On Smooth Surfaces of Degree 10 in the Projective Fourspace

Kristian Ranestad

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0 Introduction

The classification of smooth surfaces with small invariants has received renewed interest in recent years. This is primarily due to the finer study of the adjunction mapping by Reider, Sommese and Van de Ven, which provides an effective tool in the case of rational and birationally ruled surfaces. In the special case of surfaces in $\mathbf{P}^4$, where smoothness imposes additional relations among the invariants of the surface, an almost complete classification of smooth surfaces of degree less than ten has been worked out (see list and references below). This paper is the result of an attempt to give a classification of smooth surfaces of degree 10 in $\mathbf{P}^4$.

Some surfaces of degree 10 are well-known; namely the abelian surface, and surfaces linked to smooth surfaces of a lower degree, in particular the complete intersection of a quadric and a quintic hypersurface. The main result of this paper is the description of the following list of surfaces.

A) Given nine points $x_1, \ldots, x_9$ in general position in $\mathbf{F}_e$, $\quad e \leq 2$, one can choose three points y_1, y_2, y_3 such that if

$$\pi : S \to F_e$$

is the blowing-up of $\mathbf{F}_e$ in the points $x_1, \ldots, x_9,\ y_1, y_2, y_3$ and $E_1, \ldots, E_9,\ F_1, F_2, F_3$ are the exceptional divisors and B (resp. F) is a section with selfintersection e (resp. a ruling), then the linear system

$$|H_S| = |8\pi^*B + (10-4e)\pi^*F - \sum_{i=1}^{9} 4E_i - 2F_1 - F_2 - F_3|$$

is very ample and embeds S as a surface of degree 10 in $\mathbf{P}^4$. The points y_i are chosen such that there are two curves

$$C \equiv 4B + (5-2e)F - \sum_{i=1}^{9} 2x_i$$

and

$$D \equiv 6B + (7-3e)F - \sum_{i=1}^{9} 3x_i$$

which have a common tangent at a point y_1, and a transversal intersection at points y_2 and y_3.

B) Given twelve points $x_1, \ldots, x_{12}$ in general position in $\mathbf{P}^2$, one can choose six other points $y_1, \ldots, y_6$ such that if

$$\pi : S \to \mathbf{P}^2$$

is the blowing-up of $\mathbf{P}^2$ in the points $x_1, \ldots, x_{12},\ y_1, \ldots, y_6$ and $E_1, \ldots, E_{12},\ F_1, \ldots, F_6$ are the exceptional divisors then the linear system

$$|H_S| = |8\pi^*l - \sum_{i=1}^{12} 2E_i - \sum_{j=1}^{6} F_j|$$

is very ample and embeds S as a surface of degree 10 in $\mathbf{P}^4$. We may describe the choice of the points y_i as follows. The linear system

$$|D| = |4\pi^*l - \sum_{i=1}^{12} E_i|$$

of curves on S defines a morphism φ_D of degree four onto $\mathbf{P}^2$. The images of the F_i are three points n_j, such that $n_1 = \varphi_D(F_1) = \varphi_D(F_2)$ and $n_2 = \varphi_D(F_3) = \varphi_D(F_4)$ and $n_3 = \varphi_D(F_5) = \varphi_D(F_6)$. Furthermore, there is a curve $L \in |\pi^*l|$ which does not meet any of the F_i, but whose image $\varphi_D(L) \subset \mathbf{P}^2$ has three nodes at the points n_i.

C) Given a cubic hypersurface in $\mathbf{P}^4$ with an isolated quadratic singularity at a point x and a smooth quadric hypersurface which meets this singularity such that their complete intersection S_0 has a quadratic singularity at x and is smooth elsewhere. Let H be a general hyperplane section of S_0 and let Π be a plane in $\mathbf{P}^4$ which is tangent to H in three distinct points y_1, y_2, y_3. Let

$$\pi : S \to S_0$$

be the blowing-up of S_0 in the point x and the points y_i, with exceptional divisors A and E_i respectively. If C_0 denotes the total transform of a hyperplane section of S_0 on S, then the linear system

$$|H| = |2C_0 - A - \sum_{i=1}^{3} 2E_i|$$

on S is very ample and embeds S as a smooth surface of degree 10 in $\mathbf{P}^4$.

D) Let C be a plane cubic curve which meets the six nodes of four lines in the plane. Let $T \subset \mathbf{P}^3$ be a cone over C with vertex at a point p. Let D be an irreducible curve on T which has six branches through p, which are tangent to the lines connecting p and the six nodes of the four lines in the plane. Outside p, D meets any line of the ruling only once. Let

$$\pi : V \to \mathbf{P}^3$$

be the blowing-up of $\mathbf{P}^3$, first in the point p, and secondly in the strict transform of the curve D. The strict transform on V of quartic surfaces in $\mathbf{P}^3$ with triple point at the point p and which contains the curve D, form a linear system $|\Sigma|$ of divisors on V of degree 7 and projective dimension 6. Let S_0 denote the strict transform on V of a surface of degree 7 in $\mathbf{P}^3$ with quartuple point at the point p, with D as a double curve, and which is smooth elsewhere. The linear system $|\Sigma|$ has degree 13 on S_0 and maps S_0 onto a smooth surface S_1 in $\mathbf{P}^6$. The surface S_1 has a twodimensional family of trisecants, coming from the lines in $\mathbf{P}^3$ through the point p. Projecting S_1 from a general trisecant, we get an elliptic smooth surface S of degree 10 in $\mathbf{P}^4$ with three (-1)-lines and numerical invariants $p_g = 1$ and $q = 0$ and $K^2 = -3$.

E) There is a minimal smooth surface S with numerical invariants $p_g = 2$, $q = 0$, $K^2 = 3$ and exactly one irreducible (-2)-curve A, for which the linear system

$$|H| = |2K - A|$$

is very ample and embeds S as a smooth surface of degree 10 in $\mathbf{P}^4$.

F) Let T be the union of a smooth quartic Del Pezzo surface T_1 and a smooth quadric surface T_2, in the following way: Let $E_1 + E_2 + F_1 + F_2$ be one of the hyperplane sections of T_1 which consists of 4 exceptional lines, such that $E_1 \cdot E_2 = F_1 \cdot F_2 = 0$. Next let T_2 be a smooth quadric surface in the corresponding hyperplane such that F_1 and F_2 are members of one of the rulings of T_2. Then T is linked to a smooth surface S of degree 10 and $\pi = 10$ in the intersection of two quartic hypersurfaces, and S is an elliptic surface with two exceptional lines and invariants $p_g = 2$, $q = 0$ and $K^2 = -2$.

G) There is a minimal smooth surface S with numerical invariants $p_g = 3$, $q = 0$, $K^2 = 4$ and exactly three irreducible (-2)-curves A_1, A_2, A_3, for which the linear system

$$|H| = |2K - A_1 - A_2 - A_3|$$

is very ample and embeds S as a smooth surface of degree 10 in $\mathbf{P}^4$.

In terms of a classification I give the following result:

Theorem 0.1. *If S is a smooth surface of degree 10 in $\mathbf{P}^4$ and π denotes the genus of a general hyperplane section, then*

$\pi = 6$ and S is abelian or hyperelliptic, or

$\pi = 8$ and S is an Enriques surface with four (-1)-lines or a rational surface of type A, or

$\pi = 9$ and S is a rational surface (type B is an example), a K3-surface (type C is an example), an elliptic surface (type D is an example), or a surface of type E, or

$\pi = 10$ and S is an elliptic surface (type F is an example) or a surface of type G, or

$\pi = 11$ and S is linked to an elliptic quintic scroll (S lies on a cubic hypersurface) or S is linked to a Bordiga surface (S does not lie on a cubic hypersurface), or

$\pi = 12$ and S is linked to a degenerate quadric surface, or

$\pi = 16$ and S is a complete intersection of a quadric and a quintic hypersurface.

Remark. I have not been able to give examples, or give proofs that they do not exist, of hyperelliptic surfaces with $\pi = 6$, or Enriques surfaces with $\pi = 8$.

At this point it may be appropriate to recall the list of nondegenerate smooth surfaces S in $\mathbf{P}^4$ with degree less than 10. The classification in terms of numerical invariants is, as far as

I know, not complete, since there is a regular elliptic surface of degree 9 for which I do not know of any proof of existence. I list the surfaces in terms of their degree d and the genus π of a general hyperplane section. Instead of giving explicit information on the very ample linear system on S, I indicate some known facts on postulation. For further information on surfaces of degree less than 7, see Roth ([Ro1]). For degree 7 and 8, see Okonek ([O1] and [O2]) or Ionescu ([Io]), supplemented by Alexander ([A1]). For degree 9, I know of no general reference, the rational case is taken care of by Alexander [A1] and [A2], while the nonrational case has been worked out in collaboration with Aure. A construction of some of the surfaces of degree 9 will be indicated below.

If $d < 3$, then S is degenerate.
If $d = 3$, then $\pi = 0$ and S is a rational cubic scroll, cut out by a net of quadric hypersurfaces.
If $d = 4$, then $\pi = 0$ and S is a Veronese surface projected from $\mathbf{P}^5$, it is not contained in any quadric, but is cut out by cubic hypersurfaces,
or $\pi = 1$ and S is a Del Pezzo surface, a complete intersection of two quadric hypersurfaces.
If $d = 5$, then $\pi = 1$ and S is an elliptic scroll, it is not contained in any quadric hypersurfaces, but is cut out by cubic hypersurfaces,
or $\pi = 2$ and S is rational, it is linked to a plane in the complete intersection of a quadric and a cubic hypersurface.
If $d = 6$, then $\pi = 3$ and S is a Bordiga surface, it is rational and linked to a cubic scroll in the complete intersection of two cubic hypersurfaces,
or $\pi = 4$ and S is a minimal $K3-$surface, it is the complete intersection of a quadric and a cubic hypersurface.
If $d = 7$, then $\pi = 4$ and S is a rational surface, it is linked to an elliptic quintic scroll in the complete intersection of a cubic and a quartic hypersurface,
or $\pi = 5$ and S is a nonminimal $K3-$surface, it is linked to a degenerate quadric surface in the complete intersection of two cubic hypersurfaces,
or $\pi = 6$ and S is a regular elliptic surface, it is linked to a plane in the complete intersection of a quadric and a cubic hypersurface.
If $d = 8$, then $\pi = 5$ and S is rational, it does not lie on any cubic hypersurface,
or $\pi = 6$ and S is rational, it is linked to a Veronese surface in the complete intersection of a cubic and a quartic hypersurface,
or S is a nonminimal $K3-$surface, it is linked to the rational one in the complete intersection of two quartic hypersurfaces,
or $\pi = 7$ and S is a regular elliptic surface, it is linked to a plane in the complete intersection of two cubic hypersurfaces,
or $\pi = 9$ and S is of general type, it is the complete intersection of a quadric and a cubic hypersurface.
If $d = 9$, then $\pi = 6$ and S is rational, or a nonminimal Enriques surface,
or $\pi = 7$ and S is a rational surface, it lies on a net of quartic hypersurfaces,
or S may be a regular elliptic surface,
or $\pi = 8$ and S is a nonminimal $K3-$surface, it does not lie on any cubic hypersurface,

or S is of general type, it does not lie on any cubic hypersurface either,

or $\pi = 9$ and S is of general type, it is linked to a rational cubic scroll in the complete intersection of a cubic and a quartic hypersurface,

or $\pi = 10$ and S is of general type, it is the complete intersection of two cubic hypersurfaces,

or $\pi = 12$ and S is of general type, it is linked to a plane in the complete intersection of a quadric and a quintic hypersurface.

The nonminimal Enriques surface with $\pi = 6$ is the projection of a minimal smooth Enriques surface of degree 10 in $\mathbf{P}^5$ from a general point on the surface. Enriques surfaces of degree 10 in $\mathbf{P}^5$ are well-known, in fact any Enriques surface has a linear system of degree 10 and projective dimension 5 without basepoints (see Cossec [Co]), for very ampleness it suffices to require that any elliptic curve on the surface has degree at least 3 with respect to the linear system, and that there are no (-2)-curves on it.

To get the rational surface S with $\pi = 7$ one may construct a surface T of degree 7 and with $\pi(T) = 3$, such that S is linked to T in the complete intersection of two quartic hypersurfaces, as follows. Let T_0 be a Del Pezzo cubic surface in a hyperplane H_0 of $\mathbf{P}^4$. Let L_1, L_2 and L_3 be three skew lines on T_0 and let L_0 be a line meeting all three L_i, but not contained in T_0. Let P_0 be a plane through L_0 not contained in H_0, and let p_1, p_2 and p_3 be three noncolinear points in P_0 away from L_0. The lines L_i and the points p_i span three planes which we denote by P_i, $i = 1, 2, 3$. If

$$T = T_0 \cup P_0 \cup P_1 \cup P_2 \cup P_3,$$

then one may show that T is cut out by quartic hypersurfaces, and is linked to a smooth rational surface S in the complete intersection of two quartic hypersurfaces. Furthermore, one may show that the union of the planes P_1, P_2 and P_3 will be the union of 5-secants meeting S, and that

$$P_0 \cup P_1 \cup P_2 \cup P_3$$

is contained in any quartic hypersurface which contains S. Intrinsically, the linear system of hyperplane sections is given by

$$H \equiv 9\pi^* l - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{9} 2E_j - \sum_{k=10}^{15} E_k.$$

If $\pi : S \to \mathbf{P}^2$ is the blowdown map, and $x_i = \pi(E_i)$, then the $x_1, \ldots, x_6$ are in general position, while there is a pencil of curves

$$D \equiv 6l - \sum_{i=1}^{6} 2x_i - \sum_{j=7}^{9} x_j - \sum_{k=10}^{15} x_k$$

which all have a common tangent at the points x_7, x_8, x_9 ([A2]).

To get the elliptic surface S with $\pi = 7$ one may construct a surface T of degree 7 and with $\pi(T) = 3$, such that S is linked to T in the complete intersection of two quartic hypersurfaces, as follows. Let H_1, H_2, H_3 be three hyperplanes in $\mathbf{P}^4$, whose intersection is a line L. Let L_i, $i = 1, 2, 3$, be lines in the planes $H_j \cap H_k$, where $\{i, j, k\} = \{1, 2, 3\}$, such that no two of the lines L_i meet. Furthermore, let P be a plane which does not meet any of the lines L, L_1, L_2, L_3, and let $N_i = P \cap H_i$ for $i = 1, 2, 3$. If Q_i is the quadric surface in H_i which contains N_i, L_j, L_k where $\{i, j, k\} = \{1, 2, 3\}$, then we set

$$T = P \cup Q_1 \cup Q_2 \cup Q_3.$$

One may show that T is contained in a cubic hypersurface, and that it is cut out by quartic hypersurfaces. Using Proposition 0.14 we get that T is linked to a smooth surface S in the complete intersection of two quartic hypersurfaces. Furthermore $P \cap S$ will contain a canonical curve on S as the one-dimensional part, while $Q_1 \cup Q_2 \cup Q_3$ will be the union of the 5-secant lines to the surface S (cf. the secant formulas above).

To get a surface S of general type with $\pi = 8$ we may similarly construct a surface T of degree 7, such that S is linked to T in the complete intersection of two quartic hypersurfaces. In fact, let T_0 be a Del Pezzo cubic surface in a hyperplane H_0 in $\mathbf{P}^4$. Let L be a line on T_0, and let A be a smooth conic on T_0 which does not meet L. Let L_0 be a line in the plane of A, not contained in T_0 and which meets L, and let P_0 be a plane which meets H_0 along L_0. We denote the hyperplane spanned by A and P_0 by H_Q. Let Q be a smooth quadric in H_Q through the conic A, and let p be a point on P_0 away from the conic $Q \cap P_0$ and the line L_0. The line L together with the point p spans a plane which we denote by P. If

$$T = T_0 \cup Q \cup P \cup P_0,$$

then one may show that T is cut out by quartic hypersurfaces, and that T is linked to a smooth surface S of general type in the complete intersection of two quartic hypersurfaces. Furthermore, one may show that the plane P will be the union of 5-secants meeting S, and that the union of the planes $P \cup P_0$ is contained in any quartic hypersurface which contains S.

For the nonminimal $K3-$surface there is a classical construction with the grassmanian G of lines in $\mathbf{P}^5$. Let V be the threefold in $\mathbf{P}^5$ which is the union of the lines corresponding to a general member of the equivalence class of H^6, where H is a Plücker divisor. Then V is known to be smooth, its general hyperplane section is a $K3-$surface of degree 9 with five (-1)-lines.

Remark. The constructions above should be considered as examples, for the uniqueness of these constructions I know of no proof except for the rational surface.

Now, any smooth surface in $\mathbf{P}^4$, except for the Veronese surfaces, are linearly normal. This is a theorem of Severi. Therefore, by the Riemann-Roch theorem, $h^1(\mathcal{O}_S(H))$ is determined by the degree d, the genus π and the Euler characteristic $\chi(\mathcal{O}_S)$. We will call

$h^1(\mathcal{O}_S(H))$ the speciality of $|H|$ on S, and we will say that a linear system of curves $|H|$ is special if $h^1(\mathcal{O}_S(H)) > 0$. We will also say that a surface S in $\mathbf{P}^4$ is special if the linear system of hyperplane sections is special. In our list of surfaces of degree $d \leq 10$ this speciality vanishes in most cases, but for the degrees 8, 9 and 10 there are examples of special surfaces. In fact the rational surfaces of degree 8 and $\pi = 6$, of degree 9 and $\pi = 7$ and of degree 10 and $\pi = 8$ all have speciality $h^1(\mathcal{O}_S(H)) = 1$, while the rational surfaces of degree 10 and $\pi = 9$ have speciality $h^1(\mathcal{O}_S(H)) = 2$. The speciality in these cases are all reflected in the special position of the assigned basepoints of the very ample linear system. A curious fact is that in the cases with speciality one the assigned baselocus always has support on a complete intersection. The special nonrational surfaces in the list are the nonminimal $K3-$surface of degree 9 and $\pi = 8$, the nonminimal $K3-$surface of degree 10 and $\pi = 9$, the nonminimal regular elliptic surface of degree 10 and $\pi = 9$ and the nonminimal regular elliptic surface of degree 10 and $\pi = 10$. Note that these examples are also all nonminimal, and that the speciality is reflected in the special position of the (-1)-curves.

Historical note

The study of special linear systems on a surface goes back at least to Castelnuovo. In an article ([Ca]) where he studies linear systems of curves on $\mathbf{P}^2$, he shows that if

$$|D| = |al - \sum_{i=1}^{k} b_i x_i|$$

is the complete linear system of curves of degree a with multiplicities b_i at the points x_i, and $|D|$ does not have any nonassigned basepoints, then $|D|$ is special only if $k \geq 9$. Equality holds only if $a = 3b_1 = \ldots = 3b_9 > 0$. A related open problem is to find a minimal k such that $|D|$ is special and very ample.

The study of smooth surfaces in $\mathbf{P}^4$ also goes back to the Italians at the turn of the century, treating the surfaces of degree less than 7, or of genus $\pi \leq 3$. For $d \geq 7$ there are contributions by Commessati and Roth. Roth shows that any surface of degree $d \leq 10$, except for the abelian surfaces of degree 10, is regular or birationally ruled [Ro1]. He refers to Commessati for the abelian surfaces, and he gives some bounds for the arithmetic genus $p_a(S) = \chi(S) - 1$ of smooth surfaces given the degree and genus π. He also presents a list of surfaces with $\pi \leq 6$, which is incomplete since he misses the nonspecial rational surface of degree 9 in $\mathbf{P}^4$. To produce the list, he uses the adjunction mapping to get surfaces with smaller invariants that he knows already.
It is this technique that has been taken up in recent years, after Sommeses study of the adjunction mapping, in a revival in the study of surfaces with small invariants.

Notations and basic results

We use standard notations and basic results as given for instance in [Ha] and [BPV]. For the invariants of a smooth surface S in $\mathbf{P}^4$ we use the following shorthand notation:

$\pi = \pi(S)$ is the genus of a general hyperplane section
$p_g = p_g(S) = h^0(\mathcal{O}_S(K))$ is the geometric genus of S
$q = q(S)$ is the irregularity of S
$\chi = \chi(S) = \chi(\mathcal{O}_S)$
$p(C)$ is the arithmetic genus of a curve C on S
$g(C)$ is the geometric genus of a smooth curve C.

The minimal models for the rational surfaces are $\mathbf{P}^2$ and $\mathbf{F}_e$, where $\mathbf{F}_e$ is a Hirzebruch surface with $e \geq 0$. The class of a line in $\mathbf{P}^2$ will be denoted by l, while B (resp. F) will be the class of a section on $\mathbf{F}_e$ with selfintersection $B^2 = e$ (resp. a fiber in the ruling), whenever $\mathbf{F}_e$ is the minimal model involved.

In a blowing-up situation we will use the same notation for a divisor downstairs and its total transform upstairs. A rational curve C ($p(C) = 0$)) with selfintersection $C^2 = -1$ will be called a (-1)-curve, similarly if $C^2 = -2$ we call it a (-2)-curve. A (-1)-line is a (-1)-curve of degree one with respect to a given very ample linear system on the surface.

Whenever we have a nonempty linear system $|C|$ on a surface S, we will denote the rational map which it determines by φ_C. We will, by abuse of standard notation, denote by $|C - p|$ the linear subsystem of curves C in $|C|$ which contains the point p in S. We work throughout over an algebraically closed field of characteristic zero.

Let S be a smooth surface, and let $Div(S)$ be the set of linear equivalence classes of divisors on S. There is a bilinear map from $Div(S) \times Div(S)$ to the integers which defines an intersection number $C \cdot D$ between divisors on S. We set $D^2 = D \cdot D$. If K is the canonical divisor on S and C is a curve on S, then the arithmetic genus $p(C)$ is given by the

Adjunction formula 0.2. $2p(C) - 2 = C^2 + C \cdot K$.

Proof. See [Ha Prop. 1.5].□

The adjunction formula actually gives a canonical divisor on the curve C:

$$K_C \equiv (C + K)_{|C}.$$

The corresponding sheaf $\omega_C \cong \mathcal{O}_C(C + K)$ is a dualizing sheaf on S, so that we may use Riemann-Roch and Serre duality on C as if C was a smooth curve (see Mumford [Mu]). For curves C, D and $C \cup D$ on S the adjunction formula immediately gives the following addition formula for the arithmetic genus of curves on a smooth surface.

(0.3.) $$p(C \cup D) = p(C) + p(D) + C \cdot D - 1.$$

To find the number of curves linearly equivalent to C we may use the

Theorem (Riemann-Roch) 0.4.

$$\chi(\mathcal{O}_S(C)) = h^0(\mathcal{O}_S(C)) - h^1(\mathcal{O}_S(C)) + h^0(\mathcal{O}_S(K-C)) = \frac{1}{2}(C^2 - C\cdot K) + \chi(S).$$

Proof. See [Ha Th.1.6].□

If H is an ample divisor on S, then one may get a bound on the self intersection C^2 of C in terms of $H\cdot C$ and H^2 using the

Hodge index theorem 0.5. *If H is an ample divisor on S and D is a divisor on D such that $H\cdot D=0$, then $D^2<0$ or $D\equiv 0$.*

Proof. See [Ha Th.1.9].
In fact we get the following

Corollary 0.6. *Let H be a very ample divisor on a surface S. If C is a divisor on S then*

$$C^2 \leq \frac{(H\cdot C)^2}{H^2}.$$

Proof. Apply the index theorem to $C - (\frac{H\cdot C}{H^2})H$.□
When using this corollary, I refer to the index theorem throughout this paper.
For smooth surfaces in $\mathbf{P}^4$ with normal bundle N_S there is the relation,

(0.7.) $$d^2 - c_2(N_S) = d^2 - 10d - 5H\cdot K - 2K^2 + 12\chi(S) = 0,$$

which expresses the fact that S has no double points. I therefore refer to this relation as the double point formula.
The first major theorem on smooth surfaces in $\mathbf{P}^4$ is the

Theorem (Severi) 0.8. *All smooth surfaces in $\mathbf{P}^4$, except for the Veronese surfaces, are linearly normal.*

Proof. See [Se].□

Some classical numerical formulas for multisecant lines to a smooth surface in $\mathbf{P}^4$ has recently been studied again by Le Barz:

0.9. Secant Formulas (see [LB]).

Let S be a smooth surface of degree d in $\mathbf{P}^4$ with invariants π and χ. Then the number of trisecants to S which meets a general point is:

$$t = \binom{d-1}{3} - \pi(d-3) + 2\chi - 2.$$

Let

$$s = \binom{d-1}{2} - \pi$$

and

$$h = \frac{1}{2}(s(s-d+2)-3t).$$

The number of 4-secants to S which meets a general line is:

$$N_4 = 2\binom{d}{4} + t(d-3) + h - s\binom{d-2}{2}.$$

The number of 5-secants to S which meets a general plane is:

$$\begin{aligned} N_5 =& \frac{1}{24}d(d-3)(d-4)(d^2-15d+2) - \binom{s}{2}(d-4) \\ & - \frac{s}{6}(d-2)(d-4)(d-21) + h(d-8) + st - 3t(d-3) \end{aligned}.$$

The number of 6-secants to S is:

$$\begin{aligned} N_6 =& - \frac{1}{144}d(d-4)(d-5)(d^3+30d^2-577d+786) \\ & + s(2\binom{d}{4} + 2\binom{d}{3} - 45\binom{d}{2} + 148d - 317) \\ & - \frac{1}{2}\binom{s}{2}(d^2-27d+120) - 2\binom{s}{3} \\ & + h(s-8d+56) + t(9d-3s-28) + \binom{t}{2} - \sum_{i=1}^{p}\binom{7+l_i}{6}, \end{aligned}$$

where $L_i, i = 1, ..., p$ are the lines contained in S and $l_i, i = 1, ..., p$ are their respective selfintersection.

On the structure of the adjunction mapping we will use the following

Theorem (Sommese, Van de Ven) 0.10. *Let S be a smooth surface with a very ample divisor H and a canonical divisor K. Then*
1) $|H+K| = \emptyset$ if and only if S is a scroll or a Veronese surface,
2) $|H+K| \neq \emptyset$ only if $|H+K|$ has no basepoints.
In the latter case we have furthermore that
A) $(H+K)^2 = 0$ if and only if S is ruled in conics,
B) $(H+K)^2 > 0$ only if the map φ_{H+K} defined by $|H+K|$ is the blowing-down of (-1)-lines on S except for the following four cases:
i) S is $\mathbf{P}^2$ blown up in 7 points and $H \equiv 6l - \sum_{i=1}^{7} 2E_i$.
ii) S is $\mathbf{P}^2$ blown up in 8 points and $H \equiv 6l - \sum_{i=1}^{7} 2E_i - E_8$.
iii) S is $\mathbf{P}^2$ blown up in 8 points and $H \equiv 9l - \sum_{i=1}^{8} 3E_i$.

iv) $S \cong \mathbf{P}(E)$ *where* E *is an indecomposable rank 2 bundle on an elliptic curve, and* $H \equiv 3B$ *where* B *is an effective divisor with* $B^2 = 1$ *on* S.

Proof. For a proof see [SV].□

Methods

The general procedure in working out the classification of this paper has been first to use the double point formula, the Severi theorem and the index theorem to get a finite list of sets of invariants admissable for a smooth surface. Next, if S has no effective pluricanonical divisors, then we may use the adjunction mapping (several times if necessary) to get surfaces with smaller invariants that we already know, from which we may reconstruct S and the very ample linear system $|H|$. If S has effective pluricanonical divisors, then we study these to eliminate among the sets of admissable invariants, and to describe the linear system $|H|$ on S.
The next step is to find reducible hyperplane sections on S. For elimination we try to find components with an arithmetic genus too high for their degree. We use the following

Lemma 0.11. *Let* C *be a curve of degree* d *and arithmetic genus* p *on a smooth surface in* $\mathbf{P}^4$.
If $d \leq 3$, *then* $p \leq 1$ *with equality only if* C *is a plane cubic curve.*
If $d = 4$, *then* $p \leq 1$ *or* $p = 3$ *and* C *is a plane quartic curve.*
If $d = 5$, *then* $p \leq 3$ *or* $p = 6$ *and* C *is a plane sextic curve.*
If $p = 3$ *then* C *is the union of a plane quartic curve and a line which meets the plane quartic in a point.*
If $d = 6$, *then* $p \leq 6$ *or* $p = 10$ *and* C *is a plane sextic curve.*
If $p = 6$ *then* C *decomposes into a plane quintic curve and a line which meet in a point.*
If $p = 5$ *then* C *decomposes into a plane quintic and a line which do not meet.*
If $d = 7$, *then* $p \leq 6$ *unless* C *is a plane curve,*
or C *decomposes into a plane sextic curve and a line which meet in a point* $(p = 10)$ *or which do not meet* $(p = 9)$,
or C *decomposes into a plane quintic curve and a plane conic which meet along a scheme of length two* $(p = 7)$.

Proof. Straightforward using Castelnuovos bound for irreducible curves [Ha Th. 6.4], and the addition formula 0.3.□

For the special linear systems $|H|$ we find curves C on S to which $|H|$ restricts to a special linear series. We study the linear series δ_C dual to $|H_{|C}|$, and try to lift it to a special linear system of curves on S. This has been done successfully by Saint-Donat and Reid (see [SD], [R2]), to study projective embeddings of minimal $K3$-surfaces and to study regular surfaces with curves with special pencils of divisors. A special case is illustrated in the following lemma, where the trivial linear series on a curve in $|C|$ is lifted to a curve on the surface, as soon as the linear system $|C|$ is special.

Lemma 0.12. *Let $\pi : S \to \mathbf{P}^2$ be the morphism obtained by blowing up 12 points (some possibly infinitely close) in $\mathbf{P}^2$. Denote the exceptional divisors by $E_1, ..., E_{12}$ and consider the linear system*

$$|C| = |4\pi^* l - \sum_{i=1}^{12} E_i|$$

on S.

If $\dim|C| \geq 3$ and $|C|$ has a fixed curve, then there is a curve $\Gamma \equiv \pi^* l - \sum_{k=1}^{6} E_{i_k}$ or $\Gamma \equiv 2\pi^* l - \sum_{k=1}^{10} E_{i_k}$ on S, which is part of the fixed curve of $|C|$.

If $\dim|C| \geq 3$ and $|C|$ has no fixed curve, then $\dim|C| = 3$ and $|C|$ has no basepoints. Furthermore there is a curve $\Gamma \equiv 3\pi^* l - \sum_{i=1}^{12} E_i$ on S.

If $\dim|C| = 2$ and $|C|$ has a fixed curve, then there is a curve $\Gamma \equiv \pi^* l - \sum_{k=1}^{5} E_{i_k}$ or $\Gamma \equiv 2\pi^* l - \sum_{k=1}^{9} E_{i_k}$ or $\Gamma \equiv 3\pi^* l - \sum_{i=1}^{12} E_i$ on S, which is part of the fixed curve of $|C|$.

If $\dim|C| = 2$ and $|C|$ has no fixed curve, then $|C|$ has at the most one basepoint.

Proof. By Riemann-Roch we have $\dim|C| \geq 2$. Assume first that $|C|$ has a fixed curve, and denote it by Γ_0. Let

$$D \equiv C - \Gamma_0.$$

Thus we assume that $|D|$ has no fixed curve, and we may set

$$D \equiv \alpha\pi^* l - \sum_{i=1}^{12} \beta_i E_i \qquad \text{with} \quad \alpha \geq 1 \quad \text{and} \quad \beta_i \geq 0.$$

If $\alpha = 1$, then clearly all $\beta_i = 0$ and $\dim|D| = 2$. If $\alpha = 2$, then to get $\dim|D| = 2$ at the most three of the $\beta_i > 0$, and to get $\dim|D| \geq 3$ at the most two of the $\beta_i > 0$. If $\alpha = 3$, then to get $\dim|D| = 2$ at the most seven of the $\beta_i > 0$, and to get $\dim|D| \geq 3$ at the most six of the $\beta_i > 0$. If $\alpha = 4$, then Γ_0 must be supported on the exceptional set on S. But $(C - E_i - E_j)^2 < 0$ when $1 \leq i \leq j \leq 12$, so $\Gamma_0 = E_i$ and $|D|$ must have one basepoint and be composed with a pencil (if $|D|$ is basepointfree, then D would be rational, which is absurd). Since $\dim|D| \geq 2$ and S is rational, D must be a multiple divisor, which it clearly is not.

Secondly, we assume that $|C|$ has no fixed curve, and let C be a general member of $|C|$.

If $\dim|C| \geq 3$, then $\dim|C_{|C}| \geq 2$. But since C has arithmetic genus three and $C^2 = 4$, we get that $\dim|C_{|C}| \leq 2$. Thus $\dim|C_{|C}| = 2$ and $\dim|C| = 3$ and $|C_{|C}|$ is the canonical series on C, that is

$$|C_{|C}| = |K_C| = |(C + K_S)_{|C}| = |\pi^* l|.$$

In particular $|C|$ has no basepoints. If we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(-C - K_S) \longrightarrow \mathcal{O}_S(-K_S) \longrightarrow \mathcal{O}_C(C - (C + K_C)) \longrightarrow 0$$

and its associated cohomology, then since the last sheaf is trivial from the above, we get that

$$h^0(\mathcal{O}_S(-K_S)) = 1 \qquad \text{if and only if} \qquad h^1(\mathcal{O}_S(-C - K_S)) = 0.$$

But this is clearly so since $-C - K_S \equiv -\pi^* l$. Thus

$$|-K_S| = |3\pi^* l - \sum_{i=1}^{12} E_i| \neq \emptyset.$$

If $\dim|C| = 2$, and $|C|$ has more than one basepoint, then there are at the most three basepoints since $|C|$ cannot be composed with a pencil by an argument like the above. If it has three basepoints then C would be rational, which is absurd. If $|C|$ has two basepoints, then $|C_{|C}|$ would show that C is a hyperelliptic curve, which it clearly is not.□

For proofs of existence I try to find plane curves on S to be able to use the following lemma, which was communicated to me by Alexander.

Lemma 0.13. *If H has a decomposition*

$$H \equiv C + D,$$

where C and D are curves on S, such that $\dim|C| \geq 1$, and if the restriction maps $\mathrm{H}^O(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(D)H)$ and $\mathrm{H}^O(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(C)H)$ are surjective, and $|H|$ restricts to very ample linear systems on D and on every C in $|C|$, then $|H|$ is very ample on S.

Proof. We use the decomposition $H \equiv C + D$ to show that $|H|$ separates points and tangent directions on S. Let p and q be two, possibly infinitely close, points on S. By the assumptions of the lemma we may assume that $p+q$ is not contained in D or any C. In particular we may assume that $p+q$ does not meet the baselocus of $|C|$. If D contains p, then we can find a curve C which does not meet $p+q$ such that $C+D$ separates p and q. If D does not meet $p+q$, then we can find a curve C which contains one of the points p or q, such that $C+D$ separates p and q. □

Another way to get a proof of existence will be to use the

Proposition 0.14. *If T is a local complete intersection surface in $\mathbf{P}^4$, which scheme-theoretically is cut out by hypersurfaces of degree d, then T is linked to a smooth surface S in the complete intersection of two hypersurfaces of degree d.*

For a proof see [PS Proposition 4.1.].

Remark (Peskine, private communication). A slight modification of the conditions of this proposition is allowable, without changing the conclusion. Namely, at a finite set of points T need not be a local complete intersection. It suffices that it is locally Cohen-Macaulay, and that the tangent cone at that point is linked to a plane in a complete intersection.

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1 A rational surface with $\pi = 8$

Theorem A. *Given nine points $x_1, \ldots, x_9$ in general position in $\mathbf{F}_e$, $e \leq 2$. One can choose three points y_1, y_2, y_3 such that if*

$$\pi : S \to F_e$$

is the morphism obtained by blowing up the points $x_1, \ldots, x_9$, y_1, y_2, y_3 in $\mathbf{F}_e$, and $E_1, \ldots, E_9$, F_1, F_2, F_3 are the exceptional divisors and B (resp. F) is a section with selfintersetion e (resp. a ruling), then the linear system

$$|H_S| = |8\pi^*B + (10-4e)\pi^*F - \sum_{i=1}^{9} 4E_i - 2F_1 - F_2 - F_3|$$

is very ample and embeds S as a surface of degree 10 in $\mathbf{P}^4$.

Proof. We start by choosing nine points $x_1, \ldots, x_9$ in general position in $\mathbf{F}_e$, $e \leq 2$, in the sense that if $\pi_1 : S_1 \to \mathbf{F}_e$ is the morphism obtained by blowing up the points $x_1, \ldots, x_9$ in $\mathbf{F}_e$ with exceptional divisors $E_1, \ldots, E_9$, and B (resp. F) also denote the total transform of B (resp. F) on S_1, then the following conditions holds:

i) No two points x_i are infinitely close.
ii) For e=1 or 2, $h^0(\mathcal{O}_{S_1}(B - eF - E_i)) = 0$, $1 \leq i \leq 9$.
iii) $h^0(\mathcal{O}_{S_1}(F - E_i - E_j)) = 0$, for $1 \leq i < j \leq 9$.
iv) $h^0(\mathcal{O}_{S_1}(B - \sum_{k=1}^{e+2} E_{i_k})) = 0$, for $1 \leq i_1 < \ldots < i_{e+2} \leq 9$.
v) $h^0(\mathcal{O}_{S_1}(B + F - \sum_{k=1}^{e+4} E_{i_k})) = 0$, for $1 \leq i_1 < \ldots < i_{e+4} \leq 9$.
vi) $h^0(\mathcal{O}_{S_1}(B + 2F - \sum_{k=1}^{e+6} E_{i_k})) = 0$, for $1 \leq i_1 < \ldots < i_{e+2} \leq 9$.
vii) $h^0(\mathcal{O}_{S_1}(2B + (1-e)F - \sum_{k=1}^{6} E_{i_k})) = 0$, for $1 \leq i_1 < \ldots < i_6 \leq 9$.
viii) $h^0(\mathcal{O}_{S_1}(2B + (2-e)F - 2E_i - \sum_{k=1}^{6} E_{i_k})) = 0$, for $1 \leq i \leq 9$ and $1 \leq i_1 < \ldots < i_6 \leq 9$, $i \neq i_k$.
ix) $h^0(\mathcal{O}_{S_1}(2B + (2-e)F - \sum_{i=1}^{9} E_i)) = 0$.
x) $h^0(\mathcal{O}_{S_1}(4B + (4-2e)F - \sum_{i=1}^{9} 2E_i)) = 0$.

Lemma 1.1. *All the conditions i), ..., x) are open nonempty conditions for the choice of points $x_1, \ldots, x_9$.*

Proof. The conditions can in all cases be translated into conditions concerning linear systems of curves on $\mathbf{P}^2$, where the statement follows from a result of Hirschowitz (see [Hi]), which says that if $\pi_1 : S \to \mathbf{P}^2$ is the blowing-up of $\mathbf{P}^2$ in $r + s$ points in general position, with exceptional divisors E_i, and $3r + s \geq h^0(\mathcal{O}_{\mathbf{P}^2}(nl))$, then the linear system

$$|n\pi_1^* l - \sum_{i=1}^{r} 2E_i - \sum_{j=1}^{s} E_j|$$

is empty, unless $n = r = 2$ and $s = 0$ or $n = 4$, $r = 5$ and $s = 0$. As an example, in case $e = 1$, the condition x) is equivalent to

$$|6\pi_1^* l - \sum_{i=1}^{10} 2E_i| = \emptyset,$$

and this follows immediately from the result of Hirschowitz.□

On S_1 we study two linear systems of curves:

$$|C| = |4B + (5-2e)F - \sum_{i=1}^{9} 2E_i|$$

and

$$|D| = |6B + (7-3e)F - \sum_{i=1}^{9} 3E_i|.$$

Lemma 1.2. *$h^0(\mathcal{O}_{S_1}(C)) = 3$, $h^0(\mathcal{O}_{S_1}(D)) = 2$ and $|C|$ and $|D|$ have only finitely many reducible curves.*

Proof. The proof has three steps. The first step is to show that the linear systems have no fixed curves, the second step is to show that the dimensions are the given ones, and the last step is to show that no subpencil of $|C|$ has a fixed curve. The first and the last step amounts to checking possible fixed curves against the conditions $i), \ldots, x)$, and is straightforward.

We use the first step to show the second one as follows: By Riemann-Roch we get that $h^0(\mathcal{O}_{S_1}(C)) \geq 3$. Assume that $h^0(\mathcal{O}_{S_1}(C)) \geq 4$, then $|C|$ defines a rational map

$$\varphi_C : S_1 \dashrightarrow \mathbf{P}^3$$

with isolated basepoints at the most. C is not a multiple divisor, so $\varphi_C(S_1)$ is a surface. Since $C^2 = 4$, we may therefore assume that a general curve C in $|C|$ is smooth, it has genus $g_C = 3$. Now

$$h^0(\mathcal{O}_{S_1}(C)) \geq 4 \qquad \text{implies that} \qquad h^0(\mathcal{O}_C(C)) \geq 3,$$

but $\deg \mathcal{O}_C(C) = C^2 = 4$, so $\mathcal{O}_C(C)$ must be special, in fact we must have equality and that

$$\mathcal{O}_C(C) = \omega_C = \mathcal{O}_C(C + K_{S_1}),$$

where

$$K_{S_1} = -2B + (e-2)F + \sum_{i=1}^{9} E_i.$$

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1}(-K_{S_1} - C) \longrightarrow \mathcal{O}_{S_1}(-K_{S_1}) \longrightarrow \mathcal{O}_C(C - (C + K_{S_1})) \longrightarrow 0.$$

We take cohomology to get

$$h^0(\mathcal{O}_{S_1}(-K_{S_1})) = 1 \qquad \text{if and only if} \qquad h^1(\mathcal{O}_{S_1}(-K_{S_1} - C)) = 0.$$

But $h^1(\mathcal{O}_{S_1}(-K_{S_1} - C)) = h^1(\mathcal{O}_{S_1}(C + 2K_{S_1})) = h^1(\mathcal{O}_{S_1}(F)) = 0$, so we get

$$h^0(\mathcal{O}_{S_1}(-K_{S_1})) = h^0(\mathcal{O}_{S_1}(2B + (2-e)F - \sum_{i=1}^{9} E_i)) = 1,$$

which contradicts condition ix). Therefore $h^0(\mathcal{O}_{S_1}(C)) = 3$.

To show that $h^0(\mathcal{O}_{S_1}(D)) = 2$, we first get from Riemann-Roch that $h^0(\mathcal{O}_{S_1}(D)) \geq 2$. Next, we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1}(-K_{S_1}) \longrightarrow \mathcal{O}_{S_1}(D) \longrightarrow \mathcal{O}_C(D) \longrightarrow 0.$$

Since, from the above argument, $h^0(\mathcal{O}_{S_1}(-K_{S_1})) = h^1(\mathcal{O}_{S_1}(-K_{S_1})) = 0$ we get that

$$h^0(\mathcal{O}_{S_1}(D)) = h^0(\mathcal{O}_C(D)).$$

We may assume, since $C^2 = 4$, that the general C in $|C|$ is smooth of genus $g_C = 3$. Now $\deg\mathcal{O}_C(D) = D \cdot C = 4$, so if $h^0(\mathcal{O}_{S_1}(D)) \geq 3$, then we have equality and that

$$\mathcal{O}_C(D) = \omega_C = \mathcal{O}_C(C + K_{S_1}).$$

Note that $C = D + K_{S_1}$ as we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1}(-2K_{S_1} - C) \longrightarrow \mathcal{O}_{S_1}(-2K_{S_1}) \longrightarrow \mathcal{O}_C(D - C - K_{S_1}) \longrightarrow 0.$$

We take cohomology to get

$$h^0(\mathcal{O}_{S_1}(-2K_{S_1})) = 1 \qquad \text{if and only if} \qquad h^1(\mathcal{O}_{S_1}(-2K_{S_1} - C)) = 0.$$

But $h(\mathcal{O}_1(S_1))-2K_{S_1} - C = h^1(\mathcal{O}_{S_1}(-F)) = 0$, so we get

$$h^0(\mathcal{O}_{S_1}(-2K_{S_1})) = h^0(\mathcal{O}_{S_1}(4B + (4-2e)F - \sum_{i=1}^{9} 2E_i)) = 1$$

which contradicts condition x).□

Among the curves in $|C|$ and $|D|$ we want to find a smooth curve C_0 in $|C|$ and a smooth curve D_0 in $|D|$ such that

1) C_0 and D_0 intersect and have a common tangent direction p' at a point p
2) $|C-p|$ has no reducible elements
3) p is not a basepoint for $|D|$
4) $(C_0-p)\cap(D_0-p)$ is reduced
5) $|C-p|$ is a pencil with basepoints away from D_0
6) C_0 and D_0 are not hyperelliptic.

Given such a choice of curves C_0 and D_0 we set

$$C_0\cap D_0 = p+p'+q_1+q_2.$$

Then $\pi_1(p)$, $\pi_1(q_1)$ and $\pi_1(q_2)$ are, respectively, the points y_1, y_2 and y_3 in $\mathbf{F}_e$ which we choose to get S.
To see that we can make this choice of curves C_0 and D_0, we consider the incidence

$$I\subset S_1\times|D|\times|C|$$

given by

$$I=\{p\times D\times C \mid D \text{ and } C \text{ has a common tangent at p}\}.$$

Lemma 1.3. *The conditions 1),…,6) are nonempty and open in I for the choice of curves C_0 and D_0.*

Proof. By Lemma 1.2 the conditions 2) and 3) are clearly satisfied for a general choice of p. For the other conditions, we consider the following bad subsets of I.

$$\mathrm{Hyp}_C=\{p\times D\times C\in I \mid C \text{ is hyperelliptic}\}.$$

$$\mathrm{Hyp}_D=\{p\times D\times C\in I \mid D \text{ is hyperelliptic}\}.$$

$$\mathrm{Bas}=\{p\times D\times C\in I \mid |C-p| \text{ has basepoints on } D\}.$$

$$\mathrm{Iso}=\{p\times D\times C\in I \mid (C-p)\cap(D-p) \text{ is not reduced}\}.$$

It suffices to show that I is at least one-dimensional and that the bad subsets all have positive codimensions.

From Lemma 1.2 we get immediately that I is at least two-dimensional, so we check the codimension of the bad subsets. To see that Hyp_C has a positive codimension we consider the linear system

$$|C+K_{S_1}|=|2B+(3-e)F-\sum_{i=1}^{9}E_i|.$$

From the exact sequence

$$0\longrightarrow \mathcal{O}_{S_1}(K_{S_1})\longrightarrow \mathcal{O}_{S_1}(C+K_{S_1})\longrightarrow \mathcal{O}_C(C+K_{S_1})\longrightarrow 0$$

we see, taking cohomology, that $h^0(\mathcal{O}_{S_1}(C+K_{S_1}))=3$. Thus $|C+K_{S_1}|$ defines a rational map

$$\varphi_{C+K_{S_1}}: S_1 \dashrightarrow \mathbf{P}^2,$$

which is in fact basepointfree since it has no basepoints on any irreducible curve C in $|C|$. We get $\deg\varphi_{C+K_{S_1}} = (C + K_{S_1})^2 = 3$. If C is hyperelliptic, then $\varphi_{C+K_{S_1}}(C)$ is a conic Q in $\mathbf{P}^2$. Now

$$\varphi^{-1}_{C+K_{S_1}}(Q) \equiv 2(C + K_{S_1}) \equiv C + F,$$

since $C + 2K_{S_1} \equiv F$. On the other hand

$$h^0(\mathcal{O}_{S_1}(C + K_{S_1} - F)) = h^0(\mathcal{O}_{S_1}(2B + (2 - e)F - \sum_{i=1}^{9} E_i)) = 0$$

by condition ix), so $\varphi_{C+K_{S_1}}(F)$ is a conic for every member F of $|F|$. Thus every F corresponds to a hyperelliptic C giving a rational pencil $\mathcal{P}_C$ of such curves C. Let D in $|D|$ be a smooth curve and let q be a general point on D. Then there is a curve C_q in $|C|$ which is tangent to D at q. If this curve C_q is in $\mathcal{P}_C$ for every q in D, then the restriction $\mathcal{P}_C|_D$ is ramified everywhere, which is absurd. Therefore, Hyp_C has a positive codimension.

To see that Hyp_D has a positive codimension, we similarly consider the map

$$\varphi_C : S_1 \dashrightarrow \mathbf{P}^2$$

defined by $|C| = |D + K_{S_1}|$. Here $\deg\varphi_C \leq C^2 = 4$. If D in $|D|$ is hyperelliptic, then $\varphi_C(D)$ is a conic Q in $\mathbf{P}^2$. So if all D in $|D|$ are hyperelliptic, then there is a linear pencil $\mathcal{P}_Q$ of corresponding conics in $\mathbf{P}^2$. By Lemma 1.2, φ_C is a finite map, so if Z_Q is the invers image of the basepoints of $\mathcal{P}_Q$, then every D meets Z_Q in a scheme of length 8. This contradicts Lemma 1.2 which says that for any two distinct curves D_1 and D_2 in $|D|$ we have the length$(D_1 \cap D_2) = D^2 = 4$. Therefore, we may conclude that Hyp_D is of a positive codimension. This argument also yields that Bas has positive codimension since if $|C - p|$ has basepoints on D for every p in D, then $\varphi_C|_D$ must be of degree 2 which means that D is hyperelliptic.

To see that Iso has a positive codimension, we again consider the map $\varphi_C : S_1 \dashrightarrow \mathbf{P}^2$. By the above, $\varphi_C(D)$ is a smooth plane quartic curve for a general D in $|D|$. If $(C-p)\cap(D-p)$ is nonreduced, then $L = \varphi_C(C)$ is a bitangent or a flextangent of $\varphi_C(D)$. Since the number of such tangents is finite, we conclude that also Iso has a positive codimension.□

Now let C_0' and D_0' be curves satisfying the conditions 1) through 6). Then we set

$$C_0' \cap D_0' = p + p' + q_1 + q_2$$

and blow up the points p, q_1 and q_2 in S_1 to get S with exceptional curves E_{10}, E_{11} and E_{12}. We denote the blowing-up map by π_2, and the composition $\pi_1 \circ \pi_2 : S \to \mathbf{P}^2$ by π. We need to name some more curves on S. Let

$$|C_1| = |C - E_{10}|$$

and

$$C_0 \equiv C - E_{10} - E_{11} - E_{12} \equiv C_1 - E_{11} - E_{12}.$$

Thus we denote the strict transform of C_0' on S by C_0. Similarly we denote the strict transform of D_0' on S by D_0. Finally we get the system $|H_S|$ of the theorem:

$$|H_S| = |2C_1 - E_{11} - E_{12}| = |C_1 + C_0| = |8B + (10-4e)F - \sum_{i=1}^{9} 4E_i - 2E_{10} - E_{11} - E_{12}|.$$

Lemma 1.4. $dim|H_S| = 4$.

Proof. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(C_1) \longrightarrow \mathcal{O}_S(H_S) \longrightarrow \mathcal{O}_{C_0}(H_S) \longrightarrow 0.$$

Taking cohomology and noting that $h^0(\mathcal{O}_S(C_1)) = 2$ we have that and $h^1(\mathcal{O}_S(C_1)) = 0$, and hence we get that

$$h^0(\mathcal{O}_S(H_S)) = h^0(\mathcal{O}_{C_0}(H_S)) + 2.$$

Let $p' = C_0 \cap E_{10} = D_0 \cap E_{10}$, thus p' is the tangent direction of C_0' at p. Then since $C_1 \equiv D_0 + K_S - E_{10}$ and $(D_0)_{|C_0} = p'$ on C_0 we get

$$H_{S|C_0} \equiv (C_0 + K_S + D_0 - E_{10})_{|C_0} \equiv (C_0 + K_S + D_0)_{|C_0} - p' \equiv (C_0 + K_S)_{|C_0},$$

where

$$K_S \equiv -2B + (e-2)F + \sum_{i=1}^{12} E_i.$$

Thus $h^0(\mathcal{O}_S(H_S)) = h^0(\mathcal{O}_{C_0}(C_0 + K_S)) + 2 = 5.\square$

To show that H_S is very ample on S, we first consider the restriction maps

$$\mathrm{H}^0(\mathcal{O}(S)H_S) \to \mathrm{H}^0(\mathcal{O}(C_0)H_S)$$

and

$$\mathrm{H}(\mathcal{O}(0)S)H_S \to \mathrm{H}^0(\mathcal{O}(C_1)H_S).$$

The first one is surjective since $h^1(\mathcal{O}_S(C_1)) = 0$, while the surjectivity of the second map is shown in the proof of the following lemma.

Lemma 1.5. *$H_{S|C_0}$ is very ample and $H_{S|C_1}$ is very ample for every curve C_1 in $|C_1|$.*

Proof. The first statement is seen from the proof of Lemma 1.4. In fact we showed that $H_{S|C_0} \equiv K_{C_0}$, which is very ample since C_0 is not hyperelliptic.

For the second statement of the lemma we first consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(C_0) \longrightarrow \mathcal{O}_S(H_S) \longrightarrow \mathcal{O}_{C_1}(H_S) \longrightarrow 0$$

for a curve $C_1 \neq C_0 + E_{11} + E_{12}$ in $|C_1|$. Taking cohomology and noting that we have $h^1(\mathcal{O}_S(C_0)) = h^1(\mathcal{O}_S(H_S)) = 1$ and $\deg H_{S|C_1} = H_S \cdot C_1 = 6$, we get that $h^1(\mathcal{O}_{C_1}(H_S)) = 0$ and that the restriction map

$$\mathrm{H}^0(\mathcal{O}(S)H_S) \to \mathrm{H}^0(\mathcal{O}(C_1)H_S)$$

is onto. Furthermore $H_{S|C_1}$ is very ample unless there are points q and q' on C_1 such that

$$h^0(\mathcal{O}_{C_1}(H_S - q - q')) = h^0(\mathcal{O}_{C_1}(H_S)) - 1 = 3,$$

which means that

$$H_{S|C_1} - q - q' \equiv K_{C_1} \equiv (C_1 + K_S)_{|C_1}.$$

In this case let $q_t = C_1 \cap E_{10}$, and let $D_t \equiv D - E_{10} - q_t - q - q'$. Then

$$(D_t + C_1 + K_S - E_{11} - E_{12})_{|C_1} \equiv (D - 2E_{10} + C_1 + K_S)_{|C_1} - q - q' \equiv H_{S|C_1} - q - q',$$

so $D_{t|C_1}$ is trivial. Consider now the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D_t - C_1) \longrightarrow \mathcal{O}_S(D_t) \longrightarrow \mathcal{O}_{C_1}(D_t) \longrightarrow 0.$$

We take cohomology to get $h^0(\mathcal{O}_S(D_t)) = 1$ if and only if $h^1(\mathcal{O}_S(D_t - C_1)) = 0$. But

$$h^1(\mathcal{O}_S(D_t - C_1)) = h^1(\mathcal{O}_S(-K_S + E_{10} + E_{11} + E_{12})) = h^1(\mathcal{O}_S(2B + (2-e)F - \sum_{i=1}^{9} E_i)) = 0$$

by Riemann-Roch and the condition $ix)$ on the choice of points x_i. Thus D_t must be an effective curve on S. This contradicts condition 4) for the choice of C_0.

Secondly, if $C_1 = C_0 + E_{11} + E_{12}$, then we first note that $|H|$ separates points and tangents on C_0. If $|H|$ does not separate points on E_{11}, then E_{11} is mapped onto a point in the plane of C_0 by φ_H. This means that E_{11} is a fixed curve for $|C_1|$, which contradicts condition 3) for the choice of curves C_0 and D_0. Similarly, E_{12} is mapped isomorphically into $\mathbf{P}^4$. Thus we are left with two cases: a) $t_1 \epsilon E_{11}$ and $t_2 \epsilon E_{12}$ and $t_1 + t_2$ does not meet C_0,and b) $t_1 \epsilon E_{11}$ and $t_2 \epsilon C_0$.

In case b), if t_1 and t_2 are not separated by $|H|$, then E_{11} would be mapped onto a line in the plane of C_0 and thus be a fixed curve for the pencil C_1, which is impossible like above. In case a), if the points t_1 and t_2 are not seperated by $|H|$, then $E_{11} + E_{12}$ is mapped onto a plane conic by φ_H. Using the same argument as in the proof of Lemma 0.13, we would get that S has only one double point in $\mathbf{P}^4$. This contradicts the double point formula, so the lemma follows. □

The very ampleness of H_S on S now follows from Lemma 0.13, as shown in the last part of the proof of Lemma 1.5.□

Postulation. J. Alexander has shown the following

Proposition 1.6. *For a general choice of curves C_0 and D_0, the surface is not contained in a quartic hypersurface.*

Proof. Alexanders argument goes as follows. Assume that V is a quartic hypersurface containing S and let H_0 be the hyperplane whose section of S is $2C_0 + E_{11} + E_{12}$. Let P_0 be the plane of C_0. The residual pencil $|C_1|$ has three basepoints p_1, p_2, p_3 in the plane P_0, which in fact are three points on C_0 since $C_0 + E_{11} + E_{12} \in |C_1|$. The points p_1, p_2, p_3 are clairly singular points of V, so for degree reasons only $P_0 \subset V_0$. On the other hand this means that $V \cap H_0$ contains P_0 with multiplicity two, so that V is in fact singular along a cubic curve in P_0, or singular along the whole plane P_0. In the first case let V be singular along the cubic curve A in P_0. Now the general C_1 lies on a cubic surface residual to P_0 in $V \cap H_{C_1}$. This cubic surface meets P_0 along A. Now the curve C_1 is tangent to P_0 at the points p_1, p_2, p_3 and the tangent directions in P_0 sweep out the first order neighbourhoods of the points p_1, p_2, p_3. Therefore A must be singular at p_1, p_2, p_3, i.e. A consists of three lines. In P_0 there is one more singular point of V that we know, namely the point $p' = C_0 \cap D_0$, because the plane of D_0 is also contained in V_0 for reasons similar to the ones for P_0, and this plane meets P_0 in a point. For a general choice of curves C_0 and D_0 this point p' does not lie on any of the lines of A. Therefore V must be singular along all of P_0. In this case the general curve C_1 is contained in a quadric. Since it is of degree 6 and genus 3 it is hyperelliptic. But then the special curve C_0 would also be hyperelliptic which it is not.□

2 A rational surface with $\pi = 9$

Theorem B. *Given twelve points $x_1, \dots, x_{12}$ in general position in $\mathbf{P}^2$. One can choose six other points $y_1, \dots, y_6$ such that if*

$$\pi : S \to \mathbf{P}^2$$

is the blowing-up of $\mathbf{P}^2$ in the points $x_1, \dots, x_{12},\ y_1, \dots, y_6$ and $E_1, \dots, E_{12},\ F_1, \dots, F_6$ are the exceptional divisors, then the linear system

$$|H_S| = |8\pi^* l - \sum_{i=1}^{12} 2E_i - \sum_{j=1}^{6} F_j|$$

is very ample and embeds S as a surface of degree 10 in $\mathbf{P}^4$.

Proof. The proof has three parts. First, we specify the choice of points x_i and y_j in $\mathbf{P}^2$. Secondly, we show that the given choice of points implies that $\dim|H_S| = 4$, and thirdly we show that $|H_S|$ is very ample.

Part 1. We start by choosing twelve points $x_1, \dots, x_{12}$ in $\mathbf{P}^2$ which are in general position in the following sense. If $\pi_1 : S_1 \to \mathbf{P}^2$ is the blowing-up of $\mathbf{P}^2$ in the points $x_1, \dots, x_{12}$ with exceptional divisors $E_1, \dots, E_{12}$, then

i) No two of the points x_i are infinitely close
ii) No three of the points x_i are on a line
iii) No six of the points x_i are on a conic
iv) No ten of the points x_i are on a cubic
v) $|3\pi_1^* l - 2E_{i_1} - \sum_{k=2}^{8} E_{i_k}| = \emptyset$ for any $\{i_1, \dots, i_8\} \subset \{1, \dots, 12\}$
vi) $|6\pi_1^* l - \sum_{k=1}^{8} 2E_{i_k} - \sum_{k=9}^{12} E_{i_k}| = \emptyset$ for any permutation $(i_1, \dots, i_{12})$ of $(1, \dots, 12)$
vii) $|7\pi_1^* l - \sum_{i=1}^{12} 2E_i| = \emptyset$
viii) The linear system $|4\pi_1^* l - \sum_{i=1}^{12} E_i|$ has projective dimension two and is basepointfree.
ix) $\dim|4\pi_1^* l - \sum_{i=1}^{12} E_i - E_j| = 0$ for $1 \le j \le 12$.

Lemma 2.1. *The conditions i),…,ix) are satisfied for a general choice of twelve points.*

Proof. For the first four conditions this fact is well-known. That the conditions v), vi), vii) and ix) are satisfied for a general choice of twelve points, follows from a result of Hirschowitz (see [Hi]), which says that if $\pi : S \to \mathbf{P}^2$ is the blowing-up of $\mathbf{P}^2$ in $r + s$ points in general position, with exceptional divisors E_i, and $3r + s \ge h^0(\mathcal{O}_{\mathbf{P}^2}(nl))$, then the linear system

$$|n\pi_1^* l - \sum_{i=1}^{r} 2E_i - \sum_{j=1}^{s} E_j|$$

is empty. For a set of points $x_1, \dots, x_{12}$ satisfying condition viii), we can take as the twelve points that we blow up the set of points linked to four points in the complete intersection of two quartic curves in $\mathbf{P}^2$.□

To proceed with the proof of the theorem we now let

$$F \equiv 4\pi_1^* l - \sum_{i=1}^{12} E_i,$$

and let

$$\varphi_F : S_1 \to \mathbf{P}^2$$

be the map defined by $|F|$. This map has degree

$$\deg\varphi_F = F^2 = 4,$$

Lemma 2.2. *The map φ_F is finite.*

Proof. This follows from the above conditions on the choice of points x_i, $i = 1,..,12$, since otherwise a curve C would be contracted by φ_F, i.e $\dim|F - C| = 1$. Any choice of curve C would contradict one of the above conditions.□

Thus each fibre of φ_F consists of the basepoints of a subpencil of $|F|$.

Next, choose a smooth curve L_0 in the linear system $|\pi_1^* l|$ on S_1, which is general in the sense that

1) $\varphi_{F|L_0}$ is birational
2) $\varphi_F(L_0)$ has three distinct nodes n_1, n_2 and n_3
3) $\varphi_F^{-1}(n_i)$ is reduced for i=1, 2, 3
4) The nodes n_i do not lie on any of the lines $\varphi_F(E_j)$.

Lemma 2.3. *The conditions 1),...,4) are satisfied for a general choice of L_0 in $|\pi_1^* l|$.*

Proof. The first one of these conditions is automatically satisfied in view of the above conditions iv) and v). For the second condition we see that $\varphi_F(L_0)$ has cusps only if L_0 is tangent to the branch curve of φ_F. It has a triple point only if L_0 contains three basepoints of a subpencil of $|F|$, which cannot be the case for every line L_0. Thus 2) is an open condition on L_0. To see that condition 2) is not empty on $|\pi_1^* l|$, let

$$L = (\pi_1^* l - E_1 - E_2) + E_1 + E_2,$$

then $\varphi_F(L)$ has acquired three distinct nodes.
The set of curves L_0 which does not satisfy condition 3), has codimension one. In fact, they correspond to the set of curves $\varphi_F(L_0)$, whose nodes lie on the ramification curve of φ_F, and since to each point $p \in \mathbf{P}^2$ there correspond at the most six lines L_0 through pairs of points of $\varphi_F^{-1}(p)$, this set has dimension one. Similarly, the set of curves that does not satisfy the fourth condition also has codimension one. So the lemma follows.□

We proceed to name the points of $\varphi_F^{-1}(n_i) \quad i = 1, 2, 3$ on S_1: By construction two points of $\varphi_F^{-1}(n_1)$ lies on L_0. Denote them by q_1 and q_2, and denote the other two by p_1 and p_2. Similarly, let $\varphi_F^{-1}(n_2) = p_3 + p_4 + q_3 + q_4$, where q_3 and q_4 lie on L_0, and let $\varphi_F^{-1}(n_3) =$

$p_5 + p_6 + q_5 + q_6$, where q_5 and q_6 lie on L_0. The points $p_1, ..., p_6$, which we may assume are disjoint from the exceptional curves $E_1, ..., E_{12}$, are the points that we blow up to get S. Let

$$\pi_2 : S \to S_1$$

be this blowing-up map, and let $F_1, ..., F_6$ be the exceptional divisors. If we set $\pi = \pi_1 \circ \pi_2$, then the points y_i of the theorem are the points $\pi_1(p_i) \quad i = 1, ..., 6$.

Part 2. We start this part by describing some linear systems of curves on S. First consider the above maps φ_F and π_2. The three nodes n_1, n_2 and n_3 of $\varphi_F(L_0)$ define a triangle in $\mathbf{P}^2$. The inverse images under φ_F of the edges of the triangle are curves on S. If we let L_{ij} be the edge through n_i and n_j, $1 \leq i < j \leq 3$, then we set

$$F'_{ij} = \varphi_F^{-1}(L_{ij}).$$

The strict transforms of the curves F'_{ij} on S will be denoted F_{ij}. Thus as divisors on S we have

$$F_{12} \equiv \pi_2^* F - F_3 - F_4 - F_5 - F_6,$$

$$F_{13} \equiv \pi_2^* F - F_1 - F_2 - F_5 - F_6$$

and

$$F_{23} \equiv \pi_2^* F - F_1 - F_2 - F_3 - F_4.$$

The pencils of lines through the nodes n_i correspond similarly to pencils of curves on S_1 whose general member we denote by F_1^{jk}, where $\{i, j, k\} = \{1, 2, 3\}$. Their strict transforms on S will be denoted by F^{jk}, thus as divisors on S we have

$$F^{12} \equiv \pi_2^* F - F_1 - F_2,$$

$$F^{13} \equiv \pi_2^* F - F_3 - F_4$$

and

$$F^{23} \equiv \pi_2^* F - F_5 - F_6.$$

Note that

$$F_{12} + F^{12} \equiv F_{13} + F^{13} \equiv F_{23} + F^{23} \equiv H_S$$

on S.

For part 3 we will need another system of curves also: Let $\pi_0 : S_0 \to S$ be the blowing-up of S in the points $q_1, \ldots, q_6$ with exceptional divisors $G_1, \ldots, G_6$, and let $D = \pi_0^* H_S - \sum_{i=1}^{6} G_i$. Thus $|D|$ corresponds to the linear subsystem of curves in $|H_S|$ on S which is generated by the three curves

$$F_{12} + F^{12} \quad \text{and} \quad F_{13} + F^{13} \quad \text{and} \quad F_{23} + F^{23}.$$

Since these curves are linearly independent we get that $\dim|D| = 2$. In fact $|D|$ corresponds via the map $\varphi_F \circ \pi_2 \circ \pi_0$ to the conics in $\mathbf{P}^2$ through the nodes n_1, n_2, and n_3. Thus the rational map

$$\varphi_D : S_0 \dashrightarrow \mathbf{P}^2$$

is a morphism and factors into $\varphi_F \circ \pi_2 \circ \pi_0$ composed with a Cremona transformation. We have the following commutative diagram

$$\begin{array}{ccccccc} S_0 & \xrightarrow{\pi_0} & S & \xrightarrow{\pi_2} & S_1 & \xrightarrow{\pi_1} & \mathbf{P}^2 \\ {\scriptstyle\varphi_D}\downarrow & \searrow & & & {\scriptstyle\varphi_F}\downarrow & & \\ \mathbf{P}^2 & \xleftarrow{\pi'} & T & \xrightarrow{\pi''} & \mathbf{P}^2 & & \end{array}$$

where $\pi'' : T \to \mathbf{P}^2$ is the blowing up of the nodes $n_i, \quad i = 1, 2, 3$, and $\pi' : T \to \mathbf{P}^2$ is the blowing up of the points $\varphi_D(F_{ij})$. Thus the triangle defined by the nodes n_i is transformed via T into a triangle whose edges are $e_{ij} = \varphi_D(F_i + F_j + G_i + G_j)$.

We proceed to show

Lemma 2.4. *dim*$|H_S|$=*4.*

Proof. Consider the linear system of curves $|2\pi_2^* F|$ on S. It is basepointfree, since $|F|$ is basepointfree, and defines a map

$$\varphi_{2F} : S \to \mathbf{P}^8.$$

To get $\dim|H_S| = 4$ we need to show that

$$Z = \{\varphi_{2F}(F_1), \dots, \varphi_{2F}(F_6)\}$$

only spans a $\mathbf{P}^3$ in $\mathbf{P}^8$. We will prove the following

Claim. $\varphi_{2F}(F_3), \dots, \varphi_{2F}(F_6)\}$ spans a $\mathbf{P}^2$ in $\mathbf{P}^8$, and similarly for

$$\{\varphi_{2F}(F_1), \varphi_{2F}(F_2), \varphi_{2F}(F_5), \varphi_{2F}(F_6)\}$$

and

$$\{\varphi_{2F}(F_1), \dots, \varphi_{2F}(F_4)\}.$$

If this claim holds, then we have three planes Π_{12}, Π_{13} and Π_{23} in $\mathbf{P}^8$ each containing four points among the $\varphi_{2F}(F_i), \quad i = 1, \dots, 6$. The planes must meet pairwise in lines, so their union is contained in a $\mathbf{P}^3$. The points $\varphi_{2F}(F_i)$ are contained in these lines, so the lemma follows if the claim holds.

Proof of the claim: Consider the linear system of curves

$$|H_{12}| = |2\pi_2^* F - F_3 - \dots - F_6|$$

on S. For the claim we need to show that $\dim|H_{12}| = 5$. For this consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(H_{12} - F_{12}) \longrightarrow \mathcal{O}_S(H_{12}) \longrightarrow \mathcal{O}_{F_{12}}(H_{12}) \longrightarrow 0$$

of sheaves on S. Note that $H_{12} - F_{12} \equiv \pi_2^* F$, so taking cohomology we get that $\dim|H_{12}| = 5$ if and only if $\dim|H_{12|F_{12}}| = 2$. But $H_{12} \cdot F_{12} = 4$, so this is equivalent to $H_{12|F_{12}} \equiv K_{F_{12}}$. To see that $|H_{12|F_{12}}|$ is the canonical linear series on F_{12}, we use the fact that the canonical linear series is equal to the series $|\pi_2^* L_{0|F_{12}}|$. Since L_0 contains the points $q_1, \ldots, q_6$, and $H_{12} \equiv F^{13} + F^{23}$ on S, we get

$$H_{12|F_{12}} \equiv (F^{13} + F^{23})_{|F_{12}} = q_3 + \ldots + q_6 = \pi_2^* L_{0|F_{12}}.$$

Therefore the claim holds and the lemma follows.□

Part 3. For very ampleness on S we study the restrictions $\varphi_{H|F_{ij}}$ and $\varphi_{H|F^{ij}}$ for $F^{ij} \in |F^{ij}|$, to get information on the double point locus of $\varphi_H : S \to \mathbf{P}^4$. If we consider the exact sequences

$$0 \longrightarrow \mathcal{O}_S(F_{12}) \longrightarrow \mathcal{O}_S(H_S) \longrightarrow \mathcal{O}_{F^{12}}(H_S) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_S(F^{12}) \longrightarrow \mathcal{O}_S(H_S) \longrightarrow \mathcal{O}_{F_{12}}(H_S) \longrightarrow 0$$

and their cohomology, then we see that the restriction maps

$$H^0(\mathcal{O}_S(H_S)) \to H^0(\mathcal{O}_{F^{12}}(H_S)) \qquad \text{and} \qquad H^0(\mathcal{O}_S(H_S)) \to H^0(\mathcal{O}_{F_{12}}(H_S))$$

are both surjective. In fact, by Lemma 2.4 and by Riemann-Roch we see that the sequences are exact after taking H^1, therefore also on global sections. Thus $\varphi_{H|F_{12}}$ is an isomorphism if the following lemma holds.

Lemma 2.5. *$H_{S|F_{12}}$ is very ample.*

Proof. To see that the divisor $H_{S|F_{12}}$ is very ample we recall from the proof of the above claim that $H_{12|F_{12}}$ is the canonical divisor on F_{12}. But $H_{12|F_{12}} \equiv H_{S|F_{12}}$, so $H_{S|F_{12}}$ is very ample since $F_{12} \cong \pi_1(\pi_2(F_{12}))$ and therefore not hyperelliptic.□

The same argument works of course for $\varphi_{H|F_{13}}$ and $\varphi_{H|F_{23}}$. Unfortunately, we cannot give a direct proof that $H_{S|F^{12}}$ is very ample for every F^{12} in $|F^{12}|$. But

Lemma 2.6. *Let $F^{12} \in |F^{12}|$. Then*

$$\varphi_H : F^{12} \to \mathbf{P}^4$$

has at the most one double point.

Proof. For this we first note that the only reducible curves of $|F|$ on S_1 are the curves $(F - E_i) + E_i, \quad i = 1, \ldots, 12$. Thus by condition iv) on the choice of L_0, the only reducible curves of $|F^{12}|$ are the two curves

$$F_{13} + F_5 + F_6 \qquad \text{and} \qquad F_{23} + F_3 + F_4.$$

Thus we have two cases to check.

First, if F^{12} is irreducible, then

$$\varphi_H : F^{12} \to \mathbf{P}^4$$

is birational, since $H \cdot F^{12} = 6$ and F^{12} is nonhyperelliptic. Thus $\deg \varphi_H F^{12} = 6$ and $p(\varphi_H(F^{12})) \leq 4$. Since F^{12} has arithmetic genus $p(F^{12}) = 3$, the lemma follows in this case.

Secondly, if $F^{12} = F_{13} + F_5 + F_6$, then we know from the above that

$$\varphi_{H_S|F_{13}} \quad \text{and} \quad \varphi_{H_S|F_5} \quad \text{and} \quad \varphi_{H_S|F_6}$$

are isomorphisms. (The two last ones are isomorphisms since $\varphi_{D|F_5}$ and $\varphi_{D|F_6}$ are isomorphisms.) The two lines $\varphi_H(F_5)$ and $\varphi_H(F_6)$ have at the most one point in common since $\varphi_{H_S|F_{13}}$ is an isomorphism. Thus we are left with the case that a point $t_1 \epsilon F_{13}$ and a point $t_2 \epsilon F_5$ not in F_{13} are mapped onto the same point by φ_H. But we can find a curve F^{13} in $|F^{13}|$ which does not contain t_2, such that $H_S = F_{13} + F^{13}$ separates t_1 and t_2, so the lemma follows also in this case. The curve $F_{23} + F_3 + F_4$ is treated in the same way.□

We go on to study the double point locus of the morphism

$$\varphi_H : S \to \mathbf{P}^4.$$

We will show that it is finite, and then conclude from the double point formula that it is empty.

Lemma 2.7. *The morphism $\varphi_H : S \to \mathbf{P}^4$ is finite and birational.*

Proof. We have from Lemma 2.2. that $\varphi_F : S_1 \to \mathbf{P}^2$ is a finite morphism. Since φ_D factors into $\varphi_F \circ \pi_2 \circ \pi_0$ composed with a Cremona transformation with basepoints at the nodes n_1, n_2 and n_3, we get that φ_D contracts only the curves $\pi_0^{-1}(F_{12})$, $\pi_0^{-1}(F_{13})$ and $\pi_0^{-1}(F_{23})$. But the linear system $|D|$ corresponds to a linear subsystem of $|H|$, and $\varphi_{H|F_{ij}}$ is an isomorphism for $1 \leq i < j \leq 3$, so we may conclude that φ_H is finite. It is birational since it is birational when restricted to the special hyperplane sections $F_{12} + F^{12}$ above.□

Lemma 2.8. *φ_H has no double curve.*

Proof. If B is the double curve for φ_H, then clearly $B_0 = \pi_0^{-1}(B)$ is double for the map φ_D. By Lemma 2.6 we have that $\varphi_{H|F^{12}}$ has at the most one double point, so $\varphi_D(B_0)$ must have degree at the most one. Since φ_D is finite outside the curves $\pi_0^{-1}(F_{ij})$, we get that $\varphi_D(B_0)$ has at least degree one. Thus the degree of $\varphi_D(B_0)$ is one, and $B_0 \subset D_0$ for a curve $D_0 \in |D|$. Now, $\varphi_{D|B_0}$ must have degree two since otherwise $\varphi_{H|F^{12}}$ would have at least two double points. So B_0 must be a proper component of D_0. On the other hand the conditions i),…,vii) of Lemma 2.1 does not give room for any decompositions of D_0 which satisfies this condition on B_0, so the lemma follows.□

We conclude by the double point formula that $\varphi_H : S \to \mathbf{P}^4$ is an isomorphism.□

Postulation

We give the following result on the postulation of S.

Proposition 2.9. *The surface S is linked to the union of two twisted cubic surfaces T and U in two quartic hypersurfaces.*

Proof. Associated with S there are the three planes Π_{ij} of the curves F_{ij}, and their common line, the 6-secant line L. Let

$$T = \Pi_{12} \cup \Pi_{34} \cup \Pi_{56}.$$

Then T is the union of 5-secants to S, and is therefore contained in any quartic hypersurface which contains S.

Next, consider the linear subsystem of hyperplane sections which correspond to the hyperplanes of $\mathbf{P}^4$ that contains L. Recall the blowing-up $\pi_0 : S_0 \to S$ of S in the points $L \cap S$. We denote the exceptional divisors by $G_1, \ldots, G_6$. If D denotes the strict transforms $D \equiv \pi_0^* H - \sum_{i=1}^6 G_i$ on S_0, then $|D|$ defines a map $\varphi_D : S_0 \to \mathbf{P}^2$ which is of degree four. Recall, from the construction of S, the curve $L_1 \equiv \pi^* l - \sum_{i=1}^6 G_i$ on S_0, where $\pi = \pi_1 \circ \pi_2 \circ \pi_0$. L_1 is mapped isomorphically to a plane conic by φ_D, since $D \cdot L_1 = 2$ and $|D - L_1| = \emptyset$ by Lemma 2.1. Therefore, there is a curve

$$C_0 \equiv 2D - L_1$$

on S_0. Denote by C the image of C_0 on S. Then C has arithmetic genus 13 and degree $H \cdot C = 12$ on S.

Lemma 2.10. *C lies on a rational cubic scroll which contains L as a section.*

Proof. The idea is to show that C lies on three linearly independent quadric hypersurfaces. These three quadrics define a surface, call it U, since C has degree larger than 8. A careful argument will show that the surface U must be smooth.

First, consider the exact sequence

$$0 \longrightarrow \mathcal{I}_C(2) \longrightarrow \mathcal{O}_{\mathbf{P}^4}(2) \longrightarrow \mathcal{O}_C(2) \longrightarrow 0$$

of sheaves on $\mathbf{P}^4$. If we take cohomology and use Riemann-Roch on C, we get that $h^0(\mathcal{I}_C(2)) \geq 2$. Compare this with the cohomology of the exact sequence of sheaves of ideals

$$0 \longrightarrow \mathcal{I}_C(1) \longrightarrow \mathcal{I}_C(2) \longrightarrow \mathcal{I}_{C \cap H}(2) \longrightarrow 0$$

on $\mathbf{P}^4$. Since $h^0(\mathcal{I}_C(1)) = 0$ we get that

$$h^0(\mathcal{I}_{C \cap H}(2)) \geq h^0(\mathcal{I}_C(2)) \geq 2,$$

for any hyperplane section H. Let $D_0 = \pi_0(D)$ for a general $D \in |D|$. Then D_0 meets C in six points on L, and in six points $x_1, \ldots, x_6$ on S outside L.

Lemma 2.11. *The points x_i lie on the union of two lines L_1 and L_2 which both meet L.*

Proof. If we work for a moment on S_0, then, since φ_D restricts to a map of degree three on C_0, we may group the x_i on S into two sets such that say x_1, x_2, x_3 span a plane which contains L, and x_4, x_5, x_6 span another plane which contains L. Since $\varphi_D(C_0)$ is a conic, we get that $x_1 + x_2 + x_3$ and $x_4 + x_5 + x_6$ belong to the same linear series as divisors on C. Now, $L \cup \{x_1, \ldots, x_6\}$ is contained in two quadrics, so we get that either $\{x_1, x_2, x_3\}$ or $\{x_4, x_5, x_6\}$ is contained in a line. Since, as divisors on C, they belong to the same linear series, they must both be contained in lines.□

Varying the divisor D, we see that the lines L_1 and L_2 are members of a ruling of U. This is the ruling of a smooth surface U, since L_1 and L_2 cannot meet. U contains the curves C and L, and L meets each member of the ruling.□

The scroll U is rational and has a hyperplane divisor $H_U \equiv 2l - E$ on a $\mathbf{P}^2$ blown up in one point. The line L equals on U the exceptional divisor E, while C meets L in six points, so $C \cdot E = 6$. Since $H \cdot C = 12$, we get that

$$C \equiv 9l - 6E.$$

Any conic l on U meets C in nine points, therefore any quartic that contains S must also contain U.

Now, to show that S actually lies on two quartic hypersurfaces, we study more closely some cohomology groups. Let Π be a general plane which contains the line L, let H be a general hyperplane which contains Π, and consider the exact sequence of sheaves of ideals

$$0 \longrightarrow \mathcal{I}_{S\cap H}(3) \longrightarrow \mathcal{I}_{S\cap H}(4) \longrightarrow \mathcal{I}_{S\cap\Pi}(4) \longrightarrow 0.$$

We take cohomology and get that $h^1(\mathcal{I}_{S\cap\Pi}(4)) = 1$ since $S \cap \Pi$ contains six colinear points. Now, $\mathcal{O}_{S\cap H}(2)$ is nonspecial, so we get that $h^2(\mathcal{I}_{S\cap H}(2)) = 0$, and therefore, from the exact cohomology sequence, that $h^1(\mathcal{I}_{S\cap H}(4)) \geq 1$.

If we compare this with the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_S(3) \longrightarrow \mathcal{I}_S(4) \longrightarrow \mathcal{I}_{S\cap H}(4) \longrightarrow 0,$$

we get that $h^1(\mathcal{I}_S(4)) \geq 1$ as soon as $h^2(\mathcal{I}_S(3)) = 0$.

Lemma 2.12. $h^2(\mathcal{I}_S(3)) = 0$

Proof. For this we consider the cohomology of the exact sequence of sheaves of ideals

$$0 \longrightarrow \mathcal{I}_S(2) \longrightarrow \mathcal{I}_S(3) \longrightarrow \mathcal{I}_{S\cap H}(3) \longrightarrow 0.$$

Since $h^2(\mathcal{I}_{S\cap H}(3)) = 0$ from the nonspeciality of $\mathcal{O}_{S\cap H}(3)$, it suffices to show that

$$h^2(\mathcal{I}_S(2)) = 0$$

which is equivalent to

$$h^1(\mathcal{O}_S(2H)) = 0.$$

This is checked via cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(l) \longrightarrow \mathcal{O}_S(2H) \longrightarrow \mathcal{O}_C(2H) \longrightarrow 0$$

to be equivalent to $h^1(\mathcal{O}_C(2H)) = 0$. Now $\deg\mathcal{O}_C(2H) = 24$ and $p(C) = 13$, so

$$h^1(\mathcal{O}_C(2H)) \neq 0$$

only if

$$\mathcal{O}_C(2H) = \omega_C.$$

We will see that this cannot be the case by arguing on U.

Recall that $C \equiv 9l - 6E$ on U, thus

$$\mathcal{O}_C(2H) = \mathcal{O}_C(4l - 2E)$$

and

$$\omega_C = \mathcal{O}_C(C + K_U) = \mathcal{O}_C(6l - 5E).$$

Thus $\omega_C = \mathcal{O}_C(2H)$ if and only if $\mathcal{O}_C = \mathcal{O}_C(2l - 3E)$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_U(-7l + 3E) \longrightarrow \mathcal{O}_U(2l - 3E) \longrightarrow \mathcal{O}_C(2l - 3E) \longrightarrow 0$$

of sheaves on U. If we take cohomology, then we have

$$h^0(\mathcal{O}_U(2l - 3E)) = 0.$$

Therefore, $h^0(\mathcal{O}_C(2l-3E)) = 1$ only if $h^1(\mathcal{O}_U(-7l+3E)) = 1$. But this is clearly impossible since any curve in $|7l - 3E|$ is connected. □

From the lemma we get that $h^1(\mathcal{I}_S(4)) \geq 1$. If we consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_S(3) \longrightarrow \mathcal{I}_S(4) \longrightarrow \mathcal{I}_{S\cap H}(4) \longrightarrow 0$$

of sheaves of ideals, then we have that $h^2(\mathcal{I}_{S\cap H}(4H)) = 0$ since $\mathcal{O}_{S\cap H}(4H)$ is nonspecial. Thus the lemma also implies that $h^2(\mathcal{I}_S(4)) = 0$. Since $\chi(\mathcal{I}_S(4)) = 1$, this implies that $h^0(\mathcal{I}_S(4)) \geq 2$ and the proposition follows.□

Remark. Starting with two twisted cubic surfaces T and U, such that T is the union of three planes through a common line L, and U is a smooth scroll with L as a section with selfintersection $L^2 = -1$ in U, such that U meets T only along L, then one may show that $T \cup U$ is cut out by quartic hypersurfaces and is linked to a smooth rational surface S with $\pi(S) = 9$. This gives an alternative proof of the existence of S. The proposition also shows that the quartics containing S are singular along the line L.

3 A K3-surface with $\pi = 9$

Let V_3 be a cubic hypersurface in $\mathbf{P}^4$ with exactly one isolated quadratic singularity at a point x, and let V_2 be a smooth quadric hypersurface which meets the double point of V_3 such that the complete intersection $S_0 = V_2 \cap V_3$ has a quadratic singularity at x and is smooth elsewhere. Let

$$\pi_0 : X \to \mathbf{P}^4$$

be the blowing-up of $\mathbf{P}^4$ in the point x, and let S_1 be the strict transform of S_0 on X with a curve A_0 lying over the point x. Then S_1 is smooth, and the curve A_0 is an irreducible rational curve with selfintersection $A_0^2 = -2$. Next, let H be a general hyperplane section of S_0, and let Π be a plane in $\mathbf{P}^4$ which is tangent to H in three distinct points y_1, y_2, y_3, more specifically we require that the intersections $V_2 \cap \Pi$ and $V_3 \cap \Pi$ are, respectively, an irreducible plane conic and a plane cubic curve which both go through and have a common tangent at the points y_1, y_2, y_3. Denote the preimage of the points y_1, y_2, y_3 on S_1 also by y_1, y_2, y_3, and blow them up with a map $\pi_1 : S \to S_0$ to get a smooth surface S with exceptional divisors E_1, E_2, E_3.

On S we let C_0 denote the pullback (total transform) of the hyperplane divisor on S_0, and let A denote the total transform of A_0 lying over the node x on S_0. Consider the linear system of curves

$$|H| = |2C_0 - A - \sum_{i=1}^{3} 2E_i|$$

on S.

Proposition 3.1. *The above data V_2, V_3 and Π can be chosen such that the linear system of curves $|H|$ on S is very ample and embeds S as a surface of degree 10 in $\mathbf{P}^4$.*

Proof. The proof amounts to exploiting a decomposition of the divisor H. Consider the linear systems of curves

$$|C| = |C_0 - \sum_{i=1}^{3} E_i|$$

and

$$|D| = |C - A|$$

on S. First we note that we have a decomposition $H \equiv C + D$, next one sees immediately from the construction that $|C|$ is a pencil of curves S, while $|D|$ contains exactly one curve, call it D, namely the strict transform of the hyperplane section of S_0 which contains the points y_1, y_2, y_3 and x.

We will study the requirement that the linear series $|H_{|C}|$ for every curve $C \in |C|$ and $|H|_D|$ are very ample.

Since $H \cdot C = 6$ and $p(C) = 4$, we see that for this to be true, $|H_{|C}|$ must be the canonical linear series on C. It is to get this that we need the special choice of the points y_1, y_2, y_3.

Namely, for a general curve $C \in |C|$ the divisor $(y_1 + y_2 + y_3) = (\sum_{i=1}^{3} E_i)_{|C}$ is a thetacharacteristic on C. To use this fact, we do the following: On S we see that $|C_0|$ is the adjoint linear system to C, therefore

$$|H_{|C}| = |(2C_0 - A - \sum_{i=1}^{3} 2E_i)_{|C}| = |K_C| = |C_{0|C}|$$

if and only if

$$|(C_0 - A)_{|C}| = |(\sum_{i=1}^{3} 2E_i)_{|C}|.$$

But the latter holds, since A does not meet C at all, so $|C_0 - A|$ restricts to the canonical linear series on C, while $|(\sum_{i=1}^{3} E_i)_{|C}|$ is a thetacharacteristic.

Now we consider the exact sequences

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_D(H) \longrightarrow 0$$

of sheaves on S.

If we take cohomology in the first sequence, we have from the above that $h^0(\mathcal{O}_S(D)) = 1$, so we get by duality and Riemann-Roch that

$$h^1(\mathcal{O}_S(D)) = h^2(\mathcal{O}_S(D)) = 0.$$

Therefore

$$h^1(\mathcal{O}_C(H)) = h^1(\mathcal{O}_C(K_C)) = 1$$

implies that $h^1(\mathcal{O}_S(H)) = 1$ and, by Riemann-Roch again, that $h^0(\mathcal{O}_S(H)) = 5$.

If we take cohomology in the second sequence, we similarly have that

$$h^1(\mathcal{O}_S(C)) = h^2(\mathcal{O}_S(C)) = 0.$$

Thus $h^1(\mathcal{O}_S(H)) = 1$ implies that $h^1(\mathcal{O}_D(H)) = 1$ and $h^0(\mathcal{O}_D(H)) = 3$. Since $p(D) = 3$ and $H \cdot D = 4$, we get that $|H|_D|$ is the canonical linear series on D. Note also that the restriction maps

$$\mathrm{H}^0(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(C)H)$$

and

$$\mathrm{H}^0(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(D)H)$$

are both surjective.

We continue with a more detailed study of the linear series $|H|_D|$ and $|H_{|C}|$. Let $\varphi_H : S \dashrightarrow \mathbf{P}^4$ denote the rational map defined by the linear system $|H|$.

Lemma 3.2. *φ_H restricts to an embedding of the curve $D+A$ on S.*

Proof. First we check that D is not hyperelliptic. As noted above, D is the strict transform on S of the hyperplane section, call it L_0, of S_0 which contains the points y_1, y_2, y_3 and x. Since L_0 has a double point at x, we see that the canonical morphism defined by the canonical linear series on D is the projection of L_0 from the point x into a plane. Thus D is hyperelliptic if and only if this map is 2 to 1, or geometrically, any line in $\mathbf{P}^4$ through x which meets L_0 in a point away from x will meet L_0 in two (possibly infinitely close) points away from x. But by Bezout and our choice of V_2 and V_3, this means that any such line is contained in the surface S_0, which is absurd. Therefore the above considerations show that φ_H embeds the curve D.

To see that φ_H embeds the curve A, we recall that A is a smooth rational curve. Therefore, since $H \cdot A = 2$, it is enough to show that the restriction map

$$\mathrm{H}^0(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(A)H)$$

is surjective. For this we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(2D) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_A(H) \longrightarrow 0$$

of sheaves on S. If we take cohomology, we get that $h^1(\mathcal{O}_A(H)) = 0$ and from the above that $h^1(\mathcal{O}_S(H)) = 1$. Thus we get the wanted surjectivity if $h^1(\mathcal{O}_S(2D)) = 1$. In fact it suffices to show that $h^1(\mathcal{O}_S(2D)) \leq 1$. We get this by considering the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \mathcal{O}_S(2D) \longrightarrow \mathcal{O}_D(2D) \longrightarrow 0$$

of sheaves on S. If we take cohomology here, we recall from the above that $h^1(\mathcal{O}_S(D)) = h^2(\mathcal{O}_S(D)) = 0$. Therefore

$$h^1(\mathcal{O}_S(2D)) = h^1(\mathcal{O}_D(2D)).$$

Again since D is not hyperelliptic and $(2D) \cdot D = 2$, we know that $h^1(\mathcal{O}_D(2D)) \leq 1$, so $h^1(\mathcal{O}_S(2D)) \leq 1$ and $\mathrm{H}^0(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(A)H)$ is surjective.

To complete the proof of the lemma, we need to see that $|H|$ separates any two points $p \in D \setminus A$ and $q \in A \setminus D$. If we assume to the contrary that p and q are mapped to the same point by φ_H, then since $D \cdot A = 2$ and A is embedded as a smooth conic, A must be mapped into the plane of $\varphi_H(D)$. But this would imply that the residual curve $D \equiv H - (D+A)$ moves in a pencil, which contradicts the above.□

To complete the proof of the proposition, we need to check that $|H_{|C}|$ is a very ample linear series on every curve $C \in |C|$. The lemma takes care of the curve $C = D + A$. For the rest of the curves we simply recall the equalities

$$|(C+D)_{|C}| = |2C_{|C}| = |(2C_0 - \sum_{i=1}^{3} 2E_i)_{|C}| = |C_{0|C}|.$$

Since C_0 is the pullback of the hyperplane divisor on S_0, and $\pi : S \to S_0$ restricts to an isomorphism on every $C \neq D + A$, we are done.□

Postulation

If $\pi : W_2 \to V_2$ is the blowing-up of V_2 in the points y_1, y_2, y_3 and x, and S is the strict transform of S_0 on W_2, then the embedding $\varphi_H : S \to \mathbf{P}^4$ extends to a birational morphism on W_2. One may show that the image of W_2 in $\mathbf{P}^4$ is a quartic hypersurface with multiplicity two along the plane of the curve D.

4 An elliptic surface with $\pi = 9$

Proposition 4.1. *There is a smooth elliptic surface of Kodaira dimension 1, of degree 10 and with $\pi = 9$ in $\mathbf{P}^4$.*

Proof. The proof has three parts. First, we construct a rational threefold W_0 of degree 7 in $\mathbf{P}^6$ whose general hyperplane section is a Bordiga surface. Secondly, we describe a smooth regular elliptic surface S_0 on W_0 of degree 13 with $\chi(\mathcal{O}_{S_0}) = 2$, and at last we find a line L in $\mathbf{P}^6$ such that the projection of S_0 from L is a smooth surface of degree 10 in $\mathbf{P}^4$.

For the first part we study the composition of two blowing-up maps

$$W \to V \to \mathbf{P}^3,$$

where the first map, $\pi_V : V \to \mathbf{P}^3$, is the blowing-up in a point p with exceptional divisor E_V, and the second map, $\pi_W : W \to V$, is the blowing-up along a smooth curve C_V on V with exceptional divisor S_W on W. Let h denote a general hyperplane in $\mathbf{P}^3$, and as usual in a blowing-up situation, we use the same notation for its total transform on V and W.

To specify the point p and the curve C_V, we consider a cubic surface S_3 in $\mathbf{P}^3$. The surface S_3 is a cone over a smooth plane cubic curve with vertex in p. For later use we need to specify the plane cubic curve; it must contain the six points of intersection, $p_1, \ldots, p_6$, of four general lines in the plane.

Let $S_{3,V}$ be the strict transform of S_3 on V. Then $S_{3,V}$ is smooth with a morphism

$$p_S : S_{3,V} \to C_S$$

which defines the ruling of $S_{3,V}$ over an elliptic curve C_S. The Picard group of $S_{3,V}$ is

$$Pic(S_{3,V}) = < \mathcal{O}_{S_{3,V}}(B) > \oplus p_S^*(Pic(C_S)),$$

where $B = S_{3,V} \cap E_V$. If $F_1, \ldots, F_6$ are the members of the ruling which meets the points $p_1, \ldots, p_6$, then we denote the points $F_i \cap B$ for $i = 1, \ldots, 6$ by $q_1, \ldots, q_6$. The map of $S_{3,V}$ into $\mathbf{P}^3$ is defined by a linear system $|B + \alpha F|$, where αF is the pullback to $S_{3,V}$ of a divisor of degree 3 on C_S. Now, consider the linear system of curves $|B + \beta F|$, where $\beta F \equiv F_1 + \ldots + F_6 + \alpha F$.

Lemma 4.2. *$|B + \beta F|$ contains a smooth curve which meets B in the points $q_1, \ldots, q_6$.*

Proof. We blow up the points q_i, to get $S'_{3,V}$ with exceptional curves E_i, for $i = 1, \ldots, 6$. If we use the same notation for curves on $S_{3,V}$ as for their total transform on $S'_{3,V}$, then we are looking for a smooth curve in the linear system

$$|D| = |B + \beta F - \sum_{i=1}^{6} E_i|$$

on $S'_{3,V}$. If $B_0 \equiv B - \sum_{i=1}^{6} E_i$, then it follows from the choice of points q_i that $|D_{|B_0}|$ is trivial, hence has no basepoints on B_0. On the other hand βF clearly has no basepoints on $S'_{3,V}$, so $|D|$ has no basepoints on $S'_{3,V}$ as soon as the restriction map

$$\mathrm{H}^0(\mathcal{O}(S'_{3,V})D) \to \mathrm{H}^0(\mathcal{O}(B_0)D)$$

is surjective. But this holds since the cokernel of this map injects into $\mathrm{H}^1(\mathcal{O}(S'_{3,V})\beta F)$, which is trivial. The lemma now follows using Bertinis theorem.□

Let $C_V \in |B + \beta F|$ be such a curve. We blow up V along C_V to get W, with exceptional divisor S_W. We let $S_{3,W}$ and E_W denote the strict transforms of $S_{3,V}$ and E_V, respectively, on W. Then $S_{3,W}$ belongs to the linear system $|3h - S_W - 3E_W|$ of divisors on W. Consider the linear system

$$|d| = |h + S_{3,W}| = |4h - S_W - 3E_W|$$

of divisors on W.

Lemma 4.3. *$|d|$ is a basepointfree linear system of projective dimension* $\dim|d| = 6$.

Proof. Since $d \equiv h + S_{3,W}$, we get that $|d|$ has no basepoints outside $S_{3,W}$. Restricting it to $S_{3,W}$, we get

$$d_{|S_{3,W}} \equiv (4\alpha - \beta)F,$$

so that $|d_{|S_{3,W}}|$ has no basepoints. But the restriction map

$$\mathrm{H}^0(\mathcal{O}(W)d) \to \mathrm{H}^0(\mathcal{O}(S_{3,W})d)$$

is surjective; its cokernel is contained in $\mathrm{H}^1(\mathcal{O}(W)h)$, which is trivial. Thus we get that $|d|$ has no basepoints on $S_{3,W}$ either. Furthermore we get the dimension of $|d|$ if we consider the global section of the exact sequence

$$0 \longrightarrow \mathcal{O}_W(h) \longrightarrow \mathcal{O}_W(d) \longrightarrow \mathcal{O}_{S_{3,W}}(d) \longrightarrow 0$$

of sheaves on W.□

We may now describe the morphism

$$\varphi_d : W \to \mathbf{P}^6$$

defined by $|d|$. We have just seen that the restriction of $|d|$ to $S_{3,W}$ is $|(4\alpha - \beta)F|$, therefore $S_{3,W}$ is mapped to a plane cubic curve, call it C_3, by φ_d. The restriction of $|d|$ to E_W is a linear system of cubic curves with assigned basepoints at $q_1, \ldots, q_6$. Now the configuration of points q_i on E_V corresponds to the configuration of points p_i, such that the q_i are the points of intersection of four lines $L'_1, \ldots, L'_4$ in the plane E_V. The strict transforms $L_1, \ldots, L_4$ on W of these four lines are clearly collapsed by φ_d, so that φ_d maps E_W onto a cubic surface with four double points. Note that C_3 is a plane section of this cubic surface.

Let $h \in |h|$ be the strict transform on W of a general plane in $\mathbf{P}^3$. Then $|d|$ restricts to h as the linear system of quartic curves with assigned basepoints at the points of intersection of $\pi_W(h)$ and the curve C_V in V. Thus $\varphi_d(h)$ is a Bordiga surface of degree 7.

If $h_p \in |h - E_W|$ is the strict transform on W of a general plane in $\mathbf{P}^3$ through p, then $|d|$ restricts to h_p as a linear system of quartic plane curves with an assigned triple basepoint at p and with three assigned basepoints at the points of intersection of $\pi_V(h_p)$ and C_V. Therefore $\varphi_d(h_p)$ is a rational normal scroll of degree 4. The surface S_W is mapped to a scroll of degree 12 since the strict transform of a general plane in $\mathbf{P}^3$ meets S_W in nine exceptional curves which are mapped onto lines by φd, and $S_{3,W}$ meets S_W in a curve which is a section of the morphism

$$p_S \circ \pi_W : S_{3,W} \to C_S.$$

Lemma 4.4. *The morphism $\varphi_d : W \to W_0 \subset \mathbf{P}^6$ contracts $S_{3,W}$ to a plane cubic curve, it contracts the four lines $L_1, \ldots, L_4$ on E_W and it is an isomorphism elsewhere.*

Proof. It remains to show the last part, and for this we check the restriction of the morphism to any member of $|h|$. If $h \in |h|$ does not come from a plane in $\mathbf{P}^3$ through p, then, as we noted above, we may write

$$d|_h \equiv 4l - \sum_{i=1}^{9} E_i,$$

where l is the pullback to h of a line and the E_i are the exceptional curves lying over $h \cap C_V \subset V$. And if $h_p \in |h - E_W|$, then we may write

$$d_{|h_p} \equiv 4l - 3E_1 - \sum_{i=2}^{4} E_i,$$

where l is the pullback of a line, E_1 is the exceptional curve lying over p and E_2, E_3, E_4 are the exceptional curves which lie over the points $h_p \cap C_V \subset V$.

Now, the curve $\pi_V(C_V) \subset \mathbf{P}^3$ has no 4-secant line and no 7-secant conic, which does not go through the point p, therefore it is straightforward to check that any curve which is contracted by $\varphi_d|_h$ or $\varphi_{d|h_p}$ has support on the exceptional curves on h and h_p. Now, any singularity on $\varphi_d(h_p)$ comes from the contraction of a curve, since $\varphi_d(h_p)$ is a rational normal scroll. Therefore, the only scrolls that are singular are the four scrolls which meet E_W along the lines L_i. It is less obvious but may still be shown using an argument similar to that of the proof of Lemma 0.12, that any singularity on $\varphi_d(h)$ also arises from the contraction of curves on h.

Now we may use members of $|h|$ to separate any two points and tangent directions outside $S_{3,W}$ and E_W. A point on $S_{3,W}$ is seperated from any point q outside of $S_{3,W}$, since we may find an h which does not meet q. A point on E_W is separated from one outside of E_W by a similar argument.□

Next, we find a smooth elliptic surface on W which is mapped onto a surface of degree 13 in $\mathbf{P}^6$. For this, consider the linear system of divisors

$$|\Sigma| = |7h - 2S_W - 4E_W| = |d + S_{3,W} + 2E_W|$$

on W. We calculate some relevant cohomology for this linear system. First, observe that

$$\Sigma_{|E_W} \equiv L_1 + L_2 + L_3 + L_4,$$

and that

$$\Sigma_{|S_{3,W}} \equiv h_{|S_{3,W}} \equiv B + \alpha F.$$

Thus, when we consider the cohomology of the following exact sequences

$$0 \longrightarrow \mathcal{O}_W(h) \longrightarrow \mathcal{O}_W(d) \longrightarrow \mathcal{O}_{S_{3,W}}(d) \longrightarrow 0, \tag{1}$$

$$0 \longrightarrow \mathcal{O}_W(d+E_W) \longrightarrow \mathcal{O}_W(d+2E_W) \longrightarrow \mathcal{O}_{E_W}(d+2E_W) \longrightarrow 0, \tag{2}$$

$$0 \longrightarrow \mathcal{O}_W(d) \longrightarrow \mathcal{O}_W(d+E_W) \longrightarrow \mathcal{O}_{E_W}(d+E_W) \longrightarrow 0, \tag{3}$$

$$0 \longrightarrow \mathcal{O}_W(\Sigma - E_W) \longrightarrow \mathcal{O}_W(\Sigma) \longrightarrow \mathcal{O}_{E_W}(\Sigma) \longrightarrow 0, \tag{4}$$

$$0 \longrightarrow \mathcal{O}_W(d+E_W) \longrightarrow \mathcal{O}_W(\Sigma - E_W) \longrightarrow \mathcal{O}_{S_{3,W}}(\alpha F) \longrightarrow 0, \tag{5}$$

$$0 \longrightarrow \mathcal{O}_W(d+2E_W) \longrightarrow \mathcal{O}_W(\Sigma) \longrightarrow \mathcal{O}_{S_{3,W}}(B+\alpha F) \longrightarrow 0 \tag{6}$$

of sheaves on W, then we may set up the following tables for the cohomology of sheaves on W:

	$\mathcal{O}_W(h)$	$\mathcal{O}_W(d)$	$\mathcal{O}_W(d+E_W)$	$\mathcal{O}_{E_W}(d+2E_W)$	$\mathcal{O}_W(d+2E_W)$
h^0	4	7	7	0	7
h^1	0	0	0	3	0
h^2	0	0	0	0	0
h^3	0	0	0	0	0

and

	$\mathcal{O}_W(\Sigma - E_W)$	$\mathcal{O}_{E_W}(\Sigma)$	$\mathcal{O}_W(\Sigma)$
h^0	10	1	11
h^1	0	4	4
h^2	0	0	0
h^3	0	0	0

We are now ready to look for a smooth surface $S_1 \in |\Sigma|$.

Lemma 4.5. *The base locus of $|\Sigma|$ consists of the curves $L_1, \ldots, L_4$ on E_W, and the general member of $|\Sigma|$ is smooth.*

Proof. Clearly, there are no basepoints outside $S_{3,W}$ and E_W. The restriction

$$|\Sigma_{|S_{3,W}}| = |B + \alpha F|$$

has no basepoints, so $|\Sigma|$ has no basepoints on $S_{3,W}$ as soon as the restriction map

$$\mathrm{H}^0(\mathcal{O}(W)\Sigma) \to \mathrm{H}^0(\mathcal{O}(S_{3,W})\Sigma)$$

is surjective. But this is clear from the above table and the long exact sequence of cohomology associated with the exact sequence (6). The restriction $\Sigma_{|E_W} \equiv L_1 + L_2 + L_3 + L_4$, so the first part of the lemma follows.

The last part of the lemma now follows from the Bertini theorem, since the baselocus of $|\Sigma|$ is a smooth curve. □

We may now choose a smooth member $S_1 \in |\Sigma|$, and doing so we make sure that

$$(S_1 \cap S_{3,W}) \cap (S_1 \cap E_W) = \emptyset.$$

By the above Lemma 4.4, it is easy to see that $S_0 = \varphi_d(S_1)$ is smooth outside the image of the curves L_i. This could also be shown using Lemma 0.13, since C_3 lies on S_0 and the residual curves $C_1 \equiv (d - S_{3,W})_{|S_1}$ are canonical curves.

The canonical divisor of S_1 is given by adjunction:

$$K_W \equiv -4h + 2E_W + S_W$$

so

$$K_{S_1} \equiv (K_W + S_1)_{|S_1} \equiv (S_{3,W} + E_W)_{|S_1} \equiv (S_1 \cap S_{3,W}) + L_1 + L_2 + L_3 + L_4.$$

Therefore $L_1 + \ldots + L_4$ belongs to a canonical divisor of S_1. Now if $1 \leq i \leq 4$, then

$$K_{S_1} \cdot L_i = L_i^2$$

since L_i does not meet any of the other components of K_{S_1}. Thus, the curves L_i, $i = 1, \ldots, 4$, are (-1)-curves on S_W which are blown down on S_0.

The degree of S_0 is

$$deg S_0 = d^2 \cdot \Sigma = d^3 + d^2 \cdot S_{3,W} + 2d^2 \cdot E_W = 13.$$

The projection into $\mathbf{P}^4$ is the projection from the image in $\mathbf{P}^6$ of the strict transform of a general line L_0 in $\mathbf{P}^3$ which meets the point p. This image is a line L in $\mathbf{P}^6$ which meets S_0 in exactly 3 points. To show that S is smooth it suffices by Lemma 0.4 to show that the strict transform h_p on W of any plane through L_0 is mapped smoothly into $\mathbf{P}^4$. But this is straightforward to check, since they are the projections of smooth rational scrolls from a member of the ruling.

For the invariants of S we work on W and use the exact sequences

$$0 \longrightarrow \mathcal{O}_W(K_W) \longrightarrow \mathcal{O}_W(S_{3,W} + E_W) \longrightarrow \mathcal{O}_{S_1}(K_{S_1}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_W(E_W) \longrightarrow \mathcal{O}_W(S_{3,W} + E_W) \longrightarrow \mathcal{O}_{S_{3,W}}(S_{3,W} + E_W) \longrightarrow 0$$

of sheaves on W. From the last sequence we get that $\chi(\mathcal{O}_W(S_{3,W} + E_W)) = 1$, so applying this to the first sequence we get that $\chi(\mathcal{O}_{S_1}) = 2$. In particular since $h^1(\mathcal{O}_W(E_W)) = 0$ and

$$h^0(\mathcal{O}_{S_{3,W}}(S_{3,W} + E_W)) = h^0(\mathcal{O}_{S_{3,W}}((3\alpha - \beta)F)) = 0,$$

we get that $h^0(\mathcal{O}_W(S_{3,W} + E_W)) = 1$ and thus

$$p_g(S) = h^0(\mathcal{O}_{S_1}(K_{S_1})) = 1.\square$$

Postulation

Projecting W_0 from the line L we get a hypersurface in $\mathbf{P}^4$ of degree 4 with a double plane. The double plane is the image of E_W in $\mathbf{P}^4$.

5 A surface of general type with $\pi = 9$

We will construct a smooth surface S with numerical invariants $p_g = 2$, $q = 0$, $K^2 = 3$, and a linear system of curves $|H|$ on S such that $|H|$ is very ample on S and embeds S as a smooth surface of degree 10 in $\mathbf{P}^4$ with $\pi = 9$. Miles Reid gives, in [R1], equations of surfaces of the above invariants for which the linear system $|2K|$ is very ample and for which there is additionally a Z_3-action. Dropping the group action, a modification of his argument gives the following:

Proposition 5.1. *There is a minimal smooth surface S with numerical invariants $p_g = 2$, $q = 0$, $K^2 = 3$ and exactly one irreducible (-2)-curve A, for which the bicanonical linear system $|2K|$ defines a birational morphism $\varphi_{2K} : S \to \mathbf{P}^5$ which contracts the curve A and is an isomorphism elsewhere. Furthermore, the linear system*

$$|H| = |2K - A|$$

is very ample and embeds S in $\mathbf{P}^4$ as a smooth surface of degree 10.

Proof. We first describe the construction geometrically. Let S_1 be the image $\varphi_{2K}(S)$ with a node $p = \varphi_{2K}(A)$. Then S_1 lies on a hyperquadric Q of rank 3 in $\mathbf{P}^5$. Let X be the natural desingularization of Q. Then

$$X = \mathbf{P}(E) = \mathbf{P}(\mathcal{O}_{\mathbf{P}^1}(2) \oplus \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}).$$

Let $\rho : X \to Q$ be the desingularization map. On X there is a divisor corresponding to the section of the bundle $E \otimes \mathcal{O}_{\mathbf{P}^1}(-2)$, call it B, and let F denote a fiber of the natural projection $\pi : X \to \mathbf{P}^1$. B is contracted to the vertex plane of Q by ρ, in fact the map ρ is defined by the linear system of divisors $|B+2F|$. We will construct a surface S_2 on X which is mapped onto S_1 in Q by ρ. In fact there are two irreducible divisors $D_1 \in |2B + 6F|$ and $D_2 \in |3B + 6F|$ such that D_1 is smooth, and D_2 satisfies the following:

1) D_2 has a double point $p_0 \in D_1 \setminus B$
2) $D_2 \cap D_1 = S_2 + Q_0 + Q_1 + Q_2$ where Q_i, $i = 0, 1, 2$, are fibers of the projection $\pi : D_1 \to \mathbf{P}^1$
3) S_2 is tangent to B along three lines L_0, L_1 and L_2
4) S_2 is smooth except at the point p_0, where it has a quadratic singularity.

For this we give equations f, g and h for the divisors D_1 and D_2, and a divisor D_3 on X which contains S_2, together with a relation between these equations. We use the following coordinates on X. Let t be the generator of $\mathrm{H}^0(\mathcal{O}(X)B)$, let x_1, x_2 be a basis of $\mathrm{H}^0(\mathcal{O}(X)F)$ and let y_0, y_1, y_2 be sections of $\mathrm{H}^0(\mathcal{O}(X)B + 2F)$ which form a basis together with tx_1^2, tx_1x_2, tx_2^2. Thus a basis for $\mathrm{H}^0(\mathcal{O}(X)nB + mF)$ is given by monomials

$$t^d x_1^r x_2^s y_0^a y_1^b y_2^c$$

with $a+b+c+d=n$ and $2(a+b+c)+r+s=m$. Consider the following sections $f \in \mathrm{H}^0(\mathcal{O}(X)2B+6F)$, $g \in \mathrm{H}^0(\mathcal{O}(X)3B+6F)$ and $h \in \mathrm{H}^0(\mathcal{O}(X)3B+7F)$.

$$\begin{aligned}
f =&(x_1+x_2)x_1y_0y_1+(x_1+x_2)x_2y_0y_2+x_1x_2y_1y_2\\
&+t(x_1+x_2)^2x_2^2(a_0y_0+a_1y_1+a_2y_2)+t^2(x_1+x_2)^2x_2^2(x_1-x_2)(a_3x_1+a_4x_2)\\
&+t(x_1+x_2)^2x_1^2(b_0y_0+b_1y_1+b_2y_2)+t^2(x_1+x_2)^2x_1^2(x_1-x_2)(b_3x_1+b_4x_2)\\
&+tx_1^2x_2^2(c_0y_0+c_1y_1+c_2y_2)+t^2x_1^2x_2^2(x_1-x_2)(c_3x_1+c_4x_2),
\end{aligned}$$

$$\begin{aligned}
g =&y_0y_1y_2\\
&+t(b_1x_1+a_2x_2)(x_1+x_2)y_1y_2+t(b_0(x_1+x_2)+c_2x_2)x_1y_0y_2\\
&+t(c_1x_1+a_0(x_1+x_2))x_2y_0y_1\\
&+c_0tx_1x_2y_0^2+a_1t(x_1+x_2)x_2y_1^2+b_2t(x_1+x_2)x_1y_2^2\\
&+t^2(x_1+x_2)x_2(x_1-x_2)(a_3x_1+a_4x_2)y_1\\
&+t^2(x_1+x_2)x_1(x_1-x_2)(b_3x_1+b_4x_2)y_2+t^2x_1x_2(x_1-x_2)(c_3x_1+c_4x_2)y_0\\
&+t^2(x_1+x_2)x_1x_2(x_1-x_2)(\alpha_0y_0+\alpha_1y_1+\alpha_2y_2+t(x_1-x_2)(\beta_1x_1+\beta_2x_2))
\end{aligned}$$

and

$$\begin{aligned}
h =&(x_1+x_2)y_0(y_1^2+y_2^2)+x_1y_1(y_0^2+y_2^2)+x_2y_2(y_0^2+y_1^2)\\
&+t[(a_0-b_0)(x_1+x_2)^2x_2+(c_0-b_0)x_1^2x_2\\
&+(a_2-c_2)(x_1+x_2)x_2^2+(b_2-c_2)(x_1+x_2)x_1^2]y_0y_2\\
&+t[(b_0-a_0)(x_1+x_2)^2x_1+(c_0-a_0)x_1x_2^2\\
&+(a_1-c_1)(x_1+x_2)x_2^2+(b_1-c_1)(x_1+x_2)x_1^2]y_0y_1\\
&+t[(c_2-a_2)x_1x_2^2+(b_2-a_2)(x_1+x_2)^2x_1\\
&+(c_1-b_1)x_1^2x_2+(a_1-b_1)(x_1+x_2)^2x_2]y_1y_2\\
&+t[(a_0-c_0)(x_1+x_2)x_2^2+(b_0-c_0)(x_1+x_2)x_1^2]y_0^2\\
&+t[(b_1-a_1)(x_1+x_2)^2x_1+(c_1-a_1)x_1x_2^2]y_1^2\\
&+t[(a_2-b_2)(x_1+x_2)^2x_2+(c_2-b_2)x_1^2x_2)]y_2^2\\
&+t^2(x_1+x_2)(x_1-x_2)[x_1^2((a_3x_1+a_4x_2)-(c_3x_1+c_4x_2))\\
&+x_2^2((b_3x_1+b_4x_2)-(c_3x_1+c_4x_2))]y_0\\
&+t^2x_1(x_1-x_2)[(x_1+x_2)^2((b_3x_1+b_4x_2)-(a_3x_1+a_4x_2))\\
&+x_2^2((c_3x_1+c_4x_2)-(a_3x_1+a_4x_2))]y_1\\
&+t^2x_2(x_1-x_2)[(x_1+x_2)^2((a_3x_1+a_4x_2)-(b_3x_1+b_4x_2))\\
&+x_1^2((c_3x_1+c_4x_2)-(b_3x_1+b_4x_2))]y_2\\
&-t^2[((x_1+x_2)^2x_1^2+(x_1+x_2)^2x_2^2+x_1^2x_2^2)(x_1-x_2)\\
&\quad(\alpha_0y_0+\alpha_1y_1+\alpha_2y_2+t(x_1-x_2)(\beta_1x_1+\beta_2x_2))].
\end{aligned}$$

There is the following relation between these sections:

$$(x_1+x_2)x_1x_2h =$$
$$((x_1+x_2)x_1y_2+x_1x_2y_0+(x_1+x_2)x_2y_1)f-((x_1+x_2)^2x_1^2+(x_1+x_2)^2x_2^2+x_1^2x_2^2)g.$$

Now Q_0, Q_1 and Q_2 are defined by $f = x_1 + x_2 = 0$ and $f = x_1 = 0$ and $f = x_2 = 0$ respectively. It is easy to check that $g = 0$ on Q_i, $0 \le i \le 2$, thus $\{f = g = 0\} = S_2+Q_0+Q_1+Q_2$. Similarly we get $\{f = h = 0\} = S_2+Q_{p_1}+Q_{p_2}+Q_{p_3}+Q_{p_4}$, where the Q_{p_i}, $1 \le i \le 4$, are the fibres of $\pi : D_1 \to \mathbf{P}^1$ over the zeros of $(x_1+x_2)^2x_1^2+(x_1+x_2)^2x_2^2+x_1^2x_2^2$. S_2 meets B along the lines $L_i = \{t = y_j = y_k = 0\}$ where $\{i, j, k\} = \{0, 1, 2\}$. Let $p_0 = \{x_1 - x_2 = y_0 = y_1 = y_2 = 0\}$.

Lemma 5.2. *For general values of the parameters, S_2 has a quadratic singularity at p_0 and is smooth elsewhere. Furthermore the curve $S_2 \cap F_{\{x_1=x_2\}}$ does not lie on any singular quadric with vertex at p_0.*

Proof. First we use Bertinis theorem to note that D_1 is nonsingular away from B, and that D_2 is singular only at p_0 away from B, for general values of the parameters. This is so since the parameters make D_1 and D_2 move in linear systems with basepoints only in B and p_0. Thus, for general values of the parameters, the linear system $|D_2|$ restricted to D_1 has basepoints only on $t^2(x_1 + x_2)x_1x_2(x_1 - x_2) = 0$. Therefore, using Bertini again, the first part of the lemma follows if we prove the following four statements for general values of the parameters:

1) S_2 is smooth along B
2) S_2 is smooth along $(x_1 + x_2)x_1x_2 = 0$
3) S_2 is smooth along $x_1 = x_2$ except at p_0
4) S_2 has a quadratic singularity at p_0.

For 1) we check $D_1 \cap D_2 \cap B = \{f = g = t = 0\} = L_0 + L_1 + L_2$. For a point $(x_1, x_2, t, y_0, y_1, y_2) = (x_1, x_2, 0, 0, 0, 1)$ on L_2 we set up a table of partial derivatives with respect to $x_1, x_2, t, y_0, y_1, y_2$.

	f	g	h
t	$a_2(x_1+x_2)^2x_2^2$ $+b_2(x_1+x_2)^2x_1^2+c_2x_1^2x_2^2$	$b_2(x_1+x_2)x_1$	$(a_2-b_2)(x_1+x_2)^2x_2$ $+(c_2-b_2)x_1^2x_2$
x_1	0	0	0
x_2	0	0	0
y_0	$(x_1+x_2)x_2$	0	x_1+x_2
y_1	x_1x_2	0	x_1
y_2	0	0	0.

We see that D_1 is nonsingular as soon as $a_2b_2c_2 \neq 0$. For S_2 we need to check that the tangent spaces of D_1 and D_2 or, alternatively, D_1 and $D_3 = \{h = 0\}$ meet properly. For this we see that $\frac{\partial g}{dt}\frac{\partial f}{y_1} = b_2(x_1+x_2)x_1^2x_2 \neq 0$ unless $x_1x_2(x_1+x_2) = 0$, but in any of those three points there is a nonvanishing minor in the matrix

$$\left(\frac{\partial(f,h)}{\partial(t,y_0,y_1)}\right).$$

Similarly, we see that S_2 is smooth along the lines L_0 and L_1 whenever $a_0b_0c_0 \neq 0$ and $a_1b_1c_1 \neq 0$.

For 2) we see that the linear system $|D_3|$ restricted to D_1 as basepoints only on $((x_1 + x_2)^2x_1^2 + (x_1+x_2)^2x_2^2 + x_1^2x_2^2)(x_1 - x_2) = 0$, therefore we may use Bertini again to conclude that S_2 is smooth along $(x_1 + x_2)x_1x_2 = 0$.

For 3) we consider the matrix of partial derivatives

$$M = \begin{pmatrix} \frac{\partial f}{dx_1} & \frac{\partial g}{dx_1} \\ \frac{\partial f}{dt} & \frac{\partial g}{dt} \\ \frac{\partial f}{dy_0} & \frac{\partial g}{dy_0} \\ \frac{\partial f}{dy_1} & \frac{\partial g}{dy_1} \\ \frac{\partial f}{dy_2} & \frac{\partial g}{dy_2} \end{pmatrix}$$

at a point $s = (x, t, y_0, y_1, y_2)$:

$$M(s) = \begin{pmatrix} x(3y_0y_1 + y_0y_2 + y_1y_2)+ & 2t^2x^3(\alpha_0y_0 + \alpha_1y_1 + \alpha_2y_2) + \dots \\ \dots & \dots \\ 2x^2(y_1 + y_2) + \dots & y_1y_2 + \dots \\ x^2(2y_0 + y_2) + \dots & y_0y_2 + \dots \\ x^2(2y_0 + y_1) + \dots & y_0y_1 + \dots \end{pmatrix}$$

where $\dots$ only involves terms with parameters different from α_0, α_1 and α_2 as coefficients.

Lemma 5.3. *The 2×2-minors of $M(s)$ vanish simultaneously only in $p_0 = \{x_1 - x_2 = y_0 = y_1 = y_2 = 0\}$ for general values of the parameters.*

Proof. Since the statement is an open condition, it suffices to show that it is satisfied for some selected values of the parameters. Thus if we set all the parameters, except α_0, α_1 and α_2, equal to 0, then we get the following matrix.

$$M_0(s) = \begin{pmatrix} x(3y_0y_1 + y_0y_2 + y_1y_2) & 2t^2x^3(\alpha_0y_0 + \alpha_1y_1 + \alpha_2y_2) \\ 0 & 0 \\ 2x^2(y_1 + y_2) & y_1y_2 \\ x^2(2y_0 + y_2) & y_0y_2 \\ x^2(2y_0 + y_1) & y_0y_1 \end{pmatrix}$$

If $\alpha_0 = \alpha_1 = \alpha_2 = 0$, then it is easy to see that the 2×2-minors of this matrix vanish simultaneously only when $y_0 = y_1 = 0$, $y_0 = y_2 = 0$, $y_1 = y_2 = 0$ or $x = 0$. We check these loci separately, starting with $\{y_0 = y_1 = 0\}$. First we note that $x \neq 0$ on X.
Next we set all the parameters except α_2 equal to 0. Then we have the minor

$$\frac{\partial f}{dy_0}\frac{\partial g}{dy_2} - \frac{\partial f}{dy_2}\frac{\partial g}{dy_0} = 4\alpha_2t^2x^5y_2^2 + \dots$$

where $\ldots$ involves terms with y_0 and y_1. This minor vanishes only when $t = 0$, or when $y_2^2 = 0$. The locus $t = 0$ is taken care of above, so set-theoretically we are left with the point p_0. A similar argument for the loci $y_1 = y_2 = 0$ and $y_0 = y_2 = 0$ proves the lemma.□

From Lemma 5.3 we see that 3) follows.

For 4) we consider the image $S_1 = \rho(S_2)$ in $\mathbf{P}^5$. We use the basis

$$(y_0, y_1, y_2, t(x_1 - x_2)^2, t(x_1 + x_2)(x_1 - x_2), t(x_1 + x_2)^2) = (z_0, z_1, z_2, z_3, z_4, z_5)$$

of $\mathrm{H}^0(\mathcal{O}(X)B + 2F)$ as coordinates of the $\mathbf{P}^5$. We will work locally at $p = (0,0,0,0,0,1)$, so we set $z_5 \neq 0$ and use local coordinates

$$(Z_0, Z_1, Z_2, Z_3, Z_4) = (\frac{z_0}{z_5}, \frac{z_1}{z_5}, \frac{z_2}{z_5}, \frac{z_3}{z_5}, \frac{z_4}{z_5}).$$

We choose polynomials in the Z_i which restrict to f and g on X and denote them by f_1 and g_1. The quadric cone Q is given by $q_1 = Z_3 - Z_4^2 = 0$. By [Lo Prop. 4.2] p is a quadratic singularity on S_1 if the 3×3-minors of the matrix

$$J = (\begin{matrix} \frac{\partial f_1}{dZ_i} & \frac{\partial g_1}{dZ_i} & \frac{\partial q_1}{dZ_i} \end{matrix})$$

together with the equations f_1, g_1 and q_1 generate the maximal ideal of $p = (0,0,0,0,0)$ in $\mathbf{A}^5$.

Lemma 5.4. *For general values of the parameters, the 3×3-minors of the matrix*

$$J = (\begin{matrix} \frac{\partial f_1}{dZ_i} & \frac{\partial g_1}{dZ_i} & \frac{\partial q_1}{dZ_i} \end{matrix})$$

together with the equations f_1, g_1 and q_1 generate the maximal ideal of $p = (0,0,0,0,0)$.

Proof. Since the statement is an open condition on the parameters, it suffices to check that it holds for some selected values of the parameters. If we set all the parameters except a_1, b_2, c_0 and β_2 equal to 0, and calculate the matrix, we get

$$J(p) = \begin{pmatrix} \frac{1}{4}c_0 + \ldots & \frac{1}{2}c_0 Z_0 + \ldots & 0 \\ \frac{1}{4}a_1 + \ldots & a_1 Z_1 + \ldots & 0 \\ \frac{1}{4}b_2 + \ldots & b_2 Z_2 + \ldots & 0 \\ \ldots & \frac{1}{8}\beta_2 Z_4 + \ldots & 1 \\ \ldots & -\frac{1}{8}\beta_2 Z_3 + \frac{1}{4}\beta_2 Z_4 + \ldots & 2Z_4 \end{pmatrix}.$$

In the first column $(\ldots)$ involves terms vanishing at p, while $(\ldots)$ in the second column involves terms vanishing to the second order at p. We choose three linearly independent minors, say J_{35}, J_{25} and J_{13}, where J_{ij} is the 3×3-minor given by deleting the i-th and the j-th row in $J(p)$. Let j_{ij} be the linear part of J_{ij}, and let f_{10} and q_{10} be the linear part of f_1 and q_1 respectively. Then we get

$$\begin{pmatrix} j_{35} \\ j_{25} \\ j_{13} \\ f_{10} \\ q_{10} \end{pmatrix} = \begin{pmatrix} c_0a_1 & -2c_0a_1 & 0 & 0 & 0 \\ b_2c_0 & 0 & -2c_0b_2 & 0 & 0 \\ 0 & 0 & 0 & a_1\beta_2 & -2a_1\beta_2 \\ c_0 & 4a_1 & 4b_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1 \\ Z_2 \\ Z_3 \\ Z_4 \end{pmatrix},$$

which shows that the linear forms $j_{35}, j_{25}, j_{13}, f_{10}$ and q_{10} span the vector space generated by the Z_i for general values of a_1, b_2, c_0 and β_2, so Lemma 5.4 follows.□

For the last part of Lemma 5.2 we go back to X and simply note that $F_{\{x_1=x_2\}} = \{f = x_1 - x_2 = 0\}$ is smooth at p_0 for general values of the parameters, and that $S_2 \cap F_{\{x_1=x_2\}}$ is the complete intersection $\{f = g = x_1 - x_2 = 0\}$.□

We calculate the invariants of S_2. By adjunction on X we get that

$$K_{D_1} = (K_X + D_1)_{|D_1} = ((-4B - 8F) + (2B + 6F))_{|D_1} = (-2B - 2F)_{|D_1}$$

thus by adjunction again on D_1 we get

$$K_{S_2} = (K_{D_1} + S_2)_{|S_2} = ((-2B - 2F) + (3B + 6F) - Q_0 - Q_1 - Q_2)_{|S_2} = (B + F)_{S_2}.$$

Now $|B + F|$ has B as a fixed component and F as a moving part. Calculating on X, one may show that the three lines L_i are (-1)-curves on S_2. The map $\rho : X \to Q$ restricts to the blowing-down of these lines on S_2. Let $S_1 = \rho(S_2)$. Then one gets the following invariants for S_1:

$$K_{S_1}^2 = K_{S_2}^2 + 3 = 3, \quad p_g = h^0(\mathcal{O}_{S_1}(K_{S_1})) = 2, \quad q = 0.$$

The double point of S_2 is the point $p_0 = \{x_1 - x_2 = y_0 = y_1 = y_2 = 0\}$, and $p = \rho(p_0) \in S_1$. If we let

$$\pi : Y \to Q$$

be the blowing-up of Q at p, and S is the strict transform of S_1 on Y with exceptional divisor A lying over p, then S is as described in the first part of the proposition.

Consider now the linear system

$$|H| = |2K - A|$$

on S. It has dimension

$$h^0(\mathcal{O}_S(H)) = 5,$$

since it defines the projection of $S_1 = \varphi_{2K}(S)$ from the point $p = \varphi_{2K}(A)$. Furthermore we have a decomposition

$$H \equiv C + D,$$

where D is the only curve in the linear system $|K - A|$ and C is a canonical curve. We are in a situation similar to the one of the above $K3$-surface and proceed along the same lines. In this case we argue by the following lemmas.

Lemma 5.5. *Every canonical curve $C \in |K|$ except for the curve $D + A$ is canonically embedded by $|H|$.*

Proof. From the proposition we argue directly by studying the image S_1 of S in $\mathbf{P}^5$ under the map φ_{2K}. In fact every canonical curve C is mapped into a $\mathbf{P}^3$ in $\mathbf{P}^5$, and the image of S lies on a singular hyperquadric Q, a cone over a plane conic with vertex a $\mathbf{P}^2$, such that the $\mathbf{P}^3$s of the canonical curves is the pencil of $\mathbf{P}^3$s of the hyperquadric Q. Let $p \in \mathbf{P}^5$ be the image of the curve A. Then $\varphi_H : S \to \mathbf{P}^4$ is the projection of S_1 from the point $p \in S_1$. Since p is not contained in any of the $\mathbf{P}^3$s of the curves $\varphi_{2K}(C)$ except for the one of $\varphi_{2K}(D)$, this projection restricts to an isomorphism on these curves which is the statement of the lemma.□

Lemma 5.6. *The curve $D + A$ is embedded by $|H|$.*

Proof. We follow the argument of the previous lemma. Since $D \cdot A = (K - A) \cdot A = 2$ and D has arithmetic genus 3, we see that $\varphi_{2K}(D)$ has arithmetic genus 4 and has a node at p. Projecting from p we get a plane quartic curve unless D is hyperelliptic. This is assured as soon as $\varphi_{2K}(D)$ is the complete intersection of a smooth quadric and a cubic in the $\mathbf{P}^3$ of $\varphi_{2K}(D)$. Furthermore $H \cdot A = 2$ and the restriction map

$$\mathrm{H}^0(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(A)H)$$

is onto as soon as p is not in the vertex of Q. Thus A and D are embedded as plane curves by φ_H, which spans a hyperplane in $\mathbf{P}^4$. The lemma follows since A is an irreducible conic and $D \cdot A = 2$.□

The very ampleness of $|H|$ on S now follows from Lemma 0.13. This concludes the proof of Proposition 5.1.□

Postulation

Projecting from p, Q is mapped birationally onto $\mathbf{P}^4$. The divisors D_1 and D_2 on X are projected, via ρ, onto hypersurfaces which we denote by V_1 and V_2 respectively. Both $\rho(D_1)$ and $\rho(D_2)$ have degree 6 as varieties in $\mathbf{P}^5$, but only $\rho(D_2)$ is singular at p, so V_2 is a quartic hypersurface, while V_1 is a quintic hypersurface containing S. One may show that the plane that contains the curve D has multiplicity one on V_2 and multiplicity three on V_1.

6 An elliptic surface with $\pi = 10$

Proposition 6.1. *There is an elliptic surface blown up in 2 points, with invariants $p_g = 2$, $q = 0$ and $K^2 = -2$, which is embedded in $\mathbf{P}^4$ as a smooth surface of degree 10 and $\pi = 10$ such that the exceptional curves are embedded as lines.*

Proof. We will show this by a linkage argument. The first step is to construct a local complete intersection surface T of degree 6 and $\pi = 2$ which is cut out by quartics. By Proposition 0.14, we get that T is linked to a smooth surface S of degree 10 and $\pi = 10$ in the intersection of two quartic hypersurfaces. The other invariants of S will follow from a more detailed study of this linkage.

T is the union of a smooth del Pezzo surface T_1 of degree 4 and a smooth quadric surface T_2, in the following way: Let $E_1 + E_2 + F_1 + F_2$ be one of the hyperplane sections of T_1 which consists of four exceptional lines, such that $E_1 \cdot E_2 = F_1 \cdot F_2 = 0$. Next, let T_2 be a smooth quadric in the corresponding hyperplane such that F_1 and F_2 are members of one of the rulings of T_2, and E_1 and E_2 meet T_2 transversally.

Lemma 6.2. *$T = T_1 \cup T_2$ is cut out by quartic hypersurfaces, and is linked to a smooth surface S in the complete intersection of two hyperquartics.*

Proof. For the first part of the lemma we consider the exact sequence

$$0 \longrightarrow \mathcal{I}_{T_1}(3) \longrightarrow \mathcal{I}_T(4) \longrightarrow \mathcal{I}_{T\cap H_2}(4) \longrightarrow 0$$

of sheaves of ideals on $\mathbf{P}^4$, where H_2 is the hyperplane of T_2. This sequence remains exact after taking global sections, since $h^1(\mathcal{I}_{T_1}(3)) = 0$. Therefore it suffices to check whether $T \cap H_2$ is cut out by quartics, and T_1 is cut out by cubics. The latter is clear since T_1 is a complete intersection of two hyperquadrics, while $T \cap H_2$ is the union of T_2 and the lines E_1 and E_2, so it is clearly cut out by the unions of T_2 and the quadrics which define $E_1 \cup E_2$.

For the second part of the lemma one may show that T is locally a complete intersection and then use Proposition 0.14. to conclude the lemma. Instead we will focus on the bad locus, the lines F_1 and F_2 where T_1 and T_2 intersect, and work out the linkage directly. We consider the blowing-up

$$\pi : W \to \mathbf{P}^4$$

of $\mathbf{P}^4$ along the line F_1, with exceptional divisor

$$E \cong \mathbf{P}(3\mathcal{O}_{\mathbf{P}^1}) \cong \mathbf{P}^1 \times \mathbf{P}^2.$$

If a hypersurface V of degree v contains the line F_1 with multiplicity m, then the strict transform of V on W will meet E along a divisor of E which corresponds to a section

$$f : \mathcal{O}_{\mathbf{P}^1} \to Sym^m(3\mathcal{O}_{\mathbf{P}^1}(-1)) \otimes \mathcal{O}_{\mathbf{P}^1}(v).$$

If we let B be a divisor of E corresponding to a section of $(3\mathcal{O}_{\mathbf{P}^1}(1))$, and F corresponds to a fiber of the projection $\pi_E : E \to \mathbf{P}^1$, then the divisor

$$\{f = 0\} \equiv mB + (v - 2m)F.$$

The strict transform of T_1 will therefore meet E along a curve which is equivalent to $B \cdot B$ as a cycle on E, while the strict transform of T_2 will meet E along a curve equivalent to $B \cdot (B - F)$ on E. Thus the strict transform of T on W meets E along a curve equivalent to $B \cdot (2B - F)$.

If we perform a similar blowing-up of W along the $\pi^{-1}(F_2)$, we get similar intersections on the new exceptional divisor E_0. To avoid chaotic notation we use the same for divisors on E and E_0. Now the quartic hypersurface V which corresponds to a general section of $\mathrm{H}^0(\mathcal{I}(T)4)$, has multiplicity one along the lines F_1 and F_2, since this is true for the general section of $\mathrm{H}^0(\mathcal{I}(T \cap H)4)$. Thus the strict transform of V meets E and E_0 along a divisor equivalent to $B + 2F$. Linking T to a surface S in the complete intersection of quartics, we use V and a quartic hypersurface V' which corresponds to a section of $\mathrm{H}^0(\mathcal{I}(T)4)$ with multiplicity two along the lines F_1 and F_2. The strict transform of V' meets E and E_0 along divisors equivalent to $2B$. A Bertini argument shows that for a general choice of V and V', the surface S residual to T in $V \cap V'$ is smooth.□

Remark. The strict transform of S on W meets E and E_0 in curves equivalent to $(2B \cdot (B+2F) - B \cdot (2B - F)) \equiv 5B \cdot F$. These curves are blown down on S. Thus the lines F_1 and F_2 are 5-secants to S.

A general hyperplane section H_T of T has the decomposition $H_T = H_{T_1} + H_{T_2}$ such that $H_{T_1} \cap H_{T_2}$ is two closed points. Therefore, H_T has arithmetic genus $\pi(T) = p(H_{T_1}) + p(H_{T_2}) + 1 = 2$. By the formulas for linkage of curves [PS], $\pi(S) = 10$. We find further invariants of S by studying this linkage more closely.

Let X denote the complete intersection of the two hyperquartics, and consider the cohomology associated with the liaison exact sequence

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_X(3) \longrightarrow \mathcal{O}_T(3) \longrightarrow 0$$

of sheaves on X. Note that T lies on a pencil of reducible hypercubics, and that the restriction map $\mathrm{H}^0(\mathcal{O}(X)3) \to \mathrm{H}^0(\mathcal{O}(T)3)$ is onto. Therefore S is regular and has $p_g = 2$. We get that $K^2 = -2$ by the double point formula for smooth surfaces in $\mathbf{P}^4$.

To show that S is elliptic, we find the curves of intersection: $C = S \cap T$, $C_1 = T_1 \cap (S \cup T_2)$ and $C_2 = T_2 \cap (S \cup T_1)$. By the liaison exact sequences

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_S(3) \longrightarrow \mathcal{O}_C(3) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_{T_1}(K_{T_1}) \longrightarrow \mathcal{O}_{T_1}(3) \longrightarrow \mathcal{O}_{C_1}(3) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{T_2}(K_{T_2}) \longrightarrow \mathcal{O}_{T_2}(3) \longrightarrow \mathcal{O}_{C_2}(3) \longrightarrow 0$$

we get that $C_1 \equiv 3H_{T_1} - K_{T_1} \equiv 4H_{T_1}$ on T_1, $C_2 \equiv 3H_{T_2} - K_{T_2} \equiv 5H_{T_2}$ on T_2 and $C = C_1 + C_2 - 2(F_1 + F_2) \equiv 3H - K$ on S. Thus C is the union of a curve A on T_1 and a curve B on T_2, where $A \equiv 4H_{T_1} - F_1 - F_2$ and B is of type $(3,5)$ on T_2. Now, restricting the pencil of quartics to the hyperplane H_2 of T_2 we see that it has the quadric defining T_2 as a fixed component, and that the moving part is the quadrics defining $E_1+E_2+G_1+G_2$, where G_1 and G_2 are skew lines meeting both E_1 and E_2. Thus $S \cap H_{T_2} = B + G_1 + G_2$, and

$$A \equiv 3H - K - B \equiv 2H - K + G_1 + G_2.$$

Since the curve $G_1 + G_2$ does not lie on T_1, it must be a component of some canonical divisor. Thus we have effective divisors

$$K - (G_1 + G_2) \equiv 2H - A.$$

We get that the restriction to S of the pencil of quadrics defining T_2 is a pencil with the curve $A + G_1 + G_2$ as a fixed part, and a moving part which is the moving part of $|K|$. In fact G_1+G_2 must be the fixed part of $|K|$. Since $K_S^2 = -2$, they are (the only) (-1)-curves on S. The moving part of $|K|$ is a pencil of elliptic curves, since $(K - (G_1 + G_2))^2 = 0$.□

Remark. Since, according to the secant formulas 0.2., there are two 5-secants to S which meet a general plane in $\mathbf{P}^4$, and $B = T_2 \cap S$ is of type $(3,5)$ on T_2, we may characterize T_2 as the union of the 5-secants to S. Therefore, any quartic hypersurface containing S must also contain T_2.

Postulation

Let $T_0 = S \cup T_2$.

Proposition 6.3. $h^0(\mathcal{I}_S(4)) = h^0(\mathcal{I}_{T_0}(4)) = 3$.

Proof. We let Q denote the complete intersection of two quartic hypersurfaces containing S, and consider the cohomology associated with the liaison exact sequence

$$0 \longrightarrow \mathcal{O}_{T_1}(H+K) \longrightarrow \mathcal{O}_Q(4H) \longrightarrow \mathcal{O}_{T_0}(4H) \longrightarrow 0$$

of sheaves on Q. Since T_1 is a Del Pezzo surface, we get that $\mathcal{O}_{T_1}(H+K) \equiv \mathcal{O}_{T_1}$. Thus we get that $h^0(\mathcal{O}_{T_0}(4H)) = 67$, and that the map

$$\mathrm{H}^0(\mathcal{O}(Q)4H) \to \mathrm{H}^0(\mathcal{O}(T_0)4H)$$

is onto. Therefore $h^0(\mathcal{I}_{T_0}(4H)) = 3$. But we have already seen that any quartic containing S, must contain T_0, so the proposition follows.□

7 A surface of general type with $\pi = 10$

We will construct a smooth surface S with numerical invariants $p_g = 3$, $q = 0$, $K^2 = 4$, and a linear system of curves $|H|$ on S such that $|H|$ is very ample and embeds S as a smooth surface of degree 10 in $\mathbf{P}^4$ with $\pi = 10$. Via the bicanonical linear system $|2K|$, a surface S_0 of the above invariants is realized as the complete intersection on the four-dimensional cone over a Veronese surface in $\mathbf{P}^7$. Projecting from three linearly independent singular points on S_0, we get a surface S in $\mathbf{P}^4$ with the right invariants. We want to show that S can be realized as a smooth surface. The tricky part is to impose exactly three quadratic singularities on the surface S_0 such that the projection from these points into $\mathbf{P}^4$ is an embedding.

Proposition 7.1. *There is a minimal smooth surface S with numerical invariants $p_g = 3$, $q = 0$, $K^2 = 4$ and exactly three irreducible (-2)-curves A_1, A_2 and A_3, for which the bicanonical linear system $|2K|$ defines a birational morphism*

$$\varphi_{2K} : S \to \mathbf{P}^7$$

which contracts the curves A_i and is an isomorphism elsewhere.
Furthermore the linear system

$$|H| = |2K - A_1 - A_2 - A_3|$$

is very ample and embeds S as a smooth surface of degree 10 in $\mathbf{P}^4$.

Proof. For the last part we note that H has decompositions

$$H \equiv C_i + D_i,$$

where $C_i \equiv K - A_i$ and $D_i \equiv K - A_j - A_k$ with $\{i, j, k\} = \{1, 2, 3\}$. We get that $(C_i + K)_{|C_i} \equiv (2K - A_i)_{|C_i}$, and that $(D_i + K)_{|D_i} \equiv (2K - A_j - A_k)_{|D_i}$. Thus, on the general member of $|C_i|$, and on the curve D_i, $|H|$ will be restricted to the canonical linear series. For the very ampleness of $|H|$, it is therefore necessary that neither the general member of $|C_i|$ nor the curves D_i, where $i = 1, 2, 3$, are hyperelliptic. We will take careful notice of this in the first part, where we start by describing the construction geometrically.

Let S_1 be the image $\varphi_{2K}(S)$ with three nodes $p_i = \varphi_{2K}(A_i)$ for $i = 1, 2, 3$. Then S_1 lies on a fourdimensional cone X over a Veronese surface in $\mathbf{P}^7$. Let X_0 be the natural desingularization of X. Then

$$X_0 = \mathbf{P}(E) = \mathbf{P}(\mathcal{O}_{\mathbf{P}^2}(2) \oplus \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}).$$

Let $\rho : X_0 \to X$ be the desingularization map. On X_0 there is a divisor corresponding to the section of the bundle $E \otimes \mathcal{O}_{\mathbf{P}^2}(-2)$, call it B_0, and let F denote the pullback of a line by the natural projection $\pi : X_0 \to \mathbf{P}^2$. B_0 is contracted to the vertex line of X

by ρ, in fact the map ρ is defined by the linear system of divisors $|B| = |B_0 + 2F|$. On X_0 we have a canonical divisor $K_0 \equiv -3B - F$ and intersections $B^4 = 4$ and $B^3 \cdot F = 2$ and $B^2 \cdot F^2 = 1$. For two general members G_1 and G_2 of $|2B|$, the complete intersection $G_1 \cap G_2$ is a smooth surface Σ. By adjunction, $B - F$ restricts to the canonical divisor K_Σ on Σ. Thus Σ is a regular surface with $\chi(\Sigma) = 4$ and $K_\Sigma^2 = 4$. Since $B^3 \cdot B_0 = 0$, Σ is embedded by ρ in $\mathbf{P}^7$ as the complete intersection of X and two quadric hypersurfaces Q_1 and Q_2.

For the construction I give equations for X and the quadric hypersurfaces Q_1 and Q_2, such that the following conditions hold for $S_1 = X \cap Q_1 \cap Q_2$:

1) S_1 does not meet the vertex line X
2) S_1 has exactly three linearly independent isolated quadratic singularities, and is smooth elsewhere.

Let $z_0, ..., z_7$ be a basis for $\mathrm{H}^0(\mathcal{O}(\mathbf{P}^7)1)$. Then we can define X as the subvariety where the matrix

$$M_X = \begin{pmatrix} z_5 & z_2 & z_3 \\ z_2 & z_6 & z_4 \\ z_3 & z_4 & z_7 \end{pmatrix}$$

has rank one.

For Q_1 and Q_2, consider the following sections of $\mathrm{H}^0(\mathcal{O}(\mathbf{P}^7)2)$:

$$f = z_0(f_0 + h_0) + z_1(f_1 + h_1) + z_2(f_2 + h_2) + z_3(f_3 + h_3) + z_4(f_4 + h_4)$$

and

$$g = z_0(g_0 + h_0) + z_1(g_1 + h_1) + z_2(g_2 + h_2) + z_3(g_3 + h_3) + z_4(g_4 + h_4),$$

where the f_i, g_i and h_i are linear forms for $i = 0, .., 4$ such that:

f_0, f_1, g_0, g_1 involves only $z_0, .., z_4$, while h_0, h_1 involves only z_5, z_6, z_7.
f_2, g_2 involves only $z_0, .., z_4$ and z_7, while h_2 involves only z_5, z_6.
f_3, g_3 involves only $z_0, .., z_4$ and z_6, while h_3 involves only z_5, z_7.
f_4, g_4 involves only $z_0, .., z_4$ and z_5, while h_4 involves only z_6, z_7.
Let $Q_1 = \{f = 0\}$ and $Q_2 = \{g = 0\}$, and note that $S_1 = X \cap Q_1 \cap Q_2$ contains the points $p_1 = \{z_0 = .. = z_6 = 0\}$, $p_2 = \{z_0 = .. = z_5 = z_7 = 0\}$ and $p_3 = \{z_0 = .. = z_4 = z_6 = z_7 = 0\}$.

Lemma 7.2. *For a general choice of forms f_i, g_i, h_i, S_1 has isolated quadratic singularities at the points p_1, p_2, p_3, and is smooth elsewhere.*

Proof. First of all we see that if we fix f and let the parameters of g vary, then the corresponding Q_2 vary with basepoints only in the plane $\{z_0 = .. = z_4 = 0\}$ on $Q_1 = \{f = 0\}$. But X meets this plane only in the points p_1, p_2, p_3, so this means, by the Bertini theorem, that for general linear forms f_i, g_i, h_i, S_1 is smooth away from these points.

Now, for the point p_1, we consider the minors

$$m_{12} = z_2 z_7 - z_3 z_4,$$

$$m_{22} = z_5 z_7 - z_3^2$$

and

$$m_{11} = z_6 z_7 - z_4^2$$

of M_X together with f and g. We set up a table of the partial derivatives of these equations evaluated at p_1:

	m_{12}	m_{22}	m_{11}	f	g
z_0	0	0	0	h_0	h_0
z_1	0	0	0	h_1	h_1
z_2	1	0	0	f_2	g_2
z_3	0	0	0	h_3	h_3
z_4	0	0	0	h_4	h_4
z_5	0	1	0	0	0
z_6	0	0	1	0	0

Clearly the minors m_{12}, m_{22} and m_{11} define X locally at p_1, while S_1 is singular at p_1. To see that S_1 has a quadratic singularity at the point p_1, it is enough to check that the maximal minors of the above table considered as a matrix, together with the equations $m_{12}, m_{22}, m_{11}, f$ and g generate the maximal ideal which defines p_1 (see [Lo Prop. 4.4]). This is now straightforward. The points p_2 and p_3 are checked in the same manner as the point p_1.□

We will now make use of Lemma 0.13 to show that if we project S_1 from the plane spanned by the points p_1, p_2, p_3, that is from the plane $\{z_0 = .. = z_5 = 0\}$, then we get a smooth surface of degree 10 in $\mathbf{P}^4$. For this let S be the strict transform of S_1 when we blow up $\mathbf{P}^7$ in the points p_i. Then S meets the exceptional divisors in (-2)-curves, which we denote by A_1, A_2 and A_3 respectively. A hyperplane section of S_1 is a bicanonical divisor, therefore the pullback to S of a hyperplane section of S_1 is a bicanonical divisor on S. Let

$$H \equiv 2K - A_1 - A_2 - A_3.$$

Then $|H|$ defines a map of S into $\mathbf{P}^4$. That it in fact is a morphism, follows from the fact that S_1 meets the plane spanned by the points p_i schemetheoretically in those points only. Now H has the decompositions $H = C_i + D_i$, where $C_i \equiv K - A_i$ and $D_i \equiv K - A_j - A_k$ with $\{i, j, k\} = \{1, 2, 3\}$. As we exploit this decomposition, we bear in mind the geometric setting of the surface S_0 in X_0. A canonical divisor on S_0 is the restriction of a divisor $B - F$ on X_0. Via the map $\rho : X_0 \to X \subset \mathbf{P}^7$ we see that the image of $B - F \equiv B_0 + F$ on X is a threedimensional quadric of rank three. It spans a $\mathbf{P}^4$, so any canonical curve on S_1 is in this way the complete intersection of three quadrics in $\mathbf{P}^4$, such that at least one of the quadrics has rank three. With the above equations for S_1, we can now give the explicit equations for the canonical curves on S_1 corresponding to the curves D_i on S. Thus the curve on S_1 whose strict transform on S is D_3, is defined by the equations

$$z_2 = z_3 = z_5 = 0,$$

$$z_6z_7 - z_4^2 = 0$$

and

$$f = g = 0.$$

We use this explicit description to prove

Lemma 7.3. *$|H|$ restricts to a very ample linear series on the curves D_i, for $i = 1,2,3$.*

Proof. First we show that $|H|$ restricts to the canonical linear series on the curves D_i. But $\mathcal{O}_{D_i}(K_{D_i}) \cong \mathcal{O}_{D_i}(K+D_i) \cong \mathcal{O}_{D_i}(H+A_i)$, and $\mathcal{O}_{D_i}(A_i) \cong \mathcal{O}_{D_i}$, so $\mathcal{O}_{D_i}(K_{D_i}) \cong \mathcal{O}_{D_i}(H)$. Thus we are left to show that the curves D_i are not hyperelliptic.

Let D_3' denote the canonical curve on S_1 whose strict transform on S is D_3. Then D_3' is the curve cut out by the equations given above. We may assume, by the Bertini theorem, that D_3' is an irreducible curve with singularities only in the points p_1 and p_2. The restriction of $|H|$ to D_3 corresponds to hyperplane sections of D_3' which contains the line L joining the points p_1 and p_2. A general hyperplane section of this kind consists of four distinct points outside p_1 and p_2, so D_3 is hyperelliptic only if none of the six secants which join these four points meet the line L.

Now look at the equations for D_3':

$$z_2 = z_3 = z_5 = 0,$$

$$z_5z_6 - z_4^2 = 0,$$

$$f = z_0(f_0+h_0) + z_1(f_1+h_1) + z_4(f_4+h_4) = 0$$

and

$$g = z_0(g_0+h_0) + z_1(g_1+h_1) + z_4(g_4+h_4) = 0.$$

We study a special hyperplane section, namely $z_4 = 0$. Then we see that the restrictions of the equations $f = 0$ and $g = 0$ coincides with the restrictions of two minors of the matrix

$$\begin{pmatrix} z_0 & -f_1-h_1 & -g_1-h_1 \\ z_1 & f_0+h_0 & g_0+h_0 \end{pmatrix}.$$

But for general linear forms f_i, g_i, h_i these two minors define the union of a twisted cubic curve and the line L, such that L is a secant of the twisted cubic curve. Since no two secants of a twisted cubic curve are coplanar, the lemma follows.□

Lemma 7.4. *$|H|$ restricts to a very ample linear series on every curve of the linear systems $|C_i|$, for $i = 1,2,3$.*

Proof. We treat the linear system of curves $|C_1|$. The other two are treated similarly. Let $|C_1'|$ denote the linear system of canonical curves on S_1 which meet p_1. The linear system $|H|$ corresponds to hyperplanes which contain all three points p_1, p_2, p_3. Let $C_1' \in |C_1'|$ and let C_1 be its strict transform on S. Unless C_1' meets p_2 or p_3, the linear series $|H_{|C_1}|$ on C_1 fails to be very ample only if C_1' has a trisecant line L through p_1, that is a line which meets C_1' in a scheme of length three. But S_1 is cut out by quadrics, so the line would be contained in S_1, and likewise also in X. Now, any line in X meets the vertex of X, since a Veronese surface does not contain lines. On the other hand S_1 does not meet the vertex of X, so we have reached a contradiction.

For the curves in $|C_1'|$ which does meet p_2 or p_3, we need another argument. These curves correspond to the curves $D_2 + A_3$ and $D_3 + A_2$ on S. We have already seen, in the above lemma, that $|H|$ restricts to a very ample linear series on D_2 and D_3. The curves A_i are smooth rational curves and $H \cdot A_i = 2$, so $|H_{|A_i}|$ is also very ample, for $i = 1, 2, 3$. To finish the proof, we consider the cohomology associated with the exact sequences

$$0 \longrightarrow \mathcal{O}_S(H - A_1) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_{A_1}(H) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_S(D_i) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_{C_i}(H) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_S(C_i) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_{D_i}(H) \longrightarrow 0,$$

and

$$0 \longrightarrow \mathcal{O}_S(D_3) \longrightarrow \mathcal{O}_S(D_2 + D_3) \longrightarrow \mathcal{O}_{D_2}(D_2 + D_3) \longrightarrow 0$$

of sheaves on S. From our construction we have that $h^0(\mathcal{O}_S(H)) = 5$ and $h^1(\mathcal{O}_S(H)) = 0$, and similarly that $h^0(\mathcal{O}_S(D_i)) = h^2(\mathcal{O}_S(D_i)) = 1$ and $h^1(\mathcal{O}_S(D_i)) = 0$, and that $h^0(\mathcal{O}_S(C_i)) = 2$ and $h^1(\mathcal{O}_S(C_i)) = 0$ for $i = 1, 2, 3$. Therefore the restriction maps

$$\mathrm{H}^0(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(C_i)H),$$

$$\mathrm{H}^0(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(D_i)H)$$

and

$$\mathrm{H}^0(\mathcal{O}(S)D_2 + D_3) \to \mathrm{H}^0(\mathcal{O}(D_2)D_2 + D_3)$$

are all surjective. We want to show

Lemma 7.5. $h^0(\mathcal{O}_S(D_2 + D_3)) = 2$.

Proof. Since clearly $h^2(\mathcal{O}_S(D_2 + D_3)) = 0$ and $h^2(\mathcal{O}_S(D_3)) = 1$ we get from the last exact sequence above that $h^1(\mathcal{O}_{D_2}(D_2 + D_3)) \geq 1$. But $D_2 \cdot (D_2 + D_3) = 2$ and we have already seen that D_2 is not hyperelliptic, so we have equality; $h^1(\mathcal{O}_{D_2}(D_2 + D_3)) = 1$. Therefore $h^1(\mathcal{O}_S(D_2 + D_3)) = 0$ and $h^0(\mathcal{O}_S(D_2 + D_3)) = 2$.□

It now follows, when we note that $H - A_1 \equiv D_2 + D_3$, that the restriction map

$$\mathrm{H}^0(\mathcal{O}(S)H) \to \mathrm{H}^0(\mathcal{O}(A_1)H)$$

is also surjective. We put this together to conclude the proof of Lemma 7.4. If $\varphi_H : S \to \mathbf{P}^4$ denotes the map defined by $|H|$, then we have that the image of D_2 is a plane quartic, and the image of A_1 is an irreducible plane conic. Now $D_2 \cdot A_1 = 2$, so if $|H|$ is not very ample on $D_2 + A_1$, then the image of $D_2 + A_1$ is a plane curve. But this contradicts the fact that $h^0(\mathcal{O}_S(D_1)) = h^0(\mathcal{O}_S(H - D_2 - A_1)) = 1$, so the lemma follows.□

In the course of the proof of Lemma 7.4, we showed that all the conditions of Lemma 0.13 are satisfied, so we may conclude that $|H|$ is very ample on S. This ends the proof of the proposition.□

7.6 Postulation

We take a closer look at the geometry of S. On S there are three quartic plane curves D_i, $i = 1, 2, 3$. Since $D_i \cdot D_j = 2$ when $i \neq j$, we see that the planes Π_i of the three curves meet in lines. In fact since the union of the three curves are not contained in a hyperplane, these planes must have a line, call it L, in common, which in turn is a 6-secant line for the surface S. This fits with the formula of Le Barz for the number of 6-secant lines to the surface S. Furthermore, since S meets each of the planes Π_i in two points on L outside the curve D_i for $i = 1, 2, 3$, we get that each plane Π_i has two pencils of 5-secant lines for S. Altogether we have counted six 5-secant lines to S which meet a general plane in $\mathbf{P}^4$. This fits with the corresponding formula of Le Barz. Anyway, the planes must be contained in any quartic hypersurface which contains S. Let $T_0 = S \cup \Pi_1 \cup \Pi_2 \cup \Pi_3$.

Proposition 7.7. $h^0(\mathcal{I}_S(4)) = h^0(\mathcal{I}_{T_0}(4)) = 3.$

Proof. We first describe the net of quartics which contain T_0. For this we analyze the projection map

$$\rho_X : X \dashrightarrow \mathbf{P}^4$$

from the points p_1, p_2, p_3 on X. If $\pi : Y \to X$ is the blowing-up of X in the points p_i with exceptional divisors E_1, E_2 and E_3, then ρ_X extends to a morphism

$$\rho_Y : Y \to \mathbf{P}^4$$

defined by the linear system

$$|h| = |B - \sum_{i=1}^{3} E_i|$$

of divisors on Y (as before we keep the notation B and F for the pullbacks to Y of the divisors B and F on X).

The general member $F \in |F|$ is a threedimensional quadric in X, therefore the divisor $F_{ij} = F - E_i - E_j$, for $1 \leq i < j \leq 3$, is mapped by ρ_Y onto the plane Π_k where $\{i, j, k\} = \{1, 2, 3\}$. So we are interested in members of the linear system

$$|4h - 3F + \sum_{i=1}^{3} 2E_i| = |4B - 3F - \sum_{i=1}^{3} 2E_i|$$

which contains the strict transform S of S_1 on Y.

Now, let $\Sigma \in |2B - \sum_{i=1}^{3} E_i|$ be the strict transform on Y of a divisor in $|2B|$ on X which contains S_1. Then

$$S \equiv (2B - \sum_{i=1}^{3} 2E_i)_{|\Sigma}$$

as divisors on Σ, so

$$\mathcal{O}_\Sigma(4B - 3F - \sum_{i=1}^{3} 2E_i - S) \cong \mathcal{O}_\Sigma(2B - 3F).$$

If we consider the cohomology associated with the exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-3F + \sum_{i=1}^{3} E_i) \longrightarrow \mathcal{O}_Y(2B - 3F) \longrightarrow \mathcal{O}_\Sigma(2B - 3F) \longrightarrow 0$$

of sheaves on Y, then we get that

$$\mathrm{H}^0(\mathcal{O}(Y)2B - 3F) \cong \mathrm{H}^0(\mathcal{O}(\Sigma)2B - 3F).$$

But $h^0(\mathcal{O}_Y(2B - 3F)) = h^0(\mathcal{O}_Y(F)) = 3$, so we get a net of quartics which contain T_0.

This argument does not show that there does not exist any other quartic which contain S in $\mathbf{P}^4$. To show this we will use an argument via linkage. The linkage that we will describe, could in fact give a quick direct proof of existence as well, arguing as in the above chapter 6.

Now, by the secant formula of Le Barz (see 0.2), there are 36 4-secants to S which meet a general line in $\mathbf{P}^4$. Therefore S cannot lie on any cubic hypersurface, as any 4-secant line would be contained in the cubic hypersurface, while no cubic hypersurface contains as many as 12 lines through a general point. Thus S is linked to a surface T in the intersection of two quartic hypersurfaces. We know already that T contains the planes Π_i as components, so let T_1 be the residual component

$$T = \Pi_1 \cup \Pi_2 \cup \Pi_3 \cup T_1.$$

Thus T_1 has degree three. If we consider the liaison exact sequence

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_S(3H) \longrightarrow \mathcal{O}_{S\cap T}(3H) \longrightarrow 0$$

we see that

$$S \cap T \equiv 3H - K \equiv H + D_1 + D_2 + D_3 \equiv H + (\Pi_1 \cap S) + (\Pi_2 \cap S) + (\Pi_3 \cap S)$$

on S. Hence $T_1 \cap S \equiv H$ on S, and T_1 must be contained in a hyperplane. Now $T_1 \cap (S \cup \Pi_1 \cup \Pi_2 \cup \Pi_3) \equiv 3H_{T_1} - K_{T_1} \equiv 4H_{T_1}$. So $T_1 \cap S$ is linked to a curve C_0 of degree two in the intersection of a quartic and a cubic suface. By the liaison formulas (see [PS]) we get that C_0 has arithmetic genus $p(C_0) = -2$, which means that C_0 is the doubling of a line E in the cubic surface T_1. In fact this line E must coincide with the line L, since its doubling C_0 is contained in $\Pi_1 \cup \Pi_2 \cup \Pi_3$.

To conclude the proof of the proposition, we let Q denote the complete intersection of two quartic hypersurfaces containing S, and consider the liaison exact sequence

$$0 \longrightarrow \mathcal{O}_{T_1}(H + K) \longrightarrow \mathcal{O}_Q(4H) \longrightarrow \mathcal{O}_{T_0}(4H) \longrightarrow 0$$

of sheaves on Q. Since T_1 is a cubic surface in $\mathbf{P}^3$, we get that $\mathcal{O}_{T_1}(H + K) \equiv \mathcal{O}_{T_1}$. Thus, taking cohomology, we get that $h^0(\mathcal{O}_{T_0}(4H)) = 67$, and that the map

$$\mathrm{H}^0(\mathcal{O}(Q)4H) \to \mathrm{H}^0(\mathcal{O}(T_0)4H)$$

is onto. Therefore $h^0(\mathcal{I}_{T_0}(4H)) = 3$. But we have already seen that any quartic that contains S, must also contain all of T_0, so the proposition follows.□

8 Classification of rational surfaces of degree 10

Theorem 8.1. *If S is a smooth rational surface of degree 10 in $\mathbf{P}^4$, then S is of type A or B, or $\pi = 9$ and*

$$H \equiv 9l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j - \sum_{k=12}^{18} E_k,$$

where l is the pullback of a line from $\mathbf{P}^2$ and the E_i are (-1)-curves on S.

Proof. The idea is to use Sommese and Van de Ven's results, Theorem 0.10, on the adjunction mapping to generate candidates for smooth surfaces in $\mathbf{P}^4$, and then to rule out all but those described above.

Throughout this section we by S denote a smooth rational surface of degree 10 in $\mathbf{P}^4$. The double point formula (0.7) now takes the form

$$(A) \qquad 5H \cdot K + 2K^2 = 12.$$

The theorem of Severi (0.8) and Riemann-Roch (0.4) says that $5 - h^1(\mathcal{O}_S(H)) = 6 - \frac{1}{2} H \cdot K$, or

$$(B) \qquad H \cdot K = 2h^1(\mathcal{O}_S(H)) + 2.$$

Combining (A) and (B) we get

$$(C) \qquad K^2 = 1 - 5h^1(\mathcal{O}_S(H))$$

and

$$(D) \qquad \pi = 7 + h^1(\mathcal{O}_S(H)).$$

Lemma 8.2. *S is not a scroll or ruled in conics.*

Proof. If S is a scroll, then S is minimal so $K^2 = 8$ which contradicts (C). If S is ruled in conics, then by Theorem 0.10, (B) and (C), $(H + K)^2 = 15 - h^1(\mathcal{O}_S(H)) = 0$, and thus by (D), $\pi = 22$. But then a general hyperplane section is a smooth curve of degree 10 and genus 22, which is impossible by the Castelnuovo bound [Ha Th. 6.4.].□

We proceed to bound the speciality $h^1(\mathcal{O}_S(H))$. For this we need a lemma which we will also use later.

Lemma 8.3. *Let S be a rational surface, and H a very ample divisor on S. If $K^2 \geq 0$, then $H \cdot K < -2$.*

Proof. Riemann-Roch says that

$$h^0(\mathcal{O}_S(-K)) = 1 + K^2 + h^1(\mathcal{O}_S(-K)) \geq 1.$$

Let C be a member of the linear system $|-K|$. Then C has arithmetic genus 1, so since H is very ample we get that $H \cdot C = -H \cdot K > 2.\square$

The lemma combined with (B) and (C) gives

$$h^1(\mathcal{O}_S(H)) > 0,$$

since $h^1(\mathcal{O}_S(H)) = 0$ implies that $K^2 = 1$ and $H \cdot K = 2$. On the other hand we get from Theorem 0.10 and lemma 8.2 that $|H + K|$ defines a birational morphism of S onto a surface S_1 of degree $(H + K)^2$ in $\mathbf{P}^{\pi-1}$. Since S_1 is nondegenerate, we get the inequality

$$(H + K)^2 \geq \pi - 2$$

which combine with (B),(C) and (D) to imply that

$$h^1(\mathcal{O}_S(H)) \leq 5.$$

From here we treat the different possible values of $h^1(\mathcal{O}_S(H))$ case by case. In each case we will be looking for reducible hyperplane sections with components that have arithmetic genus too high for their degree.

Proposition 8.5. *If S is a smooth rational surface of degree 10 in $\mathbf{P}^4$, then $h^1(\mathcal{O}_S(H)) \neq 3, 4, 5$.*

Proof. Let S_1 be the image of S by the morphism defined by $|H + K|$. Let K_1 be its canonical divisor and π_1 the genus of its hyperplane section H_1. By Theorem 0.10 we get that $S_1 \subset \mathbf{P}^h$, $h = h^1(\mathcal{O}_S(H)) + 6$, with

$$\deg S_1 = (H + K)^2 = 15 - h^1(\mathcal{O}_S(H))$$

and

$$\pi_1 = \frac{1}{2}H_1(H_1 + K_1) + 1 = \frac{1}{2}(H + K)(H + 2K) + 1 = 10 - 2h^1(\mathcal{O}_S(H)).$$

If $h^1(\mathcal{O}_S(H)) = 5$, then S_1 is of minimal degree 10 in $\mathbf{P}^{11}$, so S_1 is a scroll. Therefore S is ruled in some possibly singular, twisted cubic curves. Let F be a fibre of the ruling. Then F is part of a hyperplane section H of S. Let C be the residual curve. Then C has degree 7 and arithmetic genus

$$p(C) = \frac{1}{2}(C^2 + C \cdot K) + 1 = 10.$$

By Lemma 0.11 C decomposes into a plane sextic curve, call it E, and a line L such that $E \cdot L = 1$. If $L^2 \geq 0$, then S would be a scroll or $\mathbf{P}^2$, so we may assume that $L^2 \leq -1$. Thus $C^2 = (E + L)^2 = 4$ implies that $E^2 \geq 3$. The index theorem applies to E to see that $E^2 = 3$ and therefore $L^2 = -1$. Now $(L + F)^2 = (H - E)^2 = 1$ implies that $L \cdot F = 1$, so L is an exceptional line on S, which is also a section of S as a ruled surface. By Theorem 0.10, L is blown down on S_1, but S_1 is a smooth scroll which spans $\mathbf{P}^1 1$, so this is absurd.

If $h^1(\mathcal{O}_S(H)) = 4$, then S_1 is of degree 11 in $\mathbf{P}^{10}$. Since $\pi_1 = 2$, S_1 must be ruled in conics, and thus S is ruled in rational quartic curves. We get that $(H_1 + K_1)^2 = -7 + K_1^2 = 0$ and therefore $K_1^2 = 7$. Since

$$K^2 = -19 < K_1^2,$$

there are exceptional lines on S. They must be components of fibres in the ruling, so there are twisted cubic curves residual to each line in a fiber. Let E be such a curve. We have $H \cdot E = 3$, $E^2 = -1$ and $p(E) = 0$. E is part of a hyperplane section H of S. Let C be the residual curve $C = H - E$. Then $\deg C = H \cdot C = 7$ and $p(C) = 8$, which is impossible by Lemma 0.11.

If $h^1(\mathcal{O}_S(H)) = 3$, then S_1 is of degree 12 in $\mathbf{P}^9$ and $\pi_1 = 4$. Since S_1 is not a scroll, we know by Theorem 0.10 that $|H_1 + K_1|$ defines a morphism of S_1 into $\mathbf{P}^3$. Let Σ be the image of S_1. Since S is rational, we now have three cases: Either i) Σ is a Del Pezzo surface, or ii) Σ is a quadric, or iii) Σ is a curve.

In case i) we get that $(H_1 + K_1)^2 = K_1^2 = 3$. Since $K^2 = -14$, we can reproduce H from H_1 and H_Σ:

$$H_\Sigma \equiv 3l - \sum_{i=1}^{6} E_i.$$

$$H_1 \equiv 6l - \sum_{i=1}^{6} 2E_i.$$

$$H \equiv 9l - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{23} E_j.$$

Now let $C_0 \equiv 2l - \sum_{i=1}^{5} E_i$, then $H \cdot C_0 = 3$, so C_0 is contained in a hyperplane section H of S. Let $C \equiv H - C_0$, then

$$C \equiv 7l - \sum_{i=1}^{5} 2E_i - 3E_6 - \sum_{j=7}^{23} E_j,$$

$\deg C = 7$ and $p(C) = 7$. By Lemma 0.11, C decomposes into a plane quintic, call it E, and a plane conic, call it Q, such that $E \cdot Q = 2$. Now $Q^2 \geq 0$ would imply that S is ruled in conics or is isomorphic to $\mathbf{P}^2$, so we may assume that $Q^2 \leq -1$. Therefore

$$C^2 = (E + Q)^2 = 3 \text{ implies that } E^2 = -1 - Q^2 \geq 0.$$

The index theorem applies to E to get $E^2 \leq 2$, and therefore $Q^2 \geq -3$. So we need to consider $-3 \leq Q^2 \leq -1$.

First note that

$$(Q + C_0)^2 = (H - E)^2 = E^2 = -1 - Q^2,$$

which implies that $Q \cdot C_0 = -Q^2$, and that $E \cdot l \geq 5$ since $p(E) = 6$. Thus since $C \cdot l = 7$ we get that $Q \cdot l \leq 2$. Now we may write

$$Q = \alpha l - \sum_{i=1}^{23} \beta_i E_i \qquad \text{with} \quad 0 \leq \alpha \leq 2, \beta_i \geq -1.$$

If $Q^2 = -1$, we get that $Q \cdot C_0 = 1$, implying $2\alpha - \sum_{i=1}^{5} \beta_i = 1$ and therefore $\alpha \geq 1$ or $\beta_j = -1$ for some i. In the latter case $H \cdot Q = 2$ implies that $\beta_k = 1$ for some $7 \leq k \leq 23$ and $Q^2 = -2$, which is a contradiction. In case $\alpha \geq 1$ we may write

$$Q \equiv l - E_i - E_j \quad \text{or} \quad Q \equiv 2l - \sum_{k=1}^{5} E_{i_k},$$

since $Q^2 = -1$. In either case $H \cdot Q = 2$ is impossible.

If $Q^2 = -2$ or $Q^2 = -3$, then we argue similarly to get contradictions.

In case ii) we get

$$(H_1 + K_1)^2 = K_1^2 = 2.$$

Since $K^2 = -14$, we again reproduce H from H_1 and H_Σ:

$$H_\Sigma \equiv F + G \qquad (F, G \quad \text{are the rulings of the quadric } \Sigma).$$

$$H_1 \equiv 3F + 3G - \sum_{i=1}^{6} E_i.$$

$$H \equiv 5F + 5G - \sum_{i=1}^{6} 2E_i - \sum_{j=7}^{22} E_j.$$

Now let $C_0 \equiv G - E_6$, then $H \cdot C_0 = 3$, so C_0 is contained in a hyperplane section H of S. Let $C = H - C_0$, thus

$$C \equiv 5F + 4G - \sum_{i=1}^{5} 2E_i - \sum_{j=6}^{22} E_j,$$

$\deg C = 7$ and $p(C) = 7$. As above we may write $C = E + Q$, where E is a plane quintic, Q is a plane conic and $E \cdot Q = 2$. Again we may assume that $Q^2 \leq -1$, therefore

$$C^2 = (E + Q)^2 = 3 \text{ implies that } E^2 = -1 - Q^2 \geq 0.$$

$E^2 \leq 2$ by the index theorem, so we need to check $-3 \leq Q^2 \leq -1$. This is done like in case i).

In case iii) we get that $(H_1+K_1)^2 = K_1^2 = 0$, therefore S_1 is ruled in conics with 8 singular fibres in the ruling. Let $\mathbf{F}_e$ be the relative minimal model of S_1. Then $\mathbf{F}_e$ has a section B with minimal nonnegative selfintersection $e \geq 0$, and a ruling F. Thus we may write, pulling back B and F to S_1,

$$H_1 \equiv 2B + aF - \sum_{i=1}^{8} E_i$$

where $a = 5 - e$ since $h^0(\mathcal{O}_{S_1}(H_1)) = h^0(Sym^2(\mathcal{O}_{\mathbf{P}^1}(e) \oplus \mathcal{O}_{\mathbf{P}^1}) \otimes \mathcal{O}_{\mathbf{P}^1}(a)) - 8 = 10$. Thus

$$H \equiv 4B + (7-2e)F - \sum_{i=1}^{8} 2E_i - \sum_{j=9}^{22} E_j.$$

Now let $C_0 \equiv F - E_9$. Then $H \cdot C_0 = 3$, so C_0 is contained in a hyperplane section H of S. Let $C = H - C_0$, then C has degree 7 and arithmetic genus 7. A procedure like the above case i) leads to a contradiction, which finishes the proof of the proposition.□

Remark. The formula for the number of 5-secants of a surface which meet a general plane in $\mathbf{P}^4$ (see section 0.3), gives a negative number in the cases $h^1(\mathcal{O}_S(H)) > 2$. This fits well with our proof, where we are left to consider the existence of plane curves of degree at least 5 on S.

The course of the proof indicates the procedure that we will follow in the remaining cases, $h^1(\mathcal{O}_S(H)) = 2$ and $h^1(\mathcal{O}_S(H)) = 1$.

Proposition 8.6. *If S is a smooth rational surface of degree 10 in $\mathbf{P}^4$ with $h^1(\mathcal{O}_S(H)) = 2$, then*

$$H \equiv 8l - \sum_{i=1}^{12} 2E_i - \sum_{j=13}^{18} E_j$$

and S is of the kind described in Theorem A, or

$$H \equiv 9l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j - \sum_{k=12}^{18} E_k.$$

Proof. The proof has two parts. In the first part we produce a number of possible candidates using Theorem 0.10, and in the second part we show that all candidates except those described in the proposition are impossible.

Part 1. With $h^1(\mathcal{O}_S(H)) = 2$ we get the following invariants for S, S_1 and Σ, where S_1 is the image of S under the adjunction map and Σ is the image of S_1 under the adjunction map defined by $|H_1 + K_1|$.

$S \subset \mathbf{P}^4$	$H^2 = 10$	$H \cdot K = 6$	$K^2 = -9$	$\pi = 9$
$S_1 \subset \mathbf{P}^8$	$H_1^2 = 13$	$H_1 \cdot K_1 = -3$	$K_1^2 = -9 + a$	$\pi_1 = 6$
$\Sigma \subset \mathbf{P}^5$	$H_\Sigma^2 = a - 2$	$H_\Sigma \cdot K_\Sigma = a - 12$	$K_\Sigma^2 = -9 + a + b$	$\pi_\Sigma = a - 6,$

where a and b are nonnegative integers, and where the invariants for Σ only make sense when Σ is a surface. Now Σ is either a curve, in which case $a = 2$, or Σ is a surface, which implies that $a \geq 6$. On the other hand $a < 10$, since $a \geq 10$ implies that $K_\Sigma^2 > 0$ and $H_\Sigma \cdot K_\Sigma \geq -2$, which is impossible by Lemma 8.3. So we need to consider $a = 2$ and $6 \leq a \leq 9$.

If $a = 2$, then S_1 is ruled in conics. $K_1^2 = -7$ so therefore S_1 has 15 singular fibres, and we may think of S_1 as the projection of a conic bundle S_1' in $\mathbf{P}^{23}$ from a $\mathbf{P}^{14}$ spanned by 15 points on S_1'. If we write

$$S_1' \cong P(\mathcal{O}_{\mathbf{P}^1}(e) \oplus \mathcal{O}_{\mathbf{P}^1}) = \mathbf{F}_e$$

and denote a section with selfintersection $e \geq 0$ by B and a fiber of the ruling by F, then we may write

$$H_1 \equiv 2B + (7 - e)F - \sum_{i=1}^{15} E_i \qquad \text{with} \quad e < 7,$$

since

$$h^0(\mathcal{O}_{S_1'}(H)) = h^0(Sym^2(\mathcal{O}_{\mathbf{P}^1}(e) \oplus \mathcal{O}_{\mathbf{P}^1}) \otimes \mathcal{O}_{\mathbf{P}^1}(7 - e)) = 3e + 3 + 3(7 - e) = 24.$$

Since $K_1 \equiv -2B + (e - 2)F + \sum_{i=1}^{15} E_i$, we get

$$H \equiv 4B + (9 - 2e)F - \sum_{i=1}^{15} 2E_i - \sum_{j=16}^{17} E_j.$$

If $a = 6$, then Σ is either a scroll or a Veronese surface.

If Σ is a scroll, say $\Sigma \cong \mathbf{F}_e$ with $e \geq 0$, then we get that

$$H_\Sigma \equiv B + (2 - \frac{1}{2}e)F \qquad \text{with} \quad e = 0 \text{ or } 2.$$

Furthermore $K_\Sigma \equiv -2B + (e - 2)F$, so we may reproduce H_1 and H.

$$H_1 \equiv 3B + (4 - \frac{3}{2}e)F - \sum_{i=1}^{11} E_i.$$

$$H \equiv 5B + (6 - \frac{5}{2}e)F - \sum_{i=1}^{11} 2E_i - \sum_{j=12}^{17} E_j.$$

If Σ is a Veronese surface, we get

$$H_\Sigma \equiv 2l \qquad \text{(where } l \text{ is a line in } \mathbf{P}^2\text{)},$$

$$H_1 \equiv 5l - \sum_{i=1}^{12} E_i$$

and

$$H \equiv 8l - \sum_{i=1}^{12} 2E_i - \sum_{j=13}^{18} E_j.$$

If $a = 7$, then Σ is a Del Pezzo surface, so we get

$$H_\Sigma \equiv 3l - \sum_{i=1}^{4} E_i,$$

$$H_1 \equiv 6l - \sum_{i=1}^{4} 2E_i - \sum_{j=5}^{11} E_j$$

and

$$H \equiv 9l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j - \sum_{k=12}^{18} E_k.$$

If $a = 8$, then Σ is of degree 6 in $\mathbf{P}^5$ with $\pi_\Sigma = 2$. Thus Σ is ruled in conics and $(H_\Sigma + K_\Sigma)^2 = 0$, which implies that $b = 3$. So $K_\Sigma^2 = 2$ and Σ has 6 singular fibers. Using the above notation, we get

$$H_\Sigma \equiv 2B + (3-e)F - \sum_{i=1}^{6} E_i \qquad \text{with} \quad e \le 3,$$

since $h^0(\mathcal{O}_\Sigma(H_\Sigma)) = h^0(Sym^2(\mathcal{O}_{\mathbf{P}^1}(e) \oplus \mathcal{O}_{\mathbf{P}^1}) \otimes \mathcal{O}_{\mathbf{P}^1}(3-e)) - 6 = (3e+3+3(3-e)) - 6 = 6$. We reproduce H_1 and H.

$$H_1 \equiv 4B + (5-2e)F - \sum_{i=1}^{6} 2E_i - \sum_{j=7}^{9} E_j.$$

$$H \equiv 6B + (7-3e)F - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{9} 2E_j - \sum_{k=10}^{17} E_k.$$

If $a = 9$, then Σ is of degree 7 in $\mathbf{P}^5$ with $\pi_\Sigma = 3$. We use $|H_\Sigma + K_\Sigma|$ to map Σ into $\mathbf{P}^2$. By Theorem 0.10 we have three cases. Either we are in the exceptional case 4) of the theorem and

$$H_\Sigma \equiv 6l - \sum_{i=1}^{7} 2E_i - E_8,$$

or $|H_\Sigma + K_\Sigma|$ defines a birational morphism, which means that

$$H_\Sigma \equiv 4l - \sum_{i=1}^{\alpha} E_i$$

where $\alpha = 9$ since Σ is nonspecial, or thirdly Σ is mapped onto a curve, which means that Σ is ruled in conics.

In the first case we get

$$H_1 \equiv 9l - \sum_{i=1}^{7} 3E_i - 2E_8 - E_9$$

and

$$H \equiv 12l - \sum_{i=1}^{7} 4E_i - 3E_8 - 2E_9 - \sum_{j=10}^{18} E_j.$$

In the second case we get

$$H_1 \equiv 7l - \sum_{i=1}^{9} 2E_i$$

and

$$H \equiv 10l - \sum_{i=1}^{9} 3E_i - \sum_{j=10}^{18} E_j.$$

In the third case we get $(H_\Sigma + K_\Sigma)^2 = 4a + b - 35 = 1 + b = 0$, which is absurd since $b \geq 0$.

Thus we have established the following list of candidates.

1) $$H \equiv 4B + (9 - 2e)F - \sum_{i=1}^{15} 2E_i - \sum_{j=16}^{17} E_j \qquad e < 7.$$

2) $$H \equiv 5B + (6 - \frac{5}{2}e)F - \sum_{i=1}^{11} 2E_i - \sum_{j=12}^{17} E_j \qquad e = 0 \text{ or } 2.$$

3) $$H \equiv 8l - \sum_{i=1}^{12} 2E_i - \sum_{j=13}^{18} E_j.$$

4) $$H \equiv 9l - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{11} 2E_j - \sum_{k=12}^{18} E_k.$$

5) $$H \equiv 6B + (7 - 3e)F - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{9} 2E_j - \sum_{k=10}^{17} E_k \qquad e < 3.$$

6) $$H \equiv 12l - \sum_{i=1}^{7} 4E_i - 3E_8 - 2E_9 - \sum_{j=10}^{18} E_j.$$

7) $$H \equiv 10l - \sum_{i=1}^{9} 3E_i - \sum_{j=10}^{18} E_j.$$

Part 2. We go on to study these candidates, excluding all but 3) and 4).

In case 1) we will study the linear systems of curves

$$|C_j| = |2B + (4-e)F - \sum_{i=1}^{15} E_i + E_j| \qquad \text{for} \quad 1 \le j \le 15$$

on S . By Riemann-Roch we have that $\dim|2B + (4-e)F| = 14$, so the linear systems $|C_j|$ are all nonempty. We get $H \cdot C_j = 6$ and $p(C_j) = 3$ for a curve C_j in $|C_j|$. The curve C_j may or may not be contained in a hyperplane section. Let us first assume that it is not contained in a hyperplane. Then by Lemma 0.11. C_j must be reducible with components $C_j = A + E$, where A is a plane quartic curve and E is two skew lines or a conic, such that each line (resp. the conic) meets A in one point.

Claim.

$$A \equiv C_j - E_{16} - E_{17} \qquad \text{or} \qquad A \equiv C_j - E_j \equiv 2B + (4-e)F - \sum_{i=1}^{15} E_i.$$

Proof of the claim. By the index theorem $A^2 \le 1$, so $C_j^2 = (A+E)^2 = 2$ implies that $E^2 \ge 1 - 2(A \cdot E) \ge -3$. If E is two skew lines L_1 and L_2, then we may assume as before that $L_i^2 \le -1$, so in this case we get that $L_1^2 = -1$ and $L_2^2 = -1$ or -2. Now one may check that the only possibility for L_1 and L_2 is that they are the lines E_{16} and E_{17}. If E is a conic, then $E^2 \ge 1 - 2(A \cdot E) = -1$, but as before we may assume that $E^2 \le -1$, so $E^2 = -1$ and E must be the conic E_j.□

If C_j is contained in a hyperplane section H, then $H - C_j$ is a plane quartic curve and thus C_j moves in a pencil. Therefore we would find a curve $A' \equiv C_j - E_j$ on S which is again a plane quartic, so we may summarize the above in the following:
On S there is either a plane quartic curve

$$A \equiv 2B + (4-e)F - \sum_{i=1}^{15} E_i,$$

or there are, for each $1 \le j \le 15$, a plane quartic curve

$$A_j \equiv 2B + (4-e)F - \sum_{i=1}^{15} E_i + E_j - E_{16} - E_{17}.$$

We proceed by showing that the second case implies the first. For this let

$$|C'_j| = |H - A_j| = |2B + (5-e)F - \sum_{i=1}^{15} E_i - E_j|$$

be the pencil residual to A_j. Since it is a pencil, we may find a curve

$$D_j \equiv C'_j - E_{16}$$

on S. Now D_j has degree 5 and arithmetic genus 3, so it decomposes by Lemma 0.11 into a plane quartic curve G and a line L with $G \cdot L = 1$. By the index theorem $G^2 \leq 1$ so

$$L^2 = D_j^2 - 2(G \cdot L) - G^2 \geq -2.$$

As before we may assume that $L^2 \leq -1$, so we get that $L^2 = -1$ and $D_i \cdot L = 0$ or $L^2 = -2$ and $D_j \cdot L = -1$.

In the first case $L = E_{17}$ since E_{17} is the only exceptional line on S which does not meet D_i. Thus $G \equiv C'_j - E_{16} - E_{17}$ is a plane quartic on S. The residual pencil

$$|H - G| = |2B + (4-e)F - \sum_{i=1}^{15} E_i + E_j|$$

will contain a reducible curve with components A and E_j, where A is the plane quartic that we are looking for.

In the other case, that is if $L^2 = -2$, then $D_j \cdot L = -1$ implies that $-1 \leq L \cdot C'_j = -1 + L \cdot E_{16} \leq 0$ since $0 \leq L \cdot E_{16} \leq 1$. Again we have two subcases.

If $L \cdot E_{16} = 0$, that is when $L \cdot C'_j = -1$, then L must be a fixed curve of the pencil $|C'_j|$, so L must lie in the plane of A_j and therefore $L \cdot A_j = 4$. But then $H \cdot L = (C'_j + A_j) \cdot L = 3$, which is absurd since L is a line.

If $L \cdot E_{16} = 1$, that is when $L \cdot C'_j = 0$, we set

$$L \equiv aB + bF - \sum_{i=1}^{15} \alpha_i E_i - E_{16} - \alpha_{17} E_{17}, \quad a, b \geq 0, \quad \alpha_i \geq -1.$$

We have

$$L \cdot D_j = 5a + 2b - ae - \sum_{i=1}^{15} \alpha_i - \alpha_j - 1 = -1$$

and

$$L \cdot A_j = L \cdot (H - C'_j) = 4a + 2b - ae - \sum_{i=1}^{15} \alpha_i + \alpha_j - 1 - \alpha_{17} = 1,$$

which yields the relation

$$-a + 2\alpha_j - \alpha_{17} = 2.$$

Thus $\alpha_j > 0$ and therefore also $a > 0$ or $b > 0$. If $a > 0$, then $\alpha_j \geq 2$, so $(L + E_j)^2 \geq 1$ which contradicts the index theorem. If $a = 0$ and $b > 0$, then, since L is irreducible, we have $b = 1$. From this follows that $L \equiv F - E_j - E_{16}$. But then

$$G \equiv D_j - L \equiv 2B + (4-e)F - \sum_{i=1}^{15} E_i \equiv A.$$

To show that the existence of a curve A on S leads to a contradiction, we study the residual pencil

$$|D| = |H - A| = |2B + (5-e)F - \sum_{i=1}^{17} E_i|.$$

First of all since $D \cdot A = 3$, $|D|$ has no fixed component since it would lie in the plane of A. Furthermore $D^2 = 3$, so a general D is smooth, of degree 6 and genus 4. Thus D is canonically embedded on S in $\mathbf{P}^4$. But D is hyperelliptic since it is a bisection on S, so the canonical series on D is not very ample. This is the contradiction we desire, which concludes case 1).

In case 2) we study curves in the linear system

$$|C| = |2B + (3-e)F - \sum_{i=1}^{11} E_i|.$$

Since dim$|2B + (3-e)F|$=11, we can find a curve C in $|C|$. It has degree 5 and arithmetic genus 2. If C is not contained in a hyperplane, then it must be the union of a plane quartic A and a line L not meeting the plane of A. By the index theorem $A^2 \leq 1$, so we get $L^2 = 1 - A^2 \geq 0$, which is impossible by Riemann-Roch. Therefore C is contained in a hyperplane section H with a residual curve $C_1 = H - C$ of degree 5 and arithmetic genus 4. This is impossible by Lemma 0.11.

In case 3) we want to show that if

$$H \equiv 8l - \sum_{i=1}^{12} 2E_i - \sum_{j=1}^{6} F_j$$

is the hyperplane section of a smooth surface in $\mathbf{P}^4$, then the surface S can be constructed as described in the proof of Theorem B.

Let

$$C \equiv 4l - \sum_{i=1}^{12} E_i,$$

$$C_{ij} \equiv C - \sum_{k=1}^{6} F_k + F_i + F_j$$

and

$$C^{ij} \equiv H - C_{ij} \qquad \text{for} \quad 1 \leq i < j \leq 6.$$

Now $h^0(\mathcal{O}_S(C^{ij})) > 0$, so there is a curve C^{ij} in $|C^{ij}|$. It has degree 6 and arithmetic genus 3, so as a curve on S it either spans a $\mathbf{P}^3$, in which case there is a residual plane quartic curve $C_{ij} \equiv H - C^{ij}$ and $h^0(\mathcal{O}_S(C^{ij})) = 2$, or C^{ij} is reducible, that is C^{ij} is the union of a plane quartic A and two skew lines L_1 and L_2, with $A \cdot L_1 = A \cdot L_2 = 1$, or a conic Q, with $A \cdot Q = 1$. One may now check that in the latter case $A \equiv C_{st}$ for some s, t and L_1 and L_2 are lines F_k and F_l.

These two possibilities for each C^{ij} fit together only if say

$$C_{12}, \quad C_{34} \quad \text{and} \quad C_{56}$$

are plane quartics and

$$|C^{12}|, \quad |C^{23}| \quad \text{and} \quad |C^{56}|$$

are their respective residual pencils.

Now $C_{12} \cdot C_{34} = C_{12} \cdot C_{56} = C_{34} \cdot C_{56} = 2$, so the planes of C_{12}, C_{34} and C_{56} meet pairwise in lines. Since the three planes span all of $\mathbf{P}^4$, they must intersect in a common line L, which is now a 6-secant for the surface S unless L lies on S.

To see that L cannot lie on S, we first note that if L is a fixed curve for the pencil $|C^{12}|$, then $L \cdot C_{12} = 4$. Therefore

$$L \cdot H = L \cdot 2C_{12} + L \cdot (F_3 + F_4 + F_5 + F_6 - F_1 - F_2) = 8 - L \cdot (F_3 + F_4 + F_5 + F_6 - F_1 - F_2) = 1.$$

But $L \cdot (F_3 + F_4 + F_5 + F_6 - F_1 - F_2) = 7$ is clearly impossible, as long as L is a line. Similarly L cannot be a fixed curve for any of the pencils $|C^{34}|$ and $|C^{56}|$. So if L lies on S, then L is a component of the curves C_{ij} and $L \cdot (C_{ij} - L) = 3$. In this case we get $p(C_{ij} - L) = 1$, so $l \cdot (C_{ij} - L) \geq 3$ and therefore $l \cdot L \leq 1$. On the other hand if $l \cdot L = 0$, then L has support on the exceptional curves. Thus L equals some F_s or some $E_r - F_s$. Both cases contradicts the assumption that $L \cdot (C_{ij} - L) = 3$ for any $\{ij\} \in \{\{12\}, \{34\}, \{56\}\}$. Therefore, $l \cdot L = 1$ and we may write

$$L \equiv l - \sum_{i=1}^{12} \alpha_i E_i - \sum_{i=1}^{6} \beta_i F_i,$$

where $0 \leq \alpha_i \leq 1$ and $0 \leq \beta_i \leq 1$. We get the relations

$$L \cdot (C_{ij} - L) = 3 - \beta_i - \beta_j = 3$$

for any $\{ij\} \in \{\{12\}, \{34\}, \{56\}\}$, so $\beta_i = 0$ for $i = 1, \ldots, 6$. But we also have the relation

$$H \cdot L = 8 - \sum_{i=1}^{12} 2\alpha_i = 1,$$

which is now impossible. Thus L cannot be contained in S.

Let $S \cap L \supseteq q_1 + \ldots + q_6$ such that $C_{12} \cap L = q_3 + q_4 + q_5 + q_6$, $C_{34} \cap L = q_1 + q_2 + q_5 + q_6$ and $C_{56} \cap L = q_1 + q_2 + q_3 + q_4$. Since C_{12} is a plane quartic curve on S, that is

$$\mathcal{O}_{C_{12}}(H) \cong \omega_{C_{12}} \cong \mathcal{O}_{C_{12}}(l),$$

we see that the colinear points $q_3, \ldots, q_6$ all lie on a curve $L_0 \equiv l$. But we get the same for C_{34} and C_{56}, so this means that all the points q_i lie on L_0. The rest now follows from the proof of Theorem B.

In case 5) let $C_0 \equiv 2B + (2-e)F - \sum_{i=1}^{8} E_i$, then $h^0(\mathcal{O}_S(C_0)) \geq 1$ by Riemann-Roch and $H \cdot C_0 = 4$ and $p(C_0) = 1$. The curve C_0 is therefore part of a hyperplane section with a residual curve $C \equiv H - C_0$. Now C has degree 6 and arithmetic genus 5, so C must, by Lemma 0.11, be the union of a plane quintic curve A and a line L which does not meet A. We get

$$C^2 = (A+L)^2 = A^2 + L^2 = 2.$$

Since we may assume that $L^2 \leq -1$, we get thet $A^2 \geq 3$, which contradicts the index theorem.

In case 6) we start off noting that there is a curve

$$C_0 \equiv 3l - \sum_{i=1}^{9} E_i$$

on S. It has degree 3 and arithmetic genus 1, so it is a plane cubic curve on S. We will study the residual pencil of curves

$$|D| = |H - C_0|.$$

This pencil may have a fixed curve in the plane of C_0, and since $D \cdot C_0 = 3$, this fixed curve must be a line, call it L. If we set

$$L \equiv \alpha l - \sum_{i=1}^{18} \beta_i E_i, \qquad \alpha \geq 0,\ \beta_i \geq -1,$$

we get the relations

$$H \cdot L = 12\alpha - \sum_{i=1}^{7} 4\beta_i - 3\beta_8 - 2\beta_9 - \sum_{j=10}^{18} \beta_j = 1$$

and

$$C_0 \cdot L = 3\alpha - \sum_{i=1}^{7} \beta_i - \beta_8 - \beta_9 = 3.$$

Thus

$$\beta_8 + 2\beta_9 - \sum_{j=10}^{18} \beta_j = -11.$$

But $\beta_j \leq 1$ for $10 \leq j \leq 18$, so we get that $\beta_9 = -1$ and $\beta_j = 1$ for $10 \leq j \leq 18$, which contradicts $H \cdot L = 1$. Therefore $|D|$ does not have a fixed curve.

Now since D is not a multiple divisor on S, we can assume that the general curve D in $|D|$ is irreducible. It has degree 7 and arithmetic genus 6, so it is of type (3,4) on a quadric surface, and $|H|$ restricts to a special linear series on D. S is rational, so there is an isomorphism $\mathrm{H}^0(\mathcal{O}(S)D+K) \cong \mathrm{H}^0(\mathcal{O}(D)K_D)$. Therefore we may write

$$|H|_D| = |K_D - \delta_D| = |(D+K)|_D - \delta_D|$$

where δ_D is a divisor of degree 3 on D. Note that δ_D moves in a pencil on D, so our choice of divisor δ_D is in no way canonical. We now blow up S in the points of δ_D to get a surface S_0 with exceptional divisors A_1, A_2 and A_3, and study the linear system

$$|B| = |D + K - A_1 - A_2 - A_3|$$

on S_0. If D' is the strict transform of D on S_0, then by the above we know that $|H_{|D'}| = |B_{|D'}|$, so $\dim|B|=3$ and $|B|$ defines a rational map $\varphi_B : S_0 \dashrightarrow \mathbf{P}^3$.

Claim. Any fixed component of $|B|$ has support on the E_j, $9 \leq j \leq 18$.

Proof of the claim. Consider the morphism

$$\varphi_{H+2K} : S \to \Sigma \subset \mathbf{P}^5$$

defined by $|D+K| = |6l - \sum_{i=1}^{7} 2E_i - E_8|$. Then $Z = \varphi_{H+2K}(\delta_D) \subset \Sigma$ is supported on a line L in $\mathbf{P}^5$ since there is a threedimensional subsystem of $|D+K|$ whose members all vanish on Z. This line L must support the image of the fixed component of $|B|$. Since φ_{H+2K} is a composition of adjunction mappings, we see from Theorem 0.10 that for the claim it is enough to show that L does not lie on Σ. If it did, then we could write

$$L \equiv \alpha l - \sum_{i=1}^{8} \beta_i E_i, \quad \alpha \geq 0, \quad \beta_i \geq -1,$$

such that

$$(D+K) \cdot L = 6\alpha - \sum_{i=1}^{7} 2\beta_i - \beta_8 = 1,$$

and

$$D \cdot L = 9\alpha - \sum_{i=1}^{7} 3\beta_i - 2\beta_8 = 3 \quad (\text{since } D \cap L = Z).$$

But these two equations yield $\beta_8 = -3$, which is impossible.□

Since all the curves E_i, $9 \leq i \leq 18$, meet D, a fixed curve of $|B|$ can only occur when δ_D has support on one of these curves. For our purpose we may assume that δ_D is chosen without support on the exceptional curves.

Now if B is a general member of $|B|$, then we may assume that B is irreducible. Since $B^2 = 4$, $p(B) = 3$ and $\dim|B|_B| = 2$, φ_B restricts to the canonical map on B. In particular, for reasons of degree only, $|B|$ is basepointfree. Thus $|B|_B| = |B + K_{S_0}|_B|$, and considering the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_0}(-K_{S_0} - B) \longrightarrow \mathcal{O}_{S_0}(-K_{S_0}) \longrightarrow \mathcal{O}_B(-K_{S_0}) \longrightarrow 0$$

we get that $\mathcal{O}_B(-K_{S_0}) \cong \mathcal{O}_B$. We take cohomology to see that $h^0(\mathcal{O}_{S_0}(-K_0)) = 1$ if and only if $h^1(\mathcal{O}_{S_0}(-K_{S_0} - B)) = h^1(\mathcal{O}_{S_O}(-3l + \sum_{i=1}^7 E_i)) = 0$. But this is so unless $\dim|3l - \sum_{i=1}^7 E_i| \geq 3$, which would imply the absurdity that there is a curve in the system $|l|$ which meets at least five of the E_i. Thus for each δ_D we get a curve $C \equiv -K_0 \equiv 3l - \sum_{i=1}^8 E_i - A_1 - A_2 - A_3$ on S_0. Using different Ds and different choices of δ_D, we may conclude that there is a curve $C_1 \equiv 3l - \sum_{i=1}^{10} E_i$ on S. But C_1 has degree 2 and arithmetic genus 1, so this is impossible.

In case 7) we follow a track very similar to the one of case 6). Let $C_0 \equiv 3l - \sum_{i=1}^9 E_i$, then C_0 has degree 3 and arithmetic genus 1, so it is a plane cubic curve on S. We will study the residual pencil $|C| = |H - C_0| = |7l - \sum_{i=1}^9 2E_i - \sum_{j=10}^{18} E_j|$. We first show

Lemma 8.9. *$|C|$ has only isolated basepoints.*

Proof. A fixed curve Γ of $|C|$ must lie in the plane of C_0. Since $\Gamma \cdot C_0 \leq C \cdot C_0 = 3$, Γ must be a line on S. We may write

$$\Gamma \equiv \alpha l - \sum_{i=1}^{18} \beta_i E_i$$

with $\alpha \geq 0$ and $\beta_i \geq -1$, and get the relations

$$\Gamma \cdot H = 10\alpha - \sum_{i=1}^{9} 3\beta_i - \sum_{j=10}^{18} \beta_j = 1$$

and

$$\Gamma \cdot C_0 = 3\alpha - \sum_{i=1}^{9} \beta_i = 3.$$

These relations combine to give

$$\alpha + 8 = \sum_{j=10}^{18} \beta_j.$$

The three relations imply, since clearly $\beta_j \leq 1$ for $10 \leq j \leq 18$, that $\Gamma \equiv l - \sum_{j=10}^{18} E_j$. Thus

$$|D| = |H - \Gamma| = |9l - \sum_{i=1}^{9} 3E_i|$$

is a net of elliptic curves with possible basepoints only on Γ. Since $D^2 = 0$, there are no basepoints and $|D|$ is composed of a rational pencil (rational since Γ dominates the base).

We may write $D \equiv nF$, where F is elliptic and $n \geq 2$, so from the class of D on S we see that $n = 3$ and $F \equiv 3l - \sum_{i=1}^{9} E_i$. Since F moves in a pencil, we can find a curve

$$C_1 \equiv F - E_{10} \equiv 3l - \sum_{i=1}^{10} E_i$$

on S. But then C_1 has degree 2 and arithmetic genus 1 on S, which is impossible.□

From the lemma we may assume that the general member C of $|C|$ is irreducible. Since the degree of C is 7 and the arithmetic genus is 6, C must be of type (3,4) on a quadric surface and $|H_{|C}|$ must be a special linear series on C. We can therefore write

$$|H_{|C}| = |K_C - \delta_C| = |(C + K)_{|C} - \delta_C|$$

where δ_C is a divisor of degree 3 on C. If we blow up S in the points of δ_C, we get a surface S_0 with exceptional divisors which we denote by A_1, A_2 and A_3. Since S_0 is rational, there is an isomorphism

$$\mathrm{H}^0(\mathcal{O}(S_0)C' + K_{S_0} - A_1 - A_2 - A_3) \cong \mathrm{H}^0(\mathcal{O}(C')K_{C'} - \delta_{C'})$$

where C' is the strict transform of C on S_0. Thus

$$|B| = |C' + K_{S_0} - A_1 - A_2 - A_3| = |4l - \sum_{i=1}^{9} E_i - A_1 - A_2 - A_3|$$

is a threedimensional linear system of curves on S_0 which restricts like $|H|$ to C. Comparing this with Lemma 0.12, we get that, for a general choice of δ_C, the linear system $|B|$ has no fixed curve, and there is a curve

$$C_1 \equiv 3l - \sum_{i=1}^{9} E_i \equiv C_0$$

on S such that C_1 contains the points of δ_C. δ_C moves in a pencil on C, so therefore C_1 must move in a pencil on S. But that means that we can find a curve

$$C'' \equiv C_1 - E_{10} \equiv 3l - \sum_{i=1}^{10} E_i$$

on S, which has degree 2 and arithmetic genus 1. This is impossible.□

Proposition 8.10. *If S is a smooth rational surface of degree 10 in $\mathbf{P}^4$ with hyperplane section H and $h^1(\mathcal{O}_S(H)) = 1$, then S must be of the kind described in theorem A.*

Proof. We follow the same procedure as in the proof of Proposition 8.5. Using the results of Sommese and Van de Ven (Theorem 0.10), we establish a list of candidates which we thereafter treat one by one. If we denote the images of S by the (iterated) adjunction mappings by S_i, and denote their respective hyperplane sections, canonical divisors and genus of H_i (when these have meaning) by H_i, K_i and π_i, we get the following invariants:

$S \subset \mathbf{P}^4$	$H^2 = 10$	$H \cdot K = 4$	$K^2 = -4$	$\pi = 8$
$S_1 \subset \mathbf{P}^7$	$H_1^2 = 14$	$H_1 \cdot K_1 = 0$	$K_1^2 = -4 + a$	$\pi_1 = 8$
$S_2 \subset \mathbf{P}^7$	$H_2^2 = 10 + a$	$H_2 \cdot K_2 = a - 4$	$K_2^2 = -4 + a + b$	$\pi_2 = 4 + a$
$S_3 \subset \mathbf{P}^{3+a}$	$H_3^2 = 4a + b - 2$	$H_3 \cdot K_3 = 2a + b - 8$	$K_3^2 = -4 + a + b + c$	$\pi_3 = 3a + b - 4$

where a, b, c are nonnegative integers.

Now S_3 is either a curve, in which case $H_3^2 = 0$, that is $a = 0$ and $b = 2$, or S_3 is a surface. By Lemma 8.3 we need to consider $0 \leq a \leq 3$.

If $a = 0$, then S_3 is a curve, as observed above, if $b = 2$, or S_3 is a smooth surface in $\mathbf{P}^3$. If S_3 is a curve, then S_2 is ruled in conics with 10 singular fibers since $K_2^2 = -2$. We may write, using notation as before,

$$H_2 \equiv 2B + (5 - e)F - \sum_{i=1}^{10} E_i \qquad \text{with} \quad e < 5$$

since

$$h^0(\mathcal{O}_{S_2}(H_2)) = h^0(Sym^2(\mathcal{O}_{\mathbf{P}^1}(e) \oplus \mathcal{O}_{\mathbf{P}^1}) \otimes \mathcal{O}_{\mathbf{P}^1}(5 - e)) - 10 = 3(e + 1 + (5 - e)) - 10 = 8.$$

Thus we can reproduce H_1 and H:

$$H_1 \equiv 4B + (7 - 2e)F - \sum_{i=1}^{10} 2E_i - \sum_{j=11}^{12} E_j$$

and

$$H \equiv 6B + (9 - 3e)F - \sum_{i=1}^{10} 3E_i - \sum_{j=11}^{12} 2E_j \qquad \text{with} \quad e < 5.$$

If S_3 is a smooth surface in $\mathbf{P}^3$, then, since S_3 is rational, we have either $H_3 \equiv B + F$ and $b = 4$, $e = 0$, or $H_3 \equiv 3l - \sum_{i=1}^{6} E_i$ and $b = 5$, thus we get

$$H_2 \equiv 3B + 3F - \sum_{i=1}^{8} E_i \quad \text{and} \quad H_1 \equiv 5B + 5F - \sum_{i=1}^{8} 2E_i - \sum_{j=9}^{12} E_j$$

and

$$H \equiv 7B + 7F - \sum_{i=1}^{8} 3E_i - \sum_{j=9}^{12} 2E_j \qquad \text{with} \quad e = 0,$$

or

$$H_2 \equiv 6l - \sum_{i=1}^{6} 2E_i - \sum_{j=7}^{8} E_j \quad \text{and} \quad H_1 \equiv 9l - \sum_{i=1}^{6} 3E_i - \sum_{j=7}^{8} 2E_j - \sum_{k=9}^{13} E_k$$

and

$$H \equiv 12l - \sum_{i=1}^{6} 4E_i - \sum_{j=7}^{8} 3E_j - \sum_{k=9}^{13} 2E_k.$$

If $a = 1$, then S_3 is a smooth surface in $\mathbf{P}^4$ of degree $H_3^2 = b + 2$, $H_3 \cdot K_3 = b - 6$ and $K_3^2 = b + c - 3$. By Lemma 8.3 we get that $b \leq 3$, and since S_3 is nondegenerate, $b \geq 1$.

In case $b = 1$ we get $H_3 \equiv 2l - E_1$ and thus

$$H_2 \equiv 5l - 2E_1 - \sum_{i=2}^{11} E_i \quad \text{and} \quad H_1 \equiv 8l - 3E_1 - \sum_{i=2}^{11} 2E_i - E_{12}$$

and

$$H \equiv 11l - 4E_1 - \sum_{i=2}^{11} 3E_i - 2E_{12} - E_{13}.$$

In case $b = 2$ we get $H_3 \equiv 3l - \sum_{i=1}^{5} E_i$, and thus

$$H_2 \equiv 6l - \sum_{i=1}^{5} 2E_i - \sum_{j=6}^{10} E_j \quad \text{and} \quad H_1 \equiv 9l - \sum_{i=1}^{5} 3E_i - \sum_{j=6}^{10} 2E_j - \sum_{k=11}^{12} E_k$$

and

$$H \equiv 12l - \sum_{i=1}^{5} 4E_i - \sum_{j=6}^{10} 3E_j - \sum_{k=11}^{12} 2E_k - E_{13}.$$

In case $b = 3$ we get $H_3 \equiv 4l - 2E_1 - \sum_{i=2}^{8} E_i$ and thus

$$H_2 \equiv 7l - 3E_1 - \sum_{i=2}^{8} 2E_i - E_9 \quad \text{and} \quad H_1 \equiv 10l - 4E_1 - \sum_{i=2}^{8} 3E_i - 2E_9 - \sum_{j=10}^{12} E_j$$

and

$$H \equiv 13l - 5E_1 - \sum_{i=2}^{8} 4E_i - 3E_9 - \sum_{j=10}^{12} 2E_j - E_{13}.$$

If $a = 2$, then S_3 is a smooth surface in $\mathbf{P}^5$ of degree $H_3^2 = 6 + b$. Furthermore we get that $H_3 \cdot K_3 = b - 4$ and $K_3^2 = b + c - 2$ and $\pi_3 = b + 2$. By Lemma 8.3 we get that $b = 0$ or 1.

If $b=0$, then $\pi_3=2$, so S_3 is ruled in conics. Thus $(H_3+K_3)^2=c-4=0$ and $K_3^2=c-2=2$ and S_3 has 6 singular fibers. We may write $H_3\equiv 2B+(3-e)F-\sum_{i=1}^6 E_i$ with $e<3$, since

$$h^0(\mathcal{O}_{S_3}(H_3))=h^0(Sym^2(\mathcal{O}_{\mathbf{P}^1}(e)\oplus\mathcal{O}_{\mathbf{P}^1})\otimes\mathcal{O}_{\mathbf{P}^1}(3-e))-6=3(e+1+(3-e))-6=6.$$

We calculate H_2, H_1 and H:

$$H_2\equiv 4B+(5-2e)F-\sum_{i=1}^{6}2E_i-\sum_{j=7}^{10}E_j,$$

$$H_1\equiv 6B+(7-3e)F-\sum_{i=1}^{6}3E_i-\sum_{j=7}^{10}2E_j$$

and

$$H\equiv 8B+(9-4e)F-\sum_{i=1}^{6}4E_i-\sum_{j=7}^{10}3E_j-\sum_{k=11}^{12}E_k.$$

If $b=1$, then $\pi_3=3$ and $(H_3+K_3)^2=c$. By Theorem 0.10 we have $0\le c\le 2$. If $c=0$ then S_3 is ruled in conics with 9 singular fibers. We may write $H_3\equiv 2B+(4-e)F-\sum_{i=1}^9 E_i$, $e<4$, since

$$h^0(\mathcal{O}_{S_3}(H_3))=h^0(Sym^2(\mathcal{O}_{\mathbf{P}^1}(e)\oplus\mathcal{O}_{\mathbf{P}^1})\otimes\mathcal{O}_{\mathbf{P}^1}(4-e))-9=3(e+1+(4-e))-9=6.$$

We can calculate H_2, H_1 and H:

$$H_2\equiv 4B+(6-2e)F-\sum_{i=1}^{9}2E_i \quad\text{and}\quad H_1\equiv 6B+(8-3e)F-\sum_{i=1}^{9}3E_i-E_{10}$$

and

$$H\equiv 8B+(10-4e)F-\sum_{i=1}^{9}4E_i-2E_{10}-\sum_{j=11}^{12}E_j.$$

If $c=1$, then $H_3+K_3\equiv l$ and thus $H_3\equiv 4l-\sum_{i=1}^9 E_i$. We get

$$H_2\equiv 7l-\sum_{i=1}^{9}2E_i-E_{10} \quad\text{and}\quad H_1\equiv 10l-\sum_{i=1}^{9}3E_i-2E_{10}-E_{11}$$

and

$$H\equiv 13l-\sum_{i=1}^{9}4E_i-3E_{10}-2E_{11}-\sum_{j=12}^{13}E_j.$$

If $c = 2$, then $H_3 \equiv 6l - \sum_{i=1}^{7} 2E_i - E_8$, so we get

$$H_2 \equiv 9l - \sum_{i=1}^{7} 3E_i - 2E_8 - \sum_{j=9}^{10} E_j \quad \text{and} \quad H_1 \equiv 12l - \sum_{i=1}^{7} 4E_i - 3E_8 - \sum_{j=9}^{10} 2E_j - E_{11}$$

and

$$H \equiv 15l - \sum_{i=1}^{7} 5E_i - 4E_8 - \sum_{j=9}^{10} 3E_j - 2E_{11} - \sum_{k=12}^{13} E_k.$$

If $a = 3$, then S_3 is a smooth surface in $\mathbf{P}^6$ of degree $H_3^2 = 10 + b$ with $H_3 \cdot K_3 = b - 2$ and $K_3^2 = b + c - 1$ and $\pi_3 = b + 5$. By Lemma 8.3 we get $b = c = 0$, thus $(H_3 + K_3)^2 = 5$ and $\pi_3 = 5$, so $H_3 + K_3 \equiv 4l - 2E_1 - \sum_{i=2}^{8} E_i$. We may reproduce H_3, H_2, H_1 and H:

$$H_3 \equiv 7l - 3E_1 - \sum_{i=2}^{8} 2E_i - \sum_{j=9}^{10} E_j,$$

$$H_2 \equiv 10l - 4E_1 - \sum_{i=2}^{8} 3E_i - \sum_{j=9}^{10} 2E_j,$$

$$H_1 \equiv 13l - 5E_1 - \sum_{i=2}^{8} 4E_i - \sum_{j=9}^{10} 3E_j$$

and

$$H \equiv 16l - 6E_1 - \sum_{i=2}^{8} 5E_i - \sum_{j=9}^{10} 4E_j - \sum_{k=11}^{13} E_k.$$

We have established the following list of candidates:

1) $$H \equiv 6B + (9 - 3e)F - \sum_{i=1}^{10} 3E_i - \sum_{j=11}^{12} 2E_j \qquad \text{with} \quad e < 5.$$

2) $$H \equiv 7B + 7F - \sum_{i=1}^{8} 3E_i - \sum_{j=9}^{12} 2E_j \qquad \text{with} \quad e = 0.$$

3) $$H \equiv 12l - \sum_{i=1}^{6} 4E_i - \sum_{j=7}^{8} 3E_j - \sum_{k=9}^{13} 2E_k.$$

4) $$H \equiv 11l - 4E_1 - \sum_{i=2}^{11} 3E_i - 2E_{12} - E_{13}.$$

5) $$H \equiv 12l - \sum_{i=1}^{5} 4E_i - \sum_{j=6}^{10} 3E_j - \sum_{k=11}^{12} 2E_k - E_{13}.$$

6) $$H \equiv 13l - 5E_1 - \sum_{i=2}^{8} 4E_i - 3E_9 - \sum_{j=10}^{12} 2E_j - E_{13}.$$

7) $$H \equiv 8B + (9-4e)F - \sum_{i=1}^{6} 4E_i - \sum_{j=7}^{10} 3E_j - \sum_{k=11}^{12} E_k.$$

8) $$H \equiv 8B + (10-4e)F - \sum_{i=1}^{9} 4E_i - 2E_{10} - \sum_{j=11}^{12} E_j.$$

9) $$H \equiv 13l - \sum_{i=1}^{9} 4E_i - 3E_{10} - 2E_{11} - \sum_{j=12}^{13} E_j.$$

10) $$H \equiv 15l - \sum_{i=1}^{7} 5E_i - 4E_8 - \sum_{j=9}^{10} 3E_j - 2E_{11} - \sum_{k=12}^{13} E_k.$$

11) $$H \equiv 16l - 6E_1 - \sum_{i=2}^{8} 5E_i - \sum_{j=9}^{10} 4E_j - \sum_{k=11}^{13} E_k.$$

Next we go case by case and show that only case 8), with $e < 3$, is possible.

In case 1) let
$$C \equiv 2B + (3-e)F - \sum_{i=1}^{11} E_i.$$
Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 4$ and $p(C) = 2$, which is impossible by Lemma 0.11.

In case 2) let
$$C \equiv 2B + 3F - \sum_{i=1}^{11} E_i.$$
Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 5$ and $p(C) = 2$. If C spans all of $\mathbf{P}^4$, then C is the union of a plane quartic curve A and a line L which does not meet the plane of A. We can assume that $L^2 \leq -1$ and get $A^2 = C^2 - L^2 \geq 2$, but this contradicts the index theorem, so C must be part of a hyperplane section. The residual curve $C_0 \equiv H - C$ has degree 5 and arithmetic genus 3, so it is the union of a plane quartic curve A_0 and a line L_0, which meets A_0 in one point. We may again assume that $L_0^2 \leq -1$ and get $A_0^2 = C_0^2 - L_0^2 - 2 = -1 - L^2 \geq 0$. The index theorem implies that $A_0^2 \leq 1$, so we need to check the two cases $L_0^2 = -1$ and $L_0^2 = -2$. Since $K^2 = K_1^2$, there are no exceptional lines on S, so we are left to check $L_0^2 = -2$. Now L_0 is a line on S, so $L_0 \cdot B \leq 1$ and $L_0 \cdot F \leq 1$. Using this, one may check that we get a few possible divisor classes for L_0 on S, none of which matches the invariants of L_0.

In case 3) let

$$C \equiv 6l - \sum_{i=1}^{7} 2E_i - \sum_{j=8}^{13} E_j.$$

Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 5$ and $p(C) = 3$, so C must be the union of a plane quartic curve A and a line L which meets A in a point. We may assume that $L^2 \leq -1$, so we get that $A^2 = C^2 - L^2 - 2 = -L^2 \geq 1$. On the other hand the index theorem implies that $A^2 \leq 1$, so L must be an exceptional line on S. This is contradicted by Theorem 0.10 and the fact that $K^2 = K_1^2$.

In case 4) let

$$C \equiv 4l - 2E_1 - \sum_{i=2}^{12} E_i.$$

Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 4$ and $p(C) = 2$, which is impossible by Lemma 0.11.

In case 5) let

$$C \equiv 4l - 2E_1 - \sum_{i=2}^{12} E_i.$$

Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 5$ and $p(C) = 2$, so we proceed as in case 2). Since S contains an exceptional line, there is one more subcase than in case 2) to check, but like the other cases it is straightforward to rule it out.

In case 6) let

$$C \equiv 4l - 2E_1 - \sum_{i=2}^{12} E_i.$$

Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 5$ and $p(C) = 2$. From here on an argument just like the one for case 2) and 5) will rule out this case too.

In case 7) let

$$C \equiv 4B + (4 - 2e)F - \sum_{i=1}^{7} 2E_i - \sum_{j=8}^{10} E_j.$$

Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 5$ and $p(C) = 2$, so we use the same argument as above to rule out this case too.

In case 8) we want to show that S has to be of the kind described in theorem B. Let

$$C \equiv 4B + (5 - 2e)F - \sum_{i=1}^{9} 2E_i - E_{10} - E_{11}.$$

Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 5$ and $p(C) = 3$. Thus C must be the union of a plane quartic curve A and a line L such that $A \cdot L = 1$. An argument like the one for case 2) shows that this is possible only if $L = E_{12}$ and

$$A \equiv 4B + (5-2e)F - \sum_{i=1}^{9} 2E_i - E_{10} - E_{11} - E_{12}.$$

Since A is a plane quartic curve on S, we get that E_{10} cannot be a component of A. Let $q = A \cap E_{10}$ and blow up S in q to get S_0 with exceptional divisor E_0. Let $H_0 \equiv H + E_{10} - E_0$ on S_0. Then, if we let A' be the strict transform of A on S_0, we get that

$$|H_{0|A'}| = |H_{|A'}| = |K_{A'}| = |(A' + K_{S_0})_{|A'}|.$$

Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_0}(H_0 - 2A' - K_{S_0}) \longrightarrow \mathcal{O}_{S_0}(H_0 - A' - K_{S_0}) \longrightarrow \mathcal{O}_{A'}(H_0 - A' - K_{S_0}) \longrightarrow 0.$$

From the above equalities $\mathcal{O}_{A'}(H_0 - A' - K_{S_0}) \cong \mathcal{O}_{A'}$, on the other hand, by Riemann-Roch, we get that

$$h^0(\mathcal{O}_{S_0}(H_0 - 2A' - K_{S_0})) = h^1(\mathcal{O}_{S_0}(H_0 - 2A' - K_{S_0})),$$

so taking cohomology of the exact sequence we see that

$$h^0(\mathcal{O}_{S_0}(H_0 - A' - K_{S_0})) \geq 1.$$

Thus there is a curve

$$D_0 \equiv H_0 - A' - K_{S_0} \equiv 6B + (7-3e)F - \sum_{i=1}^{9} 3E_i - E_{10} - E_0 - E_{11} - E_{12}$$

on S_0, and S is of the kind described in theorem B (if $e = 3$ then $H \cdot (B - eF) < 0$, which is impossible, so $e \leq 2$).

In case 9) let

$$C \equiv 7l - 3E_1 - \sum_{i=2}^{10} 2E_i - E_{11} - E_{12}.$$

Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 6$ and $p(C) = 3$. If C spans all of $\mathbf{P}^4$, then C is the union of a plane quartic curve A and two skew lines L_1 and L_2 or a conic Q, such that $L_1 \cdot A = L_2 \cdot A = 1$ or $Q \cdot A = 1$. We may assume, like before, that $L_i^2 \leq -1$ (resp. $Q^2 \leq -1$), therefore we get that

$$A^2 = C^2 - 4 - L_1^2 - L_2^2 \geq 0 \qquad (\text{resp.} \quad A^2 = C^2 - 2 - Q^2 > 0).$$

On the other hand, the index theorem implies that $A^2 \leq 1$, so we get three cases: $L_1^2 = L_2^2 = -1$ and $A^2 = 0$; $L_1^2 = -1$, $L_2^2 = -2$ and $A^2 = 1$, and $Q^2 = -1$ and $A^2 = 1$. It is straightforward to check that no divisor classes on S fit together to match these invariants. Thus we may assume that C is contained in a hyperplane. The residual curve $C_0 \equiv H - C$ has degree 4 and arithmetic genus 2, which is again impossible by Lemma 0.11.

In case 10) let

$$C \equiv 6l - \sum_{i=1}^{8} 2E_i - \sum_{j=9}^{11} E_j.$$

Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 4$ and $p(C) = 2$, which is impossible.

In case 11) let

$$C \equiv 9l - \sum_{i=1}^{8} 3E_i - \sum_{j=9}^{10} 2E_j - E_{11}.$$

Then $h^0(\mathcal{O}_S(C)) \geq 1$, $H \cdot C = 4$ and $p(C) = 2$, which is impossible.

This concludes the proof of Proposition 8.10 and of Theorem 8.1.□

9 Classification of nonrational surfaces of degree 10

(proof of Theorem 0.1)

As in the above rational case, we start by recalling the basic relations between the invariants of a smooth surface S of degree 10 in $\mathbf{P}^4$. The first relation, the double point formula (0.7), takes the form

$$5\pi + K^2 - 6\chi - 30 = 0.$$

The second relation is given by the Severi theorem (0.8) and takes the form

$$\pi = 6 + \chi + h^1(\mathcal{O}_S(H)) - h^0(\mathcal{O}_S(K-H)).$$

In addition to these relations, there is an inequality among the invariants which is given by the results on the adjunction mapping that we have summarized in Theorem 0.10. In fact, since the scrolls in $\mathbf{P}^4$ are classified (see [Au],[La]) and are of degree 3 and 5, these results provide the inequality $(H+K)^2 \geq 0$ with equality only if S is ruled in conics. For $\deg S = 10$, we get

$$4\pi - 14 + K^2 \geq 0.$$

We also get that $|H+K|$ defines a map

$$\varphi_{H+K} : S \to \mathbf{P}^n$$

where $n = h^0(\mathcal{O}_S(H+K)) - 1 = \pi - 2 + \chi$. Together with the index theorem these relations are effective in proving

Proposition 9.1. *If S is a smooth surface of degree 10 in $\mathbf{P}^4$ with $\pi \leq 6$, then $\pi = 6$ and S is an abelian surface.*

Proof. We start with the smallest values of π. If $\pi \leq 4$, then $K^2 \geq -2$, so $\chi \geq 0$, since $K^2 \leq 8\chi$ when $\chi < 0$. But $H \cdot K \leq -4$ so that $h^0(\mathcal{O}_S(K-H)) = 0$, hence the Severi theorem implies that $\pi \geq 6 + \chi \geq 6$. This is absurd.

If $5 \leq \pi \leq 6$, then the index theorem implies that

$$K^2 \leq \frac{(H \cdot K)^2}{10} < 1.$$

On the other hand the double point formula implies that

$$K^2 = 6\chi + 30 - 5\pi \geq 6\chi.$$

Therefore $\chi \leq 0$. Strict inequality here means that S is birationally ruled, which in turn means that $K^2 \leq 8\chi$. This contradicts the above, so we get that $\chi = 0$, $K^2 = 0$,

$H \cdot K = 0$, and $\pi = 6$. We proceed now to use the full force of Theorem 0.10 on the adjunction mapping. We get that

$$(H + K)^2 = H^2 = 10.$$

Thus we are in case B) of Theorem 0.10. Since S is minimal ($K^2 = 0$) and we are not in any of the exceptional cases, $\varphi_{H+K} : S \to \mathbf{P}^4$ is an isomorphism onto its image. By iteration we see that S cannot be a ruled surface. Therefore, by the Enriques-Kodaira classification, S is abelian or hyperelliptic. Now a result of Roth shows that S cannot be hyperelliptic either [Ro2 p.170-172], so S must be abelian. □

Lemma 9.2 (Roth). *If H is a very ample divisor on hyperelliptic surface, then $H^2 \geq 18$.*

Proof. A hyperelliptic surface is the quotient of the product of two elliptic curves with a group G of order 2,3,4,6,8 or 9, such that the quotient has one rational elliptic fibration and one elliptic fibration over an elliuptic curve. From the invariants of the surface we get that the Picard group is of rank 2 (see BPV V.5.). So at least over the rational numbers it is generated by the fibers of the two elliptic fibrations on S. Let A be a member of the rational one, and let B be a member of the elliptic one. Thus we may write $H \equiv aA + bB$, where a and b are rational numbers. Now the fibration of B has isomorphic smooth fibers so we may assume that b is an integer, while the fibration of A has a double, a triple, a quartuple fibre or a fibre of multiplisity 6 (see BPV V.5.), so we may assume that 2a,3a,4a,or 6a is an integer. Now $A^2 = B^2 = 0$, while $A \cdot B = v$ where v is the order of the group G. Now for H to be very ample we must have $H \cdot A = bv \geq 3v$ and $H \cdot B = av \geq 3$, since B and any reduced fiber in the firation of A are elliptic curves. We combine this to get $H^2 = 2abv \geq 18$.□

Proposition 9.3. *There are no smooth surfaces S of degree 10 with $\pi = 7$ in $\mathbf{P}^4$.*

Proof. We calculate the invariants of S. Since $H \cdot K = 2$, we get that $K^2 \leq \frac{4}{10}$ by the index theorem. Thus

$$K^2 \leq 0.$$

Since $(H + K)^2 \geq 0$, we get $K^2 \geq -14$. The double point formula says in this case that $6\chi = 5 + K^2$. Thus we get the inequalities

$$-9 \leq 6\chi \leq 5$$

which means that $\chi = -1$ or $\chi = 0$.

If $\chi = -1$, then $K^2 = -11$, $(H + K)^2 = 3$ and $h^0(\mathcal{O}_S(H + K)) = \pi - 1 + \chi = 5$. This means, by Theorem 0.10, that $\varphi_{H+K}(S)$ is a surface of degree 3 in $\mathbf{P}^4$, which is impossible since $\chi = -1$.

If $\chi = 0$, then $K^2 = -5$, $(H + K)^2 = 9$ and $h^0(\mathcal{O}_S(H + K)) = \pi - 1 + \chi = 6$. This means, by Theorem 0.10, that $S_1 = \varphi_{H+K}(S)$ is a smooth surface of degree 9 in $\mathbf{P}^5$. Since $K^2 = -5$, S must have five (-1)-curves. Therefore $H \cdot K = 2$ means that S must be birationally ruled. In fact it must be ruled over an elliptic curve since $\chi = 0$, so we may conclude the proof of the proposition with

Lemma 9.4. *There are no smooth surfaces S of degree 10 with $\pi = 7$ in $\mathbf{P}^4$ which are birationally ruled over an elliptic curve.*

Proof. By considering the adjunction on $S_1 = \varphi_{H+K}(S)$ we will see that there are two possibilities for S and that both of them lead to a contradiction.

The invariants of S_1 in $\mathbf{P}^5$ are:

$$H_1^2 = 9, \quad H_1 \cdot K_1 = (H + K) \cdot K = -3, \quad \pi_1 = 4 \quad \text{and} \quad K_1^2 = K^2 + a,$$

where a is the number of (-1)-lines on S. Thus $h^0(\mathcal{O}_{S_1}(H_1 + K_1)) = \pi_1 - 1 + \chi = 3$, which means that $|H_1 + K_1|$ defines a map

$$\varphi_{H_1+K_1} : S_1 \to \mathbf{P}^2.$$

The image of this map is either $\mathbf{P}^2$, in which case we are in the exceptional case *iv*) of Theorem 0.10, or the image is a curve, in which case S_1 is ruled in conics. Let, in both cases,

$$\rho : S \to C_e$$

denote the morphism which defines the ruling of S.

In the first case we get $(H_1 + K_1)^2 = 3$, hence $K_1^2 = 0$, so S has five (-1)-lines and S_1 is minimal. If we let B denote a section on the ruled surface S_1 with the minimal selfintersection $B^2 = 1$, then $K_1 \equiv -2B + \kappa F$, where κF is the pullback by ρ of a certain divisor of degree one on C_e, and $H_1 \equiv H + K \equiv 3B$, and thus

$$H \equiv 5B - \kappa F - \sum_{i=1}^{5} E_i,$$

where E_i, $i = 1, \ldots, 5$ are the exceptional lines on S. Now any general member $B \in |B|$ is an irreducible elliptic curve of degree $H \cdot B = 4$ on S, so it spans only a hyperplane in $\mathbf{P}^4$. Hence there is a residual curve

$$C \equiv H - B$$

on S. C has arithmetic genus 4 and degree $H \cdot C = 6$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_S(B) \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_C(H) \longrightarrow 0$$

of sheaves on S. If we take global sections, we have that $h^0(\mathcal{O}_S(B)) = 2$. Thus C must be a plane curve in $\mathbf{P}^4$. Given the degree and arithmetic genus of C, this means that $\varphi_{H|C}$ cannot be an isomorphism.

In the second case we get $(H_1 + K_1)^2 = 0$, hence $K_1^2 = -3$ and S_1 is ruled in conics with three singular fibres. If we let B denote a section on the ruled surface S_1 with the minimal selfintersection $B^2 = e \leq 1$, then $K_1 \equiv -2B + \epsilon F + \sum_{i=1}^{3} E_i$, where ϵF is the pullback to S_1 of a divisor of degree e on C_e. We may set

$$H_1 + \sum_{i=1}^{3} E_i \equiv 2B + \alpha F,$$

where αF is, say, the pullback from a divisor on C_e of degree a. Since $H_1^2 = 9$, we get $(2B + \alpha F)^2 = 4e + 4a = 12$ and therefore that $a = 3 - e$. Thus

$$H_1 \equiv 2B + \alpha F - \sum_{i=1}^{3} E_i,$$

and

$$H \equiv 4B + (\alpha - \epsilon)F - \sum_{i=1}^{3} 2E_i - E_4 - E_5.$$

Now B has arithmetic genus 1 on S and degree $H \cdot B = 3 + 2e$, thus $e = 0$ or $e = 1$.

If $e = 0$, then consider the linear system of curves $|2B + \tau F|$ on S, where τF is the pullback of a divisor of degree two on C_e. By the Riemann-Roch theorem, it has dimension

$$\dim|2B + \tau F| \geq \chi(\mathcal{O}_S(2B + \tau F)) - 1 = 5,$$

so there is a curve $D \equiv 2B + \tau F - \sum_{i=1}^{5} E_i$ on S. It has arithmetic genus $p(D) = 3$ and degree $H \cdot D = 6$, so it spans only a hyperplane in $\mathbf{P}^4$ unless D is the union of a plane quartic A and two skew lines L_1 and L_2, or D is the union of a plane quartic A and a conic Q such that each line (resp. the conic) meets A in one point. Now $D^2 = 3$ and $A^2 \leq 1$ by the index theorem, so $L_1^2 + L_2^2 \geq -2$, (resp. $Q^2 \geq 0$). But this means that L_i, $i = 1, 2$, are (-1)-lines on S (resp. S is ruled in conics), which is absurd. Thus D spans a hyperplane, and there is a residual curve $C \equiv H - D$ on S. This curve C has arithmetic genus $p(C) = 2$ and degree $H \cdot C = 4$, which is again impossible.

If $e = 1$, then consider the linear system of curves $|2B + \sigma F|$ on S, where σF is the pullback of a divisor of degree one on C_e. It has dimension

$$\dim|2B + \sigma F| \geq \chi(\mathcal{O}_S(2B + \sigma F)) - 1 = 5$$

so there is a curve $D \equiv 2B + \sigma F - \sum_{i=1}^{5} E_i$ on S. It has arithmetic genus $p(D) = 3$ and degree $H \cdot D = 6$. An argument like the one for the case $e = 0$ shows that this case also is impossible.□

Proposition 9.5. *If S is a smooth surface of degree 10 in $\mathbf{P}^4$ with $\pi = 8$, then S is rational or an Enriques surface with four (-1)-lines.*

Proof. We calculate the invariants of S. Since $H \cdot K = 4$, we get that $K^2 \leq \frac{16}{10}$ by the index theorem. Thus

$$K^2 \leq 1.$$

By Theorem 0.10., $(H + K)^2 \geq 0$, thus $K^2 \geq -18$. The double point formula says in this case that

$$6\chi = 10 + K^2.$$

Thus we get the inequalities

$$-8 \leq 6\chi \leq 11$$

which means that

$$-1 \leq \chi \leq 1.$$

If $\chi = -1$, then $K^2 = -16$, $(H+K)^2 = 2$ and $h^0(\mathcal{O}_S(H+K)) = \pi - 1 + \chi = 6$. Therefore, by Theorem 0.10, $\varphi_{H+K}(S)$ is a nondegenerate surface of degree 2 in $\mathbf{P}^5$, which is impossible.

If $\chi = 0$, then $K^2 = -10$, $(H+K)^2 = 8$ and $h^0(\mathcal{O}_S(H+K)) = \pi - 1 + \chi = 7$. This means, by Theorem 0.10, that $S_1 = \varphi_{H+K}(S)$ is a smooth surface of degree 8 in $\mathbf{P}^6$. We calculate some further invariants of S_1 and get

$$H_1 \cdot (H_1 + K_1) = (H+K) \cdot (H+2K) = 2.$$

Thus $\pi_1 = 2$, $h^0(\mathcal{O}_{S_1}(H_1+K_1)) = \pi_1 - 1 + \chi = 1$ and $(H_1+K_1)^2 = 0$, which means that $K_1^2 = 4$. This is clearly impossible since $\chi = 0$.

If $\chi = 1$, then $K^2 = -4$. Therefore S must have at least four (-1)-curves. If S is nonrational, then $h^0(\mathcal{O}_S(2K)) \neq 0$. Therefore $H \cdot 2K = 8$ means that S must be a rational surface or an Enriques surface with four exceptional lines.□

Proposition 9.6. *If S is a smooth surface of degree 10 in $\mathbf{P}^4$ with $\pi = 9$, then S is a rational surface, a nonminimal $K3-$surface, a regular elliptic surface with $\chi = 2$ and three (-1)-lines or a minimal surface of general type with $K^2 = 3, p_g = 2, q = 0$, and exactly one (-2)-curve A such that S is embedded in $\mathbf{P}^4$ by the linear system $|2K - A|$.*

Proof. We calculate the invariants of S. Since $H \cdot K = 6$, we get that $K^2 \leq \frac{36}{10}$ by the index theorem. Thus $K^2 \leq 3$. On the other hand Theorem 0.10 implies that $(H+K)^2 \geq 0$, which means that $K^2 \geq -22$. The double point formula says in this case that $6\chi = 15 + K^2$. Thus we get the inequalities

$$-7 \leq 6\chi \leq 18$$

which means that $-1 \leq \chi \leq 3$.

If $\chi = -1$, then $K^2 = -21$, $(H+K)^2 = 1$ and $h^0(\mathcal{O}_S(H+K)) = \pi - 1 + \chi = 7$. This means, by Theorem 0.10, that $\varphi_{H+K}(S)$ is a nondegenerate surface of degree 1 in $\mathbf{P}^5$, which is absurd.

If $\chi = 0$, then $K^2 = -15$, $(H+K)^2 = 7$ and $h^0(\mathcal{O}_S(H+K)) = \pi - 1 + \chi = 8$. This means, by Theorem 0.10, that $S_1 = \varphi_{H+K}(S)$ is a smooth surface of degree 7 in $\mathbf{P}^7$. We calculate some further invariants of S_1 and get

$$H_1 \cdot (H_1 + K_1) = (H+K) \cdot (H+2K) = -2.$$

Thus $\pi_1 = 0$ and S is rational, which contradicts the assumption that $\chi = 0$.

If $\chi = 1$, then $K^2 = -9$. Therefore S must have at least nine (-1)-curves which appear as fixed curves in $|2K|$ whenever this linear system is nonempty. If S is nonrational, then $h^0(\mathcal{O}_S(2K)) \neq 0$. Therefore $H \cdot 2K = 12 < 18$ means that S must be a rational surface.

If $\chi = 2$, then $K^2 = -3$, $(H+K)^2 = 19$ and $h^0(\mathcal{O}_S(H+K)) = \pi - 1 + \chi = 9$. Therefore S must have at least three (-1)-curves. Let S_0 denote the minimal model of S, and let K_0 denote its canonical divisor.

If S is of general type, then $|K_0|$ is nonempty. Since $p(K_0) = K_0^2 + 1 \geq 2$, we must have that $H \cdot K_0 \geq 4$, which means that $H \cdot (K - K_0) \leq 2$. But this is impossible since $K - K_0$ is the sum of the (-1)-curves on S.

If S_0 is elliptic and K_0 is nontrivial, then, since $p(K_0) = 1$, we must have $H \cdot K_0 \geq 3$, hence $H \cdot (K - K_0) \leq 3$. Thus S has three (-1)-lines, and $|K_0|$ contains an elliptic curve of degree 3 on S. If K_0 moves in a pencil, then we can find an elliptic curve $C \equiv K_0 - E_i$ on S, where E_i is one of the exceptional lines. But then C would have degree 2 on S, which is impossible. Therefore $p_g = h^0(\mathcal{O}_S(K)) = h^0(\mathcal{O}_S(K_0)) = 1$, and S is regular ($q = 0$).

If K_0 is trivial, then S is birational to a $K3-$surface.

If $\chi = 3$, then $K^2 = 3$. As above, let S_0 denote the minimal model of S, and let K_0 denote its canonical divisor. If S is nonminimal, then $H \cdot K_0 < 6$ while $p(K_0) > 4$. As in the above case, this leads immediately to a contradiction, so S is minimal. To see that S is regular we assume it is not. Thus $p_g = h^0(\mathcal{O}_S(K)) \geq 3$. Now, any curve $K \in |K|$ has arithmetic genus 4 and degree $H \cdot K = 6$, which means that it is contained in a hyperplane in $\mathbf{P}^4$. The residual curve $C \equiv H - K$ has degree 4 but is mapped into a line by φ_H since $h^0(\mathcal{O}_S(H - C)) = h^0(\mathcal{O}_S(K)) \geq 3$. This contradicts the very ampleness of $|H|$.

For the last statement of the proposition, we first study the pencil of canonical curves $|K|$. Now, $H \cdot K = 6$ and $p(K) = K^2 + 1 = 4$, so a general integral curve in the pencil must be canonically embedded by $|H|$. Thus, if K is a general canonical curve and we consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S(K - H) \longrightarrow \mathcal{O}_S(2K - H) \longrightarrow \mathcal{O}_K(2K - H) \longrightarrow 0$$

of sheaves on S, then we must have that $h^0(\mathcal{O}_K(2K - H)) = 1$. On the other hand $h^1(\mathcal{O}_S(K - H)) = h^1(\mathcal{O}_S(H)) = 0$ by the Severi theorem and Riemann-Roch, so we get that $h^0(\mathcal{O}_S(2K - H)) = 1$. Let A be the curve of $|2K - H|$. Then $H \cdot A = 2$ and $K \cdot A = 0$ and $A^2 = -2$, so A is a (possibly reducible) (-2)-curve of degree two on S.□

In the case of an elliptic surface we show

Proposition 9.7. *If S is a smooth elliptic surface of degree 10 and $\pi = 9$ in $\mathbf{P}^4$ and with a smooth canonical curve, then S is of the kind described in chapter 4.*

Proof. We have seen already that such a surface S would have a canonical divisor

$$K \equiv C + \sum_{i=1}^{3} E_i$$

where C is an elliptic curve of degree 3, i.e. a plane cubic curve, and the E_i are (-1)-lines on S in the embedding in $\mathbf{P}^4$. By assumption C is a smooth curve, but we will not use this until the last part of the proof. Let H denote a hyperplane section of S, and let

$$H_1 \equiv H + \sum_{i=1}^{3} E_i.$$

There is a pencil of curves

$$|D| = |H - C|$$

on S.

Lemma 9.8. *$|D|$ has no fixed component.*

Proof. A fixed component of $|D|$ would be contained in the plane of C. Since $D \cdot C = 3$, this fixed component would be a line, call it L. Thus $L \cdot C = 3$ and $L \cdot 2C = 6 = H \cdot 2C$. But $|2C|$ is a pencil of elliptic curves whose general member is irreducible, so L would be a 6-secant to such a curve, which is absurd.□

Thus the general member of $D \in |D|$ is irreducible. It has degree $H \cdot D = 7$ and arithmetic genus $p(D) = 6$, so it is linked to a line in the complete intersection of a quadric and a cubic surface. One of the rulings on the quadric surface will sweep out a g_3^1 on D, i.e. D is trigonal. Since $C \cdot E_i = 0$ for $i = 1, 2, 3$, we get that $D \cdot E_i = H \cdot E_i = 1$. Consider the adjoint linear system

$$|D + K| = |D + C + \sum_{i=1}^{3} E_i| = |H + \sum_{i=1}^{3} E_i| = |H_1|$$

of curves on S. If we consider the cohomology associated with the exact sequence

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_S(H_1) \longrightarrow \mathcal{O}_D(H_1) \longrightarrow 0$$

of sheaves of S, then we have that $h^1(\mathcal{O}_S(K)) = 0$. Therefore the sequence

$$0 \longrightarrow \mathrm{H}^0(\mathcal{O}(S)K) \longrightarrow \mathrm{H}^0(\mathcal{O}(S)H_1) \longrightarrow \mathrm{H}^0(\mathcal{O}(D)H_1) \longrightarrow 0$$

of global sections is exact, and $|H_1|$ restricts surjectively to the canonical linear series on D.

Now $h^0(\mathcal{O}_S(H_1)) = 7$, so we may consider the map

$$\varphi_{H_1} : S \to \mathbf{P}^6$$

defined by $|H_1|$. This map is in fact a morphism; it contracts the (-1)-curves E_i and restricts to an isomorphism outside the E_i. We denote the image of S in $\mathbf{P}^6$ by S_1. Let us put $x_i = \varphi_{H_1}(E_i)$ for $i = 1,\ 2,\ 3$. Since $H \equiv H_1 - \sum_{i=1}^{3} E_i$, the points x_i must be colinear; the embedding of S in $\mathbf{P}^4$ is simply the projection of S_1 from the linear span of the three points x_i. Let us denote the line spanned by x_1, x_2 and x_3 by L. We denote the image of C on S_1 by C_1, and similarly we denote the image of D on S_1 by D_1. The curve C_1 is a plane cubic curve on S_1. We denote the plane of C_1 by Π. Now the linear system $|D_1| = |H_1 - C_1|$ of curves on S_1 has projective dimension $\dim|D_1| = 3$. Denote the subpencil of $|D_1|$ which corresponds to the pencil $|D|$ on S by $\mathcal{P}$; i.e. the pencil of curves in $|D_1|$ which meet the points x_i. Note that the general member $D_p \in \mathcal{P}$ is an irreducible canonical curve in a hyperplane of $\mathbf{P}^6$. It is trigonal, so it is contained in a rational normal scroll whose ruling restricts to the g^1_3 on D_p. The line L must be a member of this ruling. L cannot meet any other member of the ruling, in which case the projection of D_p from the line L into $\mathbf{P}^3$ would be three to one, therefore the scroll must be smooth. We denote this rational normal scroll by S_D.

Although all of the curves in $\mathcal{P}$ are trigonal, this is not nessecarily the case for all the curves in $|D_1|$. But $|D_1|$ contains at least a net of trigonal curves. To see this, consider the base locus $Z_p \subset S_1$ of the pencil $\mathcal{P}$. We may write $Z_p = Z + Z_L$, where Z_L is of length three and is contained in L, while Z is of length four and has support outside L. For a general member $D_p \in \mathcal{P}$, the scheme Z is a divisor which by duality on D_p spans a $\mathbf{P}^3$ together with the plane Π. In fact

$$Z + (\Pi \cap D_p) \equiv K_{D_p} - Z_L$$

as divisors on D_p. We denote this $\mathbf{P}^3 \subset \mathbf{P}^6$ by V_p. Now there is a net of curves D in $|D_1|$ which contain Z, and, by duality again, all of these curves are trigonal. We denote this net by $\mathcal{N}$. As above, we may, to every member D of $\mathcal{N}$, find a scroll S_D whose ruling restricts to the trigonal linear series on D.

The proof is now based on a study of S_1 together with the following objects. First, a variety V_0 which we define as the intersection of the quadric hypersurfaces which contain S_1, and a variety V which is the irreducible component of V_0 which contains S_1. Secondly, the net of rational normal scrolls $\{S_D | D \in \mathcal{N}\}$. Thirdly, the projection

$$proj_\Pi : \mathbf{P}^6 \dashrightarrow \mathbf{P}^3$$

from the plane Π in $\mathbf{P}^6$, and at last the projection

$$proj_L : \mathbf{P}^6 \dashrightarrow \mathbf{P}^4$$

from the line L in $\mathbf{P}^6$, which restricts to the embedding of S in $\mathbf{P}^4$ on S_1.

First let us find the possibilities for V_0.

Lemma 9.9. *$h^0(\mathcal{I}_{S_1}(2)) \geq 3$; i.e. V_0 is cut out by at least three linearly independent quadric hypersurfaces.*

Proof. Consider the cohomology associated with the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1}(H_1) \longrightarrow \mathcal{O}_{S_1}(2H_1) \longrightarrow \mathcal{O}_{H_1}(2H_1) \longrightarrow 0$$

of sheaves on S_1 for a general smooth member H_1 of $|H_1|$. Since H_1 has genus nine and $H_1^2 = 13$, we get that $h^0(\mathcal{O}_{S_1}(2H_1)) = 25$. Now $h^0(\mathcal{O}_{\mathbf{P}^6}(2)) = 28$, so taking global sections in the exact sequence

$$0 \longrightarrow \mathcal{I}_{S_1}(2) \longrightarrow \mathcal{O}_{\mathbf{P}^6}(2) \longrightarrow \mathcal{O}_{S_1}(2H_1) \longrightarrow 0$$

of sheaves on $\mathbf{P}^6$ we see that $h^0(\mathcal{I}_{S_1}(2)) \geq 3$.□

Note that, as we have defined V_0, it is not necessarily irreducible, and that both the plane Π and the scrolls S_D are contained in V_0; they are all swept out by trisecants to S_1. Thus V_0 is at least threedimensional. On the other hand V_0 is at the most fourdimensional since it is contained in two linearly independent quadric hypersurfaces.

Lemma 9.10. *$dim V_0 < 4$.*

Proof. If V_0 is fourdimensional, then it is at least of degree three. On the other hand since it is contained in three linearly independent quadrics, it at the most of degree three, so $\deg V_0 = 3$ and V_0 is irreducible. Codimension two varieties of degree three are well understood. They are ruled by a pencil of linear spaces of codimension three. In our case V_0 is ruled by a pencil of $\mathbf{P}^3$s. This pencil must clearly restrict to the ruling of the scrolls S_D. Thus L is contained in one of the $\mathbf{P}^3$s. On the other hand these $\mathbf{P}^3$s must sweep out a pencil of curves on S_1. Projecting from L, we see that the member of this pencil which belongs to the $\mathbf{P}^3$ of L, is mapped onto a line. Thus the curves of the pencil must all be rational, which is absurd for an elliptic surface of Kodaira dimension 1.□

Thus V_0 and V are threedimensional and are contained in the complete intersection of three quadric hypersurfaces. Since the family of scrolls S_D sweep out a net of curves on S_1, we get that V must contain the scrolls S_D. In fact the scrolls S_D are parts of hyperplane sections of V: The net $\mathcal{N}$ of divisors on S_1 is the restriction to S_1 of the net of hyperplanes which contains the linear space V_p. The restriction of the same net to V has the scrolls S_D as members of the moving part, since the general such member must be irreducible. The fixed part of this net is $V \cap V_p$.

Lemma 9.11. *$V \cap V_p$ is a cubic surface.*

Proof. Let D_p be a general irreducible member of $\mathcal{N}$, and let S_D be the corresponding scroll. We will first show that $S_D \cap V_p$ is a twisted cubic curve. For this consider the intersection

$$Z_D = D_p \cap V_p = (D_p \cap C_1) \cup Z,$$

where Z is the baselocus of the net $\mathcal{N}$ as above. This intersection is a scheme of length seven on the curve D_p. On the other hand we have that

$$Z_D \subset S_D \cap V_p,$$

where S_D is a scroll of degree 4. Therefore $S_D \cap V_p$ contains a curve. The possibilities for such a curve are: 1) a line L with $L^2 \leq 0$, 2) a plane conic Q with $Q^2 = 0$, 3) two skew lines $L_1 + L_2$ with $(L_1 + L_2)^2 = 0$ and 4) a twisted cubic curve C with $C^2 = 2$. In the first case we get that $(H_D - L)^2 \leq 2$, where H_D is a hyperplane section of S_D, so that Z_D meets L in a scheme of length at least five, which is absurd. In the second case we get that $(H_D - Q)^2 = 0$, so that Z_D is contained in Q, which is absurd since Z_D spans V_p. In the third case we get that $(H_D - (L_1 + L_2))^2 = 0$, so that one of the lines meet Z_D in a scheme of length at least four. This is also absurd since the lines L_1 and L_2 are trisecants to D_p. Thus $S_D \cap V_p$ is a twisted cubic curve.

Thus we have a net of twisted cubic curves on $V \cap V_p$. If we consider the restriction of the surjective map

$$\mathrm{H}^0(\mathcal{O}(S_1)D_1) \to \mathrm{H}^0(\mathcal{O}(C_1)D_1)$$

of global sections of sheaves on S_1 to the net $\mathcal{N}$, then we see that $\mathcal{N}$ has at the most one basepoint on C_1. Therefore the net of twisted cubic curves has at the most one basepoint on C_1, so they sweep out a surface which contains C_1 and meets the plane of C_1 properly. This surface, which we denote by T, therefore has degree at least 3, and is contained in the intersection $V \cap V_p$.

To see that $\deg T = 3$ and that $T = V \cap V_p$, we go back to V_0. Since we already have that V_0 is cut out by quadrics, we get that V_p is a component of V_0. Thus $\deg V \leq 7$. But $\deg V = \deg S_D + \deg T \geq 7$, so we get equality and $\deg T = 3$ and $T = V \cap V_p$.□

Now, consider the projection $proj_\Pi : \mathbf{P}^6 \dashrightarrow \mathbf{P}^3$. The restriction to S_1 of this map is the map

$$\varphi_{D_1} : S_1 \dashrightarrow \mathbf{P}^3$$

defined by the linear system of curves $|D_1|$. We study this map. Consider the cohomology associated with the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1}(D_1 - C_1) \longrightarrow \mathcal{O}_{S_1}(D_1) \longrightarrow \mathcal{O}_{C_1}(D_1) \longrightarrow 0$$

of sheaves on S_1. Now, $D_1 - C_1 \equiv H_1 - 2C_1$, and $2C_1$ moves in a pencil whose general member is a smooth elliptic curve of degree $H_1 \cdot 2C_1 = 6$. Thus any member of the pencil $|2C_1|$ spans at the most a hyperplane in $\mathbf{P}^6$. The residual curves move in the linear system $|F| = |D_1 - C_1|$ on S_1. We get that $h^0(\mathcal{O}_{S_1}(F)) \geq 1$. Let E denote a general smooth member of $|2C_1|$, and let F be a member of $|F|$. If E does not span a hyperplane in $\mathbf{P}^6$, then, since $E + F \equiv H_1$, we get that $E \cap F$ spans at the most a $\mathbf{P}^3$. But $E \cdot F = 6 = H_1 \cdot E$ and $E \cap F$ is a scheme of finite length, so this is absurd. Thus $h^0(\mathcal{O}_{S_1}(F)) = 1$ and C_1 is mapped onto a plane cubic curve by φ_{D_1}. In particular $|D_1|$ has no basepoints on C_1.

Since C_1 is a canonical divisor on S_1, we see that $|D_1| = |F + C_1|$ restricts to the canonical linear series on F. From the global sections of the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1}(C_1) \longrightarrow \mathcal{O}_{S_1}(D_1) \longrightarrow \mathcal{O}_F(D_1) \longrightarrow 0$$

of sheaves on S_1, we get that φ_{D_1} restricts to the canonical map on F. $|D_1|$ has therefore no basepoints on F. If we combine this with the above, we see that $|D_1|$ has no basepoints on $C_1 + F \equiv D_1$.

Now $D_1^2 = 7$, so the image
$$\Sigma = \varphi_{D_1}(S_1)$$
is a surface of degree seven in $\mathbf{P}^3$. The image of a curve D_p of the net $\mathcal{N}$ must be a plane curve of degree seven, and therefore the image of the scroll S_D by the projection $proj_\Pi$ is a plane in $\mathbf{P}^3$. The baselocus Z of $\mathcal{N}$ is mapped onto a point

$$q = \varphi_{D_1}(Z).$$

Therefore $\varphi_{D_1}(D_p)$ acquires a quartuple point at q. Additionally, $\varphi_{D_1}(D_p)$ acquires three double points from the members of the ruling of S_D which meets C_1. That there are no other singularities can be checked by the genus formula of a plane curve.

Now we may sum up to conclude that the rational map

$$proj_\Pi : \mathbf{P}^6 \dashrightarrow \mathbf{P}^3$$

restricts to a generically finite map on V. Since V is contained in the complete intersection of three quadrics, it is not hard to see that the map $proj_\Pi$ must be birational. In fact, if P is a $\mathbf{P}^3 \subset \mathbf{P}^6$ which contains the plane Π, then the three quadrics will restrict to P as the union of Π and three other planes. If the intersection of the other planes is finite, then it is one point.

The family of scrolls $\{S_D\}$ is mapped onto the net of planes through the point

$$q = \varphi_{D_1}(Z),$$

and the net of curves $\mathcal{N}$ is mapped onto curves of degree seven in these planes with a quartuple point at the point q and three double points outside q.

The image $C_0 = \varphi_{D_1}(C_1)$ of the curve C_1 is a plane cubic curve; it lies on a cubic scroll with vertex at q, which we denote by S_3. This scroll is the image in $\mathbf{P}^3$ of the exceptional divisor we get by blowing up V along C_1.

If we study the inverse rational map

$$\rho : \mathbf{P}^3 \dashrightarrow V \subset \mathbf{P}^6,$$

then we see that the restriction of ρ to the plane $proj_\Pi(S_D)$ is a linear system of plane quartic curves with a triple point at q and three simple points outside q as assigned basepoints, since the curve D_p is canonically embedded in S_D. Therefore ρ is defined by a linear system $|d_0|$ of quartic surfaces with a triple point as assigned basepoint at q, and with an assigned basecurve which meets a general plane through q in three points outside q. We denote this basecurve by C_B.

If
$$\pi : U \to U_0 \to \mathbf{P}^3$$

is the composition of blowing up first q to get U_0 and then the strict transform of C_B on U_0 to get U, and we let $E_{q,0}$ be the exceptional divisor on U_0 and we let E_q and E_B be the strict transforms of the exceptional divisors on U, then ρ extends to a morphism

$$\rho_U : U \to V \subset \mathbf{P}^6$$

which is defined by the linear system

$$|d| = |\pi^* d_0 - 3E_q - E_B|$$

of divisors on U. The canonical divisor on U is

$$K_U \equiv -\pi^* d_0 + 2E_q + E_B.$$

The image of E_q by the map ρ_U is the cubic surface T. Now, E_q is the projective plane $E_{q,0}$ blown up in the points q_i, $i = 1, \ldots, n$ of the intersection with the strict transform of C_B on U_0. The restriction to E_q of the linear system $|d|$ is the linear system of plane cubic curves on $E_{q,0}$ with assigned basepoints at these points of intersection. Therefore T is a Del Pezzo surface, $n = 6$, and C_B has six branches at q. This means that C_B is a curve of degree nine in $\mathbf{P}^3$.

Going back to the original description of the variety

$$V \subset \mathbf{P}^6$$

we now get that if S_D is the scroll of a general D in $\mathcal{N}$, then $S_D \cup T$ is a hyperplane section of V with at the most isolated singularities on T beside the intersection $S_D \cap T$. Therefore, by an argument analogous to that of the proof of Lemma 0.13, a general hyperplane section H_V of V is a smooth surface of degree 7. A calculation on $S_D \cup T$ shows that the genus of a general hyperplane section of H_V is three. Using the adjunction mapping on H_V, we get, by Theorem 0.10, that H_V is a Bordiga surface, i.e. it is the rational surface defined by plane quartic curves with nine assigned basepoints.
We may use this in our further study of the map ρ.

If we denote the strict transform on U of S_3 by $S_{3,U}$, then we may consider the exact sequence

$$0 \longrightarrow \mathcal{O}_U(d - S_{3,U}) \longrightarrow \mathcal{O}_U(d) \longrightarrow \mathcal{O}_{S_{3,U}}(d) \longrightarrow 0$$

of sheaves on U. We now know that the linear system $|d - S_{3,U}|$ consists of strict transforms of planes which are mapped into $\mathbf{P}^6$ as Bordiga surfaces, therefore the basecurve C_B must be contained in S_3.

Collecting our information on $\varphi_{D_1}(D_1)$, for $D_1 \in \mathcal{N}$, we get that the surface $\Sigma = \varphi_{D_1}(S_1)$ must have a quartuple point at q and must have C_B as a double curve. If Σ_0 is the strict transform of Σ on U, then Σ_0 must meet E_q in a curve which is the strict transform of a quartic curve with double point at the points q_i, $i = 1, \ldots, 6$ on $E_{q,0}$. Thus the points q_i lie on a conic or are the points of intersection of four lines in the plane E_q, where no three lines meet in a point. We conclude the proof of the proposition by showing that the first of these cases is impossible if C_1 is smooth.

The strict transform S_{3,U_0} of S_3 on U_0 is then a smooth elliptic scroll. We denote the elliptic curve lying over q by B and let $B+\alpha F$ be the class of a hyperplane section. Let F_i denote the member of the ruling of S_{3,U_0} which meets B in q_i for $i=1,\dots,6$, and let $C_B \equiv B+\beta F$, where $\beta F \cdot B = 9$. Then, since C_B meets B in exactly the points q_i, we must have $C_B - F_1 - \dots - F_6 \equiv B + \alpha F$. Thus

$$|d_{|S_{3,U}}| = |4\alpha F - \beta F|.$$

Now, if the points q_i lie on a conic, then $|4\alpha F - \beta F| = |\alpha F|$. Going back to S_1, this would mean that $\mathcal{O}_{C_1}(D_1) \cong \mathcal{O}_{C_1}(H_1)$, i.e. that $\mathcal{O}_{C_1}(C_1)$ is trivial. But if we consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_1} \longrightarrow \mathcal{O}_{S_1}(C_1) \longrightarrow \mathcal{O}_{C_1}(C_1) \longrightarrow 0$$

of sheaves on S_1, then, taking global sections, we would get that S_1 is irregular, which is impossible.

This finishes the proof of Proposition 9.7.□

Proposition 9.12. *If S is a smooth surface of degree 10 with $\pi = 10$ in $\mathbf{P}^4$, then S is a regular elliptic surface with two (-1)-lines and $p_g = 2$, or S is a minimal regular surface of general type with three (-2)-curves A_1, A_2 and A_3 embedded as conics and $p_g = 3$, such that S is embedded by the linear system $|2K - A_1 - A_2 - A_3|$ in $\mathbf{P}^4$.*

Proof. We calculate the invariants of S. Since $H \cdot K = 8$, we get that $K^2 \le \frac{64}{10}$ by the index theorem. Thus $K^2 \le 6$. On the other hand, Theorem 0.10 implies that $(H+K)^2 \ge 0$, which means that $K^2 \ge -26$. The double point formula says in this case that $6\chi = 20 + K^2$. Thus we get the inequalities

$$-6 \le 6\chi \le 26$$

which means that $-1 \le \chi \le 4$.

If $\chi = -1$, then $K^2 = -26$, $(H+K)^2 = 0$ and $h^0(\mathcal{O}_S(H+K)) = \pi - 1 + \chi = 8$. Hence, S is ruled in conics.

Lemma 9.13. *There are no smooth surfaces S of degree 10 with $\pi = 10$ in $\mathbf{P}^4$ which are ruled in conics.*

Proof. We have just seen that S must have the following invariants:

$$\chi = -1, \quad K^2 = -26.$$

Let V denote the hypersurface which is the union of the planes of the conic fibers of S. Now, by a result of C. Segre, (see [GP2]) there is the following relation between the degree δ and genus g of the scroll $V \cap H$ and the degree d and genus π of $S \cap H$, where H is a general hyperplane:

$$2\pi - 2 = 2d - 2\delta + 2(2g-2).$$

By the genus of the scroll $V \cap H$ we mean the genus of the corresponding curve of lines in the Grassmannian of lines in $\mathbf{P}^3$. With $g = 2$, $\pi = 10$ and $d = 10$, we get $\delta = 3$. But there are no scrolls of genus 2 and degree 3, so we have a contradiction.□

If $\chi = 0$, then $K^2 = -20$, $(H+K)^2 = 6$ and $h^0(\mathcal{O}_S(H+K)) = \pi - 1 + \chi = 9$. This means, by the above Theorem 0.10, that $S_1 = \varphi_{H+K}(S)$ is a nondegenerate surface of degree 6 in $\mathbf{P}^8$, which is impossible.

If $\chi = 1$, then $K^2 = -14$. Therefore S must have at least fourteen (-1)-curves, which appear as fixed curves of $|2K|$ as soon as this linear system is nonempty. If S is nonrational, then $h^0(\mathcal{O}_S(2K)) \neq 0$. Therefore $H \cdot 2K = 16 < 28$, means that S must be a rational surface. This was ruled out in chapter 8.

If $\chi = 2$, then $K^2 = -8$. Therefore S must have at least eight (-1)-curves. Since there is an effective canonical divisor K on S and $H \cdot K = 8$, K must be the sum of eight exceptional lines and S must be birational to a $K3$-surface.

Lemma 9.14. *There are no smooth surfaces S of degree 10 with $\pi = 10$ in $\mathbf{P}^4$ which are birationally $K3$.*

Proof. Since $H \cdot K = 8$ and $K^2 = -8$, there must be eight (-1)-lines on S. The formula for the number of 6-secant lines to S gives the number -1 in this case. Therefore S must have infinitely many 6-secant lines. We need only one such line for our argument. The hyperplanes which contains a 6-secant line, induce a g^1_4 on a hyperplane section H_0 which has this 6-secant. We want to use an idea of Saint-Donat and Reid (see [SD] and [R2]) to show that this g^1_4 must be swept out by a pencil of curves on S. This will lead to a contradiction.

We fix a smooth hyperplane section H which has a 6-secant line, and denote the g^1_4 on H swept out by the hyperplanes containing this line by $|\delta|$. Now let $S_1 = \varphi_{H+K}(S) \subset \mathbf{P}^{10}$, and denote the image of H on S_1 by H_1. Then $\varphi_{H+K|H_0}$ is an isomorphism, so we may work with $|\delta|$ as a linear series on H_1. Let δ_0 be a general member of $|\delta|$. Then we may assume that δ_0 consists of 4 distinct closed points, call them $x_1, \dots, x_4$. Let $\pi : S_0 \to S_1$ be the blowing-up of S_1 in these four points. Denote the exceptional divisors by $E_1, \dots, E_4$ respectively and denote the total transform of H_1 by H_0. Let us denote the strict transform of H_1 on S_0 by C, such that $C \equiv H_0 - \sum_{i=1}^4 E_i$. Then $h^0(\mathcal{O}_{H_1}(\delta)) = 2$ implies that $h^0(\mathcal{O}_C(\sum_{i=1}^4 E_i)) = 2$. Consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_0}(-H_0 + \sum_{i=1}^{4} 2E_i) \longrightarrow \mathcal{O}_{S_0}(\sum_{i=1}^{4} E_i) \longrightarrow \mathcal{O}_C(\sum_{i=1}^{4} E_i) \longrightarrow 0$$

of sheaves on S_0. Since $h^0(\mathcal{O}_{S_0}(\sum_{i=1}^4 E_i)) = 1$, we get that

$$h^1(\mathcal{O}_{S_0}(-H_0 + \sum_{i=1}^{4} 2E_i)) = 1.$$

This means that any curve in the linear system $|H_0 - \sum_{i=1}^4 2E_i|$ must decompose into disjoint curves or be nonreduced. Since we are on a $K3$-surface, this will lead to the desired result.

Lemma 9.15. $h^0(\mathcal{O}_{S_0}(H_0 - \sum_{i=1}^4 2E_i)) \geq 1$.

Proof. We will work on $S_1 \subset \mathbf{P}^{10}$. Then H_1 is canonically embedded in S_1 as a hyperplane section. The divisor $\delta_0 = x_1 + \ldots + x_4$ is a member of a g_4^1 on H_0, therefore the points $x_1, \ldots, x_4$ only span a $\mathbf{P}^2$ in $\mathbf{P}^{10}$. Thus if the g_4^1 has no base points, then a divisor in the linear series $|2\delta|$ only spans a $\mathbf{P}^5$ in $\mathbf{P}^{10}$. (In case the g_4^1 has base points, then a divisor in the linear series $|2\delta|$ would at the most span a $\mathbf{P}^4$.) This means that

$$h^0(\mathcal{O}_{H_0}(K_{H_0} - 2\delta)) = h^0(\mathcal{O}_C(H_0 - \sum_{i=1}^{4} 2E_i)) \geq 4.$$

Consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_{S_0}(-\sum_{i=1}^{4} E_i) \longrightarrow \mathcal{O}_{S_0}(H_0 - \sum_{i=1}^{4} 2E_i) \longrightarrow \mathcal{O}_C(H_0 - \sum_{i=1}^{4} 2E_i) \longrightarrow 0$$

of sheaves on S_0. Now $h^1(\mathcal{O}_{S_0}(-\sum_{i=1}^4 E_i)) = 3$, therefore $h^0(\mathcal{O}_C(H_0 - \sum_{i=1}^4 2E_i)) \geq 4$ implies that $h^0(\mathcal{O}_{S_0}(H_0 - \sum_{i=1}^4 2E_i)) \geq 1$.□

Let A_0 be a member of $|H_0 - \sum_{i=1}^4 2E_i|$, and denote its image on S_1 by A_1.

Lemma 9.16. *A_1 has a decomposition*

$$A_1 = A + A'$$

such that A meets all the points x_i, $i = 1, \ldots, 4$, of δ_0 and $p(A) = 1$ or $p(A) = 2$. Furthermore, $H_1 \cdot A = 4$ if A is elliptic, and $H_1 \cdot A \leq 6$ if $p(A) = 2$.

Proof. Apply the argument of the proof of Lemma 3 in [R2].□

We use Lemma 9.16 to conclude the proof of Lemma 9.14. If $p(A) = 1$, then A moves in a pencil. If we denote also the total transform of A on S by A, then we can find a curve $D_i \equiv A - F_i$ on S, where F_i is an exceptional line on S. D_i must be a plane cubic curve in S. The residual curve $H - D_i$ moves in a pencil, has degree 7 and arithmetic genus 7. This is impossible by Lemma 0.11.

If $p(A) = 2$, then $h^0(\mathcal{O}_{S_1}(A)) = 3$. As above, we think of $|A|$ as a linear system of curves on S. Thus there is a curve $D \equiv A - F_i - F_j$, where F_i and F_j are exceptional lines on S. D has at the most degree 4 and arithmetic genus 2 on S. This is also impossible by Lemma 0.11.□

If $\chi = 3$, then $K^2 = -2$. Therefore S must have at least two (-1)-curves. We can use the fact that $H \cdot K = 8$ to show

Lemma 9.17. *S is a regular surface ($q(S) = 0$).*

Proof. If S is irregular then $p_g(S) \geq 3$. Thus we may find a curve $D \equiv K - 2E_1 - 2E_2$ on S, where E_1 and E_2 are (-1)-curves on S. We get that $H \cdot D \leq 4$ while D has arithmetic genus $p(D) \geq 1$. Thus D has arithmetic genus 1 or 3.

If $p(D) = 3$, then the minimal model S_0 of S has $K_0^2 = 2$, which means that S has four (-1)-curves. Hence there is a curve $D_0 \equiv K - 2E_1 - 2E_2 - E_3 - E_4$ on S of degree $H \cdot D_0 \leq 2$ and $p(D_0) = 3$. This is impossible.

If $p(D) = 1$, then the minimal model S_0 of S is an elliptic surface. The canonical linear system $|K_0|$ on S_0 has a moving part $|M|$ without basepoints. Since $p_g(S_0) = 3$, any member of this moving part must decompose into several elliptic curves. Thus the above curve D must also decompose into several elliptic curves. Since D has at the most degree 4, this is impossible. □

As we go on we let S_0 be the minimal model of S with canonical divisor K_0. Then S_0 is a minimal elliptic surface, or a surface of general type.

If S_0 is of general type, then $K_0^2 = 1$ (or $K_0^2 \geq 2$). Hence a canonical curve K_0 has arithmetic genus $p(K_0) = 2$ (resp. $p(K_0) \geq 3$). On the other hand S must have three (resp. at least four) (-1)-curves. Thus there is a curve $D \equiv K - 2E_1 - E_2 - E_3$ (resp. $D \equiv K - 2E_1 - E_2 - E_3 - E_4$) on S. Now D has degree $H \cdot D \leq 4$ (resp. $H \cdot D \leq 3$) and arithmetic genus $p(D) = 2$ (resp. $p(D) \geq 3$). This is impossible, so S must be an elliptic surface.

We go on to show that the two exceptional curves on S, call them E_1 and E_2, are lines. Now, $|K|$ is a pencil of curves on S with $E_1 + E_2$ as a fixed part. In fact, by the formulas for the canonical divisor on an elliptic surface (see [BPV]), any member of $|K - E_1 - E_2| = |F|$ is the sum of elliptic curves. Since $p_g = 2$, the moving part of $|F|$ defines an elliptic fibration on S, such that the fixed part of $|F|$ is a sum of multiple fibres of this fibration. But $H \cdot F = H \cdot (K - E_1 - E_2) \leq 6$, so $|F|$ can have no fixed part and the general member is an irreducible elliptic curve.

Assume that E_1 has degree at least 2, i.e. $H \cdot E_1 \geq 2$. Then $H \cdot F \leq 5$ and there are curves $C_i \equiv F - E_i$ on S, for $i = 1, 2$, with arithmetic genus $p(C_i) = 1$ and degrees $H \cdot C_1 = 8 - 2(H \cdot E_1) - H \cdot E_2$ and $H \cdot C_2 = 8 - H \cdot E_1 - 2(H \cdot E_2)$. Hence $H \cdot E_2 = 1$ and $H \cdot E_1 = 2$. Now, C_2 has degree 4, so it is contained in a hyperplane. The residual curve $D \equiv H - C_2$ has degree 6 and arithmetic genus 5. This is only possible if D decomposes into a plane quintic curve A and a line B which does not meet A. We have $D^2 = A^2 + B^2 = 1$, and $B^2 \leq -2$ since E_1 and E_2 are the only (-1)-curves on S. Hence $A^2 \geq 3$, which is impossible by the index theorem ($H \cdot A = 5$ implies that $A^2 \leq 2$).

If $\chi = 4$, then $K^2 = 4$, so S is of general type. We first show that S is minimal and regular.

Assume that S is not minimal, and let E be a (-1)-curve on S. Then, since $p_g = h^0(\mathcal{O}_S(K)) \geq 3$, there is a curve $C \equiv K - 2E$ on S of degree $H \cdot C \leq 6$ and $C^2 = 4$. But this contradicts the index theorem, so S must be minimal.

Now consider a general member $C_K \in |K|$. Since $H \cdot C_K = 8$ and $p(C_K) = K^2 + 1 = 5$, we want to check whether $|H_{|C_K}|$ is the canonical linear series on C_K. If it is not, then, by the Riemann-Roch theorem, C_K spans only a hyperplane in $\mathbf{P}^4$. But $p_g = h^0(\mathcal{O}_S(K)) \geq 3$, so the residual curve $D \equiv H - C_K$ has degree $H \cdot D = 2$ and is contained in a line, which is absurd. Therefore C_K is canonically embedded in $\mathbf{P}^4$ by $|H_{|C_K}|$. We use this to get a more intrinsic description of the linear system of curves $|H|$. Consider the cohomology assosiated to the exact sequence

$$0 \longrightarrow \mathcal{O}_S(K - H) \longrightarrow \mathcal{O}_S(2K - H) \longrightarrow \mathcal{O}_{C_K}(2K - H) \longrightarrow 0$$

of sheaves on S. By the Severi theorem and the Riemann-Roch theorem $h^1(\mathcal{O}_S(K - H)) = h^1(\mathcal{O}_S(H)) = 0$ and $h^0(\mathcal{O}_S(K - H)) = 0$, and by the above $\mathcal{O}_{C_K}(2K) \cong \mathcal{O}_{C_K}(H)$, so $h^0(\mathcal{O}_{C_K}(2K - H)) = 1$. Therefore $h^0(\mathcal{O}_S(2K - H)) = 1$. Let A be the curve in $|2K - H|$. Since $K \cdot A = 0$, A must be the union of (-2)-curves on S. Now $A^2 = -6$ and $p(A) = -2$, so A must be the union of three numerically disjoint (-2)-curves which we denote by A_1, A_2 and A_3. Thus $A_i \cdot A = A_i \cdot (2K - H) = -2$ for $i = 1, 2, 3$. But $K \cdot A_i = 0$, so we get that $H \cdot A_i = 2$, which means that the (-2)-curves A_1, A_2 and A_3 are embedded as conics in $\mathbf{P}^4$.

If S is irregular, then $p_g = h^0(\mathcal{O}_S(K)) \geq 4$, so there is a curve

$$C \equiv K - A \equiv K - A_1 - A_2 - A_3$$

on S. It has degree $H \cdot C = 2$ and arithmetic genus $p(C) = 2$, which is impossible. Therefore S is a regular surface embedded by the linear system

$$|2K - A_1 - A_2 - A_3|,$$

in $\mathbf{P}^4$.

This concludes the proof of Proposition 9.12.□

Proposition 9.18. *If S is a smooth surface of degree 10 in $\mathbf{P}^4$ with $\pi = 11$, then S is either*

A) contained in a cubic hypersurface, in which case S is linked to an elliptic scroll in the intersection of a cubic and a quintic hypersurface,

or

B) S is not contained in any cubic hypersurface, in which case S is linked to a Bordiga surface in the intersection of two quartic hypersurfaces.

Proof. The first step is to get the right invarants.

Lemma 9.19. *If S is linked to a Bordiga surface in the intersection of two quartic hypersurfaces, then S has the invariants $p_g = 4, K^2 = 5$ and $q = 0$.*

Proof. Since a Bordiga surface is projective Cohen-Macaulay, so is S. In fact the ideal of S is generated by the maximal minors of a $4 \times 5-$matrix with linear entries. Thus we get

$q = 0$ and $p_g = 4$ from the resolution of the ideal of S. We get $K^2 = 5$ from the double point formula.□

Lemma 9.20. *If S is linked to an elliptic scroll T in the intersection of a cubic and quintic hypersurface, then S has the invariants $p_g = 4, K^2 = 5$ and $q = 0$.*

Proof. From the cohomology of the liaison exact sequence

$$0 \longrightarrow \mathcal{O}_S(K) \longrightarrow \mathcal{O}_U(3H) \longrightarrow \mathcal{O}_T(3H) \longrightarrow 0$$

where U is the complete intersection of a cubic and a quintic hypersurface, we immediately get that $p_g(S) = 4$ and $q = 0$. From the double point formula we then get that $K^2 = 5$.□

Lemma 9.21. *If S is a smooth surface of degree 10 in $\mathbf{P}^4$ with $\pi = 11$, then S has the invariants $p_g = 4, K^2 = 5$ and $q = 0$.*

Proof. We calculate the possible range of the invariants of S. Since $H \cdot K = 10$, we get that $K^2 \leq \frac{100}{10}$ by the index theorem. Thus

$$K^2 \leq 10.$$

Since $(H + K)^2 \geq 0$, we get $K^2 \geq -30$. The double point formula says in this case that $6\chi = 25 + K^2$. Thus we get the inequalities

$$-5 \leq 6\chi \leq 35$$

which means that

$$0 \leq \chi \leq 5.$$

If $\chi = 0$, then $K^2 = -25$, $(H+K)^2 = 5$ and $h^0(\mathcal{O}_S(H+K)) = \pi - 1 + \chi = 10$. This means, by Theorem 0.10, that $\varphi_{H+K}(S)$ is a nondegenerate surface of degree 5 in $\mathbf{P}^9$, which is impossible.

If $\chi = 1$, then $K^2 = -19$. Therefore S must have at least nineteen (-1)-curves which, with multiplicity two, appear as fixed curves of $|2K|$ whenever this linear system is nonempty. If S is nonrational, then $h^0(\mathcal{O}_S(2K)) \neq 0$. Therefore $H \cdot 2K = 20 < 38$ means that S must be rational. This again was ruled out in chapter 8.

If $\chi = 2$, then $K^2 = -13$ and $p_g \geq 1$. Hence S has a canonical curve with at least 13 components. But $H \cdot K = 10$, so this is impossible.

If $\chi = 3$, then $K^2 = -7$ and $p_g \geq 2$. Hence S has a canonical curve with at least 8 components, seven of which must be rational curves. Let K_0 be the pullback to S of a canonical curve on the minimal model S_0 of S. Then $H \cdot K_0 \leq 3$, while K_0 has arithmetic genus $p(K_0) \geq 1$. Since K_0 moves in a pencil, there is a curve $C \equiv K_0 - E$, where E is a (-1)-curve on S. Thus $H \cdot C \leq 2$ and $p(C) = 1$. This is impossible.

If $\chi = 4$, then $K^2 = -1$ and $p_g \geq 3$. Hence S has a (-1)-curve E which is part of any canonical curve on S. Let S_0, with canonical divisor K_0, denote the minimal model of S. By abuse of notation we will denote also the pullback of K_0 to S by K_0. Now, S_0 is either an elliptic surface or of general type.

If S_0 is elliptic, then K_0 is numerically equivalent to $nF + \sum_{i=1}^{k} F_i$, $n > 1, k \geq 0$, where F is a fibre of an elliptic fibration, and the F_i are reduced parts of not neccessarily distinct multiple fibres. Since $H \cdot K_0 = H \cdot (K - E) \leq 9$ and $H \cdot F > H \cdot F_i$, we get that $n = 2, k = 0$ and $H \cdot F \leq 4$. Hence $H \cdot E \geq 2$. Since there is a curve $C \equiv F - E$, with $H \cdot C \leq 2$ and $p(C) = 1$ on S, we have reached a contradiction to the smoothness of S.

If S is of general type, then by Noethers inequality ([BPV])

$$K_0^2 \geq 2p_g - 4 \geq 2.$$

Thus S has at least three (-1)-curves E_i and $H \cdot K_0 \leq 7$ and $p(K_0) \geq 3$. Since $p_g \geq 3$, we can find a curve $C \equiv K_0 - E_1 - E_2$ on S. It has degree $H \cdot C \leq 5$ and arithmetic genus $p(C) \geq 3$. Hence C must have arithmetic genus 3, and $K_0^2 = p(K_0) - 1 = p(C) - 1 = 2$. We get

$$C \equiv K_0 - E_1 - E_2 \equiv K - 2E_1 - 2E_2 - E_3$$

and $H \cdot C = 4$ or 5.

If C has degree 5, then there is a residual curve $D \equiv H - C$ of degree 5. But D would have arithmetic genus $p(D) = 4$, which is impossible by Lemma 0.11.

If C has degree 4, then $H \cdot E_3 = 2$. But if we consider the curve $C_1 \equiv K_0 - E_1 - E_3$ instead of C, then we get a contradiction in this case also, since $H \cdot C_1 = 3$ while $p(C_1) = 3$.

If $\chi = 5$, then $K^2 = 5$. We first show that S is minimal. If S is not minimal, then we let S_0 be the minimal model and let K_0 be the pullback of a canonical curve on S_0 to S. Let E be a (-1)-curve on S. Now, $K_0^2 \geq 6$, so $p(K_0) = K_0^2 + 1 \geq 7$. There is on S a curve $C \equiv K_0 - E$ of degree $H \cdot C \leq 8$ and arithmetic genus $p(C) = p(K_0) \geq 7$. Since $C^2 = 5$, we must, by the index theorem, have $H \cdot C = 8$ and $K_0^2 = 6$.

Now, $p_g \geq 4$, so we may find a curve $C_0 \equiv K_0 - 2E$ on S. This curve has degree $H \cdot C_0 = 7$ and arithmetic genus $p(C_0) = 6$. If we follow the proof of Lemma 3.4, it is straightforward to see that C_0 spans only a hyperplane in $\mathbf{P}^4$ unless it contains a plane quintic curve B as a component. In that case, $C_0 \equiv A + B$, where $A \cdot B = 1$ and A is a plane conic curve, or $A \cdot B = 2$ and A is the union of two skew lines. In the first case, $A^2 \leq -2$, so $B^2 = C_0^2 - A^2 - 2 \geq 2$. But $B^2 \leq 2$ by the index theorem, so we get equality. This leads to a contradiction, since any curve D in the pencil $|H - B|$ has degree 5 and genus $p(3)$. In fact D must be reducible with a component in the plane of B, while $D \cdot B = 3$. In the second case we get $A^2 \leq -4$, so $B^2 = C_0^2 - A^2 - 4 \geq 2$, thus again we get $B^2 = 2$, and a contradiction like the one above. Thus C_0 spans only a hyperplane in $\mathbf{P}^4$.

Since $C_0 \cdot E = 2$, even $C \equiv C_0 + E$ is contained in a hyperplane. On the other hand $h^0(\mathcal{O}_S(C)) \geq p_g - 1 \geq 3$ implies that the residual curve $D \equiv H - C$ must be a line. This is absurd, since $H \cdot D \geq 2$. We conclude that S is minimal.

Now, $p(K) = 6$ and $H \cdot K = 10$, so $h^0(\mathcal{O}_S(K-H)) = h^2(\mathcal{O}_S(H)) = 0$. From Riemann-Roch and Severis theorem, we get that $h^1(\mathcal{O}_S(H)) = 0$. By the Castelnuovo bound, a curve of degree 10 and genus 11 can only span a $\mathbf{P}^3$, thus $h^0(\mathcal{O}_H(H)) = 4$. If we collect these facts and compare them with the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(H) \longrightarrow \mathcal{O}_H(H) \longrightarrow 0,$$

we get that $q = 0$.□

We are left with the two cases a) S lies on a cubic hypersurface, and b) S does not lie on any cubic hypersurface.

In case a) we denote the cubic hypersurface which contains S by V. A result of Aure ([Au Lemmas 2.1.6 and 3.1.19]) shows that V cannot be a cone or have a double plane. Therefore V must be normal, and a general hyperplane section H_V is a Del Pezzo surface or the cone over a nonsingular plane cubic curve.

Now, let H denote a general hyperplane in $\mathbf{P}^4$, and let Π denote a general plane in H. Let $\mathcal{I}_{V,S}$ denote the sheaf of ideals defining S on V, and similarly let $\mathcal{I}_{V_H,S_H}$ (resp. $\mathcal{I}_{V_\Pi,S_\Pi}$) denote the sheaf of ideals defining $S \cap H$ on $V \cap H$ (resp. $S \cap \Pi$ on $V \cap \Pi$).

Consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_{V_H,S_H}(4) \longrightarrow \mathcal{I}_{V_H,S_H}(5) \longrightarrow \mathcal{I}_{V_\Pi,S_\Pi}(5) \longrightarrow 0$$

of sheaves on $V \cap H$. Now, $V \cap \Pi$ is a plane cubic curve and $\mathcal{I}_{V_\Pi,S_\Pi}(5)$ is an invertible sheaf of degree 5 on it. Hence $h^0(\mathcal{I}_{V_\Pi,S_\Pi}(5)) = 5$. On the other hand, from the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_{V_H,S_H}(5) \longrightarrow \mathcal{O}_{V \cap H}(5) \longrightarrow \mathcal{O}_{S \cap H}(5) \longrightarrow 0,$$

we get that $h^0(\mathcal{I}_{V_H,S_H}(5)) \geq 6$ since $h^0(\mathcal{O}_{V\cap H}(5)) = 46$ and $h^0(\mathcal{O}_{S \cap H}(5)) = 40$. Therefore

$$h^0(\mathcal{I}_{V_H,S_H}(4)) \geq 1,$$

and $S \cap H$ is contained in a quartic surface. Since $\deg S \cap H = 10$, we may even assume that $S \cap H$ lies on an irreducible quartic surface.

Since $S \cap H$ lies on an irreducible cubic surface and on an irreducible quartic surface in H, we may link $S \cap H$ to a curve C in the intersection of these two surfaces. Then C has degree 2 and arithmetic genus $p(C_0) = -1$. Thus C_0 is the union of two skew lines on $V \cap H$, and $V \cap H$ must be a Del Pezzo surface.

The linkage of $S \cap H$ and C in H cannot be lifted to $\mathbf{P}^4$, since in that case S would be linked to the union U of two planes which spans $\mathbf{P}^4$ in the complete intersection of V and a quartic hypersurface, and U is not locally Cohen-Macaulay. (Being locally Cohen-Macaulay is preserved by linkage.) Therefore $h^0(\mathcal{I}_{V,S}(4)) = 0$.

Let T be the desingularization of $V \cap H$. We denote by H_T the pullback of a hyperplane section of H_V to T. If we let D_S denote the pullback of $S \cap H$ to T, we get

$$D_S \equiv 4H_T - L_1 - L_2$$

where L_1 and L_2 are the pullbacks of two the skew lines in C_0. Thus D_S is residual to a curve $C_E \equiv H_T + L_1 + L_2$ in $5H_T$, which has degree $H_T \cdot C_E = 5$ and arithmetic genus $p(C_E) = 1$.

Now, going back to S and V, we consider the map

$$\rho : \mathrm{H}^0(\mathcal{I}(S)5) \to \mathrm{H}^0(\mathcal{I}(V, S)5).$$

The kernel of this map is $\mathrm{H}^0(\mathcal{O}(\mathbf{P}^4)2)$. Since, by Riemann-Roch, $h^0(\mathcal{I}_S(5)) \geq 21$, we get that $h^0(\mathcal{I}_{V,S}(5)) \geq 6$. This means that S lies on an irreducible quintic hypersurface W. Hence S is linked to a surface S' of degree 5 in the complete intersection $V \cap W$. If we cut down to H, we see that $S' \cap H$ has arithmetic genus 1 and is isomorphic to a curve linearly equivalent to C_E on T.

Lemma 9.22. *S' is smooth for a general choice of W.*

Proof. Consider the cohomology of the exact sequences

$$0 \longrightarrow \mathcal{I}_S(4) \longrightarrow \mathcal{I}_S(5) \longrightarrow \mathcal{I}_{S\cap H}(5) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^4}(1) \longrightarrow \mathcal{I}_S(4) \longrightarrow \mathcal{I}_{S,V}(4) \longrightarrow 0$$

of sheaves of ideals on $\mathbf{P}^4$. For the latter we already have that $h^0(\mathcal{I}_{S,V}(4)) = 0$, therefore $h^0(\mathcal{I}_S(4)) = 5$, and by Riemann-Roch $h^1(\mathcal{I}_S(4)) = 0$. Therefore, taking global sections in the first sequence, we get that the restriction map

$$\mathrm{H}^0(\mathcal{I}(S)5) \to \mathrm{H}^0(\mathcal{I}(S \cap H)5)$$

is onto. Thus, we may, by the Bertini theorem, choose W such that C_E is smooth and S' is normal. To conclude the proof of the lemma, we refer to an argument of Aure (see [Au page 23]) which shows that if S' is a normal surface of degree five and $\pi = 1$, then S' is a smooth elliptic scroll.□

In case b) we use the fact that $h^0(\mathcal{I}_S(4)) \geq 5$. If we cut to a general plane Π in $\mathbf{P}^4$, we get a linear system, call it $|C|$, of quartic plane curves through 10 points whose general member is irreducible. The projective dimension of $|C|$ is at least 4, in fact, by an argument similar to that of the proof of Lemma 0.12, one may show that $|C|$ can have no more than 10 basepoints and that the dimension is equal to 4. Thus the linear system $|V|$ of quartic hypersurfaces that contains S has no base locus of codimension two outside of S. By Bertini a general subpencil of $|V|$ cuts out an irreducible surface T linked to S. The general hyperplane section H_T of T is an irreducible curve of degree 6 and arithmetic genus 3 which is not contained in any quadric surface. Therefore H_T is projective Cohen-Macaulay and the ideal of H_T is generated by the maximal minors of a 3×4-matrix.

Since H_T is projective Cohen-Macaulay, so is T and S. Hence T is a Bordiga surface and Proposition 9.17 follows.□

Remark. Surfaces of the above type A) may be realized on a Segre cubic hypersurface. This is a rational threefold whose desingularization V is isomorphic to $\mathbf{P}^3$ blown up in five points, $p_1, \ldots, p_5$, in general position. The morphism to $\mathbf{P}^4$ is given by quadrics through the five points. The Segre cubic has 10 quadratic singularities; the images of the lines through any two of the points p_i. Consider a quintic hypersurface S_0 with quadratic singularities in the five points $p_1, \ldots, p_5$, and nonsingular elsewhere. Then the strict transform S_1 of S_0 on V is nonsingular with five (-2)-curves, call them A_i, lying over the points p_i. The image S of S_1 in $\mathbf{P}^4$ is in fact isomorphic to S_1. Since S_0 is the canonical image of S, we get that

$$H_S \equiv 2K_S - \sum_{i=1}^{5} A_i.$$

We conclude the proof of Theorem 0.1 with

Proposition 9.23. *If S is a smooth surface of degree 10 in $\mathbf{P}^4$ with $\pi \geq 12$, then $\pi = 12$ and S is linked to a quadric in the intersection of a cubic and a quartic hypersurface, or $\pi = 16$ and S is the complete intersection of a quadric and a quintic hypersurface.*

Proof. Let H be a general smooth hyperplane section of S. A theorem of Gruson and Peskine (see [GP1]) says that if $\pi > 12$, then H lies on a quadric hypersurface. If we consider the cohomology of the exact sequence

$$0 \longrightarrow \mathcal{I}_S(1) \longrightarrow \mathcal{I}_S(2) \longrightarrow \mathcal{I}_H(2) \longrightarrow 0$$

of sheaves of ideals on $\mathbf{P}^4$, then we may conclude, from the Severi theorem, that

$$h^0(\mathcal{I}_S(2)) = h^0(\mathcal{I}_H(2)).$$

Therefore, S must lie on a quadric. Smooth surfaces on quadrics are either complete intersections or linked to a plane (see for instance [Au, Proposition 1.3.1]), so the latter part of the proposition follows.

Now, the theorem of Gruson and Peskine says furthermore that if $\pi = 12$, then H is linked to a conic in the complete intersection of a cubic and a quartic hypersurface. It therefore remains to show that this linkage lifts to $\mathbf{P}^4$. Now, H is projective Cohen-Macaulay, so $h^i(\mathcal{I}_H(n)) = 0$ for $i > 0$ and all n. The cohomology of the exact sequences

$$0 \longrightarrow \mathcal{I}_S(n) \longrightarrow \mathcal{I}_S(n+1) \longrightarrow \mathcal{I}_H(n+1) \longrightarrow 0$$

shows by induction that $h^i(\mathcal{I}_S(n)) = 0$ for $i > 0$ and all n. Thus we may lift the linkage, and the proposition follows.□

This completes the proof of Theorem 0.1.

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