$$\sigma_{n-k,\ldots,n-k,n-k-2,n-k-2}$$

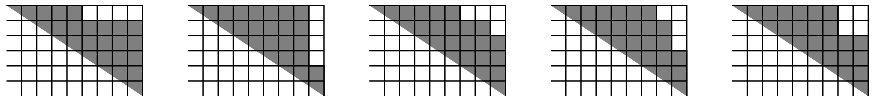


Figure 2.5: The cycles of dimension 4

Their Chow rings are shown in figure 2.5. The associated sequences are

$$\{0,1,2,\dots,k-4,k-2,k-1,k,k+1\}$$
$$\{0,1,2,\dots,k-1,k+4\}$$
$$\{0,1,2,\dots,k-3,k-1,k,k+2\}$$
$$\{0,1,2,\dots,k-2,k,k+3\}$$
$$\{0,1,2,\dots,k-2,k+1,k+2\}$$

and the formula for the degree of a Schubert cycle gives that the degrees are respectively $1, 1, 3, 3, 2$. Thus only the first and second cycles are linear spaces. The unshaded squares are the Chow rings of $G(4,5)$ and $G(1,5)$, and this proves the proposition when $r = 4$. Now, let σ be a cycle of dimension greater than four. Then σ must either be on the form

$$\sigma = \sigma_{n-k,\dots,n-k,n-k-1,\dots,n-k-1,n-k-(\dim\sigma-s+1)}$$

where $n-k-1$ occurs $s-1$ times (then the sum of the indices is $(s-1)(n-k-1)+n-k-(\dim\sigma-s+1)+(k+1-s)(n-k) = (k+1)(n-k)-\dim\sigma$, so this fits), or σ must contain the quadric Q.

In the first case, the degree of σ is

$$\begin{aligned}\deg(\sigma) &= \frac{(\dim\sigma)!}{2\cdot 3\cdots(s-2)(s-1)\cdot\dim\sigma\cdot(\dim\sigma-s)\cdot(\dim\sigma-s-1)\cdots 2}\\ &= \frac{(\dim\sigma-1)\cdot\dots\cdot(\dim\sigma-s+1)}{(s-1)!}\end{aligned}$$

and this expression is equal to one if and only if $\dim\sigma = s$ or $s = 1$. Then σ is either the cycle

$$\sigma = \sigma_{n-k,\dots,n-k,n-k-1,\dots,n-k-1}$$

where $n-k-1$ occurs s times, or the cycle

$$\sigma = \sigma_{n-k,\dots,n-k,n-k-\dim\sigma}$$

These cycles are isomorphic to $G(\dim\sigma, \dim\sigma+1)$ and $G(1,\dim\sigma+1)$ respectively. We know that Q is isomorphic to $G(2,4)$, and thus the linear span of Q is five dimensional. The cycles of dimension five containing the quadric Q are

$$\sigma_{n-k,\dots,n-k,n-k-2,n-k-3}$$

and

$$\sigma_{n-k,\ldots,n-k,n-k-1,n-k-2,n-k-2}$$

Their associated sequences are

$$(0, 1, \ldots, k-2, k+1, k+3)$$

and

$$(0, 1, \ldots, k-3, k-1, k+1, k+2)$$

and the degrees are 5 in both cases. Thus there is no linear space on G which contains the quadric Q. This completes the proof.

□

Now that we know exactly what linear spaces on the Grassmannian looks like, we will find a formula for the linear span of the union of the tangent spaces along any r-dimensional linear space on G. This formula will be a function of k, n and r. The visual parallel statement to proposition 2.5, says that any linear r-dimensional space in the Grassmannian must be a cycle of one of the two types shown in figure 2.6. Notice that there can be no $\mathbb{P}^r$ of the first type on $G(k+1, n+1)$ if $r > n-k$. Thus in the following argument, r is always less than or equal to $n-k$.

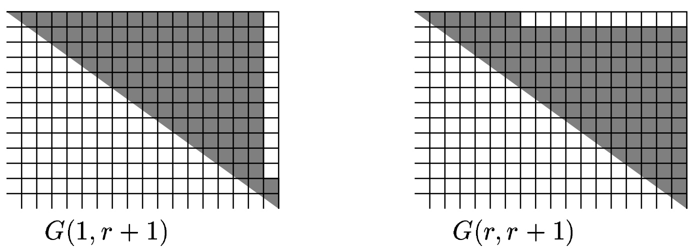


Figure 2.6: Any linear space on $G(k+1, n+1)$ is a cycle of one of these two types.

A linear r-space $G(r, r+1) \simeq \mathbb{P}^r \subset G(k+1, n+1)$ can be expressed as the cycle

$$\begin{aligned} G(r, r+1) = \sigma_{n-k,\ldots,n-k,n-k-r}(F) = \{\Lambda \in G(k+1, n+1) \,|\, & \dim(\Lambda \cap V_k) \geq k \\ & \text{and } \dim(\Lambda \cap V_{k+1+r}) \geq k+1\} \\ = \{\mathbb{P}^k \subset \mathbb{P}^n | \mathbb{P}^{k-1}_F \subset \mathbb{P}^k \subset \mathbb{P}^{k+r}_F\} & \end{aligned}$$

The union of the tangent spaces along this $\mathbb{P}^r$ is spanned by all $\mathbb{P}^k_t$ which satisfy

$$\begin{aligned} \mathbb{P}^k_t \cap \mathbb{P}^{k-1}_F &= \mathbb{P}^{k-2} \\ \mathbb{P}^k_t \cap \mathbb{P}^{k+r}_F &= \mathbb{P}^{k-1} \end{aligned} \tag{2.6}$$

Lemma 2.6. *Let $\sigma_{c_1c_2,\ldots,c_{k+1}}(F)$ be a Schubert cycle on $G(k+1,n+1)$ for some fixed flag F. Let $\mathbb{P}^k$ be such that $\mathbb{P}^k \cap \mathbb{P}_F^{n-k+i-c_i-1} \supset \mathbb{P}^{i-2}$ for $i = 1, \ldots, k+1$. Then there is a point $p \in \sigma_{c_1c_2,\ldots,c_{k+1}}(F)$ such that $\mathbb{P}^k$ is in the tangent space T_pG.*

Proof. The cycle $\sigma_{c_1c_2,\ldots,c_{k+1}}(F)$ is defined as

$$\sigma_{c_1c_2,\ldots,c_{k+1}}(F) = \{\mathbb{P}^k \in \mathbb{P}^n \mid \mathbb{P}^k \cap \mathbb{P}_F^{n-k+i-c_i-1} \supset \mathbb{P}^{i-1} \text{ for } i = 1, \ldots, k+1\}$$

In particular, a general $\mathbb{P}^k_\sigma$ in the cycle is contained in a fixed $\mathbb{P}_F^{n-c_{k+1}}$ and it intersects a fixed $\mathbb{P}^{n-k-c_1}$ in a point q. Now, let $\mathbb{P}^k$ be such that $\mathbb{P}^k \cap \mathbb{P}_F^{n-k+i-c_i-1} \supset \mathbb{P}^{i-2}$ for $i = 1, \ldots, k+1$. Then $\mathbb{P}^k$ in general does not intersect $\mathbb{P}^{n-k-c_1}$, and $\mathbb{P}^k \cap \mathbb{P}^{n-c_{k+1}}$ is in general a $\mathbb{P}^{k-1}$. The linear span $\langle q, \mathbb{P}^{k-1}\rangle$ is a $\mathbb{P}^k$, and this $\mathbb{P}^k$ is in σ. Furthermore, the $\mathbb{P}^k$ we started with intersects $\langle q, \mathbb{P}^{k-1}\rangle$ in codimension one. □

As above we may assume that $\mathbb{P}_F^{k-1}$ is represented by a matrix of the form

$$\mathbb{P}_F^{k-1} = (I_k |\ 0)$$

and that $\mathbb{P}_F^{k+r}$is represented by a matrix of the form

$$\mathbb{P}_F^{k+r} = (I_{k+r+1} |\ 0)$$

Lemma 2.6 implies that a $\mathbb{P}^k$ is in some tangent space along σ if and only if it satisfies the conditions in equation 2.6. Thus any $\mathbb{P}^k_t$ on $G(k+1,n+1)$ which is in some tangent space along this linear space, must be represented by a matrix of the form

$$\mathbb{P}^k_t = \left(\begin{array}{ccccc|ccccccc} * & * & \ldots & \ldots & * & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ * & * & \ldots & \ldots & * & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ * & * & \ldots & \ldots & * & * & \ldots & * & 0 & \ldots & 0 & 0 \\ \hline * & * & \ldots & \ldots & * & * & \ldots & * & * & & \ldots & * \end{array}\right)^{(k+1)\times(n+1)}$$

$$\uparrow \ \ldots \ \uparrow$$

columns number $(k+1), (k+2), \ldots, (k+r+1)$

where the boxed submatrix has dimension $k \times (n+1-k)$.

A $(k+1) \times (k+1)$-minor of this matrix is non-zero if and only if it involves at least $k-1$ of the first k columns, and at most one of the $n-k-r$ last columns. The total number of maximal minors not identically zero is

$$S_r^1(k,n,r) = \begin{cases} (n+1-k) + \binom{k}{k-1}\left(\binom{r+1}{2} + (r+1)(n-k-r)\right) & \text{if } k \neq 1 \\ \left(\binom{n+1}{2} - \binom{n-1-r}{2}\right) & \text{if } k = 1 \end{cases}$$

The union of the tangent spaces along a $\mathbb{P}^r$ of this type spans a linear space of dimension $S_r^1(k,n,r) - 1$.

A linear r-space $G(1,r+1) \simeq \mathbb{P}^r \subset G(k+1,n+1)$ can be expressed as the cycle

$$G(1,r+1) = \sigma_{n-k,\ldots,n-k,n-k-1,\ldots,n-k-1}(F)$$

where $n-k-1$ occurs r times. By definition, we have

$$\begin{aligned} &\sigma_{n-k,\ldots,n-k,n-k-1,\ldots,n-k-1}(F) = \\ &\{\Lambda \in G(k+1,n+1) |\, \dim(\Lambda \cap V_{k+1-r}) \geq k+1-r \text{ and } \dim(\Lambda \cap V_{k+2}) \geq k+1\} \\ &= \{\mathbb{P}^k \subset \mathbb{P}^n | \mathbb{P}_F^{k-r} \subset \mathbb{P}^k \subset \mathbb{P}_F^{k+1}\} \end{aligned}$$

The union of the tangent spaces along a linear r-space of this type is spanned by all $\mathbb{P}_t^k$ which satisfy

$$\begin{aligned} \mathbb{P}_t^k \cap \mathbb{P}_F^{k-r} &= \mathbb{P}^{k-r-1} \\ \mathbb{P}_t^k \cap \mathbb{P}_F^{k+1} &= \mathbb{P}^{k-1} \end{aligned} \tag{2.7}$$

We may assume that $\mathbb{P}_F^{k-r}$ is represented by a matrix of the form

$$\mathbb{P}_F^{k-r} = (I_{k+1-r} |\ 0)$$

and that $\mathbb{P}_F^{k+1}$ is represented by a matrix of the form

$$\mathbb{P}_F^{k+1} = (I_{k+2} |\ 0)$$

The conditions given in equation 2.7 and lemma 2.6 then give that any $\mathbb{P}_t^k$ on $G(k+1,n+1)$ and in some tangent space along this linear space, must be represented by a matrix of the form

$$\mathbb{P}_t^k = \left(\begin{array}{ccccc|ccccccc} * & * & \dots & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ * & * & \dots & \dots & * & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & * & \dots & * & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & \dots & * & * & \dots & * & 0 & \dots & 0 & 0 \\ \hline * & * & \dots & \dots & * & * & \dots & * & * & & \dots & * \end{array}\right)^{(k+1)\times(n+1)}$$

$$\uparrow \; \dots \; \uparrow$$

columns number $(k-r+2), \dots, (k+2)$ i.e $r+1$ special columns

where the boxed submatrix has dimension $k \times (n-k+r)$. The non-zero submatrix of the boxed matrix has dimension $r \times (r+1)$.

A $(k+1) \times (k+1)$ minor of the above matrix is non-zero if and only if it involves at least $k-r$ of the first $k-r+1$ columns, and at most one of the $n-k-1$ last columns. The total number of maximal minors not identically zero is

$$S_r^2(k,n,r) = \begin{cases} \binom{r+1}{r} + \binom{r+1}{r-1}(n-k-1) + \binom{k-r+1}{k-r}\left(1 + \binom{r+1}{r}(n-k-1\right) & \\ & \text{if } k \neq 1, r < k+1 \\ \binom{k+2}{k+1} + \binom{k+2}{k}(n-k-1) & \text{if } k \neq 1, r = k+1 \end{cases}$$

Notice that figure 2.6 shows that there can be no $\mathbb{P}^r$ of this type on $G(2, n+1)$ when $r > 2$. The case $r = 2$ is treated separately above. To summarize:

Proposition 2.7. *Let σ be an r-dimensional linear space on $G = G(k+1, n+1)$ isomorphic to $G(r, r+1)$. Then the dimension of the linear span of the union of tangent spaces along σ is*

$$S_r^1(k,n,r) - 1 = \begin{cases} (n+1-k) + \binom{k}{k-1}\left(\binom{r+1}{2} + (r+1)(n-k-r)\right) - 1 & \textit{if } k \neq 1 \\ (\binom{n+1}{2}) - \binom{n-1-r}{2}) - 1 & \textit{if } k = 1 \end{cases}$$

If σ is a linear space isomorphic to $G(1, r+1)$, the dimension of the linear span of the union of tangent spaces along σ is

$$S_r^2(k,n,r) - 1 = \begin{cases} \binom{r+1}{r} + \binom{r+1}{r-1}(n-k-1) + \binom{k-r+1}{k-r}\left(1 + \binom{r+1}{r}(n-k-1\right) - 1 & \\ & \textit{if } k \neq 1, r < k+1 \\ \binom{k+2}{k+1} + \binom{k+2}{k}(n-k-1) - 1 & \textit{if } k \neq 1, r = k+1 \end{cases}$$

In the next section we will study two dual filtrations. Of reasons that soon will become apparent, they are called “vertical” and “horizontal” filtrations.

2.3 The “Spanning-tangent-space”-map

In this section we will define a map from the Chow ring of G to itself. The image of any cycle σ under this map will be the cycle of points on G which spans the union of the tangent spaces along σ.

So fix a flag F as above, and take any cycle

$$\begin{aligned}\sigma_{c_1\ldots c_{k+1}} &= \{\Lambda \in G(k+1,n+1) \,|\, \dim(\Lambda \cap V_{n-k+i-c_i}) \geq i \text{ for all } i\} \\ &= \{\mathbb{P}^k \subset \mathbb{P}^n \mid \mathbb{P}^k \cap \mathbb{P}_F^{n-k+i-c_i-1} \supset \mathbb{P}^{i-1} \text{ for all } i\}\end{aligned}$$

The union of the tangent spaces along this cycle is spanned by the $\mathbb{P}^k_t$-s which intersect some element in the cycle in codimension one. Thus the union of tangent spaces is spanned by $\mathbb{P}^k_t$-s which satisfy

$$\mathbb{P}^k_t \cap \mathbb{P}^{n-k+i-c_i-1} \supset \mathbb{P}^{i-2} \tag{2.8}$$

where $i = 1, \ldots, k+1$. Lemma 2.6 implies that any $\mathbb{P}^k$ which satisfies the conditions in equation 2.8 is in some tangent space along σ. The k-spaces which satisfy the conditions in equation 2.8 form the cycle

$$\sigma_{\mathbf{a}} = \{\Lambda \in G(k+1,n+1) \,|\, \dim(\Lambda \cap V_{n-k+1-c_i}) \geq i-1 \text{ for } i = 1, \ldots, k+1\}$$

where $a_{i-1} = c_i - 1$ for $i = 2, \ldots, k+1$ and $a_{k+1} = 0$.

Definition 2.1. *The “spanning-tangent-space”-map*

$$\tau : \{ \textit{ Schubert cycles on } G\} \rightarrow \{ \textit{ Schubert cycles on } G \}$$

maps the cycle $\sigma = \sigma_{c_1\ldots c_{k+1}}$ to the cycle $\tau(\sigma) = \sigma_{c_2-1,\ldots,c_{k+1}-1,0}$. There is a visual way of looking at this map. Figure 2.7 shows an arbitrary cycle and its image under τ.

Proposition 2.8. *The linear span of $\tau(\sigma)$ equals the linear span of the union of tangent spaces along σ.*

Proof. Let $Y = \bigcup_{p\in\sigma} T_pG$. We have seen that T_pG is spanned by all $\mathbb{P}^k$-s which intersect $\mathbb{P}^k_p$ in codimension one. This linear span is contained in the linear span of $\tau(\sigma)$. Thus Y is contained in the linear span of $\tau(\sigma)$. Furthermore, by construction of τ, the linear span of $\tau(\sigma)$ is the smallest linear space which contains Y. Therefore, $\langle Y\rangle = \langle\tau(\sigma)\rangle$. □

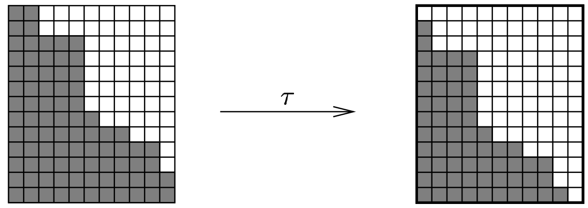


Figure 2.7: Visualization of the map τ

We can pretend that the bottom left corner of the Chow ring of G is the origin of some Cartesian coordinate system. Then the Chow ring itself is an integer grid in the first quadrant of size $(k+1) \times (n-k)$. The image of any cycle is the picture we get after moving the origin to the point $(1,1)$, and drawing an integer grid of the same size with the new origin as the bottom left corner. Figure 2.8 illustrates this idea.

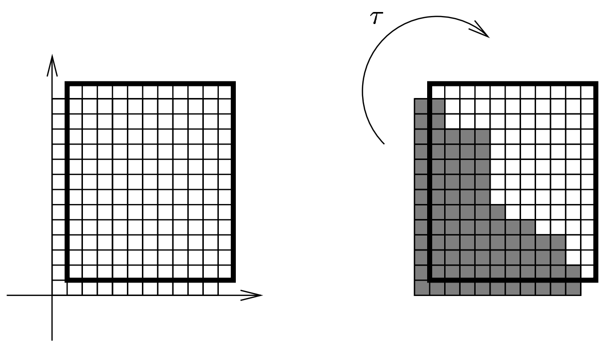


Figure 2.8: Alternative visualization of the map τ

Example 2.5. Figure 2.3 shows the two types of Schubert planes on G. The images of the two planes under the map τ are shown in figure 2.9. Notice that both cycles contain the quadric Q, and that they have the same dimension d. What about the degrees?

The associated sequences to the cycles in figure 2.9 are

$$(1, 2, \dots, k-1, k+2, n) \text{ and } (1, 2, \dots, k-2, k, k+1, n)$$

By applying the formula for the degree of a cycle to these sequences, we see that the degrees are different in general. △

Example 2.6 (Planes on $G(3,6)$). There are two types of planes on $G(3,6)$:

$$\sigma_{331} \text{ and } \sigma_{322}$$

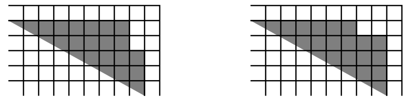


Figure 2.9: The image of the two types of planes on G under τ

The images of these two planes under τ are the cycles

$$\sigma_{200} \text{ and } \sigma_{110}$$

A simple calculations gives that the degrees of the image cycles are the same, namely 21 in both cases. $\triangle$

Example 2.7 (Planes on $G(3,7)$). The images of the two planes

$$\sigma_{442} \text{ and } \sigma_{433}$$

under τ are the cycles

$$\sigma_{310} \text{ and } \sigma_{220}$$

The degrees of the image cycles are 70 and 56. $\triangle$

Proposition 2.9. *Let $\mathbb{P}^r$ and $\tilde{\mathbb{P}}^r$ be the two types of r-planes on $G(k+1,n+1)$. Then the degrees of the images of these planes under the map τ are equal if $k+1=n-k$.*

Proof. We have already seen that the two r-planes are given by

$$\mathbb{P}^r = \{\mathbb{P}^k \subset \mathbb{P}^n | \mathbb{P}_F^{k-1} \subset \mathbb{P}^k \subset \mathbb{P}_F^{k+r}\}$$

and

$$\tilde{\mathbb{P}}^r = \{\mathbb{P}^k \subset \mathbb{P}^n | \mathbb{P}_F^{k-r} \subset \mathbb{P}^k \subset \mathbb{P}_F^{k+1}\}$$

The space $\mathbb{P}_F^{k+r} \subset \mathbb{P}^n$ corresponds to $\mathbb{P}_{\check{F}}^{n-k-r-1} \subset \check{\mathbb{P}}^n$, and $\mathbb{P}_F^{k-1} \subset \mathbb{P}^n$ corresponds to $\mathbb{P}_{\check{F}}^{n-k} \subset \check{\mathbb{P}}^n$. The conditions

$$\mathbb{P}_F^{k-1} \subset \mathbb{P}^k \subset \mathbb{P}_F^{k+r}$$

becomes the conditions

$$\mathbb{P}_{\check{F}}^{n-k} \supset \mathbb{P}^{n-k-1} \supset \mathbb{P}_{\check{F}}^{n-k-r-1}$$

in the dual space, and when $k+1=n-k$, these last conditions are exactly the same conditions defining $\tilde{\mathbb{P}}^r$ above. Thus $\tilde{\mathbb{P}}^r$ is the image of $\mathbb{P}^r$ under the isomorphism

$$\{G(k+1, V^{2k+2}), F\} \longrightarrow \{G(k+1, \check{V}^{2k+2}), \check{F}\}$$

and it follows that their images under τ have the same degree.

$\square$

Now, consider a cycle of the form

$$\sigma = \sigma_{c_1 2 c_3 \dots c_{k+1}} \tag{2.9}$$

where $2 \leq c_1 \leq n-k$ and $1 \geq c_3 \geq \cdots \geq c_{k+1} \geq 0$. For any such cycle, the image under τ is the cycle

$$\tau(\sigma) = \sigma_{10\dots0} = \sigma_1$$

By definition,

$$\begin{aligned}\sigma_1 &= \{\Lambda \in G(k+1, n+1) \mid \dim(\Lambda \cap V_{n-k}) \geq 1\} \\ &= \{\mathbb{P}^k \subset \mathbb{P}^n \mid \mathbb{P}^k \cap \mathbb{P}_F^{n-k-1} \neq \emptyset\}\end{aligned}$$

After a suitable choice of coordinates, we may assume that $\mathbb{P}_F^{n-k-1}$ is represented by a matrix of the form

$$\mathbb{P}_F^{n-k-1} = (I_{n-k} | \ 0)$$

and thus any $\mathbb{P}^k$ in this cycle must be represented by a matrix of the form

$$\mathbb{P}^k = \begin{pmatrix} * & * & \dots & \dots & * & \boxed{0 \ 0 \ \dots \ 0 \ 0} \\ * & * & \dots & \dots & * & * \ * \ \dots \ \dots \ * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\ * & * & \dots & \dots & * & * \ * \ \dots \ \dots \ * \\ * & * & \dots & \dots & * & * \ * \ \dots \ \dots \ * \end{pmatrix}^{(k+1)\times(n+1)}$$

$$\uparrow$$

column number $(n-k+1)$

where the boxed submatrix has dimension $1 \times (k+1)$. There is only one $(k+1) \times (k+1)$-minor of this matrix which is identically zero (the one involving all the $k+1$ last columns). Thus, the linear span of the points in σ_1 has dimension one less than the Plücker space.

Now, consider any cycle of the form

$$\sigma = \sigma_{c_1 2 2 c_4 \dots c_{k+1}} \tag{2.10}$$

where $2 \leq c_1 \leq n-k$ and $1 \geq c_4 \geq \cdots \geq c_{k+1} \geq 0$. For any such cycle,

$$\tau(\sigma) = \sigma_{110\dots0} = \sigma_{11}$$

By definition

$$\begin{aligned}\sigma_{11} &= \{\Lambda \in G(k+1, n+1) | \dim(\Lambda \cap V_{n-k+1}) \geq 2\} \\ &= \{\mathbb{P}^k \subset \mathbb{P}^n | \mathbb{P}^k \cap \mathbb{P}_F^{n-k} \supset \mathbb{P}^1\}\end{aligned}$$

After a suitable choice of coordinates, we may assume that $\mathbb{P}_F^{n-k}$ is represented by a matrix of the form

$$\mathbb{P}_F^{n-k} = (I_{n-k+1}|\ 0)$$

and this gives that any element in σ_{11} must be represented by a matrix of the form

$$\mathbb{P}^k = \left(\begin{array}{ccccc|ccccc|} * & * & \dots & \dots & * & 0 & 0 & \dots & 0 & 0 \\ * & * & \dots & \dots & * & 0 & 0 & \dots & 0 & 0 \\ \hline * & * & \dots & \dots & * & * & * & \dots & \dots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & \dots & \dots & * & * & * & \dots & \dots & * \end{array}\right)^{(k+1)\times(n+1)}$$

$$\uparrow$$

column number $(n-k+2)$

where the boxed submatrix has dimension $2 \times k$. A $(k+1) \times (k+1)$-minor of this matrix is identically zero if and only if it involves only one of the $n-k+1$ first columns. There are $n-k+1$ such minors, and thus the linear span of the points in σ_{11} is the intersection of $n-k+1$ hyperplanes in the Plücker space. In the next sections, we will study two filtrations of the Grassmannians.

2.4 Horizontal filtration

Consider a cycle $\sigma = \sigma_{c_1c_2\dots c_{k+1}}$ where $1 \geq c_1 \geq \cdots \geq c_{k+1} \geq 0$. The image of such a cycle under the map τ is the cycle $\sigma_{00\dots0}$, i.e. the whole Grassmannian. The linear span is thus the Plücker space itself.

Rule 1 (Rule of horizontal filtration). *To the normal intersection rules of Schubert calculus, we add the following: Filling the Chow ring of $G(k+1, n+1)$ is done by intersecting with the cycle σ_1 again and again. The rule of horizontal filtration says that we have to fill a row completely before starting filling a new one.*

In this section, we will stay true to this new law, and it will never be violated. Thus we will always fill the first row completely before starting filling the second, and so on. The smallest possible cycle of the form specified above is $\sigma_{11\dots1}$, and the largest possible cycle contained in this, is the cycle $\sigma_{21\dots1}$. These are shown in figure 2.10. Both of these are mapped to the Grassmannian itself by τ. Thus the union of the tangent spaces along $\sigma_{21\dots1}$ spans the Plücker space. Any cycle smaller than $\sigma_{21\dots1}$ (which obey the rule of horizontal filtration) does not have this property.

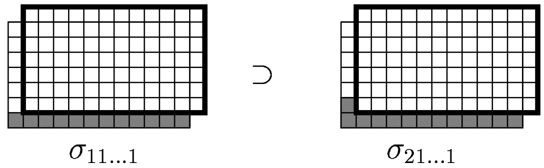


Figure 2.10: Inclusions satisfying the rule of horizontal filtration

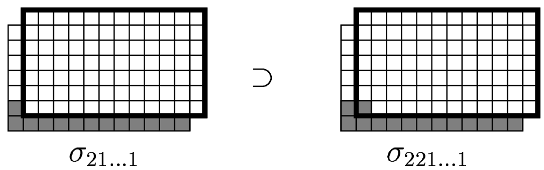


Figure 2.11: Inclusions satisfying the rule of horizontal filtration

The largest cycle contained in $\sigma_{21\ldots1}$ is $\sigma_{221\ldots1}$, which is a cycle of the type specified in equation 2.9. We have seen that the image under τ is σ_1, and if the Plücker space is a $\mathbb{P}^N$, the linear span of this image has dimension $N-1$.

We can continue, and say that the largest cycle contained in $\sigma_{221\ldots1}$ is $\sigma_{2221\ldots1}$, which is a cycle of the form specified in equation 2.10. We have seen that the image under τ is σ_{11}, and that the linear span of the image has dimension $N-(n-k+1)$.

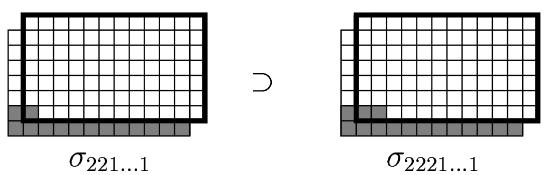


Figure 2.12: Inclusions satisfying the rule of horizontal filtration

Assume now that we have come to a stage in the filtration where we have filled up $r+1$ rows and $s+1$ boxes in row number $r+2$. The situation is illustrated in figure 2.13. The image of this cycle under the map τ is the cycle

$$\begin{aligned}\sigma &= \sigma_{r+1,\ldots,r+1,r,\ldots,r,0}\\ &= \{\Lambda \in G(k+1,n+1) \mid \dim(\Lambda \cap V_{n-k+s-r-1}) \geq s, \dim(\Lambda \cap V_{n-r}) \geq k\}\\ &= \{\mathbb{P}^k \subset \mathbb{P}^n \mid \mathbb{P}^k \cap \mathbb{P}_F^{n-k+s-r-2} \supset \mathbb{P}^{s-1} \text{ and } \mathbb{P}^k \cap \mathbb{P}_F^{n-r-1} \supset \mathbb{P}^{k-1}\}\end{aligned} \tag{2.11}$$

where $r+1$ occurs s times and r occurs $k-s$ times.

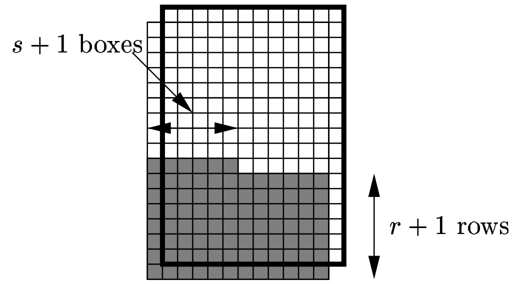


Figure 2.13: A random stage in the horizontal filtration

After a suitable choice of coordinates, we may assume that $\mathbb{P}_F^{n-k+s-r-2}$ is represented by a matrix of the form

$$\mathbb{P}_F^{n-k+s-r-2} = (I_{n-k+s-r-1}|\ 0)$$

and that $\mathbb{P}_F^{n-r-1}$ is represented by a matrix of the form

$$\mathbb{P}_F^{n-r-1} = (I_{n-r}|\ 0)$$

Then any element in the cycle specified in equation 2.11 is represented by a matrix of the form

$$\mathbb{P}^k = \left(\begin{array}{ccc|ccc|ccc} * & \dots & * & 0 & \dots & 0 & 0 & \dots & 0 \\ & & & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & * & \dots & * & 0 & \dots & 0 \\ & & & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \dots & * & * & \dots & * & 0 & \dots & 0 \\ \hline * & \dots & * & * & \dots & * & * & \dots & * \end{array}\right)^{(k+1)\times(n+1)}$$

$$\uparrow \ \dots \ \uparrow$$

columns number $n-r-k+s, \dots, n-r$

where the boxed submatrix has dimension $k\times(r+k-s+2)$, and the non-zero submatrix of the boxed matrix has dimensions $(k-s)\times(k-s+1)$.

A $(k+1)\times(k+1)$-minor of this matrix is non-zero if and only if it involves at least s of the first $n-r-k+s-1$ columns, and at most one of the $r+1$ last columns. Thus the number of maximal minors not identically zero is given by the formula

$$S(r,s,k,n)_h = \sum_{j=s}^{k+1}\binom{n-r-k+s-1}{j}\cdot\left[\binom{k-s+1}{k-j+1}+\binom{k-s+1}{k-j}\cdot(r+1)\right] \tag{2.12}$$

This formula is fairly complicated, and the terms deserve a short explanation. The matrix can be divided into submatrices of the dimensions indicated below:

<table>
<tr><td rowspan="3">$(k+1)\times$
$(n-r-k+s-1)$</td><td colspan="2">$s \times (r+k-s+2)$
zeros only</td></tr>
<tr><td>$(k-s)\times(k-s+1)$
arbitrary numbers</td><td>$(k-s)\times(r+1)$
zeros only</td></tr>
<tr><td colspan="2">$1 \times (r+k-s+2)$ arbitrary numbers</td></tr>
</table>

$\vdash$ special columns $\dashv$

(2.13)

A non-zero minor must involve at least s of the $(n-r-k+s-1)$ first columns. So fix a $j \in \{s, \dots, k+1\}$. If we pick j of the first $(n-r-k+s-1)$ columns, we need to pick $k+1-j$ of the others. We can either pick all the rest out of the $(k-s+1)$ special columns, or we can choose $k-j$ out of the special ones, and one out of the last $(r+1)$. All other choices of columns produce a minor which is identically zero. This explains the formula of equation 2.12.

Example 2.8 ($r+1 = n-k-1$ and $s+1 = k$). With $r+1 = n-k-1$ and $s+1 = k$, we have come to a stage in the filtration where only one box is left to fill. The picture of the situation is similar to the one in figure 2.2. The formula in equation 2.12 becomes

$$\begin{aligned} S(k,n) &= \sum_{j=k-1}^{k+1} \binom{k}{j} \cdot \left[\binom{2}{k-j+1} + \binom{2}{k-j} \cdot (n-k-1) \right] \\ &= \binom{k}{k-1} \left[\binom{2}{2} + \binom{2}{1} \cdot (n-k-1) \right] + \\ &\qquad\qquad \binom{k}{k} \left[\binom{2}{1} + \binom{2}{0} \cdot (n-k-1) \right] \\ &= k[1 + 2(n-k-1)] + [2 + (n-k-1)] \end{aligned}$$

Fortunately, this coincides with the formula for $S_1(k,n)$ given on page 29. $\triangle$

Proposition 2.10. *Let σ be a cycle which appears in the horizontal filtration of $G(k+1, n+1)$, namely the cycle where $r+1$ rows are filled completely and we have filled*

$s+1$ boxes in row number $r+2$. Then the dimension of the linear span of the union of tangent spaces along σ is

$$S(r,s,k,n)_h - 1 =$$
$$\sum_{j=s}^{k+1} \binom{n-r-k+s-1}{j} \cdot \left[\binom{k-s+1}{k-j+1} + \binom{k-s+1}{k-j} \cdot (r+1) \right] - 1$$

Remark: Notice that if we are at a step where $r+1$ rows are filled completely, and no other boxes are filled, we can set $s=0$ in the formula. We can do this because the cycles $\sigma_{r+2,r+1,\ldots,r+1,0,\ldots,0}$ and $\sigma_{r+1,r+1,\ldots,r+1,0,\ldots,0}$ have the same image under τ.

Example 2.9 ($k=1$). When $k=1$, $s+1$ must be either 0 or 1. When $s+1=0$, we are at a stage in the filtration where we have filled $r+1$ rows completely, and all other boxes are empty. The image under τ of such a cycle is the cycle σ_{r0}. The next step in the filtration is keeping r as it was, but setting $s=1$. The image under τ of this cycle is also σ_{r0}. The next step is increasing r by one, and the image under τ is $\sigma_{r+1,0}$. Thus when $k=1$, the filtration is proper every other step. What happens to

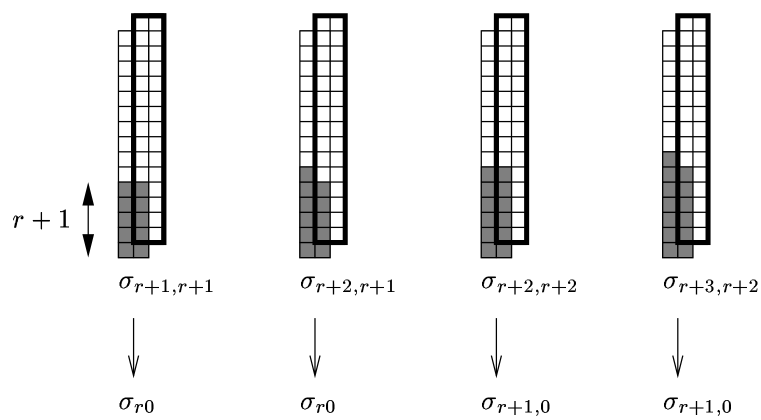


Figure 2.14: Four random steps in the filtration of $G(2,n+1)$ and their images under τ

the formula? The number s is at most zero, so the upper right submatrix in equation 2.13 is not there in this case. As figure 2.14 shows, the number the formula produces must be the same whether $s+1=0$ or $s+1=1$. Thus we can set $s=0$ to find the formula. The "dimension" matrix now becomes

<table>
<tr><td rowspan="2">$2 \times (n-r-2)$</td><td>1×2 (arbit. numbers)</td><td>$1 \times (r+1)$ (zeros only)</td></tr>
<tr><td colspan="2">$1 \times (r+3)$ arbit. numbers</td></tr>
</table>

(2.14)

$\vdash$ 2 special columns $\dashv$

and the formula becomes

$$S(r,n) = \sum_{j=0}^{2} \binom{n-r-2}{j} \left[\binom{2}{2-j} + \binom{2}{2-j-1} \cdot (r+1) \right]$$
$$= [1 + 2 \cdot (r+1)] + (n-r-2)[2 + (r+1)] + \binom{n-r-2}{2}$$

Notice that $S(n-3,n) = [1 + 2(n-2)] + [2 + (n-2)] = 3n - 3$ which is the same result we have on page 32. △

2.5 Vertical filtration

The rule of horizontal filtration states that the boxes must be filled one by one from left to right, and an entire row must be filled before we can start filling a row at a higher level. What if we decide to fill the boxes upward instead?

Rule 2 (Rule of vertical filtration). *To the normal intersection rules of Schubert calculus, we add the following: Filling the Chow ring of $G(k+1, n+1)$ is done by intersecting with the cycle σ_1 again and again. The rule of vertical filtration says that we have to fill a column completely before starting filling a new one.*

In this section, this law will always apply. This will produce a different filtration. Assume we have reached a step in the filtration where we have filled $r+1$ columns completely and $s+1$ boxes in column number $r+2$. The situation is illustrated in figure 2.15.

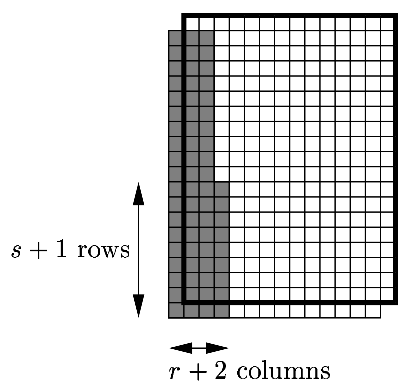


Figure 2.15: A random stage in the vertical filtration

The image of this cycle under the map τ is the cycle

$$\begin{aligned}\sigma &= \sigma_{n-k-1,\ldots,n-k-1,s,0,\ldots,0}\\ &= \{\Lambda \in G(k+1,n+1) |\, \dim(\Lambda \cap V_{r+1}) \geq r, \dim(\Lambda \cap V_{n-k+r+1-s}) \geq r+1\}\\ &= \{\mathbb{P}^k \subset \mathbb{P}^n | \mathbb{P}^k \cap \mathbb{P}^r_F \supset \mathbb{P}^{r-1} \text{ and } \mathbb{P}^k \cap \mathbb{P}^{n-k+r-s}_F \supset \mathbb{P}^r\}\end{aligned}$$

where $n-k-1$ occurs r times. After a suitable choice of coordinates, we may assume that $\mathbb{P}^r_F$ is represented by a matrix of the form

$$\mathbb{P}^r_F = (I_{r+1}|\ 0)$$

and that $\mathbb{P}^{n-k+r-s}_F$ is represented by a matrix of the form

$$\mathbb{P}^{n-k+r-s}_F = (I_{n-k+r+1-s}|\ 0)$$

Then any element in the cycle σ must be represented by a matrix of the form

$$\mathbb{P}^k = \left(\begin{array}{ccc|ccccccc|} * & \cdots & * & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \cdots & \vdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \\ \cline{4-9} * & \cdots & * & * & \cdots & * & * & \cdots & * \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * & * & \cdots & * \end{array}\right)^{(k+1)\times(n+1)}$$

$$\uparrow \ \cdots \ \uparrow$$

columns number $r+2, \ldots, (n-k+r+1-s)$

where the boxed submatrix has dimension $r \times (n-r)$. The corresponding dimension-matrix is

<table>
<tr><td rowspan="3">$(k+1)\times(r+1)$
arbit. numbers</td><td colspan="2">$r \times (n-r)$
zeros only</td></tr>
<tr><td>$(n-k-s)$ numbers</td><td>$(k-r+s)$ zeros</td></tr>
<tr><td colspan="2">$(k-r)\times(n-r)$ arbit. numbers</td></tr>
</table>

(2.15)

$\vdash$ the special columns $\dashv$

A $(k+1)\times(k+1)$-minor of the matrix above is non-zero if and only if it involves at least r out of the $(r+1)$ first columns, and at most $(k-r)$ of the $(k-r+s)$ last ones. Thus the number of maximal minors not identically zero is given by the formula

$$S(r,s,k,n)_v = \binom{n-r}{k-r} + \binom{r+1}{r}\left[\sum_{j=0}^{k-r}\binom{k-r+s}{j}\cdot\binom{n-k-s}{k+1-r-j}\right] \qquad (2.16)$$

The first binomial expression is the number of non-zero minors which involves all the $r+1$ first columns. If we choose r out of the $r+1$ first columns, we need to choose $k+1-r$ more columns to make a minor. At most $k-r$ of these can be among the last $k-r+s$. This proves

Proposition 2.11. *Let σ be a cycle which appears in the vertical filtration of $G(k+1,n+1)$, namely the cycle where $r+1$ columns are filled completely, and we have filled $s+1$ boxes in columns number $r+2$. Then the dimension of the linear span of the union of tangent spaces along σ is*

$$S(r,s,k,n)_v - 1 = \binom{n-r}{k-r} + \binom{r+1}{r}\left[\sum_{j=0}^{k-r}\binom{k-r+s}{j}\cdot\binom{n-k-s}{k+1-r-j}\right] - 1$$

Remark: When we are at a stage when $r+1$ columns are filled completely, and no other box is filled, we can set $s=0$. This is because $\sigma_{n-k,\dots,n-k,0,\dots,0}$ and $\sigma_{n-k,\dots,n-k,1,0,\dots,0}$ have the same image under τ.

2.6 Filtration on Grassmannians of lines

We will now apply the filtration technique on Grassmannians of lines in a projective space.

2.6.1 Horizontal filtration for $G(2,n+1)$

Horizontal filtration for Grassmannians of lines is treated in example 2.9 above. From the matrix in equation 2.14 it is clear that the number of minors not identically zero is

$$S(r,n) = \binom{n-r}{2} + (n-r)(r+1) = \frac{n^2+n-(r^2+r)}{2}$$

As we explained above, the filtration will alternate between proper inclusions and equalities.

Example 2.10 ($G(2,6)$). The Grassmannian $G(2,6)$ has dimension 8 inside a $\mathbb{P}^{14}$. The complete filtration is shown in figure 2.16, where the $\mathbb{P}^i$-s indicate the span of the image under τ. $\triangle$

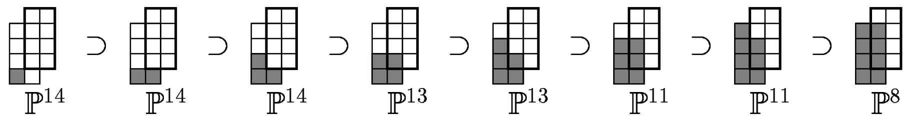


Figure 2.16: All the steps in the horizontal filtration of $G(2,6)$ and their images under τ

2.6.2 Vertical filtration for $G(2, n+1)$

The formula in equation 2.16 becomes the following in the case $k = 1$:

$$\begin{aligned}
S(s,n) &= \binom{n}{1} + \left[\binom{n-1-s}{2} + (s+1)\cdot(n-1-s)\right] \\
&= n + \frac{(n-1-s)(n-2-s)}{2} + (s+1)(n-1-s) \\
&= \frac{2n + (n-1-s)(n-2-s+2s+2)}{2} \\
&= \frac{2n + (n-1-s)(n+s)}{2} \\
&= \frac{n^2 + n - (s^2+s)}{2}
\end{aligned}$$

Thus the linear spans that occur in this filtration will be the same as for the horizontal filtration.

Example 2.11 ($G(2,6)$). The complete filtration is shown in figure 2.17. △

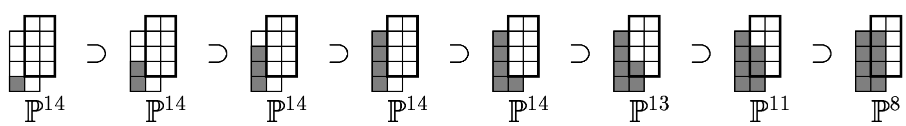


Figure 2.17: All the steps in the vertical filtration of $G(2,6)$ and their images under τ

2.7 Apolar varieties of Schubert cycles on Grassmannians of lines

We have found formulas for the linear span of Y when $Y = \bigcup_{p\in\sigma} T_pG$ for a cycle $\sigma \in G(2, n+1)$ which appears in a filtration. In this section, we will study these varieties for arbitrary cycles σ.

In section 1.2 we described the points in the Plücker space of $G(2, n+1)$ as $(n+1)\times(n+1)$ skew symmetric matrices. The points on the Grassmannian are represented by matrices of rank two, i.e. matrices for which all 4-Pfaffians are zero. How can we find the tangent space of $G(2, n+1)$ in a point p? Let p be represented by the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

that is, the point p is the point where all coordinates except x_{12} is zero. This is a point on the Grassmannian, since the matrix has rank two. The Grassmannian is defined by equations of the form

$$\sum a_{ij}^{kl} x_{ij} x_{kl}$$

where $a_{ij}^{kl} = \pm 1$ and where the indices i, j, k and l are all different numbers between 1 and $n+1$. Now, do partial differentiation:

$$\partial_{st}(a_{ij}^{kl} x_{ij} x_{kl}) = a_{ij}^{st} x_{ij}$$

Evaluating in the point p gives

$$x_{ij}(p) = \delta_{12}^{ij} = \begin{cases} 0 & \text{if } ij \neq 12 \\ 1 & \text{if } ij = 12 \end{cases}$$

When we use partial differentiation and evaluate in p, the only monomials which survive are the ones which involve the coordinate x_{12}. In the quadrics defining the Grassmannian, this coordinate occurs only together with coordinates x_{ij} where $i, j \notin \{1, 2\}$. Thus the only partial differentiations which does not evaluate to zero are the the ∂_{ij}-s where $i, j \notin \{1, 2\}$. The tangent space is therefore the linear space where all coordinates which does not have the numbers 1 or 2 in their index are zero. The points in the tangent space at p is therefore represented by matrices which look like

$$\begin{pmatrix} 0 & 1 & * & * & \cdots & * \\ -1 & 0 & * & * & \cdots & * \\ * & * & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ * & * & 0 & 0 & \cdots & 0 \\ * & * & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where a star means you can put in any complex number. Notice that this matrix has rank at most four, since the $n-1$ bottom rows can have rank at most two. Thus every 6-Pfaffian of this matrix is zero. When we discuss wedge product spaces in detail in chapter 3, it will become apparent that the opposite also is true, i.e. if a point is represented by a matrix of rank four, it is in some tangent space along the Grassmannian.

Now, let Y_σ be the union of the tangent spaces along a Schubert cycle on $G(2,n+1)$, i.e let

$$Y_{\sigma_{c_1,c_2+1}} = \bigcup_{p\in\sigma=\sigma_{c_1,c_2+1}\subset G} T_pG \tag{2.17}$$

where σ_{c_1,c_2+1} is any Schubert cycle. We know that $Y \cap G(2, V^{n+1})$ is the cycle σ_{c_2}, and a point in σ_{c_2} can be represented by a skew symmetric matrix of the form

$$M = \begin{pmatrix} 0 & * & \cdots & * & * & * & \cdots & * \\ * & 0 & \cdots & * & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & \cdots & 0 & * & * & \cdots & * \\ * & * & \cdots & * & 0 & 0 & \cdots & 0 \\ * & * & \cdots & * & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ * & * & \cdots & * & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where the submatrix of zeros has size $(c_2+1)\times(c_2+1)$, and the skew symmetric matrix in the upper left corner has size $(n-c_2)\times(n-c_2)$. Since every point in Y_σ is in some tangent space, they all have rank at most four. Thus the 6-Pfaffians must be zero. We get the inclusion

$$Y \subset Z(x_{ij} \text{ where } n+1-c_2 \le i < j \le n+1\ ,\ \text{6-Pfaffians of } M)$$

Moreover, a point in the zero locus of the 6-Pfaffians of the matrix M and in $Z(x_{ij})$, is in some tangent space, and in fact in Y_σ. Thus the opposite inclusion is true, also.

Example 2.12 ($G(2,7)$). Consider the cycle σ_{32}. By definition

$$\sigma_{32} = \{\mathbb{P}^1 \subset \mathbb{P}^6 \mid \mathbb{P}^1 \cap \mathbb{P}^2_F \supset \mathbb{P}^0 \text{ and } \mathbb{P}^1 \subset \mathbb{P}^4_F\}$$

for some fixed partial flag $\mathbb{P}^2_F \subset \mathbb{P}^4_F$. The same type of argument as above shows that a point in this cycle comes from a line in $\mathbb{P}^6$ which is spanned by the row vectors in the following matrix:

$$p \in \sigma_{32}: \quad \begin{pmatrix} * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & * & * & 0 & 0 \end{pmatrix}$$

The minors $M_{16}, M_{17}, M_{26}, M_{27}, M_{36}, M_{37}, M_{45}, M_{46}, M_{47}, M_{56}, M_{57}$ and M_{67} are all zero, and the cycle is therefore contained in intersection of the corresponding hyperplanes in the Plücker space. Therefore, a point in $\langle\sigma_{32}\rangle$ can be represented by a skew-symmetric matrix of the following form:

$$p \in \langle\sigma_{32}\rangle : \qquad \begin{pmatrix} 0 & * & * & * & * & 0 & 0 \\ * & 0 & * & * & * & 0 & 0 \\ * & * & 0 & * & * & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where a star means you can put in any complex number. The image under τ is the cycle σ_{10}, and the lines in this cycle are all represented by a (2×7)-matrix of the form

$$p \in \sigma_{10} : \qquad \begin{pmatrix} * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & * & * \end{pmatrix}$$

The only minor which is identically zero is M_{67}, and thus the corresponding (7×7) skew symmetric matrix looks like this:

$$p \in \langle\sigma_{10}\rangle : \qquad \begin{pmatrix} 0 & * & * & * & * & * & * \\ * & 0 & * & * & * & * & * \\ * & * & 0 & * & * & * & * \\ * & * & * & 0 & * & * & * \\ * & * & * & * & 0 & * & * \\ * & * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 & 0 \end{pmatrix}$$

where a star means you can put in any complex number. We have seen that the points in a tangent space are represented by matrices of rank 4, so any point of this form *and* of rank 4 is in some tangent space along σ_{32}. Equivalently, any point of this form *and* with vanishing 6-Pfaffians, is in some tangent space along σ_{32}. $\triangle$

If Y_σ is a variety of the type

$$Y_{\sigma_{c_1,c_2+1}} = \bigcup_{p \in \sigma_{c_1,c_2+1}} T_pG$$

what is the dimension of Y_σ? Consider the incidence

$$\textstyle\coprod_{p\in\sigma} T_pG = \{(p,q) \in \sigma \times \mathbb{P}^{\binom{n+1}{2}-1} \mid q \in T_pG\}$$

$$\sigma \xleftarrow[\pi_1]{\mathbb{P}^{2(n-1)}} \textstyle\coprod_{p\in\sigma} T_pG \xrightarrow[\pi_2]{3} Y = \bigcup_{p\in\sigma} T_pG$$

For a fixed point p in σ the fiber $\pi_1^{-1}(p)$ is given by

$$\pi_1^{-1}(p) = \{q \in \mathbb{P}^{\binom{n+1}{2}-1} \mid q \in T_pG\}$$

But this is the tangent space T_pG itself, and thus the fiber over p is a $\mathbb{P}^{2(n-1)}$. Next, fix a point q in Y_σ. The fiber $\pi_2^{-1}(q)$ is given by

$$\pi_2^{-1}(q) = \{p \in \sigma \mid q \in T_pG\}$$

The point q is a point in Y_σ, so it is in some tangent space to $G(2, n+1)$. Thus q is a point of rank four. Therefore it is a point in the linear span of some $G(2,4)$, i.e in a $\mathbb{P}(\wedge^2 W^4)$. As a point in this $\mathbb{P}^5$, how many tangent spaces of $G(2,4)$ is q in? The $G(2,4)$ is defined by a quadric Q, and if q has coordinates $q_0, \ldots, q_5$, the polar of Q in q is given by

$$P_q(Q) = q_0\partial_0(Q) + q_1\partial_1(Q) + \cdots + q_5\partial_5(Q)$$

The zero locus of the polar is a hyperplane in $\mathbb{P}^5$, and the intersection with $G(2,4)$ is a family of dimension three. Take any point p in this intersection, and consider the tangent space of $G(2,4)$ in p. Then

$$p \in Z(Q) \cap Z(q_0\partial_0 Q + q_1\partial_1 Q + \cdots + q_5\partial_5 Q)$$

and the tangent space of $Z(Q)$ in p is

$$Z(Q(p) + (\partial_0 Q)(p)(x_0 - p_0) + \cdots + (\partial_5 Q)(p)(x_5 - p_5))$$

Now, evaluate the polynomial defining the tangent space in q:

$$\begin{aligned}
&[Q(p) + (\partial_0 Q)(p)(x_0 - p_0) + \cdots + (\partial_5 Q)(p)(x_5 - p_5)](q) = \\
&\qquad = (\partial_0 Q)(p)(q_0 - p_0) + \cdots + (\partial_5 Q)(p)(q_5 - p_5) \\
&\qquad = q_0(\partial_0 Q)(p) + \cdots + q_5(\partial_5 Q)(p) - (p_0(\partial_0 Q)(p) + \cdots + p_5(\partial_5 Q)(p)) \\
&\qquad = 0 - (p_0(\partial_0 Q)(p) + \cdots + p_5(\partial_5 Q)(p)) \\
&\qquad = 0
\end{aligned}$$

Thus the point q lies in $T_pG(2,4)$. There was nothing special about p, so q is in a three dimensional family of tangent spaces. We are in a situation which can be illustrated as

$$\begin{array}{cccc}
q \in & \mathbb{P}(\wedge^2 W^4) & \subset & \mathbb{P}(\wedge^2 V^{n+1}) \\
& \cup & & \cup \\
& G(2,4) & \subset & G(2, n+1)
\end{array}$$

We have seen that q is in a three dimensional family of tangent spaces on the small Grassmannian. But then it is also in a three dimensional family of tangent spaces on $G(2, n+1)$. On the other hand, if q is in a tangent space for $G(2, n+1)$, this tangent space restricts to a tangent space for $G(2,4)$. Thus q is in a family of tangent spaces for $G(2,n+1)$ of dimension exactly three. This implies that the fiber dimension for π_2 in the incidence above is three, and the dimension of Y_σ is therefore as in the following proposition:

Proposition 2.12. *The dimension of the union of the tangentspaces along the cycle* σ_{c_1,c_2+1} *is* $4(n-2)-c_1-c_2$.

Proof. The union of the tangentspaces along σ_{c_1,c_2+1} is the variety $Y_{\sigma_{c_1,c_2+1}}$ and

$$\begin{aligned}\dim Y_{\sigma_{c_1,c_2+1}} &= \dim \sigma_{c_1,c_2+1} + 2(n-1) - 3\\ &= 2(n-1) - c_1 - c_2 - 1 + 2(n-1) - 3\\ &= 4(n-2) - c_1 - c_2\end{aligned}$$

This proves the statement. ☐

The dimension of $Y \cap G(2,n+1)$ equals the dimension of σ_{c_2}, which is equal to $2(n-1)-c_2$, and the codimension of $Y \cap G$ in Y_σ is therefore

$$\begin{aligned}\dim Y - \dim(Y\cap G) &= 4(n-2) - c_1 - c_2 - 2(n-1) + c_2\\ &= 2(n-3) - c_1\end{aligned}$$

The number c_1 is an index for a cycle on $G(2,n+1)$, and it is therefore less than or equal to $n-1$. The codimension of $Y\cap G$ in Y_σ is therefore greater than or equal to $2(n-3)-(n-1)=n-5$. Notice that the codimension is one only when $n=6$ and c_1 is maximal. In all other cases, the codimension is strictly greater than one.

We have seen that varieties Y_σ of the from given in equation 2.17 are apolar varieties of Schubert cycles on G(2,n+1). However, these apolar varieties are not the nicest, since their dimension generally is too big. In the next chapter, we will find apolar varieties which *are* nice.

Chapter 3

2-forms of low rank

In this chapter, we will exclusively deal with Grassmannians of lines. As mentioned above, a point on such a Grassmannian may be considered as a skew symmetric matrix. First, we will thoroughly investigate the Plücker space of Grassmannians of lines. Next, we use our new knowledge to find good apolar varieties for Schubert cycles on $G(2, n+1)$.

Recall that for a smooth variety X in $\mathbb{P}^n$, a hyperplane H in $\mathbb{P}^n$ is called a *tangent hyperplane to* X if it contains a tangent space to X. The locus of tangent hyperplanes to X is called the *dual variety of* X, and is denoted $\check{X} \subset \check{\mathbb{P}}^n$.

Theorem 3.1. *The dual variety* $\check{G}(2, n+1) \subset \check{\mathbb{P}}(\wedge^2\mathbb{C}^{n+1})$ *of the Grassmannian* $G(2, n+1)$ *consists of matrices of corank* ≥ 2 *for* n *odd and it consists of matrices of corank* ≥ 3 *when* n *is even. Moreover, for* n *odd, the dual variety is a hypersurface of degree* $\frac{n+1}{2}$*; for* n *even it is a subvariety of codimension* 3.

For a proof of this theorem, see [20]. We will also give a proof of this theorem later, when we have established the language of forms.

Let V^{n+1} be a vector space of dimension $n+1$ over $\mathbb{C}$. Every subvector space of dimension two is spanned by two independent vectors $u_1, u_2 \in V^{n+1}$. Let $G = G(2, V^{n+1})$ be the Grassmannian of all two dimensional subvector spaces of V^{n+1}. The Plücker embedding maps the subspace $U = \langle u_1, u_2 \rangle$ to the point $u_1 \wedge u_2 \in \wedge^2 V^{n+1}$.

3.1 Wedge product spaces

In this section, we will construct a filtration of the Plücker space of the Grassmannian of lines. We treat the cases when n is odd and when n is even separately. Moreover, we define what we mean by the *support* of a point in the Plücker space, and we construct a filtration in the dual space.

Let $\{v_1, v_2, \ldots, v_n, v_{n+1}\}$ be a basis for V^{n+1}. The operation $\wedge$ is defined in such a way that $v_i \wedge v_j = -v_j \wedge v_i$, and $v_i \wedge v_i = 0$. Therefore, a basis for $\wedge^2 V^{n+1}$ is

$\{v_i \wedge v_j | i < j\}$. There are $\binom{n+1}{2}$ possible ways of choosing v_i and v_j such that $i < j$, and thus the dimension of $\wedge^2 V^{n+1}$ is $\binom{n+1}{2}$. A general vector v in $\wedge^2 V^{n+1}$ can be written

$$\begin{aligned} v &= \sum_{1 \leq i < j \leq n+1} 2a_{ij} v_i \wedge v_j \\ &= 2a_{12} v_1 \wedge v_2 + 2a_{13} v_1 \wedge v_3 + \cdots + 2a_{n,n+1} v_n \wedge v_{n+1} \end{aligned} \tag{3.1}$$

Since $v_i \wedge v_j = -v_j \wedge v_i$, we can instead write v as

$$v = a_{12} v_1 \wedge v_2 - a_{12} v_2 \wedge v_1 + \cdots + a_{n,n+1} v_n \wedge v_{n+1} - a_{n,n+1} v_{n+1} \wedge v_n$$

Now the coefficients fit into an $(n+1) \times (n+1)$ skew symmetric matrix:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1,n+1} \\ -a_{12} & 0 & & & \vdots \\ -a_{13} & & 0 & & \vdots \\ \vdots & & & \ddots & \vdots \\ & & & & a_{n,n+1} \\ -a_{1,n+1} & \cdot\cdot & \cdot\cdot & -a_{n,n+1} & 0 \end{pmatrix}$$

Thus $v \in \wedge^2 V^{n+1}$ can be represented by a $(n+1) \times (n+1)$ skew symmetric matrix. On the other hand, every such matrix represents a vector in $\wedge^2 V^{n+1}$, so we can say that the vector v in $\wedge^2 V^{n+1}$ *is* a $(n+1) \times (n+1)$ skew symmetric matrix. The number n may be either odd or even, and we treat these two cases separately:

The number n is odd

If n is an odd number, $n+1$ is even. A general skew symmetric matrix of even size $n+1$ has rank $n+1$. A matrix has maximal rank if and only if its determinant is non-zero.

Recall the fact that if M is a skew symmetric matrix of even size $2k \times 2k$, its determinant is the square of a polynomial. This polynomial is called the *Pfaffian* of M, or the $2k$-Pfaffian, and it is denoted $\mathrm{Pf}(M)$ or $\mathrm{Pf}_{2k}(M)$.

$$\det(M) = \mathrm{Pf}(M)^2$$

Since the degree of the determinant is $2k$, the degree of the Pfaffian is k. Let M_{ij} be the skew symmetric submatrix of M obtained by deleting rows and columns number i and j. Then M_{ij} has maximal rank if and only if its Pfaffian is non-zero. We call this Pfaffian $\mathrm{Pf}_{ij}(M)$, and think of it as a $(2k-2)$-Pfaffian of M. Notice that the square of this Pfaffian is a $(2k-2)$-minor of M. If $\mathrm{Pf}(M)$ is zero, the matrix M has rank at most $2k-2$. If all the $(2k-2)$-Pfaffians of M are zero, M has rank at most $(2k-4)$. By deleting pairs of rows and columns, we can define Pfaffians of any even size.

The general point in $\wedge^2 V^{n+1}$ is represented by a skew symmetric matrix of maximal rank. If n is an odd number, this maximal rank is $n+1$. Some of the points in $\wedge^2 V^{n+1}$ are represented by matrices of rank $n-1$, and these all have the property that their $(n+1)$-Pfaffian is zero, but not all $(n-1)$-Pfaffians are zero. Inside this subset are the points represented by matrices of rank $n-3$. These all have the property that all their $(n-1)$-Pfaffians are zero, but they have at least one $(n-3)$-Pfaffian which is non-zero.

The number n is even

When n is even, $n+1$ is an odd number. Since a skew symmetric matrix always has even rank, such a matrix of odd size must have determinant zero. Thus when n is even, a general point in $\wedge^2 V^{n+1}$ is represented by a matrix M of rank n. If M_i is the skew symmetric submatrix obtained by deleting the i-th row and column of M, let $\mathrm{Pf}_i(M)$ denote the maximal Pfaffian of M_i. Notice that this is an n-Pfaffian. Then all general points in $\wedge^2 V^{n+1}$ are represented by matrices with the property that their determinant is zero, but not all n-Pfaffians are zero. By deleting an odd number of corresponding rows and columns, we can define Pfaffians of any size also in this case.

The matrices or rank $n-2$ form a subset inside $\wedge^2 V^{n+1}$. These all have the property that all n-Pfaffians are zero, but there is at least one $(n-2)$-Pfaffian which is non-zero. A subset of these matrices are the matrices of rank $n-4$. These all have the property that all the $(n-2)$-Pfaffians are zero, but they all have at least one $(n-4)$-Pfaffian which is non-zero.

We are now ready to define the projectivized version of the vector space $\wedge^2 V^{n+1}$. A 2-vector in $\wedge^2 V^{n+1}$ as defined in equation 3.1 is only defined up to multiplication with a scalar. If U is a two dimensional linear subspace spanned by the vectors u_1 and u_2, we associate to U the 2-vector $u_1 \wedge u_2 \in \wedge^2 V^{n+1}$. If we choose a different basis for U, the corresponding 2-vector is equal to $u_1 \wedge u_2$ multiplied with the determinant of the change of basis matrix.

Example 3.1. Let U be spanned by the vectors u_1 and u_2. To U we associate the 2-vector $u_1 \wedge u_2$. The vectors $v_1 = 2(u_2 + u_2)$ and $v_2 = u_1 - u_2$ span the same two dimensional linear space, so $v_1 \wedge v_2$ should define the same 2-vector. Now

$$\begin{aligned} v_1 \wedge v_2 &= 2(u_1 + u_2) \wedge (u_1 - u_2) \\ &= -2u_1 \wedge u_2 + 2u_2 \wedge u_1 \\ &= -2u_1 \wedge u_2 - 2u_1 \wedge u_2 \\ &= -4u_1 \wedge u_2 \end{aligned}$$

and the change of basis matrix is the (2×2)-matrix

$$\begin{pmatrix} 2 & 1 \\ 2 & -1 \end{pmatrix}$$

This matrix has determinant -4. $\triangle$

Since the elements in $\wedge^2 V^{n+1}$ are only defined up to multiplication with a scalar, the projectivized version $\mathbb{P}(\wedge^2 V^{n+1})$ is well defined. This projective space has dimension $\binom{n+1}{2} - 1$. This gives a well defined map of sets

$$G(2, v^{n+1}) \to \mathbb{P}(\wedge^2 V^{n+1})$$

sending $\langle u_1, u_2 \rangle$ to the point $u_1 \wedge u_2$ as we have seen above. Since $\langle u_1, u_2 \rangle$ is two dimensional, the skew symmetric matrix corresponding to the point $u_1 \wedge u_2 \in \mathbb{P}(\wedge^2 V^{n+1})$ has rank exactly two. Thus the points in $\mathbb{P}(\wedge^2 V^{n+1})$ which lies on the Grassmannian, are all represented by matrices of rank two. All matrices of rank two have the property that all their 4-Pfaffians are zero. Thus the Grassmannian is actually a variety in $\mathbb{P}(\wedge^2 V^{n+1})$ defined by the vanishing of all 4-Pfaffians of the matrix

$$\begin{pmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1,n+1} \\ -x_{12} & 0 & & & \vdots \\ -x_{13} & & 0 & & \vdots \\ \vdots & & & \ddots & \vdots \\ & & & & x_{n,n+1} \\ -x_{1,n+1} & \cdot\cdot & \cdot\cdot & -x_{n,n+1} & 0 \end{pmatrix} \tag{3.2}$$

where x_{ij} is the coordinate in $\mathbb{P}(\wedge^2 V^{n+1})$ corresponding to the basisvector $v_i \wedge v_j$ of $\wedge^2 V^{n+1}$.

Now, let X_i be the set of all skew symmetric matrices in $\wedge^2 V^{n+1}$ of rank exactly i. We have seen that i can be any even number between 0 and $n+1$. The matrices in X_i all have the property that their $(i+2)$-Pfaffians are zero, and X_i is therefore a variety in $\mathbb{P}(\wedge^2 V^{n+1})$ defined by the vanishing of all $(i+2)$-Pfaffians in the matrix given in equation 3.2.

When n is odd, we get the following chain of inclusions in $\mathbb{P}(\wedge^2 V^{n+1})$:

$$G(2, V^{n+1}) = X_2 \subset X_4 \subset \cdots \subset X_i \subset \cdots \subset X_{n-3} \subset X_{n-1} \subset \mathbb{P}(\wedge^2 V^{n+1}) \tag{3.3}$$

and when n is even the chain looks like this:

$$G(2, V^{n+1}) = X_2 \subset X_4 \subset \cdots \subset X_i \subset \cdots \subset X_{n-4} \subset X_{n-2} \subset \mathbb{P}(\wedge^2 V^{n+1}) \tag{3.4}$$

In both cases, i is an even number. A matrix of rank i in $\wedge^2 V^{n+1}$ comes from a subspace of dimension i inside V^{n+1}. The family of all such subspaces of V^{n+1} have dimension

$\dim G(i, n+1)$. Inside each subspace of dimension i there is a family of dimension $\dim \mathbb{P}(\wedge^2 U^i)$ of matrices of rank i. Thus the codimension of X_i in $\mathbb{P}(\wedge^2 V^{n+1})$ is

$$\begin{aligned} \text{codim} X_i &= \binom{n+1}{2} - 1 - \left[\dim G(i, n+1) + \binom{i}{2} - 1\right] \\ &= \binom{n+1-i}{2} \end{aligned} \tag{3.5}$$

The codimensions are listed below for some choices of i:

n odd	codimX_i	n even	codimX_i
$i = n-1$	1	$i = n-2$	3
$i = n-3$	6	$i = n-4$	10
$i = n-5$	15	$i = n-6$	21
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$i = 2$	$\frac{(n-1)(n-2)}{2}$	$i = 2$	$\frac{(n-1)(n-2)}{2}$

The chains of inclusions of equations 3.3 and 3.4 can now be made even more complete.

Proposition 3.2. *Let X_i be the set of all skew matrices in $\wedge^2 V^{n+1}$ of rank i. Then we have the following filtrations, where the lower index indicates the rank as above, and the upper index gives the codimension of the variety:*

$$G(2, V^{n+1}) = X_2 \subset X_4 \subset \cdots \subset X_{n-3}^6 \subset X_{n-1}^1 \subset \mathbb{P}(\wedge^2 V^{n+1}) \qquad n \textbf{ is odd} \tag{3.6}$$

$$G(2, V^{n+1}) = X_2 \subset X_4 \subset \cdots \subset X_{n-4}^{10} \subset X_{n-2}^3 \subset \mathbb{P}(\wedge^2 V^{n+1}) \qquad n \textbf{ is even} \tag{3.7}$$

We have seen that every point in $\mathbb{P}(\wedge^2 V^{n+1})$ is represented by a skew symmetric matrix of size $n+1$. This matrix can have any even rank between 2 and $n+1$, and the points on $G(2, V^{n+1})$ inside $\mathbb{P}(\wedge^2 V^{n+1})$ are all represented by matrices of rank two. If a point $p = \sum_{1 \le i < j \le n+1} 2a_{ij} v_i \wedge v_j \in \mathbb{P}(\wedge^2 V^{n+1})$ is represented by a matrix of rank i we will say that the *point has rank i* or *the vector $\sum_{1 \le i < j \le n+1} 2a_{ij} v_i \wedge v_j$ has rank i.* A vector in $\wedge^2 V^{n+1}$ is sometimes called a 2-vector or a 2-form.

We have seen that any element in $\wedge^2 V^{n+1}$ has rank at most $n+1$. If $\{v_1, \ldots, v_{n+1}\}$ is a basis for V^{n+1}, let L_i be linear forms in these basis vectors. Then a general element ω in $\wedge^2 V^{n+1}$ can be written as

$$\omega = L_1 \wedge L_2 + L_3 \wedge L_4 + \cdots + L_n \wedge L_{n+1}$$

Definition 3.1 (Support). *If ω has rank i, the linear span $\langle L_1, \ldots, L_{n+1}\rangle$ is a linear subspace of V^{n+1} of dimension i. We call this linear space the* support *of ω. Similarly, if we view ω as a point in $\mathbb{P}(\wedge^2 V^{n+1})$, the support of ω is a projective space of dimension $i-1$, which we will denote by $\mathbb{P}^{i-1}_\omega$ or simply $|\omega|$.*

Notice that a point on the Grassmannian $G(2, V^{n+1})$ inside $\mathbb{P}(\wedge^2 V^{n+1})$ has rank two, so its support is a projective line. This makes sense, since a point on the Grassmannian corresponds to a line in the space $\mathbb{P}(V^{n+1})$. There is a map

$$\theta : \wedge^2 V^{n+1} \times \wedge^{n-1} V^{n+1} \to \wedge^{n+1} V^{n+1} \simeq \mathbb{C}$$

First of all, the $(n+1)$-th wedge product of a vector space over $\mathbb{C}$ of dimension $n+1$ has dimension $\binom{n+1}{n+1} = 1$. If $\{v_1, \ldots v_{n+1}\}$ is a basis for V^{n+1}, every element in $\wedge^{n+1}V^{n+1}$ can be written as $av_1 \wedge v_2 \wedge \cdots \wedge v_{n+1}$ for some $a \in \mathbb{C}$. Thus the map θ above takes the pair $(v_{i_1} \wedge v_{i_2}, v_{j_1} \wedge \cdots \wedge v_{j_{n-1}})$ of basis vectors to $v_{i_1} \wedge v_{i_2} \wedge v_{i_1} \wedge \cdots \wedge v_{j_{n-1}})$ which is either zero or $\pm v_1 \wedge v_2 \wedge \cdots \wedge v_{n+1}$.The isomorphism $\wedge^{n+1}V^{n+1} \simeq \mathbb{C}$ identifies $av_1 \wedge \cdots \wedge v_{n+1}$ with a.

Now, let ω_i be elements in $\wedge^2 V^{n+1}$ and let ω'_i be elements in $\wedge^{n+1}V^{n+1}$. Then

$$(\omega_1, \omega'_1 + \omega'_2) \overset{\theta}{\mapsto} \omega_1 \wedge (\omega'_1 + \omega'_2) = \omega_1 \wedge \omega'_1 + \omega_1 \wedge \omega'_2$$

and

$$(\omega_1 + \omega_2, \omega'_1) \overset{\theta}{\mapsto} (\omega_1 + \omega_2) \wedge \omega'_1 = \omega_1 \wedge \omega'_1 + \omega_2 \wedge \omega'_1$$

so θ is bilinear.

Consider the basis vector $v_1 \wedge v_2$ of $\wedge^2 V^{n+1}$. A form $\omega' \in \wedge^{n-1}V^{n+1}$ is such that $\theta(v_1 \wedge v_2, \omega') = 0$ if and only if the basis vector $v_3 \wedge \cdots \wedge v_{n+1}$ is not involved in ω'. Thus the right kernel of the map is the vector space $\wedge^{n-1}V^{n+1}/(v_3 \wedge \cdots \wedge v_{n+1})$. This vector space has dimension $\binom{n+1}{n-1} - 1$, and we can think of it as a hyperplane in $\mathbb{P}(\wedge^{n-1}V^{n+1})$.

Similarly, consider the basisvector $v_1 \wedge \cdots \wedge v_{n-1}$ in $\wedge^{n-1}V^{n+1}$. A 2-form ω is such that $\theta(\omega, v_1 \wedge \cdots \wedge v_{n-1}) = 0$ if and only if the basis vector $v_n \wedge v_{n+1}$ is not involved in ω. Thus the left kernel of θ is the vector space $\wedge^2 V^{n+1}/(v_n \wedge v_{n+1})$. This vector space has dimension $\binom{n+1}{2} - 1$, and we can think of it as a hyperplane in $\mathbb{P}(\wedge^2 V^{n+1})$.

This shows that a point in $\mathbb{P}(\wedge^2 V^{n+1})$ corresponds to a hyperplane in $\mathbb{P}(\wedge^{n-1}V^{n+1})$ and a point in $\mathbb{P}(\wedge^{n-1}V^{n+1})$ corresponds to a hyperplane in $\mathbb{P}(\wedge^2 V^{n+1})$. Thus the projective spaces $\mathbb{P}(\wedge^2 V^{n+1})$ and $\mathbb{P}(\wedge^{n-1}V^{n+1})$ are naturally dual to each other.

$$\check{\mathbb{P}}(\wedge^2 V^{n+1}) = \mathbb{P}(\wedge^{n-1}V^{n+1}) \quad \text{and} \quad \check{\mathbb{P}}(\wedge^{n-1}V^{n+1}) = \mathbb{P}(\wedge^2 V^{n+1})$$

Let $\omega \in \wedge^2 V^{n+1}$ be an element of even rank i. Then the support $|\omega|$ is a projective space of dimension $i-1$, denoted $\mathbb{P}^{i-1}_\omega$. Remember that every basisvector in $\wedge^2 V^{n+1}$ has a line as its support, and the support of ω is the linear span of the union of the supports of the basisvectors involved in the expression of ω. Since a projective space of dimension $i-1$ is spanned by $\frac{i}{2}$ disjoint lines, the form ω can be written as a sum of $\frac{i}{2}$ forms of rank two, all with disjoint supports.

$$\omega = \omega_1 + \cdots + \omega_{\frac{i}{2}}$$

where all ω_j have rank two and where $\mathbb{P}^1_{\omega_{j_1}} \cap \mathbb{P}^1_{\omega_{j_2}} = \emptyset$ for all $j_1, j_2 \in \{1, \ldots, \frac{i}{2}\}$. Since the supports are disjoint, we can change the basis of V^{n+1} in such a way that the ω_j-s become basisvectors of $\wedge^2 V^{n+1}$.

The discussion above gives that every basisvector in $\wedge^2 V^{n+1}$ corresponds to a basisvector in $\wedge^{n-1} V^{n+1}$. Thus after a suitable change of basis, every ω_j corresponds to a basisvector $\check{\omega}_j \in \wedge^{n-1} V^{n+1}$. The support of $\check{\omega}_j$ is a projective space of dimension $n-2$ which we denote by $\mathbb{P}^{n-2}_{\check{\omega}_j}$. This space is the dual of $\mathbb{P}^1_{\omega_j}$ in $\mathbb{P}^n = \mathbb{P}(V^{n+1})$. The dual of the union of the lines $\mathbb{P}^1_{\omega_j}$ is the intersection of the spaces $\mathbb{P}^{n-2}_{\check{\omega}_j}$:

$$\bigcap_{j=1}^{\frac{i}{2}} \mathbb{P}^{n-2}_{\check{\omega}_j} = \mathbb{P}^{\frac{i}{2}(n-2)-(\frac{i}{2}-1)n} = \mathbb{P}^{n-i}$$

This is exactly the dual of $\mathbb{P}^{i-1}_{\omega}$. We can therefore to ω associate a $(n-1)$-form $\check{\omega}$ defined as

$$\check{\omega} = \check{\omega}_1 + \cdots + \check{\omega}_{\frac{i}{2}}$$

where none of the $\check{\omega}_j$-s are sums of other $(n-1)$-forms, and $|\check{\omega}| = \cap_{j=1}^{\frac{i}{2}} |\check{\omega}_j| = \check{|\omega|}$. For every even i, an element in $\omega \in X_i$ can be associated to a $(n-1)$-form $\check{\omega}$ using the above argument. Let $\check{X}_i$ denote the collection of such $(n-1)$-forms. The correspondence $\omega \leftrightarrow \check{\omega}$ is one to one by construction, so $\check{X}_i$ has the same codimension in $\mathbb{P}(\wedge^{n-1} V^{n+1})$ as X_i has in $\mathbb{P}(\wedge^2 V^{n+1})$. Remember the chain of inclusions in $\mathbb{P}(\wedge^2 V^{n+1})$ given in equations 3.6 and 3.7. Using the correspondence we just constructed, we get a similar chain of inclusions in $\mathbb{P}(\wedge^{n-1} V^{n+1}) = \check{\mathbb{P}}(\wedge^2 V^{n+1})$:

Proposition 3.3. *Let X_i be the set of all skew symmetric matrices in $\wedge^2 V^{n+1}$ of rank i. To an element $\omega \in X_i$ we can associate an element $\check{\omega}$ in $\wedge^{n-1} V^{n+1}$ for which $|\check{\omega}| = \check{|\omega|}$. Let $\check{X}_i$ be the collection of such $(n-1)$-forms. Then we have the following filtrations:*

$$G(n-1, V^{n+1}) = \check{X}_2 \subset \check{X}_4 \subset \cdots \subset \check{X}^6_{n-3} \subset \check{X}^1_{n-1} \subset \mathbb{P}(\wedge^{n-1} V^{n+1}) \; n \textbf{ is odd} \quad (3.8)$$

$$G(n-1, V^{n+1}) = \check{X}_2 \subset \check{X}_4 \subset \cdots \subset \check{X}^{10}_{n-4} \subset \check{X}^3_{n-2} \subset \mathbb{P}(\wedge^{n-1} V^{n+1}) \; n \textbf{ is even} \quad (3.9)$$

Definition 3.2. *Let M be a linear space in $\mathbb{P}(\wedge^r V^s)$ for some natural numbers $r \leq s$. The* orthogonal space *of M is denoted $M^{\perp}$, and is defined as*

$$M^{\perp} := \{\check{\omega} \in \mathbb{P}(\wedge^{s-r} V^s) | \check{\omega} \wedge \omega = 0 \text{ for all } \omega \in M\}$$

Example 3.2 **($G(2,8)$).** Let $\{v_1,\dots,v_8\}$ be a basis for V^8. The Grassmannian $G(2,V^8)$ sits inside $\mathbb{P}(\wedge^2 V^8)$, and a general point in the Plücker space is a 2-form of rank 8. Inside $\mathbb{P}(\wedge^2 V^8)$ the forms of rank 6 form a variety X_6 of codimension one. Every element ω in X_6 can be written as a sum of three forms of rank two. Inside X_6 is the variety X_4 of forms of rank four. All elements here can be written as a sum of two forms of rank two. Finally, inside X_4 are the forms of rank two. The collection of these forms is exactly the Grassmannian $G(2,8)$. Let ω_j be forms of rank two. The first row in the following equation gives the chain of inclusions in the Plücker space in this case. The second row indicates what a general element in the corresponding variety looks like. The third row indicates the support of a general element:

$$\begin{array}{ccccccc} G(2,8)^{15} & \subset & X_4^6 & \subset & X_6^1 & \subset & \mathbb{P}(\wedge^2 V^8) \\ \omega=\omega_1 & & \omega=\omega_1+\omega_2 & & \omega=\omega_1+\omega_2+\omega_3 & & \omega=\omega_1+\cdots+\omega_4 \\ |\omega|=\mathbb{P}^1_\omega & & |\omega|=\mathbb{P}^3_\omega & & |\omega|=\mathbb{P}^5_\omega & & |\omega|=\mathbb{P}^7_\omega \end{array}$$

△

If ω_1 and ω_2 are two elements in $\wedge^2 V^{n+1}$ of rank i, their supports $\mathbb{P}^{i-1}_{\omega_1}$ and $\mathbb{P}^{i-1}_{\omega_2}$ are contained in $\mathbb{P}(V^{n+1})=\mathbb{P}^n$. Thus they generally intersect in a $\mathbb{P}^{2(i-1)-n}$. Suppose the supports intersects in a $\mathbb{P}^{2i-1-n}$. Then they are both contained in $\langle\mathbb{P}^{i-1}_{\omega_1},\mathbb{P}^{i-1}_{\omega_2}\rangle=\mathbb{P}^{n-1}\subset\mathbb{P}^n$, and there exists a n-dimensional subvector space $W^n\subset V^{n+1}$ such that ω_1 and ω_2 are elements in $\wedge^2 W^n$.

3.2 The dual variety and families of $G(2,4)$-s

In this section, we will describe the dual variety of $G(2,n+1)$. We treat the cases when n is odd and when n is even separately. Furthermore, we investigate the linear spaces on the dual variety. Finally, we look for families of $G(2,4)$-s on $G(2,n+1)$.

We already know that there are a lot of $G(2,4)$-s on G, and each $G(2,4)$ has codimension one in its linear span. Thus, we can set $Y=\cup_{\gamma\in\Gamma}\langle G(2,4)_\gamma\rangle$, where the union is over some set Γ. Then, $G\cap Y=\cup_{\gamma\in\Gamma}G(2,4)_\gamma$. The subvariety $G\cap Y$ of G is connected if and only if each $G(2,4)$ in the union intersects some other $G(2,4)$ non-empty.

So how may two $G(2,4)$-s intersect? Let $G(2,V_1^4)$ be the Grassmannian of two dimensional subspaces of the four dimensional vector space V_1^4. Equivalently, $G(2,V_1^4)$ is the Grassmannian of lines in $\mathbb{P}(V_1^4)=\mathbb{P}^3_1$. The Grassmannian $G(2,V_2^4)$ is all the lines $\mathbb{P}(V_2^4)=\mathbb{P}^3_2$. The two Grassmannians of lines intersect if and only if the two projective spaces $\mathbb{P}^3_1$ and $\mathbb{P}^3_2$ have at least a line in common.

If they have exactly a line in common, the two vector spaces V_1^4 and V_2^4 have exactly one two dimensional subspace in common. If w_1, w_2 is a basis for this subspace, the common point is the point $w_1 \wedge w_2$, and this is also the only common point for the Plücker spaces $\mathbb{P}(\wedge^2 V_1^4)$ and $\mathbb{P}(\wedge^2 V_2^4)$.

If the projective spaces have a plane in common, the entire $\mathbb{P}^2$ of lines in this plane will be in both $G(2,4)$-s. The Plücker spaces only have this plane in common. If the projective spaces coincide, the Grassmannians are equal, of course.

In this section we will look for families of $G(2,4)$-s on $G(2,n+1)$ by using what we have just learned about forms. We will start by investigating the case when $Y \cap G$ is the intersection locus of G and some tangent hyperplanes, all tangent at the same point. Before we do the general case, we look closely at two examples.

Example 3.3 ($G(2,7)$). The Grassmannian $G(2,7)$ sits inside $\mathbb{P}(\wedge^2 V^7)$, and it has dimension 10 inside this 20-dimensional projective space. Since 7 is an odd number, the general point in $\mathbb{P}(\wedge^2 V^7)$ has rank 6, and then there are only two inclusions in the chain:

$$G(2, V^7) \subset X_4^3 \subset \mathbb{P}(\wedge^2 V^7)$$

A general point can be written as the sum of three forms of rank two, while a point in X_4 can be written as a sum of two rank two forms. If ω_i are elements in $\wedge^2 V^7$ of rank two, we thus have

$$\begin{array}{llllll} \omega \in G(2,V^7) & \Rightarrow & \omega = \omega_1 & \text{and} & |\omega| = \mathbb{P}^1_\omega \\ \omega \in X_4 \setminus G(2,V^7) & \Rightarrow & \omega = \omega_1 + \omega_2 & \text{and} & |\omega| = \mathbb{P}^3_\omega \\ \omega \in \mathbb{P}(\wedge^2 V^7) \setminus X_4 & \Rightarrow & \omega = \omega_1 + \omega_2 + \omega_3 & \text{and} & |\omega| = \mathbb{P}^5_\omega \end{array}$$

Also in the dual space there are only two inclusions:

$$G(5, V^7) \subset \check{X}_4^3 \subset \mathbb{P}(\wedge^5 V^7)$$

If $\check{\omega}_i$ are simple 5-forms (i.e. not a sum of two or more), we have

$$\begin{array}{llllll} \check{\omega} \in G(5,V^7) & \Rightarrow & \check{\omega} = \check{\omega}_1 & \text{and} & |\check{\omega}| = \mathbb{P}^4_{\check{\omega}} \\ \check{\omega} \in \check{X}_4 \setminus G(5,V^7) & \Rightarrow & \check{\omega} = \check{\omega}_1 + \check{\omega}_2 & \text{and} & |\check{\omega}| = \mathbb{P}^2_{\check{\omega}} \\ \check{\omega} \in \mathbb{P}(\wedge^5 V^7) \setminus \check{X}_4 & \Rightarrow & \check{\omega} = \check{\omega}_1 + \check{\omega}_2 + \check{\omega}_3 & \text{and} & |\check{\omega}| = \mathbb{P}^0_{\check{\omega}} \end{array}$$

Let u and v be vectors in V^7, and let $p = u \wedge v$ be a point in $G(2,7)$. The tangent space T_p to $G(2,V^7)$ at the point p is spanned by all the lines on $G(2,V^7)$ through p, i.e. all the points on the form $u \wedge v'$ or $u' \wedge v$ for vectors u', v' in V^7. A form $\check{\omega} \in \wedge^5 V^7$ is in the orthogonal space to T_p if and only if it can be written as

$$\check{\omega} = u \wedge v \wedge \overline{\check{\omega}}$$

where $\overline{\breve{\omega}}$ is an element in $\wedge^3(V^7/\langle u,v\rangle)$. But we have seen that no point in the dual space has support on a line, and therefore, $\breve{\omega}$ can be written as

$$\breve{\omega} = u \wedge v \wedge w \wedge \overline{\breve{\omega}'}$$

Every point on the dual variety of $G(2,V^7)$ corresponds to a hyperplane in $\mathbb{P}(\wedge^2 V^7)$ tangent at some point $p \in G(2,V^7)$. But if H is a hyperplane tangent at $p = u \wedge v$ we have the inclusion $H \supset T_p$, and by the definition of $T_p^{\perp}$, this implies that the point $[H]$ in $\wedge^5 V^7$ lies in $T_p^{\perp}$. Thus every point on the dual variety lies in the orthogonal of some tangent space, and the discussion above implies that every such point is in $\check{X}_4$. On the other hand, every point in $\check{X}_4 \setminus G(5,V^7)$ has support the intersection of two $\mathbb{P}^4$-s inside $\mathbb{P}^6$, i.e. a plane, so every point $\breve{\omega} \in \check{X}_4 \setminus G(5,V^7)$ can be written as

$$\breve{\omega} = u_1 \wedge u_2 \wedge u_3 \wedge \overline{\breve{\omega}} \qquad (3.10)$$

where $\overline{\breve{\omega}}$ is an element in $\wedge^2(V^7/\langle u_1,u_2,u_3\rangle)$. But then $\breve{\omega}$ is in $T_p^{\perp}$ for all points p corresponding to lines in $\mathbb{P}^2_{\breve{\omega}}$. This shows that the dual variety is equal to $\check{X}_4$ in this case. A point $\breve{\omega}$ on the dual variety therefore has support on a plane $\mathbb{P}^2_{\breve{\omega}}$, and the corresponding hyperplane is tangent at all points coming from lines in this plane. From equation 3.10 we see that any $\breve{\omega} = u_1 \wedge u_2 \wedge u_3 \wedge \overline{\breve{\omega}}$ on the dual variety comes with a vector space $\wedge^2 V^7/\langle u_1,u_2,u_3\rangle$, so $\breve{\omega}$ lies in the linear span of a $G_{\mathbb{P}^2_{\breve{\omega}}}(2,4)$. More specific, $\breve{\omega}$ lies in the linear span of

$$G_{\mathbb{P}^2_{\breve{\omega}}}(2,4) = \{\mathbb{P}^4 \subset \mathbb{P}^6 = \mathbb{P}(V^7) \mid \mathbb{P}^2_{\breve{\omega}} \subset \mathbb{P}^4 \subset \mathbb{P}^6\}$$

For a form $\breve{\omega} \in \wedge^5 V^7$, let $H_{\breve{\omega}}$ be the hyperplane in $\mathbb{P}(\wedge^2 V^7)$ corresponding to $\breve{\omega}$. Assume $H_{\breve{\omega}}$ is a tangent hyperplane. We will now describe the intersection $H_{\breve{\omega}} \cap G(2,V^7)$. The definition of $H_{\breve{\omega}}$ gives us that

$$H_{\breve{\omega}} \cap G(2,V^7) = \{u_1 \wedge u_2 \in \wedge^2 V^7 \mid \breve{\omega} \wedge u_1 \wedge u_2 = 0\}$$

Since $H_{\breve{\omega}}$ is a tangent hyperplane, $\breve{\omega}$ is a point on $\check{X}_4$. Thus we can write $\breve{\omega}$ as $\omega_0 \wedge (\omega_1 + \omega_2)$ where $\omega_0 \in \wedge^3 V^7$ and $\omega_1, \omega_2 \in \wedge^2(V^7/|\omega_0|)$ are all simple. Thus the intersection above can be specified:

$$\begin{aligned}
H_{\breve{\omega}} \cap G(2,V^7) &= \{u_1 \wedge u_2 \in \wedge^2 V^7 \mid \omega_0 \wedge u_1 \wedge u_2 \wedge (\omega_1 + \omega_2) = 0\} \\
&= \{u_1 \wedge u_2 \in \wedge^2 V^7 \mid \omega_0 \wedge u_1 \wedge u_2 = 0\} \\
&\qquad \cup \{u_1 \wedge u_2 \in \wedge^2 V^7 \mid (\omega_1 + \omega_2) \wedge \mathbb{P}^4_{\langle \omega_0, u_1, u_2\rangle} = 0\} \\
&= \{\mathbb{P}^1 \subset \mathbb{P}(V^7) \mid \mathbb{P}^1 \cap \mathbb{P}^2_{\breve{\omega}} \neq \emptyset\} \\
&\qquad \cup \{\mathbb{P}^1 \subset \mathbb{P}^4_{\langle \omega_0, u_1, u_2\rangle} \mid (\omega_1 + \omega_2) \wedge \mathbb{P}^4_{\langle \omega_0, u_1, u_2\rangle} = 0\}
\end{aligned}$$

This might need a little explanation: The wedge product $\omega_0 \wedge u_1 \wedge u_2 \wedge (\omega_1 + \omega_2)$ is zero if $\omega_0 \wedge u_1 \wedge u_2$ is zero. If $\omega_0 \wedge u_1 \wedge u_2$ is *not* zero, the support $|\omega_0 \wedge u_1 \wedge u_2|$ is a four dimensional projective space denoted $\mathbb{P}^4_{\langle\omega_0,u_1,u_2\rangle}$. The other possibility for the big wedge product $\omega_0 \wedge u_1 \wedge u_2 \wedge (\omega_1 + \omega_2)$ to be zero is that $\omega_1 + \omega_2$ kills the entire $\mathbb{P}^4_{\langle\omega_0,u_1,u_2\rangle}$. Once we fix $\mathbb{P}^4_{\langle\omega_0,u_1,u_2\rangle}$, a general line inside will have the same property as the line spanned by u_1 and u_2 because a line and a plane inside a $\mathbb{P}^4$ generally does not intersect. Since we are looking for families of $G(2,4)$-s, we can write the intersection $H_{\breve{\omega}} \cap G(2,V^7)$ as

$$H_{\breve{\omega}} \cap G(2,V^7) = \{\mathbb{P}^1 | \mathbb{P}^1 \cap \mathbb{P}^2_{\breve{\omega}} \neq \emptyset\} \cup \{\mathbb{P}^1 | \mathbb{P}^1 \subset \mathbb{P}^3 \subset \mathbb{P}^4_{\langle\omega_0,u_1,u_2\rangle}\}$$

We get exactly the same lines, this is just a way of writing the set of lines so that the $G(2,4)$-s are "visible".

What about the intersection of $G(2,n+1)$ with more than one tangent hyperplane, all tangent at the same point? For instance, look for families of $G(2,4)$-s inside three tangent hyperplanes, all tangent at the same point. We have to inspect the intersection of $G(2,V^7)$ by three such tangent hyperplanes. Let

$$\begin{aligned}
\breve{\omega} &= \breve{\omega}_0 \wedge (\breve{\omega}_1 + \breve{\omega}_2) && \text{where } \breve{\omega}_0 \in \wedge^3 V^7 && \text{and } \breve{\omega}_1, \breve{\omega}_2 \in \wedge^2 (V^7/|\breve{\omega}_0|),\\
\breve{\omega}' &= \breve{\omega}'_0 \wedge (\breve{\omega}'_1 + \breve{\omega}'_2) && \text{where } \breve{\omega}'_0 \in \wedge^3 V^7 && \text{and } \breve{\omega}'_1, \breve{\omega}'_2 \in \wedge^2 (V^7/|\breve{\omega}'_0|),\\
\breve{\omega}'' &= \breve{\omega}''_0 \wedge (\breve{\omega}''_1 + \breve{\omega}''_2) && \text{where } \breve{\omega}''_0 \in \wedge^3 V^7 && \text{and } \breve{\omega}''_1, \breve{\omega}''_2 \in \wedge^2 (V^7/|\breve{\omega}''_0|)
\end{aligned}$$

be three points on the dual variety of $G(2,V^7)$. If $H_{\breve{\omega}}$, $H_{\breve{\omega}'}$ and $H_{\breve{\omega}''}$ are all tangent at the point $p = u \wedge v \in G(2,V^7)$, we must have

$$\begin{aligned}
\breve{\omega}_0 &= u \wedge v \wedge w\\
\breve{\omega}'_0 &= u \wedge v \wedge w'\\
\breve{\omega}''_0 &= u \wedge v \wedge w''
\end{aligned}$$

where w, w' and w'' are vectors (1-forms). Thus

$$\begin{aligned}
H_{\breve{\omega}} \cap G(2,V^7) &= \{u_1 \wedge u_2 \in \wedge^2 V^7 \mid u \wedge v \wedge w \wedge u_1 \wedge u_2 \wedge (\breve{\omega}_1 + \breve{\omega}_2) = 0\}\\
H_{\breve{\omega}'} \cap G(2,V^7) &= \{u'_1 \wedge u'_2 \in \wedge^2 V^7 \mid u \wedge v \wedge w' \wedge u'_1 \wedge u'_2 \wedge (\breve{\omega}'_1 + \breve{\omega}'_2) = 0\}\\
H_{\breve{\omega}''} \cap G(2,V^7) &= \{u''_1 \wedge u''_2 \in \wedge^2 V^7 \mid u \wedge v \wedge w'' \wedge u''_1 \wedge u''_2 \wedge (\breve{\omega}''_1 + \breve{\omega}''_2) = 0\}
\end{aligned}$$

A $\mathbb{P}^3$ that appears in all three intersections must be contained in all three

$$\mathbb{P}^4_{\langle u,v,w,u_1,u_2\rangle}, \quad \mathbb{P}^4_{\langle u,v,w',u'_1,u'_2\rangle} \quad \text{and} \quad \mathbb{P}^4_{\langle u,v,w'',u''_1,u''_2\rangle}$$

Suppose a common $\mathbb{P}^3$ does not contain the line $\langle u,v\rangle$. Since all three $\mathbb{P}^4$-s obviously contain the line, this implies that all three $\mathbb{P}^4$-s must be equal. Thus all three forms

$$\breve{\omega}_1 + \breve{\omega}_2, \qquad \breve{\omega}'_1 + \breve{\omega}'_2 \qquad \text{and} \qquad \breve{\omega}''_1 + \breve{\omega}''_2$$

must lie in the orthogonal space of the $\mathbb{P}^4$. This orthogonal space has dimension one, and when we choose random forms, these three will not all lie on a line. Thus such $\mathbb{P}^3$-s can not occur in the intersection

$$\Delta := H_{\check{\omega}} \cap H_{\check{\omega}'} \cap H_{\check{\omega}''} \cap G(2, V^7)$$

Thus a common $\mathbb{P}^3$ must contain the line $\langle u, v\rangle$. All common $\mathbb{P}^3$-s are therefore spanned by u, v, z_1, z_2 for some vectors z_1 and z_2, and the conditions on a given $\mathbb{P}^3$ become

$$u \wedge v \wedge z_1 \wedge z_2 \wedge w \wedge (\check{\omega}_1 + \check{\omega}_2) = 0$$
$$u \wedge v \wedge z_1 \wedge z_2 \wedge w' \wedge (\check{\omega}_1' + \check{\omega}_2') = 0$$
$$u \wedge v \wedge z_1 \wedge z_2 \wedge w'' \wedge (\check{\omega}_1'' + \check{\omega}_2'') = 0$$

Since $u \wedge v \wedge z_1 \wedge z_2$ is not zero, we must have

$$z_1 \wedge z_2 \wedge w \wedge (\check{\omega}_1 + \check{\omega}_2) = 0$$
$$z_1 \wedge z_2 \wedge w' \wedge (\check{\omega}_1' + \check{\omega}_2') = 0$$
$$z_1 \wedge z_2 \wedge w'' \wedge (\check{\omega}_1'' + \check{\omega}_2'') = 0$$

Thus all the $\mathbb{P}^3$-s we get are spanned by u and v and a line $\langle z_1, z_2\rangle$ where z_1 and z_2 are vectors in $V^7/\langle u, v\rangle$. The conditions above gives us that every $\mathbb{P}^3$ corresponds to a point on

$$G(2, V^7/\langle u, v\rangle) \cap H_1 \cap H_2 \cap H_3 \tag{3.11}$$

because $w\wedge(\check{\omega}_1+\check{\omega}_2)$, $w'\wedge(\check{\omega}_1'+\check{\omega}_2')$ and $w''\wedge(\check{\omega}_1''+\check{\omega}_2'')$ are all 3-forms in $V^7/\langle u, v\rangle$ which is a sum of two simple ones. Thus these three forms represents (tangent) hyperplanes. The vector space $V^7/\langle u, v\rangle$ has dimension five, and the intersection in equation 3.11 has dimension $\dim G(2, 5) - 3 = 3$. From this we get a family of $\mathbb{P}^3$-s of dimension three.

Now take three points $\check{\omega}_1, \check{\omega}_2, \check{\omega}_3 \in \check{X}_4$. If the corresponding hyperplanes $H_{\check{\omega}_i}$ are tangent at $p \in G(2, V^7)$ for $i = 1, 2, 3$, we have

$$\check{\omega}_1, \check{\omega}_2, \check{\omega}_3 \in T_p^{\perp} \tag{3.12}$$

Since $T_p^{\perp}$ is a linear space, this implies that $\{\lambda_1\check{\omega}_1+\lambda_2\check{\omega}_2+\lambda_3\check{\omega}_3\} \subset T_p^{\perp}$, thus the whole plane $\mathbb{P}^2_{\langle\check{\omega}_1,\check{\omega}_2,\check{\omega}_3\rangle}$ lies in $T_p^{\perp}$. We have seen that $\check{X}_4$ is the dual variety of $G(2, V^7)$, and that the dual variety is the union of all $T_p^{\perp}$-s where p is a point in $G(2, V^7)$. Thus the plane $\mathbb{P}^2_{\langle\check{\omega}_1,\check{\omega}_2,\check{\omega}_3\rangle}$ is a plane in $\check{X}_4$. Thus if we pick three tangent hyperplanes, all tangent at the same point, the plane spanned by their corresponding points in $\check{X}_4$ is contained in $\check{X}_4$. We may ask if the opposite statement is true:

Question: Is every plane in $\check{X}_4$ spanned by three points whose corresponding hyperplanes are all tangent at the same point?

For the moment, we have no answer to this question, and this may be an interesting subject for further research.
△

We will now look at the Grassmannian $G(2,8)$, and see if we can find similar results.

Example 3.4 ($G(2,8)$)**.** The Grassmannian $G(2,V^8)$ sits inside $\mathbb{P}(\wedge^2 V^8)$ and it has dimension 12 inside this 27-dimensional projective space. Since 8 is an even number, the general point in $\mathbb{P}(\wedge^2 V^8)$ has rank 8. There are three inclusions in the chain:

$$G(2,V^8) \subset X_4^6 \subset X_6^1 \subset \mathbb{P}(\wedge^2 V^8)$$

A general point can be written as a sum of four 2-forms of rank two, a point in X_6 as a sum of three, and a point on X_4 can be written as a sum of two. We have

$$\begin{array}{llll}
\omega \in G(2,V^8) & \Rightarrow \ \omega = \omega_1 & \text{and} & |\omega| = \mathbb{P}^1_\omega \\
\omega \in X_4 \setminus G(2,V^8) & \Rightarrow \ \omega = \omega_1 + \omega_2 & \text{and} & |\omega| = \mathbb{P}^3_\omega \\
\omega \in X_6 \setminus X_4 & \Rightarrow \ \omega = \omega_1 + \omega_2 + \omega_3 & \text{and} & |\omega| = \mathbb{P}^5_\omega \\
\omega \in \mathbb{P}(\wedge^2 V^8) \setminus X_6 & \Rightarrow \ \omega = \omega_1 + \omega_2 + \omega_3 + \omega_4 & \text{and} & |\omega| = \mathbb{P}^7_\omega
\end{array}$$

In the dual space, the chain becomes

$$G(6,V^8) \subset \check{X}_4^6 \subset \check{X}_6^1 \subset \mathbb{P}(\wedge^6 V^8)$$

If $\check{\omega}_i$ are simple 6-forms (i.e. not a sum of two of more), we have

$$\begin{array}{llll}
\check{\omega} \in G(5,V^8) & \Rightarrow \ \check{\omega} = \check{\omega}_1 & \text{and} & |\check{\omega}| = \mathbb{P}^5_{\check{\omega}} \\
\check{\omega} \in \check{X}_4 \setminus G(5,V^8) & \Rightarrow \ \check{\omega} = \check{\omega}_1 + \check{\omega}_2 & \text{and} & |\check{\omega}| = \mathbb{P}^3_{\check{\omega}} \\
\check{\omega} \in \check{X}_6 \setminus \check{X}_4 & \Rightarrow \ \check{\omega} = \check{\omega}_1 + \check{\omega}_2 + \check{\omega}_3 & \text{and} & |\check{\omega}| = \mathbb{P}^1_{\check{\omega}} \\
\check{\omega} \in \mathbb{P}(\wedge^6 V^8) \setminus \check{X}_6 & \Rightarrow \ \check{\omega} = \check{\omega}_1 + \check{\omega}_2 + \check{\omega}_3 + \check{\omega}_4 & \text{and} & |\check{\omega}| = \emptyset
\end{array}$$

Let u and v be vectors in V^8, and let $p = u \wedge v$ be a point on $G(2,V^8)$. A form $\check{\omega} \in \wedge^6 V^8$ is in the orthogonal space of the tangent space T_p if and only if it can be written as

$$\check{\omega} = u \wedge v \wedge \overline{\check{\omega}}$$

where $\overline{\check{\omega}}$ is an element in $\wedge^4(V^8/\langle u,v\rangle)$. Thus $\overline{\check{\omega}}$ is a 4-form in six variables, dual to a 2-form in 6 variables. Such a 2-form has rank at most six, so $\overline{\check{\omega}}$ is in general a sum a three simple 4-forms. So if $\check{\omega}$ is in $T_p^\perp$ for some $p \in G(2,V^8)$, the form is also in $\check{X}_6$. On the other hand, a point $\check{\omega}$ in $\check{X}_6 \setminus \check{X}_4$ has support on a line $\mathbb{P}^1_p$, and then $\check{\omega}$ is in $T_p^\perp$. This implies that so $\check{X}_6$ equals the dual variety.

For a form $\check{\omega} \in \wedge^6 V^8$, let $H_{\check{\omega}}$ be the corresponding hyperplane in $\mathbb{P}(\wedge^6 V^8)$. The definition of $H_{\check{\omega}}$ gives that

$$H_{\check{\omega}} \cap G(2,V^8) = \{u_1 \wedge u_1 \in \wedge^2 V^8 |\ \check{\omega} \wedge u_1 \wedge u_2 = 0\}$$

Now if $H_{\breve{\omega}}$ is a *tangent* hyperplane, we know that $\breve{\omega}$ is an element in $\check{X}_6$. Such forms have support on a line, and we have

$$\breve{\omega} \in \check{X}_6 \setminus \check{X}_4 \Rightarrow \breve{\omega} = \omega_0 \wedge (\omega_1 + \omega_2 + \omega_3)$$

where ω_0 is an element in $\wedge^2 V^8$ or rank two, and $\omega_1, \omega_2, \omega_3 \in \wedge^4(V^8/|\omega_0|)$ are all simple. If $H_{\breve{\omega}}$ is tangent at the point $p = u \wedge v$, we have:

$$\begin{aligned} H_{\breve{\omega}} \cap G(2, V^8) &= \{u_1 \wedge u_2 \in \wedge^2 V^8 | \; u \wedge v \wedge u_1 \wedge u_2 \wedge (\omega_1 + \omega_2 + \omega_3) = 0\} \\ &= \{u_1 \wedge u_2 \in \wedge^2 V^8 | \; u \wedge v \wedge u_1 \wedge u_2 = 0\} \\ &\qquad \cup \{u_1 \wedge u_2 \in \wedge^2 V^8 | \; (\omega_1 + \omega_2 + \omega_3) \wedge \mathbb{P}^3_{\langle u,v,u_1,u_2 \rangle} = 0\} \\ &= \{\mathbb{P}^1 \subset \mathbb{P}(\wedge^2 V^8) | \; \mathbb{P}^1 \cap \mathbb{P}^1_{\langle u,v \rangle} \neq \emptyset\} \\ &\qquad \{\mathbb{P}^1 \subset \mathbb{P}^3_{\langle u,v,u_1,u_2 \rangle} | \; (\omega_1 + \omega_2 + \omega_3) \wedge \mathbb{P}^3_{\langle u,v,u_1,u_2 \rangle} = 0\} \end{aligned}$$

As in the previous example, this might need an explanation: The wedge product $u \wedge v \wedge u_1 \wedge u_2 \wedge (\omega_1 + \omega_2 + \omega_3)$ is zero if $u \wedge v \wedge u_1 \wedge u_2$ is zero. If $u \wedge v \wedge u_1 \wedge u_2$ is *not* zero, the union of the lines $\langle u, v \rangle$ and $u_1 \wedge u_2$ is a three dimensional space denoted $\mathbb{P}^3_{\langle u,v,u_1,u_2 \rangle}$. In this case, the big wedge product is zero if $(\omega_1 + \omega_2 + \omega_3)$ is in $(\mathbb{P}^3_{\langle u,v,u_1,u_2 \rangle})^{\perp}$. Any general line in $\mathbb{P}^3_{\langle u,v,u_1,u_2 \rangle}$ does not intersect the line $\langle u, v \rangle$, and all such lines $\langle u_1', u_2' \rangle$ give the same $\mathbb{P}^3$.

The forms ω_1, ω_2 and ω_3 are elements in $\wedge^2(V^8/\langle u, v \rangle)$, so they do not involve u and v. Thus $(\omega_1 + \omega_2 + \omega_3) \wedge u \wedge v \neq 0$. Thus we can think of the statement $(\omega_1 + \omega_2 + \omega_3) \wedge \mathbb{P}^3_{\langle u,v,u_1,u_2 \rangle} = 0$ as $(\omega_1 + \omega_2 + \omega_3) \wedge u_1 \wedge v_1 = 0$. But this is the same as saying that the line $\langle u_1, v_1 \rangle$ is in $H_{(\omega_1 + \omega_2 + \omega_3)}$, and the lines in this hyperplane is precisely the intersection

$$H_{(\omega_1 + \omega_2 + \omega_3)} \cap G(2, V^8/\langle u, v \rangle)$$

Also in this case, we want to see what happens if we intersect with more that one hyperplane, all tangent at the same point. As above, we consider the intersection of three such hyperplanes:

$$H_{\breve{\omega}} \cap H_{\breve{\omega}'} \cap H_{\breve{\omega}''} \cap G(2, V^8)$$

From the above argument, we see that we obtain a family of $\mathbb{P}^3$-s of dimension

$$\dim(H_{\breve{\omega}} \cap H_{\breve{\omega}'} \cap H_{\breve{\omega}''} \cap G(2, V^8/\langle u, v \rangle) = 2(8 - 2 - 2) - 3 = 5$$

If we take three tangent hyperplanes, all tangent at the same point p, the corresponding 6-forms are all points on $T_p^{\perp}$. This is a linear space, so the plane spanned by the three points is also in $T_p^{\perp}$. This is a plane in $\check{X}_6$. We can ask the same question as we did for $G(2, V^7)$:

Question: Is every plane in $\check{X}_6$ spanned by three points whose corresponding hyperplanes are all tangent at the same point?

In this case, the answer is "no": Any plane is spanned by three points $\check{\omega}, \check{\omega}'$ and $\check{\omega}''$. Their supports are lines $\mathbb{P}^1_{\check{\omega}}$, $\mathbb{P}^1_{\check{\omega}'}$ and $\mathbb{P}^1_{\check{\omega}''}$. Suppose two supporting lines coincide and intersect the third in a point. The corresponding situation in the space of 2-forms is that two supporting $\mathbb{P}^5$-s coincide, and the union of all three span a $\mathbb{P}^6$. But then all three 2-forms are in $\wedge^2 W^7$ for some W^7, and therefore any linear combination has rank at most six. Thus the plane spanned by $\check{\omega}$,$\check{\omega}'$ and $\check{\omega}''$ *is* contained in $\check{X}_6$, *and this is not a plane spanned by points whose corresponding hyperplanes are tangent at the same point.* Furthermore, if the supporting lines all have a point in common, the supporting $\mathbb{P}^5$-s in the space of 2-forms are all contained in a $\mathbb{P}^6$. As above, this implies that the plane *is* contained in $\check{X}_6$. $\triangle$

We will now generalize to $G(2, n+1)$. The number n is either even or odd, and the two cases will be treated separately.

3.2.1 The dual variety of $G(2, n+1)$ when n is even

In this case, $n+1$ is an odd number, so an element in $\wedge^2 V^{n+1}$ has rank less than or equal to n. The chain of inclusions in $\mathbb{P}(\wedge^2 V^{n+1})$ is given in equation 3.7, but we repeat it here to refresh our memory:

$$G(2, V^{n+1}) = X_2 \subset X_4 \subset \cdots \subset X_i \subset \cdots \subset X^{10}_{n-4} \subset X^3_{n-2} \subset \mathbb{P}(\wedge^2 V^{n+1}) \qquad (3.13)$$

The variety X_i is defined by the vanishing of all $(i+2)$-Pfaffians.

Let $p = u \wedge v$ be the point on $G(2, n+1)$ corresponding to the line $\langle u, v\rangle$ in $\mathbb{P}(V^{n+1})$. Then the tangent space T_p of $G(2, n+1)$ at p is a linear space, and by definition of the orthogonal of a linear space,

$$T_p^{\perp} = \{\check{\omega} \in \wedge^{n-1} V^{n+1}) |\ \check{\omega} \wedge \omega = 0 \text{ for all } \omega \in T_p\}$$

The tangent space T_p is spanned by all the lines on $G(2, n+1)$ through $p = u \wedge v$, i.e of all points of the form

$$u \wedge v' \quad \text{or} \quad u' \wedge v$$

for vectors $u', v' \in V^{n+1}$. Thus for $\check{\omega}$ to be an element in $T_p^{\perp}$, $\check{\omega}$ must be an $(n-1)$-form of the type

$$\check{\omega} = u \wedge v \wedge \overline{\check{\omega}} \qquad \text{where } \overline{\check{\omega}} \in \wedge^{n-3}(V^{n+1}/\langle u, v\rangle)$$

We see that $\overline{\check{\omega}}$ is an element on $\wedge^{n-3} W^{n-1}$ where $W^{n-1} \simeq V^{n+1}/\langle u, v\rangle$. A general element in $\wedge^2 W^{n-1}$ has rank $n-2$. This implies that $\overline{\check{\omega}}$ is an element in $\check{X}_{n-2}$. Thus we have the inclusion $T_p^{\perp} \subset \check{X}_{n-2}$. There is nothing special about the point p, so the orthogonal of every tangent space along $G(2, n+1)$ is contained in $\check{X}_{n-2}$. Now, a general point in $\check{X}_{n-2}$ can be written as a sum of $\frac{n-2}{2}$ simple $(n-1)$-forms $\check{\omega}_j$. Thus any general form in $\check{X}_{n-2}$ has support

$$\check{\omega} \in \check{X}_{n-2} \Rightarrow |\check{\omega}| = \bigcap_{j=1}^{\frac{n-2}{2}} \mathbb{P}^{n-2}_{\check{\omega}_j} = \mathbb{P}^{(\frac{n-2}{2})\cdot(n-2)-(\frac{n-2}{2}-1)\cdot n} = \mathbb{P}^2_{\check{\omega}}$$

The hyperplane corresponding to $\check{\omega}$ is tangent at every point corresponding to lines in $\mathbb{P}^2_{\check{\omega}}$. So every general point in $\check{X}_{n-2}$ is in a $\mathbb{P}^2$ of $T_p^{\perp}$-s. We have proved the theorem

Theorem 3.4. *When n is even, the variety $\check{X}_{n-2}$ is the dual variety of $G(2, V^{n+1})$, and a general point on the dual variety corresponds to a hyperplane which is tangent along a plane.*

Theorem 3.4 implies that a general point $\check{\omega}$ on the dual variety of the Grassmannian of lines in a projective space of dimension n can be written as

$$\check{\omega} = \check{\omega}_0 \wedge (\check{\omega}_1 + \cdots + \check{\omega}_{\frac{n-2}{2}})$$

where $\check{\omega}_0 \in \wedge^3 V^{n+1}$ and $\check{\omega}_j \in \wedge^{n-4}(V^{n+1}/|\check{\omega}_0|)$ are all simple. The quotient $V^{n+1}/|\check{\omega}_0|$ is a vector space of dimension $n-2$. By a general point we mean a point in $\check{X}_{n-2} \backslash \check{X}_{n-4}$. Let $H_{\check{\omega}}$ be the hyperplane in $\mathbb{P}(\wedge^2 V^{n+1})$ corresponding to the point $\check{\omega}$ on the dual variety. Then

$$\begin{aligned}
H_{\check{\omega}} \cap G(2, V^{n+1}) = & \\
& \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ \check{\omega} \wedge u_1 \wedge u_2 = 0\} \\
&= \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ \check{\omega}_0 \wedge u_1 \wedge u_2 \wedge (\check{\omega}_1 + \cdots + \check{\omega}_{\frac{n-2}{2}}) = 0\} \\
&= \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ \check{\omega}_0 \wedge u_1 \wedge u_2 = 0\} \\
&\quad \cup \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ (\check{\omega}_1 + \cdots + \check{\omega}_{\frac{n-2}{2}}) \wedge \mathbb{P}^4_{\langle \mathbb{P}^2_{\check{\omega}}, u_1, u_2\rangle} = 0\} \\
&= \{\mathbb{P}^1 \subset \mathbb{P}(V^{n+1}) |\ \mathbb{P}^1 \cap \mathbb{P}^2_{\check{\omega}} \neq \emptyset\} \\
&\quad \cup \{\mathbb{P}^1 \subset \mathbb{P}^4_{\langle \mathbb{P}^2_{\check{\omega}}, u_1, u_2\rangle} |\ (\check{\omega}_1 + \cdots + \check{\omega}_{\frac{n-2}{2}}) \wedge \mathbb{P}^4_{\langle \mathbb{P}^2_{\check{\omega}}, u_1, u_2\rangle} = 0\}
\end{aligned}$$

We will now examine the intersection

$$\Delta = H_{\check{\omega}_1} \cap \cdots \cap H_{\check{\omega}_\alpha} \cap G(2, V^{n+1})$$

where α is some natural number and the hyperplanes $H_{\check{\omega}_j}$ are all tangent at the same point. So pick α points $\check{\omega}_1, \ldots, \check{\omega}_\alpha$ on $\check{X}_{n-2} \setminus \check{X}_{n-4}$, i.e. α general points on the dual variety. If the hyperplanes $H_{\breve{\omega}_j}$ are all tangent at the point $p = u \wedge v$, the forms $\breve{\omega}_j$ can all be written as

$$\breve{\omega}_j = u \wedge v \wedge w_j \wedge (\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}})$$

where w_j is a vector in V^{n+1} and $\breve{\omega}_{j,k}$ is an element in $\wedge^{n-4}(V^{n+1}/\langle u, v, w_j\rangle)$ for $j = 1, \ldots, \alpha$. Thus the intersection Δ can be found explicitly:

$$\begin{aligned}
\Delta &= H_{\breve{\omega}_1} \cap \cdots \cap H_{\breve{\omega}_\alpha} \cap G(2, V^{n+1}) \\
&= \bigcap_{j=1}^{\alpha} \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ u \wedge v \wedge u_1 \wedge u_2 \wedge w_j \wedge (\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}}) = 0\} \\
&= \bigcap_{j=1}^{\alpha} \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ u \wedge v \wedge u_1 \wedge u_2 \wedge w_j = 0\} \\
&\qquad\qquad \cup \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ (\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}}) \wedge \mathbb{P}^4_{\langle u,v,w_j,u_1,u_2\rangle} = 0\} \\
&= \bigcap_{j=1}^{\alpha} \{\mathbb{P}^1 \subset \mathbb{P}(V^{n+1}) |\ \mathbb{P}^1 \cap \mathbb{P}^2_{\breve{\omega}_j} \neq \emptyset\} \\
&\qquad\qquad \cup \{\mathbb{P}^1 \subset \mathbb{P}^4_{\langle u,v,u_1,v_1,w_j\rangle} |\ (\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}}) \wedge \mathbb{P}^4_{\langle u,v,w_j,u_1,u_2\rangle} = 0\}
\end{aligned}$$

Since we are looking for families of $G(2,4)$-s on $G(2, V^{n+1})$, we choose to write Δ as

$$\begin{aligned}
\Delta = \bigcap_{j=1}^{\alpha} &\{\mathbb{P}^1 \subset \mathbb{P}(V^{n+1}) |\ \mathbb{P}^1 \cap \mathbb{P}^2_{\breve{\omega}_j} \neq \emptyset\} \\
&\cup \{\mathbb{P}^1 \subset \mathbb{P}^3 \subset \mathbb{P}^4_{\langle u,v,u_1,v_1,w_j\rangle} |\ (\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}}) \wedge \mathbb{P}^4_{\langle u,v,w_j,u_1,u_2\rangle} = 0\}
\end{aligned}$$

The forms $\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}}$ are elements in $\wedge^{n-4}(V^{n+1}/\langle u, v, w_j\rangle)$. A $\mathbb{P}^3$ can only occur in the intersection if the $\mathbb{P}^4$-s have this $\mathbb{P}^3$ in common. If all the $\mathbb{P}^4$-s are equal, all the forms $\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}}$ are elements in the same $\wedge^{n-4} W^{n-2}$ (the vector spaces $V^{n+1}/\langle u, v, w_j\rangle$ are all isomorphic in this case). Thus these are α forms in $\langle G(n-4, n-2)\rangle$, and since they are a sum of $\frac{n-2}{2}$ simple forms, they are all general points in this space. To say that $(\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}}) \wedge \mathbb{P}^4_{\langle u,v,w_j,u_1,u_2\rangle} = 0$ equals saying that all these forms are points on $\mathbb{P}^4$, and if we choose our forms randomly, there is no reason why this should be true for a fixed $\mathbb{P}^4$.

If the $\mathbb{P}^4$-s intersect in $\mathbb{P}^3_{\langle u,v,u_1,u_1\rangle}$ only, we must have

$$u \wedge v \wedge u_1 \wedge u_2 \wedge \left[w_j \wedge (\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}})\right] = 0 \text{ for } j = 1, \ldots, \alpha$$

We know that the form in the square brackets does not involve u and v, so modulo u, v we have

$$u_1 \wedge u_2 \wedge \left[w_j \wedge (\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}})\right] = 0 \quad \text{ for } j = 1, \ldots, \alpha$$

Thus every $\mathbb{P}^3$ in Δ is spanned by $\langle u, v\rangle$ and a line $\langle u_1, u_2\rangle$ in

$$\begin{aligned}&\{u_1 \wedge u_2 \in \wedge^2(V^{n+1}/\langle u,v\rangle)|\ u_1 \wedge u_2 \wedge \left[w_j \wedge (\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}})\right] = 0\ \forall\ j\}\\ &= G(2, n-1) \cap H_1 \cap \cdots \cap H_\alpha\end{aligned}$$

where H_j is the hyperplane in $\mathbb{P}(\wedge^2 Z^{n-1})$ corresponding to the point $w_j \wedge (\breve{\omega}_{j,1} + \cdots + \breve{\omega}_{j,\frac{n-2}{2}})$ in $\mathbb{P}(\wedge^{n-3} Z^{n-1})$ (the vector space Z is isomorphic to $V^{n+1}/\langle u,v\rangle$). We get a family of $\mathbb{P}^3$-s if dimension $2(n-3) - \alpha$, and thus a family of $G(2,4)$-s on $G(2, V^{n+1})$ of dimension $2(n-3) - \alpha$.

Recall that we are investigating apolar varieties of the form

$$Y = \bigcup_{\gamma\in\Gamma} \langle G(2,4)_\gamma\rangle$$

where Γ is is some family of $\mathbb{P}^3$-s. The $\mathbb{P}^3$-s in the family obtained here are all the $\mathbb{P}^3$-s in $\mathbb{P}^n$ which contain the line $\mathbb{P}^1_{\langle u,v\rangle}$. Such $\mathbb{P}^3$-s make a cycle on $G(4, V^{n+1})$:

$$\{\Lambda \in G(4, V^{n+1})|\ \dim(\Lambda \cap V^2) \geq 2\} = \sigma_{n-3,n-3,0,0}$$

All lines in $\mathbb{P}(V^{n+1})$ lies in such a $\mathbb{P}^3$, because any line in $\mathbb{P}(V^{n+1})$ will, together with $\mathbb{P}^1_{\langle u,v\rangle}$, span such a $\mathbb{P}^3$. Intersecting with α hyperplanes, we get that

$$\dim\langle Y\rangle = \binom{n+1}{2} - 1 - \alpha$$

Now, pick α points $\breve{\omega}_1, \ldots, \breve{\omega}_\alpha$ on $\check{X}_{n-2} \setminus \check{X}_{n-4}$. If $H_{\breve{\omega}_j}$ are all tangent at the point $p = u \wedge v$, the forms $\breve{\omega}_1, \ldots, \breve{\omega}_\alpha$ are all points in $T_p^\perp$. We have previously seen that this implies that the $\mathbb{P}^{\alpha-1}$ in $\mathbb{P}(\wedge^{n-1}V^{n+1})$ spanned by these points is contained in $\check{X}_{n-2}$.

A point $\breve{\omega}$ in $\check{X}_{n-4} \cap T_p^\perp$ can we written as

$$\breve{\omega} = u \wedge v \wedge (\breve{\omega}_1 + \cdots + \breve{\omega}_{\frac{n-4}{2}})$$

where the $\breve{\omega}_j$ in $\wedge^{n-3}(V^{n+1}/\langle u,v\rangle)$ are all simple . The elements in the wedge product space $\wedge^{n-3}(V^{n+1}/\langle u,v\rangle)$ which can be written in this way corresponds to 2-forms in $n-1$ variables of rank $n-4$, i.e. second to maximal rank. Thus $\check{X}_{n-4} \cap T_p^\perp$ has codimension three in $T_p^\perp$. We have

$$\begin{array}{llll}
\check{X}_{n-4} \cap T_p^\perp & \text{has codimension} & 3 & \text{in}\ \ T_p^\perp\\
\check{X}_{n-6} \cap T_p^\perp & \text{has codimension} & 10 & \text{in}\ \ T_p^\perp\\
\check{X}_{n-8} \cap T_p^\perp & \text{has codimension} & 21 & \text{in}\ \ T_p^\perp\\
 & \vdots & & \\
\check{X}_{n-k} \cap T_p^\perp & \text{has codimension} & \binom{k-1}{2} & \text{in}\ \ T_p^\perp
\end{array}$$

where k is an even number. Thus a $\mathbb{P}^{\alpha-1}$ in $\check{X}_{n-2}$ spanned by α points whose corresponding hyperplanes are all tangent at the same point, will intersect $\check{X}_{n-k}$ in dimension

$$\alpha - 1 + 2(n-1) - \binom{k-1}{2} - 2(n-1) = \alpha - 1 - \binom{k-1}{2}$$

So as long as $\alpha - 1 \geq \binom{k-1}{2}$, such a $\mathbb{P}^{\alpha}$ will actually intersect $\check{X}_{n-k}$.

Example 3.5 ($G(2,V^7)$). In the case of $G(2,V^7)$, we studied the case when $\alpha = 3$. A plane in $\check{X}_4$ spanned by three points whose corresponding hyperplanes are all tangent at the same point will *not* intersect $\check{X}_2 = G(2,V^7)$, since $\check{X}_2 \cap T_p^{\perp}$ has codimension 3 in $T_p^{\perp}$. △

3.2.2 The dual variety of $G(2,n+1)$ when n is odd

In this case, $n+1$ is an even number, so an element in $\wedge^2 V^{n+1}$ has rank less than or equal to $n+1$. We repeat the chain of inclusions given in equation 3.6:

$$G(2,V^{n+1}) = X_2 \subset X_4 \subset \cdots \subset X_{n-3}^6 \subset X_{n-1}^1 \subset \mathbb{P}(\wedge^2 V^{n+1}) \qquad n \textbf{ is odd} \qquad (3.14)$$

Let $p = u \wedge v$ be the point on $G(2,n+1)$ corresponding to the line $\langle u,v\rangle$ in $\mathbb{P}(V^{n+1})$. Then the tangent space T_p of $G(2,n+1)$ at p is a linear space, and by definition of the orthogonal of a linear space,

$$T_p^{\perp} = \{\check{\omega} \in \wedge^{n-1}V^{n+1} |\ \check{\omega} \wedge \omega = 0 \text{ for all } \omega \in T_p\}$$

The tangent space T_p is spanned by all the lines on $G(2,n+1)$ through $p = u \wedge v$, i.e of all points of the form

$$u \wedge v' \quad \text{or} \quad u' \wedge v$$

for vectors $u', v' \in V^{n+1}$. Thus for $\check{\omega}$ to be an element in $T_p^{\perp}$, $\check{\omega}$ must be an $(n-1)$-form of the type

$$\check{\omega} = u \wedge v \wedge (\overline{\check{\omega}}) \qquad \text{where } \overline{\check{\omega}} \in \wedge^{n-3}(V^{n+1}/\langle u,v\rangle)$$

We see that $\overline{\check{\omega}}$ is an element on $\wedge^{n-3}W^{n-1}$ where $W^{n-1} \simeq V^{n+1}/\langle u,v\rangle$. A general element in $\wedge^2 W^{n-1}$ has rank $n-1$. This implies that $\overline{\check{\omega}}$ is an element in $\check{X}_{n-1}$. Thus we have the inclusion $T_p^{\perp} \subset \check{X}_{n-1}$. There is nothing special about the point p, so the orthogonal of every tangent space along $G(2,n+1)$ is contained in $\check{X}_{n-1}$. Now, a general point in $\check{X}_{n-1}$ can be written as a sum of $\frac{n-1}{2}$ simple $(n-1)$-forms $\check{\omega}_j$. Thus for any general form in $\check{X}_{n-1}$ we have

$$\check{\omega} \in \check{X}_{n-1} \Rightarrow |\check{\omega}| = \bigcap_{j=1}^{\frac{n-1}{2}} \mathbb{P}^{n-2}_{\check{\omega}_j} = \mathbb{P}^{(\frac{n-1}{2})\cdot(n-2)-(\frac{n-1}{2}-1)\cdot n} = \mathbb{P}^1_{\check{\omega}}$$

The hyperplane corresponding to $\check{\omega}$ is tangent at the point corresponding to the line $\mathbb{P}^1_{\check{\omega}}$. So every general point in $\check{X}_{n-1}$ is in exactly one $T_p^{\perp}$. We have proved the theorem

Theorem 3.5. *When n is odd, the variety $\check{X}_{n-1}$ is the dual variety of $G(2, V^{n+1})$, and a general point on the dual variety corresponds to a hyperplane which is tangent in exactly one point.*

Notice that theorem 3.5 implies that a general point $\check{\omega}$ on the dual variety of $G(2, V^{n+1})$ can be written as

$$\check{\omega} = \check{\omega}_0 \wedge (\check{\omega}_1 + \cdots + \check{\omega}_{\frac{n-1}{2}})$$

where $\check{\omega}_0 \in \wedge^2 V^{n+1}$ and $\check{\omega}_j \in \wedge^{n-3}(V^{n+1}/|\check{\omega}_0|)$ are all simple. The quotient $V^{n+1}/|\check{\omega}_0|$ is a vector space of dimension $n+1-2 = n-1$. By a general point we mean a point in $\check{X}_{n-1} \setminus \check{X}_{n-3}$. Let $H_{\check{\omega}}$ be the hyperplane in $\mathbb{P}(\wedge^2 V^{n+1})$ corresponding to the point $\check{\omega}$ on the dual variety. Then

$$\begin{aligned}
H_{\check{\omega}} \cap G(2, V^{n+1}) = & \\
& \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} | \ \check{\omega} \wedge u_1 \wedge u_2 = 0\} \\
= & \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} | \ \check{\omega}_0 \wedge u_1 \wedge u_2 \wedge (\check{\omega}_1 + \cdots + \check{\omega}_{\frac{n-1}{2}}) = 0\} \\
= & \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} | \ \check{\omega}_0 \wedge u_1 \wedge u_2 = 0\} \\
& \cup \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} | \ (\check{\omega}_1 + \cdots + \check{\omega}_{\frac{n-1}{2}}) \wedge \mathbb{P}^3_{\langle \mathbb{P}^1_{\check{\omega}}, u_1, u_2 \rangle} = 0\} \\
= & \{\mathbb{P}^1 \subset \mathbb{P}(V^{n+1}) | \ \mathbb{P}^1 \cap \mathbb{P}^1_{\check{\omega}} \neq \emptyset\} \\
& \cup \{\mathbb{P}^1 \subset \mathbb{P}^3_{\langle \mathbb{P}^1_{\check{\omega}}, u_1, u_2 \rangle} | \ (\check{\omega}_1 + \cdots + \check{\omega}_{\frac{n-1}{2}}) \wedge \mathbb{P}^3_{\langle \mathbb{P}^1_{\check{\omega}}, u_1, u_2 \rangle} = 0\}
\end{aligned}$$

We will now examine the intersection

$$\Delta = H_{\check{\omega}_1} \cap \cdots \cap H_{\check{\omega}_\alpha} \cap G(2, V^{n+1})$$

where α is some natural number and the hyperplanes $H_{\check{\omega}_j}$ are all tangent at the same point. So pick α points $\check{\omega}_1, \ldots, \check{\omega}_\alpha$ on $\check{X}_{n-1} \setminus \check{X}_{n-3}$, i.e. α general points on the dual variety. If the hyperplanes $H_{\check{\omega}_j}$ are all tangent at the point $p = u \wedge v$, the forms $\check{\omega}_j$ can all be written as

$$\check{\omega}_j = u \wedge v \wedge (\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}})$$

where $\check{\omega}_{j,k}$ is an element in $\wedge^{n-3}(V^{n+1}/\langle u, v\rangle)$ for $j = 1, \ldots, \alpha$ and $k = 1. \ldots, \frac{n-1}{2}$. Notice that $\mathbb{P}^1_{\check{\omega}_j}$ is the line $\langle u, v\rangle$ for all j. Thus the intersection Δ can be found explicitly:

$$\begin{aligned}
\Delta &= H_{\check{\omega}_1} \cap \cdots \cap H_{\check{\omega}_\alpha} \cap G(2, V^{n+1}) \\
&= \bigcap_{j=1}^{\alpha} \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ u \wedge v \wedge u_1 \wedge u_2 \wedge (\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}}) = 0\} \\
&= \bigcap_{j=1}^{\alpha} \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ u \wedge v \wedge u_1 \wedge u_2 = 0\} \\
&\qquad\qquad \cup \{u_1 \wedge u_2 \in \wedge^2 V^{n+1} |\ (\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}}) \wedge \mathbb{P}^3_{\langle u,v,u_1,u_2\rangle} = 0\} \\
&= \bigcap_{j=1}^{\alpha} \{\mathbb{P}^1 \subset \mathbb{P}(V^{n+1}) |\ \mathbb{P}^1 \cap \langle u, v\rangle \neq \emptyset\} \\
&\qquad\qquad \cup \{\mathbb{P}^1 \subset \mathbb{P}^3_{\langle u,v,u_1,v_1\rangle} |\ (\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}}) \wedge \mathbb{P}^3_{\langle u,v,u_1,u_2\rangle} = 0\}
\end{aligned}$$

The intersection is all the lines which is contained in some $\mathbb{P}^3$ where the $\mathbb{P}^3$ contains $\langle u, v\rangle$ and is killed by $(\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}})$ for all j.

The forms $\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}}$ are elements in $\wedge^{n-4}(V^{n+1}/\langle u, v\rangle)$. A $\mathbb{P}^3$ occurs in the intersection if it is spanned by $\langle u, v\rangle$ and a disjoint line $\langle u_1, v_1\rangle$ and have the property

$$u \wedge v \wedge u_1 \wedge u_2 \wedge \left(\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}}\right) = 0 \text{ for } j = 1, \ldots, \alpha$$

We know that the form in the brackets does not involve u and v, so modulo u, v we have

$$u_1 \wedge u_2 \wedge \left(\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}}\right) = 0 \quad \text{for } j = 1, \ldots, \alpha$$

Thus every $\mathbb{P}^3$ in Δ is spanned by $\langle u, v\rangle$ and a line $\langle u_1, u_2\rangle$ satisfying

$$\begin{aligned}
&\{u_1 \wedge u_2 \in \wedge^2(V^{n+1}/\langle u, v\rangle) |\ u_1 \wedge u_2 \wedge \left(\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}}\right) = 0\ \forall\ j\} \\
&= G(2, n-1) \cap H_1 \cap \cdots \cap H_\alpha
\end{aligned}$$

where H_j is the hyperplane in $\mathbb{P}(\wedge^2 Z^{n-1})$ corresponding to the point $(\check{\omega}_{j,1} + \cdots + \check{\omega}_{j,\frac{n-1}{2}})$ in $\mathbb{P}(\wedge^{n-3} Z^{n-1})$ (the vector space Z is isomorphic to $V^{n+1}/\langle u, v\rangle$). We get a family of $\mathbb{P}^3$-s of dimension $2(n-3) - \alpha$, and thus a family of $G(2,4)$-s on $G(2, V^{n+1})$ of dimension $2(n-3) - \alpha$.

Now, pick α points $\check{\omega}_1, \ldots, \check{\omega}_\alpha$ on $\check{X}_{n-1} \setminus \check{X}_{n-3}$. If $H_{\check{\omega}_j}$ are all tangent at the point $p = u \wedge v$, the forms $\check{\omega}_1, \ldots, \check{\omega}_\alpha$ are all points in $T_p^\perp$. We have previously seen that this implies that the $\mathbb{P}^{\alpha-1}$ in $\mathbb{P}(\wedge^{n-1} V^{n+1})$ spanned by these points is contained in $\check{X}_{n-1}$.

A point $\check{\omega}$ in $\check{X}_{n-3} \cap T_p^\perp$ can we written as

$$\check{\omega} = u \wedge v \wedge (\check{\omega}_1 + \cdots + \check{\omega}_{\frac{n-3}{2}})$$

where the $\check{\omega}_j$ in $\wedge^{n-3}(V^{n+1}/\langle u,v\rangle)$ are all simple . The elements in the wedge product space $\wedge^{n-3}(V^{n+1}/\langle u,v\rangle)$ which can be written in this way corresponds to 2-forms in $n-1$ variables of rank $n-3$, i.e. second to maximal rank. Thus $\check{X}_{n-3}\cap T_p^{\perp}$ has codimension one in $T_p^{\perp}$. We have

$$\begin{array}{llcl} \check{X}_{n-3}\cap T_p^{\perp} & \text{has codimension} & 1 & \text{in } T_p^{\perp} \\ \check{X}_{n-5}\cap T_p^{\perp} & \text{has codimension} & 6 & \text{in } T_p^{\perp} \\ \check{X}_{n-8}\cap T_p^{\perp} & \text{has codimension} & 15 & \text{in } T_p^{\perp} \\ & & \vdots & \\ \check{X}_{n-k}\cap T_p^{\perp} & \text{has codimension} & \binom{k-1}{2} & \text{in } T_p^{\perp} \end{array}$$

where k is an odd number. Thus a $\mathbb{P}^{\alpha-1}$ i $\check{X}_{n-1}$ spanned by α points whose corresponding hyperplanes are all tangent at the same point, will intersect $\check{X}_{n-k}$ in dimension

$$\alpha-1+2(n-1)-\binom{k-1}{2}-2(n-1)=\alpha-1-\binom{k-1}{2}$$

So as long as $\alpha-1\geq\binom{k-1}{2}$, such a $\mathbb{P}^{\alpha}$ will actually intersect $\check{X}_{n-k}$.

Example 3.6 ($G(2,V^8)$). In the case of $G(2,V^8)$, we have examined the case $\alpha=3$. A plane in $\check{X}_6$ spanned by three points whose corresponding hyperplanes are all tangent at the same point *will* intersect $\check{X}_4$, since $\check{X}_4\cap T_p^{\perp}$ has codimension 1 in $T_p^{\perp}$. The plane will not intersect $G(2,V^8)$ since $G(2,V^8)\cap T_p^{\perp}$ has codimension six in $T_p^{\perp}$. △

We are ready to prove the following proposition:

Proposition 3.6. *There exist $\mathbb{P}^{\alpha-1}$-s in $\check{X}_{n-1}$ which are not spanned by α points whose corresponding hyperplanes are all tangent at the same point.*

Proof. Any $\mathbb{P}^{\alpha-1}$ is spanned by α points, $\check{\omega}_1,\ldots,\check{\omega}_\alpha$. Their supports are the lines $\mathbb{P}^1_{\check{\omega}_1},\ldots,\mathbb{P}^1_{\check{\omega}_\alpha}$. Suppose all the supports pass through the same point. This translates to having α forms in $\wedge^2V^{n+1}$ of rank $n-1$ whose supports are all contained in a $\mathbb{P}^{n-1}$. Then all the forms are elements in $\wedge^2W^n$. Since n is an odd number, any linear combination has rank at most $n-1$. Thus the span of these forms is contained in X_{n-1}. In the dual space, $\mathbb{P}^{\alpha-1}=\langle\check{\omega}_1,\ldots,\check{\omega}_\alpha\rangle$ is contained in $\check{X}_{n-1}$. □

Recall that we have seen that the hyperplanes of the type $\langle\sigma_{10}\rangle$, are all tangent hyperplanes. Now that we know exactly what the dual varieties to Grassmannians of lines are, we may investigate whether all tangent hyperplanes are of this type. In $G(2,n+1)$ the cycle σ_{10} looks like this

$$\sigma_{10}=\{\Lambda\in G(2,n+1)|\dim(\Lambda\cap V_{n-1})\geq 1\}$$

i.e we have a tangent hyperplane section for each linear subspace of codimension 2. Furthermore, $\dim G(n-1,n+1) = 2(n-1)$, so the family of points on the dual variety corresponding to tangent hyperplanes of this type, has dimension $2(n-1)$. The dual variety $\check{G}(2,n+1)$ sits inside $\check{\mathbb{P}}^{\binom{n+1}{2}-1}$, and theorems 3.4 and 3.5 says that it is either a hypersurface or a variety of codimension 3. The codimension of the family of points of the type $\langle\sigma_{10}\rangle$ on the dual variety is

$$\binom{n+1}{2} - 1 - 1 - 2(n-1) = \frac{n^2-3n}{2} \quad \text{when } n \text{ is odd}$$

and

$$\binom{n+1}{2} - 1 - 3 - 2(n-1) = \frac{n^2-3n}{2} - 2 \quad \text{when } n \text{ is even}$$

Thus when $n = 4$ the tangent hyperplanes coming from cycles σ_{10} form the whole dual variety. Furthermore, $\frac{n^2-3n}{2} > 0$ whenever $n > 3$, and $\frac{n^2-3n}{2} - 2 > 0$ whenever $n > 4$, so for all other Grassmannians of lines, except $G(2,5)$ there are other types of tangent hyperplanes beside the ones of the type σ_{10}. This proves

Proposition 3.7. *When $n = 4$, the cycles σ_{10} form the whole dual variety. These tangent hyperplanes does not form the whole dual variety for any $n \geq 5$.*

3.3 Apolar varieties

We have already glanced at varieties of the type

$$Y = \bigcup_{\gamma\in\Gamma} \langle G(2,4)_\gamma \rangle$$

They have promising properties as candidates for apolar varieties for Grassmannians of lines. In this section, we will study such Y-s more carefully. In particular, we will study Y-s parametrized by cycles on $G(4,n+1)$. We will define a map v from the Schubert cycles on $G(4,n+1)$ to the Schubert cycles on $G(2,n+1)$ where $v(\sigma) = G(2,n+1) \cap (\bigcup_{W^4\in\sigma} \langle G(2,W^4)\rangle)$. When σ is a linear space, we find the degree of Y and find the ideal of Y explicitly. Finally, we prove that all these Y-s are cones.

The way we constructed Y was by writing the intersection $\Delta = H_1 \cap \cdots \cap H_\alpha \cap G(2,V^{n+1})$ (where all the H_i-s are hyperplanes, tangent at the same point) as a union of $G(2,4)$-s. The union is ACM because it is a linear intersection of the Grassmannian, which is ACM. By construction, all the $G(2,4)$-s in the union have a point in common, namely the point of tangency. Call this common point p. The tangent cone at p over the intersection Δ is cut out by the leading terms in the ideal of Δ when expanded around p. This is a complete intersection, and the tangent cone is therefore ACM and generated by the leading terms.

Example 3.7 **($G(2,6)$).** Let p be the point $(1,0,\ldots,0)$. The hyperplanes $H^1 = Z(x_{46})$ and $H^2 = Z(x_{56})$ are both tangent at p. When expanded around p (by setting $x_{12} = 1$), the leading terms of the generators in the ideal of $H^1 \cap H^2 \cap G(2,6)$ is

$$x_{34}, \quad x_{35}, \quad x_{36}, \quad x_{45}, \quad -x_{14}x_{26} + x_{16}x_{24}, \quad -x_{15}x_{26} + x_{16}x_{25}$$

The tangent cone at p has dimension $\dim G(2,6) - 2 = 6$ inside $\mathbb{P}^{14-2}$, and therefore it must be a complete intersection. △

Now consider the cone over $H_1 \cap \cdots \cap H_\alpha \cap G(2, V^{n+1})$ with vertex in p. We will now show that this cone equals the union of the linear spans of the $G(2,4)$-s.

Let $G(2,4)_\gamma$ be one of the $G(2,4)$-s in the union. The cone

$$C_p G(2,4)_\gamma$$

must be contained in the cone over $H_1 \cap \cdots \cap H_\alpha \cap G(2, V^{n+1})$ with vertex in p. Call this big cone C_p. Now, pick a point q in $\langle G(2,4)_\gamma \rangle$. The line through p and q is a line which intersects $G(2,4)_\gamma$, and therefore it has two points in common with $G(2,4)_\gamma$. Thus this line is in $C_p G(2,4)_\gamma$. This gives the inclusion

$$\langle G(2,4)_\gamma \rangle \subset C_p G(2,4)_\gamma$$

But $\langle G(2,4)_\gamma \rangle$ and $C_p G(2,4)_\gamma$ both have dimension five, and thus they are equal. There was nothing special about the chosen $G(2,4)$, so this implies the inclusion

$$Y = \bigcup_{\gamma \in \Gamma} \langle G(2,4)_\gamma \rangle \subset C_p$$

Moreover, any line in C_p is a line through p and some other point on $\cup_{\gamma \in \Gamma} G(2,4)_\gamma$. Thus any line in C_p is in $C_p G(2,4)_\gamma$ for some $\gamma \in \Gamma$. This implies that we also have the opposite inclusion, and it all summarizes to the equality

$$Y = \bigcup_{\gamma \in \Gamma} \langle G(2,4)_\gamma \rangle = C_p$$

We have proved that the Y-s of the type discussed here are cones, and proposition 1.7 gives that they are ACM.

3.3.1 What about more general unions?

When we consider unions of $G(2,4)$-s on the Grassmannian $G(2, V^{n+1})$, we automatically get a set of points on $G(4, V^{n+1})$. To be precise, the particular $G(2, W^4)$ corresponds to the point $W^4 \in G(4, V^{n+1})$. Conversely, any set of point on $G(4, V^{n+1})$ gives a union of $G(2,4)$-s on $G(2, V^{n+1})$. We want to consider the unions of $G(2,4)$-s coming from Schubert cycles on $G(4, V^{n+1})$.

Example 3.8 ($G(2,7)$)**.** Consider the Schubert cycle σ_{3221} in $G(4,7)$. By definition

$$\sigma_{3221} = \{\Lambda \in G(4,7) \mid \dim(\Lambda \cap V_1) \geq 1, \dim(\Lambda \cap V_4) \geq 3, \dim(\Lambda \cap V_6) \geq 4\}$$

Projectively, the cycle is

$$\sigma_{3221} = \{\mathbb{P}^3 \subset \mathbb{P}^6 \mid \mathbb{P}^0_F \subset \mathbb{P}^3 \subset \mathbb{P}^5_F \text{ and } \mathbb{P}^3 \cap \mathbb{P}^3_F \supset \mathbb{P}^2\}$$

A line in such a $\mathbb{P}^3$ is contained in $\mathbb{P}^5_F$ and it intersects $\mathbb{P}^3_F$ in at least a point. Thus a line in such a $\mathbb{P}^3$ is an element in the cycle

$$\{\Lambda \in G(2,7) \mid \dim(\Lambda \cap V_4) \geq 1, \dim(\Lambda \cap V_6) \geq 2\}$$

As a cycle, this has indices c_1 and c_2 given by the equalities

$$7 - 2 + 1 - c_1 = 4 \text{ and } 7 - 2 + 2 - c_2 = 6$$

Thus the union of lines in the given $\mathbb{P}^3$-s is contained in the cycle σ_{21} on $G(2,7)$. Now, pick a line in σ_{21} on $G(2,7)$. The line and $\mathbb{P}^0_F$ will span a plane, and this plane together with a point in $\mathbb{P}^3_F$ spans a $\mathbb{P}^3$. This $\mathbb{P}^3$ contains the point $\mathbb{P}^0_F$, it is contained in $\mathbb{P}^5_F$ and it intersects $\mathbb{P}^3_F$ in a plane. Thus any line in σ_{21} corresponds to a $\mathbb{P}^3$ in σ_{3221}. We have defined a correspondence

$$\begin{array}{ccc} \{\text{Schubert cycles on } G(4,7)\} & \leftrightarrow & \{\text{Schubert cycles on } G(2,7)\} \\ \sigma_{3221} & \leftrightarrow & \sigma_{21} \end{array}$$

Consider an arbitrary cycle $\sigma_{c_1c_2c_3c_4}$ on $G(4,7)$. It is defined as

$$\sigma_{c_1c_2c_3c_4} = \{\Lambda \in G(4,7) \mid \dim(\Lambda \cap V_{3+i-c_i}) \geq i \text{ for all } i\}$$

A line in a $\mathbb{P}^3$ in this cycle will intersect V_{3+i-c_i} in at least dimension $i-2$ for all i. Thus a line in such a $\mathbb{P}^3$ is contained in the cycle

$$\{\Lambda \in G(2,7) \mid \dim(\Lambda \cap V_{3+i-c_i}) \geq i - 2\} \tag{3.15}$$

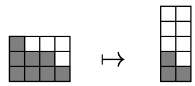
Notice that only the indices c_3 and c_4 will matter, as the expression $\dim(\Lambda \cap V_{3+i-c_i}) \geq i-2$ is meaningful only for $i \geq 3$. The expression in equation 3.15 forms the cycle $\sigma_{a_1a_2}$ on $G(2,7)$, where $5 + 1 - (3 + 3 - c_3) = a_1$ and $5 + 2 - (3 + 4 - c_4) = a_2$. That is, $a_1 = c_3$ and $a_2 = c_4$. Equation 3.15 can therefore be rewritten as

$$\sigma_{c_3c_4} = \{\Lambda \in G(2,7) \mid \dim(\Lambda \cap V_{6-c_3}) \geq 1 \text{ and } \dim(\Lambda \cap V_{7-c_4}) \geq 2\}$$

Pick a line L in $\sigma_{c_3c_4}$ on $G(2,7)$. The line together with a line in $\mathbb{P}(V_{5-c_2})$ which intersects $\mathbb{P}(V_{4-c_1})$ in a point, will span a $\mathbb{P}^3$. This $\mathbb{P}^3$ is an element in $\sigma_{c_1c_2c_3c_4}$. We have defined a map

$$\begin{array}{rccc} \upsilon : & \{\text{Schubert cycles on } G(4,7)\} & \to & \{\text{Schubert cycles on } G(2,7)\} \\ & \sigma_{c_1c_2c_3c_4} & \to & \sigma_{c_3c_4} \end{array}$$

Notice that every time we pick a line in $\mathbb{P}(V_{5-c_2})$ which intersects $\mathbb{P}(V_{4-c_1})$ in a point, we will get a $\mathbb{P}^3$, and L is in all these $\mathbb{P}^3$-s. These lines form the cycle $\sigma_{c_1-c_2}$ on $G(2, V_{5-c_2})$, and L is therefore in a family of $\mathbb{P}^3$-s of dimension $2(3-c_2)-c_1+c_2 = 6-c_1-c_2$. The line L is in a unique $\mathbb{P}^3$ if and only if $c_1 = c_2 = 3$. △



σ_{3221} σ_{21}

Figure 3.1: Illustration for example 3.8

We will now generalize this map to arbitrary n. Consider an arbitrary cycle $\sigma_{c_1c_2c_3c_4}$ on $G(4, V^{n+1})$. It is defined as

$$\sigma_{c_1c_2c_3c_4} = \{\Lambda \in G(4, V^{n+1}) \mid \dim(\Lambda \in V_{n-3+i-c_i}) \geq i \text{ for all } i\}$$

A line in a $\mathbb{P}^3$ in this cycle will intersect $V_{n-3+i-c_i}$ in at least dimension $i-2$ for all i. Thus a line in such a $\mathbb{P}^3$ is contained in the cycle

$$\{\Lambda \in G(2, V^{n+1}) \mid \dim(\Lambda \cap V_{n-3+i-c_i}) \geq i-2\} \tag{3.16}$$

Again, only the indices c_3 and c_4 matter, as the expression in equation 3.16 is meaningful only when $i \geq 3$. The expression in equation 3.16 forms the cycle $\sigma_{a_1a_2}$ on $G(2, V^{n+1})$, where $n-1+1-(n-3+3-c_3) = a_1$ and $n-1+2-(n-3+4-c_4) = a_2$. That is, $a_1 = c_3$ and $a_2 = c_4$. Equation 3.16 can therefore be rewritten as

$$\sigma_{c_3c_4} = \{\Lambda \in G(2, V^{n+1}) \mid \dim(\Lambda \cap V_{n-c_3}) \geq 1 \text{ and } \dim(\Lambda \cap V_{n+1-c_4}) \geq 2\}$$

Pick a line L in $\sigma_{c_3c_4}$ on $G(2, V^{n+1})$. Take any line in $\mathbb{P}(V_{n-1-c_2})$ which intersects $\mathbb{P}(V_{n-2-c_1})$ in a point. The two lines will span a $\mathbb{P}^3$ in $\sigma_{c_1c_2c_3c_4}$.

The lines in $\mathbb{P}(V_{n-1-c_2})$ which intersects $\mathbb{P}(V_{n-2-c_1})$ in a point forms the cycle

$$\{\Lambda \in G(2, n-1-c_2) \mid \dim(\Lambda \cap V_{n-2-c_1}) \geq 1\} = \sigma_{c_1-c_2}$$

on $G(2, n-1-c_2)$. Thus the line L lies in a family of $\mathbb{P}^3$-s of dimension $2(n-3-c_2)-c_1+c_2 = 2(n-3)-c_1-c_2$. Therefore, the line L is in a unique $\mathbb{P}^3$ if and only if $c_1 = c_2 = n-3$.

Definition 3.3. *The map*

$$\upsilon : \{\textit{Schubert cycles on } G(4, V^{n+1})\} \rightarrow \{\textit{Schubert cycles on } G(2, V^{n+1})\}$$

is defined by

$$\upsilon(\sigma_{c_1c_2c_3c_4}) = \sigma_{c_3c_4}$$

The image of υ *is the union of all cycles* $\sigma_{a_1a_2}$ *where* $0 \leq a_2 \leq a_1 \leq n-3$.

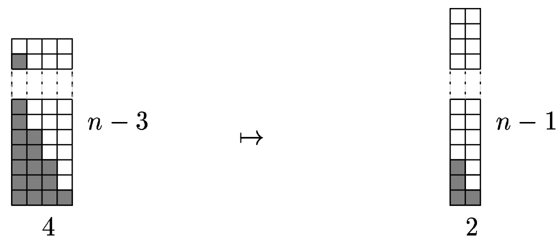


Figure 3.2: Illustration of the map υ

We have seen that any cycle $\sigma_{c_3c_4}$ in the image of υ can be written as a union of $G(2,4)$-s where a general line lies in only one $G(2,4)$. The cycle $\sigma_{c_3c_4}$ is determined by the partial flag $(\mathbb{P}_F^{n-1-c_3} \subset \mathbb{P}_F^{n-c_4})$, and since $n - c_3 - 1 \geq 2$, there is enough space to extend the flag to a partial flag of four linear spaces: $(\mathbb{P}_F^{n-3-c_1} \subset \mathbb{P}_F^{n-2-c_2} \subset \mathbb{P}_F^{n-1-c_3} \subset \mathbb{P}_F^{n-c_4})$. The extended flag defines a cycle $\sigma_{c_1c_2c_3c_4}$ on $G(4, n+1)$. For a general line in $\sigma_{c_3c_4}$ to lie in exactly one $\mathbb{P}^3$ from $\sigma_{c_1c_2c_3c_4}$, we have seen that c_1 and c_2 must be equal to $n-3$. Thus the extended flag is given by $(\mathbb{P}_F^0 \subset \mathbb{P}_F^1 \subset \mathbb{P}_F^{n-1-c_3} \subset \mathbb{P}_F^{n-c_4})$. Since $c_1 = c_2$, it is actually only the $\mathbb{P}_F^1$ that gives a condition on the $\mathbb{P}^3$-s. Precisely, the cycle of $\mathbb{P}^3$-s is given by

$$\sigma_{n-3,n-3,c_3,c_4} = \{\mathbb{P}^3 \in \mathbb{P}^n \mid \mathbb{P}_F^1 \subset \mathbb{P}^3 \subset \mathbb{P}^{n-c_4} \text{ and } \mathbb{P}^3 \cap \mathbb{P}^{n-c_3-1} \supset \mathbb{P}^2\}$$

So let $\sigma_{c_1c_2c_3c_4}$ be an arbitrary Schubert cycle on $G(4, V^{n+1})$, and consider the union of the linear spans of the $G(2,4)$-s coming from this cycle:

$$Y = \bigcup_{W^4 \in \sigma_{c_1c_2c_3c_4}} \langle G(2, W^4) \rangle$$

Then

$$Y \cap G(2, n+1) = \bigcup_{W^4 \in \sigma_{c_1c_2c_3c_4}} G(2, W^4)$$

We have found a way of describing this union as a Schubert cycle on $G(2, V^{n+1})$. If σ is the cycle $\sigma_{c_1c_2c_3c_4}$, the union $Y \cap G(2, n+1)$ is the cycle $\sigma_{c_3c_4}$ on $G(2, V^{n+1})$. By definition, this cycle consists of all the lines l such that l intersects a fixed $\mathbb{P}_F^{n-c_3-1}$ in at least a point and such that l is contained in a fixed $\mathbb{P}_F^{n-c_4}$. The $\mathbb{P}_F^{n-c_3-1}$ can be represented by a matrix of the form

$$\mathbb{P}_F^{n-c_3-1} = (I_{n-c_3}|0)$$

and the fixed $\mathbb{P}_F^{n-c_4}$ can be represented by a matrix of the form

$$\mathbb{P}_F^{n-c_4} = (I_{n+1-c_4}|0)$$

Thus a line l in $\sigma_{c_3c_4}$ can be represented by a matrix

$$\begin{pmatrix} * & \cdots & * & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & \cdots & * & * & * & \cdots & * & 0 & \cdots & 0 \end{pmatrix} \tag{3.17}$$
$$\uparrow \quad \cdots \quad \uparrow$$
$$\text{columns number } n-c_3+1,\ldots,n+1-c_4 \text{ are special}$$

where a star means an arbitrary number. The first row has $n - c_3$ stars, and the second row has $n + 1 - c_4$ stars. Both rows have zeros only on the c_4 last entries, so any (2×2)-minor which involves at least one of the c_4 last columns are zero. Thus the Plücker coordinates x_{ij} where $j \geq n + 2 - c_4$ are zero. Moreover, any (2×2)-minor which involves two of the special columns is zero. Thus the Plücker coordinates x_{ij} where

$$n - c_3 + 1 \leq i < j \leq n + 1 - c_4$$

are also zero.

Example 3.9 ($G(2,7)$ continued). In example 3.8, we considered the cycle σ_{3221} and its image under v. An element in σ_{21} on $G(2,7)$ can be represented by a matrix

$$\begin{pmatrix} * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 \end{pmatrix}$$

The (2×2)-minors which involve the last column are all zero, and so is the minor involving columns five and six. Thus the following Plücker coordinates are all zero:

$$x_{17},\ x_{27},\ x_{37},\ x_{47},\ x_{57},\ x_{67} \text{ and } x_{56}$$

Any point on $G(2,7)$ can be represented by a (7×7) skew symmetric matrix, and the above argument shows that a point in σ_{21} can be represented by a skew symmetric matrix of the form

$$\begin{pmatrix} 0 & * & * & * & * & * & 0 \\ * & 0 & * & * & * & * & 0 \\ * & * & 0 & * & * & * & 0 \\ * & * & * & 0 & * & * & 0 \\ * & * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.18}$$

△

In the general case, a line represented by a matrix as in equation 3.17, can also be represented by an $(n+1) \times (n+1)$ skew symmetric matrix:

$$\begin{pmatrix} 0 & * & * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \\ * & 0 & * & \cdots & * & * & \cdots & * & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & 0 & * & \cdots & * & 0 & \cdots & 0 \\ * & * & * & \cdots & * & \boxed{0} & \cdots & \boxed{0} & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & * & \boxed{0} & \cdots & \boxed{0} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.19}$$

$$\uparrow \cdots \uparrow$$

columns number $n+2-c_4, \ldots, n+1$

The boxed submatrix is a square matrix and it consists of the common entries on rows and columns number $n-c_3+1, \ldots, n+1-c_4$.

Next, we need to find the dimension of the linear span of $Y \cap G(2, n+1)$. This dimension equals the number of non-zero minors of the matrix in equation 3.17 minus one. The number of non-zero minors is exactly equal to the number of non-zero entries in the matrix in equation 3.19. But the number of non-zero entries 3.19 is the number of non-zero entries in the first row, plus the number of non-zero entries in the second row and so on, i.e.

$$\begin{aligned} &(n-c_4)+(n-c_4-1)+\cdots+(n-c_4-(n-c_3-1)) \\ &=(n-c_4)+(n-c_4-1)+\cdots+(c_3-c_4+1) \\ &=\sum_{i=1}^{n-c_4} i-\sum_{j=1}^{c_3-c_4} j \\ &=\frac{(n-c_4)(n-c_4+1)}{2}-\frac{(c_3-c_4)(c_3-c_4+1)}{2} \\ &=\frac{(n-c_4)^2+(n-c_4)-(c_3-c_4)^2-(c_4-c_4)}{2} \end{aligned}$$

This proves

Proposition 3.8. *Thus the linear span of*

$$Y \cap G(2, n+1) = \bigcup_{W^4 \in \sigma_{c_1 c_2 c_3 c_4}} G(2, W^4)$$

where $\sigma_{c_1c_2c_3c_4}$ is a Schubert cycle on $G(4, V^{n+1})$ is

$$\begin{aligned}\dim\langle Y \cap G(2,n+1)\rangle &= \frac{(n-c_4)^2 + (n-c_4) - (c_3-c_4)^2 - (c_3-c_4)}{2} - 1 \\ &=: d_{Y\cap G}(c_3,c_4) - 1\end{aligned}$$

We have a cycle $\sigma_{c_3c_4}$ on $G(2, n+1)$, where $c_4 \le c_3 \le n-3$. We have seen that this cycle is a union of $G(2,4)$-s. In fact, we can write the cycle $\sigma_{c_3c_4}$ as a union of $G(2,4)$-s where a general line is in only one $G(2,4)$, by choosing the cycle $\sigma_{n-3,n-3,c_3,c_4}$ on $G(4, n+1)$. We defined a variety Y to be the union of the linear spans of these $G(2,4)$-s. Since every $\mathbb{P}^5$ in Y is the linear span of a part of $\sigma_{c_3c_4}$, every $\mathbb{P}^5$ in Y must be contained in the linear span of $\sigma_{c_3c_4}$ itself. Thus Y is a variety which habitat is a $\mathbb{P}^{d_{Y\cap G}-1}$.

What is the dimension of Y? We can find the dimension of Y by considering the incidence

$$\begin{array}{ccc} \coprod_{W^4\in\sigma}\langle G(2,W^4)\rangle = \{(q,W^4) \in Y\times\sigma \mid q \in \langle G(2,W^4)\rangle\} & & \\ \text{birational}\big\downarrow \pi_1 & \overset{\pi_2}{\underset{5}{\searrow}} & \\ Y = \bigcup_{W^4\in\sigma}\langle G(2,W^4)\rangle & & \sigma \end{array}$$

Fix a point Z^4 in σ. The fiber $\pi_2^{-1}(Z^4)$ is given by

$$\pi_2^{-1}(Z^4) = \{q \in Y \mid q \in \langle G(2,Z^4)\rangle\}$$

Thus the fiber over a fixed point in σ is five dimensional. Next, fix a point p in Y. The fiber $\pi_1^{-1}(p)$ is given by

$$\pi_1^{-1}(p) = \{W^4 \in \sigma \mid p \in \langle G(2,W^4)\rangle\}$$

We have previously seen that we can choose the cycle σ in such a way that this fiber consists of exactly one element, by setting $c_1 = c_2 = n-3$. This gives

Proposition 3.9. *The dimension of Y is*

$$\begin{aligned}\dim Y = \dim \bigcup_{W^4\in\sigma}\langle G(2,W^4)\rangle &= \dim\sigma + 5 \\ &= 4(n-3) - 2(n-3) - c_3 - c_4 + 5 \\ &= 2(n-1) - c_3 - c_4 + 1 \\ &= \dim(Y \cap G(2,n+1)) + 1\end{aligned}$$

We will now describe the ideal of Y. We have seen that the index c_4 gives the condition that a line in $\sigma_{c_3 c_4}$ is actually contained in a $\mathbb{P}_F^{n-c_4}$. The $\mathbb{P}^3$-s in the corresponding minimal cycle on $G(4, n+1)$ are all contained in $\mathbb{P}_F^{n-c_4}$, too. So we loose no generality by setting $c_4 = 0$. To make the notation as simple as possible, we will from now on write σ_c instead of $\sigma_{c_3 0}$. The corresponding minimal cycle on $G(4, n+1)$ is the cycle $\sigma_{n-3,n-3,c,0}$. From equation 3.19 a line in the cycle σ_c is represented by a $(n+1) \times (n+1)$ matrix of the form

$$\begin{pmatrix} 0 & * & * & \cdots & * & * & \cdots & * \\ * & 0 & * & \cdots & * & * & \cdots & * \\ \vdots & & & & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & 0 & * & \cdots & * \\ * & * & * & \cdots & * & \boxed{0} & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * & 0 & \cdots & 0 \end{pmatrix} \tag{3.20}$$

$$\uparrow \cdots \uparrow$$

columns number $n-c+1, \ldots, n+1$

The boxed submatrix is a square matrix and it consists of the common entries on rows and columns number $n-c+1, \ldots, n+1$.

The cycle σ_c consists of the lines which intersect the fixed $\mathbb{P}_F^{n-c-1}$ in at least a point. In particular, the lines contained in this $\mathbb{P}_F^{n-c-1}$ is in σ_c. All such lines are represented by $2 \times (n+1)$-matrices of the form

$$\begin{pmatrix} * & * & \cdots & * & 0 & \ldots & 0 \\ * & * & \cdots & * & 0 & \ldots & 0 \end{pmatrix}$$

where both rows have $n-c$ stars. Thus all Plücker coordinates x_{ij} where $j \geq n-c+1$ are zero. We can therefore identify the fixed $\mathbb{P}_F^{n-c-1}$ as the upper left $(n-c) \times (n-c)$ submatrix of the matrix in equation 3.20.

We constructed the corresponding minimal cycle on $G(4, n+1)$ by choosing a line $\mathbb{P}_F^1$ in the fixed $\mathbb{P}_F^{n-c-1}$. Choosing a line in the fixed $\mathbb{P}_F^{n-c-1}$ corresponds to choosing one coordinate in the matrix defining $\mathbb{P}_F^{n-c-1}$ to be different from zero:

$$\begin{pmatrix} 0 & 1 & * & \cdots & * & * & \cdots & * \\ -1 & 0 & * & \cdots & * & * & \cdots & * \\ \vdots & & & & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & 0 & * & \cdots & * \\ * & * & * & \cdots & * & 0 & \cdots & 0 \\ \vdots & & & & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * & 0 & \cdots & 0 \end{pmatrix} \tag{3.21}$$

$$\uparrow \cdots \uparrow$$
columns number $n-c+1,\ldots,n+1$

The cycle σ_c is defined by the 4-Pfaffians of this matrix, and the coordinates which are zero:

$$\sigma_c = Z(\text{4-Pfaffians }, x_{ij} \text{ where } n-c+1 \leq i < j \leq n+1)$$

Since σ_c is contained in Y, the ideal of Y must be contained in the ideal of σ_c:

$$I_Y \subset (\text{4-Pfaffians }, x_{ij} \text{ where } n-c+1 \leq i < j \leq n+1)$$

We have already seen that Y lives in the linear span of σ_c, so the x_{ij}-s are contained in I_Y. Now, $Y = \bigcup_{W^4 \in \sigma_{n-3,n-3,c}} \langle G(2,W^4)\rangle$, so pick one particular $W_0^4 \in \sigma_{n-3,n-3,c}$. Then $\mathbb{P}(W_0^4)$ is a $\mathbb{P}^3$ which contains the chosen line $\mathbb{P}^1_F$, and intersects $\mathbb{P}^{n-c-1}$ in at least a plane, so the 4-Pfaffian of the matrix in equation 3.21 must involve both the first and the second rows and columns, and at least three of the first $n-c$ rows and columns in total. There are

$$Q_1 = (n-c-2)(c+1) + \binom{n-c-2}{2}$$

such 4-Pfaffians. The $\langle G(2,W_0^4)\rangle$ is contained in Y, so I_Y is contained in the ideal of $\langle G(2,W_0^4)\rangle$. The ideal of Y restricted to $\langle G(2,W_0^4)\rangle$ must therefore be zero. But there are points in $\langle G(2,W_0^4)\rangle$ which are not in the zero locus of the 4-Pfaffian defining $G(2,W_0^4)$, and therefore this 4-Pfaffian can not be in the ideal I_Y. In this way, all the Q_1 quadrics of this type are excluded from the ideal of Y.

Notice that all the Q_1 quadrics discussed above come from $\mathbb{P}^3$-s in the cycle

$$\{\mathbb{P}^3 \subset \mathbb{P}^n \mid \mathbb{P}^3 \supset \mathbb{P}^1_F \text{ and } \mathbb{P}^3 \cap \mathbb{P}^{n-c-1}_F \supset \mathbb{P}^2\} = \sigma_{n-3,n-3,c,0}$$

Every element in the cycle above can be represented by a $4 \times (n+1)$ matrix of the form

$$\begin{pmatrix} * & * & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & * & * & \cdots & * & 0 & \cdots & 0 \\ * & * & * & \cdots & * & * & \cdots & * \end{pmatrix}$$

where the third row has $n-c$ stars. There are

$$\binom{n-c-2}{2}+(n-c-2)(c+1)$$

non-zero maximal minors in this matrix, and the dimension of the linear span of the cycle $\sigma_{n-3,n-3,c,0}$ is this number minus one.

What about the rest of the quadrics? We know that there are $\binom{n+1}{4}$ quadrics in the ideal of σ_c. Thus we can think of the quadrics as points in the Plücker space of a $G(4,n+1)$. Recall that each quadric corresponds to a point on $G(4,n+1)$ itself. We have excluded the quadrics corresponding to points in the cycle $\sigma_{n-3,n-3,c,0}$ on $G(4,n+1)$.

Fix a flag F' where $\mathbb{P}^0_{F'}$ is the last column vector in the matrix in equation 3.21, where $\mathbb{P}^1_{F'}$ is spanned by the last two column vectors of the matrix and so on.

The 4-Pfaffians of the matrix in equation 3.21 coming from $\mathbb{P}^3$-s which intersect $\mathbb{P}^c_{F'}$ in at least a line, are not among the excluded ones. There are

$$Q_2=\binom{c+1}{2}\binom{n-c}{2}+\binom{c+1}{3}(n-c)+\binom{c+1}{4}$$

such 4-Pfaffians. Notice that the $\mathbb{P}^3$-s which intersect $\mathbb{P}^c_{F'}$ in a plane or a 3-space gives 4-Pfaffians which are identically zero. The $\mathbb{P}^3$-s which intersect $\mathbb{P}^c_{F'}$ in exactly a line, gives quadrics which are actually the (2×2)-minors of the $(n-c)\times(c+1)$-matrix directly above the boxed submatrix. Such minors are called *rank four* quadrics, because they only involve four coordinates. A general point in Y lies in exactly one $\langle G(2,4)\rangle$, thus a general point lies in a unique $\mathbb{P}^5$. A particular $\mathbb{P}^5$ is defined by only six coordinates, and they form a rank six quadric which defines the $G(2,4)$. Thus all the rank four quadrics are identically zero, and each point in Y is contained in the zero locus defined by them. Thus the rank four quadrics are contained in the ideal of Y.

These quadrics are parametrized by the $\mathbb{P}^3$-s which intersect $\mathbb{P}^c_{F'}$ in at least a line. Such $\mathbb{P}^3$-s form the cycle

$$\{\mathbb{P}^3\subset\mathbb{P}^n \mid \mathbb{P}^3\cap\mathbb{P}^c_{F'}\supset\mathbb{P}^1\}=\sigma_{n-2-c,n-2-c,0,0}$$

All elements in this cycle are represented by $4\times(n+1)$-matrices of the form

$$\begin{pmatrix} * & * & \cdots & * & 0 & \cdots & 0\\ * & * & \cdots & * & 0 & \cdots & 0\\ * & * & \cdots & * & * & \cdots & *\\ * & * & \cdots & * & * & \cdots & * \end{pmatrix}$$

where the two top rows has $c+1$ stars. A (4×4)-minor is zero if it involves three or four of the last $n-c$ columns, which implies that the linear span of this cycle has projective dimension

$$\binom{n+1}{4} - (c+1)\binom{n-c}{3} - \binom{n-c}{4} - 1$$

There are also quadrics parametrized by $\mathbb{P}^3$-s which intersect $\mathbb{P}^c_{F'}$ in at least a point and $\mathbb{P}^{n-2}_{F'}$ in at least a plane. There are

$$Q_3 = 2\binom{n-c-2}{2}(c+1) + \binom{n-c-2}{3}(c+1) + \binom{n-c-2}{2}\binom{c+1}{2} + 2(n-c-2)\binom{c+1}{2} + 2\binom{c+1}{3} + (n-c-2)\binom{c+1}{3} + \binom{n-c-2}{4}$$

These $\mathbb{P}^3$-s form the cycle

$$\{\mathbb{P}^3 \subset \mathbb{P}^n \mid \mathbb{P}^3 \cap \mathbb{P}^c_{F'} \supset \mathbb{P}^0 \text{ and } \mathbb{P}^3 \cap \mathbb{P}^{n-2}_{F'} \supset \mathbb{P}^2\} = \sigma_{n-3-c,1,1,0}$$

All elements in this cycle are represented by $4 \times (n+1)$-matrices of the form

$$\begin{pmatrix} * & * & \cdots & * & 0 & \cdots & 0 & 0 & 0 \\ * & * & \cdots & * & * & \cdots & * & 0 & 0 \\ * & * & \cdots & * & * & \cdots & * & 0 & 0 \\ * & * & \cdots & * & * & \cdots & * & * & * \end{pmatrix}$$

where the top row has $c+1$ stars. There are

$$\binom{c+1}{4} + \binom{c+1}{3}(n-c) + \binom{c+1}{2}\left[\binom{n-c-2}{2} + 2(n-c-2)\right] + (c+1)\left[\binom{n-c-2}{3} + 2\binom{n-c-2}{2}\right]$$

non-zero maximal minors of this matrix, and the linear span of this cycle is the number of non-zero maximal minors minus one.

The quadrics which the Q_3 ones have in common with the Q_2 ones are in the ideal of Y, since we have already seen that all the Q_2 ones are. The quadrics common to the two groups come from $\mathbb{P}^3$-s in the intersection of the two cycles on $G(4, n+1)$. The intersection of the cycles $\sigma_{n-2-c,n-2-c,0,0}$ and $\sigma_{n-3-c,1,1,0}$ is given by

$$\{\mathbb{P}^3 \subset \mathbb{P}^n \mid \mathbb{P}^3 \cap \mathbb{P}^c_{F'} \supset \mathbb{P}^1 \text{ and } \mathbb{P}^3 \cap \mathbb{P}^{n-2}_{F'} \supset \mathbb{P}^2\} = \sigma_{n-2-c,n-2-c,1,0}$$

All elements in this cycle are represented by a $4 \times (n+1)$-matrix of the form

$$\begin{pmatrix} * & * & \cdots & * & 0 & \cdots & 0 & 0 & 0 \\ * & * & \cdots & * & 0 & \cdots & 0 & 0 & 0 \\ * & * & \cdots & * & * & \cdots & * & 0 & 0 \\ * & * & \cdots & * & * & \cdots & * & * & * \end{pmatrix}$$

where the top two rows have $c+1$ stars. The number of non-zero maximal minors of this matrix is

$$\binom{c+1}{4} + \binom{c+1}{3}(n-c) + \binom{c+1}{2}\left[\binom{n-2-c}{2} + 2(n-2-c)\right]$$

The linear span of the intersection of the cycle is this number minus one.

The quadrics among the Q_3 ones which are not in the Q_2 ones come from $\mathbb{P}^3$-s in $\sigma_{n-3-c,1,1,0}$ which intersect $\mathbb{P}^c_{F'}$ in exactly a point. There are

$$\begin{aligned} Q_3' &= 2\binom{n-c-2}{2}(c+1) + \binom{n-c-2}{3}(c+1) \\ &= (n-k+1)\left((2-k) + \binom{k}{3}\right) \quad \text{when } c = n-k \\ &= \begin{cases} 0 & \text{when } k=3 \\ 2(n-3) & \text{when } k=4 \\ 7(n-4) & \text{when } k=5 \text{ and so on} \end{cases} \end{aligned}$$

such quadrics. Thus if there exists such quadrics, there are at least $2(n-3)$ of them.

These Q_3' quadrics have rank six, but they do not define $G(2,4)$-s which appear in the union which defines Y. Each point in $Y \cap G(2, n+1)$ is in the zero locus of these quadrics. Since the $G(2,4)$-s does not appear in the union which defines Y, there are no points in Y which lie in the linear spans of these $G(2,4)$-s but not on $G(2,4)$ itself. Therefore, all these quadrics are elements in the ideal of Y, that is

Proposition 3.10. *The Q_2 and Q_3' quadrics described above are all in the ideal of Y.*

The dimension of the union of the cycles $\sigma_{n-2-c,n-2-c,0,0}$ and $\sigma_{n-3-c,1,1,0}$ is

$$\begin{aligned} &\dim\langle \sigma_{n-2-c,n-2-c,0,0} \cup \sigma_{n-3-c,1,1,0}\rangle = \\ &\quad \binom{n+1}{4} - (c+1)\binom{n-c}{3} - \binom{n-c}{4} - 1 \\ &\quad + \binom{c+1}{4} + \binom{c+1}{3}(n-c) + \binom{c+1}{2}\left[\binom{n-c-2}{2} + 2(n-c-2)\right] \\ &\quad + (c+1)\left[\binom{n-c-2}{3} + 2\binom{n-c-2}{2}\right] - 1 \\ &\quad - \binom{c+1}{4} - \binom{c+1}{3}(n-c) - \binom{c+1}{2}\left[\binom{n-2-c}{2} + 2(n-2-c)\right] + 1 \\ &\quad = \binom{n+1}{4} - \binom{n-c}{4} - 1 + (c+1)\left[\binom{n-c-2}{3} + 2\binom{n-c-2}{2} - \binom{n-c}{3}\right] \end{aligned}$$

Now, consider the number

$$\dim\langle \sigma_{n-2-c,n-2-c,0,0} \cup \sigma_{n-3-c,1,1,0}\rangle + \dim\langle \sigma_{n-3,n-3,c,0}\rangle - \left(\binom{n+1}{4} - 1\right) \qquad (3.22)$$

This is often a negative number. This implies that the union of the cycles on $G(n+1)$ not always spans the whole Plücker space, and we can not guarantee that there are no other quadrics in the ideal of Y.

The case $c = n - 3$

In the extreme case $c = n - 3$, the number in equation 3.22 is -1. Thus the three cycles span the entire space. If we ad another quadric to the ones we know are in the ideal of Y, the linear span of included ones will intersect the linear span of excluded ones non-empty. This is a contradiction. Thus the ideal of Y is generated by no more than the $Q_2 + Q'_3$ quadrics discussed above. Notice that, when $c = n - 3$, the number Q'_3 is zero.

Recall that the basis for the ideal of $G(2, n+1)$ is the quadrics $\{Q_{ijkl}\}_{1\le i<j<k<l\le n+1}$, where Q_{ijkl} is the 4-Pfaffian of the skew symmetric matrix of basisvectors for $\wedge^2 V^{n+1}$ involving rows and columns number i, j, k and l. Recall also that each of these Pfaffians defines a $G(2, W^4)$ where W^4 is a point on $G(4, n+1)$. Keeping this in mind, we have proved the following theorem:

Theorem 3.11. *Fix a basis $B = (e_1, \dots, e_{n+1})$ for a vector space V^{n+1}, and fix a complete flag $F : \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \langle e_1, e_2, e_3\rangle \subset \cdots \subset \langle e_1, \dots, e_{n+1}\rangle = V^{n+1}$. Let $\sigma = \sigma_{n-3,n-3,n-3,0}(F)$ be a linear space on $G(4, n+1)$, and let*

$$Y = \bigcup_{W^4 \in \sigma} \langle G(2, W^4)\rangle \subset \mathbb{P}(\wedge^2 V^{n+1})$$

Then $Y \cap G(2, n+1) = \sigma_{n-3,0}$. Let $\{Q_{ijkl}\}$ where $1 \le i < j < k < l \le n+1$ be the standard basis for the ideal of $G(2, n+1)$ with respect to the basis B. Then the ideal of Y is generated by the quadrics Q_{ijkl} where $ijk \ne 123$.

Theorem 3.11 gives that when $c = n - 3$, the ideal of Y is generated by the Q_2 quadrics which are (2×2)-minors as mentioned above. In this case, we can find the degree of Y.

We can go back to the general case when the cycle on $G(2, n+1)$ is $\sigma_{c_3 c_4}$ where c_4 may be non-zero. The reason we do this, is to get a formula which always applies. When $c_3 = n - 3$, the variety Y is the zero locus of the (2×2)-minors of a $(n - c_3) \times (c_3 - c_4 + 1)$-matrix. We may view this matrix as a map

$$\mathcal{O}_{\mathbb{P}^N}^{c_3 - c_4 + 1} \longrightarrow \mathcal{O}_{\mathbb{P}^N}^{n - c_3}(1)$$

where $N = d_{Y\cap G} - 1$. The total Chern class of $\mathscr{O}_{\mathbb{P}^N}^{c_3-c_4+1}$ is $c(\mathscr{O}_{\mathbb{P}^N}^{c_3-c_4+1}) = 1$, and the total Chern class of $\mathscr{O}_{\mathbb{P}^N}^{n-c_3}(1)$ is $c(\mathscr{O}_{\mathbb{P}^N}^{n-c_3}(1)) = (1+H)^{n-c_3}$. Also $c(\mathscr{O}_{\mathbb{P}^N}^{n-c_3}(1) - \mathscr{O}_{\mathbb{P}^N}^{c_3-c_4+1})$ equals (by definition) $c(\mathscr{O}_{\mathbb{P}^N}^{n-c_3}(1))/c(\mathscr{O}_{\mathbb{P}^N}^{c_3-c_4+1})$. We will use the following formula:

Theorem 3.12 ([6], Thom-Porteous formula). *Let M be an $(f \times e)$ matrix which defines a homomorphism of vectorbundles over a purely n-dimensional scheme:*

$$E \xrightarrow{M} F$$

where E has rank e and F has rank f. Let c denote the total Chern class of $F - E$. Let $k \leq min(e, f)$ and assume a variety Y is defined as the zero locus of the $(k+1)$-minors of M. Then the degree of Y equals the coefficient of H in the determinant

$$|c_{f-k+j-i}|_{1\leq i,j\leq e-k} = \begin{vmatrix} c_{f-k} & c_{f-k+1} & c_{f-k+1} & \cdots & c_{f+e-2k-1} \\ c_{f-k-1} & c_{f-k} & c_{f-k+1} & \cdots & c_{f+e-2k-2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{f-e+1} & c_{f-e+2} & c_{f-e+3} & \cdots & c_{f-k} \end{vmatrix}$$

In our case, k is always equal to 1, and $e = c_3 - c_4 + 1$ and $f = n - c_3$. If C denotes the total Chern class of $\mathscr{O}_{\mathbb{P}^N}^{n-c_3}(1)$, where $N = d_{Y\cap G} - 1$ we have

$$C = 1 + \binom{n-c_3}{1} H + \binom{n-c_3}{2} H^2 + \cdots + \binom{n-c_3}{n-c_3-1} H^{n-c_3-1} + H^{n-c_3}$$

and the degree of Y is equal to the coefficient of H in the determinant

$$\begin{vmatrix} C_{n-c_3-1} & C_{n-c_3} & C_{n-c_3+1} & \cdots & C_{n-c_4-2} \\ C_{n-c_3-2} & C_{n-c_3-1} & C_{n-c_3} & \cdots & C_{n-c_4-3} \\ \vdots & \vdots & \vdots & & \vdots \\ C_{n-2c_3+c_4} & C_{n-2c_3+c_4+1} & C_{n-2c_3+c_4+2} & \cdots & C_{n-c_3-1} \end{vmatrix}$$

This determinant equals

$$\begin{vmatrix} \binom{n-c_3}{n-c_3-1} H^{n-c_3-1} & H^{n-c_3} & 0 & 0 & \cdots & 0 \\ \binom{n-c_3}{n-c_3-2} H^{n-c_3-2} & \binom{n-c_3}{n-c_3-1} H^{n-c_3-1} & H^{n-c_3} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ \binom{n-c_3}{n-2c_3+c_4} H^{n-2c_3+c_4} & \binom{n-c_3}{n-2c_3+c_4+1} H^{n-2c_3+c_4+1} & \cdots & & \cdots & \binom{n-c_3}{n-c_3-1} H^{n-c_3-1} \end{vmatrix} \tag{3.23}$$

This determinant equals $\beta \cdot H^{(n-c_3-1)(c_3-c_4)}$ for some number β, and this β is the degree of Y. The number $(n-c_3-1)(c_3-c_4)$ is the codimension of Y in $\mathbb{P}^{d_{Y\cap G}-1}$, thus the dimension of Y is

$$\begin{aligned}\dim Y &= d_{Y\cap G(2,n+1)} - 1 - (n-c_3-1)(c_3-c_4)\\ &= \frac{(n-c_4)^2 + (n-c_4) - (c_3-c_4)^2 - (c_3-c_4)}{2} - (n-c_3-1)(c_3-c_4) - 1\\ &= n+2-c_4\end{aligned}$$

when $c_3 = n-3$. This fits, since the dimension of $Y \cap G(2, n+1)$ is $2(n-1)-(n-3)-c_4 = n+1-c_4$ in this case.

Example 3.10 ($G(2, 11)$)**.** Consider the cycle σ_{7772} on $G(4, 11)$, and let Y be the variety

$$Y = \bigcup_{W^4 \in \sigma_{7772}} \langle G(2, W^4)\rangle$$

Then Y is the zero locus of the (2×2)-minors of a (3×6)-matrix. The degree of Y equals the coefficient of H in the determinant

$$\begin{vmatrix} 3H^2 & H^3 & 0 & 0 & 0\\ 3H & 3H^2 & H^3 & 0 & 0\\ 1 & 3H & 3H^2 & H^3 & 0\\ 0 & 1 & 3H & 3H^2 & H^3\\ 0 & 0 & 1 & 3H & 3H^2 \end{vmatrix}$$

A simple calculation shows that this determinant is equal to $21H^{10}$, and the degree of Y is therefore 21. The dimension of Y is 10 $\triangle$

The coefficient of $H^{(n-c_3-1)(c_3-c_4)}$ in the determinant in equation 3.23 is the determinant

$$\begin{vmatrix} \binom{n-c_3}{n-c_3-1} & 1 & 0 & 0 & 0 & \cdots & 0\\ \binom{n-c_3}{n-c_3-2} & \binom{n-c_3}{n-c_3-1} & 1 & 0 & 0 & \cdots & 0\\ \binom{n-c_3}{n-c_3-3} & \binom{n-c_3}{n-c_3-2} & \binom{n-c_3}{n-c_3-1} & 1 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & & & & \vdots\\ \binom{n-c_3}{n-2c_3+c_4} & \binom{n-c_3}{n-2c_3+c_4+1} & \cdots & & & \cdots & \binom{n-c_3}{n-c_3-1} \end{vmatrix} \tag{3.24}$$

If $r = n - c_3$ and $s = c_3 - c_4$, this matrix equals

$$D_s = \begin{vmatrix} \binom{r}{r-1} & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \binom{r}{r-2} & \binom{r}{r-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ \binom{r}{r-3} & \binom{r}{r-2} & \binom{r}{r-1} & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & & & \vdots & \vdots \\ \binom{r}{r-s+1} & \binom{r}{r-s+2} & \cdots & & \cdots & & \binom{r}{r-1} & 1 \\ \binom{r}{r-s} & \binom{r}{r-s+1} & \cdots & & \cdots & & \binom{r}{r-2} & \binom{r}{r-1} \end{vmatrix} \tag{3.25}$$

Expanding this matrix on the first row, gives

$$\begin{aligned} D_s = \binom{r}{r-1} D_{s-1} - \binom{r}{r-2} D_{s-2} + \binom{r}{r-3} D_{s-3} - \binom{r}{r-4} D_{s-4} \\ + \cdots + (-1)^s \binom{r}{r-(s-1)} D_1 + (-1)^{s+1} \binom{r}{r-s} \end{aligned} \tag{3.26}$$

Notice that

$$D_1 = \binom{r}{r-1} = \binom{r}{1}$$

and that

$$D_2 = \binom{r}{r-1}^2 - \binom{r}{r-2} = r^2 - \frac{r(r-1)}{2} = \frac{r^2+r}{2} = \binom{r+1}{2}$$

Continuing, it is not hard to see that $D_3 = \binom{r+2}{3}$. Now, assume

$$D_j = \binom{r+j-1}{j} \text{ for } j = 1, \ldots, i$$

Then we can use the sum in equation 3.26 and write D_{i+1} as

$$\begin{aligned} D_{i+1} &= \left[\sum_{j=1}^{i} (-1)^{i+j} \binom{r}{r-i-1+j} D_j \right] + (-1)^{i+2} \binom{r}{r-i-1} \\ &= \left[\sum_{j=1}^{i} (-1)^{i+j} \binom{r}{r-i-1+j} \binom{r+j-1}{j} \right] + (-1)^{i+2} \binom{r}{r-i-1} \end{aligned}$$

Evaluating this sum in Maple ([18]), gives

$$D_{i+1} = \binom{r+i}{i+1} + \left[(-1)^{i+1} \frac{(r-i)\binom{r}{r-i}}{i+1} + (-1)^{i+2} \binom{r}{r-i-1} \right]$$

and playing with this equation for a while, gives

$$\begin{aligned} D_{i+1} &= \binom{r+i}{i+1} + \left[(-1)^{i+1}\frac{r!}{(r-i-1)!(i+1)!} + (-1)^{i+2}\binom{r}{r-i-1}\right] \\ &= \binom{r+i}{i+1} + \left[(-1)^{i+1}\binom{r}{r-i-1} + (-1)^{i+2}\binom{r}{r-i-1}\right] \\ &= \binom{r+i}{i+1} \end{aligned}$$

We have proved the following theorem:

Theorem 3.13 (The degree of Y). *When $\sigma_{n-3,n-3,n-3,c_4}$ is a Schubert cycle on $G(4,n+1)$, and Y is a variety given by*

$$Y = \bigcup_{W^4 \in \sigma_{n-3,n-3,n-3,c_4}} \langle G(2,W^4)\rangle$$

the degree of Y is

$$\deg Y = \binom{n-c_4-1}{2}$$

When Y is the union of the linear spans of some $G(2,W^4)$-s where W^4 is in a Schubert cycle $\sigma_{n-3,n-3,n-3,c_4}$ on $G(4,n+1)$, we have seen that $Y \cap G(2,n+1)$ is equal to the cycle σ_{n-3,c_4}. The degree of this cycle is (see the formula in equation 1.4)

$$\begin{aligned} \deg(Y \cap G(2,n+1)) &= \frac{(n+1-c_4)!}{2(n-c_4)!} \cdot (n-2-c_4) \\ &= \frac{(n+1-c_4)(n-2-c_4)}{2} \end{aligned}$$

Example 3.11 ($G(2,11)$ continued). The Y defined in example 3.10 has degree 21, and the degree of σ_{72} on $G(2,11)$ is

$$\frac{9!}{2!8!} \cdot 6 = \frac{9 \cdot 6}{2} = 27$$

Thus the degree of Y is less than the degree of the Schubert cycle. △

Consider the quotient

$$\begin{aligned} \frac{\deg Y \cap G(2,n+1)}{\deg Y} &= \frac{(n+1-c_4)(n-2-c_4)}{(n-1-c_4)(n-2-c_4)} \\ &= \frac{n+1-c_4}{n-1-c_4} \end{aligned}$$

The quotient is strictly bigger than one, which implies that the degree of Y is strictly less than the degree of $Y \cap G(2,n+1)$.

The case $c = 0$

We have treated the one extreme case when $c_3 = n-3$. Another extreme case is when $c_3 = 0$. In this case, the variety Y is given by

$$Y = \bigcup_{W^4 \in \sigma_{n-3,n-3,0,0}} \langle G(2, W^4) \rangle$$

The intersection $Y \cap G(2, n+1)$ is the cycle σ_{00}, i.e the whole $G(2, n+1)$. The total number of quadrics in the ideal of $G(2, n+1)$ is $\binom{n+1}{4}$, and

$$Q_1 = (n-2) + \binom{n-2}{2} = \binom{n-1}{2}$$

quadrics are excluded from the ideal of Y. All these quadrics come from $\mathbb{P}^3$-s which contain $\mathbb{P}^1_F$. Such $\mathbb{P}^3$-s form the cycle $\sigma_{n-3,n-3,0,0}$ on $G(4, n+1)$. This cycle is isomorphic to a $G(2, n-1)$, and its linear span therefore has dimension $\binom{n-2}{2} - 1$.

In this particular case, there are no quadrics of rank four in the ideal of $Y \cap G(2, n+1)$, and the number Q_2 is therefore zero.

Let F' be a flag where $\mathbb{P}^0_{F'}$ is the last column vector in the matrix representing $G(2, n+1)$, where $\mathbb{P}^1_{F'}$ is spanned by the two last vectors in the matrix and so on. The quadrics which come from $\mathbb{P}^3$-s which intersect $\mathbb{P}^{n-2}_{F'}$ in at least a plane are not among the excluded ones. They are in fact included in the ideal of Y, as we saw above. These $\mathbb{P}^3$-s form the cycle

$$\{\mathbb{P}^3 \in \mathbb{P}^n \mid \mathbb{P}^3 \cap \mathbb{P}^{n-2}_{F'} \supset \mathbb{P}^2\} = \sigma_{1110}$$

Each point in this cycle can be represented by a matrix of the form

$$\begin{pmatrix} * & * & \cdots & * & 0 & 0 \\ * & * & \cdots & * & 0 & 0 \\ * & * & \cdots & * & 0 & 0 \\ * & * & \cdots & * & * & * \end{pmatrix}$$

There are

$$2\binom{n-1}{3} + \binom{n-1}{4}$$

non-zero maximal minors for this matrix, and the linear span of σ_{1110} has dimension one less than the number of non-zero maximal minors. Since

$$\begin{aligned} &\dim\langle \sigma_{n-3,n-3,0,0} \rangle + \dim\langle \sigma_{1110} \rangle - \left(\binom{n+1}{4} - 1\right) \\ &= \binom{n-1}{2} - 1 + 2\binom{n-1}{3} + \binom{n-1}{4} - 1 - \binom{n+1}{4} + 1 \\ &= \binom{n}{3} + \binom{n}{4} - \binom{n+1}{4} - 1 \\ &= -1 \end{aligned}$$

The union of excluded quadrics and included quadrics therefore span the whole space, and we can conclude that the ideal of Y is generated by precisely these quadrics.

Theorem 3.14. *Fix a basis $B = (e_1, \dots, e_{n+1})$ for a vector space V^{n+1}, and fix a complete flag $F : \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \cdots \subset \langle e_1, \dots, e_{n+1} \rangle = V^{n+1}$. Let $\sigma = \sigma_{n-3,n-3,0,0}(F)$ be a cycle on $G(4, n+1)$, and let*

$$Y = \bigcup_{W^4 \in \sigma} \langle G(2, W^4) \rangle \subset \mathbb{P}(\wedge^2 V^{n+1})$$

Then $\bigcup_{W^4 \in \sigma} G(2, W^4) = G(2, n+1)$. Let $\{Q_{ijkl}\}$ where $1 \leq i < j < k < l \leq n+1$ be the standard basis for the ideal of $G(2, n+1)$ with respect to the basis B. Then the ideal of Y is generated by Q_{ijkl} such that $ij \neq 12$.

By writing $G(2, n+1)$ as a union of $G(2,4)$-s in such a way that a general point on $G(2, n+1)$ lies in exactly one $G(2,4)$, we have constructed a variety Y which contains $G(2, n+1)$, and it has dimension one more than the dimension of $G(2, n+1)$. Recall that during the construction, we chose a line $\mathbb{P}^1_F$ in the Plücker space of $G(2, n+1)$. That is, we chose a point p_F on $G(2, n+1)$, and this point is common to all the $G(2,4)$-s. Now, let q be a point on $G(2, n+1)$ and let L be the line spanned by p_F and q. Then there is a $G(2,4)$ such that p_F and q lies on it, and the line L is contained in the linear span of it. Thus L is contained in Y. This implies that every line spanned by p_F and another point on $G(2, n+1)$ is contained in Y, and thus the whole cone over $G(2, n+1)$ with vertex in p_F is contained in Y. Since Y and the cone has equal dimension, they must be equal.

Thus in this case, the variety Y is a cone over $G(2, n+1)$ with vertex in p_F.

Proposition 3.15. *Let $\sigma = \sigma_{n-3,n-3,c,0}$ be a cycle on $G(4, n+1)$. All the Y-s of the type*

$$Y = \bigcup_{W^4 \in \sigma} \langle G(2, W^4) \rangle$$

are cones over a point in $\sigma_{c,0}$.

Proof. It is not only in the case $c = 0$ we choose a line in the Plücker space. In all cases $0 \leq c < n-3$ we choose a line $\mathbb{P}^1_F$ in the Plücker space (or, equivalently, a point p_F on $G(2, n+1)$), and construct a variety Y using this line. A point q in σ_c comes from a line $\mathbb{P}^1_q$ in $\mathbb{P}^n$, and this line lies in the $\mathbb{P}^3$ spanned by $\mathbb{P}^1_q$ and $\mathbb{P}^1_F$. The union of all lines in this $\mathbb{P}^3$ is a $G(2,4)$, and the linear span of this $G(2,4)$ is contained in Y. The line spanned by p_F and q is contained in this linear space, and therefore it is contained in Y. This implies that the entire cone over σ_c is contained in Y, and since Y and the cone has equal dimensions, the variety Y *is* this cone.

In the special case when $c = n-3$, the cycle on $G(2, n+1)$ is

$$\{\mathbb{P}^1 \subset \mathbb{P}^n \mid \mathbb{P}^1 \cap \mathbb{P}^2_F \neq \emptyset\}$$

and the cycle on $G(4, n+1)$ is

$$\sigma_{n-3,n-3,n-3,0} = \{\mathbb{P}^3 \subset \mathbb{P}^n \mid \mathbb{P}^3 \supset \mathbb{P}^2_F\}$$

Thus, in this case, choosing a line $\mathbb{P}^1_F$ inside $\mathbb{P}^2_F$ does not give any extra condition on the $\mathbb{P}^3$-s. A particular line $\mathbb{P}^1_q$ in σ_{n-3} on $G(2, n+1)$ lies in the $\mathbb{P}^3$ spanned by the plane $\mathbb{P}^2_F$ and the line $\mathbb{P}^1_q$ itself. The entire plane of lines in $\mathbb{P}^2_F$ is common to all the $G(2,4)$-s, and a line spanned by a point in this plane and another point in σ_{n-3} is contained in Y. This implies that the cone over σ_{n-3} is contained in Y. Notice that every cone with vertex in a point in the plane $\check{\mathbb{P}}^2_F \subset \sigma_{n-3}$ is contained in Y. This cone has the same dimension as Y, and they are therefore equal. $\square$

Recall that the ideal of σ_c is generated by the 4-Pfaffians of the matrix

$$\left(\begin{array}{ccccc|ccc|}
0 & 1 & * & \cdots & * & * & \cdots & * \\
-1 & 0 & * & \cdots & * & * & \cdots & * \\
\hline
* & * & 0 & \cdots & * & * & \cdots & * \\
\vdots & & & & \vdots & \vdots & & \vdots \\
* & * & * & \cdots & 0 & * & \cdots & * \\
\hline
* & * & * & \cdots & * & 0 & \cdots & 0 \\
\vdots & & & & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & * & 0 & \cdots & 0 \\
\end{array}\right) \tag{3.27}$$

$$\uparrow \cdots \uparrow$$

$$\text{columns number } n-c+1, \ldots, n+1$$

The cycle σ_c is defined by the 4-Pfaffians of this matrix, and the coordinates which are zero:

$$\sigma_c = Z(\text{4-Pfaffians }, x_{ij} \text{ where } n-c+1 \leq i < j \leq n+1)$$

Notice that the generators are expanded around p_F by setting $x_{12} = 1$. All the 4-Pfaffians Q_{12jk} where $3 \leq j \leq n-c$ has linear leading terms, and the rest of the 4-Pfaffians has quadratic leading term. The tangent cone is the common zero locus of these leading terms inside the linear span of σ_c. Notice that the linear leading terms are all coordinates x_{ij} where $3 \leq i \leq n-c$ and $4 \leq j \leq n+1$. A consequence of setting these equal to zero is that all 4-Pfaffians which involve three of these coordinates are identically zero. Also, the 4-Pfaffians Q_{13jk} and Q_{23jk} where $4 \leq j < k \leq n+1$ are identically zero. Thus the 4-Pfaffians in the ideal of the tangent cone are exactly Q_{12jk} where $n-c+1 \leq j < k \leq n+1$. These quadrics are the (2×2) minors of the boxed $2 \times (c+1)$ matrix in the upper right corner of the matrix.

We have found that the tangent cone over σ_c at the point p_F is defined by the zero locus of the minors of a matrix of coordinates. This fits perfectly: The tangent cone sits inside the linear span of σ_c intersected with the linear leading terms. There are $\sum_{i=c+1}^{n+1} i$ linear leading terms, and therefore, the tangent cone sits inside a projective space of dimension

$$\dim\langle\sigma_c\rangle - \sum_{i=c+1}^{n+1} i = \frac{n^2+n-c^2-c}{2} - 1 - \frac{n^2-3n-c^2-c}{2} - 2 = 2(n-1)$$

and the Thom-Porteous formula gives that the dimension of the zero locus of the maximal minors of a $2 \times (c+1)$ matrix has codimension c. Thus this zero locus has dimension $2(n-1)-c$ which is equal to the dimension of σ_c. The ideal generated by these minors has depth c. But then the tangent cone is ACM by [4], corollary 5.4. Now, propositions 1.7 and 3.15 gives that Y_σ is ACM in this case.

In the case $c=0$, the variety Y_σ is the entire $G(2,n+1)$. The ideal of $G(2,n+1)$ is generated by all the 4-Pfaffians Q_{ijkl}. Expanding these around p_F gives $\frac{n^2-3n}{2}+1$ linear leading terms, and $\binom{n+1}{4} - \frac{n^2-3n}{2}+1$ quadratic leading terms. The tangent cone $TC_{p_F}G(2,n+1)$ is cut out by these leading terms. In fact, inside the zero locus of the linear leading terms, the quadratic leading terms are identically zero, and this implies that the tangent cone is equal to the tangent space in this case. Thus the tangent cone is a linear space, and a complete intersection. In particular, it is ACM and generated by the leading terms. Propositions 1.7 and 3.15 gives that Y_σ is ACM in this case. Finally,

Proposition 3.16. *All these Y-s are minimal, in the sense that there is no variety Y_0 such that $Y \cap G(2,n+1) \subset Y_0 \subset Y$.*

Proof. Assume now that there is a variety Y_0 such that

$$Y \cap G(2,n+1) \subset Y_0 \subset Y$$

Then

$$I_{Y\cap G(2,n+1)} \supset I_{Y_0} \supset I_Y$$

i.e

$$\begin{array}{c}(x_{ij}, Q_2 \text{ rank four quadrics }, Q_1+Q_3' \text{ rank six quadrics })\\ \cup \\ I_{Y_0} \\ \cup \\ (x_{ij}, Q_2+Q_3' \text{ rank four quadrics })\end{array}$$

Thus Y_0 must consist of some of the $\mathbb{P}^5$-s but not all. On the other hand, we have seen that $Y \cap G(2,n+1)$ can be written as a union of $G(2,4)$-s where every line is in exactly one $G(2,4)$, so a variety which consists of only some of the $\langle G(2,4)\rangle$-s can not contain the entire $Y \cap G(2,n+1)$. Thus such a Y_0 can not exist.

□

Chapter 4

Power sums

When dealing with Grassmannians of lines, a lot of interesting questions appear. One of them is related to the topic of power sums, and this chapter will treat the power sum problem and Grassmannians of lines. We start by some general theory to establish the tools we need.

If f is a homogeneous form of degree d in $n+1$ variables, we know that f can be written as a sum of powers of linear forms l_i:

$$f = l_1^d + l_2^d + \cdots + l_r^d \tag{4.1}$$

for r large enough. If N_d is the number of monomials of degree d in $n+1$ variables, consider the d-th Veronese embedding $\nu_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^{N_d-1}$. A point in $\mathbb{P}^n$ corresponds to a hyperplane in $\check{\mathbb{P}}^n$, which corresponds to a linear form l in the coordinate ring $\mathbb{C}[x_0, \ldots, x_n]$. We can identify ν_d with the map $l \mapsto l^d$, and since every $f \in \mathbb{C}[x_0, \ldots, x_n]_d$ can be written in the form given in equation 4.1, we know that the image of the map $l \mapsto l^d$ spans $\mathbb{P}^{N_d-1}$. Now, if we fix n and d, the number of summands r will of course vary with f, but for general f we have the following theorem:

Theorem 4.1 (Alexander, Hirschowitz). *A general form f of degree d in $n+1$ variables, is a sum of $r = \lceil \frac{1}{n+1}\binom{n+d}{n} \rceil^*$ powers of linear forms, unless*

1 *$d = 2$, when $r = n+1$ instead of $\lceil \frac{n+2}{2} \rceil$, or*

2 *$d = 4$ and $n = 2, 3, 4$, when $r = 6, 10, 15$ instead of $5, 9, 14$ respectively, or*

3 *$d = 3$ and $n = 4$, when $r = 8$ instead of 7*

Proof. This is a result of Alexander and Hirschowitz [1] combined with Terracinis Lemma [13]. □

*For any real number a, the notation $\lceil a \rceil$ means the smallest integer bigger than or equal to a.

Now, let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be of degree d, and let $F = Z(f) \subset \mathbb{P}^n$. If $l \in \mathbb{C}[x_0, \ldots, x_n]_1$ †, we have $L = Z(l) \subset \mathbb{P}^n$. By double use of notation, we can say that L is the point in $\check{\mathbb{P}}^n$ that corresponds to the hyperplane L in $\mathbb{P}^n$. We are ready for the first definition

Definition 4.1. *The* **v***ariety of* **s***ums of* **p***owers is defined to be the closure*

$$VSP(F, s) = \overline{\{\{L_1, \ldots, L_s\} \in \mathrm{Hilb}_s(\check{\mathbb{P}}^n)|\exists \lambda_i \in \mathbb{C} : f = \lambda_1 l_1^d + \cdots + \lambda_s l_s^d\}}$$

of the set of powersums representing f in the Hilbert scheme.

From now on, let $S = \mathbb{C}[x_0, \ldots, x_n]$ and let $T = \mathbb{C}[\partial_0, \ldots, \partial_n]$. We know (see [21]) that T acts on S by differentiation, and S acts on T in the same way. This action defines a perfect pairing between forms of degree d and homogeneous differential operators of order d. In particular, S_1 and T_1 are natural dual vector spaces. Therefore, the projective spaces with coordinate rings S and T are natural dual to each other. We denote them $\mathbb{P}^n$ and $\check{\mathbb{P}}^n$. A point $a = (a_0, \ldots, a_n) \in \check{\mathbb{P}}^n$ defines a form $l_a = \sum_{i=0}^n a_i x_i \in S_1$, and for a form $D \in T_e$,

$$D \cdot l_a^d = e!\binom{d}{e} D(a) l_a^{d-e}$$

when $e \leq d$. In particular,

$$D \cdot l_a^d = 0 \Leftrightarrow D(a) = 0$$

Definition 4.2 (Apolar forms). *We say that homogeneous forms $f \in S$ and $D \in T$ are* **apolar** *if $f \cdot D = D \cdot f = 0$*

Now, let $f \in S_d$ and let $F = Z(f) \subset \mathbb{P}^n$ be the corresponding hypersurface . We define

$$F^{\perp} = \{D \in T | D \cdot f = 0\}$$

and

$$A^F = T/F^{\perp}$$

We also have the notion of apolar schemes:

Definition 4.3 (Apolar schemes). *Let $F = Z(f) \subset \mathbb{P}^n$ be a hypersurface of degree d. A subscheme $\Gamma \subset \check{\mathbb{P}}^n$ is called* **apolar** *to F if $I_\Gamma \subset F^{\perp} \subset \mathbb{C}[\partial_0, \ldots, \partial_n] = T$, where I_Γ is the ideal of Γ.*

When working with powersum-related problems, the following lemma plays an important role:

† We will use this notation for the linear forms in the polynomial ring $\mathbb{C}[x_0, \ldots, x_n]$. Generally, $\mathbb{C}[x_0, \ldots, x_n]_d$ denotes the d-th graded part

Lemma 4.2 (Apolarity lemma, [21]). *Let $l_1, \dots, l_s$ be linear forms in S, and let $L_i \in \check{\mathbb{P}}^n$ be the corresponding points in the dual space. Then $f = \lambda_1 l_1^d + \cdots + \lambda_s l_s^d$ for some $\lambda_i \in \mathbb{C} \setminus \{0\}$ if and only if $\Gamma = \{L_1, \dots, L_s\} \subset \check{\mathbb{P}}^n$ is apolar to $F = Z(f)$, i.e. if and only if $I_\Gamma \subset F^\perp$.*

Recall that the socle of A^F is $(0 : T_1) = \{D \in A^F | D' \cdot D = 0 \quad \forall D' \in T_1\} \subset A^F$. In detail

$$\begin{aligned}(0:T_1) =& \{D \in A^F | \partial_i D = 0 \quad \text{in} \quad A^F \quad \forall i\} \\ =& \{D \in T/F^\perp | \partial_i D \in F^\perp \quad \forall i\} \\ =& \{D \in T/F^\perp | \partial_i D \cdot f = 0 \quad \forall i\} \\ =& \{D \in T/F^\perp | D \cdot f \in \mathbb{C}\} \\ =& \{D \in T/F^\perp | D \in f^\perp \quad \text{or} \quad \deg D = \deg f = d\} \\ =& A_d^F\end{aligned}$$

and this implies that the socle of A^F is A_d^F and is one dimensional. In particular, A^F is Gorenstein with socle degree d. In fact, A^F is an Artinian Gorenstein graded quotient ring of T, and is called the **apolar Artinian Gorenstein ring of F**. Thus a form $f \in R_d$ gives us an Artinian Gorenstein graded quotient ring $A^F = T/F^\perp$ with socle degree d. We have the following theorem:

Theorem 4.3 (Macaulay). *The map $F \mapsto A^F$ is a bijection between hypersurfaces $F = Z(f) \subset \mathbb{P}^n$ of degree d and graded Artinian Gorenstein quotient rings $A = T/I$ of T with socle degree d.*

Proof. See [17]. □

Let $X \subset \mathbb{P}^{n+m+1}$ be an m-dimensional arithmetically Gorenstein variety. Let $S(X)$ be the homogeneous coordinate ring of X, and let $h_1, \dots, h_{m+1}$ be general linear forms and set $L = Z(h_1, \dots, h_{m+1})$. Then $S(X)/(h_1, \dots, h_{m+1})$ is Artinian Gorenstein, and the theorem above says that $S(X)/(h_1, \dots, h_{m+1})$ is equal to A^{f_L} for some $T' = \mathbb{C}[z_0, \dots, z_n]$ and some $f_L \in T'$ of degree d equal to the socle degree of the ring (see [14]).

The space L is a linear space of dimension $(n+m+1)-(m+1) = n$, and $F_L = Z(f_L)$ is a hypersurface of degree d in the dual space to L. We say that F_L is apolar to (the empty) intersection $L \cap X =: \emptyset_{f_L}$. Thus, L is an n-dimensional linear subspace of $\mathbb{P}^{n+m+1}$, and we can associate a hypersurface of degree d in a $\mathbb{P}^n$ to L. Hence, we have a map

$$\begin{array}{ccc} G(n+1, n+m+2) & \longrightarrow & \tilde{H}_{n,d} \\ L & \mapsto & F_L \end{array} \tag{4.2}$$

where $\tilde{H}_{n,d}$ is the space of hypersurfaces of degree d in $\mathbb{P}^n$. We want to treat hypersurfaces that only differ by a change of coordinates as equal. Thus, we consider $\tilde{H}_{n,d}$ modulo the action of $PGL(n+1,\mathbb{C})$ and call it $H_{n,d}$. We get a rational map

$$\begin{array}{rccc} \alpha_X: & G(n+1,n+m+2) & - - - \rightarrow & H_{n,d} \\ & L & \mapsto & F_L \end{array} \tag{4.3}$$

The map α_X is only defined for those L which are intersections of $m+1$ general hyperplanes in $\mathbb{P}^{n+m+1}$.

4.1 Powersums and Grassmannians of lines

Let $G(2,n+1)$ be the Grassmannian of lines in a n-dimensional projective space. Recall theorem 1.6 of chapter 1, which says: All Grassmannians varieties have homogeneous coordinate rings which are Gorenstein. Any Schubert subvariety of a Grassmannian has homogeneous coordinate ring which is Cohen-Macaulay.

Let $G = G(2,n+1) \subset \mathbb{P}^N$, where $N = \binom{n+1}{2} - 1$ and let $L \subset \mathbb{P}^N$ be a linear space such that $G(2,n+1)$ and L does not intersect. Then the intersection $G \cap L$ is empty, but by theorem 4.3 and theorem 1.6, we can associate a form f_L to it. From the apolarity lemma (lemma 4.2) we see that we want to find a variety Y such that $I_{Y\cap L} \subset I_{G\cap L}$ and such that $Y \cap L$ is a finite set of points. Recall that the empty intersection $G \cap L$ be denoted $\emptyset_{f_L}$.

Let the linear span of Y be denoted by $\langle Y\rangle$. If $L \subset \langle Y\rangle \subset \mathbb{P}^N$, the intersection $G\cap\langle Y\rangle$ must be non-empty (remember that $\dim L+\dim G = N-1$, so if $\langle Y\rangle$ is strictly bigger than L, then G and $\langle Y\rangle$ must intersect). If $G\cap\langle Y\rangle \subset Y$, then $I_Y \subset I_{G\cap\langle Y\rangle}$ and if Y is arithmetically Cohen-Macaulay (ACM), this implies that $I_{Y\cap L} \subset I_{G\cap\langle Y\rangle\cap L} = I_{G\cap L}$. The assumption $G \cap \langle Y\rangle \subset Y$ is true when $G \cap \langle Y\rangle = G \cap Y$, as is the case for the apolar varieties discussed in the previous chapters.

The intersection $Y \cap L$ is a finite set of points if

$$\dim L = \text{codim} Y \text{ in } \langle Y\rangle$$

We found apolar varieties Y for Schubert cycles on $G(2,n+1)$ which have the property that $Y \cap G(2,n+1)$ has codimension one in Y. For those, we also had the property that $\langle Y\rangle = \langle Y \cap G\rangle$. Adopting these as nice properties, the intersection is a finite set of points if

$$\dim L = \text{codim}(G(2,n+1) \cap Y) - 1 \text{ in } \langle Y \cap G(2,n+1)\rangle$$

which is equivalent to

$$\text{codim} G(2,n+1) \text{ in } \langle G(2,n+1)\rangle = \text{codim}(G(2,n+1) \cap Y) \text{ in } \langle Y \cap G(2,n+1)\rangle$$

But then the intersection of Y with $G(2, n+1)$ must be proper, which is generally not the case when $Y \cap G(2, n+1)$ is a Schubert cycle.

We have seen that the Grassmannian $G(2, n+1)$ is a subvariety of $\mathbb{P}^{\binom{n+1}{2}-1}$, and now consider the polynomial rings

$$S = \mathbb{C}[x_0, \ldots, x_{\binom{n+1}{2}-1}] \quad \text{and} \quad T = \mathbb{C}[\partial_0, \ldots, \partial_{\binom{n+1}{2}-1}]$$

The ring T acts on S by differentiation as above.

In our case, $G(2, n+1) \subset \mathbb{P}^{\binom{n+1}{2}-1}$ is a variety of dimension $2(n-1)$, and $S(G(2, n+1))$ is its homogeneous coordinate ring. We also have that $\frac{n^2-3n}{2} + 2(n-1) + 1 = \binom{n+1}{2} - 1$, so if $h_1, \ldots, h_{2n-1}$ are general linear forms, theorem 4.3 says that $S(G(2, n+1))/(h_1, \ldots, h_{2n-1})$ is equal to $\mathbb{C}[z_0, \ldots, z_{\frac{n^2-3n}{2}}]/f_L^{\perp}$ for some f_L of degree equal to the socle degree of the ring. We also know that there is $\binom{n+1}{4}$ quadrics in the ideal $f_L^{\perp}$. In the following section, we will determine the degree of the polynomial f_L.

4.1.1 The socle degree

By using Macaulay 2 ([8]) we get the resolution of the ideal of the Grassmannian:

Example 4.1 ($G(2,5)$). The Grassmannian $G(2,5)$ sits inside $\mathbb{P}^9$ and the codimension is 3. The resolution of the ideal of $G(2,5)$ is given by

$$\mathcal{O}_{\mathbb{P}^9} \leftarrow \mathcal{O}_{\mathbb{P}^9}(-2) \leftarrow \mathcal{O}_{\mathbb{P}^9}(-3) \leftarrow \mathcal{O}_{\mathbb{P}^9}(-5)$$

and the Betti numbers are

1	–	–	–
–	5	5	–
–	–	–	1

The shift in the last piece is 5. △

Example 4.2 ($G(2,6), G(2,7)$). From the resolutions of the ideal of $G(2,6)$ we get the Betti numbers

1	–	–	–	–	–	–
–	15	35	21	–	–	–
–	–	–	21	35	15	–
–	–	–	–	–	–	1

and from the resolution of the ideal of $G(2,7)$ we get the Betti numbers

1	–	–	–	–	–	–	–	–	–	–
–	35	140	189	84	–	–	–	–	–	–
–	–	–	196	735	1080	735	196	–	–	–
–	–	–	–	–	–	84	189	140	35	–
–	–	–	–	–	–	–	–	–	–	1

From this we can see that the shift in the last piece of the resolution of $G(2,6)$ is $6+3=9$ and $10+4=14$ for $G(2,7)$. △

We know that the Grassmannian $G(k+1,n+1)$ sits inside $\mathbb{P}^{\binom{n+1}{k+1}-1}$. If $N := \binom{n+1}{k+1}$, the resolution looks like

$$0 \leftarrow \mathscr{O}_G \leftarrow \mathscr{O}_{\mathbb{P}^{N-1}} \leftarrow \cdots \leftarrow \mathscr{O}_{\mathbb{P}^{N-1}}(-d) \tag{4.4}$$

for some d. The length of the resolution (starting counting with a 0 at the first $\mathscr{O}_{\mathbb{P}^{N-1}}$) is equal to the codimension m of $G(k+1,n+1)$. We know that $\mathscr{H}om(-,\omega_{\mathbb{P}^{N-1}})$ is a contravariant left exact functor, and applying this to the exact sequence in equation 4.4 gives

$$0 \to \mathscr{H}om(\mathscr{O}_{\mathbb{P}^{N-1}},\omega_{\mathbb{P}^{N-1}}) \to \cdots \xrightarrow{\phi} \mathscr{H}om(\mathscr{O}_{\mathbb{P}^{N-1}}(-d),\omega_{\mathbb{P}^{N-1}}) \to \mathrm{coker}\phi \to 0$$

But $\mathrm{coker}\phi = \mathscr{E}xt^m(\mathscr{O}_G,\omega_{\mathbb{P}^{N-1}})$, which is exactly ω_G. Thus we have

$$0 \to \mathscr{H}om(\mathscr{O}_{\mathbb{P}^{N-1}},\omega_{\mathbb{P}^{N-1}}) \to \cdots \xrightarrow{\phi} \mathscr{H}om(\mathscr{O}_{\mathbb{P}^{N-1}}(-d),\omega_{\mathbb{P}^{N-1}}) \to \omega_G \to 0$$

It is a well known fact that $\omega_{\mathbb{P}^{N-1}} = \mathscr{O}_{\mathbb{P}^{N-1}}(-N+1-1) = \mathscr{O}_{\mathbb{P}^{N-1}}(-N)$, so we have

$$0 \to \mathscr{O}_{\mathbb{P}^{N-1}}(-N) \to \cdots \to \omega_G \to 0$$

Furthermore, we will prove later (lemma 4.6) that $\omega_G = \mathscr{O}_G(-n-1)$, so twisting by $n+1$ gives

$$0 \to \mathscr{O}_{\mathbb{P}^{N-1}}(-N+n+1) \to \cdots \to \mathscr{O}_G \to 0 \tag{4.5}$$

Equations 4.4 and 4.5 are two resolutions of $\mathscr{O}_G$, both minimal, and they have the same Betti numbers. Thus the shift in the last piece must be the same, and we get the equality

$$d = N - n - 1$$

Observe that $\emptyset_{f_L}$ sits inside $\mathbb{P}^{m-1}$, and the canonical line bundle on $\mathbb{P}^{m-1}$ is $\mathscr{O}(-m)$. Following the same procedure as above, we get a resolution

$$0 \leftarrow \omega_{\emptyset_{f_L}} \leftarrow \cdots \leftarrow \mathscr{O}(-m) \leftarrow 0$$

We also know that $\omega_{\emptyset_{f_L}} = \mathscr{O}_G(s)$ ([5], p 549) where s is the socle degree, so if we twist by $-s$, we get a resolution of $\mathscr{O}_G$:

$$0 \leftarrow \mathscr{O}_G \leftarrow \cdots \leftarrow \mathscr{O}(-m-s) \leftarrow 0 \tag{4.6}$$

By comparing equation 4.4 and equation 4.6 we see that

$$d = m + s$$

Thus if d is the shift in the last piece of the resolution of the ideal of $G(k+1,n+1)$ and m is the codimension of $G(k+1,n+1) \subset \mathbb{P}^{\binom{n+1}{k+1}-1} = \mathbb{P}^{N-1}$, then

1. $d = N - n - 1$
2. $s = d - m = N - n - 1 - m$

where s is the socle degree.

We know that $\dim G(2, n+1) = 2(n-1)$, so the codimension m is equal to

$$\begin{aligned}\binom{n+1}{2} - 1 - 2(n-1) &= \frac{(n+1)n}{2} - 1 - 2n + 2 \\ &= \frac{n^2 + n - 4n + 2}{2} \\ &= \frac{n^2 - 3n + 2}{2} \\ &= \frac{(n-1)(n-2)}{2} \\ &= \binom{n-1}{2}\end{aligned} \tag{4.7}$$

Proposition 4.4. *The socle degree of* $\mathbb{C}[z_0, \ldots, z_{\frac{n^2-3n}{2}}]/f_L^\perp$ *is* $n-2$

Proof. The discussion above gives that the socle degree is

$$\begin{aligned}s = N - n - 1 - m &= \binom{n+1}{2} - (n+1) - \binom{n-1}{2} \\ &= \frac{(n+1)n}{2} - (n+1) - \frac{(n-1)(n-2)}{2} \\ &= \frac{(n+1)(n-2)}{2} - \frac{(n-1)(n-2)}{2} \\ &= \frac{n-2}{2}(n+1-n+1) \\ &= n-2\end{aligned} \tag{4.8}$$

□

4.1.2 The number of summands for a general f

As we saw in proposition 4.4 above, the form f we are considering is a form of degree $n-2$ in $\frac{n^2-3n}{2}+1$ variables, i.e $f \in \mathbb{C}[z_0, \ldots, z_{\frac{n^2-3n}{2}}]_{n-2}$. By the Alexander/Hirschowitz-theorem (theorem 4.1), the number of summands in the power sum presentation of a *general* such f is

$$r = \left\lceil \frac{1}{\frac{n^2-3n}{2}+1} \binom{\frac{n^2-3n}{2} + (n-2)}{\frac{n^2-3n}{2}} \right\rceil \tag{4.9}$$

$G(2,n+1)$	$A=\frac{n^2-3n}{2}$	$B=n-2$	$s=\lceil\frac{1}{A+1}\binom{A+B}{A}\rceil$
$G(2,4)$	0	1	1
$G(2,5)$	2	2	3
$G(2,6)$	5	3	10
$G(2,7)$	9	4	72
$G(2,8)$	14	5	776

Table 4.1: The table shows the expected number of summands r for general f

except when $n=4$, then $r=\frac{n^2-3n}{2}+1$ instead.

We have seen that if there is a variety Y with the properties listed above, we get an inclusion $I_{Y\cap L}\subset I_{G\cap L}$, which by the apolarity lemma gives a point in the variety of power sums for the form f we get from G. Now, we are ready to examine some of the possible Y-s.

4.1.3 The apolar variety is a cone over X

Let $X\subset\mathbb{P}^{n+m+1}$ be a reduced and irreducible $m-$dimensional non-degenerate variety of degree $\delta\geq 3$, and codimension $n+1\geq 2$.

Remark:

1 $\deg G(2,n+1)=\frac{1}{n}\binom{2n-2}{n-1}>3$ when $n\geq 4$

2 $\operatorname{codim} G(2,n+1)=\binom{n+1}{2}-1-2(n-1)=\frac{n^2-3n}{2}+1>2$ when $n\geq 4$

Now, let $p\in X$ be a general smooth point, and let C_pX be the cone over X with vertex at p. Since p is a smooth point, C_pX has dimension $m+1$ and degree $\delta-1$, and clearly $X\subset C_pX$.

We will apply this simple construction to describe power sum presentations of hypersurfaces in the image of the map α_X of equation 4.3. Let again $X\subset\mathbb{P}^{n+m+1}$ be an $m-$dimensional arithmetically Gorenstein variety of degree δ. Fix a general $n-$dimensional linear subspace $L\subset\mathbb{P}^{n+m+1}$, in particular, fix the hypersurface F_L in the image of the map α_X. Let p be a smooth point on X. We know that $L\cap X=\emptyset_{f_L}$, but since $\dim C_pX+\dim L=m+1+n$, the intersection $L\cap C_pX$ is non-empty. In fact, if the intersection is proper, it is zero dimensional of the same degree as C_pX, i.e. of degree $\delta-1$. We may assume that this intersection is proper and smooth for general L or general p. Thus if $\Gamma=L\cap C_pX$, then Γ is $(\delta-1)$ points and $\Gamma\supset L\cap X$. By the apolarity lemma, $I_\Gamma\subset F_L^\perp$. Hence, we get an apolar subscheme of degree $\delta-1$ to F_L, i.e. a point in $VSP(F_L,d-1)$. This proves

Proposition 4.5 ([14]). *Let $X \subset \mathbb{P}^{n+m+1}$ be an $m-$dimensional arithmetically Gorenstein variety of degree δ and let $L \subset \mathbb{P}^{n+m+1}$ be an $n-$dimensional linear subspace such that $L \cap X = \emptyset$. Let F_L be the associated apolar hypersurface. Then there is a rational map $X - - \to VSP(F_L, d-1)$ defined by $p \mapsto L \cap C_p X$.*

The case $X = G(2, n+1)$

The Grassmannian $G(2, n+1)$ of lines in $\mathbb{P}^n$ sits inside $\mathbb{P}^{\frac{n^2-3n}{2}+2(n-1)+1}$ and has degree $\frac{1}{n}\binom{2n-2}{n-1}$. So we fix a linear space of dimension $\frac{n^2-3n}{2}$, and pick a point $p \in G := G(2, n+1)$. Then $L \cap C_p G$ is $\frac{1}{n}\binom{2n-2}{n-1} - 1$ points and this is an apolar subscheme to F_L in the image of α_G. Thus, f_L of degree $n-2$ (the socle degree) can be written as a sum of $\frac{1}{n}\binom{2n-2}{n-1} - 1$ powers of linear forms! Also notice that every smooth point on G (i.e. every point in G, since G is smooth) gives rise to an apolar subscheme. Hence, f_L can be presented as a sum of powers of linear forms in as many ways as there are points on G. That is, in a $2(n-1)$-dimensional family of ways.

Recall that equation 4.9 gives us the expected r if f_L was general. The argument above, says that the s we are looking for is no bigger than $\frac{1}{n}\binom{2n-2}{n-1} - 1$. Table 4.2 compares these two numbers for different n.

n	expected r	r from cone argument
4	3	4
5	10	13
6	72	41
7	771	131

Table 4.2: The table shows the expected r for general f, and the r we get from the cone argument.

It is not hard to see that when $n \geq 6$, the expected r from the Alexander/Hirschowitz theorem is much larger than the r we get from the cone argument. Can we get an even better upper bound for r?

4.1.4 The apolar variety is a cone over tangent hyperplane sections

Let, as before, $X \in \mathbb{P}^{n+m+1}$ be a reduced and irreducible variety. Also, X is non-degenerate and m-dimensional of degree δ and codimension $n+1 \geq 2$. We assume in addition that X satisfies the following condition (see [14]):

Condition 1. A general tangent hyperplane section of X has a double point at the point of tangency, and the projection of the tangent hyperplane section from the point of tangency is birational

So what does this mean? Let $p \in X$ be a general smooth point, and let H_p be a general hyperplane tangent to X at p. Then condition 1 says that the intersection $H_p \cap X$ has multiplicity 2 in p and the projection

$$\pi_p : H_p \cap X \to \overline{X}$$

is birational. The image of π_p has dimension $m-1$ and degree $\delta - 2$. Thus, the cone $C_p(H_p \cap X)$ has degree $\delta - 2$ and dimension m, and it contains $H_p \cap X$.

Similarly, if H_p and H_p' are two general hyperplanes tangent to X at p, the intersection $H_p \cap H_p' \cap X$ has a singularity at p of multiplicity 4, and it is a complete intersection of two singularities of multiplicity 2. We say that the codimension 2 space $H_p \cap H_p'$ is **doubly tangent** to X at p. Now, assume that the following condition is satisfied:

Condition 2. The projection of $H_p \cap H_p' \cap X$ from p is birational

Now, the image of the projection is $(m-2)-$dimensional, and of degree $\delta - 4$. Thus, the intersection $H_p \cap H_p' \cap X$ is contained in the $(m-1)-$dimensional cone $C_p(H_p \cap H_p' \cap X)$ of degree $d-4$ with vertex in p.

As before, let $X \subset \mathbb{P}^{n+m+1}$ be as $m-$dimensional arithmetically Gorenstein variety of degree δ. In addition, assume that Condition 1 is satisfied. Now, fix an $n-$dimensional linear subspace $L \subset \mathbb{P}^{n+m+1}$, in particular, fix the hypersurface F_L in the image of α_X. If $L \subset H_p$ (H_p as defined above), there is an $m-$dimensional variety $Y \supset H_p \cap X$ of degree $\delta - 2$. We know from the discussion above that Y is the cone over $H_p \cap X$ with vertex in $p \in X$, i.e all lines $\langle p, x \rangle$, where $x \in H_p \cap X$. Since $p \in H_p \cap X$ and $x \in H_p \cap X$, and H_p is a hyperplane, the entire line $\langle p, x \rangle$ sits inside H_p. Thus $Y \subset H_p$ has dimension m, $L \subset H_p$ has dimension n and H_p has dimension $n+m$. Therefore, $L \cap Y \neq \emptyset$. In fact, since no lines on Y lies inside L (because then L would intersect X non-empty, and this is not the case), we know that $L \cap Y$ is $0-$dimensional of degree $\delta - 2$. Thus we get a point in $VSP(F_L, \delta - 2)$.

Now, let $\check{X} \subset \check{\mathbb{P}}^{n+m+1}$ be the dual variety of X, i.e. the collection of hyperplanes tangent in some point p on X. Let $\check{X}_L = \{[H] \in \check{X} : H \supset L\} \subset \check{X}$. We get a rational map (only defined for general tangent hyperplanes)

$$\begin{array}{ccc} \check{X}_L & \dashrightarrow & VSP(F_L, \delta - 2) \\ {[H]} & \mapsto & L \cap Y \end{array}$$

In the same way, assume Conditions 1 and 2 are satisfied, and let $L \subset (H_p \cap H_p')$, where H_p and H_p' are two general hyperplanes tangent at p. There is an $(m-1)-$dimensional variety $Y \supset (H_p \cap H'_p \cap X)$ of degree $\delta - 4$. As before, Y is the cone over $H_p \cap H'_p \cap X$ with vertex in p, i.e. all lines $\langle p, x\rangle$ with $x \in H_p \cap H'_p \cap X$. As above, since $p, x \in H_p \cap H'_p \cap X$, the entire line sits inside $H_p \cap H'_p$, and thus Y sits inside $H_p \cap H'_p$ and is of dimension $(m-1)$. Furthermore, $L \subset H_p \cap H'_p$ has dimension n, and since $\dim(H_p \cap H'_p) = n + m - 1$, the intersection $L \cap Y$ is non-empty. For general L the intersection is 0-dimensional and of degree $\delta - 4$. Thus, we get a point in $VSP(F_L, d-4)$.

We know that $H_p \cap H'_p$ is a codimension 2 space, doubly tangent in $p \in X$. Let $Z_X \subset G(n+m, n+m+2)$ be the set of codimension 2 spaces, doubly tangent at some point $p \in X$, and let $Z_L = \{[V] \in Z_X : V \supset L\} \subset G(n+m, n+m+2)$. We get a rational map

$$\begin{array}{ccc} Z_L & ---\rightarrow & VSP(F_L, \delta - 4) \\ [V] & \mapsto & L \cap Y \end{array}$$

The case $X = G(2, n+1)$, continued

We already know that $G(2, n+1)$ sits inside

$$\mathbb{P}^{\frac{n^2-3n}{2}+2(n-1)+1} =: \mathbb{P}^N$$

and that $\dim G(2, n+1) = 2(n-1)$. Thus, we fix a linear space L of dimension $\frac{n^2-3n}{2}$. The dual variety of $G(2, n+1)$ is defined by

$$\check{G} = \{[H] \in \check{\mathbb{P}}^N | H \text{ is a tangent hyperplane to } G(2, n+1)\}$$

The dual variety consists of all tangent hyperplanes to G, and we know that it is a hypersurface of degree three when $n+1$ is even, that it has codimension three when $n+1$ is odd (see theorems 3.4 and 3.5). By definition, $H \subset \mathbb{P}^N$ is a *tangent hyperplane, tangent at the point* $p \in G(2, n+1)$ if $H \supset T_pG$, where T_pG is the tangent space to $G(2, n+1)$ at the point p.

Step 1: Fix a point p

Now, fix a point $p \in G(2, n+1)$. As the "strategy" above indicates, we are looking for hyperplanes H such that $H \supset T_pG$ *and* $H \supset L$. Since $\dim T_pG + \dim L = N - 1$, there is always one hyperplane which contains them both. The linear space L corresponds to an orthogonal linear space $L^\perp \subset \check{\mathbb{P}}^N$, and the tangent space at p corresponds to an orthogonal linear space $T_pG^\perp$ in the same dual space.

Note that the dimensions of $L^\perp$ and $T_pG^\perp$ are given by

$$\dim L^\perp = \operatorname{codim} L - 1 = 2(n-1)$$
$$\dim T_pG^\perp = \operatorname{codim} T_pG - 1 = N - \dim T_pG - 1 = \frac{n^2 - 3n}{2}$$

Remark: If H_1 and H_2 are tangent hyperplanes, both tangent at a point p and both supsets of L, then every linear combination of them will also satisfy these specifications. Thus, when we speak of *the number of tangent hyperplanes* we mean the dimension of the vector space consisting of such hyperplanes.

We are looking for points $[H]$ such that

$$[H] \in \check{G} \cap L^{\perp} \cap T_pG^{\perp}$$

If $[H] \in T_pG^{\perp}$, then, by duality, $H \supset T_pG$. Thus, H is a tangent hyperplane, and $[H]$ therefore automatically is a point on the dual variety. Thus, we can simplify, and say that we are looking for points $[H]$ such that

$$[H] \in L^{\perp} \cap T_pG^{\perp} \tag{4.10}$$

Step 2: Unfix the point p

Until now, the point p has been a fixed point on the Grassmannian $G(2, n+1)$. It is natural to assume that the dimension of the intersection in equation 4.10 may vary with p. Thus, we now unfix the point p, and define $m(p) :=\dim(L^{\perp} \cap T_pG^{\perp})$, where $p \in G(2, n+1)$ is a variable. We know that $m(p) \geq 0$ ($= 0$ in the general case where the two linear spaces intersect in a point). Our goal is to find the maximal "number of" hyperplanes which contain both T_pG and L, and the natural way of thinking is that if the two linear spaces intersect more than in the general case, they are probably subsets of more hyperplanes. Thus we examine the number $m(p)$.

Reformulation of the problem

We have a linear space A (corresponding to $L^{\perp}$) of dimension a, and a linear space B_p (corresponding to $T_pG^{\perp}$) of dimension b. The space B_p varies with p, where p is a point on a variety X of dimension a. The variety X is isomorphic to the Grassmannian $G(2, n+1)$, and sits inside a $\mathbb{P}^N$ (to be correct, we should write $\check{\mathbb{P}}^N$ here) together with A and B_p. The dimensions a and b are such that $a + b = N - 1$. We know that A and X are fixed, while B_p is variable, parametrized by X. The question we want to answer is the following:

For $\alpha = 1, 2, 3 \ldots$ does there exist a B_p such that $\dim(B_p \cap A) = \alpha$?

Since B_p is a linear space of projective dimension b inside $\mathbb{P}^N$, we know that $B_p \in G(b+1, N+1) := \mathbf{G}$ for all $p \in X$. Furthermore, every B_p corresponds to a point p on X, so $X \subset G(b+1, N+1)$ as a subvariety. To answer the above question, we need to involve some Schubert calculus.

Let $V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_{N+1} = \mathbb{C}^{N+1}$ be a flag in $\mathbb{C}^{N+1}$. Now, let A be the $(a+1)$-dimensional piece (remember that A has *projective* dimension a) of the flag in $\mathbb{C}^{N+1}$, and define

$$U_\alpha = \{\Lambda \in G |\, \dim(\Lambda \cap A) \geq \alpha\}$$

Now, $U_\alpha \cap X = \{T_pG(2, n+1)^\perp |\, \dim(T_pG(2, n+1)^\perp \cap L^\perp) \geq \alpha\}$ in the original problem. Thus, if $U_\alpha \cap X \neq \emptyset$ there will be a point of the type B_p inside U_α, and thus the answer to the above (reformulated) question is "yes". Our goal is thus to find the maximal α such that $U_\alpha \cap X \neq \emptyset$.

Example 4.3 ($G(2,5)$)**.** We know that $G(2,5)$ sits inside $\mathbb{P}^9$, and that $\dim G(2,5) = 2 \cdot (5-2) = 6$. Thus, we fix a linear space of dimension 2. Then, $\dim L^\perp = 6$ and $\dim T_pG^\perp = 2$. We wish to examine how $\dim(L^\perp \cap T_pG^\perp)$ varies when we vary the point p. We now have two linear spaces in $\check{\mathbb{P}}^9$; $A = L^\perp$ of dimension 6 and $B_p = T_pG^\perp$ of dimension 2. Thus , B_p in an element in $G(3,10)$ and a variety X isomorphic to $G(2,5)$ sits inside $G(3,10)$ as a subvariety. Fix a flag in $\mathbb{C}^{10}$ such that A is the 7$-$dimensional piece. As above, let

$$U_\alpha = \{\Lambda \in G(3,10) |\, \dim(\Lambda \cap V_7) \geq \alpha\} \tag{4.11}$$

where V_7 is the 7$-$dimensional piece of the fixed flag, i.e. V_7 is equal to A. Then,

$$U_1 = \{\Lambda \in G(3,10) |\, \dim(\Lambda \cap V_7) \geq 1\} = \sigma_{100}$$

Furthermore, $\dim G(3,10) = 3 \cdot (10-3) = 21$, while $\dim X = 6$, so the codimension of X as a subvariety of $G(3,10)$ is $21 - 5 = 15$. The cycle σ_{100} is a hyperplane section in $G(3,10)$, and thus the intersection $U_1 \cap X$ is non-empty.

What about the intersection $U_2 \cap X$? According to equation 4.11, $U_2 = \{\Lambda \in G(3,10) |\, \dim(\Lambda \cap V_7) \geq 2\}$, which we recognize as the cycle σ_{220}. Thus, U_2 is a cycle of codimension $2+2=4$ in $G(3,10)$. Therefore, we may expect that U_2 intersects X non-empty.

We continue in the same way, and see that $U_3 = \sigma_{333}$. Thus, U_3 is a cycle of codimension 9, and we may expect that it does not intersect X.

We have now found that we may expect that there is a point p such that $\dim(T_pG^\perp \cap L^\perp) = 2$. This is the maximal possible dimension of this intersection (since $\dim T_pG^\perp = 2$), so we conclude that $\alpha = 2$ is the maximal α in this case. Notice: We still need to figure out how many tangent hyperplanes contain T_pG and L when the intersection is this big. $\triangle$

We now return to our original problem. Remember that we fixed a linear space A of projective dimension a and we have a variable linear space B_p of dimension b (the dimension b is not dependent of the point p). We fix a flag in $\mathbb{C}^{N+1}$ where A is the

$(a+1)-$dimensional piece. We call this piece V_{a+1} to use the same notation as in the example. We will try to describe $U_1, U_2, \ldots$ in terms of cycles in $G = G(b+1, N+1)$.

$\alpha = 1$:
By definition (equation 4.11), $U_1 = \{\Lambda \in G | \dim(\Lambda \cap V_{a+1}) \geq 1\}$. Furthermore, $N - b + 1 - 1 = N - b = a + 1$ (remember that $a + b = N - 1$), and therefore, we recognize U_1 as the cycle $\sigma_{10\ldots 0}$. Thus , U_1 is a hyperplane section as in the example, and intersects X non-empty.

$\alpha = 2$:
By definition $U_2 = \{\Lambda \in G | \dim(\Lambda \cap V_{a+1}) \geq 2\}$. To describe this in terms of cycles, we examine the indices, and find that $N - b + 2 - c_2 = a + 1$, which implies that $c_2 = N - b + 2 - a - 1 = N + 1 - (a + b) = N + 1 - (N - 1) = 2$. This makes the cycle $\sigma_{220\ldots 0}$, and thus U_2 is a cycle of codimension $2 + 2 = 4$.

Remark: The dimension of the big Grassmannian $G(b+1, N+1) = \mathbf{G}$ is $\dim \mathbf{G} = (b+1)(N-b) = (N-a)(N-b)$. The variety $X \subset \mathbf{G}$ has dimension a, and thus $\operatorname{codim} X = (N-a)(N-b) - a$. As long as the sum of the codimensions of U_α and X is less than or equal to the dimension of $G(b+1, N+1)$, the intersection $U_\alpha \cap X$ is expected to be non-empty. That is, as long as

$$((N-a)(N-b)) - a + \operatorname{codim} U_\alpha \leq (N-a)(N-b)$$

that is, as long as

$$\operatorname{codim} U_\alpha \leq a \tag{4.12}$$

the intersection $U_\alpha \cap X$ is expected to be non-empty.

Now, for a general α,

$$U_\alpha = \{\Lambda \in G | \dim(\Lambda \cap V_{a+1}) \geq \alpha\}$$

As above, we want the following equation to be true: $N - b + \alpha - c_\alpha = a + 1$, and this implies that $c_\alpha = N - b + \alpha - a - 1 = a + 1 + \alpha - (a+1) = \alpha$. This makes the Schubert cycle $\sigma_{\alpha\alpha\ldots\alpha 0\ldots 0}$ where the last α is index number α. In particular, this means that U_α is a cycle of codimension $\alpha \cdot \alpha = \alpha^2$. Comparing with equation 4.12, we expect the following implication to be true:

$$\alpha^2 \leq a \Rightarrow U_\alpha \cap X \neq \emptyset \tag{4.13}$$

Conclusion: We have found that as long as α^2 is less than or equal to a, we may expect U_α to intersect X. To translate this back to our original problem, we see that as long as α^2 is less than or equal to $2(n-1)$ we expect an intersection. We still need to prove that there actually is an intersection, and finally, find the number γ of tangent hyperplanes. Then, the discussion above says that we get a point in $VSP(G(2,n+1), \frac{1}{n}\binom{2n-2}{n-1} - 2^{\gamma})$.

How many tangent hyperplanes?

Assume $U_\alpha \cap X \neq \emptyset$ when $\alpha^2 \leq 2(n-1)$. Then $\alpha = \lfloor \sqrt{2(n-1)} \rfloor$ is the maximal α. Now, look at

$$I = \{(T_pG, H_t) \mid (T_pG \cup L) \subset H_t, \dim(T_pG \cap L) \geq \alpha\}$$

The first condition gives that H_t is a tangent hyperplane, tangent at p, and that H_t contains L. Thus we have a projection from I to the set of all tangent hyperplanes of $G(2,n+1)$ containing L. Furthermore, the second condition gives that we also have a projection to $U_\alpha \cap X$ (Why? $U_\alpha \cap X$ is precisely the set of tangent spaces to $G(2,n+1)$ such that the vector space dimension of the intersection with L is greater then or equal to α). To summarize, we have projections

$$\begin{array}{ccc} & \{(T_pG, H_t) \mid (T_pG \cup L) \subset H_t,\ \dim(T_pG \cap L) \geq \alpha\} & \\ \swarrow & & \searrow^{\mathbb{P}^{\alpha-1}} \\ H_{t,L} & & (U_\alpha \cap X)^{2(n-1)-\alpha^2} \end{array}$$

First of all, the dimension of $U_\alpha \cap X$ is $2(n-1) - \alpha^2$, since the codimension of U_α is α^2. To find the dimension of the fibers over $U_\alpha \cap X$, fix a tangent space T_pG where $\dim(T_pG \cap L) \geq \alpha$. How many tangent hyperplanes contains L and T_pG? We know that the dimension of L is $\frac{n^2-3n}{2}$, and the dimension of T_pG is $2(n-1)$, and $\dim(L \cup T_pG) = \frac{n^2-3n}{2} + 2(n-1) - (\alpha - 1)$. Furthermore, the dimension of a tangent hyperplane is $\frac{n^2-3n}{2} + 2(n-1)$, and thus the fiber is a $\mathbb{P}^{\alpha-1}$. There are therefore α tangent hyperplanes, tangent at the point p which contains L. Thus the γ in the conclusion above is equal to α.

4.1.5 The intersection $U_\alpha \cap X$

We want to show that the intersection $U_\alpha \cap X$ is non-empty. In a given example, the intersection can be described in terms of Chern classes. In this section, we will give the general setting, and do the calculations in three examples.

The Grassmannian $G = G(2,n+1)$ consists of two dimensional subvector spaces of a $(n+1)$-dimensional vector space. Therefore, it comes with an exact sequence of vectorbundles:

$$0 \to U^2 \to V^{n+1} \to Q^{n-1} \to 0$$

This is called *the universal sequence over* G. The cohomology ring of the Grassmannian is generated by the classes of the special Schubert cycles ([9]), and the cycle σ_{r0} equals the r-th Chern class of the universal quotient bundle Q^{n-1} ([6], section 14.7):

$$\sigma_r = c_r(Q^{n-1}) \qquad \text{for } 1 \leq r \leq n-1$$

Notice that V^{n+1} is a vector bundle over $G(2, n+1)$:

$$\begin{array}{c} 0 \to U^2 \to V^{n+1} \to Q^{n-1} \to 0 \\ \downarrow \\ G(2, n+1) \end{array}$$

Over a point $p \in G(2, n+1)$ lies the vector space U_p^2. The tangent bundle T_G is given by $\mathrm{Hom}(U^2, Q^{n-1}) = Q^{n-1} \otimes (U^2)^*$, where A^* indicates the dual of the vector bundle A. The differentials Ω_G is the dual of T_G, and this implies that $\Omega_G = (Q^{n-1})^* \otimes U^2$.

Lemma 4.6. *Let* $G = G(k+1, n+1)$. *For any* k, *the canonical sheaf* ω_G *of* G *is equal to* $\mathscr{O}_G(-n-1)$.

Proof. The Grassmannian $G(k+1, n+1)$ comes with an exact sequence of vector bundles

$$0 \to U^{k+1} \to V^{n+1} \to Q^{n-k} \to 0$$

The line bundle which embeds G in the Plücker space is $\wedge^{n-k} Q^{n-k}$ (or $\wedge^{k+1}(U^{k+1})^*$). As above, $\Omega_G = (Q^{n-k})^* \otimes U^{k+1}$, and by definition,

$$\begin{aligned} \omega_G = \wedge^{\dim G} \Omega_G &= \wedge^{(k+1)(n-k)}((Q^{n-k})^* \otimes U^{k+1}) \\ &= \mathscr{O}_G(-(n-k) - (k+1)) \\ &= \mathscr{O}_G(-n-1) \end{aligned}$$

□

Let E_G be a bundle such that $\mathbb{P}(E_G^*)$ is the variety of tangent spaces to $G(2, n+1)$. Let H be the line bundle $\wedge^2(U^2)^*$ which embeds $G(2, n+1)$ in the Plücker space $\mathbb{P}^{\binom{n+1}{2}-1}$. Then it is well known (see [6]) that E_G fits in the following exact sequence:

$$0 \to \Omega_G(H) \to E_G \to \mathscr{O}_G(H) \to 0 \tag{4.14}$$

The bundle E_G is such that $\mathbb{P}(E_G^*)$ is the variety of tangent spaces to $G(2, n+1)$. Thus $\mathbb{P}(E_G^*)$ is a collection of $\mathbb{P}^{2(n-1)}$-s The bundle E_G^* is a bundle over $G(2, n+1)$, where the fiber over a point $p \in G(2, n+1)$ is the tangent space $T_pG(2, n+1)$:

$$\begin{array}{ccc} E_G^* & & E_{G,p}^* = T_pG(2,n+1) \\ \downarrow & & \uparrow \\ G(2,n+1) & & p \end{array}$$

The fibers $E_{G,p}^* = T_pG(2,n+1)$ are contained in $\mathbb{P}(\wedge^2 V^{n+1})$, and from this we get an exact sequence

$$0 \to E_G^* \to \wedge^2 V^{n+1} \to K \to 0$$

where K is the quotient. We can think of E_G^* as a subset of $G(2(n-1)+1, \binom{n+1}{2})$ and use it to put $G(2,n+1)$ inside the bigger Grassmannian:

$$\begin{array}{ccc} G(2,n+1) & \xrightarrow{\gamma} & G(2(n-1)+1, \binom{n+1}{2}) \\ p & \mapsto & E_{G,p}^* \end{array}$$

Of course, the bigger Grassmannian comes with an exact sequence of its own, so now we have two exact sequences:

$$\begin{gathered} 0 \to E_G^* \to \wedge^2 V^{n+1} \to K \to 0 \\ 0 \to U^{2(n-1)+1} \to \wedge^2 V^{n+1} \to Q \to 0 \end{gathered} \tag{4.15}$$

We are trying to answer the question

For $\alpha = 1, 2, 3, \ldots$ does there exist a $T_pG^\perp$ such that $\dim(T_pG^\perp \cap L^\perp) \geq \alpha$?

where $L^\perp$ is a fixed linear space in the dual Plücker space $\check{\mathbb{P}}^{\binom{n+1}{2}-1}$. We have a composition of maps

$$\begin{array}{ccc} & G(2,n+1) & \\ {}^{\gamma}\swarrow & & \searrow^{\overline{\gamma}} \\ G(2(n-1)+1, \wedge^2 V^{n+1}) & \xrightarrow{\perp} & G(\frac{n^2-3n}{2}+1, (\wedge^2 V^{n+1})^*) =: \mathbf{G} \end{array}$$

where $\overline{\gamma}(p) = T_pG^\perp$. The universal sequence on $\mathbf{G}$ is

$$0 \to Q^* \to (\wedge^2 V^{n+1})^* \to (U^{2(n-1)+1})^* \to 0$$

and $\overline{\gamma}$ gives a sequence

$$0 \to E_G \to (\wedge^2 V^{n+1})^* \to K^* \to 0 \tag{4.16}$$

To ease the notation, let U^* denote $(U^{2(n-1)+1})^*$. The cycle U_α is a cycle on $\mathbf{G}$, and we want to see if the intersection cycle $G(2,n+1) \cap U_\alpha$ is different from zero on $\mathbf{G}$.

We have seen that U_α is a cycle of codimension α^2, and Giambelli's formula (lemma 1.9) gives that

$$U_\alpha = \sigma_{\underbrace{\alpha\cdots\alpha}_{\alpha \text{ times}} 0\cdots 0} = \begin{vmatrix} c_\alpha & c_{\alpha+1} & c_{\alpha+1} & \cdots & c_{2\alpha} \\ c_{\alpha-1} & c_\alpha & c_{\alpha+1} & \cdots & c_{2\alpha-1} \\ \vdots & & & & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_\alpha \end{vmatrix}$$

where c_i are the Chern classes on the bundle U^* on $\mathbf{G}$. The pull-back formula for Chern classes gives that

$$\begin{aligned} \overline{\gamma}^*(c_i(U^*) \cap G) &= c_i(\overline{\gamma}^* U^*) \cap \overline{\gamma}^* G \\ &= c_i(K^*) \cap G \end{aligned}$$

If $c(A)$ denotes the total Chern class of a vector bundle A, the Whitney sum formula and equation 4.16 gives that

$$c(K^*) \cdot c(E_G) = 1$$

and using the identity

$$\frac{1}{c(A)} = 1 - (c_1(A) + c_2(A) - c_3(A) + \dots) + ((c_1(A) + c_2(A) - c_3(A) + \dots)^2 - \cdots$$

where A is some vector bundle, we are able to write the Chern classes of K^* as polynomials in the Chern classes of E_G. But Whitney sum and equation 4.14 gives that

$$\begin{aligned} c(E_G) &= c(\Omega_G(H)) \cdot c(\mathscr{O}_G(H)) \\ &= c(\Omega_G(H)) \cdot (1 + H) \end{aligned}$$

Altogether, we have

$$c(K^*) = \frac{1}{c(\Omega_G(H)) \cdot (1 + H)}$$

Recall that $\Omega_G = (Q^{n-1})^* \otimes U^2$, and the universal bundle on $G(2, n+1)$ gives that

$$c(Q^{n-1})^*) \cdot c((U^2)^*) = 1$$

and thus the Chern classes of U^2 can be expressed as polynomials in the Chern classes of $(Q^{n-1})^*$, and so can the Chern classes of Ω_G. We use this to write the Chern classes of K^* as a polynomial in the Chern classes of $(Q^{n-1})^*$, and if this polynomial evaluates to something positive, we know the intersection $U_\alpha \cap G(2, n+1)$ is non-empty.

Example 4.4 ($G(2,6)$). We will now do the calculations on the Grassmannian $G(2,6)$. In this case, $\alpha = \lfloor\sqrt{8}\rfloor = 2$, and

$$U_2 = \begin{vmatrix} c_2 & c_3 \\ c_1 & c_2 \end{vmatrix}$$

Let α_1 and α_2 be Chern roots for U^2, i.e. let

$$c(U^2) = (1+\alpha_1)(1+\alpha_2)$$

Then

$$c_1(U^2) = \alpha_1 + \alpha_2 = -H \quad \text{and} \quad c_2(U^2) = \alpha_1\alpha_2 =: t$$

We need the following formula:

Lemma 4.7 (Formula for Chern roots of tensor product, [17] p37,[6] p54). *If E and F are vector bundles of rank e and f, and the total Chern classes are*

$$c(E) = \prod_{i=1}^{e}(1+x_i) \quad \textit{and} \quad \prod_{j=1}^{f}(1+y_i)$$

the total Chern class of $E \otimes F$ is

$$c(E\otimes F) = \prod_{i,j}(1+x_i+y_j)$$

It follows that the rank of $E\otimes F$ is $e \cdot f$. Furthermore, the Chern classes of the dual bundle E^* is given by

$$c_i(E^*) = (-1)^i c_i(E)$$

The Chern roots of E^* are $-x_1, \ldots, -x_e$.

The lemma implies that the total Chern class of $U^2 \otimes H$ is

$$\begin{aligned} c(U^2\otimes H) &= (1+\alpha_1+H)(1+\alpha_2+H) \\ &= 1 + (\alpha_1+\alpha_2+2H) + (\alpha_1\alpha_2 + \alpha_1 H + \alpha_2 H + H^2) \end{aligned}$$

and

$$\begin{aligned} c_1(U^2\otimes H) &= -H + 2H = H \\ c_2(U^2\otimes H) &= t - H^2 + H^2 = t \end{aligned}$$

For the following calculations, let $c(U^2 \otimes H) = (1+v_1)(1+v_2)$. The Chern roots v_1 and v_2 are only temporary, to make the notation easier. Remember that we need to find the Chern classes of $\Omega_G(H)$, and we know that $\Omega_G(H) = (Q^4)^* \otimes U^2 \otimes H$. We find the Chern classes of $(Q^4)^*$:

$$\begin{aligned} c((Q^4)^*) &= \frac{1}{c((U^2)^*)} \\ &= \frac{1}{1+H+t} \\ &= 1-(H+t)+(H+t)^2-(H+t)^3+(H+t)^4 \\ &= 1-H-t+H^2+2Ht+t^2-H^3-3H^2t-3Ht^2-t^3+\cdots \end{aligned} \tag{4.17}$$

i.e

$$\begin{aligned} c_1((Q^4)^*) &= -H \\ c_2((Q^4)^*) &= -t+H^2 \\ c_3((Q^4)^*) &= 2Ht-H^3 \end{aligned}$$

For the following calculations, let $c((Q^4)^*) = \prod_{i=1}^4 (1+q_i)$. As with the v_i-s, the Chern roots q_i are temporary, to make the notation easier. We are now ready to find the Chern classes of $\Omega_G(H)$:

$$\begin{aligned} c(\Omega_G(H)) &= c((Q^4)^* \otimes U^2 \otimes H) \\ &= (1+v_1+q_1)\cdots(1+v_1+q_4)(1+v_2+q_1)\cdots(1+v_2+q_4) \\ &= 1+\left(2\sum_i q_i + 4(v_1+v_2)\right) \\ &\quad + \left(\sum_i q_i^2 + 4\sum_{i\neq j} q_iq_j + 7\sum_{i,j} q_iv_j + 6(v_1^2+v_2^2) + 16v_1v_2\right) \\ &\quad + 2\sum_{i\neq j} q_i^2 q_j + 3\sum_{i,j} q_i^2 v_j + 8\sum_{i\neq j\neq k} q_iq_jq_k + 12\sum_{k,i\neq j} q_iq_jv_k + 9\sum_{i,j} q_iv_j^2 \\ &\quad + 24\sum_i q_iv_1v_2 + 4(v_1^3+v_2^3) + 24v_2v_1^2 + 24v_1v_2^2 + \cdots \end{aligned}$$

Now, $\sum_i q_i = c_1((Q^4)^*)$ and $v_1 + v_2 = c_1(U^2 \otimes H)$, and the linear term of the sum is

$$c_1(\Omega_G(H)) = 2(-H) + 4H = 2H$$

Continuing writing the terms as polynomials in the elementary symmetric functions of the q_i-s and v_1 and v_2, we get that

$$\begin{aligned} c_2(\Omega_G(H)) &= 2t + 16H^2 \\ c_3(\Omega_G(H)) &= 18Ht - 4H^3 \end{aligned}$$

Thus

$$c(\Omega_G(H)) \cdot (1+H) = 1 + \underbrace{3H + 2t + 4H^2 + 20tH - 4H^3 + \cdots}_{R}$$

and

$$\begin{aligned} c(K^*) &= \frac{1}{c(\Omega_G(H)) \cdot (1+H)} \\ &= 1 - R + R^2 - R^3 + \cdots \\ &= 1 - 3H - 2t + 5H^2 - 8tH + H^3 + \cdots \end{aligned}$$

This implies that

$$\begin{aligned} c_1(K^*) &= -3H \\ c_2(K^*) &= -2t + 5H^2 \\ c_3(K^*) &= -8tH + H^3 \end{aligned}$$

and

$$\begin{aligned} U_2 \cap X &= \begin{vmatrix} -2t + 5H^2 & -8tH + H^3 \\ -3H & -2t + 5H^2 \end{vmatrix} \\ &= 4t^2 - 44tH^2 + 28H^4 \end{aligned}$$

We need to evaluate this polynomial, by multiplying with H^4 to find the degree in the Plücker space. A similar calculation to the one in equation 4.17 shows that

$$\begin{aligned} c_1(Q^4) &= H \\ c_2(Q^4) &= -t + H^2 \\ c_3(Q^4) &= -2Ht + H^3 \end{aligned}$$

Recall that $c_i(Q^4)$ is exactly the special Schubert cycle σ_i on $G(2, n+1)$. This implies that

$$\begin{aligned} 4t^2 - 44tH^2 + 28H^4 &= 4(\sigma_1^2 - \sigma_2)^2 - 44(\sigma_1^2 - \sigma_2)\sigma_1^2 + 28\sigma_1^4 \\ &= -12\sigma_1^4 + 36\sigma_1^2\sigma_2 + 4\sigma_2^2 \end{aligned}$$

Multiplying with H^4 gives

$$-12\sigma_1^8 + 36\sigma_1^6\sigma_2 + 4\sigma_1^4\sigma_2^2$$

The degree of $G(2,6)$ is 14, so σ_1^8 is 14 times the class of a point. Thus this polynomial equals

$$\begin{aligned}
& -12 \cdot 14\sigma_{44} + 36\sigma_1^6\sigma_2 + 4\sigma_1^4\sigma_2^2 \\
= & -168\sigma_{44} + 36\sigma_1^5(\sigma_3 + \sigma_{21}) + 4\sigma_1^4(\sigma_{22} + \sigma_{31} + \sigma_4) \\
= & -168\sigma_{44} + 36\sigma_1^4(\sigma_4 + 2\sigma_{31} + \sigma_{22}) + 4\sigma_1^3(2\sigma_{32} + 2\sigma_{41}) \\
= & -168\sigma_{44} + 36\sigma_1^3(3\sigma_{41} + 3\sigma_{32}) + 4\sigma_1^2(4\sigma_{42} + 2\sigma_{33}) \\
= & -168\sigma_{44} + 36\sigma_1^2(6\sigma_{42} + 3\sigma_{33}) + 24\sigma_{44} \\
= & -168\sigma_{44} + 324\sigma_{44} + 24\sigma_{44} \\
= & 180\sigma_{44}
\end{aligned}$$

Thus the degree of the intersection $U_2 \cap X$ in the Plücker space is strictly positive, and we may conclude that this intersection is non-empty. $\triangle$

Example 4.5 ($G(2,7)$). In the case $n = 6$, we have $\alpha = \lfloor\sqrt{10}\rfloor = 3$, and

$$U_3 = \begin{vmatrix} c_3 & c_4 & c_5 \\ c_2 & c_3 & c_4 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

As above, $c_1(U^2 \otimes H) = H$ and $c_2(U^2 \otimes H) = t$, and

$$\begin{aligned}
c((Q^5)^*) &= \frac{1}{c((U^2)^*)} \\
&= \frac{1}{1 + H + t} \\
&= 1 - H - t + H^2 + 2Ht - H^3 + t^2 - 3H^2t + H^4 - 3Ht^2 + 4H^3t - H^5 + \cdots
\end{aligned}$$

and thus

$$\begin{aligned}
c_1((Q^5)^*) &= -H \\
c_2((Q^5)^*) &= -t + H^2 \\
c_3((Q^5)^*) &= 2Ht - H^3 \\
c_4((Q^5)^*) &= t^2 - 3H^2t + H^4 \\
c_5((Q^5)^*) &= -3Ht^2 + 4H^3t - H^5
\end{aligned}$$

For the following calculations, let $c((Q^5)^*) = \prod_{i=1}^{5}(1+q_i)$. Then

$$
\begin{aligned}
c(\Omega_G(H)) &= c((Q^5)^* \otimes U^2 \otimes H) \\
&= \prod_{i,j}(1 + v_i + q_j) \\
&= 1 + 2\sum_i q_i + 5(v_1+v_2) \\
&+ \sum_i q_i^2 + 9\sum_{i,j} q_i v_j + 10(v_1^2+v_2^2) + 4\sum_{i\neq j} q_i q_j + 25 v_1 v_2 \\
&+ 10(v_1^3+v_2^3) + 16\sum_{i,j} v_i^2 q_j + 50(v_1^2 v_2 + v_1 v_2^2) + 4\sum_{i,j} v_i q_j^2 \\
&\quad + 16(v_1+v_2)\sum_{i\neq j} q_i q_j + 40 v_1 v_2 \sum_i q_i + 2\sum_{i\neq j} q_i^2 q_j + 8\sum_{i\neq j\neq k} q_i q_j q_k \\
&+ 5(v_1^4+v_2^4) + 14(v_1^3+v_2^3)\sum_i q_i + 50(v_1^3 v_2 + v_1 v_2^3) + 25(v_1^2+v_2^2)\sum_{i\neq j} q_i q_j \\
&\quad + 70(v_1^2 v_2 + v_1 v_2^2)\sum_i q_i + 6(v_1^2+v_2^2)\sum_i q_i^2 + 100 v_1^2 v_2^2 + 16 v_1 v_2 \sum_i q_i^2 \\
&\quad + 7(v_1+v_2)\sum_{i\neq j} q_i^2 q_j + 28(v_1+v_2)\sum_{i\neq j\neq k} q_i q_j q_k + 62 v_1 v_2 \sum_{i\neq j} q_i q_j \\
&\quad + \sum_{i\neq j} q_i^2 q_j^2 + 4\sum_{i\neq j\neq k} q_i^2 q_j q_k + 16\sum_{i\neq j\neq k\neq l} q_i q_j q_k q_l \\
&+ 4(v_1^3+v_2^3)\sum_i q_i^2 + 6(v_1^4+v_2^4)\sum_i q_i + (v_1^5+v_2^5) + 9(v_1^2+v_2^2)\sum_{i\neq j} q_i^2 q_j \\
&\quad + 19(v_1^3+v_2^3)\sum_{i\neq j} q_i q_j + 3(v_1+v_2)\sum_{i\neq j} q_i^2 q_j^2 + 12(v_1+v_2)\sum_{i\neq j\neq k} q_i^2 q_j q_k \\
&\quad + 2\sum_{i\neq j\neq k} q_i^2 q_j^2 q_k + 38(v_1^2+v_2^2)\sum_{i\neq j\neq k} q_i q_j q_k + 8\sum_{i\neq j\neq k\neq l} q_i^2 q_j q_k q_l \\
&\quad + 48(v_1+v_2)\sum_{i\neq j\neq k\neq l} q_i q_j q_k q_l + 60(v_1^3 v_2 + v_1 v_2^3)\sum_i q_i + 24 v_1 v_2 \sum_{i\neq j} q_i^2 q_j \\
&\quad + 93(v_1^2 v_2 + v_1 v_2^2)\sum_{i\neq j} q_i q_j + 92 v_1 v_2 \sum_{i\neq j\neq k} q_i q_j q_k + 120 v_1^2 v_2^2 \sum_i q_i \\
&\quad + 100(v_1^2 v_2^3 + v_1^3 v_2^2) + 25(v_1^4 v_2 + v_1 v_2^4) + 24(v_1^2 v_2 + v_1 v_2^2)\sum_i q_i^2 \\
&\quad + 32\sum_{i\neq j\neq k\neq l\neq m} q_i q_j q_k q_l q_m + \cdots
\end{aligned}
$$

which gives that

$$\begin{aligned}
c_1(\Omega_G(H)) &= 3H \\
c_2(\Omega_G(H)) &= 4H^2 + 3t \\
c_3(\Omega_G(H)) &= -4H^3 + 22Ht \\
c_4(\Omega_G(H)) &= -6H^4 + 6H^2t + 23t^2 \\
c_5(\Omega_G(H)) &= -18H^5 - 92Ht^2 + 46H^3t
\end{aligned}$$

Furthermore,

$$\begin{aligned}
&c(\Omega_G(H))(1+H) \\
&= 1 + 4H + 3t + 7H^2 + 25tH - 10H^4 + 23t^2 + 28H^2t - 69Ht^2 - 24H^5 + 52H^3t + \cdots
\end{aligned}$$

Setting $c(\Omega_G(H))(1+H) - 1 = R$, we have

$$\begin{aligned}
c(K^*) &= \frac{1}{c(\Omega_G(H))(1+H)} \\
&= 1 - R + R^2 - R^3 + R^4 - R^5 + \cdots \\
&= 1 - 4H - 3t + 9H^2 - tH - 8H^3 - 14t^2 + 70tH^2 - 21H^4 \\
&\qquad + 295t^2H - 414tH^3 + 124H^5 + \cdots
\end{aligned}$$

This implies that

$$U_3 \cap X = \begin{vmatrix} -tH - 8H^3 & -14t^2 + 70tH^2 - 21H^4 & 295t^2H - 414tH^3 + 124H^5 \\ -3t + 9H^2 & -tH - 8H^3 & -14t^2 + 70tH^2 - 21H^4 \\ -4H & -3t + 9H^2 & -tH - 8H^3 \end{vmatrix}$$

$$= 776H^9 + 1955t^4H - 5200tH^7 + 13617t^2H^5 - 12997t^3H^3$$

Now, the Chern classes of the bundle Q^5 are

$$\begin{aligned}
c_1(Q^5) &= H \\
c_2(Q^5) &= -t + H^2 \\
c_3(Q^5) &= -2Ht + H^3 \\
c_4(Q^5) &= t^2 - 3H^2t + H^4 \\
c_5(Q^5) &= 3Ht^2 - 4H^3t + H^5
\end{aligned}$$

and these Chern classes are exactly the special Schubert cycles on $G(2,7)$. Thus

$$\begin{aligned}
&776H^9 + 1955t^4H - 5200tH^7 + 13617t^2H^5 - 12997t^3H^3 \\
&= 776\sigma_1^9 + 1955\sigma_1(\sigma_1^2 - \sigma_2)^4 - 5200\sigma_1^7(\sigma_1^2 - \sigma_2) \\
&\qquad + 13617\sigma_1^5(\sigma_1^2 - \sigma_2)^2 - 12997\sigma_1^3(\sigma_1^2 - \sigma_2)^3 \\
&= -1849\sigma_1^9 + 9137\sigma_1^7\sigma_2 - 13644\sigma_1^5\sigma_2^2 + 5177\sigma_1^3\sigma_2^3 + 1955\sigma_1\sigma_2^4
\end{aligned}$$

This is a variety of codimension 9 on $G(2,7)$, i.e. a variety of dimension one. We find its degree by intersecting with σ_1:

$$\begin{aligned}
&-1849\sigma_1^{10} + 9137\sigma_1^8\sigma_2 - 13644\sigma_1^6\sigma_2^2 + 5177\sigma_1^4\sigma_2^3 + 1955\sigma_1^2\sigma_2^4 \\
&= -77658\sigma_{55} + 9137\sigma_1^7(\sigma_3 + \sigma_{21}) - 13644\sigma_1^6(\sigma_4 + \sigma_{31} + \sigma_{22}) \\
&\quad + 5177\sigma_1^4(2\sigma_{51} + 3\sigma_{42} + \sigma_{33}) + 1955\sigma_1^2(6\sigma_{53} + 3\sigma_{44}) \\
&= -77658\sigma_{55} + 9137\sigma_1^6(\sigma_4 + 2\sigma_{31} + \sigma_{22}) - 13644\sigma_1^5(\sigma_5 + 2\sigma_{41} + 2\sigma_{32}) \\
&\quad + 5177\sigma_1^3(5\sigma_{52} + 4\sigma_{43}) + 1955 \cdot 9\sigma_{55} \\
&= -60063\sigma_{55} + 9137\sigma_1^5(\sigma_5 + 3\sigma_{41} + 3\sigma_{32}) - 13644\sigma_1^4(3\sigma_{51} + 4\sigma_{42} + 2\sigma_{33}) \\
&\quad + 5177\sigma_1^2(9\sigma_{53} + 4\sigma_{44}) \\
&= -60063\sigma_{55} + 9137\sigma_1^4(4\sigma_{51} + 6\sigma_{42} + 3\sigma_{33}) - 13644\sigma_1^3(7\sigma_{52} + 6\sigma_{43}) \\
&\quad + 5177 \cdot 13\sigma_{55} \\
&= 7238\sigma_{55} + 9137\sigma_1^3(10\sigma_{52} + 9\sigma_{43}) - 13466\sigma_1^2(13\sigma_{53} + 6\sigma_{44}) \\
&= 7238\sigma_{55} + 9137\sigma_1^2(19\sigma_{53} + 9\sigma_{44}) - 13466 \cdot 19\sigma_{55} \\
&= -248616\sigma_{55} + 9137 \cdot 28\sigma_{55} \\
&= 7220\sigma_{55}
\end{aligned}$$

△

This type of calculation may be done for any n. Of course, it quickly gets a lot more complicated.

Example 4.6 ($G(2,5)$). In this case $\alpha = \lfloor\sqrt{6}\rfloor = 2$. Thus U_2 is as in the previous example. Now, the universal quotient Q^3 bundle has rank 3, and assuming $c((Q^3)^*) = \prod_{i=1}^3(1+q_i)$, we get

$$\begin{aligned}
c(\Omega_G(H)) &= \prod_{i,j}(1 + v_i + q_j) \\
&= 1 + 3(v_1 + v_2) + 2(q_1 + q_2 + q_3) \\
&\quad + 3(v_1^2 + v_2^2) + 5(v_1 + v_2)(\sum_{i=1}^3 q_i) + \sum_{i=1}^3 q_i^2 + 4\sum_{i,j} q_iq_j + 9v_1v_2 \\
&\quad + (v_1^3 + v_2^3) + 4(v_1^2 + v_2^2)(\sum_{i=1}^3 q_i) + 2(v_1 + v_2)(\sum_{i=1}^3 q_i^2) \\
&\quad + 8(v_1 + v_2)(\sum_{i\neq j} q_iq_j) + 2\sum_{i\neq j} q_i^2q_j + 12v_1v_2\sum_{i=1}^3 q_i \\
&\quad + 9v_1v_2(v_1 + v_2) + 8q_1q_2q_3 + \cdots
\end{aligned}$$

Thus

$$c_1(\Omega_G(H)) = H$$
$$c_2(\Omega_G(H)) = H^2 + t$$
$$c_3(\Omega_G(H)) = 16tH - 7H^3$$

Furthermore,

$$c(K^*) = \frac{1}{c(\Omega_G(H))(1+H)} = 1 - H - t - H^2 - 15Ht + 9H^3 \cdots$$

and from this we get that

$$c_1(K^*) = -H \qquad c_2(K^*) = -t - H^2 \qquad c_3(K^*) = -15Ht + 9H^3$$

and

$$\begin{aligned} U_2 \cap X = &\begin{vmatrix} c_2 & c_3 \\ c_1 & c_2 \end{vmatrix} \\ &= (-t - H^2)^2 - (-H)(-15Ht + 9H^3) \\ &= t^2 - 13H^2t + 10H^4 \end{aligned}$$

As in the previous example

$$c_1(Q^3) = H \qquad c_2(Q^3) = -t + H^2 \qquad c_3(Q^3) = -2Ht + H^3$$

and the intersection $U_2 \cap X$ expressed in terms of special Schubert cycles is

$$\begin{aligned} &(\sigma_2 - \sigma_1^2)^2 + 13\sigma_1^2(\sigma_2 - \sigma_1^2) + 10\sigma_1^4 \\ = &- 2\sigma_1^4 + 11\sigma_1^2\sigma_2 + \sigma_2^2 \end{aligned}$$

Multiplying with σ_1^2 gives the degree of $U_2 \cap X$ in the Plücker space:

$$\begin{aligned} &- 2\sigma_1^6 + 11\sigma_1^4\sigma_2 + \sigma_1^2\sigma_2^2 \\ = &- 10\sigma_{33} + 11\sigma_1^3(\sigma_3 + \sigma_{21}) + \sigma_1^2(\sigma_{31} + \sigma_{22}) \\ = &- 10\sigma_{33} + 11\sigma_{33} + 11\sigma_1^2(\sigma_{31} + \sigma_{22}) + 2\sigma_{33} \\ = &25\sigma_{33} \end{aligned}$$

△

In the next section, we will study the Grassmannian $G(2,5)$ in more detail.

4.2 The method works for all $n \geq 5$

In this section, we will show that the projection of $H_p^1 \cap \cdots \cap H_p^\alpha \cap G(2, n+1)$ from p is birational whenever $n \geq 5$.

4.2.1 The Grassmannian of lines in $\mathbb{P}^4$

This section is about $G(2,5)$, and some of the special properties of this particular Grassmannian.

We know that the cycle σ_{10} is a tangent hyperplane section, and its linear span is a hyperplane tangent at p, where p is a point on $G(2,5)$ corresponding to a line l_p in the plane $\mathbb{P}(V_3)$ which determines the cycle. In fact, $\langle\sigma_{10}\rangle$ is tangent along an entire plane, since 5 is an odd number. The plane is precisely the plane of lines in $\mathbb{P}(V_3)$. Recall also that $G(2,5)$ is the only Grassmannian of lines for which every point on the dual variety is of this type.

4.2.2 Powersums and $G(2,5)$

As we have seen many times before, $\dim G(2,5) = 2 \cdot (5-2) = 6$, and it sits inside $\mathbb{P}^9$ as a subvariety. We fix a linear space L of dimension 2. In the dual space, we thus have a linear space $L^\perp$ of dimension 6. Inside $\check{\mathbb{P}}^9$ lies the union D of all tangent hyperplanes to $G(2,5)$. The variety D is called the dual variety of $G(2,5)$, and $D \simeq G(3,5)$. So, how many tangent hyperplanes contain L? If we translate this to the dual space, the question becomes: How many points on D lies on $L^\perp$? Or, how many points are common to D and $L^\perp$? Remember that we want to figure out how many tangent hyperplanes, *tangent at the same point*, contain L.

Powersums as above

A dimension count shows that as long as $\alpha^2 \leq 6$ we expect U_α to intersect $X = G(2,5)$ non-empty. The example above shows that the intersection really is non-empty.

This implies that we get a point in $VSP(G(2,5), 5-2^2) = VSP(G(2,n+1),1)$. We have shown that a form of degree $n-2 = 4-2 = 2$ in 3 variables can be written as l^2 where l is a linear form. This must be wrong. Thus, the conditions for using this method is not satisfied. So where does this go wrong?

We need to study the intersection $H_{t,p} \cap H'_{t,p} \cap G(2,5)$ more carefully. The intersection has codimension 2 inside $G(2,5)$, and we know that *all* tangent hyperplanes to $G(2,5)$ is of the type $\langle\sigma_{10}(V_3)\rangle$ for some plane V_3 in $\mathbb{P}^4$. So let l_p be the line in $\mathbb{P}^4$ corresponding to the point p of tangency. Every tangent hyperplane to $G(2,5)$, tangent at p, comes from a plane containing the line l_p. The double tangent hyperplane section we must study is thus $\sigma_{10}(V_3) \cap \sigma_{10}(V_3')$. Remember that $\sigma_{10}(V_3)$ are all lines in $\mathbb{P}^4$ intersecting a fixed plane $\mathbb{P}(V_3)$, so

$$\begin{aligned}\sigma_{10}(V_3) \cap \sigma_{10}(V_3') &= \{\text{lines in } \mathbb{P}^4 \text{ that intersects both } \mathbb{P}(V_3) \text{ and } \mathbb{P}(V_3')\} \\ &= \sigma_{11} + \sigma_{20} \\ &= \{\text{lines } l \text{ in } \mathbb{P}^4 \mid l \subset \mathbb{P}^3\} \cup \{\text{lines } l \text{ in } \mathbb{P}^4 \mid l \cap l_p \neq \emptyset\}\end{aligned}$$

We see that the first part of this intersection is $G(2,4)$, which has degree 2. Since the degree of $G(2,5)$ is 5, the degree of the second part of the intersection must be 3. Condition 2 from page 110 says that the projection of $\sigma_{10}(V_3) \cap \sigma_{10}(V_3')$ from the point p (coming from the line l_p), must be birational.

So take a line in $\mathbb{P}^4$ from p to a point on $G(2,4)$, not contained in $G(2,4)$. Since the degree of $G(2,4)$ is 2, a line not contained in the variety can only meet it at two points (counting with multiplicity). The point p is one of these points, and thus the line will meet the Grassmannian in only one more point. This shows that projection from p of the $G(2,4)$-part of the tangent hyperplanes section is birational. So what about projection from the other part of the intersection?

The line l_p defines the cycle σ_{20}, and let l' be a line intersecting l_p. This is mapped by the Plücker embedding to a point $p' \in G(2,5)$, and the one-dimensional family of lines in the plane $\langle l', l_p \rangle$ through the point $l' \cap l_p$ is mapped to the line $\langle p', p \rangle$ in $G(2,5)$. For every line l' which intersects l_p we get a line in $G(2,5)$ through p. This shows that this component is a cone of degree 3, and thus projection from p is *not* birational (the projection collapses all the lines).

4.2.3 When will the projection go wrong in the general case?

Let $H_p^1 \cap \cdots \cap H_p^r \cap G(2, n+1)$ be a tangent hyperplane section of the Grassmannian of lines in $\mathbb{P}^n$, where all the tangent hyperplanes are tangent at the same point p. When is the projection from p not birational? Suppose one of the components of the intersection is a smaller Grassmannian, i.e the Grassmannian of lines in a smaller space than $\mathbb{P}^n$. A line from p can only meet this Grassmannian in one other point p', because both p and p' lie on a certain number of quadrics, and a line can only meet a quadric twice. Thus the projection is birational from components of this type.

Now, assume a line through p and a point p' in the intersection meets the intersection in a finite number of additional points. All these points lie on the Grassmannian, and thus they are common zeros of a certain number of quadrics. As in the case above, a line can only meet a quadric twice, and p and p' are two points. Thus, the line will not meet the intersection in any other distinct points. We conclude that the projection will never be $m:1$ where $m > 1$. Thus he projection must collapse lines, and in that case the intersection is a cone.

We have found that if none of the components in the intersection is a cone, projecting from the point of tangency will be a birational map.

So when is one of the components in the intersection a cone? A cone in the Grassmannian comes from (as we have seen above) all lines intersecting a fixed line. In other words, a cone comes from

$$\{\Lambda \in G(2, n+1) \,|\, \dim(\Lambda \cap V_2) \geq 1\} = \sigma_{n-2,0}$$

The dimension of this cycle is $2(n-1)-(n-2)=n$. Furthermore, we have seen that $\alpha = \lfloor\sqrt{2(n-1)}\rfloor$ is the maximal number of tangent hyperplanes for $G(2,n+1)$, and $H_p^1 \cap \cdots \cap H_p^\alpha \cap G(2,n+1)$ has dimension $2(n-1)-\alpha$. To summarize, if one of the components of the intersection is a cone, it must have dimension n, and the intersection itself has dimension $2(n-1)-\alpha$. If we are able to show that the dimension of the possibly existing cone is less than the dimension of the intersection, we have in fact shown that the cone does not exist. Thus we examine the equation

$$n < 2(n-1) - \alpha \tag{4.18}$$

This inequality is false only when $n = 2, 3$ or 4. We have seen that in the case $n = 4$, the intersection actually contains a cone, and the cases $n = 2$ and $n = 3$ is excluded from the discussion (see page 108). We have proved

Proposition 4.8. *Let α be the largest integer less than or equal to $\sqrt{2(n-1)}$. The projection of $H_p^1 \cap H_p^2 \cap \cdots \cap H_p^\alpha \cap G(2,n+1)$ from p is birational whenever $n \geq 5$.*

4.2.4 When is the intersection non-empty?

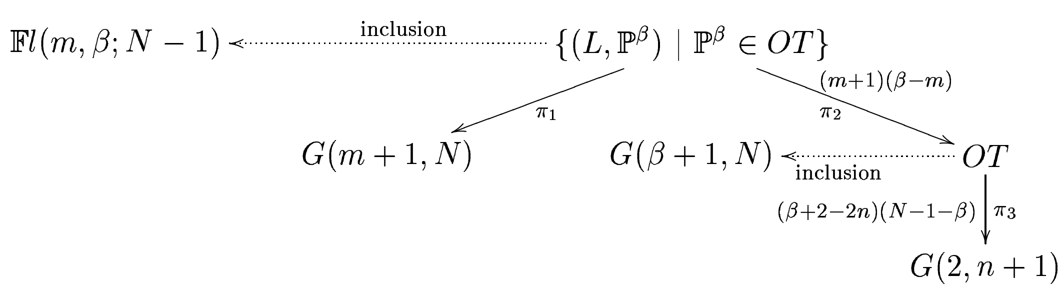
Assume α is maximal, i.e assume $\alpha = \lfloor\sqrt{2(n-1)}\rfloor$. Let $m+1$ be the codimension of $G(2,n+1)$ in the Plücker space $\mathbb{P}^{N-1}$. Recall that L is a linear space of dimension m in $\mathbb{P}^{N-1}$, and define

$$OT = \{\mathbb{P}^\beta = H_1 \cap \cdots \cap H_\alpha \mid \text{all } H_i \text{ are tangent at the same point on } G(2,n+1)\}$$

It follows that $\beta = N-1-\alpha$. We can define an incidence

$$\{(L,\mathbb{P}^\beta) \mid \mathbb{P}^\beta \in OT\} \subset \mathbb{F}l(m,\beta;N-1)$$

where $\mathbb{F}l(m,\beta;N-1)$ is the flag variety of $\mathbb{P}^m$-s in $\mathbb{P}^\beta$-s in the Plücker space. The flag variety itself has projections to $G(m+1,N)$ and $G(\beta+1,N)$, and from the incidence, we have the projections



If $\mathbb{P}^\beta$ is the intersection of α tangent hyperplanes, all tangent at the point p, the projection π_3 takes $\mathbb{P}^\beta \in OT$ to the point p. This is well defined as long as $n \geq 5$, as we have seen above. The fiber over a point $q \in G(2, n+1)$ is

$$\pi_3^{-1}(q) = \{\mathbb{P}^\beta \mid \mathbb{P}^\beta \in OT \text{ and } \mathbb{P}^\beta \supset T_pG\}$$

This is the cycle

$$\{\mathbb{P}^\beta \in \mathbb{P}^{N-1} \mid \mathbb{P}^\beta \supset \mathbb{P}^{2(n-1)}_{\text{fixed}}\} = \sigma_{N-\beta-1,\ldots,N-\beta-1,0,\ldots,0}$$

where $N-\beta-1$ occurs $2n-1$ times. Hence, this cycle is isomorphic to the Grassmannian $G(\beta+2-2n, N+1-2n)$. This implies that the fiber dimension of π_3 is $(\beta+2-2n)(N-1-\beta)$. The dimension of OT is

$$\dim(OT) = 2(n-1) + (\beta+2-2n)(N-1-\beta)$$

The fiber over an element $\mathbb{P}^\beta$ in OT is

$$\pi_2^{-1}(\mathbb{P}^\beta) = \{L \mid L \subset \mathbb{P}^\beta\}$$

But this is the Grassmannian $G(m+1, \beta+1)$ which has dimension $(m+1)(\beta-m)$. This implies that the dimension of the incidence is

$$2(n-1) + (\beta+2-2n)(N-1-\beta) + (m+1)(\beta-m)$$

Using the identities $\beta = N-1-\alpha$ and $m+1 = N-1-2(n-1)$, the dimension of the incidence is

$$6n - 3 - \alpha^2 - N + 2Nn - 4n^2$$

The fiber over a point L in $G(m+1, N)$ is

$$\pi_1^{-1}(L) = \{\mathbb{P}^\beta \mid \mathbb{P}^\beta \supset L\}$$

The expected fiber dimension of π_1 is

$$2(n-1)+(\beta+2-2n)(N-1-\beta)+(m+1)(\beta-m)-(m+1)(N-m-1) = 2(n-1)-\alpha^2$$

Notice that the intersection $U_\alpha \cap X$ is non-empty precisely when π_1 is surjective. And π_1 is surjective if there is an element in $G(m+1, N)$ such that $\pi_1^{-1}(L)$ has the expected dimension.

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