THE DEGREE OF THE SECANT VARIETY AND THE JOIN OF MONOMIAL CURVES

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Abstract. A monomial curve is a curve parametrized by monomials. The degree of the secant variety of a monomial curve is given in terms of the sequence of exponents of the monomials defining the curve. Likewise, the degree of the join of two monomial curves is given in terms of the two sequences of exponents.

Contents

1. Introduction

A monomial curve C is the image of an injective morphism of $f : \mathbb{P}^1 \to \mathbb{P}^r$ defined by monomials. After ordering the monomials by ascending degree it is therefore given by

$$[s : t] \mapsto [s^d : s^{d-a_1}t^{a_1} : \ldots : s^{d-a_{r-1}}t^{a_{r-1}} : t^d]$$

where $a_1 < a_2 < \ldots < a_r = d$. So this latter sequence completely determines C. We define the first secant variety $SecC$ to be the closure of the union of lines that meet C in two distinct points. The first aim of this note is to compute the degree of this secant variety as a subvariety of $\mathbb{P}^r$. According to a well-known argument using a general projection $\pi : C \to \overline{C} \subset \mathbb{P}^2$ this degree is given by the formula

$$\deg SecC = \binom{d-1}{2} - \delta_p - \delta_q$$

where δ_p and δ_q are the genus contributions of the cusps at $p = \pi([1 : 0 \ldots : 0])$ and $q = \pi([0 : \ldots : 0 : 1])$ on $\overline{C}$. Equivalently, $2\delta_p$ and $2\delta_q$ are the Milnor numbers of the cusps at p and q. To compute δ_p and δ_q given C, we analyze the Puiseux expansions of $\overline{C}$ at the cusps, and apply an algorithm due to Chisini and Enriques, eventually refined and given a closed form by Casas-Alvero.

Date: September 14, 2005.

1991 *Mathematics Subject Classification.* Primary 14; Secondary 14.

* Supported by MSRI.

Given two curves C and D in $\mathbb{P}^r$ we define their join $Join(C, D)$ to be the closure of the union of lines that meet C and D in two distinct points. We consider the join of two monomial curves C and D: In the notation of the previous section we ask that the two curves are defined by

$$C : [s : t] \mapsto [s^{d_C} : s^{d_C - a_1} t^{a_1} : \ldots : s^{d_C - a_{r-1}} t^{a_{r-1}} : t^{d_C}]$$

where $a_1 < a_2 < \ldots < a_r = d_C$, and

$$D : [s : t] \mapsto [s^{d_D} : s^{d_D - b_1} t^{b_1} : \ldots : s^{d_D - b_{r-1}} t^{b_{r-1}} : t^{d_D}]$$

where $b_1 < b_2 < \ldots < b_r = d_D$. Again the two sequences

$$a_1 < a_2 < \ldots < a_r = d_C, \quad b_1 < b_2 < \ldots < b_r = d_D$$

determine the two curves completely, and our second goal is to compute the degree of the join of C and D as a subvariety of $\mathbb{P}^r$. In this case the general projection of the two curves to $\mathbb{P}^2$ gives the formula

$$\deg Join(C, D) = d_C \cdot d_D - I_s(C, D),$$

where $I_s(C, D)$ is the sum of the intersection multiplicities in $\mathbb{P}^2$ of the two curves at the images of intersection points between the two curves in $\mathbb{P}^r$. Algorithms computing the sum of intersection multiplicities $I_s(C, D)$ are given in section 4.

The author thanks MSRI for excellent working conditions, Bernd Sturmfels for his inspired interest in the problem, Eduardo Casas-Alvero for giving the solution a nicer form and the referee for pointing out inaccuracies in an early version.

2. The multiplicity sequence of a plane curve singularity

A crucial ingredient in the two algorithms below is the multiplicity sequence of a plane curve singularity. Given a point p in the plane and a sequence of blowups at simple points $(p = p_0, p_1, p_2, ..., p_s)$, such that all exceptional divisors lie over p, i.e. is mapped to p by the natural map to the original plane, and such that the strict transform of the curve is smooth. The multiplicities $m_0(C)$, (resp. $m_i(C), i > 0$) of C at p (respectively its strict transforms at p_i), form the multiplicity sequence of C at p with respect to the sequence of blowups. Equivalently, the multiplicity sequence coincides with the sequence of intersection numbers of the strict transform of C with the exceptional divisor of each blow up. The multiplicity sequence may contain 1's, but these would not appear in a blowup that provides a minimal resolution of the singularity. In the latter case we say that the multiplicity sequence is minimal. Note that by the unicity of a minimal resolution of a plane curve singularity, the minimal multiplicity sequence is unique. Both minimal and nonminimal cases will however occur in our setting.

We shall use the multiplicity sequence of plane singularities with given Puiseux series. Consider the parameterized affine plane curve

$$C : t \mapsto (t^m, t^{k_1} + t^{k_2} + \cdots).$$

This plane curve has a cusp at the origin, where $t = 0$. The multiplicity sequence is computed from the sequence $m, k_1, k_2, \ldots$ as described in a result of Enriques and Chisini

[1] Theorem 8.4.12. This algorithm is presented below, and the genus contribution δ at $t = 0$ is subsequently computed from the multiplicity sequence.

Algorithm 2.1. *(***Multiplicity sequence.***)*
Consider the strictly increasing sequence

$$m < k_1 < k_2 < \dots$$

Step 1. The **gcd-sequence** *and* **characteristic terms.** *(This step is not necessary to compute the multiplicity sequence, but clarifies the role of the different terms k_i.) Let $g_0 = m$ and $g_i = gcd\{m, k_1, \dots, k_i\}$ for $i > 0$. The g_i form the gcd-sequence of $m, k_1, \dots, k_r$:*

$$g_0 \geq g_1 \geq g_2 \geq g_3 \dots.$$

Clearly, in the gcd-sequence, $g_i = 1$ for some i, since otherwise the parameterization is not $1:1$. The characteristic terms in the sequence $k_1, k_2, \dots$ are the terms

$$k_{i_1}, ..., k_{i_s}$$

where $i_1 = \min\{i|g_i < m\}$, $i_2 = \min\{i|g_i < g_{i_1}\}$ etc. Thus $m = g_0 > g_{i_1}$ and

$$g_{i_1} > ... > g_{i_s} = 1.$$

In particular the number of characteristic terms is finite and bounded by the number of prime factors in m.

Step 2. Given the sequence

$$m < k_1 < k_2 < \dots$$

let $\kappa_i = k_i - k_{i-1}$ where $k_0 = 0$ and $i = 1, 2, \dots.$ We call

$$\kappa_1, \kappa_2, \dots$$

the **difference sequence** *of the cusp. In our applications we will always have a finite number of terms in this sequence, so we assume we have a difference sequence with s terms.*

Apply the Euclidean algorithm successively to the elements of the difference sequence: Let

$$\kappa_i = e_{i,1} r_{i,1} + r_{i,2}$$

$$r_{i,1} = e_{i,2} r_{i,2} + r_{i,3}$$

$$\dots$$

$$r_{i,w(i)-1} = e_{i,w(i)} r_{i,w(i)}$$

with $0 \leq r_{i,j+1} < r_{i,j}$ and $r_{1,1} = m, r_{i,1} = r_{i-1,w(i-1)}, i > 1$. Note that

$$r_{i+1,1} = g_i = \gcd(m, k_1, ..., k_i).$$

The **multiplicity sequence of the sequence** *$\{m < k_1 < k_2 < \dots\}$ is $e_{i,j}$ times the multiplicity $r_{i,j}$, with $1 \leq i \leq s$ and $1 \leq j \leq w(i)$.*

We write the multiplicity sequence in the order it is computed, and with repetitions in stead of the numbers $e_{i,j}$. Note that the overall sequence is nonincreasing. The genus contribution or δ-invariant of the sequence is given by

$$\delta = \sum_{i,j} e_{i,j} \binom{r_{i,j}}{2}.$$

This sum is given a closed form in terms of the original sequence and its gcd-sequence by the following result due to Casas-Alvero:

Proposition 2.2. *Given the sequence*

$$m < k_1 < k_2 < \dots$$

Let

$$g_0 \geq g_1 \geq g_2 \geq g_3 \cdots$$

be its gcd-sequence. Then the δ-invariant of the sequence is

$$\delta = \frac{1}{2}\left(\sum_{i \geq 1} k_i(g_{i-1} - g_i) - m + 1\right).$$

In particular, the δ-invariant depend only on the characteristic terms of the sequence $m, k_1, k_2, \dots$.

Proof. See [2, p. 194, ex. 5.6]. First, we may assume that the difference sequence is finite, say with s terms. From the Euclidean algorithm applied to the elements of the difference sequence we note that

$$\sum_{j=1}^{w(i)} e_{i,j} r_{i,j} = k_i - k_{i-1} + \sum_{j=1}^{w(i)-1} (r_{i,j} - r_{i,j+1})$$

$$= k_i - k_{i-1} + r_{i,1} - r_{i,w(i)} = k_i - k_{i-1} + g_{i-1} - g_i.$$

Secondly,

$$\sum_{j=1}^{w(i)} e_{i,j} r_{i,j}^2 = (k_i - k_{i-1}) r_{i,1} + \sum_{j=1}^{w(i)-1} (r_{i,j} r_{i,j+1} - r_{i,j+1} r_{i,j})$$

$$= (k_i - k_{i-1}) g_{i-1}.$$

Thus

$$\sum_{i=1}^{s} \sum_{j=1}^{w(i)} e_{i,j} r_{i,j} = k_s + g_0 - g_s,$$

$$\sum_{i=1}^{s} \sum_{j=1}^{w(i)} e_{i,j} r_{i,j}^2 = \sum_{i=1}^{s} (k_i - k_{i-1}) g_{i-1}$$

and

$$2\delta = \sum_{i=1}^{s} (k_i - k_{i-1}) g_{i-1} - k_s - m + 1$$

$$= \sum_{i=1}^{s} k_i(g_{i-1} - g_i) - m + 1$$

□

3. The degree of the secant variety of a monomial curve

Let $C \subset \mathbb{P}^r$ be a monomial curve defined by the sequence of positive integers $a_1 < a_2 < \ldots < a_r = d$ as above. Consider the secant variety $SecC$ of C. This is a threefold, so its degree is counted by the intersection of this variety with a general codimension three subspace, or equivalently by the number of ordinary double points of the general projection $\pi : C \to \mathbb{P}^2$. For a general projection the only other singularities on $\overline{C} = \pi(C)$ are possible cusps at the image of the points $\pi(p)$ and $\pi(q)$ where $p = [1 : \ldots : 0]$ and $q = [0 : \ldots : 1]$ in $\mathbb{P}^r$. The formula for the arithmetic genus of a plane curve of degree d and the computation of the genus contribution at these cups provides a formula for the degree of $SecC$.

Proposition 3.1. *Let $C \subset \mathbb{P}^r$ be a monomial curve defined by the sequence of positive integers $a_1 < a_2 < \ldots < a_r = d$. Let $b_i = d - a_{r-i}$, for $i = 1, ..., r-1$ and $b_r = d$. Let $g_i = gcd(a_1, ..., a_i)$ and $h_i = gcd(b_1, ..., b_i)$, then*

$$\deg SecC = \binom{d-1}{2} - \frac{1}{2}(\sum_i a_{i+1}(g_i - g_{i+1}) - a_1 + \sum_i b_{i+1}(h_i - h_{i+1}) - b_1) - 1.$$

Proof. The arithmetic genus $p(C)$ for a curve C on a smooth surface S is given by the adjunction formula [3] on the surface:

$$2p(C) - 2 = C \cdot C + C \cdot K_S$$

where K_S is the canonical divisor on S. If C has multiplicity m at a point q on S, and $S' \to S$ is the blowup of S at q, then the adjunction formula on S' says

$$2p(C') - 2 = C' \cdot C' + C' \cdot K_{S'} =$$

$$= (C^* - mE) \cdot (C^* - mE) + (C^* - mE) \cdot (K_S + E) = 2p(C) - 2 - m^2 + m$$

where E is the exceptional divisor and C^* is the total transform and C' is the strict transform of C (cf. [3] chapter V). Thus

$$p(C') = p(C) - \binom{m}{2},$$

so $\binom{m}{2}$ is the genus contribution of a point of multiplicity m. After resolving all singularities on $\overline{C} \subset \mathbb{P}^2$ by a series of blow ups centered at singular points of $\overline{C}$ or its strict transform, the arithmetic genus of the strict transform C' is 0 since it is a rational curve. At the ordinary double points the difference between the arithmetic genus of the curve and its strict transform after blowing up the point is $\binom{2}{2} = 1$. The points $\pi(p)$ and $\pi(q)$ are the only other singularities on $\overline{C}$. The contribution δ_p is by definition the difference between the arithmetic genus of $\overline{C}$ and a strict transform that is smooth at the inverse image of $\pi(p)$ and isomorphic to $\overline{C}$ outside the point $\pi(p)$. Likewise for δ_q. Since $K_{\mathbb{P}^2} \cong -3L$, where

L is a line in the plane, the arithmetic genus of $\overline{C}$ is given by $2p(\overline{C})-2=d_C(d_C-3)$, i.e. $p(\overline{C})=\binom{d-1}{2}$. Adding all genus contributions we get the formula:

$$\deg SecC=\binom{d-1}{2}-\delta_p-\delta_q,$$

where $\delta_p=\sum\binom{m_i-1}{2}$, $\delta_q=\sum\binom{n_i-1}{2}$ and $\{m_1,m_2,\ldots\}$ and $\{n_1,n_2,\ldots\}$ are the multiplicity sequences of $\overline{C}$ at $\pi(p)$ and $\pi(q)$ respectively.

The algorithm 2.1 computes these multiplicity sequences from the exponents of the Puiseux expansion. By Proposition 2.2 it is enough for this algorithm to know the characteristic terms of the Puiseux expansion. Therefore we need only to find these terms of the Puiseux expansion of $\overline{C}\subset\mathbb{P}^2$ at $\pi(p)$ and $\pi(q)$. The projection $\pi:C\to\mathbb{P}^2$ is determined by the choice of three projective coordinates $(X:Y:Z)$ in $\mathbb{P}^r$, two of which, say X,Y vanish at p, and two, say X,Z, vanish at q. In particular in terms of the parameterization of $\overline{C}\subset\mathbb{P}^2$ we may choose

$$X=t^{a_1}+b_{13}t^{a_3}+...+b_{1(r-1)}t^{a_{r-1}},$$

$$Y=t^{a_2}+b_{23}t^{a_3}+...+b_{2(r-2)}t^{a_{r-2}}+t^{a_r},$$

$$Z=1+b_{32}t^{a_2}+b_{33}t^{a_3}+...+b_{3(r-2)}t^{a_{r-2}}.$$

The coefficients b_{ij} are independant and determine the projection. Clearly the characteristic terms are determined by the projection, and therefore by the b_{ij}. The meaning of "general" in general projection is that the characteristic terms are constant for an open set of choices of coefficients b_{ij}. The following lemma says that a particularly simple choice of coefficients belong to this open set, so that the characteristic terms can be computed from this choice.

Lemma 3.2. *The characteristic terms in the Puiseux expansion of $\overline{C}$ at p for a general projection coincides with the characteristic terms in the Puiseux expansion*

$$x=t^{a_1},y=t^{a_2}+t^{a_3}+...+t^{a_r}.$$

Proof. The characteristic terms of the latter Puiseux expansion is precisely the characteristic terms of the sequence $a_1,a_2,\ldots,a_r$ as computed by the algorithm 2.1. So we compare these characteristic terms with those in a Puiseux expansion of a curve parameterized by

$$X=t^{a_1}+b_{13}t^{a_3}+...+b_{1(r-1)}t^{a_{r-1}},\quad Y=t^{a_2}+b_{23}t^{a_3}+...+b_{2(r-2)}t^{a_{r-2}}+t^{a_r}.$$

This Puiseux expansion is computed by a formal quotient $\frac{X}{Y}$ and has the form

$$t^{\frac{\beta_1}{\alpha}}+c_2t^{\frac{\beta_2}{\alpha}}+\ldots c_nt^{\frac{\beta_n}{\alpha}}\ldots.$$

For our purposes it can also be done step by step, by iterated reparameterizations substituting t with $t+ut^k$ for suitable u and k. We want to compare the characteristic terms of the sequence $\alpha,\beta_1,\beta_2,...$ with those of the sequence $a_1,a_2,\ldots,a_r$.

Since the characteristic terms are finite in number there is a largest one, say N_0. Clearly then the curve parameterized by

$$X=t^{\alpha}+b_Nt^N+\ldots,\quad Y=t^{\beta_1}+b_2t^{\beta_2}+...,$$

with $N \geq N_0 + \alpha$, has the same characteristic terms as $\overline{C}$. So it is enough to find a reparameterization of this kind. We do this step by step and reparameterize $\overline{C}$ by substituting t with $t + ut^{a_3 - a_1 + 1}$ for suitable u to cancel the coefficient of t^{a_3} in X. In the new parameterization we get:

$$X = t^{a_1} + b'_{14}t^{a'_4} + ... + b'_{1r}t^{a'_{r'}}, \quad Y = t^{a_2} + b_{23}t^{a_3} + ... + b_{2r}t^{a_r} + c_1 t^{b_1} + ...$$

where $a'_4 > a_3$, and all new exponents appearing are of the form $a_i + k(a_3 - a_1)$ for some positive integer k. Compare the greatest common divisors $g_i, i = 1, 2, ...$ of a_1 and the i lowest exponents of t occurring in Y, before and after the reparameterization. The difference is a possible repetition of some terms, and some possible cancellations. We similarly reparameterize until the second exponent of t in the X-coordinate is bigger than $N_0 + a_1$ and conclude that the characteristic terms of sequence

$$a_1, a_2, \ldots, a_r$$

include the characteristic terms of the general projection $\overline{C}$. But specialization can only result in fewer characteristic terms, so the inclusion must be an equality.

□

Since non-characteristic terms do not contribute to the δ-invariant the proposition follows from Proposition 2.2. □

Example 3.3. Consider the monomial curve C given by the sequence $(0, 30, 45, 55, 78)$. At $p = [1 : 0]$, we may compute the δ-invariant from the Puiseux expansion with exponents $m = 30$, $(a_3, a_4, a_5) = (45, 55, 78)$ The gcd-sequence is $(30, 15, 5, 1)$ and the δ-invariant is

$$\delta_p = \frac{1}{2}(45(30 - 15) + 55(15 - 5) + 78(5 - 1) - 30 + 1) = 754.$$

At $q = [0 : 1]$ we compute the δ-invariant from the Puiseux expansion with exponents $m = 23$ and $(a_3, a_4, a_5) = (33, 48, 78)$. Since m is prime and coprime to 33, the only characteristic term is 33 with gcd-sequence $(23, 1)$. The δ-invariant is

$$\delta_q = \frac{1}{2}(33(23 - 1) - 23 + 1) = 352$$

The degree of the secant variety of C is

$$\deg SecC = \binom{77}{2} - \delta_p - \delta_q = 2926 - 754 - 352 = 1820$$

4. The degree of the join of two monomial curves

Consider the join of two monomial curves C and D in $\mathbb{P}^r$ defined by

$$C : [s : t] \mapsto [s^{d_C} : s^{d_C - a_1}t^{a_1} : \ldots : s^{d_C - a_{r-1}}t^{a_{r-1}} : t^{d_C}]$$

where $a_1 < a_2 < \ldots < a_r = d_C$, and

$$D : [s : t] \mapsto [s^{d_D} : s^{d_D - b_1}t^{b_1} : \ldots : s^{d_D - b_{r-1}}t^{b_{r-1}} : t^{d_D}]$$

where $b_1 < b_2 < \ldots < b_r = d_D$. The two sequences

$$a_1 < a_2 < \ldots < a_r = d_C, \quad b_1 < b_2 < \ldots < b_r = d_D$$

therefore determine the two curves completely. For the parameterizations to be $1-1$ onto the image, we ask that the a_i have no common factor, and likewise for the b_i. The join is a threefold, so its degree coincides with the number of lines meeting the two curves in distinct points that also meet a given codimension 3 linear space L in $\mathbb{P}^r$. But this number equals the number of new intersection points obtained by projecting the union of the two curves from L to a plane. Denote by π_L the projection from L, and let $\overline{C} = \pi_L(C)$ and $\overline{D} = \pi_L(D)$ be the images of C and D respectively. The total intersection number

$$\overline{C} \cdot \overline{D} = d_C \cdot d_D$$

by Bezout's theorem, so to get the degree we have to subtract the intersection multiplicity at the points of $\pi_L(C \cap D)$. In our special situation there certainly are points in $C \cap D$:

$$\{p = [1 : 0 \ldots : 0], q = [0 : \cdots : 0 : 1], u = [1 : \ldots : 1]\} \subset C \cap D.$$

There may be more:

Lemma 4.1. *Let $p_{ij} = a_i b_j - a_j b_i \quad 1 \leq i < j \leq r$, and let $g = gcd\{p_{ij})|1 \leq i < j \leq r\}$, then C and D intersect in exactly g points beside p and q, and the intersection in these g points is transversal. For a general plane projection π_L the total intersection multiplicity of $\overline{C}$ and $\overline{D}$ at the points $\pi_L(C \cap D \setminus \{p,q\})$ is g.*

Proof. The intersection points $C \cap D$ are, besides p and q, precisely the solutions (t_1, t_2) to the equations

$$t_1^{a_i} = t_2^{b_i} \quad i = 1, \ldots, r.$$

To find these we first consider the equations of the absolute values. Since $t_i \neq 0$, these are real positive numbers, so we may take logarithms and get a system of linear equations

$$a_i log(|t_1|) = b_i log(|t_2|) \quad i = 1, \ldots, r.$$

Now,

$$gcd(a_1, \ldots, a_r) = gcd(b_1, \ldots, b_r) = 1$$

and a_i is different from b_i for some i, so there is at least one nonzero p_{ij}. The corresponding pair of homogeneous equations of logarithms has a unique solution, i.e. only the zero-solution, in particular $|t_1| = |t_2| = 1$.

Therefore we may write $t_1 = exp(2\pi i x) = e^{2\pi i x}$ and $t_2 = exp(2\pi i y) = e^{2\pi i y}$, and the equations translate into the linear conditions

$$a_i x - b_i y \in \mathbb{Z} \quad i = 1, \ldots, r$$

Again, one of the p_{ij} must be non-zero, and xp_{ij} and yp_{ij} are integers. In fact, since $g = gcd\{p_{ij})|1 \leq i < j \leq r\}$, the real numbers xg and yg must be integers. Therefore $a_i xg - b_i yg$ is an integer divisible by g, so we set $X = xg$ and $Y = yg$ and have reduced the above equations to the modular equations

$$A_i X - B_i Y \equiv 0 \quad (g) \quad i = 1, \ldots, r,$$

where $A_i \equiv a_i \quad (g)$ and $B_i \equiv b_i \quad (g)$. If $A_i \equiv 0 \quad (g)$, then $p_{ij} \equiv B_i A_j \equiv 0 \quad (g), j \neq i$. Let $g_i = gcd(b_i, g)$, then $\frac{g}{g_i}|a_j \quad j = 1, ..., r$. But $gcd(a_1, \ldots, a_r) = 1$, so $\frac{g}{g_i} = 1$ and $g|b_i$. By symmetri we get $A_i \equiv 0 \quad (g)$ if only if $B_i \equiv 0 \quad (g)$.

Since $gcd(a_1, \ldots, a_r) = 1$ there is an integral combination of these equations on the form: $X \equiv dY \quad (g)$, for some d. The above argument applies again to show that d is nonzero. Now, any of the pairs $(X_\alpha, Y_\alpha) = (-d\alpha, \alpha)$ $\alpha = 1, \dots, g$ is a solution to this equation. Since all equations are proportional modulo g these are the solutions to all r equations. Consequently, the pairs

$$(exp(\frac{2\pi i X_\alpha}{g}), exp(\frac{2\pi i Y_\alpha}{g})), \quad \alpha = 1, ..., g$$

are the solutions of the original equations. For transversality, we need to check that the tangent directions of the two curves at intersection points are distinct. At the intersection points the absolute value of each coordinate is 1, so taking the absolute value of the tangent directions at these points we get $(1, a_1, ..., a_r)$ and $(1, b_1, ..., b_r)$. But these are clearly distinct by assumption, so transversality follows. Thus at every point of intersection the two tangents span a plane. A general codimension 3 subspace $L \subset \mathbb{P}^r$ does not intersect any of these planes, so the intersections remain transversal after the projection π_L to a plane. At each point of $\pi_L(C \cap D \setminus \{p, q\})$ the intersection multiplicity is 1, so they add up to g for the g points. □

For the points $\pi_L(p)$ and $\pi_L(q)$ the intersection multiplicity is at least two, since the two curves have the same tangent(cone) at those points. In fact, since the curves are unibranched, there is a unique tangent direction at the point, i.e. if they are singular they have a cusp there. The intersection multiplicity at these points is determined by a procedure similar to the one given in the previous section. More precisely consider say the point $\pi_L(p)$. Blow it up and let p_1 be the common intersection point of the strict transforms of the two curves on the exceptional divisor. There is a unique such intersection point since the two curves are unibranched and the tangents to $\overline{C}$ and $\overline{D}$ at $\pi_L(p)$ coincide. Now blow up in the point p_1. If the strict transforms meet on the new exceptional divisor, then denote it by p_2 and blow up in this point. Continue, until the strict transforms do not intersect on the exceptional divisor. Thus we get a finite sequence $p_0 = \pi_L(p), p_1, \dots, p_k$, and together with it the multiplicities of the strict transforms of the two curves at each p_i. We denote these multiplicity sequences by $m_0(C), \dots, m_k(C)$ and $m_0(D), \dots, m_k(D)$. The intersection multiplicity between the two curves at the point $\pi_L(p)$ is

$$I_{\pi_L(p)}(\overline{C}, \overline{D}) = \sum_{i=0}^{k} m_i(C) m_i(D).$$

The multiplicity sequences $m_0(C), \dots, m_k(C)$ and $m_0(D), \dots, m_k(D)$ are decreasing and similar to the multiplicity sequences constructed in the previous section. There are however a main difference in that the we need to consider nonminimal multiplicity sequences, i.e. sequences that contain 1's since these terms contribute to the intersection multiplicity, while they do not contribute to the δ-invariant. Because of the unibranch property these 1's will only be appear at the end of the sequences though.

The problem is how to compute these sequences from sequences $a_1, ..., a_r$ and $b_1, ..., b_r$ of the curves C and D. In this case the non-characteristic terms are as important as the characteristic ones, since the intersection point of the strict transforms with the exceptional divisor is crucial. Some special cases may illustrate the issue:

Example 4.2. Consider monomial curves $C : (1, 2, 3, 4)$ and $D : (1, 2, 3, 5)$. Then the two curves separate over $\pi_L(p)$ after four blowups and the multiplicity sequences are $m_i(C)$: $1, 1, 1, 1$ and $m_i(D) : 1, 1, 1, 1$. The intersection multiplicity at $\pi_L(p)$ is $1 + 1 + 1 + 1 = 4$.

Example 4.3. The monomial curves $C : (1, 2, 3, 4)$ and $D : (2, 4, 6, 9)$ separate after four blowups starting at $\pi_L(p)$, the multiplicity sequences are $m_i(C) : 1, 1, 1, 1$ and $m_i(D)$: $2, 2, 2, 2$. The intersection multiplicity at $\pi_L(p)$ is $2 + 2 + 2 + 2 = 8$.

Example 4.4. For $C : (1, 2, 3, 4)$ and $D : (b_0, b_1, b_2, b_3)$ where $b_0 > 1$ and $b_1 \neq 2b_0$, then the strict transforms over $\pi_L(p)$ separate after two blowups and the multiplicities that contribute to the intersection multiplicity are $m_i(C) : 1, 1$ and $m_i(D) : b_0, min\{(b_1 - b_0), b_0\}$. The intersection multiplicity at $\pi_L(p)$ is $min\{b_1, 2b_0\}$.

With these examples in mind we formulate the algorithm computing the degree of the join.

Algorithm 4.5. (Intersection multiplicity algorithm I) *Given two monomial curves C and D defined by the sequences*

$$a_1 < a_2 < \ldots < a_r = d_C, \quad b_1 < b_2 < \ldots < b_r = d_D$$

respectively, and assume that for some $j \geq 1$ $b_i \geq a_i$ for $i < j$ while $b_j > a_j$. The following two steps computes the intersection multiplicity of the general projection π of the two curves to a plane in the point $\pi([1 : 0 : ... : 0])$.

Step 1. Let $\alpha = \frac{b_1}{a_1}$. If α is not an integer, then set $k = 0$, otherwise let

$$k = max\{i|b_i = \alpha a_i\}$$

If $k \geq 2$, let $m_1, m_2, \ldots, m_s$ be the multiplicity sequence, the outcome of step 2 of the algorithm 2.1, of the sequence $(a_1, a_2, \ldots, a_k)$, and set

$$\delta_k = \alpha(m_1^2 + \cdots + m_s^2).$$

If $k < 2$, set $\delta_k = 0$.

Step 2. If $k < 2$, apply step 2 of the algorithm 2.1 to the sequences (a_1, a_2) and (b_1, b_2), with outcome

$$(e_1, r_1), (e_2, r_2), ..., (e_m, r_m) \quad \text{and} \quad (e'_1, r'_1), ..., (e'_n, r'_n)$$

respectively.

If $k \geq 2$, let $g = \gcd(a_1, ..., a_k)$, and apply the multiplicity algorithm in section 2.1 to the sequences $(g, a_{k+1} - a_k)$ and $(g\alpha, b_{k+1} - b_k)$, with outcome

$$(e_1, r_1), (e_2, r_2), ..., (e_m, r_m) \quad \text{and} \quad (e'_1, r'_1, ..., (e'_n, r'_n)$$

respectively.

Let $l = \max\{j|e_i = e'_i, i = 1, \ldots, j\}$ and let

$$\epsilon = \sum_j^l e_j \cdot r_j r'_j + e'_{l+1} r_{l+1} r'_{l+1} + r_{l+1} r'_{l+2},$$

if $e'_{l+1} < e_{l+1}$ *and*

$$\epsilon = \sum_{j}^{l} e_j \cdot r_j r'_j + e_{l+1} r_{l+1} r'_{l+1} + r_{l+2} r'_{l+1},$$

if $e_{l+1} < e'_{l+1}$.

Proposition 4.6. *The intersection multiplicity between the curves* $\pi(C)$ *and* $\pi(D)$ *at* $\pi([1 : 0... : 0])$ *is*

$$I(C, D) = \delta_k + \epsilon.$$

Proof. To start we project C and D into the plane and may choose coordinates such that $\pi(C)$ and $\pi(D)$ have the parameterizations

$$\pi(C) : x = t^{a_1} + c_{1,3}t^{a_3} + ... + c_{1,r}t^{a_r}, y = t^{a_2} + c_{2,3}t^{a_3} + ... + c_{2,r}t^{a_r}$$

and

$$\pi(D) : x = t^{b_1} + c_{1,3}t^{b_3} + ... + c_{1,r}t^{b_r}, y = t^{b_2} + c_{2,3}t^{b_3} + ... + c_{2,r}t^{b_r}.$$

By assumption $a_1 < a_2$ and $b_1 < b_2$, so both curves are tangent along the x-axis. Now, we blow up the plane in the origin. The strict transforms of these curves on the blowup intersect the exceptional curve in the x-chart (with coordinates (x, xy)). In this chart the strict transforms $\pi(C)'$ and $\pi(D)'$ have local parameterizations:

$$\pi(C)' : x = t^{a_1} + c_{1,3}t^{a_3} + ... + c_{1,r}t^{a_r}, y = t^{a_2-a_1} + c_{2,3}t^{a_3-a_1} + ... + c_{2,r}t^{a_r-a_1} - c_{1,3}t^{a_2+a_3-2a_1} + \dots$$

and

$$\pi(D)' : x = t^{b_1} + c_{1,3}t^{b_3} + ... + c_{1,r}t^{b_r}, y = t^{b_2-b_1} + c_{2,3}t^{b_3-b_1} + ... + c_{2,r}t^{b_r-b_1} - c_{1,3}t^{b_2+b_3-2b_1} + \dots.$$

The tangent at the origin is $y = 0$ if $a_1 < a_2 - a_1$, it is $x = 0$ if $a_1 - a_2 < a_1$ and it is $x = y$ if $a_1 = a_2 - a_1$.

Notice, that the terms of order less than $a_k - a_1$ and $b_k - b_1$ respectively, have the same coefficients and differ only in the exponent by the factor α. Therefore, if $k > 0$ the two curves $\pi(C)'$ and $\pi(D)'$ have the same tangent direction at the origin, and their strict transform on the blow up in the origin intersect. Proceeding we need to know after how many blowups, the strict transforms does not intersect, and keep track of the multiplicities of the two strict transforms up to that point. Computing the number of blowups needed to separate the two curves, comes down to keeping track of first terms of the parametrizations of the strict transforms after successive blowups. The tangent direction decides the parametrization of the strict transform: If the tangent direction is $y = 0$ then the strict transform is parametrized by $x, \frac{y}{x}$, if the tangent direction is $x = 0$, then the strict transform is parametrized by $\frac{x}{y}, y$, and if the tangent direction is $x = y$, then the strict transform is parametrized by $x, \frac{y-x}{x}$. Now, the multiplicities of the strict transforms at the origin form the multiplicity sequence obtained by step 2 of the algorithm 2.1, but keeping track of the tangent directions at each point, we actually also control the intersection between the two curves. The change from $y = 0$ to $x = 0$ of tangent direction correspond to going from (i, j) to $(i, j+1)$ in the Euclidean algorithm, while the third kind of tangent corresponds to going from $(i, w(i))$ to (i+1,1) or to non-characteristic terms. In this algorithm, as long as $i \leq k$, the leading terms of the parametrizations differ only by a factor of t^α. So the corresponding tangent directions coincide. Assume first $k \geq 2$. If

$i = k+1$ and $j = 1$ we have parametrizations $t^g + ..., t^{a_{k+1}-a_k} + ...$ and $t^{\alpha g} + ..., t^{b_{k+1}-b_k} + ...$ To see when these two curves separate, we apply again the Euclidean algorithm to the pairs $(g, a_{k+1} - a_k)$ and $(g\alpha, b_{k+1} - b_k)$ and get

$$a_{k+1} - a_k = e_1 r_1 (= g) + r_2 \dots r_m = e_m r_{m+1}$$

and

$$b_{k+1} - b_k = e'_1 r'_1 (= \alpha g) + r'_2 \dots r'_n = e'_n r'_{n+1}$$

So here we compare the coefficients e_i and e'_i. The curves split after

$$e_1 + e_2 + ... + e_l + \min\{e_{l+1}, e'_{l+1}\} + 1$$

blowups if $e_i = e'_i$ for $i \leq l$, while $e_{l+1}, \neq e'_{l+1}\}$. In fact, the above argument says that after $e_1 + e_2 + ... + e_l + \min\{e_{l+1}, e'_{l+1}\}$ blowups, the tangent directions of the two strict transforms still coincides, while after one more blowup they do not, and after two more blowups the two strict transforms separate.

If $k < 2$ the number s of blowups needed to separate the two curves is determined by the initial pairs of exponents (a_1, a_2) and (b_1, b_2) and the above procedure starting with these pairs in stead of $(g, a_{k+1} - a_k)$ and $(g\alpha, b_{k+1} - b_k)$ clearly determines s. □

The intersection multiplicity algorithm may be shortened.

Lemma 4.7. *Let (g, h) and (g', h') be pairs of integers and consider the Euclidean algorithm applied to each pair:*

$$h = e_1 r_1 + r_2 \dots r_m = e_m r_{m+1}$$

and

$$h' = e'_1 r'_1 + r'_2 \dots r'_n = e'_n r'_{n+1}$$

where $r_1 = g$ and $r'_1 = g'$. Let $l = \max\{j | e_i = e'_i, i = 1, \dots, j\}$ and set

$$\epsilon = \begin{cases} \sum_j^l e_j \cdot r_j r'_j + e'_{l+1} r_{l+1} r'_{l+1} + r_{l+1} r'_{l+2} & \text{if } e'_{l+1} < e_{l+1} \\ \sum_j^l e_j \cdot r_j r'_j + e_{l+1} r_{l+1} r'_{l+1} + r_{l+2} r'_{l+1} & \text{if } e_{l+1} < e'_{l+1}. \end{cases} \tag{4.1}$$

Then

$$\epsilon = \begin{cases} gh' & \text{if } l \text{ is even and } e'_{l+1} < e_{l+1}, \text{ or } l \text{ is odd and } e'_{l+1} > e_{l+1} \\ g'h & \text{if } l \text{ is odd and } e'_{l+1} < e_{l+1}, \text{ or } l \text{ is even and } e'_{l+1} > e_{l+1}. \end{cases} \tag{4.2}$$

Proof. Rewrite the sum $\sum_j^l e_j \cdot r_j r'_j$ using the identity $(r_{i-1} - r_{i+1}) r'_i = e_i r_i r'_i$ or $(r'_{i-1} - r'_{i+1}) r_i = e'_i r'_i r_i$ from the Euclidean algorithm. □

Algorithm 4.8. (Intersection multiplicity II) *Given two monomial curves C and D defined by the sequences*

$$A : a_1 < a_2 < \dots < a_r = d_C, \quad B : b_1 < b_2 < \dots < b_r = d_D$$

respectively. Let $\alpha = \frac{b_1}{a_1}$. If α is not an integer, then set $k = 0$, otherwise let

$$k = max\{i | b_i = \alpha a_i\}$$

If $k < 2$, consider the Euclidean algorithm applied to the pairs (a_1, a_2) and (b_1, b_2), with factors and residues

$$(e_1, r_1(= a_1)), (e_2, r_2), ..., (e_m, r_m) \quad \text{and} \quad (e'_1, r'_1(= b_1)), ..., (e'_n, r'_n)$$

respectively. Let $l = \max\{j | e_i = e'_i, i = 1, \ldots, j\}$.

Set

$$\epsilon = \begin{cases} a_1 b_2 & \text{if } l \text{ is even and } e'_{l+1} < e_{l+1} \text{ or } l \text{ is odd and } e_{l+1} < e'_{l+1} \\ a_2 b_1 & \text{if } l \text{ is even and } e'_{l+1} > e_{l+1} \text{ or } l \text{ is odd and } e_{l+1} > e'_{l+1}. \end{cases} \tag{4.3}$$

If $k \geq 2$, let

$$g_1 = a_1, \quad g_i = gcd\{a_1, ..., a_i\} \quad i = 2, \ldots, k$$

and consider the Euclidean algorithm applied to the pairs $(g_k, a_{k+1} - a_k)$ and $(\alpha g_k, b_{k+1} - b_k)$, with factors and residues

$$(e_1, r_1(= g_k)), (e_2, r_2), ..., (e_m, r_m) \quad \text{and} \quad (e'_1, r'_1(= \alpha g_k)), ..., (e'_n, r'_n)$$

respectively. Let $l = \max\{j | e_i = e'_i, i = 1, \ldots, j\}$.

Set

$$\epsilon = \begin{cases} g_k(b_{k+1} - b_k) & \text{if } l \text{ is even and } e'_{l+1} < e_{l+1} \text{ or } l \text{ is odd and } e'_{l+1} > e_{l+1} \\ \alpha g_k(a_{k+1} - a_k) & \text{if } l \text{ is even and } e'_{l+1} > e_{l+1} \text{ or } l \text{ is odd and } e_{l+1} > e'_{l+1}. \end{cases} \tag{4.4}$$

Then the intersection multiplicity of the general plane projection of the two curves C and D, given by the sequences A and B respectively, at the image of $p = [1 : 0... : 0]$ is

$$I(A, B) = \begin{cases} \epsilon & \text{if } k < 2 \\ \alpha(\sum_{i=2}^{k-1} a_i(g_{i-1} - g_i) + a_k g_{k-1}) + \epsilon & \text{if } k \geq 2. \end{cases} \tag{4.5}$$

Proof. The proof follows the argument of [2, p. 194, ex. 5.6] applied to the algorithm 4.5. According to Proposition 4.6 the intersection multiplicity at $\pi([1 : 0... : 0])$ is

$$I(C, D) = \alpha(m_1^2 + \cdots + m_s^2) + \epsilon,$$

where $m_1, \ldots, m_s$ is the multiplicity sequence of the sequence $\{a_1 < ... < a_k\}$. The sum

$$\sum_{i=1}^{s} \alpha m_i^2 = \alpha \sum_{i=1}^{s} m_i^2$$

may now be computed as in the proof of Proposition 2.2, i.e.

$$\alpha \sum_{i=1}^{s} m_i^2 = \alpha a_2 g_1 + \alpha \sum_{i=2}^{k-1} a_{i+1} - a_i) g_i = \alpha(\sum_{i=1}^{k-1} (a_{i+1}(g_i - g_{i+1}) + a_k g_{k-1}).$$

For the residual contribution ϵ we apply the Lemma 4.7. So the intersection multiplicity $I(C, D)$ coincides with $I(A, B)$. $\square$

Using 4.8 to compute $I(A, B)$ for a pair of sequences A and B we conclude:

Proposition 4.9. *Given two monomial curves C and D defined by the sequences*

$$A : a_1 < a_2 < \ldots < a_r = d_C, \quad B : b_1 < b_2 < \ldots < b_r = d_D$$

respectively. Set

$$A' : a_r - a_{r-1} < a_r - a_{r-2} < \ldots < a_r - a_1 < d_C, \quad B' : b_r - b_{r-1} < b_r - b_{r-2} < \ldots < b_r - b_1 < d_D,$$

and let $g = gcd\{a_i b_j - a_j b_i | 1 \leq i < j \leq r\}$. Then the degree of the join of C and D is

$$\deg Join(C, D) = d_C \cdot d_D - I(A, B) - I(A', B') - g.$$

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