THE SPECTRAL SEQUENCE OF AN INCLUSION

M.G.Barratt

Manchester

Introduction.

In the absence of special circumstances, the study of homotopy groups of a complex is an uncertain business. The most flexible procedure available (and potentially as successful as any) is the suspension method: it is also the least systematically developed, so that the unsympathetic regard it more as a way of life. The basic idea of the suspension method is to study the effects on the homotopy groups of an inclusion $(X,A) \longrightarrow (X',A')$, where the relative homology is mapped isomorphically, such as occurs when $X' = X \cup A'$ and $A' = TA$ is a cone on base A. This example shows that in the category of complexes the suspension method is equivalent to the study of the homotopy groups of a cofibration

$$A \longrightarrow X \longrightarrow X/A .$$

It is, however, more convenient to retain the space $X \cup TA$, rather than pinch A to a point, for then more cones can be added to the same space A - a device first exploited by S.C.Chang (unpublished).

This process leads to an exact couple, which defines a filtration of, and a spectral sequence converging to, $\pi_*(X,A)$ or $\pi_*(A)$. In particular, when X is contractible, we have a spectral sequence for $\pi_*(A)$ whose E_1 terms depend on the suspension EA (or, alternatively, an exact couple for computing suspension). This machine generalizes G.W.Whitehead's EHP sequence, and is most convenient for the calculation of homotopy groups of spheres by induction on stems, or for the deduction of the unstable groups from the stable ones.

The appropriate language to describe the E_2 terms of the spectral sequence is that of the cohomology of analysers [3]. In particular, the cohomology of the Lie ring analyser is important for the homotopy groups of spheres.

Although this spectral sequence is in no sense dual to the cohomology spectral sequence of a fibration, it has an exact dual converging to the cohomology of the base space, involving the cohomology of the total space, the fibre, and the loops on the base. In the case of the classifying bundle of a group, this seems to be the spectral sequence which arises from the obvious filtration of the classifying space.

Remark: spaces are supposed to be simply-connected complexes.

1. The exact couple of an inclusion.

Let $P_1, \ldots, P_{n-1}$ be spaces such that any two meet in Q, and let $X = \cup P_i$, $X_t = \bigcup_{t \neq i} P_i$. Then $(X; X_1, \ldots, X_{n-1})$ is an n-ad, and in fact is excisive, so that if (P_i, Q) is m_i-connected, the n-ad is $\sum m_i$-connected. Also, if Q is a point, or is contractible,

$$(1.1) \qquad \pi_*(X; X_1, \ldots, X_{n-1}) \approx \tilde{\pi}_*(P_1 \vee P_2 \vee \ldots \vee P_{n-1}) \subset \pi_*(\vee P_i),$$

the subgroups of cross-terms. Since the only n-ads we shall deal with are of this kind, it is convenient to make the

Definition: $ad(P_1, \ldots, P_{n-1}) = (X; X_1, X_2, \ldots, X_{n-1})$;

$$ad(P_1) = (P_1, Q) .$$

Associated with each (n+1)-ad are n exact sequences. In particular, there is an exact sequence

$$(1.2) \quad \ldots \longrightarrow \pi_q(ad(P_1, \ldots, P_{n-1})) \longrightarrow \pi_q(ad(P_1 \cup P_n, \ldots, P_{n-1} \cup P_n)) \longrightarrow$$

$$\pi_q(ad(P_1, \ldots, P_{n-1}, P_n)) \longrightarrow \ldots \ .$$

Definition. The exact couple of an inclusion $A \subset X$ is the exact sequence of the pair (X, A) together with all exact sequences (1.2) with $P_1 = X$, and P_i a cone on A $(i > 1)$. Thus, in the usual notation for exact couples,

$$\underline{A} = \pi_*(A) \oplus \sum_{n \geq 2} \pi_*(ad(X, T_2A, \ldots, T_{n-1}A)) ,$$

$$\underline{C} = \pi_*(X) \oplus \pi_*(X/A) \oplus \sum_{n > 2} \tilde{\pi}_*(X/A \vee E_2A \vee \ldots \vee E_{n-1}A) ,$$

on replacing $\pi_*(ad(X \cup TA, T_2A \cup TA, \ldots, T_{n-1}A \cup TA)$ by first shrinking TA to a point and using (1.1).

<u>Definition</u>. <u>The spectral sequence of an inclusion</u> is the spectral sequence associated with the exact couple.

There is some freedom of choice in the indexing of the spectral sequence: we choose

$$E_1 = \underline{C} \; ; \quad E_1^{0,n} = \pi_n(X) \, , \quad E_1^{1,n-1} = \pi_{n+1}(X/A) \, ,$$

$$E_1^{p,n-p} = \tilde{\pi}_{n+p}(X/A \vee E_2A \vee \ldots \vee E_pA) \, .$$

Thus the differential d_r raises the filtration p by r, and lowers the total degree by 1; it might be more usual to reverse the sign of the total degree.

The spectral sequence is associated with a filtration of $\pi_*(A)$ derived from the exact couple:

$$F^0\pi_n(A) = \pi_n(A); \quad F^1\pi_n(A) = \partial\pi_{n+1}(X,A) = \mathrm{Ker}(\pi_n(A) \to \pi_n(X));$$

$$F^p\pi_n(A) = \mathrm{Im}(\pi_{n+p}(ad(X,T_2A,\ldots,T_pA)) \longrightarrow \pi_n(A)) \, .$$

2. The first differential.

The differential $d_1 : \pi_n(X) \longrightarrow \pi_n(X/A)$ is obviously that induced by the map which pinches A to a point. In general

$$d_1 \; : \; \tilde{\pi}_n(X/A \vee \ldots \vee E_pA) \longrightarrow \tilde{\pi}_n(X/A \vee \ldots \vee E_{p+1}A)$$

is also induced by a geometric map. The natural maps $EA \to EA \vee EA$, $X/A \to X/A \vee EA$ (the latter is obtained from $X \cup TA$ by pinching A to a point) induce an addition of maps (track addition):

$$\left.\begin{array}{l} f,g : EA \longrightarrow Y \\ h : X/A \longrightarrow Y \end{array}\right\} \text{ define } \left\{\begin{array}{l} f+g : EA \longrightarrow Y \\ h+f : X/A \longrightarrow Y \, . \end{array}\right.$$

<u>Lemma</u>. $d_1 : E_1^{p,n-p} \longrightarrow E_1^{p+1,n-p-2}$ is induced by the map

$$f : X/A \vee \ldots \vee E_pA \longrightarrow X/A \vee \ldots \vee E_{p+1}A$$

such that

$$f \mid E_iA = 1 - \theta \, , \qquad f \mid X/A = 1 - \theta \, ,$$

where 1 denotes the appropriate identity map, and θ is the identity map $EA \longrightarrow E_{p+1}A$.

Thus $d_1 : \pi_*(EA) \longrightarrow \tilde{\pi}_*(EA \vee EA)$ is equivalent to the generalized Hopf invariant.

3. Cohomology of Analysers.

Let P be a functor from the category of finitely generated free abelian groups and homomorphisms to the category of abelian groups and homomorphisms, and let F_n denote the free abelian group on generators $x_1, \ldots, x_n$. Furthermore, let homomorphisms $\partial_i, \partial : F_{n-1} \to F_n$ be defined by

$$\partial_i \, x_j = \begin{cases} x_j & (i<j) \\ x_{j+1} & (i \geq j) \end{cases} , \qquad \partial x_j = x_j - x_{n+1} .$$

Let

$$\delta_P = \sum_1^n (-1)^{i-1} P\partial_i + (-1)^n P\partial \; : PF_{n-1} \to PF_{n+1}$$

then $\delta_P \delta_P = 0$. (Note: P might not be additive).

Definition. The cohomology of the analyser of P is the homology of the differential group $(\sum PF_n , \delta_P)$.

This is based on a definition of Lazard's, equivalent to that given in [3] (see [2, § 6]).

The cohomology measures the non-additivity of P in some sense; it is PF_1 if P is additive.

Now, let X in § 1 be contractible, so that $X/A = EA$.

Theorem. There is a functor P such that $PF_n = \pi_*(EA \vee \ldots \vee E_{n-1}A)$, and the E_2 term of the spectral sequence of the inclusion $A \subset X$ is the cohomology of the analyser of P.

The homorphisms $P\phi$ are induced by geometric maps realising ∂_i, ∂. A slight but important modification of the theorem is needed when X is not contractible. The proof is based on a similar argument in § 7 of [2], and derives from an unpublished theorem of Lazard's.

4. Homotopy groups of spheres.

The cohomology (H^n) of the Lie ring analyser (discussed in [3] and [1,2]) has a second grading derived from the degree of the Lie polynomials. The groups H^n_k are finite, except for $H^1_1 \cong H^2_2 \cong Z$. Let p be an odd prime, and let K^n_k denote the p-primary part of H^n_k ; take a free resolution

$$0 \longrightarrow M^n_k \xrightarrow{\Psi} L^n_k \longrightarrow K^n_k \longrightarrow 0 ,$$

such that L^n_k has the smallest possible number of generators. In the following theorem (which generalizes the EHP sequence), all groups are reduced to their p-primary parts.

<u>Theorem.</u> There is a spectral sequence converging to $\sum \pi_n(S^{2q})$, graded by n , and suitably filtered, whose E_1 terms are

$$E_1^{t,n-t} = \sum_{k \geq t} L^t_k \otimes \pi_{n+t}(S^{2kq+1})$$

$$+ \sum_{k \geq t+1} M^{t+1}_k \otimes \pi_{n+t}(S^{2kq+1}) ,$$

in which d_1 is $\Psi \otimes 1$ on the latter summands, and 0 on the former. *)

(Note: the differentials raise the filtration t , and lower n by 1 .) Although there is a corresponding theorem for $p = 2$, it includes the EHP sequence, and probably reduces to it.

Up to stems $2p^2(p-1)-4$ the only relevant cohomology in the Lie ring analyser is

$$H^1_1 \cong H^2_2 \cong Z , \qquad H^3_p \cong H^4_{2p} \cong H^5_{p^2} \cong Z_p .$$

In this range the spectral sequence splits into two (associated with alternate cohomology groups) yielding $E\pi_*(S^{2q-1})$ and $P\pi_*(S^{4q-1})$. Here we have a simplified spectral sequence in this range, converging to $E\pi_{n-1}(S^{q-1})$, with E_1 terms

$$E_1^{1,n-1} = \pi_{n+1}(S^{2q+1}), \quad E_1^{2,n-2} = \pi_{n+2}(S^{2qp+1}), \quad E_1^{3,n-3} = \pi_{n+3}(S^{2pq+1}),$$

$$E_1^{4,n-4} = \pi_{n+4}(S^{2p^2q+1}), \quad E_1^{5,n-5} = \pi_{n+5}(S^{2p^2q+1}) .$$

*) The nature of d_1 has been oversimplified; d_1 does not respect the grading of E_1 by k . In general, for a homogeneous element $x = u \otimes \theta$ of degree k $(\theta \in \pi_*(S^{2kq+1}))$, $d_1 x$ will contain extra terms of degrees $p^s k (s \geq 1)$, determined by $d_1(\theta)$ in the spectral sequence of the suspension $S^{2kq} \longrightarrow S^{2kq+1}$ (i.e. by the Hopf invariants of θ) .

References.

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