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ON BSG AND THE SYMMETRIC GROUPS

by

J. Tornehave

Introduction

It is known that the permutative category $\mathcal{E}_{\oplus}$ of finite sets and bijective maps with disjoint union gives rise to an infinite loop space equivalent to $QS^{0} = \Omega^{\infty}S^{\infty}$ (Anderson [1], Segal [16]). We will here prove a multiplicative analogue of this result, based on the permutative category $\mathcal{E}_{\otimes}$ of finite sets and bijective maps with cross product. Let S be a multiplicative monoid of positive integers with $1 \in S$ and $S \neq \{1\}$, and let $\mathcal{E}_{\otimes}^{S}$ be the full subcategory of $\mathcal{E}_{\otimes}$ of sets E, such that card $E \in S$. This is a permutative category, whose associated infinite loop space may be written

$$\Omega B \coprod_{s \in S} B\Sigma_s .$$

Here Σ_s denotes the symmetric group on s letters, and the disjoint union $\coprod_{s \in S} B\Sigma_s$ has a monoid structure induced by cross product of sets. In the following theorem BSG denotes the classifying space of stable oriented spherical fibrations and $BSG[S^{-1}]$ its localization at the set of primes p, such that $(s, p) = 1$ for all $s \in S$.

<u>Theorem 6.1</u> The simply connected covering of $B \coprod_{s \in S} B\Sigma_s$ is homotopy equivalent as an infinite loop space to $BSG[S^{-1}]$.

The reader should be warned that the infinite loop space structure on $BSG[S^{-1}]$ referred to, comes from a new infinite loop structure an BSG, and it is not known, whether it agrees with any of the other known infinite loop structures (Boardman-Vogt [3], May [9], Segal [16] and Anderson [1]).

This is the first part of the construction of a splitting of BSG localized at an odd prime p into a product of two infinite loop spaces

$$BSG_{(p)} = BImJ_{(p)} \times B\,CokerJ_{(p)} .$$

The remaining part of the construction will be presented in [20].

Another consequence of the theorem is the existence of infinite loop maps

$$SG \longrightarrow BU_{\otimes} , \qquad SG \longrightarrow BO_{\otimes} .$$

This will be dealt with in [19].

In §1 we define (simplicial and topological) permutative categories and give a construction of the associated spectrum (a slight modification of D. Anderson's construction). Here and many times later we use the following result, whose proof is given in the appendix: If a bisimplicial map induces a weak homotopy equivalence an each vertical (or horizontal) simplical set, then it induces a weak homotopy equivalence on the diagonal.

In §2 we give a number of different constructions of Dyer-Lashof maps and compare them. This is used in §3 to get a geometric description of the composite

$$\coprod B\Sigma_n \longrightarrow \Omega B \coprod B\Sigma_n \longrightarrow QS^0 ,$$

where the middle space is the infinite loop space associated to $\mathcal{E}_\oplus$, and the right hand map is an infinite loop map and a homotopy equivalence. This should provide some motivation for the constructions in §6 . We also derive a lemma, which will be used in the multiplicative situation.

We define the notion of a Boardman-Vogt functor in §4 . This is a continuous functor from a topological permutative category $\mathcal{C}$ (the base category) to topological spaces equipped with a "Whitney sum". We proceed to construct a functor from Boardman-Vogt functors to topological permutative categories. In the applications to be given the base category will always be the permutative category $\mathcal{J}$ of countably generated real pre-Hilbert spaces and inner product preserving linear maps with orthogonal direct sum. Many examples of Boardman-Vogt functors on $\mathcal{J}$ can be found in [3] .

In §5 we prove a result analogous to Theorem B of [3] on monoid valued Boardman-Vogt functors. The section closes with the construction of a natural transformation of monoid valued - Boardman-Vogt functors $G^1 \to F^S$, which induces the localization map

$$BSG \rightarrow BSG\,[S^{-1}] .$$

The proof of the main theorem, given in §6, proceeds by constructing permutative functors

$$\mathcal{E}^S_\otimes \longleftarrow \mathcal{M}^S \longrightarrow \mathcal{J}(P^S) \longleftarrow \mathcal{J}(D^S) \longrightarrow \mathcal{J}(\dot{F}^S)$$

inducing homotopy equivalences of spectra. Here the last two arrows come from natural transformation of Boardman-Vogt functors

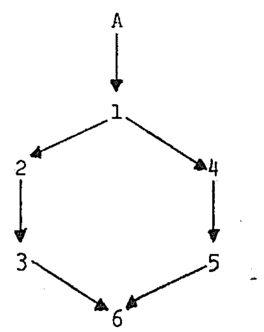
$$P^S \longleftarrow D^S \longrightarrow F^S .$$

Acknowledgment

It is a pleasure to thank D. W. Anderson, from whom I learned about permutative categories, and also D. Quillen and D. Sullivan for several illuminating conversations.

Contents

Logical interdependence



1. Permutative categories. Let Top be the category of compactly generated weak Hausdorff spaces as defined in [6] . All topological spaces encountered in this paper will be objects of Top.

By a topological category we will understand a small category $\mathcal{C}$ with topologies on the set of objects $Ob(\mathcal{C})$ and the set of morphisms $Mor(\mathcal{C})$, such that the structure maps

$$\begin{aligned} \text{source:} &\quad Mor(\mathcal{C}) \to Ob(\mathcal{C}) \\ \text{target:} &\quad Mor(\mathcal{C}) \to Ob(\mathcal{C}) \\ \text{identity:} &\quad Ob(\mathcal{C}) \to Mor(\mathcal{C}) \\ \text{composition:} &\quad Mor(\mathcal{C}) \times_{Ob(\mathcal{C})} Mor(\mathcal{C}) \to Mor(\mathcal{C}) \end{aligned}$$

are continuous. A functor $\mathcal{C} \to \mathcal{C}'$ is continuous if $Mor(\mathcal{C}) \to Mor(\mathcal{C}')$ is continuous (it follows that $Ob(\mathcal{C}) \to Ob(\mathcal{C}')$ is continuous). A natural transformation of two continuous functors $\mathcal{C} \rightrightarrows \mathcal{C}'$ is given by a map $Ob(\mathcal{C}) \to Mor(\mathcal{C}')$ and is continuous, provided this map is continuous.

The singular complex $\Delta\mathcal{C}$ of a topological category $\mathcal{C}$ is a simplicial category with

$$Ob(\Delta_n\mathcal{C}) = \Delta_n Ob(\mathcal{C}) ,$$

$$Mor(\Delta_n\mathcal{C}) = \Delta_n Mor(\mathcal{C}) ,$$

where $\Delta_n X$ for a space X denotes the set of singular n-simplices in X . The four structure maps for the category $\Delta\mathcal{C}_n$ are induced by the structure maps of $\mathcal{C}$ and the simplical operators of $\Delta\mathcal{C}$ are given by the simplicial operators in $\Delta\, Ob(\mathcal{C})$ and $\Delta\, Mor\,(\mathcal{C})$.

<u>The morphism complex</u> or <u>nerve</u> of a topological category $\mathcal{C}$ is the simplicial space $M(\mathcal{C})$ given by

$$M_n(\mathcal{C}) = \{(f_1,\ldots,f_n) \in Mor\,(\mathcal{C})^n \mid f_n \cdots f_1 \text{ is defined}\}$$

for $n \geq 1$ and $M_o(\mathcal{C}) = Ob(\mathcal{C})$, with simplicial operators defined for $n \geq 2$ by

$$(1)\quad \begin{aligned} \partial_o(f_1,\ldots, f_n) &= (f_2,\ldots,f_n) \\ \partial_i(f_1,\ldots,f_n) &= (f_1,\ldots,f_{i+1}f_i,\ldots,f_n),\ i = 1,\ldots,n-1 \\ \partial_n(f_1,\ldots,f_n) &= (f_1,\ldots,f_{n-1}) \\ s_i(f_1,\ldots,f_{n-1}) &= (f_1,\ldots,1,\ldots,f_{n-1}), \quad i = 0,\ldots,n-1 \end{aligned}$$

where the identity map is in the $(i+1)$'st place. For $n = 1$ we have ∂_o = target, ∂_1 = source and s_o = identity.

We can form the bisimplicial set $\Delta M(\mathcal{C})$ by applying the singular complex functor, and define the classifying space of $\mathcal{C}$ to be the geometric realization of its diagonal

$$B\mathcal{C} = |\text{diag}\ \Delta M(\mathcal{C})| .$$

Notice that this is different from the classifying space defined in Segal [15]

$$B_{top}\,\mathcal{C} = |M(\mathcal{C})|$$

obtained by taking the geometric realization of the simplicial space $M(\mathcal{C})$.

The morphism complex of a small simplicial category $\mathcal{C}$ is the bisimplicial set $M(\mathcal{C})$ obtained by applying M in each degree. The classifying space is defined to be

$$B\mathcal{C} = |\text{diag}\ M(\mathcal{C})| .$$

With these definitions we have $B\mathcal{C} = B\Delta\mathcal{C}$ for a topological category $\mathcal{C}$.

All the functors introduced above commute with product. In particular we have

$$B(\mathcal{C}_1 \times \mathcal{C}_2) = B\mathcal{C}_1 \times B\mathcal{C}_2$$

for topological or small simplicial categories $\mathcal{C}_1$ and $\mathcal{C}_2$.

We have the "unit interval category" $[0\to1]$ with two objects 0 , 1 and only one non-identity morphism going from 0 to 1 . Its classifying space $B[0\to1]$ can be identified with the unit interval $I = [0,1]$. A continuous natural transformation of two continuous functors $\mathcal{C}_1 \rightrightarrows \mathcal{C}_2$ can be considered as a continuous functor

$$\mathcal{C}_1 \times [0 \to 1] \to \mathcal{C}_2 .$$

Hence it induces a homotopy

$$B\mathcal{C}_1 \times I \longrightarrow B\mathcal{C}_2 .$$

between the two induced maps $B\mathcal{C}_1 \rightrightarrows B\mathcal{C}_2$.

A topological monoid H with unit may be regarded as a topological category with one object and H as the space of morphisms. We can therefore talk about the classifying spaces BH and $B_{top}H$.

A <u>topological monoid category</u> $\mathcal{C}$ is a topological category $\mathcal{C}$ with a continuous functor

$$+ : \mathcal{C}^2 = \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

such that the diagram

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{(+,1)} & \mathcal{C} \times \mathcal{C} \\ \downarrow{\scriptstyle (1,+)} & & \downarrow{\scriptstyle +} \\ \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \end{array}$$

is strictly commutative. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be topological monoid categories. Then a continuous monoid functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a continuous functor such that the diagram

$$\begin{array}{ccc} \mathcal{C}_1 \times \mathcal{C}_1 & \xrightarrow{F \times F} & \mathcal{C}_2 \times \mathcal{C}_2 \\ \downarrow{\scriptstyle +} & & \downarrow{\scriptstyle +} \\ \mathcal{C}_1 & \xrightarrow{F} & \mathcal{C}_2 \end{array}$$

is strictly commutative. The zero category 0 has one object 0 and one morphism. A zero in $\mathcal{C}$ is defined to be a functor $0 \xrightarrow{\epsilon} \mathcal{C}$ such that the diagram

$$\begin{array}{ccccc} \mathcal{C} \times 0 & \xrightarrow{(1,\epsilon)} & \mathcal{C} \times \mathcal{C} & \xleftarrow{(\epsilon,1)} & 0 \times \mathcal{C} \\ & \searrow & \downarrow{\scriptstyle +} & \swarrow & \\ & & \mathcal{C} & & \end{array}$$

is commutative. The image of 0 under ϵ is called the zero object of $\mathcal{C}$. In the following all topological monoid categories will be assumed to have a zero, and monoid functors will be assumed to commute with the zeros. Similar remarks apply to simplicial monoid categories.

A <u>topological permutative category</u> $\mathcal{C}$ is a topological monoid category with a continuous natural transformation $\gamma : + \to + \tau$, where $\tau : \mathcal{C}^2 \to \mathcal{C}^2$ interchanges the factors, satisfying the following conditions

(I) For all objects A,B, $\gamma(A,B) = \gamma(B,A)^{-1}$

(II) For all objects A,B,C, the diagram

$$\begin{array}{ccc} A + B + C & \xrightarrow{1_A + \gamma(B,C)} & A + C + B \\ {\scriptstyle \gamma(A+B,\, C)} \searrow & & \swarrow {\scriptstyle \gamma(A,C)+1_B} \\ & C + A + B & \end{array}$$

commutes. A zero in $\mathcal{C}$ is required to satisfy $\gamma(A,0) = 1_A$ for every object A .

A continuous permutative functor $F: \mathcal{C}_1 \to \mathcal{C}_2$ is a continuous monoid functor such that

$$F(\gamma(A,B)) = \gamma(FA,FB)$$

for all objects A, B of $\mathcal{C}_1$.

Let Σ_n denote the symmetric group on $\{1,\dots,n\}$. As shown by D. Anderson in [1] it is possible to define for $\pi \in \Sigma_n$ a natural transformation

$$\gamma_\pi : A_1 + \dots + A_n \longrightarrow A_{\pi^{-1}(1)} + \dots + A_{\pi^{-1}(n)}$$

constructed from a factorisation of π as a product of transpositions. It is clear that γ_π is continuous when we have a topology on $\mathcal{C}$.

Of course similar remarks apply to simplical permutative categories. It is clear that the singular complex $\Delta\mathcal{C}$ of a topological permutative category is a simplicial permutative category.

In [1] D. Anderson constructs a functor from the category of simplicial permutative categories to the category of Ω-spectra. We will now give a slight variation of this construction.

First consider a small simplicial monoid category $\mathcal{C}$. Then $+ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ induces a monoid structure on $B\mathcal{C}$ with a unit $BO = pt. \to B\mathcal{C}$ induced by the zero of $\mathcal{C}$. We can then form $B_{top} B\mathcal{C}$.

On the other hand we can form a bisimplicial category $W\mathcal{C}$, the bar-construction on $\mathcal{C}$, with the n-fold product $\mathcal{C}^n$ in degree n (in particular 0 in degree 0), and with simplicial operators defined similarly to (1) above. Next we apply the morphism complex functor to obtain a trisimplicial set $MW\mathcal{C}$.

<u>Lemma 1.1</u> We have

$$B_{top} B\mathcal{C} = |\ \mathrm{diag}\ MW\mathcal{C}\ |$$

<u>Proof</u> Since the morphism complex functor M commutes with product, we have $MW\mathcal{C} = WM\mathcal{C}$. Here $M\mathcal{C}$ is a bisimplicial monoid and $WM\mathcal{C}$ its bar-construction. By Lemma A2 of the appendix $|\mathrm{diag}\ MW\mathcal{C}|$ can be obtained by first taking geometric realisation of the diagonals of the bisimplicial sets defined by fixing the "bar-degree", and then realize the resulting simplicial space geometrically. The first step gives us the bar-construction on the topological monoid $B\mathcal{C}$, and the second step gives us $B_{top}B\mathcal{C}$.

In order to iterate the classifying space construction in the case, where $\mathcal{C}$ is permutative, we have to replace $W\mathcal{C}$ with the additive bar-construction $W_\oplus \mathcal{C}$ as follows.

Let $\mathcal{C}$ and $\mathcal{D}$ be small permutative categories with objects $C = Ob(\mathcal{C})$, $D = Ob(\mathcal{D})$. The direct sum $\mathcal{C} \oplus \mathcal{D}$ is a permutative category with objects $C * D$, i.e., the free product of the monoids C and D. Let $\varphi : C * D \to C \times D$ be the obvious map. Then the morphisms in $\mathcal{C} \oplus \mathcal{D}$ between two elements of $C * D$ are the $\mathcal{C} \times \mathcal{D}$ - morphisms between their images under φ. The permutation $\gamma(A,B)$ is just $\gamma(\varphi A, \varphi B)$. It is easy to show that $\mathcal{C} \oplus \mathcal{D}$ is the direct sum of $\mathcal{C}$ and $\mathcal{D}$ in the category of small permutative categories ([1]). The permutative functor $\mathcal{C} \oplus \mathcal{D} \to \mathcal{C} \times \mathcal{D}$ is clearly an equivalence of categories.

The direct sum $\mathcal{C} \oplus \mathcal{D}$ of two small simplicial permutative categories is obtained by applying the above in each degree. This is again the categorical direct sum, and it follows that the composite $\mathcal{C} \oplus \mathcal{C} \to \mathcal{C} \times \mathcal{C} \xrightarrow{+} \mathcal{C}$ is a simplicial permutative functor, even though + is usually not permutative.

Now let $\mathcal{C}$ be a small simplicial permutative category and define the additive bar-construction $W_\oplus \mathcal{C}$ to be the bisimplicial set with the n-fold sum $\mathcal{C}^{\oplus n}$ in degree n , and with simplicial operators similar to (1). These simplicial operators are simplicial permutative functors. This shows that $W_\oplus \mathcal{C}$ is a bisimplicial permutative category. Its diagonal is a simplicial permutative category to be denoted by $S^1 \otimes \mathcal{C}$.

The additive bar-construction maps naturally into the bar-construction and we obtain a trisimplicial map $MW_\oplus \mathcal{C} \to MW\mathcal{C}$. Since diag $MW_\oplus \mathcal{C}$ = diag $M(S^1 \otimes \mathcal{C})$ we have

$$B(S^1 \otimes \mathcal{C}) = |\text{diag } MW_\oplus \mathcal{C}| \ .$$

<u>Lemma 1.2</u> The trisimplicial map $MW_\oplus \mathcal{C} \to MW\mathcal{C}$ induces a homotopy equivalence

$$B(S^1 \otimes \mathcal{C}) \to B_{top} B\mathcal{C}$$

<u>Proof</u> By fixing the degree coming from the simplicial structure on $\mathcal{C}$ at k and the "bar-degree" at n we obtain the simplicial map $M(\mathcal{C}_k^{\oplus n}) \to M(\mathcal{C}_k^n)$. This is a homotopy equivalence, since the functor $\mathcal{C}_k^{\oplus n} \to \mathcal{C}_k^n$ is an equivalence of categories. The lemma follows now from Theorem A.1 of the appendix.

By taking the 1-skeleton with respect to the "bar degree" $W^1\mathcal{C} = W^1_\oplus \mathcal{C}$ we obtain a commutative triangle

$$\begin{array}{ccc} & MW^1\mathcal{C} & \\ \swarrow & & \searrow \\ MW_\oplus \mathcal{C} & \longrightarrow & MW\mathcal{C} \end{array}$$

This induces a commutative triangle of spaces

$$\begin{array}{ccc} & SB\mathcal{C} & \\ \swarrow & & \searrow \\ B(S^1 \otimes \mathcal{C}) & \longrightarrow & B_{top}B\mathcal{C} \end{array}$$

It is well known that if $B\mathcal{C}$ is connected then the adjoint map $B\mathcal{C} \to \Omega B_{top} B\mathcal{C}$ is a homotopy equivalence (it actually suffices that $\pi_0(B\mathcal{C})$ is a group). The simplicial analogue of this map is

$$\text{diag } M(\mathcal{C}) \to G\overline{W} \text{ diag } M(\mathcal{C})$$

where

$$\overline{W} \text{ diag } M(\mathcal{C}) = \text{diag } W \text{ diag } M(\mathcal{C}) = \text{diag } WM(\mathcal{C}) .$$

The group completion theorem of Barratt and Priddy [2] or Quillen [14] can be applied to this map. Note that the natural transformation γ induces a homotopy showing that diag $M(\mathcal{C})$ is homotopy commutative.

The group completion theorem as stated in [14] (see also the remark at the end of section 5 of that paper) gives us the following result, where we use the notation $B^o\mathcal{C} = \Omega B(S^1 \otimes \mathcal{C})$.

Theorem 1.3 The map $B\mathcal{C} \to B^o\mathcal{C}$ induces a group completion $\pi_0 = \pi_0(B\mathcal{C}) \to \pi_0(B^o\mathcal{C})$. Moreover, if $B^o\mathcal{C}_0$ denotes the zero-component of $B^o\mathcal{C}$ and $B\mathcal{C}_s$ the component corresponding to $s \in \pi_0$, then we have for any coefficient ring k an isomorphism

$$\varinjlim H_*(B\mathcal{C}_s;k) \to H_*(B^o\mathcal{C}_0;k) .$$

Here the direct limit is taken over the translation category of π_0 with objects $s \in \pi_0$ and one morphism $s \to s'$ for every $t \in \pi_0$, such that $s + t = s'$. The homomorphisms of the direct limit system are induced by the translations $B\mathcal{C}_s \to B\mathcal{C}_{s+t}$.

We now define iterated classifying spaces

$$B^n\mathcal{C} = B(S^n \otimes \mathcal{C}) \qquad (n \geq 1) ,$$

where $S^n \otimes \mathcal{C}$ is defined inductively by

$$S^n \otimes \mathcal{C} = S^1 \otimes (S^{n-1} \otimes \mathcal{C}) ,$$

and we have maps

$$SB^{n-1}\mathcal{C} \to B^n\mathcal{C} \qquad (n \geq 2) ,$$

defined in the same way as $SB\mathcal{C} \to B(S^1 \otimes \mathcal{C})$. The adjoints

$$B^{n-1}\mathcal{C} \to \Omega B^n\mathcal{C} \qquad (n \geq 2)$$

are homotopy equivalences, since $B^{n-1}\mathcal{C}$ is connected for $n \geq 2$. This completes the construction of the Ω-spectrum

$$B^o\mathcal{C}, B^1\mathcal{C}, \ldots, B^n\mathcal{C}, \ldots .$$

For a topological permutative category $\mathcal{C}$ we can construct an Ω-spectrum as above from $\Delta\mathcal{C}$. We will write $B^n\mathcal{C} = B^n\Delta\mathcal{C}$. This defines a functor from the category of topological permutative categories to the category of 0-connected Ω-spectra.

As remarked by D. Anderson [1] a small category $\mathcal{C}$ with a coherent unitary associative and commutative addition can be blown up to a permutative category by introducing the free monoid F on $Ob(\mathcal{C})$ as a new set of objects, and defining $Mor\ (H_1, H_2)$ for $H_1, H_2 \in F$ to be $Mor_{\mathcal{C}}\ (\varphi H_1, \varphi H_2)$, where $\varphi : F \to Ob(\mathcal{C})$ is defined by adding in $Ob(\mathcal{C})$ in some fixed order. In case $\mathcal{C}$ is not small it can usually be replaced by an equivalent, full, small subcategory closed under addition. We can then by abuse of language talk about such permutative categories as the category of finitely generated projective modules over a given ring R and isomorphisms with direct sum as the addition (the homotopy groups of the associated spectrum are Quillen's algebraic K-groups as explained in [14]).

2. Dyer-Lashof maps. Let $\mathcal{C}$ be a simplicial permutative category. We have a Σ_k-equivariant simplicial functor $\mathcal{C}^{\oplus k} \overset{+}{\to} \mathcal{C}$, where Σ_k acts trivially on $\mathcal{C}$ and by permutation of factors on $\mathcal{C}^{\oplus k}$. This gives us maps

$$(1) \qquad \begin{gathered} E\,\Sigma_k \times_{\Sigma_k} B\mathcal{C}^{\oplus k} \longrightarrow B\mathcal{C} \\ E\,\Sigma_k \times_{\Sigma_k} B^{o}\mathcal{C}^{\oplus k} \longrightarrow B^{o}\mathcal{C} \end{gathered}$$

Moreover the simplicial permutative functor $\mathcal{C}^{\oplus k} \to \mathcal{C}^k$ is an equivalence of categories in each degree, so that a simple application of Theorem A1 shows that the induced map

$$B\mathcal{C}^{\oplus k} \longrightarrow B\mathcal{C}^k$$

is a homotopy equivalence. By Theorem 1.3 the map

$$B^{o}\mathcal{C}^{\oplus k} \longrightarrow B^{o}\mathcal{C}^k$$

is also a homotopy equivalence. Since these maps are Σ_k-equivariant, they induce homotopy equivalences

$$E\,\Sigma_k \times_{\Sigma_k} B\mathcal{C}^{\oplus k} \longrightarrow E\,\Sigma_k \times_{\Sigma_k} B\mathcal{C}^k = E\,\Sigma_k \times_{\Sigma_k} (B\mathcal{C})^k$$

$$E\,\Sigma_k \times_{\Sigma_k} B^{o}\mathcal{C}^{\oplus k} \longrightarrow E\,\Sigma_k \times_{\Sigma_k} B^{o}\mathcal{C}^k = E\,\Sigma_k \times_{\Sigma_k} (B^{o}\mathcal{C})^k$$

by choosing homotopy inverses and composing with (1) we get Dyer-Lashof maps

$$(2) \qquad \begin{array}{l} E\,\Sigma_k \times_{\Sigma_k} (B\mathscr{C})^k \longrightarrow B\mathscr{C} \\ E\,\Sigma_k \times_{\Sigma_k} (B^o\mathscr{C})^k \longrightarrow B^o\mathscr{C} \end{array}$$

well defined up to homotopy.

Proposition 2.1 The group completion map $B\mathscr{C} \to B^o\mathscr{C}$ commutes up to homotopy with the Dyer-Lashof maps.

Proof Trivial.

Another way of obtaining Dyer-Lashof maps for $B\mathscr{C}$ is the following. We assume first that $\mathscr{C}$ is a (discrete) permutative category. Let $\tilde{\Sigma}_k$ be the category with the elements of Σ_k as objects and just one morphism $\sigma_1 \to \sigma_2$ for every pair σ_1, σ_2 of objects. A functor

$$F : \tilde{\Sigma}_k \times \mathscr{C}^k \to \mathscr{C}$$

is defined on objects by

$$F(\sigma ; H_1, \dots, H_k) = H_{\sigma^{-1}(1)} + \dots + H_{\sigma^{-1}(k)}$$

and on morphisms by

$$F(\sigma_1 \to \sigma_2 ; f_1, \dots, f_k) = \gamma_{\sigma_1\sigma_2^{-1}}(f_{\sigma_1^{-1}(1)} + \dots + f_{\sigma_1^{-1}(k)}) .$$

If we let Σ_k act by right mutiplication on $\tilde{\Sigma}_k$, by permutation of factors on $\mathscr{C}^k$, and trivially on $\mathscr{C}$, then F is Σ_k - equivariant.

In case $\mathscr{C}$ is a simplicial permutative category we get a Σ_k - equivariant simplicial functor F by applying the above construction in each degree. Now F induces a map of classifying spaces

$$B\,\tilde{\Sigma}_k \times (B\,\mathscr{C})^k \longrightarrow B\mathscr{C},$$

where $B\,\tilde{\Sigma}_k = E\,\Sigma_k$ is contractible with a free Σ_k - action. This gives us the Dyer-Lashof map

$$(3) \qquad E\,\Sigma_k \times_{\Sigma_k} (B\mathscr{C})^k \longrightarrow B\mathscr{C}$$

Proposition 2.2 The two Dyer-Lashof maps (2) and (3) for $B\mathscr{C}$ are homotopic.

Proof. It is easily seen that the diagram of Σ_k - equivariant functors

$$\begin{array}{ccc} \tilde{\Sigma}_k \times \mathscr{C}^{\oplus k} & \longrightarrow & \tilde{\Sigma}_k \times \mathscr{C}^k \\ \downarrow{\scriptstyle \text{proj.}} & & \downarrow{\scriptstyle F} \\ \mathscr{C}^{\oplus k} & \longrightarrow & \mathscr{C} \end{array}$$

is commutative. This induces a commutative diagram

$$\begin{array}{ccc} E\,\Sigma_k \times_{\Sigma_k} B\mathscr{C}^{\oplus k} & \longrightarrow & E\,\Sigma_k \times_{\Sigma_k} (B\mathscr{C})^k \\ \downarrow & & \downarrow \\ E\mathscr{C}^{\oplus k} & \longrightarrow & B\mathscr{C} \end{array}$$

which shows that the map (2) can be chosen to be equal to the map (3).

Next we will show that the Dyer-Lashof map (2) for $B^o\mathscr{C}$ can be obtained directly from the spectrum $\{B^n\mathscr{C}\}$.

Suppose given a spectrum $X_* = \{X_n, g_n\}$, where $g_n : SX_n \to X_{n+1}$ is a base point preserving map $(n \geq 0)$. We will always assume that each X_n $(n \geq 0)$ has the homotopy type of a CW-complex.

By taking the k-fold wedge product of all the spaces and maps we obtain the spectrum $X_*^{vk} = \{X_n^{vk}, g_n^{vk}\}$. There is also a k-fold product $X_*^k = \{X_n^k, g_n^k\}$, where the adjoint to g_n^k

$$X_n^k \longrightarrow \Omega(X_{n+1}^k) = (\Omega X_{n+1})^k$$

is the k-fold product of the adjoint $\tilde{g}_n : X_n \to \Omega X_{n+1}$ to g_n . We have an obvious spectrum map $X_*^{vk} \xrightarrow{i} X_*^k$, which is Σ_k-equivariant with respect to the obvious actions. Moreover the spectrum map $X_*^{vk} \xrightarrow{s} X_*$, which is the identity on each copy of X_* is equivariant if Σ_k acts trivially on X_* .

We recall that there is a construction, which to X_* associates an Ω-spectrum LX_* , where the n^{th} space L_nX_* is the reduced mapping telescope of the sequence

$$X_n \xrightarrow{\tilde{g}_n} \Omega X_{n+1} \xrightarrow{\Omega\tilde{g}_{n+1}} \Omega^2 X_{n+2} \xrightarrow{\Omega^2\tilde{g}_{n+2}} \Omega^3 X_{n+3} \longrightarrow \cdots .$$

The maps are obtained by applying the reduced mapping telescope functor (which commutes with reduced suspension) to the diagram

$$\begin{array}{ccccccccc}
SX_n & \xrightarrow{S\tilde{g}_n} & S\Omega X_{n+1} & \longrightarrow & S\Omega^2 X_{n+2} & \longrightarrow & S\Omega^3 X_{n+3} & \longrightarrow & \cdots \\
\downarrow{\scriptstyle g_n} & & \downarrow & & \downarrow & & \downarrow & & \\
X_{n+1} & \xrightarrow{\tilde{g}_{n+1}} & \Omega X_{n+2} & \longrightarrow & \Omega^2 X_{n+3} & \longrightarrow & \Omega^3 X_{n+4} & \longrightarrow & \cdots
\end{array}$$

where the map $S\Omega^i X_{n+i} \to \Omega^i X_{n+i+1}$ is adjoint to $\Omega^i \tilde{g}_{n+i}$: $\Omega^i X_{n+i} \to \Omega^{i+1} X_{n+i+1}$. There is an obvious spectrum map $X_* \to LX_*$.

The above can be combined to form a commutative diagram of Σ_k-equivariant spectrum maps

$$(4) \qquad \begin{array}{ccccc}
X_* & \xleftarrow{s} & X_*^{vk} & \xrightarrow{i} & X_*^k \\
\downarrow & & \downarrow & & \downarrow \\
LX_* & \xleftarrow{Ls} & LX_*^{vk} & \xrightarrow{Li} & (LX_*)^k
\end{array}$$

If X_n is $(n-1)$-connected for every n , the map

$$L_oX_*^{vk} \longrightarrow (L_oX_*)^k$$

is a homotopy equivalence, so it induces a homotopy equivalence

$$E\,\Sigma_k \times_{\Sigma_k} L_oX_*^{vk} \longrightarrow E\,\Sigma_k \times_{\Sigma_k} (L_oX_*)^k .$$

By choosing a homotopy inverse and composing with the map

$$E\,\Sigma_k \times_{\Sigma_k} L_oX_*^{vk} \longrightarrow L_oX_*$$

induced by Ls , we get a Dyer-Lashof map

$$(5) \qquad E\,\Sigma_k \times_{\Sigma_k} (L_o X_*)^k \longrightarrow L_o X_*$$

This applies in particular, when X_* is a o-connected Ω-spectrum. In that case the map $X_o \to L_o X_*$ is a homotopy equivalence, and we get a Dyer-Lashof map

$$(6) \qquad E\,\Sigma_k \times_{\Sigma_k} X_o \longrightarrow X_o$$

unique up to homotopy by requiring the diagram

$$\begin{array}{ccc} E\,\Sigma_k \times_{\Sigma_k} X_o & \longrightarrow & X_o \\ \downarrow & & \downarrow \\ E\,\Sigma_k \times_{\Sigma_k} L_o X_* & \longrightarrow & L_o X_* \end{array}$$

to be homotopy commutative.

<u>Remark</u> There is a Top-valued homotopy chain functor (i.e. Γ-space in the terminology of Segal [16]) with the space $L_o X_*^{\vee k}$ in degree k . The map (5) is defined entirely in terms of this.

<u>Proposition 2.3</u> Let $\mathcal{C}$ be a simplicial permutative category. Then the Dyer-Lashof map (2) for $B^o\mathcal{C}$ is homotopic to the map (6) arising from the Ω-spectrum $\{B^n\mathcal{C}\}$ by the above construction.

<u>Proof</u> Let the Ω-spectrum associated to a simplicial permutative category be denoted by B_*. Then we have a commutative diagram of Σ_k-equivariant spectrum maps

$$\begin{array}{ccc} & & B_*\mathcal{C} \\ & \overset{s}{\nearrow} & \uparrow B_*+ \\ B_*\mathcal{C}^{\vee k} & \longrightarrow & B_*\mathcal{C}^{\oplus k} \\ & \underset{i}{\searrow} & \downarrow \\ & & (B_*\mathcal{C})^k \end{array}$$

where the restriction of the horizontal map to the i'th copy of $B_*\mathcal{C}$ is induced by the i'th inclusion $\mathcal{C} \to \mathcal{C}^{\oplus k}$. From this we get the left hand portion of the following commutative diagram of Σ_k-equivariant maps

$$\begin{array}{ccccc} & & L_0 B_*\mathcal{C} & \longleftarrow & B^0\mathcal{C} \\ & \nearrow & \uparrow & & \uparrow \\ L_0(B_*\mathcal{C}^{\vee k}) & \longrightarrow & L_0 B_*\mathcal{C}^{\oplus k} & \longleftarrow & B^0\mathcal{C}^{\oplus k} \\ & \searrow & \downarrow & & \downarrow \\ & & (L_0 B_*\mathcal{C})^k & \longleftarrow & (B^0\mathcal{C})^k \end{array}$$

where all the horizontal maps are homotopy equivalences. Now we apply the functor $E\Sigma_k \times_{\Sigma_k}(-)$ to this diagram and the result is easily read off.

Sometimes the mapping telescopes used above can be replaced by actual direct limits. Consider for example the sphere spectrum $S_* = \{S^n\}$. If we modify the construction of LS_*^{vk} by using actual direct limits we obtain the Ω-spectrum $\{Q(S^n)^{vk}\}_{n\geq 0}$. We have a commutative diagram of Σ_k-equivariant maps

$$\begin{array}{ccccc} L_0S_* & \longleftarrow & L_0S_*^{vk} & \longrightarrow & (L_0S_*)^k \\ \downarrow & & \downarrow & & \downarrow \\ QS^0 & \longleftarrow & Q(S^0)^{vk} & \longrightarrow & (QS^0)^k \end{array}$$

where the vertical maps collapse telescopes to direct limits. The vertical maps are homotopy equivalences due to the fact that the spaces $\Omega^n(S^n)^{vk}$ are absolute neighborhood retracts, so that the maps $\Omega^n(S^n)^{vk} \to \Omega^{n+1}(S^{n+1})^{vk}$ are cofibrations. It follows that an appropriate Dyer-Lashof map

(7) $$E\Sigma_k \times_{\Sigma_k} (QS^0)^k \to QS^0$$

is determined up to homotopy by the homotopy commutativity of the diagram

$$\begin{array}{ccc} E\Sigma_k \times_{\Sigma_k} (QS^0)^k & \searrow & \\ \uparrow & & QS^0 \\ E\Sigma_k \times_{\Sigma_k} Q(S^0)^{vk} & \nearrow & \end{array}$$

<u>Remark</u> It is not hard to verify that the Dyer-Lashof maps constructed above define H^∞-structures in the sense of Dyer and Lashof [8]. This sets the stage for introduction of Dyer-Lashof operations.

We conclude this section with a construction of the trace

$$\pi_*:[\overline{X};B] \to [X,B]$$

for a k-sheeted covering map $\pi:\overline{X} \to X$ (with base points) where B is a space with a Dyer-Lashof map

(8) $$E\Sigma_k \times_{\Sigma_k} B^k \to B .$$

The covering $\overline{X} \to X$ can be regarded as a fiber bundle with fiber the discrete space $\{1,\ldots,k\}$ and structure group Σ_k acting on $\{1,\ldots,k\}$ by permutation. Let $\tilde{X} \to X$ be the associated principal fiber bundle. This is a $k!$-sheeted regular covering with Σ_k as the group of covering transformations. Let $\Sigma_{k-1} \subseteq \Sigma_k$ be the subgroup fixing k. Then $\bar{X}$ can be identified with the quotient $\tilde{X}/\Sigma_{k-1}$. We have a map of principal Σ_k-bundles

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{h}} & E\Sigma_k \\ \downarrow & & \downarrow \\ X & \xrightarrow{h} & B\Sigma_k \end{array}$$

where h is unique up to homotopy and $\tilde{h}$ is Σ_k-equivariant.

Now let $f:\bar{X} \to B$ be a map. There is a unique Σ_k-equivariant map $\tilde{X} \to B^k$, whose k'th component is the composite $\tilde{X} \to \bar{X} \xrightarrow{f} B$. We combine this with $\tilde{h}$ to get $\tilde{X} \to E\Sigma_k \times B^k$ and divide out by the group action to obtain a map well-defined up to homotopy.

(9) $$X \to E\Sigma_k \times_{\Sigma_k} B^k.$$

Finally compose with (8) to get a map $X \to B$. Obviously a homotopy of f induces a homotopy of this map, so we get a well-defined trace π_*.

By applying the above construction in the case $B = \bar{X}$ and $f = 1_{\bar{X}}$ we get a map well-defined up to homotopy

(10) $$X \to E\Sigma_k \times_{\Sigma_k} \bar{X}^k.$$

We see that π_* sends the homotopy class of $f:\bar{X} \to B$

into the homotopy class of the composite

$$X \to E\Sigma_k \times_{\Sigma_k} \bar{X}^k \xrightarrow{(1,f^k)} E\Sigma_k \times_{\Sigma_k} B^k \to B.$$

It is clear from this description of π_* that if $g:B \to B'$ is a map commuting with Dyer-Lashof maps for B and B', then the diagram

$$\begin{array}{ccc} [\bar{X},B] & \xrightarrow{g_*} & [\bar{X},B'] \\ \downarrow \pi_* & & \downarrow \pi_* \\ [X,B] & \xrightarrow{g_*} & [X,B'] \end{array}$$

is commutative.

<u>Remark</u> It can be shown that, if B has an H^∞-structure, then the trace is a homomorphism (of abelian monoids) and it is functorial on the category of finite coverings.

3. The infinite loop space QS^0. Let $\mathscr{E}_\oplus$ be the permutative category of finite sets and bijective maps with disjoint union $\amalg$ as the sum, and with $\gamma: A \amalg B \to B \amalg A$ being the obvious map. The classifying space $B\mathscr{E}_\oplus$ can be thought of as being the disjoint union

$$\coprod_{n=0}^{\infty} B\Sigma_n,$$

and we can write

$$B^0\mathscr{E}_\oplus = \Omega B \coprod_{n=0}^{\infty} B\Sigma_n.$$

It follows from the group completion theorem 1.3 that $\pi_0(B^0\mathscr{E}_\oplus) = Z$ generated by the component of one-point sets. Also by shifting the inclusion $B\Sigma_n \to \Omega B \coprod B\Sigma_n$ into the zero-component, we get maps that can be fitted into a map

$$(1) \qquad B\Sigma_\infty \to \Omega_0 B \coprod B\Sigma_n,$$

where we have the zero-component on the right. This map induces in isomorphism on homology.

The map $S^0 \to B^0\mathscr{E}_\oplus$ sending the non-base point into the component of one-point sets extends to an infinite loop map

$$(2) \qquad QS^0 \to B^0\mathscr{E}_\oplus.$$

Note that the results of §2 show that this commutes with the Dyer-Lashof maps. So does the group completion map $B\mathscr{E}_\oplus \to B^0\mathscr{E}_\oplus$. We shall apply the trace maps of §2 to the covering constructed below.

Let R^n be the standard n-dimensional Euclidean space and $P^k(R^n)$ the k-fold deleted symmetric product of R^n (i.e. a point in $P^k(R^n)$ is a subset of R^n consisting of k distinct points). There is a principal Σ_k-bundle over $P^k(R^n)$ with total space $\tilde{P}^k(R^n)$ the space of ordered sequences of k distinct points in R^n. Let $\bar{P}^k(R^n)$ be the total space of the corresponding k-sheeted covering. A point of $\bar{P}^k(R^n)$ can be identified with a subset of R^n consisting of k distinct points, one of which is distinguished.

The space $\tilde{P}^k(R^n)$ can be identified with the complement in R^{kn} of the union of finitely many linear subspaces of codimension n. It is therefore (n-2)-connected, and the classifying map

$$P^k(R^n) \xrightarrow{\ k\ } B\Sigma_k$$

is an (n-2)-equivalence.

Next we construct a map

$$P^k(R^n) \xrightarrow{\ \phi\ } \Omega^n S^n,$$

where the standard n-sphere S^n is chosen to be the one-point compactification of R^n. First choose a homeomorphism

$$h:[0,1) \to [0,\infty).$$

For $y \in R^n$ and $r > 0$ let $D = D(y,r)$ be the open disc with center y and radius r in R^n. We associate to this disc the homeomorphism

$$h_D:D(y,r) \to R^n$$

given by

$$h_D(y+trz) = h(t)rz,$$

where $0 \leq t < 1$ and $z \in R^n$ of norm 1. Now let $x \in P^k(R^n)$ consist of the k points $x_1,\ldots,x_k$. We put

$$r(x) = \frac{1}{2} \min_{i \neq j} \|x_i - x_j\| ,$$

and define $\phi(x):S^n \to S^n$ as follows:

$$\phi(x)(y) = \begin{cases} h_{D(x_i,r(x))}(y) & \text{if } y \in D(x_i,r(x)) \\ \infty & \text{otherwise} \end{cases}.$$

This defines $\phi:P^k(R^n) \to \Omega^n S^n$.

In the proposition below $i:B\Sigma_k \to B^0\mathscr{C}_\oplus$ denotes the restriction of the group completion map $B\mathscr{C}_\oplus \to B^0\mathscr{C}_\oplus$ to the k'th component.

Proposition 3.1 The following diagram commutes up to homotopy:

$$\begin{array}{ccccc} P^k(R^n) & \xrightarrow{\phi} & \Omega^n S^n & \longrightarrow & QS^0 \\ \downarrow{\scriptstyle \varkappa} & & & & \downarrow \\ B\Sigma_k & & \xrightarrow{\quad i \quad} & & B^0\mathscr{C}_\oplus \end{array}$$

Proof Let 1 denote a constant map of $\bar{P}^k(R^n)$ into the 1-component of $B^0\mathscr{C}_\oplus$, and let $\pi:\bar{P}^k(R^n) \to P^k(R^n)$ be the projection. We shall compute $\pi_*(1)$ in two different ways.

First 1 factors through

$$\bar{P}^k(R^n) \xrightarrow{1} B\Sigma_1 \longrightarrow B\mathscr{C}_\oplus \longrightarrow B^0\mathscr{C}_\oplus$$

and it is clear that the trace of $1:\bar{P}^k(R^n) \to B\Sigma_1$ is represented by $\varkappa$. Since the group completion map commutes with Dyer-Lashof maps (Prop. 2.1), it follows that $\pi_*(1)$ is represented by $i\varkappa$.

Next we define a map

$$\iota : \bar{P}^k(R^n) \to \Omega^n S^n \subseteq QS^0$$

as follows. Let $x \in \bar{P}^k(R^n)$ be given by the k distinct points $x_1,\ldots,x_k$, where x_k is distinguished. Define $r(x)$ as above. Then $\iota(x)$ regarded as a map $S^n \to S^n$ is given by

$$\iota(x)(y) = \begin{cases} h_{D(x_k,r(x))}(y) & \text{if } y \in D(x_k,r(x)) \\ \infty & \text{otherwise}. \end{cases}$$

We have a locally trivial bundle

$$\bar{P}^k(R^n) \to R^n$$

which sends x into the distinguished point x_k. Since R^n is contractible, the inclusion of fiber over some x_k

$$F \to \bar{P}^k(R^n)$$

is a homotopy equivalence. A homotopy from the restriction $\iota|F$ to a constant in the 1-component of $\Omega^n S^n$ can be defined by

$$H(x,t)(y) = \begin{cases} h_{D(x_k,r(x,t))}(y) & \text{if } y \in D(x_k,r(x,t)) \\ \infty & \text{otherwise} \end{cases}$$

where

$$r(x,t) = (1-t)r(x) + t \qquad (0 \leqq t \leqq 1).$$

This shows that the composite

$$\bar{P}^k(R^n) \xrightarrow{\iota} \Omega^n S^n \subseteq QS^0 \longrightarrow B^0\mathscr{E}_\oplus$$

represents 1.

In order to find the trace of ι, we first note that there is a well-defined Σ_k-equivariant map

$$\tilde{P}^k(R^n) \xrightarrow{\tilde{\iota}} (\Omega^n S^n)^k \subseteq (QS^0)^k$$

whose k'th component is the projection $\tilde{P}^k(R^n) \to \bar{P}^k(R^n)$ composed with ι. We observe that $\tilde{\iota}$ actually factors through the Σ_k-equivariant map

$$\tilde{P}^k(R^n) \xrightarrow{\iota'} \Omega^n(S^n)^{\vee k} \subseteq Q(S^0)^{\vee k}$$

given by

$$\iota'(x)(y) = h_{D(x_i,r(x))}(y)$$

considered as a point in the i'th copy of S^n, if $y \in D(x_i,r(x))$, and

$$\iota'(x)(y) = \infty$$

otherwise (think of $(S^n)^{vk}$ as being the one-point compactification of the disjoint union of k copies of R^n). If we now compose ι' with the folding map $\Omega^n(S^n)^{vk} \to \Omega^n S^n$, we obtain the composite of the projection $\tilde{P}^k(R^n) \to P^k(R^n)$ with ϕ. It is now clear from the construction of the trace map in §2 that the trace of ι is given by ϕ. Since the map $QS^0 \to B^0\mathcal{E}_\oplus$ commutes with Dyer-Lashof maps, $\pi_*(1)$ is represented by the composite

$$P^k(R^n) \xrightarrow{\phi} \Omega^n S^n \subseteq QS^0 \longrightarrow B^0\mathcal{E}_\oplus.$$

This completes the proof.

The maps ϕ and i can be shifted into the zero components of QS^0 and $B^0\mathcal{E}_\oplus$ giving us a new homotopy commutative diagram

(3)
$$\begin{array}{ccc} P^k(R^n) & \xrightarrow{\phi'} & (QS^0)_0 \\ \kappa \downarrow & & \downarrow \\ B\Sigma_k & \xrightarrow{i'} & (B^0\mathcal{E}_\oplus)_0 \end{array}.$$

Since the map (1) induces an isomorphism in homology and κ is an $(n-2)$-equivalence every homology class in $(B^0\mathcal{E}_\oplus)_0$ comes from $P^k(R^n)$ for k and n sufficiently

large. This shows that the right hand map induces an epimorphism in homology.

It follows from the results of Nakaoka [12] and Dyer and Lashof [8] that $(QS^0)_0$ and $B\Sigma_\infty$ have isomorphic homology mod p for every prime p (of finite dimension in each degree). Therefore the right hand map in (3) induces an isomorphism in homology mod p. Since both spaces have trivial rational homology and are simple, the map is a homotopy equivalence. Moreover the map $QS^0 \to B^0\mathcal{E}_\oplus$ induces an isomorphism on π_0. This proves the following theorem (due in various forms to Barratt, Priddy, Quillen and Segal, [2], [16]):

<u>Theorem 3.2</u> The infinite loop map $QS^0 \to B^0\mathcal{E}_\oplus$ is a homotopy equivalence.

Nakaoka ([11]) has shown that the inclusion $B\Sigma_k \to B\Sigma_{k+1}$ induces an isomorphism

$$H_j(B\Sigma_k;Z_p) \to H_j(B\Sigma_{k+1};Z_p)$$

for $j < \frac{k+1}{2}$. Combining this with the above, we easily obtain the following result, which will be used in §6:

<u>Lemma 3.3</u> The map $\phi: P^k(R^n) \to \Omega^n S^n_k$ going into the degree k component induces an isomorphism

$$\phi_*: H_j(P^k(R^n);Z_p) \to H_j(\Omega^n S^n_k;Z_p)$$

for $j < \min(n-1, \frac{k+1}{2})$.

<u>Proof</u> It suffices to show that all the other maps in the diagram

$$\begin{array}{ccccc} P^k(R^n) & \xrightarrow{\phi} & \Omega^n S^n_k & \longrightarrow & QS^0_k \\ \downarrow{\scriptstyle \mathscr{K}} & & & & \downarrow \\ B\Sigma_k & & \xrightarrow{i} & & (B^0\mathscr{C}_\oplus)_k \end{array}$$

induce homology isomorphisms in this range. For i this follows from the fact that the map (1) induces an isomorphism on homology together with the above stability result of Nakaoka. It holds for $\mathscr{K}$, since it is an $(n-2)$-equivalence. For the map on the right we use Theorem 3.2. Finally the map $\Omega^n S^n \to QS^0$ is an $(n-2)$-equivalence as a consequence of the suspension theorem.

<u>4. Boardman-Vogt functors.</u> Let $\mathfrak{J}$ be the category of countably generated (in the algebraic sense) pre-Hilbert spaces over the real numbers with morphism $\mathfrak{J}(H_1,H_2)$ from H_1 to H_2 the set of inner product preserving linear maps $H_1 \to H_2$. Every object of $\mathfrak{J}$ is isomorphic to some R^n, where $n = 0,1,2\ldots$ or ∞, with the standard inner product. Orthogonal direct sum $H_1 + H_2$ makes $\mathfrak{J}$ a monoid category, and the permutation maps

$$\gamma_\pi: H_1+\cdots+H_n \to H_{\pi^{-1}(1)}+\cdots+H_{\pi^{-1}(n)}$$

make $\mathfrak{J}$ a permutative category. We topologize a pre-Hilbert space H_1 as the direct limit

$$H_1 = \lim_{\to} H_\alpha$$

over the finite dimensional subspaces $H_\alpha \subseteq H$. The compactly generated mapping space tolopogy on $H_2^{H_1}$ ([17]) induces a topology on $\mathfrak{J}(H_1,H_2)$. Well-known properties of compactly generated spaces show that

(1) $$\mathfrak{J}(H_1,H_2) = \lim_{\leftarrow \alpha} \mathfrak{J}(H_\alpha,H_2)$$

$$= \lim_{\leftarrow \alpha} \lim_{\beta \to} \mathfrak{J}(H_\alpha,H_\beta)$$

where the limits are in the category Top and are taken over the finite dimensional subspaces of H_1 and H_2.

Now $\mathfrak{J}$ becomes a topological permutative category with the discrete topology on $\mathrm{Ob}(\mathfrak{J})$ and the disjoint union topology on $\mathrm{Mor}(\mathfrak{J}) = \coprod \mathfrak{J}(H_1,H_2)$.

The definition below of a Boardman-Vogt functor in the case $\mathcal{C} = \mathfrak{J}$ is essentially the same as in §1 of [3].

Definition 4.1 Let $\mathcal{C}$ be a topological permutative category. A Boardman-Vogt functor on $\mathcal{C}$ is a functor

$$T:\mathcal{C} \to \mathrm{Top}$$

from $\mathcal{C}$ to base pointed spaces with a base point preserving natural transformation

$$\omega:T(H_1)\times T(H_2) \to T(H_1+H_2)$$

satisfying the following conditions:

(A) T is continuous (note that Top or rather a sufficiently large small subcategory of Top is a topological category with the discrete topology on the objects and the compactly generated mapping space topology on morphisms).

(B) ω is continuous.

(C) For every object H of $\mathcal{C}$ the map

$$T(H)\times T(0) \xrightarrow{\omega} T(H+0) = T(H)$$

is projection on $T(H)$.

(D) For all objects H_1,H_2,H_3 of $\mathcal{C}$ the diagram

$$\begin{array}{ccc} T(H_1)\times T(H_2)\times T(H_3) & \xrightarrow{\omega\times 1} & T(H_1+H_2)\times T(H_3) \\ \downarrow{\scriptstyle 1\times\omega} & & \downarrow{\scriptstyle \omega} \\ T(H_1)\times T(H_2+H_3) & \xrightarrow{\omega} & T(H_1+H_2+H_3) \end{array}$$

is commutative.

(E) For any two objects H_1,H_2 of $\mathcal{C}$ we have a commutative diagram

$$\begin{array}{ccc} T(H_1)\times T(H_2) & \xrightarrow{\omega} & T(H_1+H_2) \\ \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle T(\gamma)} \\ T(H_2)\times T(H_1) & \xrightarrow{\omega} & T(H_2+H_1) \end{array}$$

where τ interchanges the two factors.

It is easy to show that condition (D) automatically generalizes to the case of any finite number of objects and arbitrary arrangements of parentheses. In particular for any $n \geq 1$ we get a continuous natural transformation

$$\omega:T(H_1)\times\ldots\times T(H_n) \to T(H_1+\ldots+H_n).$$

Similarly condition (E) generalizes to the case of n objects $H_1,\ldots,H_n$ and any permutation σ of $1,\ldots,n$.

<u>Definition 4.2</u> Let $F:\mathcal{C}_1 \to \mathcal{C}_2$ be a continuous permutative functor, T_1 a Boardman-Vogt functor on $\mathcal{C}_1$ and T_2 a Boardman-Vogt functor on $\mathcal{C}_2$. A <u>natural transformation of Boardman-Vogt functors</u> $\Phi:T_1 \to T_2$ over F is a continuous base point preserving natural transformation $\Phi:T_1 \to T_2F$, such that the diagram

$$\begin{array}{ccc} T_1(H_1)\times T_1(H_2) & \xrightarrow{\omega} & T_1(H_1+H_2) \\ \downarrow \Phi & & \downarrow \Phi \\ T_2(FH_1)\times T_2(FH_2) & \xrightarrow{\omega} & T_2(F(H_1+H_2)) \end{array}$$

is commutative for all objects H_1,H_2 of $\mathcal{C}_1$.

It follows automatically that the analogous diagram for n objects is commutative. Note that T_2F is a Boardman-Vogt functor on $\mathcal{C}_1$, and that a natural transformation $\Phi:T_1 \to T_2$ is the same as a natural transformation $T_1 \to T_2F$. We will be mostly concerned with the case, where $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}$ and F is the identity. The natural transformations of Boardman-Vogt functors on $\mathcal{C}$ (i.e. over the identity on $\mathcal{C}$) form a category $BV(\mathcal{C})$. The category of all Boardman-Vogt functors will be denoted by BV. There is a forgetful functor

$$BV \xrightarrow{\text{proj.}} \text{Topperm}$$

where Topperm is the category of topological permutative categories. We will now construct for $T \in \text{Ob BV}$ a topological permutative category $\mathcal{C}(T)$, thus defining a functor

$$BV \xrightarrow{\text{total}} \text{Topperm}.$$

<u>Remark</u> The category BV is fibered over Topperm by the projection. The fiber over a topological permutative category $\mathcal{C}$ is $BV(\mathcal{C})$. Every fiber $BV(\mathcal{C})$ has a final object $*$, which sends every object of $\mathcal{C}$ into a point. It will be clear from the construction below that $\mathcal{C}(*)$ can be identified with $\mathcal{C}$, so that the natural transformation $T \to *$ over $\mathcal{C}$ induces a continuous permutative functor $\mathcal{C}(T) \to \mathcal{C}$. In the language of fibered categories it is natural to refer to this as the projection, $\mathcal{C}$ as the base category and $\mathcal{C}(T)$ as the total category.

An object of $\mathcal{C}(T)$ will be a pair (H,x), where $H \in \text{Ob}(\mathcal{C})$ and $x \in T(H)$. A morphism in $\mathcal{C}(T)$ from (H,x) to (H',x') is defined to be a morphism of $\mathcal{C}$ $A:H \to H'$, such that $T(A)x = x'$. Composition is defined using the composition in $\mathcal{C}$. Since we have the discrete topology on the objects of Top, we can write $\text{Ob}\,\mathcal{C}$ as a disjoint union $\coprod O_\alpha$ of open subsets, such that T is constant on each O_α. We topologize Ob $\mathcal{C}(T)$ as the disjoint union

$$\coprod T(H_\alpha)\times O_\alpha,$$

where $H_\alpha \in O_\alpha$. It is easy to show that this topology is independent of the splitting Ob $\mathcal{C} = \coprod O_\alpha$. There is an obvious cartesian square in the category of sets

$$(2)\qquad \begin{array}{ccc} \text{Mor } \mathcal{C}(T) & \longrightarrow & \text{Mor } \mathcal{C} \\ \downarrow \text{source} & & \downarrow \text{source} \\ \text{Ob } \mathcal{C}(T) & \longrightarrow & \text{Ob } \mathcal{C} \end{array}$$

We topologize Mor $\mathcal{C}(T)$ so as to make this a cartesian square in Top. The sum of two objects $(H_1,x_1),(H_2,x_2)$ is defined to be $(H_1+H_2,\omega(x_1,x_2))$. Two morphisms in $\mathcal{C}(T)$ are added by adding the corresponding morphisms in $\mathcal{C}$. Finally the permutation morphism γ_σ $(\sigma \in \Sigma_n)$ for the objects $(H_1,x_1),\ldots,(H_n,x_n)$ of $\mathcal{C}(T)$ is given by the morphism γ_σ for the objects $H_1,\ldots,H_n$ of $\mathcal{C}$.

Now let $\Phi:T_1 \to T_2$ be a natural transformation of Boardman-Vogt functors over $F:\mathcal{C}_1 \to \mathcal{C}_2$. The induced continuous permutative functor

$$F(\Phi):\mathcal{C}_1(T_1) \to \mathcal{C}_2(T_2)$$

is defined by sending the object (H,x) into $(FH,\Phi(x))$, and the morphism $A:(H,x) \to (H',x')$ into the morphism determined by FA.

<u>Theorem 4.3</u> The construction above defines a functor total:BV $\to$ Topperm.

<u>Proof</u> This is mainly a matter of straightforward verification, most of which is left to the reader. As to the permutativity of $\mathcal{C}(T)$ let us just remark that the basepoint of $T(0)$ is the zero object. We will prove that

$$(3)\qquad \text{target}:\text{Mor } \mathcal{C}(T) \to \text{Ob } \mathcal{C}(T)$$

is continuous and leave the continuity of the remaining structure maps to the reader. There is a commutative diagram analogous to (2) with source replaced by target. This shows that the composite of (3) with the projection Ob $\mathcal{C}(T) \to$ Ob $\mathcal{C}$ is continuous. We can write Mor $\mathcal{C}(T)$ as a disjoint union of open subsets U, where the two composites

$$\text{Mor } \mathcal{C}(T) \underset{\text{target}}{\overset{\text{source}}{\rightrightarrows}} \text{Ob } \mathcal{C}(T) \longrightarrow \text{Ob } \mathcal{C}$$

map into two of the open sets O_α, O_β introduced above. Let $H \in O_\alpha$ and $H' \in O_\beta$. Then the restrictions of source and target to U map into $T(H)\times O_\alpha$ and $T(H')\times O_\beta$ respectively. By projection on the first factors we obtain maps

$$s:U \to T(H),\ t:U \to T(H') .$$

We know that s is continuous, and it suffices to prove that t is continuous. However t can be written as a composite of continuous maps

$$U \xrightarrow{(s,\phi)} T(H)\times T(H')^{T(H)} \longrightarrow T(H')$$

where the last map is evaluation, and ϕ is the restriction to U of the composite

$$\text{Mor}\ \mathcal{C}(T) \longrightarrow \text{Mor}\ \mathcal{C} \xrightarrow{T} \text{Mor Top}.$$

The construction of §1 associates to a Boardman-Vogt functor $T \in BV(\mathcal{C})$ a spectrum

$$B^0\mathcal{C}(T),\ B^1\mathcal{C}(T),\ B^2\mathcal{C}(T),\ldots \quad .$$

We are generally interested in determining the homotopy type of $B^0\mathcal{C}(T)$. This usually proceeds via the group

completion theorem 1.3 and a study of the classifying space $B\mathcal{C}(T)$. In this paper we will always use $\mathfrak{I}$ as our base category, and we will now prove that $B\mathfrak{I}(T)$ is weakly homotopy equivalent to $T(R^\infty)$ in that case. For this we need the following lemma (which was also used by Boardman and Vogt [3]):

<u>Lemma 4.4</u> The space $\mathfrak{I}(R^\infty,R^\infty)$ is contractible.

<u>Proof</u> Let $e_1,e_2,\ldots$ be the standard orthonormal basis for R^∞. Let H_1 be the subspace generated by e_{2n-1} $(n \geq 1)$, and H_2 the subspace generated by e_{2n} $(n \geq 1)$. A homotopy $R^\infty\times I \to R^\infty$ through linear maps from the identity D_0 on R^∞ to the map D_1 sending e_n into e_{2n} is given by

$$D_t e_n = (1-t)e_n + te_{2n}.$$

Each D_t is injective and maps the subspace generated by $e_1,\ldots,e_n$ into the subspace generated by $e_1,\ldots,e_{2n}$. We can now apply the well-known Gram-Schmidt orthonormalization procedure to get a path $\overline{D}_t$ in $\mathfrak{I}(R^\infty,R^\infty)$ from $\overline{D}_0 = D_0$ to $\overline{D}_1 = D_1$. Continuity is checked by looking at the finite dimensional subspaces.

A homotopy from the identity on $\mathfrak{I}(R^\infty,R^\infty)$ is given by $\overline{D}_tA$. We can write the map $\overline{D}_1A$ in matrix form

$$\begin{pmatrix} 0 \\ G(A) \end{pmatrix}$$

with respect to the splitting $R^\infty = H_1 \oplus H_2$. Let $B: R^\infty \to H_1$ be the linear map sending e_n into e_{2n-1}. Then a homotopy from $\bar{D}_1 A$ to a constant is given by

$$\begin{pmatrix} \sin \theta \cdot B \\ \cos \theta \cdot G(A) \end{pmatrix} \qquad (0 \leq \theta \leq \tfrac{\pi}{2}) .$$

The two homotopies combine to give a contraction of $\mathcal{J}(R^\infty, R^\infty)$.

Let $\mathcal{J}_\infty$ be the full subcategory of $\mathcal{J}$ with objects the infinite dimensional objects of $\mathcal{J}$ together with the zero object. This is again a topological permutative category and any $T \in BV(\mathcal{J})$ restricts to an object of $BV(\mathcal{J}_\infty)$ also denoted by T. We have an obvious continuous natural transformation $\mathcal{J}_\infty(T) \to \mathcal{J}(T)$ over the inclusion $\mathcal{J}_\infty \to \mathcal{J}$.

<u>Lemma 4.5</u> For $T \in BV(\mathcal{J})$ the induced map

$$B\mathcal{J}_\infty(T) \to B\mathcal{J}(T)$$

is a homotopy equivalence. Consequently $\mathcal{J}_\infty(T) \to \mathcal{J}(T)$ induce a homotopy equivalence of spectra.

<u>Proof</u> The last statement follows trivially from the first. Define a continuous functor

$$\mathcal{J}(T) \longrightarrow \mathcal{J}_\infty(T)$$

by sending (H,x) into $(H+R^\infty, \omega(x,*))$, where $* \in T(R^\infty)$ is the base point, and the morphism $A:(H,x) \to (H',x')$ into the morphism determined by $A + 1_{R^\infty}$. There are obvious continuous natural transformations from the identities on $\mathcal{J}(T)$ and $\mathcal{J}_\infty(T)$ to the two composits

$$\mathcal{J}(T) \rightleftarrows \mathcal{J}_\infty(T) \quad .$$

The lemma follows by applying the functor B.

Notice that the same argument applies to the full subcategory $\mathcal{J}'_\infty(T)$ of objects (H,x) with $\dim H = \infty$ (this is not permutative, however, since we have removed the zero object). We can even do more and consider the full subcategory $\mathcal{J}'(T)$ of objects (R^∞, x), where $x \in T(R^\infty)$. This category is clearly continuously equivalent to $\mathcal{J}'_\infty(T)$, and it follows that the inclusion $\mathcal{J}'(T) \to \mathcal{J}(T)$ induces a homotopy equivalence

$$(4) \qquad B\mathcal{J}'(T) \to B\mathcal{J}(T) .$$

We can regard $T(R^\infty)$ as a category with $T(R^\infty)$ as the space of objects and only identity morphisms. Its classifying space can be identified with $|\Delta T(R^\infty)|$. Therefore the inclusion $T(R^\infty) \to \mathcal{J}'(T)$ induces a map

$$(5) \qquad |\Delta T(R^\infty)| \to B\mathcal{J}'(T) .$$

<u>Theorem 4.6</u> The inclusion $T(R^\infty) \to \mathcal{J}(T)$, where $T(R^\infty)$ is regarded as a topological category with only identity morphisms, induces a homotopy equivalence

$$|\Delta T(R^\infty)| \to B\mathcal{J}(T) .$$

<u>Proof</u> This map is the composite of (4) and (5), so it suffices to prove that (5) is a homotopy equivalence. The functor $T(R^\infty) \to \mathcal{J}'(T)$ induces a continuous simplicial map of morphism complexes

$$MT(R^\infty) \to M\mathcal{J}'(T).$$

The map in degree n can be described as

$$T(R^\infty) \to T(R^\infty)\times\mathcal{J}(R^\infty,R^\infty)^n$$

sending x into $(x,1,\dots,1)$. By Lemma 4.4 this is a homotopy equivalence, so Theorem A.1 of the appendix can be applied to the induced bisimplicial map

$$\Delta MT(R^\infty) \to \Delta M\mathcal{J}'(T).$$

We conclude that the induced map

$$\Delta T(R^\infty) = \text{diag } \Delta MT(R^\infty) \to \text{diag } \Delta M\mathcal{J}'(T)$$

is a weak homotopy equivalence, and the theorem follows by taking geometric realizations.

We can define a multiplication

$$\mu : T(R^\infty)\times T(R^\infty) \to T(R^\infty)$$

by composing $\omega : T(R^\infty)\times T(R^\infty) \to T(R^\infty + R^\infty)$ with the map $T(R^\infty + R^\infty) \to T(R^\infty)$ induced by some morphism $A : R^\infty + R^\infty \to R^\infty$. Lemma 4.4 shows that μ is well-defined up to homotopy. It is easy to show that μ is homotopy associative and homotopy commutative.

<u>Lemma 4.7</u> The following diagram commutes up to homotopy

$$\begin{array}{ccc} |\Delta T(R^\infty)|\times|\Delta T(R^\infty)| & \longrightarrow & B\mathcal{J}(T)\times B\mathcal{J}(T) \\ \downarrow |\Delta\mu| & & \downarrow B+ \\ |\Delta T(R^\infty)| & \longrightarrow & B\mathcal{J}(T) \end{array} ,$$

i.e. the homotopy equivalence of Theorem 4.6 is an H-map.

<u>Proof</u> We can define a continuous functor $T(R^\infty+R^\infty) \xrightarrow{F'} \mathcal{J}(T)$ analogous to $T(R^\infty) \xrightarrow{F} \mathcal{J}(T)$. It is then obvious that the diagram

$$\begin{array}{ccc} T(R^\infty)\times T(R^\infty) & \xrightarrow{F\times F} & \mathcal{J}(T)\times\mathcal{J}(T) \\ \downarrow \omega & & \downarrow + \\ T(R^\infty+R^\infty) & \xrightarrow{F'} & \mathcal{J}(T) \end{array}$$

is commutative. The diagram

$$\begin{array}{ccc} T(R^\infty+R^\infty) & \xrightarrow{F'} & \\ \downarrow T(A) & & \mathcal{J}(T) \\ T(R^\infty) & \xrightarrow{F} & \end{array}$$

is not commutative, but there is a continuous natural transformation from F' to $FT(A)$ given by A. The lemma is proved by applying the functor B to these two diagrams and combining them.

The next result follows directly from this lemma and the group completion theorem 1.3. Recall that the canonical map $|\Delta T(R^\infty)| \to T(R^\infty)$ is a weak homotopy equivalence (Milnor [10]).

<u>Proposition 4.8</u> Let $T \in BV(\mathcal{J})$ with the associated infinite loop space $B^0\mathcal{J}(T)$. There is a natural isomorphism

$$\varinjlim H_*(T(R^\infty)_s;k) \to H_*(B^0\mathcal{J}(T)_0;k),$$

where the lower indices indicate components and the direct limit is taken over the translation category of π_0.

<u>Corollary 4.9</u> Let $\Phi:T_1 \to T_2$ be a natural transformation of Boardman-Vogt functors over $\mathcal{J}$. If the map $\Phi:T_1(R^\infty) \to T_2(R^\infty)$ is a homology equivalence (in particular if it is a weak homotopy equivalence), then Φ induces a homotopy equivalence of spectra $\{B^n\mathcal{J}(T_1)\} \to \{B^n\mathcal{J}(T_2)\}$.

<u>Proof</u> Obvious from the proposition.

We will call a Boardman-Vogt functor $T \in BV(\mathcal{J})$ <u>convergent</u> if it satisfies the following condition:

(F) For every $H \in \mathrm{Ob}\,\mathcal{J}$

$$T(H) = \varinjlim T(H_\alpha),$$

where the direct limit is taken over the finite dimensional subspaces $H_\alpha \subseteq H$, and the inclusions $H_\alpha \to H_\beta$ induce homeomorphisms $T(H_\alpha) \to T(H_\beta)$ onto closed subsets of $T(H_\beta)$.

An equivalent condition is:

(F') $T(R^\infty) = \varinjlim T(R^n)$ and for every $n \geq 0$ the inclusion $R^n \to R^{n+1}$ induces a homeomorphism of $T(R^n)$ onto a closed subset of $T(R^{n+1})$.

Let $\mathcal{J}_0$ be the full subcategory of $\mathcal{J}$ with objects the Euclidean spaces, i.e. finite dimensional objects of $\mathcal{J}$. Obviously $\mathcal{J}_0$ is a topological permutative category.

Proposition 4.10 Let $T \in BV(\mathcal{J}_0)$ be a Boardman-Vogt functor on $\mathcal{J}_0$ and assume that the inclusion $R^n \to R^{n+1}$ for every $n \geq 0$ induces a homeomorphism of $T(R^n)$ onto a closed subset of $T(R^{n+1})$. Then T extends uniquely to a convergent Boardman-Vogt functor on $\mathcal{J}$.

Proof The uniqueness is obvious. To prove existence we define $T(H) = \varinjlim T(H_\alpha)$ for $T \in \mathrm{Ob}\,\mathcal{J}$ (similarly on morphisms). The conditions on the map $T(R^n) \to T(R^{n+1})$ carry over to the maps $T(H_\alpha) \to T(H_\beta)$, and it follows that the direct limit exists in Top (this is where weak Hausdorff spaces are more convenient than Hausdorff spaces; notice that it is still true that the image of a compact space by a continuous map into $T(H)$ is contained in some $T(H_\alpha)$). The verification that T is a Boardman-Vogt functor on $\mathcal{J}$ is straightforward. To check continuity of T one uses the identity

$$Y^{\varinjlim X_\alpha} = \varprojlim Y^{X_\alpha}$$

that holds in Top. This follows from the identity

$$(6) \qquad Y \times \varinjlim X_\alpha = \varinjlim Y \times X_\alpha$$

by adjointness of the two functors

$$X \longmapsto Y \times X, \; X \longmapsto Y^X.$$

We remark that (6) follows from Theorem 4.4 of [17], which easily generalizes to weak Hausdorff spaces (cf. [6]), applied to the proclusion $\coprod X_\alpha \to \varinjlim X_\alpha$. Continuity of ω requires the identity

$$(\varinjlim_\alpha X_\alpha) \times (\varinjlim_\beta Y_\beta) = \varinjlim_{(\alpha,\beta)} (X_\alpha \times Y_\beta),$$

which follows from (6) applied twice.

There is the following companion to Prop. 4.10, whose proof is left to the reader:

Proposition 4.11 Let T_1 and T_2 be convergent Boardman-Vogt functors on $\mathcal{J}$, and Φ a natural transformation of their restrictions to $\mathcal{J}_0$, $\Phi: T_1|\mathcal{J}_0 \to T_2|\mathcal{J}_0$. Then Φ has a unique extension to a natural transformation of Boardman-Vogt functors $\Phi: T_1 \to T_2$.

Let $\mathcal{O}$ be the subcategory of $\mathcal{J}_0$ with the same objects as $\mathcal{J}_0$ and the isomorphisms of $\mathcal{J}_0$ as morphisms.

This is again a topological permutative category. The existence of complements in $\mathcal{J}_0$ implies that a Boardman-Vogt functor on $\mathcal{O}$ can be extended to $\mathcal{J}_0$.

<u>Proposition 4.12</u> Any $T \in BV(\mathcal{O})$ has a canonical extension to $\mathcal{J}_0$. This leads to a functor $BV(\mathcal{O}) \to BV(\mathcal{J}_0)$.

<u>Proof</u> Let $T \in BV(\mathcal{O})$. A complement for a morphism $A:H \to H'$ of $\mathcal{J}_0$ is a morphism $A^{\perp}:H^{\perp} \to H'$ of $\mathcal{J}_0$ such that $A + A^{\perp}:H + H^{\perp} \to H'$ is a morphism of $\mathcal{O}$, i.e. an orthogonal transformation. The complement is unique up to canonical isomorphism in the sense that, for any other complement $\tilde{A}:\tilde{H} \to H'$, there is a unique $\mathcal{O}$-morphism $H^{\perp} \xrightarrow{S} \tilde{H}$ making the diagram

$$\begin{array}{ccc} H + H^{\perp} & \xrightarrow{(1,S)} & H + \tilde{H} \\ & {\scriptstyle A + A^{\perp}} \searrow \quad \swarrow {\scriptstyle A + \tilde{A}} & \\ & H' & \end{array}$$

commutative. This implies that the composite $T(A)$,

$$T(H) \longrightarrow T(H)\times T(H^{\perp}) \xrightarrow{\omega} T(H+H^{\perp}) \xrightarrow{T(A+A^{\perp})} T(H'),$$

where the first map sends x into $(x,*)$, is independent of the choice of complement. Next we observe that $T(A)$

depends continuously on $A \in \mathcal{J}(H,H')$. This follows easily from the fact that A has a neighborhood U in $\mathcal{J}(H,H')$ on which the complement can be chosen as a continuous map $U \longrightarrow \mathcal{J}(H^{\perp},H')$ (here we use local triviality of the fibering of the Stiefel manifold $\mathcal{J}(H^{\perp},H')$ over the corresponding grassmannian). We leave to the reader the task of writing down the commutative diagrams needed to verify that this extends T to an element of $BV(\mathcal{J}_0)$, and that we get a functor as claimed.

5. Monoid valued Boardman-Vogt functors. The definition of a monoid valued Boardman-Vogt functor on a topological permutative category $\mathscr{C}$ is the same as Definition 4.1, except that Top is replaced by the category Mon of topological monoids with unit. In particular, the sum

$$\omega: T(H_1)\times\cdots\times T(H_n) \to T(H_1+\cdots+H_n)$$

is a monoid map. Recall that we have the classifying space functor B_{top}: Mon $\to$ Top (see §1).

Proposition 5.1 If T is a monoid valued Boardman-Vogt functor, then $B_{top}\, T$ is a Boardman-Vogt functor.

Proof Since B_{top} commutes with product all the conditions of Definition 4.1, except (A), are satisfied. To verify (A) we must prove that B_{top} is a continuous functor (Mon has the topology induced from Top).

For two simplicial spaces X, Y we can regard the set of continuous simplicial maps $X \to Y$ as a subspace

$$S \subseteq \prod_{n=0}^{\infty} Y_n^{X_n}.$$

This introduces a topology on the category of simplicial spaces, and the morphism complex functor from Mon into this category is clearly continuous. To prove that geometric realization is a continuous functor from simplicial

spaces to Top we must show that it induces a continuous map

$$S \to |Y|^{|X|}.$$

It suffices to prove that its adjoint $S\times|X| \to |Y|$ is continuous. However this is seen from the commutative diagram

$$\begin{array}{ccc}
S\times X_n\times\Delta_n & \xrightarrow{(proj.,1,1)} & Y_n^{X_n}\times X_n\times\Delta_n \\
\downarrow & & \downarrow (ev.,1) \\
S\times\coprod X_n\times\Delta_n & & Y_n\times\Delta_n \\
\downarrow & & \downarrow \\
S\times|X| & \longrightarrow & |Y|
\end{array}$$

where the lower left hand map is a proclusion (Steenrod [17], Theorem 4.4). The composite $B_{top} = |M(?)|$ is therefore continuous.

Remark The other classifying space functor B is not continuous.

The main result of this section is the following theorem (analogous to Theorem B of [3]).

Theorem 5.2 Let T be a monoid valued Boardman-Vogt functor on $\mathcal{J}$. Assume that $T(R^\infty)$ has the homotopy type of a CW-complex and that its basepoint $*$ is non-degenerate (i.e. the inclusion $* \to T(R^\infty)$ is a cofibration). Then there is a homotopy equivalence of spectra

$$\{B^{n+1}\mathcal{J}(T)\} \to \{B^n\mathcal{J}(B_{top}T)\}.$$

Proof We can regard T as a monoid in the category $BV(\mathcal{J})$. Then the bar-construction on T is a simplicial object WT in $BV(\mathcal{J})$ with the ℓ-fold product T^ℓ in degree ℓ (T^0 being a point). The functor $BV(\mathcal{J}) \to$ Topperm of §4 gives us a simplicial topological permutative category $\mathcal{J}(WT)$ with $\mathcal{J}(T^\ell)$ in degree ℓ. Finally we get a bisimplicial permutative category $\Delta\mathcal{J}(WT)$ with $\Delta_k\mathcal{J}(T^\ell)$ in bidegree (k,ℓ).

The bisimplicial permutative functor $W_\oplus\Delta\mathcal{J}(T)$ has $[\Delta_k(T)]^{\oplus\ell}$ in bidegree (k,ℓ). The ℓ inclusions $T \to T^\ell$ are natural transformations of Boardman-Vogt functors, and they induce permutative functors $\Delta_k\mathcal{J}(T) \to \Delta_k\mathcal{J}(T^\ell)$. These combine to permutative functors

$$[\Delta_k\mathcal{J}(T)]^{\oplus\ell} \to \Delta_k\mathcal{J}(T^\ell).$$

It can now be verified that this defines a bisimplicial

permutative functor

(1) $$W_\oplus\Delta\mathcal{J}(T) \longrightarrow \Delta\mathcal{J}(WT).$$

The induced simplicial permutative functor on the diagonal is

$$S^1 \otimes \Delta\mathcal{J}(T) \longrightarrow \operatorname{diag} \Delta\mathcal{J}(WT),$$

and we claim that this induces a homotopy equivalence of spectra

(2) $$\{B^{n+1}\mathcal{J}(T)\} \longrightarrow \{B^n \operatorname{diag} \Delta\mathcal{J}(WT)\}.$$

The ℓ projections $T^\ell \to T$ induce simplicial permutative functors $\Delta\mathcal{J}(T^\ell) \to \Delta\mathcal{J}(T)$, that combine to a simplicial permutative functor

(3) $$\Delta\mathcal{J}(T^\ell) \to (\Delta\mathcal{J}(T))^\ell,$$

whose composite with

(4) $$[\Delta\mathcal{J}(T)]^{\oplus\ell} \to \Delta\mathcal{J}(T^\ell)$$

is the canonical map from the ℓ-fold sum to the ℓ-fold product. This is an equivalence of categories in each degree, so it follows from Theorem A.1 of the appendix that it induces a homotopy equivalence

$$B[\Delta\mathcal{J}(T)]^{\oplus\ell} \to B(\Delta\mathcal{J}(T))^{\ell}.$$

It follows from Theorem 4.6 applied to the vertical maps of the diagram

$$\begin{array}{ccc} |\Delta T^{\ell}(R^{\infty})| & \longrightarrow & |\Delta T(R^{\infty})|^{\ell} \\ \downarrow & & \downarrow \\ B\mathcal{J}(T^{\ell}) & \longrightarrow & B\mathcal{J}(T)^{\ell} \end{array}$$

that (3) induces a homotopy equivalence of classifying spaces. Hence (4) induces a homotopy equivalence of classifying spaces. This shows that, if we fix the "bar-degree" in the trisimplicial map induces by (1)

$$MW_{\oplus}\Delta\mathcal{J}(T) \to M\Delta\mathcal{J}(WT)$$

we obtain a bisimplicial map inducing a weak equivalence on the diagonal. Our claim follows now from Theorem A.1.

For a space X we define the simplicial space X^{Δ} with the mapping space X^{Δ_ℓ} in degree ℓ as the singular complex of X with the mapping space topology on the singular simplices. This is a continuous functor of X,

and it commutes with products. We can therefore compose with $B_{top}\,T$ to get a simplicial Boardman-Vogt functor $B_{top}\,T^{\Delta}$ on $\mathcal{J}$. The adjoints of the natural maps

$$T^{\ell}\times\Delta_{\ell} \to B_{top}\,T$$

define a simplicial natural transformation of Boardman-Vogt functors

(5) $$WT \to B_{top}\,T^{\Delta}$$

Let $B_{top}\,T^*$ denote the degenerate simplicial Boardman-Vogt functor with $B_{top}\,T$ in each degree. There is a simplicial natural transformation of Boardman-Vogt functors

(6) $$B_{top}\,T^* \to B_{top}\,T^{\Delta},$$

which sends $x \in B_{top}\,T(H)$ of degree ℓ into the constant ℓ-simplex at x.

At this point we use the following lemma, whose proof is given below.

<u>Lemma 5.3</u> Let U and V be simplicial objects in $BV(\mathcal{J})$ and $\Phi: U \to V$ a simplicial natural transformation of Boardman-Vogt functors. Assume that Φ induces

a weak homotopy equivalence

$$\operatorname{diag} \Delta U(R^\infty) \to \operatorname{diag} \Delta V(R^\infty).$$

Then the induced map of spectra

$$\{B^n \operatorname{diag} \Delta \mathfrak{J}(U)\} \to \{B^n \operatorname{diag} \Delta \mathfrak{J}(V)\}$$

is a homotopy equivalence.

To finish the proof of Theorem 5.2 it suffices to show that (5) and (6) satisfy the assumptions of the lemma. For (6), this is clear from Theorem A.1, since the maps

$$B_{top}\, T(R^\infty) \to B_{top}\, T(R^\infty)^{\Delta_\ell}$$

are homotopy equivalences. For (5) it follows from this lemma:

Lemma 5.4 Let $X \in \operatorname{Ob} \mathrm{Mon}$ have the homotopy type of a CW-complex, and assume that the unit is non-degenerate as a base point. Then the adjoints of the canonical map $X^\ell \times \Delta_\ell \to B_{top} X$ define a continuous simplicial map $WX \to B_{top} X^\Delta$, which induces a weak equivalence

$$\operatorname{diag} \Delta WX \to \operatorname{diag} \Delta B_{top} X^\Delta.$$

Proof Put $\tilde X = |\Delta X|$. Then $\tilde X$ is a topological monoid satisfying the same conditions as X. The canonical map $\tilde X \to X$ is a monoid map and a homotopy equivalence. Our assumptions imply that the simplicial spaces $W\tilde X$ and WX are cofibered (see the appendix), so Theorem A.3 shows that the map $B_{top} \tilde X = BX \to B_{top} X$ is a homotopy equivalence.

The bisimplicial set $\Delta B_{top} X^\Delta$ is equal to the "singular bicomplex" of $B_{top} X$, which has the set of singular bisimplices, i.e. continuous maps $\Delta_k \times \Delta_\ell \to B_{top} X$, in bidegree (k,ℓ). Restriction to the diagonal of $\Delta_k \times \Delta_k$ defines a map

$$\operatorname{diag} \Delta B_{top} X^\Delta \to \Delta B_{top} X,$$

which is easily shown to be a homotopy equivalence of Kan complexes. Now one checks that the composite

$$|\operatorname{diag} \Delta WX| \to |\operatorname{diag} \Delta B_{top} X^\Delta| \to |\Delta B_{top} X| \to B_{top} X$$

can be identified with the homotopy equivalence $BX \to B_{top} X$, and the lemma follows.

Proof of Lemma 5.3 Theorem 4.6 has a straightforward generalization to the case of a simplicial Boardman-Vogt functor T on $\mathfrak{J}$: The inclusion $T(R^\infty) \to \mathfrak{J}(T)$, where $T(R^\infty)$ is thought of as a simplicial topological category with only identity morphisms, induces a homotopy equivalence

$$|\mathrm{diag}\ \Delta T(R^\infty)| \to B\ \mathrm{diag}\ \Delta\mathcal{J}(T).$$

This can be applied to U and V, and we conclude that Φ induces a homotopy equivalence

$$B\ \mathrm{diag}\ \Delta\mathcal{J}(U) \to B\ \mathrm{diag}\ \Delta\mathcal{J}(V).$$

The lemma follows now from the group completion theorem.

This completes the proof of Theorem 5.2. As a consequence we have:

Corollary 5.5 Under the assumptions of Theorem 5.2 $B^1\mathcal{J}(T)$ is homotopy equivalent as an H-space to $B_{top}\ T(R^\infty)$.

Proof Theorem 5.2 gives us a homotopy equivalence of H-spaces between $B^1\mathcal{J}(T)$ and $B^0\mathcal{J}(B_{top}\ T)$. By Lemma 4.7 we have a homotopy equivalence of H-spaces

$$|\Delta B_{top}\ T(R^\infty)| \to B\mathcal{J}(B_{top}\ T).$$

We see from this that $B\mathcal{J}(B_{top}\ T)$ is connected, and so the group completion map

$$B\mathcal{J}(B_{top}\ T) \to B^0\mathcal{J}(B_{top}\ T)$$

is a homotopy equivalence of H-spaces. Finally the canonical map

$$|\Delta B_{top}\ T(R^\infty)| \to B_{top}\ T(R^\infty)$$

is a homotopy equivalence of H-spaces, since $B_{top}\ T(R^\infty)$ is homotopy equivalent to the CW-complex $BT(R^\infty)$.

In the remaining part of this section we study some particular monoid valued Boardman-Vogt functors F^S, G^S giving rise to localizations of BSG, the classifying space for stable oriented spherical fibrations.

Let S be a set of natural numbers containing 1 and closed under multiplication. We first define a monoid valued Boardman-Vogt functor F^S on $\mathcal{O}$, the subcategory of $\mathcal{J}$ of Euclidean spaces and orthogonal transformations. Let H be a Euclidean space and $H^+ = H \cup \infty$ its one point compactification with ∞ chosen to be its basepoint. For an integer s let $F^s(H)$ be the space of base point preserving maps $H^+ \to H^+$ of degree s, and let $F^S(H)$ be the disjoint union

$$F^S(H) = \coprod_{s \in S} F^s(H).$$

This is a topological monoid under composition. We let

$$\omega: F^S(H_1) \times F^S(H_2) \to F^S(H_1 + H_2)$$

be smash product (identifying $(H_1+H_2)^+$ in the obvious way with $H_1^+ \wedge H_2^+ = H_1^+ \times H_2^+ / H_1^+ \vee H_2^+$). An orthogonal transformation $A: H_1 \to H_2$ induces the map

$$F^S(H_1) \to F^S(H_2)$$

given by

$$f \longmapsto A^+ f (A^+)^{-1}.$$

The conditions of Definition 4.1 are now easily verified. By Proposition 4.12 F^S extends to a Boardman-Vogt functor on $\mathfrak{J}_0$, and it is easily shown that the extension is monoid valued. The assumptions of Proposition 4.11 are now satisfied, so F^S extends to a convergent Boardman-Vogt functor on $\mathfrak{J}$. This extension is again monoid valued (e.g. by applying 4.11 to the product $F^S \times F^S \to F^S$). In particular for $S = \{1\}$ we get the convergent monoid valued Boardman-Vogt functor F^1 on $\mathfrak{J}$. The inclusion $F^1 \to F^S$ extends to a morphism in $BV(\mathfrak{J})$.

Another monoid valued Boardman-Vogt functor G^S on $\mathfrak{J}$ can be obtained as follows. For a Euclidean space H let $G^s(H)$ be the space of maps of degree s of the unit sphere in H to itself, and let $G^S(H)$ be the disjoint union

$$G^S(H) = \coprod_{s \in S} G^s(H).$$

A map $G^S(H) \to F^S(H)$ is given by radial extension, i.e. $f \in G^S(H)$ is sent into $\overline{f} \in F^S(H)$, given by

$$\overline{f}(y) = \|y\| f\left(\frac{y}{\|y\|}\right)$$

for $y \in H - \{0\}$ and $\overline{f}(0) = 0$, $\overline{f}(\infty) = \infty$. This embeds $G^S(H)$ as a subspace of $F^S(H)$. All the structure defined above for F^S restricts to G^S, and G^S becomes a monoid valued object of $BV(\mathcal{O})$. As above G^S extends to a monoid valued Boardman-Vogt functor on $\mathfrak{J}$. Moreover we get a commutative diagram in $BV(\mathfrak{J})$

$$\begin{array}{ccc} G^1 & \longrightarrow & G^S \\ \downarrow & & \downarrow \\ F^1 & \longrightarrow & F^S \end{array} \qquad (7)$$

where all the arrows respect the monoid structures.

<u>Proposition 5.6</u> The infinite loop space $B^1\mathfrak{J}(G^1)$ is homotopy equivalent as an H-space to BSG with the H-structure coming from Whitney join of spherical fibrations.

<u>Proof</u> G^1 satisfies the assumptions of Theorem 5.2 ($G^1(R^\infty) = \varinjlim G^1(R^n)$ is a direct limit of absolute neighborhood retracts), so by Corollary 5.5 $B^1\mathfrak{J}(G^1)$ is homotopy equivalent as an H-space to $B_{top}\, G^1(R^\infty)$, which is equal to the space usually denoted BSG. That it has the right H-structure is clear from the definition of the product

$$B_{top}\, G^1(R^\infty)^2 \xrightarrow{\ \omega\ } B_{top}\, G^1(R^\infty + R^\infty) \longrightarrow B_{top}\, G^1(R^\infty),$$

where the last map is induced by any morphism $R^\infty + R^\infty \to R^\infty$.

This introduces a new infinite loop space structure on BSG similar in spirit to the one constructed by Boardman and Vogt. We could have done this for BG using maps of degree ± 1.

<u>Lemma 5.7</u> The vertical arrows in (7) induce homotopy equivalences of spectra.

<u>Proof</u> It suffices to prove this for $G^S \to F^S$. By Corollary 4.9 it suffices to show that $G^S(R^\infty) \to F^S(R^\infty)$ is a weak homotopy equivalence. We shall in fact show that $G^s(R^{n+1}) \to F^s(R^{n+1})$ is an $(n-2)$-equivalence for every integer s. Via a homeomorphism g of $S^n = (R^n)^+$ onto the unit sphere in R^{n+1} we can regard $F^s(R^n)$ as the subspace of $G^s(R^{n+1})$ fixing $g(\infty)$. The fibration

$$F^s(R^n) \to G^s(R^{n+1}) \to S^n$$

shows that the inclusion $F^s(R^n) \to G^s(R^{n+1})$ is an $(n-2)$-equivalence. The composite

$$F^s(R^n) \to G^s(R^{n+1}) \to F^s(R^{n+1})$$

can be identified with the suspension map $\Omega^n S^n \to \Omega^{n+1} S^{n+1}$

restricted to the degree s components. This is an $(n-2)$-equivalence, and the lemma follows.

The results of D. Sullivan [18] on localization imply that one can localize all the spaces and maps of an Ω-spectrum $\{B^n\}$ and obtain an Ω-spectrum $\{B^n[S^{-1}]\}$. Here $[S^{-1}]$ stands for localization at the set of primes p such that $(s,p) = 1$ for every $s \in S$ (i.e. the smallest localization that inverts all $s \in S$).

The horizontal arrows of (7) induce spectrum maps

$$\{B^n\mathcal{J}(G^1)\} \to \{B^n\mathcal{J}(G^S)\}$$
$$\{B^n\mathcal{J}(F^1)\} \to \{B^n\mathcal{J}(F^S)\}.$$

Since the spectra on the left are 0-connected we can lift to the 0-connected coverings of the right hand spectra

$$\{B^n\mathcal{J}(G^1)\} \to \{\tilde{B}^n\mathcal{J}(G^S)\}$$
$$\{B^n\mathcal{J}(F^1)\} \to \{\tilde{B}^n\mathcal{J}(F^S)\}. \tag{8}$$

<u>Proposition 5.8</u> The spectrum maps (8) extend to homotopy equivalences

$$\{B^n\mathcal{J}(G^1)[S^{-1}]\} \to \{\tilde{B}^n\mathcal{J}(G^S)\}$$
$$\{B^n\mathcal{J}(F^1)[S^{-1}]\} \to \{\tilde{B}^n\mathcal{J}(F^S)\}.$$

Proof Because of Lemma 5.7 it suffices to consider $F^1 \to F^S$. Also it is enough to show that

$$B^0\mathcal{J}(F^1) \to \tilde{B}^0\mathcal{J}(F^S) \tag{9}$$

extends to a homotopy equivalence

$$B^0\mathcal{J}(F^1)[S^{-1}] \to \tilde{B}^0\mathcal{J}(F^S).$$

By Proposition 4.8 the horizontal homomorphisms of the commutative diagram (with any coefficient field k)

$$\begin{array}{ccc} H_*(F^1(R^\infty)) & \text{———} & H_*(B^0\mathcal{J}(F^1)) \\ | & & | \\ \varinjlim H_*(F^S(R^\infty)) & \text{———} & H_*(\tilde{B}^0\mathcal{J}(F^S)) \end{array}$$

are isomorphisms. The homomorphisms of the direct limit are induced by s-fold smash powers

$$F^t(R^\infty) \to F^{st}(R^\infty) \quad (s,t \in S)\ . \tag{10}$$

The map (10) induces multiplication by s on the homotopy groups (isomorphic to stable homotopy of spheres), so it induces a homology isomorphism for $k = Q$ and $k = Z_p$, if p is prime to s. Hence (9) induces an isomorphism on $H_*(\ ;k)$ for $k = Q$ and $k = Z_p$, if p is prime to every $s \in S$.

Now consider the case $k = Z_p$, where p divides $s \in S$. We claim that $\tilde{H}_*(\tilde{B}^0\mathcal{J}(F^S)) = 0$ in this case. This will imply that $\tilde{B}^0\mathcal{J}(F^S)$ is local, so that the extension of (9) exists. It then follows from the above that this extension is a homotopy equivalence.

Let $\alpha \in \tilde{H}_i(F^t(R^\infty))$, where $t \in S$. Let Y be the mapping telescope of the sequence

$$F^t(R^\infty) \to F^{st}(R^\infty) \to F^{s^2t}(R^\infty) \to \dots\ .$$

Since all the maps induce multiplication by s on the homotopy groups, the homotopy groups of Y are finite with no p-torsion. Hence $\tilde{H}_*(Y) = 0$, and α goes to 0 in $H_i(F^{s^jt}(R^\infty))$ for a sufficiently high j. This shows that $\tilde{H}_*(\tilde{B}^0\mathcal{J}(F^S)) = 0$.

By combining the above results we get

Corollary 5.9 The simply connected covering of $B^1\mathcal{J}(F^S)$ is homotopy equivalent as an infinite loop space to $BSG\,[S^{-1}]$.

6. The main theorem As in §5 let S be a set of natural numbers containing 1 and closed under multiplication. Let $\mathcal{E}^S_\otimes$ be the permutative category of finite sets E with card $E \in S$ and bijective maps, the monoid structure being given by cross product. The classifying space $B\mathcal{E}^S_\otimes$ is homotopy equivalent to the disjoint union

$$\coprod_{s\in S} B\Sigma_s .$$

We can think of $B^1\mathcal{E}^S_\otimes$ as being $B \coprod_{s\in S} B\Sigma_s$, where the product on $\coprod B\Sigma_s$ is induced by the cross product homomorphisms $\Sigma_s \times \Sigma_t \to \Sigma_{st}$. For the theorems below we assume that S does not consist of 1 alone.

Theorem 6.1 The simply connected covering $\tilde{B}^1\mathcal{E}^S_\otimes$ of $B^1\mathcal{E}^S_\otimes = B\coprod B\Sigma_s$ is homotopy equivalent as an infinite loop space to the localization $BSG[S^{-1}]$.

This follows from Corollary 5.9 and the next theorem.

Theorem 6.2 The spectra $\{B^n\mathcal{E}^S_\otimes\}$ and $\{B^n\mathcal{J}(F^S)\}$ are homotopy equivalent.

The proof will occupy the rest of this section. It involves the construction of two elements of $BV(\mathcal{J})$, P^S, D^S, with natural transformations

$$P^S \leftarrow D^S \rightarrow F^S,$$

and a topological permutative category $\mathcal{H}^S$ together with continuous permutative functors

$$\mathcal{E}^S_\otimes \leftarrow \mathcal{H}^S \rightarrow \mathcal{J}(P^S).$$

For a pre-Hilbert space H let $P^s(H)$ be the s-fold deleted symmetric product of H (as in §3), and put

$$P^S(H) = \coprod_{s\in S} P^s(H) .$$

A morphism of $\mathcal{J}$ $H \to H'$ induces the obvious map $P^S(H) \to P^S(H')$. The "sum"

$$\omega : P^S(H_1)\times P^S(H_2) \to P^S(H_1+H_2)$$

sends a pair of finite subsets of H_1 and H_2 into their cross product regarded as a finite subset of $H_1 + H_2$. This defines a convergent Boardman-Vogt functor P^S on $\mathcal{J}$.

An object of $\mathcal{J}(P^S)$ is then a pair (H,E) where $H \in \mathrm{Ob}\,\mathcal{J}$ and E is a finite subset of H with card $E \in S$. A morphism $(H,E) \to (H',E')$ is an inner product preserving linear map $H \to H'$, which maps E onto E'. Addition of objects is given by

$$(H_1,E_1) + (H_2,E_2) = (H_1+H_2, E_1\times E_2),$$

and morphisms are added by taking direct sum of the two associated morphisms in $\mathcal{J}$. The permutative structure is given by interchange $H_1 + H_2 \to H_2 + H_1$.

We would like to construct directly a permutative functor $\mathcal{J}(P^S) \to \mathcal{E}^S_{\otimes}$ by sending (H,E) into E, but there are difficulties in doing this continuously. This difficulty is circumvented by introducing the intermediate topological permutative category $\mathcal{H}^S$.

An object of $\mathcal{H}^S$ is a triple (H,E,i), where $H \in \mathrm{Ob}\,\mathcal{J}$, $E \in \mathrm{Ob}\,\mathcal{E}^S_{\otimes}$ and $i:E \to H$ an injective map. For fixed H and E we introduce the topology on the set of inclusions $i:E \to H$ induced from the mapping space, and we topologize $\mathrm{Ob}\,\mathcal{H}^S$ as the disjoint union over all pairs (H,E) of these spaces. A morphism $(H,E,i) \to (H',E',i')$ is an inner product preserving linear map $H \to H'$, which maps $i(E) \subseteq H$ bijectively onto $i'(E') \subseteq H'$. We topologize $\mathrm{Mor}\,\mathcal{H}^S$ so that the diagram

$$\begin{array}{ccc} \mathrm{Mor}\,\mathcal{H}^S & \longrightarrow & \mathrm{Mor}\,\mathcal{J} \\ \downarrow{\scriptstyle \text{source}} & & \downarrow{\scriptstyle \text{source}} \\ \mathrm{Ob}\,\mathcal{H}^S & \longrightarrow & \mathrm{Ob}\,\mathcal{J} \end{array}$$

is a cartesian square in Top. Addition of objects in $\mathcal{H}^S$ is given by

$$(H_1,E_1,i_1) + (H_2,E_2,i_2) = (H_1+H_2,E_1\times E_2,i_1\times i_2),$$

where $E_1\times E_2$ is the "sum" in $\mathcal{E}^S_{\otimes}$. We add morphisms by adding in $\mathcal{J}$, and the permutative structure is given by interchange $H_1 + H_2 \to H_2 + H_1$. We leave the verification that $\mathcal{H}^S$ is a topological permutative category to the reader.

The continuous permutative functor $\mathcal{H}^S \to \mathcal{J}(P^S)$ is defined by sending (H,E,i) into $(H,i(E))$, and the morphism $(H,E,i) \to (H',E',i')$ given by $A:H \to H'$ into the morphism $(H,i(E)) \to (H',i(E))$ given by A.

Finally we define a continuous (i.e. locally constant) permutative functor $\mathcal{H}^S \longrightarrow \mathcal{E}^S_{\otimes}$ by sending (H,E,i) into E and the morphism $(H,E,i) \to (H',E',i')$ into the bijection $E \to E'$ making the diagram

$$\begin{array}{ccc} E & \longrightarrow & E' \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle i'} \\ H & \longrightarrow & H' \end{array}$$

commutative.

Choose for each $s \in S$ some set $E^s \in \mathrm{Ob}\,\mathcal{E}^S_{\otimes}$ with s elements, and let $\mathcal{N}^S$ be the subcategory of $\mathcal{H}^S$ with objects (R^∞,E^s,i) and morphisms being given by the identity on R^∞.

<u>Lemma 6.3</u> The inclusion $\mathcal{N}^S \to \mathcal{H}^S$ induces a homotopy equivalence $B\mathcal{N}^S \to B\mathcal{H}^S$.

<u>Proof</u> Let $\mathcal{H}_\infty$ be the full subcategory of $\mathcal{H}^S$ with objects (H,E,i), where $\dim H = \infty$. The proof of Lemma 4.5

can be repeated to show that $B\mathcal{M}_\infty \to B\mathcal{N}^S$ is a homotopy equivalence. Let $\mathcal{L}$ be the full subcategory of $\mathcal{M}_\infty$ with objects (R^∞, E^S, i). Then $\mathcal{L}$ is continuously equivalent to $\mathcal{M}_\infty$, so we get a homotopy equivalence $B\mathcal{L} \to B\mathcal{M}_\infty$.

The remaining part of the proof is analogous to the proof of Theorem 4.6. To show that the inclusion $\mathcal{N}^S \to \mathcal{L}$ induces a homotopy equivalence $B\mathcal{N}^S \to B\mathcal{L}$ it suffices by Theorem A.1 to show that the induced map of the degree n pieces of the morphism complexes

$$M_n\mathcal{N}^S \to M_n\mathcal{L}$$

is a homotopy equivalence.

An element of $M_n\mathcal{L}$ is given by inclusions $i_\nu : E^S \to R^\infty$ $(\nu = 0,\ldots,n)$ and elements of $\mathcal{J}(R^\infty, R^\infty)$ $A_1,\ldots,A_n$ satisfying

$$A_\nu i_{\nu-1}(E^S) = i_\nu(E^S) \qquad (\nu = 1,\ldots,n).$$

We write this element of M_n as

$$(i_0,\ldots,i_n;A_1,\ldots,A_n),$$

and we define bijections $\pi_\nu : E^S \to E^S$ $(\nu = 1,\ldots,n)$ by $A_\nu i_{\nu-1} = i_\nu \pi_\nu$. By Lemma 4.4 we can choose a homotopy $K_t : \mathcal{J}(R^\infty, R^\infty) \to \mathcal{J}(R^\infty, R^\infty)$ from the identity to the constant

1. Then a homotopy $H_t : M_n\mathcal{L} \to M_n\mathcal{L}$ from the identity to a retraction $M_n\mathcal{L} \to M_n\mathcal{N}^S$ can be defined by

$$H_t(i_0,\ldots,i_n,A_1,\ldots,A_n) = (j_0(t),\ldots,j_n(t);K_tA_1,\ldots,K_tA_n)$$

where

$$j_\nu(t) = (K_tA_\nu)\cdots(K_tA_1)i_0\pi_1^{-1}\cdots\pi_\nu^{-1}.$$

Lemma 6.4 The functor $\mathcal{M}^S \to \mathcal{J}(P^S)$ induces a homotopy equivalence $B\mathcal{M}^S \to B\mathcal{J}(P^S)$.

Proof We have a commutative diagram of continuous functors

$$\begin{array}{ccc} \mathcal{N}^S & \longrightarrow & P^S(R^\infty) \\ \downarrow & & \downarrow \\ \mathcal{M}^S & \longrightarrow & \mathcal{J}(P^S) \end{array}$$

where $P^S(R^\infty)$ is regarded as a topological category with only identity morphisms. Because of Theorem 4.6 and Lemma 6.3 it suffices to show that $\mathcal{N}^S \to P^S(R^\infty)$ induces a homotopy equivalence

$$B\mathcal{N}^S \to |\Delta P^S(R^\infty)|.$$

A point of $M_n\mathcal{N}^S$ may be described as a subset $E \subseteq R^\infty$ together with $(n+1)$ bijections $i_\nu : E^S \to E$. It is degenerate if and only if $i_{\nu-1} = i_\nu$, for some ν. This shows that the degenerate points of $M_n\mathcal{N}^S$ is a union of connectedness components. In particular $M\mathcal{N}^S$ is a cofibered simplicial space. We can therefore apply Theorem A.3 to the continuous simplicial map $\tilde{M}\mathcal{N}^S \to M\mathcal{N}^S$, where $(\tilde{M}\mathcal{N}^S)_n = |\Delta M_n\mathcal{N}^S|$, and conclude that the canonical map $B\mathcal{N}^S \to B_{top}\mathcal{N}^S$ is a homotopy equivalence. Hence it suffices to prove that the map

$$(1) \qquad B_{top}\mathcal{N}^S \to P^S(R^\infty)$$

is a homotopy equivalence. This map is locally trivial and the fiber over a set of s elements $E \subseteq R$ can be identified with the classifying space of the category with one object for each bijection $E^S \to E$ and just one morphism for any pair of objects. Hence the fibers are contractible and (1) is a homotopy equivalence.

<u>Lemma 6.5</u> The functor $\mathcal{M}^S \to \mathcal{E}^S_\otimes$ induces a homotopy equivalence $B\mathcal{M}^S \to B\mathcal{E}^S_\otimes$.

<u>Proof</u> Because of Lemma 6.3 it is enough to prove that the composite $\mathcal{N}^S \to \mathcal{M}^S \to \mathcal{E}^S_\otimes$ induces a homotopy equivalence $B\mathcal{N}^S \to B\mathcal{E}^S_\otimes$. Since the map $B\mathcal{N}^S \to B_{top}\mathcal{N}^S$ is a homotopy equivalence, it suffices to prove that the composite

$$(2) \qquad B_{top}\mathcal{N}^S \to B_{top}\mathcal{E}^S_\otimes = B\mathcal{E}^S_\otimes$$

is a homotopy equivalence. We have seen in §3 that $P^S(R^n)$ is $(n-2)$-equivalent to a $K(\Sigma_s,1)$. It follows that $P^S(R^\infty)$ is a $K(\Sigma_s,1)$. The homotopy equivalence (1) shows that $B_{top}\mathcal{N}^S$ is homotopy equivalent to the disjoint union $\coprod_{s\in S} K(\Sigma_s,1)$. The same is true for $B\mathcal{E}^S_\otimes = \coprod_{s\in S} B\Sigma_s$, so it suffices to show that (2) induces an epimorphism on the fundamental groups of the components corresponding to $s \in S$.

The set E^S represents a point in the s-component of $B\mathcal{E}^S_\otimes$, and a permutation $\sigma : E^S \to E^S$ represents a 1-simplex running from E^S to E^S. This 1-simplex becomes a loop γ_σ in $B\mathcal{E}^S_\otimes$ based at the point E^S, representing a typical element in the fundamental group of the s-component. Let $i : E^S \to R^\infty$ be an inclusion of E^S as part of an orthonormal basis. It follows easily from the contractibility of $\mathcal{J}(R^\infty,R^\infty)$, that there exists a path γ_1 in the space of inclusions $E^S \to R^\infty$ from i to $i\sigma$. Then γ_1 is a path in $Ob\,\mathcal{N}^S \subseteq B_{top}\mathcal{N}^S$. There is another path γ_2 in $B_{top}\mathcal{N}^S$ coming from the morphism $(R^\infty,E^S,i\sigma) \to (R^\infty,E^S,i)$. The composite $\gamma_1\gamma_2$ is a loop in $B_{top}\mathcal{N}^S$ based at the point (R^∞,E^S,i). Its image in $B\mathcal{E}^S_\otimes$ is the composite $c\gamma_\sigma$, where c is a constant loop. This completes the proof.

We will now define the Boardman-Vogt functor D^S.

We define it first on the category $\mathcal{O}$ of Euclidean spaces and orthogonal transformations. For a Euclidean space H, we let H^+ be its one-point compactification. We introduce the sphere compactification $\overline{H}$, which has one point at infinity for each ray in H starting at 0. Hence $\overline{H}$ is a closed disc with interior H.

For a natural number s we define a space $\tilde{D}^s(H)$. A point in $\tilde{D}^s(H)$ is an s-tuple $(f_1,\ldots,f_s)$ of continuous maps $f_i:\overline{H} \to H^+$ such that

i) The restriction of f_i to H is a homeomorphism of H onto a subset of H.

ii) The images $f_i(H)$, $f_j(H)$ are disjoint for $i \neq j$.

iii) There exists a neighborhood U of $0 \in H$, such that $f_i(x) = f_i(0) + x$ for $x \in U$.

For a fixed neighborhood U of 0 we let $\tilde{D}^s_U(H)$ denote the subset of $\tilde{D}^s(H)$ for which iii) holds. We give $\tilde{D}^s_U(H)$ the topology induced from

$$(H^+)^{\overline{H}}\times\cdots\times(H^+)^{\overline{H}}$$

and $\tilde{D}^s(H)$ the direct limit topology

$$\tilde{D}^s(H) = \lim_{\to} \tilde{D}^s_U(H),$$

where U runs through all neighborhoods of 0 (or equivalently the balls around 0 of radius $\frac{1}{n}$). The symmetric group Σ_s acts on $\tilde{D}^s(H)$ by permuting $(f_1,\ldots,f_s)$.

Let the quotient space be $D^s(H)$. A point of $D^s(H)$ is an unordered collection of s continuous functions $f_i:\overline{H} \to H^+$ satisfying the above conditions. We put $D^S(H) = \coprod_{s\in S} D^s(H)$.

An orthogonal transformation $A:H_1 \to H_2$ induces the map $\tilde{D}^s(H_1) \to \tilde{D}^s(H_2)$ sending $(f_1,\ldots,f_s)$ into

$$(A^+f_1\overline{A}^{-1},\ldots,A^+f_s\overline{A}^{-1}).$$

This is Σ_s-equivariant, so we can divide out by the action and get a map $D^s(H_1) \to D^s(H_2)$. These combine to a map $D^S(H_1) \to D^S(H_2)$.

The sum

$$\omega:D^S(H_1)\times D^S(H_2) \to D^S(H_1+H_2)$$

is defined by sending the unordered collections of functions (f_i), (g_j) into $(f_i\cdot g_j)$, where $f_i\cdot g_j$ is the composite

$$\overline{H_1+H_2} \to \overline{H}_1\times\overline{H}_2 \xrightarrow{f_i\times g_j} H_1^+\times H_2^+ \to (H_1+H_2)^+.$$

In other words $f_i\cdot g_j$ is the unique map $\overline{H_1+H_2} \to (H_1+H_2)^+$ which agrees with $f_i\times g_j$ on $H_1 + H_2$.

Observe that we have a canonical map $\overline{H} \to H^+$, which is the identity on H. This defines the basepoint in

$D^1(H) \subseteq D^s(H)$. We leave the proof that $D^S \in BV(\mathcal{O})$ and that it extends to $D^S \in BV(\mathcal{J})$ by 4.12 and 4.10 to the reader.

A natural transformation of Boardman-Vogt functors $D^S \to P^S$ on $\mathcal{O}$ is defined by sending the unordered collection (f_i) of maps $\bar{H} \to H^+$ into the finite subset $\{f_i(0)\}$ of H. Since P^S clearly is obtained from its restriction to $\mathcal{O}$ by the extension procedure of §4, we get a natural transformation of Boardman-Vogt functors $D^S \to P^S$ on $\mathcal{J}$.

In order to get a natural transformation $D^S \to P^S$ we first consider a Euclidean space H, and define

$$\Psi : \tilde{D}^s(H) \to F^s(H)$$

by

$$\Psi(f_1,\dots,f_s)(x) = \begin{cases} y \in H & \text{if } x = f_i(y) \\ \infty & \text{if } x \notin \bigcup f_i(H) \end{cases}$$

<u>Lemma 6.6</u> The above map Ψ is well-defined and continuous.

<u>Proof</u> It suffices to prove that the adjoint

$$\psi : \tilde{D}^s(H) \times H^+ \to H^+$$

is continuous, since this in particular shows that Ψ maps into $F^s(H)$. We can here consider the finer topology on $\tilde{D}^s(H)$ induced from

$$(H^+)^{\bar{H}} \times \cdots \times (H^+)^{\bar{H}} .$$

Since the spaces are now metrizable we can prove continuity by considering sequences

$$F^{(n)} = (f_1^{(n)},\dots,f_s^{(n)}) \in \tilde{D}^s(H), \; x^{(n)} \in H^+$$

with $$\lim_{n\to\infty} F^{(n)} = F = (f_1,\dots,f_n), \; \lim_{n\to\infty} x^{(n)} = x$$

and proving that the sequence $\psi(F^{(n)},x^{(n)})$ converges to $\psi(F,x)$ in H^+. By compactness of H^+ it is enough that every subsequence of $\psi(F^{(n)},x^{(n)})$ has a further subsequence, which converges to $\psi(F,x)$. By a change of notation this reduces to finding some subsequence of $\psi(F^{(n)},x^{(n)})$ converging to $\psi(F,x)$.

Assume first that there is an index i, such that $x^{(n)} \in f_i^{(n)}(\bar{H})$ for infinitely many n. By a change of notation we can assume this happens for all n. We put $x^{(n)} = f_i^{(n)}(y^{(n)})$, where $y^{(n)} \in \bar{H}$. We can pass to a further subsequence and assume that $y = \lim y^{(n)}$ exists in $\bar{H}$. Then

$$x = \lim x^{(n)} = \lim f_i^{(n)}(y^{(n)}) = f_i(y).$$

We have now

$$\psi(F^{(n)},x^{(n)}) = \pi(y^{(n)}),\ \psi(F,x) = \pi(y),$$

where $\pi:\bar{H} \to H^+$ is the canonical map. It follows that $\psi(F^{(n)},x^{(n)})$ converges to $\psi(F,x)$, and we are done in this case.

In the other case we can assume that $x^{(n)} \notin \bigcup f_i^{(n)}(\bar{H})$ for all n. Then $\psi(F^{(n)},x^{(n)}) = \infty$, and it suffices to prove that $\psi(F,x) = \infty$. If not, we must have $x = f_i(y)$ for some i and some $y \in H$. By the open mapping theorem and condition i) there exists a closed disc D around x, such that $D \subseteq f_i(H)$. For sufficiently large n we have $x^{(n)} \in D$ and $D \subseteq f_i^{(n)}(H)$ contradicting the assumption on $x^{(n)}$.

The map Ψ factors through $D^s(H) \to F^s(H)$ and we can combine them to a map $D^S(H) \to F^S(H)$. It is easily verified that this is a natural transformation of Boardman-Vogt functors on $\mathcal{O}$. Finally we extend it to $\mathcal{J}$ using the results in §4.

<u>Lemma 6.7</u> For a Euclidean space H the map

$$D^s(H) \to P^s(H)$$

is a weak homotopy equivalence.

<u>Proof</u> Let $\tilde{P}^s(H)$ be the space of ordered sets of s distinct points in H topologized as a subspace of $H\times\cdots\times H$. We have a pull back diagram of finite coverings

$$\begin{array}{ccc} \tilde{D}^s(H) & \xrightarrow{\Phi} & \tilde{P}^s(H) \\ \downarrow & & \downarrow \\ D^s(H) & \longrightarrow & P^s(H) \end{array}$$

where Φ is defined by

$$\Phi(f_1,\ldots,f_s) = (f_1(0),\ldots,f_s(0)).$$

It suffices to prove that Φ is a weak homotopy equivalence.

We will now define a map

$$\Theta:\tilde{P}^s(H) \to \tilde{D}^s(H).$$

First choose a homeomorphism

$$h:[0,1) \to [0,\infty)$$

equal to the identity on $[0,\frac{1}{2}]$, and define a homeomorphism

$$h_D:D = D(y,r) \to H,$$

where $D(y,r)$ is the open disc with center y and radius

r, by

$$h_D(y+trz) = h(t)rz$$

for $0 \leqq t \leqq 1$ and $z \in H$ with $\|z\| = 1$. Clearly h_D extends to a homeomorphism of the closed disc

$$h_D : \bar{D} = \bar{D}(y,r) \to \bar{H}.$$

For $x = (x_1,\dots,x_s) \in \tilde{P}^s(H)$ put

$$r(x) = \frac{1}{2}\min_{i \neq j} \|x_i - x_j\| ,$$

and define Θ by

$$\Theta(x) = (\Theta_1(x),\dots,\Theta_s(x)),$$

where $\Theta_i(x) : \bar{H} \to H^+$ is determined by the commutative diagram

$$\begin{array}{ccc} \bar{D}(x_i,r(x)) & \hookrightarrow & \bar{H} \\ {\scriptstyle h_{D(x_i,r(x))}} \downarrow & & \downarrow \\ \bar{H} & \xrightarrow{\Theta_i(x)} & H^+ \end{array} .$$

It is easy to verify that Θ maps into $\tilde{D}^s(H)$ and is continuous. Clearly $\Phi\Theta$ is the identity on $\tilde{P}^s(H)$, so that

Θ embeds $\tilde{P}^s(H)$ as a retract of $\tilde{D}^s(H)$.

Now consider a map $g : S^n \to \tilde{D}^s(H)$ sending the base point $*$ into $\tilde{P}^s(H)$. All that remains to be done is to find a homotopy rel. $*$ from g to a map that goes into $\tilde{P}^s(H)$.

We extend the definition of h_D to the case $r = \infty$, by letting $h_{D(y,\infty)} : \bar{H} \to \bar{H}$ be the homeomorphism given on H by sending x into $x - y$. Then h_D and its inverse h_D^{-1} depends continuously on $(y,r) \in H\times(0,\infty]$.

Let $g_i(u)$ and $\Theta_i\Phi g(u)$ denote the i'th components of $g(u)$ and $\Theta\Phi g(u)$ respectively for $u \in S^n$. By compactness of S^n we can find a neighborhood U of $0 \in H$ such that $g(S^n) \subseteq \tilde{D}^s_U(H)$, and we choose an $\varepsilon > 0$ with $D(0,2\varepsilon) \subseteq U$. Now we define two homotopies

$$K, K' : S^n \times [\varepsilon,\infty] \to \tilde{D}^s(H)$$

as follows:

$$K(u,t) = (\dots, g_i(u)h^{-1}_{D(0,t)}, \dots)$$

$$K'(u,t) = (\dots, \Theta_i\Phi g(u)h^{-1}_{D(0,t)}, \dots).$$

We have $r(\Phi g(u)) \geqq 2\varepsilon$, from which we see that the two homotopies map into $\tilde{D}^s_V(H)$, where $V = \bar{D}(0,\frac{\varepsilon}{2})$. The continuity of K and K' is now obvious. Since

$$K(u,\varepsilon) = K'(u,\varepsilon)$$

for every $u \in S^n$, we can combine K and K' to a homotopy K'' from g to a map g' into $\tilde{P}^s(H)$. However this does not fix the basepoint $*$. Using the assumption that $g(*) \in \tilde{P}^s(H)$, we see that $K(*,t) = K'(*,t)$ for $t \in [\varepsilon,\infty]$, so that the homotopy K'' sends $*$ along a contractible loop. The homotopy can therefore be changed to a new homotopy rel. $*$ from g to g'.

Note that θ factors to give a map

$$P^s(H) \to D^s(H)$$

also denoted by θ. This is a right inverse to the map of Lemma 6.7, and is therefore also a weak homotopy equivalence.

<u>Lemma 6.8</u> For a Euclidean space H the map

$$D^s(H) \to F^s(H)$$

induces a homology isomorphism in dimensions $< \min(n-1, \frac{s+1}{2})$, where $n = \dim H$.

<u>Proof</u> We can assume that $H = R^n$. Then the composite

$$P^s(H) \xrightarrow{\theta} D^s(H) \longrightarrow F^s(H)$$

can be identified with the map ϕ of Lemma 3.3. Since θ is a weak homotopy equivalence, the result follows from that

lemma.

<u>Lemma 6.9</u> The natural transformation $D^S \to P^S$ induces a homotopy equivalence $B\mathcal{J}(D^S) \to B\mathcal{J}(P^S)$.

<u>Proof</u> It follows from Lemma 6.7 that the map $D^S(R^\infty) \to P^S(R^\infty)$ is a weak homotopy equivalence using convergence of D^S and P^S. The result follows now from Theorem 4.6

We have now shown that the first three continuous permutative functors in the sequence

$$\mathcal{E}^S_{\otimes} \leftarrow \mathcal{M}^S \to \mathcal{J}(P^S) \leftarrow \mathcal{J}(D^S) \to \mathcal{J}(F^S)$$

induce homotopy equivalences of classifying spaces. It follows from the group completion theorem (1.3) that they induce homotopy equivalences of spectra. Hence to prove Theorem 6.2 only the lemma below remains.

<u>Lemma 6.10</u> The natural transformation $D^S \to F^S$ induces a homotopy equivalence of spectra.

<u>Proof</u> By Proposition 4.8 it suffices to show that the homomorphism

$$\varinjlim H_*(D^S(R^\infty)) \to \varinjlim H_*(F^S(R^\infty))$$

is an isomorphism. From Lemma 6.8 it is seen that the map $D^s(R^\infty) \to F^s(R^\infty)$ induces a homology isomorphism in dimensions $< \frac{s+1}{2}$. The assertion is now obvious (remember that S contains some integers > 1).

Appendix This appendix contains some simple results on weak homotopy equivalences of simplicial sets and homotopy equivalences of geometric realizations of simplicial spaces. First we have a comparison theorem for bisimplicial maps (in the case of bisimplicial groups it follows from Quillen [13]).

Theorem A.1 Let X_{**} and Y_{**} be bisimplicial sets and $f: X_{**} \to Y_{**}$ a bisimplicial map. Assume that the simplicial map $f: X_{*q} \to Y_{*q}$ is a weak homotopy equivalence for every $q \geqq 0$. Then the simplicial map

$$\text{diag } f: \text{diag } X_{**} \to \text{diag } Y_{**}$$

is a weak homotopy equivalence.

Notice that this theorem generalizes to multisimplicial sets by iterated application. Theorem A.1 has been proved by Bousfield and Kan using mapping "spaces" and cosimplicial methods ([4]). We give another proof based on geometric realization.

Recall that the geometric realization $FX_* = |X_*|$ of a simplicial space X_* is defined to be the quotient space

$$\left(\coprod_{n=0}^{\infty} X_n \times \Delta^n\right)/\sim$$

where Δ^n is the standard n-simplex and $\sim$ the familiar equivalence relation (Milnor [10], Segal [15]). In case the spaces X_n are discrete this is the usual geometric realization of a simplicial set as defined by Milnor [10].

If X_{**} is a bisimplicial set we can take the horizontal geometric realization $F_h X_{**}$ with $F(X_{*q})$ in degree q. The vertical simplicial structure on X_{**} induces on $F_h X_{**}$ the structure of a simplicial space. We can then take the vertical geometric realization $F_v F_h X_{**}$. The following is well known to the workers in the field:

<u>Proposition A.2</u> There is a natural homeomorphism

$$F \operatorname{diag} X_{**} \to F_v F_h X_{**}$$

defined for every bisimplicial set (or space) X_{**}.

Let it suffice to remark that this generalizes Theorem 2 of Milnor [10], and the proof given there generalizes in a straightforward way.

For a simplicial space X_* we write

$$sX_{n-1} = \coprod_{i=0}^{n-1} s_i X_{n-1}.$$

This is the space of degenerate n-simplices. We say that X_* is cofibered (resp. weakly cofibered) if the inclusion $sX_{n-1} \to X_n$ is a cofibration (resp. weak cofibration) for every $n \geq 1$. The concept "weak cofibration" is dealt with in [7] under the name "h-Cofaserung".

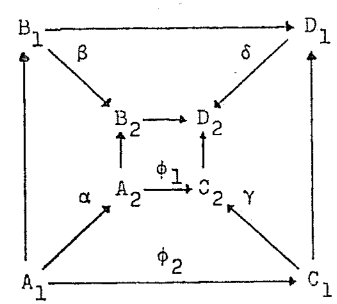
If X_{**} is a bisimplicial set, then $F_h X_{**}$ is a cofibered simplicial space, since $F_h X_{*n}$ is a CW-complex and the subspace of degenerate n-simplices a subcomplex.

The assumption of Theorem A.1 means that the simplicial map $F_h X_{**} \to F_h Y_{**}$ is a homotopy equivalence in each degree. Theorem A.1 is therefore a consequence of Prop. A.2 and the following.

<u>Theorem A.3</u> Let $f: X_* \to Y_*$ be a continuous simplicial map of weakly cofibered simplicial spaces, such that $f_n: X_n \to Y_n$ is a homotopy equivalence for every n. Then the induced map $Ff: FX_* \to FY_*$ is a homotopy equivalence.

This was proved under some additional assumptions in May [9], §11. The proof given below follows May's proof in outline. However, we have replaced his use of Mayer-Vietoris sequences by the following gluing lemma of R. Brown [5].

<u>Lemma A.4</u> Consider a commutative diagram of spaces



where the squares are cocartesian and ϕ_1 and ϕ_2 are weak cofibrations. If α, β and γ are homotopy equivalences, so is δ.

Proof of Theorem A.3 For $0 \leqq k \leqq n-1$ define the subspace $s^kX_{n-1} \subseteq sX_{n-1}$ by

$$s^kX_{n-1} = \bigcup_{i=0}^{k} s_iX_{n-1},$$

and similarly for Y. We will prove by induction on n the following statement:

(S_n) In the following diagram the horizontal maps are weak cofibrations, and the vertical maps are homotopy equivalences:

$$(1) \qquad \begin{array}{ccccccccc} s^0X_n & \to & s^1X_n & \to & \cdots & \to & s^{n-1}X_n & \to & s^nX_n \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ s^0Y_n & \to & s^1Y_n & \to & \cdots & \to & s^{n-1}Y_n & \to & s^nY_n \end{array} .$$

Since s_i maps X_n homeomorphically onto s_iX_n for all i and n, the map $s_iX_n \to s_iY_n$ is a homotopy equivalence. In particular $s^0X_n \to s^0Y_n$ is a homotopy equivalence for all n. This proves (S_0), and we assume (S_{n-1}). For $k = 1,\ldots,n$ we have a cocartesian square

$$(2) \qquad \begin{array}{ccc} s^{k-1}X_n & \longrightarrow & s^kX_n \\ \uparrow & & \uparrow \\ s^{k-1}X_n \cap s_kX_n & \longrightarrow & s_kX_n \end{array}$$

and a commutative square with homeomorphisms vertically induced by s_k

$$(3) \qquad \begin{array}{ccc} s^{k-1}X_n \cap s_kX_n & \longrightarrow & s_kX_n \\ \uparrow & & \uparrow \\ s^{k-1}X_{n-1} & \longrightarrow & X_n \end{array} .$$

By (S_{n-1}) the bottom map of (3) and hence the bottom map of (2) is a weak cofibration. Since (2) is cocartesian its upper map is a weak cofibration. This shows that the horizontal maps of (1) are weak cofibration. Suppose now that f induces a homotopy equivalence on $s^{k-1}X_n$. By (S_{n-1}) f induces a homotopy equivalence on $s^{k-1}X_{n-1}$, and hence on $s^{k-1}X_n \cap s_kX_n$. Lemma A.4 can now be applied to the map from (2) into the analogue for Y, and we see that $s^kX_n \to s^kY_n$ is a homotopy equivalence. This proves (S_n) by induction on k. In particular f induces homotopy equivalences $sX_{n-1} \to sY_{n-1}$ $(n \geqq 1)$.

Lemma A.4 can now be applied to the cocartesian square

$$\begin{array}{ccc} sX_{n-1}\times\Delta^n & \longrightarrow & sX_{n-1}\times\Delta^n \cup X_n\times\dot{\Delta}^n \\ \uparrow & & \uparrow \\ sX_{n-1}\times\dot{\Delta}^n & \longrightarrow & X_n\times\dot{\Delta}^n \end{array}$$

and the analogue for Y, to get a homotopy equivalence

$$sX_{n-1}\times\Delta^n \cup X_n\times\dot{\Delta}^n \to sY_{n-1}\times\Delta^n \cup Y_n\times\dot{\Delta}^n.$$

Let $F_nX_* \subseteq FX_*$ be the image of $\coprod_{i=0}^{n} X_i\times\Delta_i$ under the identification map $\coprod_{i=0}^{\infty} X_i\times\Delta^i \to FX_*$. We have a cocartesian square

$$(4)\qquad \begin{array}{ccc} F_{n-1}X_* & \longrightarrow & F_nX_* \\ \uparrow & & \uparrow \\ sX_{n-1}\times\Delta^n \cup X_n\times\dot{\Delta}^n & \longrightarrow & X_n\times\Delta^n \end{array}$$

where the bottom map is a weak cofibration. Hence the upper map is a weak cofibration. Since $F_0X_* = X_0$ we have a homotopy equivalence $F_0X_* \to F_0Y_*$. Lemma A.4 can now be applied inductively to (4) and the analogue for Y, and we conclude that f induces homotopy equivalences

$$F_nX_* \to F_nY_* \quad (n \geq 0).$$

Finally since $FX_* = \lim\limits_{\rightarrow} F_nX_*$ and the maps of the direct limit system are weak cofibrations, the induced map

$$FX_* \to FY_*$$

is a homotopy equivalence.

An easy consequence is the following:

Corollary A.5 Let X_* be a weakly cofibered simplicial space. If every X_n has the homotopy type of a CW-complex, then the geometric realization FX_* has the homotopy type of a CW-complex.

Proof Let $\tilde{X}_n$ be the geometric realization of the singular complex ΔX_n of X_n. The assumption implies that the canonical map $\tilde{X}_n \to X_n$ is a homotopy equivalence. Theorem A.3 can now be applied to the continuous simplicial map $\tilde{X}_* \to X_*$, and we get a homotopy equivalence $F\tilde{X}_* \to FX_*$. By Prop. A.2 $F\tilde{X}_*$ is homeomorphic to $F(\mathrm{diag}\ \Delta X_*)$, which is a CW-complex.

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