# André-Quillen (Co-)homology for Simplicial Algebras over Simplicial Operads

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July 10, 1998

In his work on homotopical algebra [19], Quillen was led to consider the homotopy theory and homology of simplicial modules over simplicial algebras. These problems arise naturally in the context of the homotopical algebra of $R$-algebras over some commutative ring $R$, as a cofibrant model (or resolution) of any $R$-algebra is a non-constant simplicial $R$-algebra. Since operads are to algebras what algebras are to modules, one might expect to encounter non-constant simplicial operads and, then, one must consider the homotopy theory of simplicial algebras over simplicial operads. We are especially interested in the André-Quillen homology of such simplicial algebras, as these are the groups that arise in the context of stable homotopy theory.

In fact, we were led to consider the simplicial operads because of the following computational difficulty that arises in homotopy theory. If $\mathcal{F}$ is an $E_\infty$ operad over the linear isometries operad $\mathcal{L}$ – for example, $\mathcal{F}$ could be $\mathcal{L}$ itself, which is the case in [6] – and $X$ is a spectrum, one would like to calculate with the free $\mathcal{F}$-algebra $\mathcal{F}(X)$. In particular, if $E$ is some generalized homology theory, one would like to compute $E_*\mathcal{F}(X)$. In general, this is very difficult, as it requires, at the very least, a thorough knowledge of the $E_*$ Dyer-Lashof algebra. However, it is possible to resolve $\mathcal{F}$ by a simplicial operad $T$ over $\mathcal{L}$ so that if $E_*$ has a Künneth spectral sequence and $E_*X$ is projective as an $E_*$ module, then the simplicial object $E_*T(X)$ is a free algebra over the simplicial operad $E_*T$ on $E_*X$. This makes computations accessible.

As this last paragraph indicates, we will not only be interested in simplicial operads in the category of $R$ modules for some commutative ring $R$ (e.g., $E_*$), but also in simplicial operads in the category of $\Gamma$ comodules where $(R, \Gamma)$ is some Hopf algebroid over $R$ (e.g., $(E_*, E_*E)$). Because this category of comodules doesn't have enough projectives, the homotopical algebra in this situation is a bit tantalizing, and requires care. This is the point of several of the side trips in the body of this paper. See sections 3 and 4, especially.

This paper is the second in a series of three papers designed to come to terms with existence of multiplicative structures on spectra and with the homotopy types of spaces of multiplicative maps between multiplicative spectra. The other two ([11],[12]) are more

*The authors were partially supported by the National Science Foundation.

homotopy theoretic in nature; this paper is something of an algebraic intermezzo. This paper, and, indeed, all of the work of these papers should be compared with the work of Robinson and Whitehouse; see [21] and [22]

Surprisingly, the theory of André and Quillen adapts in a fairly straightforward manner to define the homology and cohomology of simplicial algebras over a simplicial operad. This is mostly because most of the familiar algebraic notions of ring theory – modules, derivations, Kähler differentials and the like – have close generalizations in the operadic setting. The point of sections 1 and 2 is to explain this, and we suspect that there is very little new here.

There is, however, one significant difficulty in adapting the André-Quillen cohomology of commutative rings; namely, we can only prove an attenuated form of the transitivity sequence. The root of the problem is that, in general, the natural closed model category structure on the category of simplicial algebras over a simplicial operad is not left proper; that is, given a push-out diagram of simplicial algebras

$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ {\scriptstyle j}\downarrow & & \downarrow \\ B & \xrightarrow[g]{} & Y \end{array}$$

with $f$ a weak equivalence and $j$ a cofibration, it is not necessarily true that $g$ is a weak equivalence. This can be seen even in the case of non-commutative $R$-algebras over some commutative ring $R$ which is not a field: see Example 6.3 below. A great deal of the technical work of this paper is spent in defining a weaker notion of left proper, showing that this weaker notion is good enough for our applications, and then showing that the category of algebras over a *cofibrant* simplicial operad satisfy this notion. This naturally leads to a discussion of the homotopical algebra of simplicial operads (just as Quillen was lead to simplicial rings). All of this takes place in sections 5 and 6, and it crucial for section 7, which displays a Hurewicz theorem, and section 8, which discusses Postnikov towers. Both of these sections are standard in ideas and proofs, but both are crucial for homotopy theoretic applications. There is an appendix on free operads, mostly derived from [7], which spells out some of the grubbier internal workings necessary for our proofs in section 6.

Section 8 focuses on computational techniques of $E_\infty$ operads. Since an $E_\infty$ operad is a resolution of the commutative algebra operad, it's not surprising that the homology of algebras over an $E_\infty$ operad can be computed with ordinary André-Quillen homology as input.

**Contents**

# 1 Algebraic Preliminaries for Operads

Here we review the basic definitions of operads, algebras over operads, modules, and the like. Our primary purpose is to establish language; none of this material is new, and can certainly be found in standard sources (e.g, citeGK, [15]).

We fix an abelian category $\mathcal{C}$ with all limits and colimits equipped with a symmetric monoidal product

$$\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

We will assume the symmetric monoidal structure is closed in the sense there is a functor $\hom : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ so that

$$\mathrm{Hom}_{\mathcal{C}}(M \otimes N, K) \cong \mathrm{Hom}_{\mathcal{C}}(M, \hom(N, K)).$$

The first two examples we have in mind are the category of $R$ modules, where $R$ is some commutative ring and the category of ($\Gamma$-comodules where $(R, \Gamma)$ is a Hopf algebroid ([20] §A.1). In the latter case, the symmetric monoidal structure sends

$$(M, N) \mapsto M \otimes_R N.$$

We now let $\mathcal{C}_\Sigma$ be the category with objects $T = \{T_n\}_{n \geq 0}$ where $T_n \in \mathcal{C}$ is equipped with a $\Sigma_n$ action in $\mathcal{C}$. The morphisms in $\mathcal{C}_\Sigma$ are the obvious ones. This is the category of $\Sigma$ objects in $\mathcal{C}$. The category $\mathcal{C}_\Sigma$ inherits a pairing

$$\cdot \otimes_\Sigma \cdot : \mathcal{C}_\Sigma \times \mathcal{C}_\Sigma \to \mathcal{C}$$

from the symmetric monoidal structure in $\mathcal{C}$ with the formula

$$S \otimes_\Sigma T = \bigoplus_{n \geq 0} S_n \otimes_{\Sigma_n} T_n. \tag{1.1}$$

Here $S_n \otimes_{\Sigma_n} T_n$ denotes the colimit of the diagonal $\Sigma_n$ action on $S_n \otimes T_n$. Note that for fixed $T \in \mathcal{C}_\Sigma$, the functor $\cdot \otimes_\Sigma T : \mathcal{C}_\Sigma \to \mathcal{C}$ has right adjoint

$$M \to \{\hom(T_n, M)\}_{n \geq 0} = \hom_*(T, M)$$

where $\Sigma_n$ acts on the source of $\hom(T_n, M)$.

Here are some common uses of these constructions.

**Example 1.2** Let $M \in \mathcal{C}$. Define $\Phi(M) \in \mathcal{C}_\Sigma$ by the formula

$$\Phi(M)_n = M \underset{\leftarrow n \rightarrow}{\otimes \cdots \otimes} M = M^{\otimes n}$$

with the permutation action. If $T \in \mathcal{C}_\Sigma$ we write

$$T(M) = T \otimes_\Sigma \Phi(M) = \bigoplus_n T_n \otimes_{\Sigma_n} M^{\otimes n}.$$

**Example 1.3** Let $X$ and $M$ be in $\mathcal{C}$. Define $\Psi(X, M) \in \mathcal{C}_\Sigma$ by

$$\Psi(X, M)_n = \bigoplus_{i=1}^{n} X \otimes \cdots \otimes \underset{i}{M} \otimes \cdots \otimes X$$

again with the permutation action. If $T \in \mathcal{C}_\Sigma$ we write

$$\begin{aligned} T(X, M) &= T \otimes_\Sigma \Psi(X, M) \\ &\cong \bigoplus_n T_n \otimes_{\Sigma_{n-1}} X^{\otimes n-1} \otimes M. \end{aligned}$$

This last formula suggests we introduce an operator on $\mathcal{C}_\Sigma$

(1.4) $$D_* : \mathcal{C}_\Sigma \to \mathcal{C}_\Sigma$$

with $(D_*T)_n = T_{n+1}$ with $\Sigma_n$ action induced by $\Sigma_n \subseteq \Sigma_{n+1}$ fixing $n+1 \in \{1, 2, \ldots, n, n+1\}$. Then

$$T(X, M) = D_*T(X) \otimes M.$$

**Example 1.5** Let $T \in \mathcal{C}_\Sigma$ and let $\mathcal{C}^{\Sigma_m}$ denote the category of $\Sigma_m$ objects in $\mathcal{C}$. We define an object $T(m) \in \mathcal{C}_\Sigma^{\Sigma_m}$; thus $T(m) = \{T(m)_n\}$ and each $T(m)_n$ is a $\Sigma_m$ object equipped with a $\Sigma_n$ action that commutes with the $\Sigma_m$ action. Let $R_{n,m}$ be the set of $n$-tuples $(m_1, \ldots, m_n)$ with $m_i \geq 0$ and $m_1 + \cdots + m_n = m$. Define

$$T(m)_n = \bigoplus_{R_{n,m}} \Sigma_m \times_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_n}} T_{m_1} \otimes \cdots \otimes T_{m_n}.$$

Here we write, for a subgroup $H \subseteq \Sigma_m$,

$$\Sigma_m \times_H (\cdot) : \mathcal{C}^H \to \mathcal{C}^{\Sigma_m}$$

for the functor left adjoint to the forgetful functor. If $\sigma \in \Sigma_n$ the map switching factors extends uniquely to a $\Sigma_m$ equivariant map

$$\sigma_* : \Sigma_m \times_{\Sigma_{m_1} \times \cdots \times \Sigma_{m_n}} B_{m_1} \times \cdots \times B_{m_n} \to \Sigma_m \times_{\Sigma_{m_{\sigma(1)}} \times \cdots \times \Sigma_{m_{\sigma(n)}}} B_{m_{\sigma(1)}} \times \cdots \times B_{m_{\sigma(n)}}$$

and gives $B(m)$ the structure of an object in $\mathcal{C}_\Sigma^{\Sigma_m}$.

If $S \in \mathcal{C}_\Sigma$ and $T \in \mathcal{C}_\Sigma^{\Sigma_m}$, then

$$S \otimes_\Sigma T = \bigoplus_n S_n \otimes_{\Sigma_n} T_n$$

is an object in $\mathcal{C}^{\Sigma_m}$

**Definition 1.6** *The circle product* $(\cdot) \circ (\cdot) : \mathcal{C}_\Sigma \times \mathcal{C}_\Sigma \to \mathcal{C}_\Sigma$ *is defined by the formula*

$$(S \circ T)_m = S \otimes_\Sigma T(m).$$

**Proposition 1.7** *With the circle product, the category* $\mathcal{C}_\Sigma$ *becomes a closed monoidal category. The unit is given by* $\mathbf{1} \in \mathcal{C}_\Sigma$ *where*

$$\mathbf{1}_n = \begin{cases} e & n = 1 \\ 0 & n \neq 1 \end{cases}$$

*with* $e \in \mathcal{C}$ *the unit of the symmetric monoidal structure on* $\mathcal{C}$. *The internal function object of* $\mathcal{C}_\Sigma$ *is given by the formula*

$$\hom_{\mathcal{C}_\Sigma}(S, T) = \prod_{m \geq 0} \hom_{\Sigma_m}(S(m), T_m). \qquad \blacksquare$$

Note that this monoidal structure is not a symmetric monoidal structure; in particular $S \circ T$ need not be isomorphic to $T \circ S$.

**Proposition 1.8** *Let* $S, T \in \mathcal{C}_\Sigma$ *and* $M, N \in \mathcal{C}$. *Then there are natural isomorphisms*

1) $S(T(M)) \cong (S \circ T)M$, *and*

2) $S(T(M), T(M, N)) \cong S \circ T(M, N)$.

**Proof.** By are proved by regrouping parentheses. $\blacksquare$

**Definition 1.9** *An* operad *in* $\mathcal{C}$ *is a monoid object* $T \in \mathcal{C}_\Sigma$ *in the monoidal structure given by the circle product.*

Thus if $T$ is an operad, there are morphisms

$$\mu : T \circ T \to T$$

and $\eta : \mathbf{1} \to T$ in $\mathcal{C}_\Sigma$ so that the evident associativity and unitary diagrams commute. Let $\mathcal{O}$ be the category of operads on $\mathcal{C}$.

If $T$ is an operad, then the functor

$$M \mapsto T(M)$$

becomes the functor of a triple (i.e., monad) of $\mathcal{C}$ and a $T$-*algebra* is an algebra over this triple. In general, we will write $\mathcal{C}^T$ for the category of $T$-algebras in $\mathcal{C}$; however, if $\mathcal{C} = \mathcal{M}_R$ is the category of modules over a commutative ring $R$, then we will write $\mathcal{A}^T$ for the category of $T$-algebras.

**Example 1.10** If $M \in \mathcal{C}$, then the object

$$\mathrm{End}(M) = \{\hom(M^{\otimes n}, M)\}_{n \geq 0}$$

becomes an operad in $\mathcal{C}$, called the endomorphism operad of $M$. If $T$ is any operad, then the set of $T$-algebra structures on $M$ is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}}(T, \mathrm{End}(M)).$$

**Definition 1.11** *Let $T$ be an operad and $A$ a $T$-algebra. An object $M \in \mathcal{C}$ is a $T$-module over $A$ if there is a morphism in $\mathcal{C}$*

$$\epsilon_M : T(A, M) \to M$$

*so that the following diagrams commute:*

$$\begin{tikzcd}
T(TA, T(A,M)) \arrow[r, "T(s_A, \epsilon_M)"] \arrow[dd, "\cong"] & T(A,M) \arrow[d, "\epsilon_M"] \\
 & M \\
(T \circ T)(A,M) \arrow[r, "\mu(A,M)"] & T(A,M) \arrow[u, "\epsilon_M"']
\end{tikzcd}$$

*and*

$$\begin{tikzcd}
M = \mathbf{1}(A,M) \arrow[r] \arrow[dr, "="'] & T(A,M) \arrow[d] \\
 & M.
\end{tikzcd}$$

The first diagram is an associativity condition, and uses Proposition 1.8. The second is a unital condition.

If we fix $T$ and $A$, let $\mathcal{M}_A^T$ denote the $T$-modules over $A$. The summation map $\Psi(A, A) \to \Phi(A)$ defines a composite

$$T(A, A) \to T(A) \xrightarrow{\epsilon_A} A$$

making $A \in \mathcal{M}_A^T$. Also, if $f : A \to B$ is a morphism of $T$-algebras, and $M$ is a $T$-module over $B$, then $M$ is a $T$-module over $A$ via the composite

$$T(A, M) \xrightarrow{T(f,M)} T(B, M) \to M.$$

Before going further, it will be useful to lay out the concepts of various kinds of special coequalizers. A *reflexive coequalizer* is a coequalizer diagram

$$M_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} M_0$$

equipped with a morphism $s_0 : M_0 \to M_1$ so that $d_0 s_0 = d_1 s_0 = 1$. We will write $d_0$ for the map

$$d_0 : M_0 \to \mathrm{colim}(M_1 \rightrightarrows M_0).$$

In the closed symmetric monoidal category $\mathcal{C}$, the product $\otimes$ preserves reflexive coequalizers in the strongest possible sense: if

$$M_{1i} \rightrightarrows M_{0i} \to M_i, \qquad i = 1, 2$$

are reflexive coequalizers then so is

$$M_{11} \otimes M_{12} \rightrightarrows M_{01} \otimes M_{02} \to M_1 \otimes M_2.$$

To see this note that the functors $M_1 \otimes \cdot$ and $\cdot \otimes M_2$ both preserve all coequalizers, being left adjoints, so one need only check the coequalizer of $M_{11} \otimes M_{12} \rightrightarrows M_{01} \otimes M_{02}$ is the colimit of the diagram

$$\begin{tikzcd} M_{11} \otimes M_{12} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M_{11} \otimes M_{02} \arrow[d, shift left] \arrow[d, shift right] \\ M_{01} \otimes M_{12} \arrow[r, shift left] \arrow[r, shift right] & M_{01} \otimes M_{02}. \end{tikzcd}$$

To do this, note that any map $f : M_{01} \otimes M_{02} \to N$ which equalizes $d_0 \otimes d_0$, $d_1 \otimes d_1$ : $M_{11} \otimes M_{12} \rightrightarrows M_{01} \otimes M_{02}$ must also equalize

$$1 \otimes d_0, 1 \otimes d_1 : M_{01} \otimes M_{12} \rightrightarrows M_{01} \otimes M_{12}$$

and

$$d_0 \otimes 1, d_1 \otimes 1 : M_{11} \otimes M_{02} \rightrightarrows M_{01} \otimes M_{12},$$

since, for example,

$$f(1 \otimes d_0) = f(d_0 \otimes d_0)(s_0 \otimes 1) = f(d_1 \otimes d_1)(s_0 \otimes 1) = f(1 \otimes d_1),$$

where $s_0 : M_{01} \to M_{11}$ is any splitting.

Because $\otimes$ preserves reflexive coequalizers, so does

$$T(\cdot) : \mathcal{C} \to \mathcal{C}$$

where $T \in \mathcal{C}_\Sigma$ is any object. From this comes the following observation. Let $T$ is an operad and let

$$d_0, d_1 : A_1 \rightrightarrows A_0$$

be two maps in $\mathcal{C}^T$ that can be split by $s_0 : A_0 \to A_1$ in $\mathcal{C}$. Then if $A \in \mathcal{C}$ is the coequalizer $d_0$ and $d_1$ in $\mathcal{C}$, there is a unique map $T(A) \to A$ filling in the following diagram

$$\begin{tikzcd} T(A_1) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & T(A)_0 \arrow[r] \arrow[d] & T(A) \arrow[d] \\ A_1 \arrow[r, shift left] \arrow[r, shift right] & A_0 \arrow[r] & A. \end{tikzcd}$$

This gives $A$ the structure of a $T$-algebra and with this structure, $A$ is the coequalizer of $d_0, d_1$ in $\mathcal{C}^T$. In shorthand: the forgetful functor $\mathcal{C}^T \to \mathcal{C}$ makes reflexive coequalizers. It follows from this that $\mathcal{C}^T$ has all colimits. Since the forgetful functor makes limits, $\mathcal{C}^T$ has a limits and colimits.

If $A \in \mathcal{C}^T$ is a $T$-algebra, the diagram,

$$(T \circ T)A \overset{\epsilon_{T_A}}{\underset{T_{\epsilon_A}}{\rightrightarrows}} T(A) \overset{\epsilon_A}{\to} A \tag{1.12}$$

is a replexive coequalizer diagram with $s_0 = T_{\eta_A} : TA \to (T \circ T)(A)$. Write $d_0 = \epsilon_{TA}$, $d_1 = T\epsilon_A$ and $d_0 = \epsilon_A$. Notice that in $\mathcal{C}$, this diagram comes equipped with $s_{-1} = \eta : A \to TA$ and $s_{-1} : \eta : TA \to T \circ T(A)$ so that $d_0 s_{-1} = 1$ and $d_1 s_{-1} = s_{-1} d_0$. It follows that (1.12) is an equalizer diagram in $\mathcal{C}$, hence in $\mathcal{C}^T$, since it is split.

We now list a technical result needed below.

**Lemma 1.13** *Let $A_1 \rightrightarrows A_0 \to A$ be a reflexive coequalizer of $T$-algebras, and let $M_1 \rightrightarrows M_0 \to M$ be a reflexive coequalizer in $\mathcal{C}$. Suppose $M_i$, $i = 1, 2$, is $T$-module over $A_i$, $i = 1, 2$, $d_0, d_1 : M_1 \to M_0$ are $A_1$ modules maps and $s_0 : M_0 \to M_1$ is an $A_0$-module map. Then $M$ has a unique structure as a $T$-module over $A$ so that $d_0 : M_0 \to M$ is an $A_0$ module map.*

**Proof.** The functor $T(\cdot, \cdot)$ preserves reflexive coequalizers in the strong sense of both variables at once; hence there is a unique map $T(A, M) \to M$ in $\mathcal{C}$ making the following diagram commute.

$$\begin{tikzcd} T(A_1, M_1) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & T(A_0, M_0) \arrow[r] \arrow[d] & T(A, M) \arrow[d] \\ M_1 \arrow[r, shift left] \arrow[r, shift right] & M_0 \arrow[r] & M. \end{tikzcd}$$ ∎

**Proposition 1.14** *The forgetful functor $\mathcal{M}_A^T \to \mathcal{C}$ has a left adjoint*

$$M \mapsto A \overset{T}{\otimes} M.$$

**Proof.** First assume $A = T(N)$ for some $N$. Then

$$A \overset{T}{\otimes} M \cong T(N, M)$$

with $T$-module structure over $A$ defined by

$$T(A, A \overset{T}{\otimes} M) \cong (T \circ T)(N, M) \to T(N, M) = A \overset{T}{\otimes} M.$$

The natural map, with $K \in \mathcal{M}_T^A$

$$\mathrm{Hom}_{\mathcal{C}}(M, K) \to \mathrm{Hom}_{\mathcal{M}_A^T}(A \overset{T}{\otimes} M, K)$$

sending $f : M \to K$ to the composite

$$A \overset{T}{\otimes} M = T(N, M) \to T(T(N), M) \xrightarrow{T(A,f)} T(A, K) \to K$$

is an isomorphism with inverse sending $g : A \overset{T}{\otimes} M \to K$ to

$$M = \mathbf{1}(N, M) \to A \overset{T}{\otimes} M \xrightarrow{g} K.$$

Note that if $f : A_1 = T(N_1) \to A_2 = T(N_2)$ is any morphism of $T$-algebras, the natural map $M \to A_2 \overset{T}{\otimes} M$ and the fact that $f$ gives $A_2 \overset{T}{\otimes} M$ the structure of an $A_1$ module yields, by adjointness, a map of $T$-modules over $A_1$

$$f \overset{T}{\otimes} M : A_1 \overset{T}{\otimes} M \to A_2 \overset{T}{\otimes} M.$$

To define $A \overset{T}{\otimes} M$ for more general $M$, we use the reflexive coequalizer diagram of 1.12 of $T$-algebras

$$T^2 A \underset{d_1}{\overset{d_0}{\rightrightarrows}} T(A) \xrightarrow{d_0} A$$

and define $A \overset{T}{\otimes} M$ by the coequalizer diagram in $\mathcal{C}$

$$T^2(A) \overset{T}{\otimes} M \rightrightarrows T(A) \overset{T}{\otimes} M \to A \overset{T}{\otimes} M. \tag{1.15}$$

Lemma 1.13 supplies the $A$-module structure on $A \overset{T}{\otimes} M$.

To get the adjointness property note that under the situation described by the hypotheses of Lemma 1.13, there is a split equalizer diagram of abelian groups

$$\mathrm{Hom}_{\mathcal{M}_A^T}(M, N) \to \mathrm{Hom}_{\mathcal{M}_{A_0}^T}(M_0, N) \rightrightarrows \mathrm{Hom}_{\mathcal{M}_{A_1}^T}(M_1, N)$$

for any $N \in \mathcal{M}_A^T$. Apply this to the sequence of 1.15. ∎

**Lemma 1.16** *The forgetful functor $\mathcal{M}_A^T \to \mathcal{C}$ makes all limits and colimits, and $\mathcal{M}_A^T$ is an abelian subcategory of $\mathcal{C}$.*

**Proof.** Because of the equation $T(A,M) \cong D_*T(A) \otimes M$, the functor $T(A,\cdot)$ preserves all colimits $\mathcal{C}$. Hence, if $M : I \to \mathcal{M}_A^T$ is any diagram, the colimit in $\mathcal{C}$, $\operatorname{colim}_I M$, acquires a module structure via

$$T(A, \operatorname{colim}_I M) = \operatorname{colim}_I T(A,M) \to \operatorname{colim}_I M$$

with their structure, $\operatorname{colim}_I M \in \mathcal{M}_T^A$ becomes the colimit in $\mathcal{M}_T^A$. The case of limits is easier.

Since the direct sum in $\mathcal{M}_T^A$ of two objects $M_1, M_2 \in \mathcal{M}_T^A$ is made in $\mathcal{C}$, $\mathcal{M}_T^A$ is an additive subcategory. Since cokernels and kernels are, respectively, a colimit and limit, the result follows from the fact that colimits and limits are made in $\mathcal{C}$. ∎

**Lemma 1.17** *The category $\mathcal{M}_A^T$ is equivalent to the category of algebras over a triple with underlying functor*

$$A \overset{T}{\otimes} (\cdot) : \mathcal{C} \to \mathcal{C}.$$

**Proof.** This follows from Lemma 1.16 and Beck's Theorem. ([16], VI §7). ∎

It is standard to note that $\mathcal{M}_A^T$ is, in fact, the category of some left modules over some algebra in $\mathcal{C}$. This is proved by the following sequence of ideas. First note that if $K \in \mathcal{M}_A^T$ and $M \in \mathcal{C}$, then the mapping object $\hom(M,K) \in \mathcal{C}$ has a natural structure as an object of $\mathcal{M}_A^T$; the evaluation formula

$$T(A, \hom(M,K)) \to \hom(M,K)$$

is adjoint to the composite

$$T(A, \hom(M,K)) \otimes M \cong D_*T \otimes \hom(M,K) \otimes M \to D_*T \otimes K \cong T(A,K) \to K.$$

Similarly, $K \otimes M \in \mathcal{M}_A^T$ with evaluation map

$$T(A, K \otimes M) \cong T(A,K) \otimes M \to K \otimes M.$$

**Lemma 1.19** *There is a natural adjunction isomorphism, $M, K \in \mathcal{M}_A^T$, $N \in \mathcal{C}$*

$$\operatorname{Hom}_{\mathcal{M}_A^T}(M \otimes N, K) \cong \operatorname{Hom}_{\mathcal{M}_A^T}(M, \hom(N,K)).$$

**Proof.** This is a result of the comparison of equalizers

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{M}_A^T}(M \otimes N, K) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{C}}(M \otimes N, K) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \operatorname{Hom}_{\mathcal{C}}(T(A, M \otimes N), K) \arrow[d, "\cong"] \\
\operatorname{Hom}_{\mathcal{M}_A^T}(M, \hom(N,K)) \arrow[r] & \operatorname{Hom}_{\mathcal{C}}(M, \hom(N,K)) \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Hom}_{\mathcal{C}}(T(M), \hom_{\mathcal{C}}(N,K)).
\end{tikzcd}$$

∎

An immediate consequence of this result is that the natural map of $T$-modules over $A$

$$A \overset{T}{\otimes} (M \otimes N) \to (A \overset{T}{\otimes} M) \otimes N \tag{1.20}$$

is an isomorphism, as

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{M}_A^T}((A \overset{T}{\otimes} M) \otimes N, K) &\cong \mathrm{Hom}_{\mathcal{M}_A^T}(A \overset{T}{\otimes} M, \mathrm{hom}(N, K)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(M, \mathrm{hom}(N, K)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(M \otimes N, K) \\
&\cong \mathrm{Hom}_{\mathcal{M}_A^T}(A \overset{T}{\otimes} (M \otimes N), K).
\end{aligned}$$

Thus the object $A \overset{T}{\otimes} e$ gets a multiplication in $\mathcal{C}$

$$(A \overset{T}{\otimes} e) \otimes (A \overset{T}{\otimes} e) \cong A \overset{T}{\otimes} A \overset{T}{\otimes} e \to A \overset{T}{\otimes} e,$$

using the fact that $A$ is a $T$ module over itself. This multiplication is associative and has a unit $e \to A \overset{T}{\otimes} e$. This object is the enveloping algebra of $A$. If $M$ is a $T$-module over $A$, then $M$ is naturally a left $A \overset{T}{\otimes} e$ module via

$$(A \overset{T}{\otimes} e) \otimes M \cong A \overset{T}{\otimes} (e \otimes M) \cong A \overset{T}{\otimes} M \to M.$$

With this assignment of left $A \overset{T}{\otimes} e$ module structure we have

**Proposition 1.21** *The category $\mathcal{M}_A^T$ of $T$-modules over $A$ is equivalent to the category of left $A \overset{T}{\otimes} e$ modules.*

**Proof.** If $M$ is $A \overset{T}{\otimes} e$ module, then $M$ becomes an algebra over the triple $M \to A \overset{T}{\otimes} M$ via $A \overset{T}{\otimes} M \cong (A \overset{T}{\otimes} e) \otimes M \to M$. The result follows from Lemma 1.17. ∎

We next give a result allowing extension of scalars and change of operads. We have already noted that if $A \to B$ is a morphism of $T$-algebras we get a functor $\mathcal{M}_B^T \to \mathcal{M}_A^T$ which might be called restriction of scalars. See after Definition 1.11. Similarly if $S \to T$ is a morphism of operads, any $T$-algebra $A$ becomes an $S$-algebra via

$$S(A) \to T(A) \xrightarrow{\epsilon} A.$$

Furthermore if $M$ is a $T$-module over $A$, then $M$ is an $S$-module over $A$ via

$$S(A, M) \to T(A, M) \to M.$$

Thus we obtain functors $\mathcal{C}^T \to \mathcal{C}^S$ and $\mathcal{M}_A^T \to \mathcal{M}_A^S$.

**Lemma 1.22** *1) If $A \to B$ is a morphism of $T$-algebras, the restriction of scalars functor $\mathcal{M}_B^T \to \mathcal{M}_A^T$ has a left adjoint $B \otimes_A (\cdot) : \mathcal{M}_A^T \to \mathcal{M}_B^T$ characterized by the natural isomorphism*

$$B \otimes_A (A \overset{T}{\otimes} M) \cong B \overset{T}{\otimes} M.$$

*2) If $S \to T$ is a morphism of operads the forgetful functor $\mathcal{C}^T \to \mathcal{C}^S$ has a left adjoint $T \otimes_S (\cdot)$ characterized by*

$$T \otimes_S S(M) \cong T(M).$$

*3) If $S \to T$ is a morphism of operads and $A$ is a $T$-algebra, the functor $\mathcal{M}_A^T \to \mathcal{M}_A^S$ has left adjoint $T \otimes_S (\cdot)$ characterized by*

$$T \otimes_S (A \overset{S}{\otimes} M) \cong A \overset{T}{\otimes} M.$$

**Proof.** All three follow from [1] ■

# 2 Derivations, Differentials, and André-Quillen (Co-)homology.

André-Quillen cohomology is obtained as the right derived functors of derivations. We introduce and explore the relevant concepts. For fixed $M \in \mathcal{C}$, there is a natural diagonal map

$$M^{\otimes n} = \Phi(M)_n \to \Psi(M, M)_n = \bigoplus_n M^{\otimes n}$$

and hence, for any operad $T$, a diagonal map

$$\Delta : T(M) \to T(M, M).$$

**Definition 2.1** *Let $A$ be a $T$-algebra and $M$ a $T$-module over $A$. Then a $T$-derivation over $A$ is a morphism of $T$-modules over $A$*

$$d : A \to M$$

*so that the following diagram commutes*

$$\begin{tikzcd} T(A) \arrow[r, "\Delta"] \arrow[d, "\epsilon_A"'] & T(A,A) \arrow[r, "T(A{,}d)"] & T(A,M) \arrow[d, "\epsilon_M"] \\ A \arrow[rr, "d"] & & M. \end{tikzcd}$$

Write $\mathrm{Der}_T(A, M)$ for the set of derivatives. Since the functor $T(A, \cdot)$ is additive, $\mathrm{Der}_T(A, M)$ is a subgroup of $\mathrm{Hom}_{\mathcal{M}_A^T}(A, M)$.

We next produce Kähler differentials. For fixed $A \in \mathcal{C}^T$, the Kähler differentials on $A$ is a module $\Omega_T A \in \mathcal{M}_A^T$ representing the functor $M \mapsto \mathrm{Der}_T(A, M)$. The next result shows this object exists.

**Lemma 2.2** *Let $A$ be a $T$-algebra. Then there is a module $\Omega_T A \in \mathcal{M}_A^T$ and a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{M}_A^T}(\Omega_T A, M) \cong \mathrm{Der}_T(A, M).$$

**Proof.** First assume $A = T(N)$ for some $N \in \mathcal{C}$. Then the map, for $M \in \mathcal{M}_A^T$

$$\mathrm{Hom}_{\mathcal{C}}(N, M) \to \mathrm{Der}_T(A, M)$$

sending $f : N \to M$ to the composite

$$A = T(N) \xrightarrow{\Delta} T(N, N) \xrightarrow{T(\eta, f)} T(T(N), M) \xrightarrow{\epsilon} M.$$

is an isomorphism. Hence we have

$$\Omega_T(N) \cong T(N) \overset{T}{\otimes} N \cong T(N, N). \tag{2.3}$$

If $T(N_1) \to T(N_2)$ is any morphism of $T$-algebras, we get a natural homomorphism for $M \in \mathcal{M}_{T(N_2)}^T$

$$\mathrm{Der}_T(T(N_2), M) \to \mathrm{Der}_T(T(N_1), M)$$

hence a morphism of $T(N_2)$ modules

$$T(N_2) \otimes_{T(N_1)} \Omega_T T(N_1) \to \Omega_T T(N_2).$$

For general $A \in \mathcal{C}^T$, there is a coequalizer diagram of $T$-algebras

$$T^2(A) \rightrightarrows T(A) \to A$$

and we define $\Omega_T(A)$ by the coequalizer diagram

$$A \otimes_{T^2 A} \Omega_T T^2 A \rightrightarrows A \otimes_{TA} \Omega_T TA \to \Omega_T A. \qquad \blacksquare$$

Implicit in the preceding proof is the following result.

**Lemma 2.4** *Let $A \to B$ be a morphism of $T$-algebras. Then there is an induced map of $T$-modules over $B$*

$$B \otimes_A \Omega_T A \to \Omega_T B$$

*inducing the natural (in $\mathcal{M}_B^T$) map*

$$\mathrm{Der}_T(B, M) \to \mathrm{Der}_T(A, M). \qquad \blacksquare$$

Also, if $S \to T$ is a morphism of operads we get, for any $T$-algebra $A$, and $M \in \mathcal{M}^T_A$, a map

$$\mathrm{Der}_T(A, M) \to \mathrm{Der}_S(A, M)$$

which is, in turn, induced by a map $T$-modules over $A$

$$T \otimes_S \Omega_S A \to \Omega_T A. \tag{2.5}$$

Note that if $S(A) \to T(A)$ is surjective, $\mathrm{Der}_T(A, M) \to \mathrm{Der}_S(A, M)$ is an isomorphism; hence $T \otimes_S \Omega_S A \cong \Omega_T A$.

We now come to simplicial $T$-algebras. If $T$ is an operad in $\mathcal{C}$, we can form the category $s\mathcal{C}^T$ of simplicial $T$-algebras; however, for our applications, we need something more general. If $T = \{T_n\}$ is a simplicial operad and $X \in s\mathcal{C}$, then one can form the object

$$T(X) = \{T_n(X_n)\} \in s\mathcal{C}$$

and a simplicial operad over $T$ is an object $X \in s\mathcal{C}$ equipped with a multiplication $\epsilon : T(X) \to X$ so that the usual diagrams commute. Write $s\mathcal{C}^T$ for the category of simplicial $T$-algebras. If $T$ is any operad $\mathcal{C}$, we may regard $T$ as a constant simplicial operad and the two definitions of $s\mathcal{C}^T$ agree.

In a similar vein, if $X \in s\mathcal{C}^T$ a simplicial $T$-module over $X$ is an object $M \in s\mathcal{C}$ equipped with a multiplication in $s\mathcal{C}$

$$\epsilon : T(X, M) \to M$$

subject to the usual diagrams.

Finally, as a bit of notation, an augmented simplicial object $Z$ is a simplicial object $Z = \{Z_n\}$ equipped with a morphism $d_0 : Z_0 \to Z_{-1}$ so that $d_0 d_1 = d_0 d_0 : Z_1 \to Z_{-1}$. With this condition, all possible compositions of face maps

$$d_{i_0} d_{i_1} \cdots d_{i_n} : Z_n \to Z_{-1}$$

are equal. The coequalizer of the two maps $d_0, d_1 : Z_1 \to Z_0$, if it exists, is denoted $\pi_0 Z$ and the obvious map $d_0 : Z_0 \to \pi_0 Z$ is initial among all augmentations.

In particular, if $T \to T_{-1}$ is an augmented simplicial operad, $X \to X_{-1}$ are algebra over over $T \to T_{-1}$ and $M$ a $T_{-1}$ module over $X_{-1}$, then $M$ has a unique induced $T_n$ module structure over $X_n$.

The following assumption simplifies the homological algebra around André-Quillen homology.

**Assumption 2.6** The category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$, with the standard simplicial structure (cofibrantly generated by $s$-free morphisms) ([19], §II.2) has a simplicial model category structure so that $X \to Y$ is a weak equivalence if and only if

$$\pi_* X \to \pi_* Y$$

is an isomorphism of graded objects in $\mathcal{C}$. If $T$ is a simplicial operad over $\mathcal{C}$ then the category $(s\mathcal{C})^T$ of simplicial $T$-algebras, with its usual simplicial model category structure ([19], §II.1) inherits a simplicial model category structure from $s\mathcal{C}$ in the sense that $f : X \to Y$ is a weak equivalence (or fibration) if and only if $f$ is a weak equivalence (or fibration) in $s\mathcal{C}$.

The assumption will hold both in the case of $\mathcal{C} = \mathcal{M}_R$, the category of $R$-modules and $\mathcal{C} = \mathcal{C}_\Gamma$, the category of $\Gamma$-comodules where $(R, \Gamma)$ is a decent Hopf algebroid. The latter assertion is the topic of the next two sections. We will assume this holds for the rest of this section.

We make some comments on the model category structure on $s\mathcal{C}^T$. Since $s\mathcal{C}$ is cofibrantly generated, a morphism $A \to X$ in $s\mathcal{C}^T$ can be factored

$$A \xrightarrow{j} Z \xrightarrow{q} X$$

where $Z = \operatorname{colim}_{\alpha<\kappa} Z_\alpha$ for some ordinal $\kappa$ and $Z_0 = A$, $Z_\beta = \operatorname{colim}_{\alpha<\beta} Z_\alpha$ if $\beta < \mathcal{K}$ is a limit ordinal, and there are push-out diagrams

$$\begin{tikzcd} \coprod T(C_i) \arrow[r] \arrow[d] & Z_\alpha \arrow[d] \\ \coprod T(D_i) \arrow[r] & Z_{\alpha+1}. \end{tikzcd}$$

The $C_i \to D_i$ are the generating trivial cofibrations and the coproduct is over all squares in $s\mathcal{C}$

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] & Z_\alpha \arrow[d] \\ D_i \arrow[r] & X. \end{tikzcd}$$

If $\kappa$ is large enough $j : A \to Z$ has the left lifting property with respect to all fibrations in $s\mathcal{C}^T$; hence $j$ is a trivial cofibration. If the morphisms $C_i \to D_i$ are $s$-free, then $A \to Z$ is also $s$-free in $(s\mathcal{C})^T$. Indeed the underlying degeneracy diagram is isomorphic to

$$A \to A \sqcup T(Z^0)$$

for some free degeneracy diagram $Z^0$ in $\mathcal{C}$. We call such a morphism *free.*

Similar remarks apply to $s\mathcal{M}_A^T$ using the adjoint

$$A \otimes^T (\cdot) : s\mathcal{C} \to s\mathcal{M}_A^T$$

to the forgetful functor.

**Definition 2.7** *Let $A = s\mathcal{C}^T$ be a simplicial $T$-algebra. The cotangent complex of $A$ is defined by the equation*

$$\mathbf{L}\Omega_T A \cong A \otimes_X \Omega_T X$$

*where $X \to A$ is a trivial fibration in $(s\mathcal{C})^T$ with $X$ cofibrant.*

*The André-Quillen homology of $A$ with coefficients in $M$ is defined by*

$$D_*(A) \cong \pi_* \mathbf{L}\Omega_T A.$$

*If $T \to T_{-1}$ is an augmented simplicial operad, $A \to A_{-1}$ is an augmented $T \to T_{-1}$ algebra and $M$ a $T_{-1}$ module over $A_{-1}$, then the André-Quillen cohomology of $A$ with coefficients in $M$ is defined by*

$$D^*(A, M) \cong \pi^* \operatorname{Der}_T(X, M) \cong \pi^* \operatorname{Hom}_{\mathcal{M}_A^T}(\mathbf{L}\Omega_T A, M). \qquad \blacksquare$$

The definition of $D^*(A, M)$ will be extended to more general modules below.

We make some remarks on the definition. If we assume $X$ is free, then we may write the underlying degeneracy diagram of $X$ as $T(Z^0)$ where $Z^0$ is some free degeneracy diagram in $\mathcal{C}$, and then the underlying degeneracy diagram of $A \otimes_X \Omega_T X = \mathbf{L}\Omega_T A$ is

$$A \otimes_X \Omega_T X \cong A \overset{T}{\bigotimes} Z^0$$

which is, in turn, free in $s\mathcal{M}_A^T$.

Since every cofibrant object $X$ in $(s\mathcal{C})^T$ is a retract of a free object, $\Omega_T X$ is a retract of a free object in $s\mathcal{M}_X^T$; hence $A \otimes_X \Omega_T X$ is cofibrant in $s\mathcal{M}_A^T$. It is also well-defined up to homotopy equivalence. Thus $D_*(A, M)$ is well-defined. Also, since

$$\operatorname{Der}_T(X, M) \cong \operatorname{Hom}_{s\mathcal{M}_{A_{-1}}^T}(A_{-1} \otimes_T \Omega_T X, M)$$

we often get a composite functor spectral sequence for computing $D^*(A, M)$. See Lemma 7.5 below.

The next point to make is that André-Quillen cohomology is representable. For this, we need to have the notion of a "square-zero" extension. If $T$ is an operad, $A$ a $T$-algebra, and $M$ a $T$-module over $A$, we need to define the square-zero extension of $A$ by $M$. Note that if $\Phi(A)$ and $\Psi(A, M)$ are the functors of Examples 1.1 and 1.2, then, in $\mathcal{C}_\Sigma$, there is a decomposition,

$$\Phi(A \oplus M) \cong \Phi(A) \oplus \Psi(A, M) \oplus L(A, M)$$

hence a projection $T(A \oplus M) \to T(A) \oplus T(A, M)$.

The $T$-algebra $M \rtimes A$ is the object $A \oplus M$ with algebra structure

$$T(A \oplus M) \to T(A) \oplus T(A, M) \xrightarrow{\epsilon_A \oplus \epsilon_M} A \oplus M$$

where $\epsilon$ denotes the evaluation map.

Now let $X \in \mathcal{C}^T/A$ be a $T$-algebra over $A$. We leave it as an exercise to show there is a natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}^T/A}(X, A \rtimes M) \cong \operatorname{Der}_T(X, M) \cong \operatorname{Hom}_{\mathcal{M}_A^T}(A \otimes_X \Omega X, M) \tag{2.8}$$

defined as follows: if $f : X \to A$ is the given $T$-algebra map any $T$-algebra map $g : X \to A \rtimes M$ in $\mathcal{C}^T/A$ may written as a composite

$$X \stackrel{\Delta}{\to} X \oplus X \xrightarrow{f \oplus d} A \oplus M$$

and the isomorphism sends $g$ to $d$.

Now let $A$ be a simplicial algebra over a simplicial operad $T$. If $M$ is a simplicial $T$-module over $A$ one can form the simplicial algebra $A \rtimes M$. If $M$ is a $\pi_0 T$-module over $\pi_0 A$ we may form a simplicial $A$-module $K(M,n)$. As a simplicial object in $\mathcal{C}$, $NK(M,n) \cong M$ concentrated in degree $n$ (where $N$ is the normalization functor) and $K(M,n)_k$ acquires a $T_k$ module over $A_k$ structure via the augmentation. Write $K_A(M,n)$ for $K(M,n) \rtimes A$.

**Proposition 2.10** *Let $Y \in s\mathcal{A}^T/A$. Then there is a natural isomorphism*

$$[Y, K_A(M,n)]_{s\mathcal{A}^T/A} \cong D^n(Y,M).$$

**Proof.** By (2.8) the functor $(\cdot) \rtimes A : s\mathcal{M}_A^T \to s\mathcal{A}^T/A$ has left adjoint $Y \mapsto A \otimes_Y \Omega_T Y$. Note $(\cdot) \rtimes A$ preserves all weak equivalences and fibrations. Hence $A \otimes_Y \Omega_T(\cdot)$ preserves cofibrations and weak equivalences between cofibrant objects and

$$\begin{aligned} [Y, K_A(M,n)]_{s\mathcal{A}^T/A} &\cong [\mathbf{L}\Omega_T Y, K(M,n)]_{s\mathcal{M}_A^T} \\ &\cong D^n(Y,M). \end{aligned}$$ ∎

This result allows us to expand the definition of André-Quillen homology. Let $M \in s\mathcal{M}_A^T$. Since $s\mathcal{M}_A^T$ is a simplicial category, we may construct $\Delta^n/\partial\Delta^n \otimes M$. Define $\Delta^n/\partial\Delta^n M$ to be the cokernel of

$$* \otimes M \to \Delta^n/\partial\Delta^n \otimes M.$$

If $M$ is a $\pi_0 T$ module over $\pi_0 A$ regarded as a constant object in $s\mathcal{M}_A^T$, then $\Delta^n/\partial\Delta^n M \cong K(M,n)$. Now define the André-Quillen cohomology in $Y \in sC^T/A$ by

$$D^n(Y,M) \cong [Y, \Delta^n/\partial\Delta^n M \rtimes A]_{se^T/A} i \cong [\mathbf{L}\Omega_T Y, \Sigma^n M]_{s\mathcal{M}_A^T}. \tag{2.11}$$

For computational purposes one has the transitivity sequence and related ideas. For this one needs a relative version of the above concepts. Suppose $A \to X$ is a morphism of simplicial $T$-algebras and $M$ is a $\pi_0 T$-module over $\pi_0 X$. We define the derivations from $X$ to $M$ over $A$ by the exact sequence

$$0 \to \mathrm{Der}_A(X,M) \to \mathrm{Der}_T(X,M) \to \mathrm{Der}_T(A,M).$$

The functor $M \mapsto \mathrm{Der}_A(X,M)$ is representable; indeed, if we define $\Omega_T(X/A) \in s\mathcal{M}_A^T$ by the exact sequence in $s\mathcal{M}_A^T$

$$X \otimes_A \Omega_T A \to \Omega_T X \to \Omega_T(X/A) \to 0$$

then there is a natural isomorphism

$$\mathrm{Hom}_{s\mathcal{M}_A^T}(\Omega_T(X/A), M) \cong \mathrm{Der}_A(X, M).$$

Then $\mathbf{L}\Omega_T(X/A)$ is defined as follows. Choose a cofibrant model $A_0 \to A$ for $A$ in $(s\mathcal{C})^T$ and factor the composite $A_0 \to A \to X$ as $A_0 \xrightarrow{j} X_0 \xrightarrow{q} X$ where $j$ is a cofibration and $q$ is a weak equivalence. Then

$$\mathbf{L}\Omega_T(X/A) = X \otimes_{X_0} \Omega(X_0/A_0).$$

The arrow $A_0 \to X_0$ is a cofibrant model for the arrow $A \to X$ is the arrow category of $(s\mathcal{C})^T$ with the Bousfield-Kan model category structure: level-wise weak equivalences and fibrations. Since every object in this arrow category is fibrant, $\mathbf{L}\Omega_T(X/A)$ is well-defined in the homotopy category of simplicial $X$-modules.

One could also attempt to define $\mathbf{L}\Omega_T(X/A)$ without recourse to resolving $A$; that is, factor $A \to X$ as $A \xrightarrow{j'} Y \xrightarrow{q} X$ where $q$ is a weak equivalence and $j$ is cofibration, and consider $\mathbf{L}'\Omega_T(X/A) = X \otimes_Y \Omega_T(Y/A)$. This is again well-defined in $\mathbf{Ho}(s\mathcal{M}_X^T)$; however, this object may not be weakly equivalent to $\mathbf{L}\Omega_T(X/A)$. We have the following result.

**Proposition 2.12** *There is a natural map in* $\mathbf{Ho}(s\mathcal{M}_X^T)$

$$\mathbf{L}\Omega_T(X/A) \to \mathbf{L}'\Omega_T(X/A)$$

*and this map is an isomorphism in* $\mathbf{Ho}(s\mathcal{M}_X^T)$ *if the simplicial model category* $s\mathcal{C}^T$ *is left proper.*

**Proof.** We use the notation above. There is a sequence of arrows

$$A \xrightarrow{i} A \sqcup_{A_0} X_0 \xrightarrow{p} X$$

and $i$ is a cofibration. If $s\mathcal{C}^T$ is left proper, then $p$ is a weak equivalence. In any case, factor $p$ as

$$X_0 \sqcup_{A_0} A \xrightarrow{f} Y \xrightarrow{g} X$$

where $f$ is a cofibration and $g$ is a weak equivalence. Then we have morphisms

$$\begin{aligned} \mathbf{L}\Omega_T(X/A) = X \otimes_{X_0} \Omega_T(X_0/A_0) &\to X \otimes^T_{(X_0 \sqcup_{A_0} A)} \Omega_T(X_0 \sqcup_{A_0} A/A) \\ &\to X \otimes_Y \Omega_T(Y/A) \simeq \mathbf{L}'\Omega_T(X/A). \end{aligned}$$

If $s\mathcal{C}^T$ is left proper, we may choose $f$ to be the identity; hence, the second map is an isomorphism in $\mathbf{Ho}(s\mathcal{M}_X^T)$. Also

$$\Omega_T(X_0 \sqcup_{A_0} A/A) \cong (X_0 \sqcup_{A_0} A) \otimes^T_{X_0} \Omega_T(X_0/A_0),$$

so the first map may be rewritten

$$X \otimes_{X_0} \Omega_T(X_0/A_0) \to X \otimes_{(X_0 \sqcup_{A_0} A)} (X_0 \sqcup_A A) \otimes_{X_0} \Omega_1(X_0/A_0),$$

which is an isomorphism. ■

We now come to the transitivity sequence.

**Proposition 2.13** *Let $A \to B \to C$ be a sequence of morphisms in $(se)^T$. Then there is a homotopy cofiber sequence in $s\mathcal{M}_C^T$.*

$$C \otimes_B \mathbf{L}\Omega_T(B/A) \to \mathbf{L}\Omega_T(C/A) \to \mathbf{L}\Omega_T(C/B).$$

**Proof.** Choose a free model $A_0 \to A$ of $A$ in $s\mathcal{C}^T$, factor the composition $A_0 \to A \to B$ as $A_0 \to B_0 \to B$ where the first map is free and the second a weak equivalence, then factor the composition $B_0 \to B \to C$ as $B_0 \to C_0 \to C$ where the first map is free and the second a weak equivalence. We thus obtain a diagram

$$\begin{tikzcd} A_0 \arrow[r, tail] \arrow[d] & B_0 \arrow[r, tail] \arrow[d] & C_0 \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

where the vertical maps a weak equivalence, $A_0$ is free and the arrows labeled $\rightarrowtail$ are free. We will show that the resulting composition of simplicial $C$-modules

$$C \otimes_{B_0} \Omega_T(B_0/A_0) \to C \otimes_{C_0} \Omega_T(C_0/A_0) \to C \otimes_{C_0} \Omega_T(C_0/B_0) \tag{2.14}$$

is level-wise split short exact. The result will follow.

For a fixed integer $n \geq 0$, $(A_0)_n \cong T_n(V_n)$ for some $V_n \in \mathcal{C}$,

$$(B_0)_n \cong (A_0)_n \sqcup T_n(W_n)$$

for some $W_n \in \mathcal{C}$ and

$$(C_0)_n \cong (B_0)_n \sqcup T_n(Z_n)$$

for some $Z_n \in \mathcal{C}$. It follows that

$$\Omega_T(B_0/A_0)_n \cong B_0 \overset{T_n}{\otimes} W_n$$

$$\Omega_T(C_0/A_0)_n \cong C_0 \overset{T_n}{\otimes} (W_n \oplus Z_n)$$

and

$$\Omega_T(C_0/B_0)_n \cong C_0 \overset{T_n}{\otimes} (Z_n).$$

Thus, at level $n$, (2.14) becomes

$$0 \to C \overset{T_n}{\otimes} W_n \to C \overset{T_n}{\otimes} (W_n \oplus Z_n) \to C \overset{T_n}{\otimes} (Z_n) \to 0.$$

■

**Remark 2.15** If $s\mathcal{C}^T$ is proper, one can replace $\mathbf{L}\Omega_T(X/A)$ by $\mathbf{L}'_T\Omega(X/A)$. This is the case if $T$ is the commutative algebra operad, as in Quillen's original argument. Note that the spectral sequence Quillen invokes at the end of his proof implies that that category of simplicial commutative algebras is proper.

A similar proof leads to the following result.

**Proposition 2.16** (Meyer-Vietoris) *Let $A$ be a fixed object in $s\mathcal{C}^T$ and*

$$Y_1 \leftarrow X \rightarrow Y_2$$

*be a two source of objects under $A$. If $Z$ is the homotopy pushout then there is a homotopy cofiber sequence in $s\mathcal{M}_Z^T$*

$$Z \otimes_X \mathbf{L}\Omega_T(X/A) \rightarrow Z \otimes_{Y_1} \mathbf{L}\Omega_T(Y_1/A) \oplus Z \otimes_{Y_2} \mathbf{L}\Omega_T(Y_2/A) \rightarrow \mathbf{L}\Omega_T(Z/A).$$

**Proof.** There are no new ideas in the proof. ∎

Again, if $s\mathcal{C}^T$ is proper both the homotopy pushout and the objects $\mathbf{L}\Omega_T(X/A)$, etc., simplify considerably.

# 3 Comodules over a Hopf algebroid.

This section and the next are devoted to proving that under a standard hypotheses (Assumption 3.1 below), comodules over a Hopf algebroid fit the rubric of the previous two sections, so that one has good homotopy theoretic properties for algebras over operads and related categories. One begins with a $\mathbb{Z}$ graded commutative ring $R$, which is typically the coefficient ring of some homology theory. If $(R, \Gamma)$ is a Hopf algebroid and $\Gamma$ is a flat $R$ module, then the category $\mathcal{C}_\Gamma$ of left comodules over $\Gamma$ is abelian. However, it doesn't have enough projectives, so one must think a bit to get a good closed model category structure on $s\mathcal{C}_\Gamma$. This we do in this section.

It is extremely useful to place the following restriction on $(R, \Gamma)$.

**Assumption 3.1** There is a filtered system $\{\Gamma_\alpha\}$ of left $\Gamma$ comodules so that each $\Gamma_\alpha$ is a finitely generated projective $R$-module and there is an isomorphism in $\mathcal{C}_\Gamma$:

$$\operatorname*{colim}_\alpha \Gamma_\alpha \cong \Gamma.$$

**Example 3.2** Let $E_*$ be a Landweber exact homology theory. Then $(E_*, E_*E)$ is a Hopf algebroid and the representing spectrum satisfies Adams's conditions that there be a universal coefficient spectral sequence: if $E$ is the representing spectrum that there is a weak equivalence

$$\operatorname*{colim}_\alpha E_\alpha \rightarrow E$$

so that $E_*DE_\alpha$ is projective as an $E_*$ module, and for all $E$ module spectra $F$, the Künneth map

$$\mathrm{Hom}^*_{E_*}(E_*DE_\alpha, F_*) \cong F^*DE_\alpha.$$

In particular, $E_{-*}DE_\alpha \cong E^*DE_\alpha \cong \mathrm{Hom}^*_{E_*}(E_*DE_\alpha, E_*)$ is a finitely generated projective $E_*$ module. For if $\oplus\Sigma^i E_* \to E_*DE_\alpha$ is a surjection, it splits as an $E_*$ module, so

$$\mathrm{Hom}_{E_*}(E_*^*DE_\alpha, E_*) \to \mathrm{Hom}_{E_*}(\bigoplus_i \Sigma^i E_*, E_*) \cong \oplus\Sigma^{-i}E_*$$

splits as an $E_*$ module. Both $E^*DE_\alpha$ and $E_*DE_\alpha$ are necessarily finitely generated since $DE_\alpha$ is weakly equivalent to a finite CW spectrum.

Note also that $E_*E$ is a projective $E_*$ module, by Landweber's Theorem. ■

There is also a notion of a right comodule; that is, a graded $R$-module $M$ with coproduct

$$\psi : M \to M \otimes_R \Gamma.$$

The two categories are equivalent: if $M$ is a right $\Gamma$-comodule, we define a left comodule structure of $\Gamma$ via the composition, called $\psi^*_M$,

$$M \xrightarrow{\psi_M} M \otimes_R \Gamma \xrightarrow{t} \Gamma \otimes_R M \xrightarrow{c\otimes 1} \Gamma \otimes M$$

where $t$ is the graded sorted map $t(a \otimes b) = (-1)^{|a||b|} b \otimes a$ and $c$ is the conjugation of $\Gamma$. The resulting functor from right comodules to left comodules has an obvious inverse.

It is worth pointing out that $\Gamma$ is both a left and right $\Gamma$-comodule via $\Delta : \Gamma \to \Gamma \otimes_R \Gamma$ and these two structures are different under the obvious correspondence; indeed, the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[d, "c"'] & \Gamma \otimes_R \Gamma \arrow[r, "t"] & \Gamma \otimes_R \Gamma \arrow[d, "c\otimes c"] \\
\Gamma \arrow[rr, "\Delta"] & & \Gamma \otimes_R \Gamma
\end{tikzcd}$$

shows that $c : \Gamma \to \Gamma$ defines an isomorphism between $\Gamma$ with the right comodule structure obtained from its left comodule structure and $\Gamma$ with the right comodule structure obtained from $\Delta$.

Also note that if $i_\alpha : \Gamma_\alpha \to \Gamma$ are the maps of assumption 3.1, then $c \circ i_\alpha : \Gamma_\alpha \to \Gamma$ define a system of right comodule maps so that $\mathrm{colim}_\alpha \Gamma_\alpha \cong \Gamma$ as right comodules.

For the next construction we suppose $M \in \mathcal{C}_\Gamma$ is a left comodule and finitely generated and projective as an $R$-module. We define a comodule $DM$, called the dual of $M \in \mathcal{C}_\Gamma$, as follows. As an $R$-module

$$(DM)_k = \mathrm{Hom}_R(M_{-k}, R)$$

or, succinctly, $DM = \mathrm{Hom}^*_R(M, R)$. This $R$-module has a right $\Gamma$-comodule structure via

$$\mathrm{Hom}^*_R(M, R) \to \mathrm{Hom}^*_R(M, \Gamma) \xleftarrow{\cong} \mathrm{Hom}^*_R(M, R) \otimes_R \Gamma$$

where the first map sends $\phi$ to $(\phi \otimes 1) \circ \psi_M$ and the second sends $\phi \otimes x$ to $\phi(\cdot)x$. This last map is an isomorphism because $M$ is finitely generated and projective. Let $DM \in \mathcal{C}_\Gamma$ be the associated left comodule. Note that since $M$ is a retract of $\bigoplus_i R_{*+n_i}$ for some finite set $\{i\}$, $DM$ is a retract of $\bigoplus_i R_{*-n_i}$; hence, $DM$ is finitely generated and projective as an $R$-module. The following results lists the properties of this construction.

**Lemma 3.3** *Let $M \in \mathcal{C}_\Gamma$ be finitely generated and projective of $R$.*

1) *The natural map $M \to D(DM)$ is an isomorphism.*

2) *Let $N_1, N_2 \in \mathcal{C}_\Gamma$. Then the adjoint isomorphism*

$$\mathrm{Hom}_R(N_1 \otimes M, N_2) \cong \mathrm{Hom}_R(N_1, \mathrm{Hom}^*_R(M, R) \otimes N_2)$$

*restricts to an isomorphism*

$$\mathrm{Hom}_\Gamma(N_1 \otimes M, N_2) \cong \mathrm{Hom}_\Gamma(N_1, DM \otimes N_2). \qquad \blacksquare$$

We belabored these ideas so that we could prove the following result. The proof is essentially the same as [18]. Thanks to Ethan Devinatz for pointing this out.

**Lemma 3.4** *The left comodules $\Sigma^n D\Gamma_\alpha$, $n \in \mathbb{Z}$, generate the category $\mathcal{C}_\Gamma$. In particular, every left comodule is the filtered colimit of its sub-comodules which are finitely generated over $R$.*

**Proof.** Let $M \in \mathcal{C}_\Gamma$ and let $x \in M$ have degree $n$. We show there is a map $g : \Sigma^n D\Gamma_\alpha \to M$ in $\mathcal{C}_\Gamma$ for some $n$ so that $x$ is in the image. Both claims follow.

Consider $\Gamma \otimes_R M$ with the diagonal comodule action

$$\Gamma \otimes_R M \xrightarrow{\Delta\otimes\psi_M} \Gamma \otimes_R \Gamma \otimes_R \Gamma \otimes_R M \xrightarrow{1\otimes t\otimes 1} \Gamma \otimes_R \Gamma \otimes_R \Gamma \otimes_R M \xrightarrow{m\otimes 1} \Gamma \otimes_R \Gamma \otimes_R M.$$

This is isomorphic to the extended $\Gamma$ comodule $\Gamma \otimes_R M$ with action

$$\Gamma \otimes_R M \xrightarrow{\Delta\otimes 1} \Gamma \otimes_R \Gamma \otimes_R M;$$

in fact, the isomorphism is given by

$$\Gamma \otimes_R M \xrightarrow{1\otimes\psi_M} \Gamma \otimes_R \Gamma \otimes_R M \xrightarrow{m\otimes 1} \Gamma \otimes_R M.$$

Hence (with the diagonal action)

$$M_n \cong \mathrm{Hom}_\Gamma(\Sigma^n R, \Gamma \otimes_R M) \cong \operatorname*{colim}_\alpha \mathrm{Hom}_\Gamma(\Sigma^n R, \Gamma_\alpha \otimes M).$$

Let $f : \Sigma^n R \to \Gamma_\alpha \otimes M$ map to $x$ under this isomorphism and let

$$g : \Sigma^n R \otimes D\Gamma_\alpha = \Sigma^n D\Gamma_\alpha \to M$$

be the adjoint guaranteed by Lemma 3.3. $\blacksquare$

We use the generators $\Sigma^n D\Gamma_\alpha \in \mathcal{C}_\Gamma$ to build a simplicial model category on the category of simplicial objects $s\mathcal{C}_\Gamma$. This category is naturally a simplicial category by ([19] §II.2).

**Definition 3.5** *Define a morphism $f : X \to Y$ in $s\mathcal{C}_\Gamma$ to be a weak equivalence (or fibration) if and only if*

$$f_* : \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X) \to \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, Y)$$

*is a weak equivalence (or fibration) of simplicial sets for all $(n, \alpha)$. Define $f : X \to Y$ to be a cofibration if and only if it has the left lifting property with respect to all trivial fibrations.*

**Remark 3.6** Since the object $\Sigma^n D\Gamma_\alpha$ is constant

$$\mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X)_m \cong \mathrm{Hom}_{\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X_m).$$

In particular, $\mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X)$ is a simplicial abelian group. Thus every object in $s\mathcal{C}_\Gamma$ is fibrant. Also note that

$$\begin{aligned} \operatorname*{colim}_\alpha \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X) &\cong \operatorname*{colim}_\alpha \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n R, \Gamma_\alpha \otimes_R X) \\ &\cong \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n R, \Gamma \otimes_R X) \cong X_n. \end{aligned}$$

Hence, if $f : X \to Y$ is a weak equivalence, then

$$\pi_* f : \pi_* X \to \pi_* Y$$

is an isomorphism of graded objects in $\mathcal{C}_\Gamma$. The converse is not true however.

**Proposition 3.7** *With the definitions of 3.5, the category $s\mathcal{C}_\Gamma$ becomes a simplicial model category.*

**Proof.** This follows from the arguments of ([19] §II.4) using the fact that every object is fibrant. ■

**Remark 3.8** One can also characterize cofibrations. If $X \in s\mathcal{C}_\Gamma$, the normalization $NX$ is a chain complex of left $\Gamma$-comodules. A morphism $f : X \to Y$ is a cofibration if it is a retract of a *free map* and a free map if $Nf : NX_m \to NY_m$ is the inclusion of a summand in $\mathcal{C}_\Gamma$ and there is an isomorphism in $\mathcal{C}_\Gamma$

$$NY_m / NX_m \cong \bigoplus_A \Sigma^{ni} D\Gamma_{\alpha_i}$$

where $A$ runs over some indexing set of pairs $(n_i, \alpha_i)$. We refer to a map of chain complexes as “free”, also. ■

Because of the type of weak equivalences that arise in this model category structure (see Remark 3.6), these results are somewhat unsatisfactory. Hence:

**Definition 3.9** *Define a morphism $f : X \to Y$ to be*

*1) a $\pi_*$-weak equivalence if $\pi_* f : \pi_* X \to \pi_* Y$ is an isomorphism;*

*2) a $\pi_*$-cofibration if $f$ is a cofibration;*

*3) a $\pi_*$-fibration if $f$ has the right lifting property with respect to all morphisms which are at once cofibrations and $\pi_*$ weak equivalences.*

**Proposition 3.10** *With these definitions, and its usual simplicial structure, $s\mathcal{C}_\Gamma$ becomes a simplicial model category.*

**Proof.** The argument goes back to [2], §11, and is now standard: it is a kind of localization. We won't give all the details, but merely supply the crucial lemma, which follows. ■

Let $\lambda$ be an infinite regular cardinal greater than the cardinality of $\bigoplus_n R_n$. If $N$ is a simplicial $R$-module, let $\#N$ be the cardinality of $\oplus(N_n)_m$, with $n$ the internal degree and $m$ the simplicial degree.

**Lemma 3.11** *Let $f : A \to B$ be a $\pi_*$-trivial cofibration in $s\mathcal{C}_\Gamma$, and let $x \in B$. Then there is a sub-simplicial comodule $B' \subseteq B$ so that $A \cap B' \to B'$ is a $\pi_*$-trivial cofibration, $x \in B'$, and $\#B' \leq \lambda$.*

**Proof.** We may, by Remark 3.8, assume $f$ is free. We will then prove the following stronger assertion about chain complexes. Suppose $C \to D$ is an $\mathbb{N}$-graded chain complex in $\mathcal{C}_\Gamma$ so that $H_*C \cong H_*D$ and for all $m \geq 0$ there is an indexing set $I_m$ so that $C_m \to D_m$ is isomorphic to

$$C_m \to C_m \oplus \bigoplus_{i \in I_m} \Sigma^{n_i} D\Gamma_{\alpha_i}.$$

Let $x \in D_m$ for some $m$. Then there is a subchain complex $D' \subseteq D$ so that $\#D' \leq \lambda$, $x \in D'$, $H_*(C \cap D') \cong H_*D'$, and for all $m \geq 0$ there is a sub-indexing set $I'_m \subseteq I_m$ so that $C_m \cap D'_m \to D'_m$ is isomorphic to

$$C_m \cap D'_m \to C_m \cap D'_m \oplus \bigoplus_{I_m} \Sigma^{n_i} D\Gamma_{\alpha_i}.$$

We begin with the following observation.

Let $M \subseteq D_m$ be a finitely generated sub-comodule. Then there is a sub-chain complex $E(M) \subseteq D$ so that $\#E(M) \leq \lambda$, $M \subseteq E(M)_m$ and $C \cap E(M) \to E(M)$ is free. To see this, let $\{y_j\} \subseteq M$ be a set of generators. Then there is a unique expression

$$y_j = y_{0j} + \Sigma y_{ij}$$

where $y_{0j} \in C_m$ and $y_{ij} \in \Sigma^{n_i} D\Gamma_{\alpha_i}$. Then $y_{ij} \neq 0$ for only finitely many pairs $(i,j)$, hence for only finitely many $i$. Let $N \subseteq C_m$ be a finitely generated sub-comodule containing $\{y_{0j}\}$. Set

$$E(M)_m = N \oplus \bigoplus_{I(M)} \Sigma^{n_i} D\Gamma_{\alpha_i}$$

where $I(M) = \{i \in I_m | y_{ij} \neq 0 \text{ for some } (i,j)\}$. Since $E(M)_m$ is finitely generated, so is $\partial D(M)_m \subseteq D_{m-1}$. Choose $D(M)_{m-1}$ in the same manner. A downward regression finishes the argument.

To finish the proof of the lemma, we define sub-complexes

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$$

so that $\# D_k \leq \lambda$, $C \cap D_k \to D_k$ is free, $x \in D_0$ and $H_*(D_k / C \cap D_k) \to H_*(D_{k+1}/C \cap D_{k+1})$ is zero.

To define $D_0$, let $M \subseteq D_m$ be any finitely generated sub-comodule with $x \in M$ and let $D_0 = D(M)$. Having defined $D_k$, let

$$a_t \in H_*(D_k / C \cap D_k)$$

be a set of generators and let $x_t \in D_k$ pass to $a_t$. Since $H_*(D/C) = 0$ there exists $y_t \in D$ so that $\partial y_t = x_t \mod C$. Let $M_t \subseteq D_k$ be any finitely generated sub-module containing $y_t$ and let

$$D_{k+1} = D_k + \sum_t E(M_t). \qquad \blacksquare$$

**Remark 3.12** If $X \in s\mathcal{C}_\Gamma$ it is possible to give a concrete model of a $\pi_*$-fibrant model for $X$ is follows. Let $s\mathcal{M}_R$ be the category of simplicial $R$-modules, in its usual model category structure. The forgetful functor $s\mathcal{C}_\Gamma \to s\mathcal{M}_R$ preserves weak equivalences and cofibrations, so the right adjoint $M \mapsto \Gamma \otimes_R M$ preserves fibrations and weak equivalences, the last since every object of $s\mathcal{M}_R$ is fibrant. Write

$$M \mapsto \Gamma \otimes_R M$$

for the obvious triple on $s\mathcal{C}_\Gamma$; if $M \in s\mathcal{C}_\Gamma$ there is a cosimplicial resolution

$$M \to \Gamma^{(s)} \otimes_R M \tag{3.13}$$

where

$$\Gamma^{(n)} \otimes_R M = \underbrace{\Gamma \otimes_R \cdots \otimes_R \Gamma}_{\longleftarrow n+1 \longrightarrow} \otimes_R M.$$

Since $s\mathcal{C}_\Gamma$ is a simplicial category we can form the total object of any cosimplicial object over $s\mathcal{C}_\Gamma$; the claim is that

$$M \to \mathrm{Tot}_{s\mathcal{C}_\Gamma}(\Gamma^{(\cdot)} \otimes_R M)$$

is a $\pi_*$-weak equivalence and the target is fibrant. To see this is a weak equivalence note that (3.13) has a retraction as a cosimplicial object in $s\mathcal{M}_R$. Since $\mathrm{TOT}_{s\mathcal{C}_\Gamma}(\Gamma^{(\cdot)} \otimes_R M) \cong \mathrm{TOT}_{s\mathcal{M}_R}(\Gamma^{(\cdot)} \otimes_R M)$ as simplicial $R$-modules, this fact follows. To see that it's fibrant, we must check the canonical map

$$s : \Gamma^{(n)} \otimes_R M \to M^{n-1}(\Gamma^{(\cdot)} \otimes_R M) \tag{3.14}$$

to the matching object is a fibration. But, neglecting $d^0$, there is an isomorphism in $s\mathcal{C}$

$$\Gamma^{(\cdot)} \otimes_R M \cong \Gamma \otimes_R (\Gamma^{\otimes(\cdot)} \otimes_R M)$$

where $\Gamma^{\otimes n} \otimes_R M$ means the obvious thing. Since $s$ and the matching object depend only on codegeneracies, (3.14) is isomorphic to

$$\Gamma \otimes_R s : \Gamma \otimes_R (\Gamma^{\otimes n} \otimes_R M) \to \Gamma \otimes_R M^{n-1}(\Gamma^{\otimes(\cdot)} \otimes_R M).$$

The map $s : \Gamma^{\otimes n} \otimes_R M \to M^{n-1}(\Gamma^{\otimes(\cdot)} \otimes_R M)$ is a fibration of simplicial $R$-modules, indeed a split surjection, and $\Gamma \otimes_R (\cdot)$ preserves fibrations. ■

# 4 Simplicial algebras over a simplicial operad in comodules

In this section we promote the model category structures on $s\mathcal{C}_\Gamma$ to model category structures on $s\mathcal{C}_\Gamma^T$, where $T$ is a simplicial operad in $\mathcal{C}_\Gamma$ and a simplicial $T$ modules over a simplicial $T$-algebra $A$. We concentrate on the $\pi_*$-weak equivalence structure of Proposition 3.10. The other structure lifts also, but appears less useful.

Fix a simplicial operad $T$ in $\mathcal{C}_\Gamma$ and let $s\mathcal{C}_\Gamma^T$ denote the algebras in $s\mathcal{C}_\Gamma$ over $T$.

**Definition 4.1** *Let $f : X \to Y$ be a morphism in $s\mathcal{C}_\Gamma^T$. Then*

*1) $f$ is a weak equivalence if $\pi_* f : \pi_* X \to \pi_* Y$ is a weak equivalence;*

*2) $f$ is a fibration if it is a $\pi_*$-fibration in $s\mathcal{C}_\Gamma$;*

*3) a cofibration if it has the left lifting property with respect to all trivial fibrations.*

The category $s\mathcal{C}_\Gamma^T$ has a canonical simplicial structure (see [12], §1), and the first result is the following.

**Proposition 4.2** *With these definitions $s\mathcal{C}_\Gamma^T$ becomes a simplicial model category.*

Since the $\pi_*$-model category $s\mathcal{C}_\Gamma$ is cofibrantly generated and the forgetful functor $U_T : s\mathcal{C}_\Gamma^T \to s\mathcal{C}_\Gamma$ preserves filtered colimits and is continuous in the sense that for all $K \in \mathbf{S}$

$$U_T(X^K) \cong U_T(X)^K,$$

this falls into the standard situation where one is trying to lift a model category structure ([4], [13], §II.6). There are two crucial lemmas which we now state and prove.

Let $I = \{\Sigma^n D\Gamma_\alpha \otimes \Delta^k \to \Sigma^n D\Gamma_\alpha \otimes \Delta^k\}$ be the generating cofibrations of $s\mathcal{C}_\Gamma$ and let $J = \{i_\beta : C_\beta \to D_\beta\}$ be the generating trivial cofibrations of $s\mathcal{C}_\Gamma$. Let $\mathcal{M}_{T(I)}$ be the class of morphisms in $s\mathcal{C}_\Gamma^T$ generated by the morphisms $\{T(\Sigma^n D\Gamma_\alpha) \otimes \partial\Delta^k \to T(\Sigma^n D\Gamma_\alpha) \otimes \Delta^k\}$; thus $\mathcal{M}_{T(I)}$ is the smallest class of morphisms containing the listed morphisms and closed under isomorphism, coproducts, co-base change, sequential colimits, and retracts. Similarly, we may define $\mathcal{M}_{T(J)}$. If $i_\beta : C_\beta \to D_\beta$ is in $J$, then $T(i_\beta) \in \mathcal{M}_{T(I)}$; hence $\mathcal{M}_{T(J)} \subseteq \mathcal{M}_{T(I)}$. Here are the crucial lemmas.

**Lemma 4.3** *Every morphism in $\mathcal{M}_{T(J)}$ is a weak equivalence in $s\mathcal{C}_\Gamma^T$.*

**Proof.** Since $i_\beta : C_\beta \to D_\beta$ is a trivial cofibration of simplicial $R$-modules, and every simplicial $R$-module is fibrant, $i_\beta$ is a strong deformation retraction of simplicial $R$-modules. Hence $i_\beta$ has a retraction $r_\beta : D_\beta \to C_\beta$ and there is a homotopy $D_\beta \to D_\beta^{\Delta^1}$ from $1_{D_\beta}$ to $i_\beta r_\beta$. It follows from the natural map $T(D_\beta^{\Delta^1}) \cong T(D_\beta)^{\Delta^1}$ that $Ti_\beta : TC_\beta \to TD_\beta$ is a strong deformation retraction of simplicial algebras over $T$, regarded as a simplicial operad in $R$-modules. Thus $TC_\beta \to TD_\beta$ is a weak equivalence of simplicial $R$-modules, which is all that is required. ■

**Lemma 4.4** *Let $M \in s\mathcal{C}_\Gamma$. Then $T(M) \in s\mathcal{C}_\Gamma^T$ is small with respect to the class of morphisms $\mathcal{M}_{T(I)}$.*

**Proof.** The comodule $M$ is closed with respect to the class of morphisms $X \to Y$ in $s\mathcal{C}_\Gamma$ which are level-wise inclusions. Since $\mathrm{Hom}_{s\mathcal{C}_\Gamma^T}(T(M), X) = \mathrm{Hom}_{s\mathcal{C}_\Gamma}(M, X)$, we need only show that the morphisms in $\mathcal{M}_{T(I)}$ are level-wise inclusions.

Since the generating morphisms $T(\Sigma^n D\Gamma_\alpha \otimes \partial\Delta^k) \to T(\Sigma^n D\Gamma_\alpha \otimes \Delta^k)$ are free (see [10]), every morphism in $\mathcal{M}_{T(I)}$ is a retract of a free morphism. Since retracts of inclusions are inclusions, we need only show $s$-free morphisms in $s\mathcal{C}_\Gamma^T$ are level-wise inclusions. If $A \to B$ in $s\mathcal{C}_\Gamma^T$ is free there is, for each $k \geq 0$, a comodule $M_k \in \mathcal{C}_\Gamma$ and an isomorphism of $A_k \to B_k$ with

$$i_1 : A_k \to A_k \sqcup T_k(M_k)$$

where the coproduct is as $T_k$-algebras. The unique map of comodules $M_k \to 0$ defines a map

$$r : A_k \sqcup T_k(M_k) \to A_k \sqcup T_k(0) \cong A_k$$

which is a retraction of $i_1$. The result follows. ■

**4.5: Proof of Proposition 4.2**: A morphism $f : X \to Y$ in $(s\mathcal{C}_\Gamma)^T$ is a trivial fibration if and only if it has the left lifting property with respect to the morphisms $T(\Sigma^n D\Gamma_\alpha \otimes \partial\Delta^k) \to T(\Sigma^n D\Gamma_\alpha \otimes \Delta^k)$, and $f$ is a fibration if and only if it has the left lifting property with respect to the morphisms $T(C_\beta) \to T(D_\beta)$. The small object argument now applies, because of

Lemma 4.4. That the small object argument to produce the "trivial cofibration fibration" factorization works follows from Lemma 4.3. The Axiom SM7 holds because SM7a (citeHA, §II.2) holds in $(s\mathcal{C}_\Gamma^T)$, which follows from the facts that it holds in $s\mathcal{C}_\Gamma$ and that the forgetful functor preserves exponential objects. ■

**Remark 4.6** It is possible to characterize the cofibrations in $(s\mathcal{C}_\Gamma)^T$ as retracts of the *free* morphisms. If we write $(n\mathcal{C}_\Gamma)^T$ as the non-negatively graded $T$-algebras in $\mathcal{C}_\Gamma$, where $T$ is regarded as a discrete operad, and $(s_+\mathcal{C}_\Gamma)^T$ for the $T$-algebras in $s_+\mathcal{C}_\Gamma$ (degeneracy diagrams) with $T$ regarded as an operad with only its degeneracy morphisms, then the restriction functor

$$r_* : (s_+\mathcal{C})^T \to (n\mathcal{C})^T$$

has left adjoint $r^*$ with

$$r^*\{Z_*\}_k = T_k \otimes_{r^*T} (\coprod_{\phi:k\to j} Z_j)$$

with $\phi$ running over the morphisms of $\Delta_+$ and

$$r_*T = \coprod_{\phi:k\to j} T_j \to T_k$$

the evident map of operads.

A morphism $A \to B$ in $s\mathcal{C}_\Gamma^T$ is $s$-free if it is isomorphic to

$$i_1 : A \to A \sqcup r^*Z$$

for some $Z \in (n\mathcal{C})^T$. It is *free* if $Z_j = T_j(M_j)$ where $M_j$ is a sum of the comodules $\Sigma^n D\Gamma_\alpha$. Certainly, every cofibration created by the small object argument is free; conversely, if a morphism is free a skeletal decomposition as in [12],§1) shows the morphism is a cofibration; if in fact $A \to \operatorname{colim} sk_n^A B \cong B$ where $sk_{n-1}^A B = A$ and there are push-out diagrams

$$\begin{tikzcd} T(M_n) \otimes \partial\Delta^n \arrow[r] \arrow[d] & sk_{n-1}^A B \arrow[d] \\ T(M_n) \otimes \Delta^n \arrow[r] & sk_n^A B. \end{tikzcd}$$

Note that every cofibration in $(s\mathcal{C}_\Gamma)^T$ is a cofibration in $(s\mathcal{M}_R)^T$. ■

If we fix a $T$-algebra $A$, the same sort of discussion puts a model category structure on the category of simplicial $A$-modules in $\mathcal{C}_\Gamma$. Recall that $T$ is a simplicial operad, and let $(s\mathcal{C}_\Gamma)_A^T$ be the category of $T$-modules over $A$. Then $(s\mathcal{C}_\Gamma)_A^T$ acquires a simplicial structure and we have:

**Proposition 4.7** *There is a simplicial model category structure on $(s\mathcal{C}_\Gamma)^T_A$ where $f : M \to N$ is a weak equivalence or fibration if and only if it is a $\pi_*$-weak equivalence or $\pi_*$-fibration in $s\mathcal{C}_\Gamma$.*

**Proof.** The forgetful functor $(s\mathcal{C}_\Gamma)^T_A \to s\mathcal{C}_\Gamma$ has a left adjoint, so again the small object argument will work, provided one proves the requisite analogs of Lemmas 4.3 and 4.4. The proofs remain virtually unchanged and we leave them to the reader. ∎

**Remark 4.8** Again, the cofibrations are retracts of free morphisms, which we now define. Let $(n\mathcal{C}_\Gamma)^T_A$ and $(s_+\mathcal{C}_\Gamma)^T_A$ be the evident restricted categories. Then the restriction functor

$$r_* : (s_+\mathcal{C}_\Gamma)^T_A \to (n\mathcal{C}_\Gamma)^T_A$$

has a left adjoint. If $M = \{M_k\} \in (n\mathcal{C}_\Gamma)^T_A$, then each $M_k$ is a $T_k$ module over $A_k$. If $\phi : k \to j$ is any surjection in $\Delta$, then $\phi^* : A_j \to A_k$ is a morphism of $T_j$ algebras, and indicates a morphism of $T_k$ algebras

$$T_k \otimes_{T_j} A_j \to A_k.$$

Thus we may a $T_k$ module over $A_k$.

$$T_k \otimes_{T_j} (A_k \otimes_{A_j} M_j) \cong A_k \otimes_{(T_k \otimes_{T_j} A_k)} (T_k \otimes_{T_j} M_j).$$

Then the left adjoint to $r_*$ is defined by

$$(r_* M)_k = \bigoplus_{k \to j} T_k \otimes_{T_j} (A_k \otimes_{A_j} M_j).$$

In particular, if $M_j = A_j \overset{T_j}{\bigotimes} N_j$, then

$$\begin{aligned} (r_* M)_k &\cong \bigoplus_{k \to j} A_k \overset{T_k}{\bigotimes} N_j \\ &\cong A_k \overset{T_k}{\bigotimes} \Big(\bigoplus_{k \to j} N_j\Big). \end{aligned}$$

A morphism $X \to Y$ in $(s\mathcal{C})^A_\Gamma$ is free if there exist $N_k \in \mathcal{C}_\Gamma$, $k \geq 1$, so that each $N_k$ is a direct sum of comodules of the form $\Sigma^n D\Gamma_\alpha$, and so that in $(S_+\mathcal{C}_\Gamma)^T_A$, $X \to Y$ is isomorphic to

$$X \to X \oplus r^k\{A_k \overset{T_k}{\bigotimes} N_k\}.$$

In particular, every cofibration in $(s\mathcal{C}_\Gamma)^T_A$ is a cofibration in $(s\mathcal{M})^T_A$. ∎

We finish the section by writing down a comparison result for André-Quillen cohomology.

The forgetful functor from $\Gamma$ comodules to $R$ modules has a right adjoint which sends an $R$ module to the extended $\Gamma$ comodule:

$$M \to \Gamma \otimes_R M.$$

If $T$ is a simplicial operad in $\Gamma$ comodules, then $T$ is an operad in $R$ modules. If $A$ is a simplicial algebra over $T$ in $\Gamma$ comodules and $M$ is a simplicial $T$ module over $A$ in $R$ modules, then $\Gamma \otimes_R M$ is a simplicial $T$ module over $A$ in $\Gamma$ comodules; the structure map

$$T(A, \Gamma \otimes_R M) \to \Gamma \otimes_R M$$

is adjoint to the map

$$T(A, \Gamma \otimes_R M) \to T(A, M) \to M$$

induced by the unit of the adjunction $\Gamma \otimes_R M \to M$; in fact, the extended comodule functor is the left adjoint to the forgetful functor

$$(s\mathcal{C}_\Gamma)^T_A \to (s\mathcal{M}_R)^T_A.$$

This extends to derivations. An examination of the construction of the Kähler differentials (see Lemma 2.2) shows that if $\Omega_T A \in (s\mathcal{C}_\Gamma)^T_A$ is the $T$ module over $A$ of differentials in $\Gamma$ comodules, and $\Omega'_T A \in (s\mathcal{M}_R)^T_A$ is the $T$ module over $A$ of differentials in $R$ modules, then there is an isomorphism of $R$ modules

$$\Omega_T A \cong \Omega'_T A.$$

Thus we may as well identify these two. This has ramifications. If $T \to T_{-1}$ is an augmented $E_*E$ comodule operad, $A \to A_{-1}$ an augmented $T \to T_{-1}$ algebra in $\Gamma$ comodules and $M$ an $T_{-1}$ module over $A_{-1}$ in $R$ modules, then

$$\begin{aligned} \mathrm{Der}_{R/T}(A, M) &\cong \mathrm{Hom}_R(\Omega_T A, M) \\ &\cong \mathrm{Hom}_\Gamma(\Omega_T A, E_*E \otimes_A M) \\ &\cong \mathrm{Der}_{\Gamma/T}(A, E_*E \otimes_A M). \end{aligned}$$

For such augmented $A$ write $D^*_\Gamma(A, N)$ and $D^*_R(A, M)$ for the appropriate André-Quillen cohomologies.

**Lemma 4.9** *There is a natural isomorphism*

$$D^*_\Gamma(A, \Gamma \otimes_R M) \cong D^*_R(A, M).$$

**Proof.** Choose a weak equivalence in $(s\mathcal{C}_\Gamma)^T$

$$X \to A$$

where $X$ is free. Then

$$\begin{aligned} D^q_\Gamma(A, \Gamma \otimes_R M) &\cong \pi^q \operatorname{Der}_\Gamma(X, \Gamma \otimes_R M) \\ &\cong \pi^q \operatorname{Der}_R(X, M). \end{aligned}$$

Thus we need only check that $X$ is cofibrant is $(s\mathcal{M}_R)^T$. Since each of the generating cofibrations of $(s\mathcal{C}_\Gamma)^T$

$$T(\Sigma^n D\Gamma_\alpha \otimes \partial\Delta^k) \to T(\Sigma^n D\Gamma_\alpha \otimes \Delta^k)$$

is a cofibration in $(s\mathcal{M}_R)^T$—since if $\Sigma^n D\Gamma_\alpha$ is a projective $R$-module—we have the result.■

# 5 Model Category Structures for Categories of Operads

The purpose of this section is to explore model category structures on simplicial operads of $R$ modules, for some commutative ring $R$, simplicial operads in $\Gamma$ comodules for some Hopf algebroid $(R, \Gamma)$, and for operads in topological spaces. Our goal is to discuss to what extend cofibrations of $T$-algebras are cofibrations in the underlying category. See Propositions 5.5, 5.8 and 5.16

Let $\mathcal{M}_R$ be the category of $R$ modules for some commutative ring $R$, and let $n\mathcal{M}_R$ be the category of non-negatively graded $R$-modules. The category $s(n\mathcal{M}_R)$ of simplicial graded $R$ modules is a simplicial category in the sense of [19], §II.2 and simplicial model category with $f : X \to Y$ a weak equivalence if the map of normalized chain complexes

$$Nf : NX \longrightarrow NY$$

is a homology isomorphism and a fibration if $Nf$ is surjective is positive chain degrees. This is the same as stipulating that a fibration or weak equivalence in $s(n\mathcal{M}_R) = n(s\mathcal{M}_R)$ be a degree-wise (or level-wise) fibration or weak equivalence in the standard structure on $s\mathcal{M}_R$. Cofibrations are morphisms which are injections with projective cokernel.

This model category structure is cofibrantly generated. If $X$ is a simplicial $R$-module, write $X(n) \in s(n\mathcal{M}_R)$ for the corresponding graded simplicial $R$-module concentrated in degree $n$. Then the generating cofibrations of $s(n\mathcal{M}_r)$ are

$$R[\partial\Delta^m](n) \to R[\Delta^m](n), \qquad n \geq 0, m \geq 0, \tag{5.1}$$

and the generating trivial cofibrations are

$$(5.2) \qquad R[\Delta^m_k](n) \to R[\Delta^m]i(n), \qquad m \geq 1, 0 \leq k \leq m, n \geq 0.$$

If $X$ is a set $R[X]$ is the free $R$-module on $X$. The object $\Delta^m$ is the simplicial $k$ simplex, and

$$\Delta^m_k = \bigcup_{i \neq k} d_i \Delta^{m-1} \subseteq \Delta^m.$$

It now follows, essentially from [19] §II.4. that the category of simplicial operads inherits a simplicial structure.

**Proposition 5.3** *Define a morphism $f : A \to B$ of simplicial operads to be a fibration (or weak equivalence) if it is a fibration or weak equivalence in $s(n\mathcal{M}_R)$. Then with its standard simplicial structure, $s\mathcal{O}_R$ becomes a simplicial model category.*

**Proof.** We use the free operad functor $V \mapsto G_V$ (see [7] or the appendix to this paper) to promote the model category structure on $s(nM_R)$. This is the small object argument. Since every object of $s\mathcal{O}_R$ is fibrant, the essential point to check is that each of the functors

$$\mathrm{Hom}_{s\mathcal{O}_R}(G_X, \cdot)$$

with $X$ a source of one of the maps of 5.1 or 5.2, commutes with directed colimits. Since, by [7], the forgetful functor $s\mathcal{O}_R \to s(n\mathcal{M}_R)$ makes filtered colimits, the result follows. ■

Cofibrations in $s\mathcal{O}_R$ are retracts of of free maps in the sense of [19], §II.4. An object $X$ in $s\mathcal{O}_R$ is free if the underlying degeneracy diagram is of the form

$$X_n \cong \bigsqcup_{\phi:[n]\to[k]} G_{Z_k}$$

where $G_{(\cdot)}$ is the free operad functor, $\phi$ runs over the surjections in the ordinal number category and $Z_k \subseteq X_k$ is a free sub-$R$ module. The degeneracy maps in $X$ must be the obvious ones. A morphism $X \to Y$ is free if the underlying morphism of degeneracy diagrams is of the form

$$X \longrightarrow X \sqcup Y'$$

where $Y'$ is free. Combining these definitions with formula for the free operad functor given in the appendix and the formula for the coproduct of two operads given in [7] (see after their Theorem 1.13), we have the following result.

**Proposition 5.4** *Let $X \to Y$ be cofibration of simplicial operads in $R$-modules with $X$ cofibrant, Then $X(n) \to Y(n)$ is a cofibration of simplicial $R[\Sigma_n]$ modules, again with cofibrant source.*

■

Similar methods prove the following result.

**Proposition 5.5** *Let $T$ be a cofibrant simplicial operad in $R$-modules and $A \to X$ a cofibration of simplicial $T$-algebras with cofibrant source. Then $A \to X$ is a cofibration of simplicial $R$ modules, again with cofibrant source.*

**Proof.** First assume the $T$ is free in the sense described above. We may assume that $A \to X$ is the kind of cofibration produced by the small object argument in simplicial $T$-algebras, as any cofibration is a retract of such. Since directed colimits of $T$-algebras are made in $R$-modules, we need only show that if we have a push-out diagram

$$\begin{tikzcd} T(V) \arrow[r] \arrow[d] & B \arrow[d] \\ T(W) \arrow[r] & Y \end{tikzcd}$$

with $V \to W$ a cofibration of simplicial $R$ modules and $B$ cofibrant in $T$-algebras, then $B \to Y$ is a cofibration of simplicial $R$-modules. Since $V \to W$ is a level-wise injection with projective cokernel, in simplicial degree $n$ we have an isomorphism

$$X_n \cong A_n \sqcup T_n(Z_n)$$

where $Z_n$ is a projective $R$ module. First assume that $B_n$ is also a projective $R$-module. Then the fact that $T_n$ is a free operad and the formula for the Rezk decomposition given in Proposition A.4 give the result. Second, using the small object argument constructions, one now concludes that any cofibrant $T$ algebra $A$ is a cofibrant simplicial $R$ module. Finally, the same argument proves the result, at least for free operads $T$.

For more general cofibrant operads $T$, we use the fact that there is a free operad $S$ and split trivial fibration of operads $S \to T$. This implies that any morphism $A \to X$ of T-algebras splits off the morphism of $S$ algebras $S \otimes_T A \to S \otimes_T X$. Since the functor $Z \mapsto S \otimes_T Z$ preserves cofibrations, the result follows. ∎

We next turn to operads on comodules over a Hopf algebroid $(R, \Gamma)$. Write $\mathcal{C}_\Gamma$ for this category of comodules. We assume that Assumption 3.1 holds for $\mathcal{C}_\Gamma$. Under this hypothesis, we proved in section 3, that the category $s\mathcal{C}_\Gamma$ is a cofibrantly generated simplicial model category where the weak equivalences are the $\pi_*$ isomorphisms; that is, $A \to B$ is a weak equivalence in $s\mathcal{C}_\Gamma$ if and only if it is a weak equivalence in $s\mathcal{M}_R$. The forgetful functor $s\mathcal{C}_\Gamma \to s\mathcal{M}_R$ preserves cofibrations, but does not reflect them.

The generating cofibrations in $s\mathcal{C}_\Gamma$ are of the form

$$N \otimes \partial\Delta^m \longrightarrow N \otimes \Delta^m, \qquad m \geq 0$$

where $N$ runs over a set of $\Gamma$ comodules which are finitely generated and projective as $R$ modules. Trivial cofibrations are generated by the morphisms produced by Lemma 3.11. We can now give the category of simplicial graded objects $s(n\mathcal{C}_\Gamma) = n(s\mathcal{C}_\Gamma)$ the induced level-wise simplicial model category structure. It too will be cofibrantly generated. It is not clear that every object in $s\mathcal{C}_\Gamma$ is fibrant. See Remark 3.12.

Let $\mathcal{O}_\Gamma$ be the category of operads in $\mathcal{C}_\Gamma$. The free operad functor $G : n\mathcal{C}_\Gamma \to \mathcal{O}_\Gamma$ covers the free operad functor $G : n\mathcal{M}_R \to \mathcal{O}_R$ in the sense that the following diagram commutes

$$\begin{tikzcd} n\mathcal{C}_\Gamma \arrow[r, "G"] \arrow[d] & \mathcal{O}_\Gamma \arrow[d] \\ n\mathcal{M}_R \arrow[r, "G"] & \mathcal{O}_R \end{tikzcd}$$

where the unlabeled arrows are forgetful functors. Furthermore, by the formulas of [7], the forgetful functor $\mathcal{O}_\Gamma \to \mathcal{O}_R$ makes all colimits.

**Proposition 5.6** *Define a morphism $A \to B$ is $s\mathcal{O}_\Gamma$ to be a fibration (or weak equivalence) if it is a fibration (or weak equivalence) in $s(n\mathcal{C}_\Gamma)$. Then, with its standard simplicial structure, $s\mathcal{O}_\Gamma$ becomes a simplicial model category.*

**Proof.** Because not every object in $s\mathcal{O}_\Gamma$ is fibrant, one must expand on the proof given for Proposition 5.3. As in that argument, one must check that $\mathrm{Hom}_{s\mathcal{O}_\Gamma}(G_X, \cdot)$, where $X$ is the source of a generating cofibration or trivial cofibration, commutes with long enough directed colimits. The remarks about $G$ and colimits made immediately preceding the statement of this result, imply that this need only be checked in $R$ modules, where it is clear.

The other point one must check is that the class of trivial cofibrations in $s\mathcal{O}_\Gamma$ is closed under coproducts, cobase change, directed colimits, and retracts, and that this class contains the morphisms

$$G_A \to G_B$$

where $A \to B$ is a trivial cofibration in $s(n\mathcal{C}_\Gamma)$. However, since the forgetful functor $s\mathcal{O}_\Gamma \to s\mathcal{O}_R$ makes and reflects weak equivalences and colimits, and every cofibration in $s(n\mathcal{C}_\Gamma)$ is a cofibration in $s(n\mathcal{M}_R)$, the result follows. ■

As in the $R$ module case, cofibrations in $s\mathcal{O}_\Gamma$ are retracts of free morphisms. In this setting, $X \in s\mathcal{O}_\Gamma$ is free if the underlying degeneracy diagram is of the form

$$X_n \cong \bigsqcup_{\phi:[n]\to[k]} G_{Z_k}$$

where $Z_n$ is a direct sum of $\Gamma$ comodules which are finitely generated and projective as $R$ modules. As a result, the following analog of Proposition 5.4:

**Proposition 5.7** *Let $X \to Y$ be cofibration of simplicial operads in $\Gamma$ comodules with $X$ cofibrant, Then $X(n) \to Y(n)$ is a cofibration of simplicial $R[\Sigma_n]$ modules, again with cofibrant source.*

■

We also have the analog of Proposition 5.5:

**Proposition 5.8** *Let $T$ be a cofibrant simplicial operad in $\Gamma$ comodules and $A \to X$ a cofibration of simplicial $T$-algebras with cofibrant source. Then $A \to X$ is a cofibration of simplicial $R$ modules, again with cofibrant source.*

**Proof.** This actually follows from Proposition 5.5 once one recognizes that the forgetful functor $s\mathcal{O}_\Gamma \to s\mathcal{O}_R$ preserves cofibrations. ∎

The final category we wish to consider – more for later papers than this one – is the category **Top** of topological spaces, meaning compactly generated weak Hausdorff spaces. The model category structure we take on this category is the Serre structure: weak equivalences are topological weak equivalences and fibrations are Serre fibrations. This is cofibrantly generated. The generating cofibrations are

$$S^{m-1} = |\partial\Delta^m| \to |\Delta^m| = D^m, \qquad m \geq 0 \tag{5.9}$$

and the generating trivial cofibrations are

$$D^m_k = |\Delta^m_k| \to |\Delta^m| = D^m, \qquad m \geq 1, 0 \leq k \leq m. \tag{5.10}$$

The category **Top** with this model category structure, also satisfies the corner axiom SM7 over itself; that is, to say, **Top** is a topological model category.

It follows immediately that the category $n$**Top** of graded spaces is a cofibrantly generated topological model category.

Now let $\mathcal{O}_{\mathbf{Top}}$ be the category of operads in **Top**. By [7], $\mathcal{O}_{\mathbf{Top}}$ is the category of algebras in $n$**Top** over the triple obtained by composing the free operad functor with the forgetful functor. Thus $\mathcal{O}_{\mathbf{Top}}$ is enriched over **Top**; indeed, if $X$ and $Y$ are object in $O_{\mathbf{Top}}$, there is a coequalizer diagram

$$\mathrm{map}_{\mathcal{O}_{\mathbf{Top}}}(X,Y) \longrightarrow \mathrm{map}_{\mathbf{Top}}(X,Y) \rightrightarrows \mathrm{map}_{\mathbf{Top}}(G_X,Y)$$

Also, $\mathcal{O}_{\mathbf{Top}}$ has exponential and tensor objects over **Top**. If $X \in \mathcal{O}_{\mathbf{Top}}$ and $K \in$ **Top**, then $X^K = \{X(n)^K\}$ with the obvious operad structure and $X \Box K$ is defined as follows: if $X = G_Z$ for some $X$, then

$$X \Box K = G_{(Z \times K)};$$

otherwise there is a coequalizer diagram

$$G_{G_X} \Box K \rightrightarrows G_X \Box K \longrightarrow X \Box K.$$

One easily checks the adjoint homeomorphisms

$$\mathrm{map}_{\mathcal{O}_{\mathbf{Top}}}(X \Box K, Y) \cong \mathrm{map}_{\mathcal{O}_{\mathbf{Top}}}(X, Y^K) \cong \mathrm{map}_{\mathbf{Top}}(K, \mathrm{map}_{\mathcal{O}_{\mathbf{Top}}}(X,Y)). \tag{5.11}$$

We would now like to prove that $\mathcal{O}_{\mathbf{Top}}$ inherits a topological model category structure from $n$**Top**. As always in topological settings, one must show that the putative cofibrations in $\mathcal{O}_{\mathbf{Top}}$ are closed inclusions in $n$**Top**; this is necessary for the small object argument to work. A simple way to check whether a continuous function is a closed inclusion is prove it is a Hurewicz cofibration, meaning that is has the homotopy extension property. This is the point of the following result.

**Lemma 5.12** *Let $A \to B$ be a cofibration in $n$**Top** with the property that one can choose a homeomorphism*

$$h : B \to B \sqcup_A (A \times [0,1])$$

*so that $1_A = h|_A : A \to A \times \{1\}$. Given a push-out diagram of operads*

$$\begin{tikzcd} G_A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ G_B \arrow[r] & Y, \end{tikzcd}$$

*then $f : X(n) \to Y(n)$ is a Hurewicz cofibration for all $n$. If $X(n)$ and $A(n)$ are cofibrant in* **Top** *for all $n$, then $f : X(n) \to Y(n)$ is a cofibration in* **Top**.

**Proof.** The existence of $h$ implies that there is an isomorphism in $\mathcal{O}_{\mathbf{Top}}$

$$Y = G_B \sqcup_{G_A} X \xrightarrow{\cong} G_B \sqcup_{G_A} G_A \square [0,1] \sqcup_{G_A} X.$$

The target of this isomorphism is the geometric realization of a simplicial operad, namely the bar construction $\mathbb{B}(G_B, G_A, X)$, which in simplicial degree $k$ is

$$\mathbb{B}(G_B, G_A, , X)_k = G_B \sqcup G_A \sqcup \ldots G_A \sqcup X$$

with $G_A$ appearing $k$ times and the usual face and degeneracy operators. Since the forgetful functor from $\mathcal{O}_{\mathbf{Top}}$ to $n$**Top** makes the geometric realization, the result will follow of we can show that

$$X \longrightarrow \mathbb{B}(G_B, G_A, X)$$

is a Reedy Hurewicz cofibration in $n$**Top** (i.e., "proper") or a Reedy Serre cofibration in $n$**Top**, depending. Interpreting this statement and using that $X$ is constant, we require

$$X(n) \longrightarrow [G_B \sqcup X](n) \tag{5.13}$$

to be a cofibration and

$$L_k\mathbb{B}(G_B, G_A, X)(n) \longrightarrow \mathbb{B}(G_B, G_A, X)_k(n) \tag{5.14}$$

to be cofibrations for $k \geq 1$. The latching object is formed in **Top**. Using the formula for $G_B$ developed in the appendix and the formula for the coproduct of operads in [7], we have that 5.13 is the inclusion of a coproduct summand where the other summands are of the form

$$Y(m) = Y(m_1) \times \ldots \times Y(m_t)$$

with $Y(m_i) = B(m_i)$ or $X(m_i)$. This is trivially a Hurewicz cofibration and a Serre fibration if $B(m_i)$ and $X(m_i)$ are cofibrant.

In a similar manner, one discovers that

$$\mathbb{B}(G_B, G_A, X)_k(n) = \sqcup Y(m_1) \times \ldots \times Y(m_t)$$

where $Y(m_i) = G_B(m_i)$, $X(m_i)$, or $(G_A)_j(m_i)$, where the subscript $j$ means that $(G_A)_j(m_i)$ arise from the $j^\theta$ factor of $G_A$. A calculation now shows that 5.14 is the inclusion of a coproduct summand where the other summands are of the form just stipulated and with the additional condition that for all $j$, $0 \leq j \leq k$ there is an $m_i$ so that the factor $(G_A)_j(m_i)$ appears. The result then follows as above. ■

This allows us to prove:

**Proposition 5.15** *Define a morphism $X \to Y$ in $\mathcal{O}_{\mathbf{Top}}$ to be a fibration (or weak equivalence) of it is a fibration (or weak equivalence) in $n\mathbf{Top}$. Then, with the topological structure defined above, $\mathcal{O}_{\mathbf{Top}}$ is a topological model category.*

**Proof.** Since every object in $\mathcal{O}_{\mathbf{Top}}$ is fibrant, we need only check that if $A \to B$ is any of the generating cofibrations

$$S^{m-1}(n) \longrightarrow D^m(n)$$

or

$$D^m_k(n) \longrightarrow D^m(n)$$

of $n\mathbf{Top}$, then $\mathrm{Hom}_{\mathcal{O}_{\mathbf{Top}}}(G_A, \cdot)$ commutes with the type of directed colimits produced by the small object argument. But any coproduct of these generating cofibration satisfies the hypotheses of Lemma 5.12, hence each of the maps in such a directed colimit will be a Hurewicz cofibration in $n\mathbf{Top}$, hence a closed inclusion. Since $A$ is compact, the result follows. ■.

A refined version of Lemma 5.12 and the small object argument would imply the analog of Proposition 5.4 for $\mathcal{O}_{\mathbf{Top}}$; we leave this to the interested reader. We are more interested in the following result.

**Proposition 5.16** *Let $T$ be an operad over the linear isometries operad which is cofibrant as an operad in $\mathcal{O}_{\mathbf{Top}}$. Let $A$ be a $T$-algebra in spectra which is cofibrant as a $T$-algebra. Then $A$ is cofibrant as spectrum.*

**Proof.** This argument is due to Charles Rezk. Let $\mathcal{E}(A)$ be the endomorphism operad of $A$. Choose a fibration of spectra $Z \to A$ where $Z$ is cofibrant as a spectrum and let $\mathcal{E}(Z \to A)$ be the endomorphism operad of the map $Z \to A$. The analog of the corner axiom in this setting shows that

$$\mathcal{E}(Z \to A) \longrightarrow \mathcal{E}(A)$$

is a trivial fibration of operads. Since $T$ is cofibrant, the $T$-algebra structure map $T \to \mathcal{E}(A)$ lifts to a morphism of operads $T \to \mathcal{E}(Z \to A)$; thus, $Z \to A$ becomes a trivial fibration of $T$-algebras. Since $A$ is cofibrant in $T$-algebras, this map has a section. Since $Z$ is cofibrant in spectra, and $A$ is a retract of $Z$, $A$ is cofibrant in spectra. ■

# 6 Homotopy push-outs of simplicial algebras over a simplicial operad

If a model category is left proper, then homotopy push-outs can be calculated in a relatively simple manner. However, the category of simplicial algebras over a simplicial operad is often not left proper, and we seek to give a condition which serves as an acceptable substitute. We begin with some examples, state the condition, then show the condition applies when the simplicial operad is cofibrant.

Recall that a model category is left proper if whenever there is a push-out square

(6.1)
$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j}\downarrow & & \downarrow \\ X & \xrightarrow{g} & Y \end{array}$$

with $j$ a cofibration and $f$ a weak equivalence, then $g$ is a weak equivalence.

**Example 6.2** Let $R$ be a commutative ring and $s\,\mathbf{Comm}_R$ the category of simplicial commutative $R$-algebras. Then $s\,\mathbf{Comm}_R$ is left proper. Indeed, by [19], §II.6), there is a spectral sequence

$$\mathrm{Tor}^{\pi_* A}(\pi_* B, \pi_* X) \Rightarrow \pi_* Y. \qquad \blacksquare$$

**Example 6.3** Let $R$ be a commutative ring and $s\,\mathbf{Alg}_R$ the category of simplicial algebras over $R$. Then $s\,\mathbf{Alg}_R$ is not left proper. To see what fails, let $T(\cdot) : s\mathcal{M}_R \to s\,\mathbf{Alg}_R$ be the free $R$-algebra functor, and let $f : A \to B$ be a weak equivalence. Then if $Z \in s\mathcal{M}_R$ is cofibrant, then $A \sqcup T(Z) \to B \sqcup T(Z)$ would be a weak equivalence, where we mean the coproduct is $s\mathcal{M}_{R\,\mathrm{alg}}$; this is because there is a push-out

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ {\scriptstyle j}\downarrow & & \downarrow \\ A \sqcup T(Z) & \longrightarrow & B \sqcup T(Z) \end{array}$$

and $j$ is a cofibration. However the functor

$$Z \mapsto A \sqcup T(Z)$$

is the left adjoint to the forgetful functor $A/s\,\mathbf{Alg}_R \to s\mathcal{M}_R$; hence there is a natural isomorphism of simplicial $R$-modules

$$A \sqcup T(Z) = \bigoplus_n A \otimes Z \underbrace{\otimes A \otimes Z \cdots \otimes}_{n} Z \otimes A$$

where the $n^{th}$ summand has $n$ copies of $Z$ and $n+1$ copies of $A$. If $A$ and $B$ are not cofibrant as simplicial $R$-modules, then there is no reason why

$$A \otimes \underbrace{Z \otimes A \otimes \cdots \otimes Z}_{n} \otimes A \to B \otimes \underbrace{Z \otimes \cdots \otimes Z}_{n} \otimes B$$

should be an isomorphism for $n \geq 1$. ∎

This example prompts the following definition.

**Proposition 6.4** *Fix a commutative ring $R$ and a simplicial operad $T$ of $R$-modules. The model category $s\mathcal{A}^T$ of simplicial $T$-algebras is* weakly left proper *if*

*1) whenever $Z \in s\mathcal{M}_R$ is cofibrant and $f : A \to B$ is a weak equivalence between simplicial $T$-algebras which are cofibrant as simplicial $R$-modules, then*

$$A \sqcup T(Z) \to B \sqcup T(Z)$$

*is a weak equivalence and*

*2) any cofibrant $X \in s\mathcal{A}^T$ is cofibrant in $s\mathcal{M}_R$.*

In order to explore the implications of this idea, we will use the following observation.

**Lemma 6.5** *Let $X \in s(s\mathcal{A}^T)$ be a simplicial object in the category of simplicial $T$ algebras. Then the geometric realization of $X$ is the diagonal:*

$$|X| \simeq \operatorname{diag} X = \{X_{n,n}\}.$$

**Proof.** The proof is the standard one. Write $X_n \in (s\mathcal{A}^T)$ for the $n^{th}$ object in $X$ and $X_{n,k}$ for the $k$-simplices in $X_n$. Then $|X|$ is defined by a coequalizer diagram

$$\coprod_{\psi: n \to m} X_m \otimes \Delta^n \rightrightarrows \coprod_n X_n \otimes \Delta^n \to |X|$$

where $\psi$ runs over the morphism in the ordinal number category. Since

$$(X_n \otimes \Delta^n)_k = \coprod_{\phi: k \to n} X_{n,k}$$

we can define a morphism

$$\coprod_n X_n \otimes \Delta^n \to \operatorname{diag} X$$

by taking the coproduct of the maps

$$\phi^* : X_{n,k} \to X_{k,k}.$$

One then checks that $\operatorname{diag} X$ has the universal property necessary to be $|X|$. ∎

The nomenclature "relatively left proper" is justified by the next result.

**Lemma 6.6** *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

*be a push-out square in $s\mathcal{A}^T$ with $j$ a cofibration and $f$ a weak equivalence between objects which are cofibrant in $s\mathcal{M}_R$. Then $g$ is a weak equivalence.*

**Proof.** In the simplicial model category $s\mathcal{M}_R$, define objects $R \wedge \Delta^n/\partial\Delta^n$, $n \geq 0$, by the push-out diagram

$$\begin{tikzcd} R \cong R \otimes * \arrow[r] \arrow[d] & R \otimes \Delta^n/\partial\Delta^n \arrow[d] \\ 0 \arrow[r] & R \wedge \Delta^n/\partial\Delta^n. \end{tikzcd}$$

This forms a collection of cofibrant cogroup objects in $s\mathcal{M}_R$; hence the objects

$$T(R \wedge \Delta^n/\partial\Delta^n) \in s\mathcal{A}^T$$

form a set of cofibrant cogroup objects. Let

$$W^A_\cdot(X) \to X$$

be the Stover resolution under $A$ of $X$ with respect to these objects. Then; by the previous result

$$|W^A_\cdot(X)| \to X \tag{6.7}$$

is a weak equivalence of cofibrant objects in the under category $A/s\mathcal{A}^T$. Since every object of this under category is fibrant, the map of (6.7) is necessarily a homotopy equivalence under $A$. It follows that

$$B \sqcup_A |W^A_\cdot(X)| \to B \sqcup_A X \cong Y$$

is a homotopy equivalence. Thus we need only show

$$|W^A_\cdot(X)| \to B \sqcup_A |W^A_\cdot(X)| \cong |B \sqcup_A W^A_\cdot(X)|$$

is a weak equivalence. By the previous lemma, it is enough to show

$$W^A_n(X) \to B \sqcup_A W^A_n(X)$$

is a weak equivalence. This is homotopy equivalent, as maps, to a morphism of the form

$$A \sqcup T(Z) \to B \sqcup T(Z)$$

where $Z \cong \bigoplus_\alpha R \wedge \Delta^{n_\alpha}/\partial\Delta^{n_\alpha}$. This map is a weak equivalence by the definition of relatively left proper. ■

We now prove:

**Proposition 6.8** *Suppose $s\mathcal{A}^T$ is relatively left proper and*

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle j}\downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

*is a push-out diagram. If $j$ is a cofibration and $A, B$ are cofibrant in $s\mathcal{M}_R$, then $Y$ is weakly equivalent to the homotopy push-out in $s\mathcal{A}^T$.*

**Proof.** Choose a surjective weak equivalence $A_0 \to A$ with $A_0$ cofibrant and, as in the proof of Lemma 6.6, let

$$W^{A_0}(X) \to X$$

be the Stover resolution for $X$ under $A_0$. By Lemma 6.5, $|W^{A_0}(X)| \to X$ is a weak equivalence; furthermore $A_0 \to |W^{A_0}(X)|$ is a cofibration. Factor $A_0 \to B$ as $A_0 \to B_0 \to B$ where the first map is a cofibration and the second a weak equivalence. Then

$$B_0 \sqcup_{A_0} |W^{A_0}(X)| \cong |B_0 \sqcup_{A_0} W^{A_0}(X)|$$

is a model for the homotopy push-out.

Since there is a homotopy equivalence under $A_0$

$$W_n^{A_0}(X) \simeq A_0 \sqcup T(Z_n) \tag{6.10}$$

where $Z_n$ is a sum of copies of $R \wedge \Delta^k/\partial\Delta^k$, there is a homotopy equivalence under $A$

$$A \sqcup_{A_0} W_n^{A_0}(X) \simeq A \sqcup T(Z_n)$$

and, hence,

$$W_n^{A_0}(X) \to A \sqcup_{A_0} W_n^{A_0}(X)$$

is a weak equivalence. Thus, the natural map

$$|A \sqcup_{A_0} W_n^{A_0}(X)| \cong A \sqcup_{A_0} |W_n^{A_0}(X)| \to A \sqcup_{A_0} X \cong X$$

is a weak equivalence. The last isomorphism uses that $A \to A_0$ is surjective. Since $A \to A \sqcup_{A_0} |W_n^{A_0}(X)|$ is a cofibration and every object of $s\mathcal{A}^T$ is fibrant we have (as in Lemma 6.6) that

$$|B \sqcup_{A_0} W^{A_0}(X)| = B \sqcup_A (A \sqcup_{A_0} |W_n^{A_0}(X)|) \to Y$$

is a weak equivalence. By (6.10) we have that for each $n$

$$B_0 \sqcup_{A_0} W_n^{A_0}(X) \to B \sqcup_{A_0} W_n^{A_0}(X)$$

is a homotopy equivalent to

$$B_0 \sqcup T(Z_n) \to B \sqcup T(Z_n).$$

This map is a weak equivalence and the result follows. ■

We now provide a class of examples of relatively left proper categories. Because the forgetful functor from simplicial operads to simplicial $R$-modules has a left adjoint (see the appendix for a detailed analysis), the category of simplicial operads over $R$, with the standard simplicial structure, is a simplicial model category, where a morphism $S \to T$ is a weak equivalence or fibration if and only if it is so in $s\mathcal{M}_R$. We will show $T$ is a cofibrant simplicial operad, then $s\mathcal{A}^T$ is relatively left proper.

We begin with the following recollections. The *Rezk decomposition* assigns to each operad $T$, each $T$-algebra $A$, and each $k \geq 0$, a $R[\Sigma_k]$ module

$$D_T^k A$$

so that there is an isomorphism of $R$-modules

$$A \sqcup T(W) \cong \bigoplus_k D_T^k A \otimes_{R[\Sigma_k]} W^{\Sigma k}.$$

The isomorphism is natural in $A$, $T$, and $W$. We have $D_T^0 A \cong A$.

The key lemma for analyzing cofibrant operads is the following.

**Proposition 6.10** *Let $T$ be a cofibrant simplicial operad and $A$ a simplicial $T$-algebra which is cofibrant as a simplicial $R$-module. Then for $k \geq 1$, $D_T^k A$ is a cofibrant simplicial $R[\Sigma_k]$ module; that is, in simplicial degree $n$, $(D_T^k A)_n$ is a projective $R[\Sigma_k]$ module. If $A \to B$ is a weak equivalence of simplicial $T$ algebras which are cofibrant in $s\mathcal{M}_R$, then*

$$D_T^k A \to D_T^k B$$

*is a weak equivalence of simplicial $R[\Sigma_k]$ modules.*

**Proof.** This result certainly holds if $T = G_V$ where $V \in s\mathcal{M}_R$ is a cofibrant simplicial $R$-module and $F_{(\cdot)}$ is the free operad functor; this follows from A.3.

If $T$ is any operad, we may take the Stover resolution with respect to the homotopy cogroup objects

$$F_\Sigma(R \wedge \Delta^t/\partial\Delta^t), \qquad t \geq 0.$$

We choose, as the cone objects used to build the Stover resolutions, the obvious:

$$F_\Sigma(R \wedge \Delta^{t+1}/\Delta_0^{t+1}), t \geq 0.$$

If $V(T) \to T$ is the Stover resolution, then

$$|V(T)| \to T$$

is a weak equivalence; indeed, since the realization can be computed at the level of simplicial $R$-modules

$$|V(T)| \cong \operatorname{diag} V(T)$$

and the standard spectral sequence shows $\pi_*|V(T)| \cong \pi_* T$.

Each $V_n(T)$ is of the form $F_\Sigma(W_n)$ where $W_n$ can be described as follows. Let $R(k)$ be the graded $R$-module which is a copy of $R$ in degree $k$. Then $W_n = \{W_n(k)\}$ and each $W_n(k)$ fits into a push-out square of simplicial $R$-modules

(6.11)
$$\begin{tikzcd}
\bigoplus_\alpha R(k) \wedge \Delta^{t_\alpha}|/\partial\Delta^{t_\alpha} \arrow[r] \arrow[d] & \bigoplus_\beta R(k) \wedge \Delta^{t_\alpha}/\partial\Delta^{t_\alpha} \arrow[d] \\
\bigoplus_\alpha R(k) \wedge \Delta^{t_\alpha+1}/\Delta_0^{t_\alpha+1} \arrow[r] & W_n(k).
\end{tikzcd}$$

Since $|V(t)| \cong \operatorname{diag} V(T)$

$$D^k_{|V(T)|}A \cong \operatorname{diag} D^k_{V(t)}A.$$

Since each $W_n$ is cofibrant, the result holds for $|V(T)|$.

To complete the argument, we claim $|V(T)| \to T$ is surjective, hence a trivial fibration of simplicial operads. Assuming the claim, we would then have that if $T$ were cofibrant, this fibration would have a section and $D^k_T A$ is a retract of $D^k_{|V(T)|}A$ and this retraction is natural in $A$. The result follows.

To see that $|V(T)| \to T$ is surjective, we need only show

$$V_0(T) \to T$$

is surjective. Since $V_0(T) = F_\Sigma(W_0)$ we need only show the inducted map $W_0 \to T$ is onto. The index $\alpha$ of (6.11) runs over the elements of

$$\operatorname{Hom}_{s\mathcal{M}_R}(R(k) \wedge \Delta^{t+1}/\Delta_0^{t+1} T(k)) \cong N_{t+1}T(k)$$

and the map $f$ of (6.11) is evaluation; hence

$$N_{t+1}W_0(k) \to N_{t+1}T(k)$$

is onto for $t \geq 0$. The index $\beta$ runs over

$$\operatorname{Hom}_{s\mathcal{M}_R}(R(k) \wedge \Delta^t/\partial\Delta^t, T(k)) = Z_t W_0(k)$$

and $g$ is again evaluation; hence

$$N_0 W_0(k) = Z_0 W_0(k) \to Z_0 T(k) = N_0 T(k)$$

is onto. Since normalization has exact inverse, we have proved our claim. ∎

We now draw corollaries from this result.

**Corollary 6.12** *Let $T$ be a cofibrant operad. Then each of the modules $T(k)$ is a projective $R[\Sigma_k]$ module.*

**Proof.** Let $A = T(0) \in s\mathcal{A}^T$. Since $T(0)$ is the initial object of $s\mathcal{A}^T$

$$T(0) \sqcup T(W) \cong T(W)$$

and $D_T^k T(0) = T(k)$. ■

**Corollary 6.13** *Let $A \in s\mathcal{A}^T$ and suppose $A$ is cofibrant as a simplicial $R$-module. Then any cofibration $A \to X$ is $s\mathcal{A}^T$ is also a cofibration in $s\mathcal{M}_R$.*

**Proof.** Using the homotopy cogroup objects $T(R \wedge \Delta^t/\partial\Delta^t)$ and cones $T(R \wedge \Delta^{t+1}/\Delta_0^{t+1})$, build the relative Stover resolution

$$W^A(X) \to X.$$

As in the proof of Proposition 6.10, $|W^A(X)| \to X$ is a trivial cofibration under $A$, so has a section under $A$. Thus it is sufficient to show $A \to |W^A(X)| = \operatorname{diag} W^A(X)$ is a cofibration in $s\mathcal{M}_R$; that is

$$A_n \to [W_n^A(X)]_n$$

is an inclusion with projective cokernel. Now $W_n^A(X) \cong A \sqcup T(U_n)$ for some cofibrant simplicial $R$-module $U_n$, so

$$W_n^A(X) \cong A \oplus \bigoplus_{k \geq 1} D_A^k T \otimes_{R\Sigma_k} U_n^{\otimes k}.$$

By proposition 6.10, $D_A^k T$ is a projective $R\Sigma_k$ module, so

$$[D_A^k T \otimes_{R\Sigma_k} U_n^{\otimes k}]_n$$

is a projective $R$ module. ■

**Proposition 6.14** *If $T$ is a cofibrant simplicial operad, then $s\mathcal{A}^T$ is a relatively left proper.*

**Proof.** By Corollary 6.13 any cofibrant $X \in s\mathcal{A}^T$ is cofibrant in $s\mathcal{M}_R$. This is half the condition. If $A \to B$ is a weak equivalence in $s\mathcal{A}^T$ with $A$ and $B$ cofibrant in $s\mathcal{M}_R$, and $W \in s\mathcal{M}_R$ is cofibrant, then

$$A \sqcup T(W) \to B \sqcup T(W) \tag{6.15}$$

is isomorphic to

$$\bigoplus_{k \geq 0} D_T^k A \otimes_{R\Sigma_k} W^{\otimes k} \to \bigoplus_{k \geq 0} D_T^k B \otimes_{R\Sigma_k} W^{\otimes k}.$$

Since, by 6.10, $D_T^k A$ is cofibrant as a simplicial $R\Sigma_k$ module; there is by [HA,§II.6] a spectral sequence

$$\mathrm{Tor}_{R[\Sigma_k]}(\pi_* D_T^k A, \pi_*(W \otimes \cdots \otimes W)) \Rightarrow \pi_* D_T^k A \otimes_{R\Sigma_k} W^{\otimes k}.$$

Since $\pi_* D_T^k A \cong \pi_* D_T^k B$, we have that (6.15) is a weak equivalence, as was to be shown. ∎

# 7 Skeletal maps and a Hurewicz Theorem.

In the category of simplicial algebras over a simplicial operad $T$, we describe a notion of skeletal map akin to the notion of relative CW complex for topological spaces, then we show that every map can be factored as a skeletal map followed by a weak equivalences. This technology makes it easy to prove a relative Hurewicz Theorem for André-Quillen homology. This is crucial to our discussion of Postnikov towers.

We begin by discussing relative homotopy groups. If $A \to X$ is a morphism of simplicial $R$-modules, we can define $\pi_*(X, A)$ in several ways, both convenient. The simplest way is as follows: factor $A \to X$ as $A \xrightarrow{j} Y \xrightarrow{q} X$ where $j$ is a cofibration in $s\mathcal{M}_R$ and $q$ is a weak equivalence, and set, for $n \geq 0$

$$\pi_n(X, A) = \pi_n(Y/A). \tag{7.1}$$

¿From this definition it is easy to show that if $A \to B \to X$ is a sequence of morphisms in $s\mathcal{M}_R$, then there is a long exact sequence

$$\cdots \to \pi_n(B, A) \to \pi_n(X, A) \to \pi_n(X, B) \to \pi_{n-1}(B, A) \to \cdots \to \pi_0(X, B) \to 0. \tag{7.2}$$

The disadvantage of this definition is that it is not clear whether the functor $\pi_n(X, A)$ is representable or not. This we now rectify.

Let $I = \{0 \to 1\}$ be the category with two objects and one non-trivial morphism. The diagram category $(s\mathcal{M}_R)^I$ is the "arrow category" with objects the morphisms $A \to X$ in $s\mathcal{M}_R$. The category $(s\mathcal{M}_R)^I$ has a simplicial model category structure with level-wise weak equivalences and fibrations (cf. [3]). Cofibrations in $(s\mathcal{M}_R)^I$ are diagrams

$$\begin{array}{ccc} A_1 & \longrightarrow & A_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \end{array}$$

with $A_1 \to A_2$ a cofibration and $X_1 \sqcup_{A_1} A_2 \to X_2$ a cofibration. Note every object in $(s\mathcal{M}_R)^I$ is fibrant and $A \to X$ is cofibrant if $A$ is cofibrant in $s\mathcal{M}_R$ and $A \to X$ is a cofibration in $s\mathcal{M}_R$. In particular if $K(R, n)$ and $WK(R, n)$ are the objects in $s\mathcal{M}_R$ with

$$NK(R, n) \cong R$$

concentrated in degree $n$ and

$$NWK(R,n)_k \cong \begin{cases} R, & k = n, n+1; \\ 0, & \text{otherwise} \end{cases}$$

and with $\pi_* WK(R,n) = 0$, then the obvious map

$$K(R,n) \to WK(R,n)$$

is a cofibrant object in $(s\mathcal{M}_R)^I$.

**Proposition 7.2** *There is a natural isomorphism, for $n \geq 1$:*

$$\begin{bmatrix} K(R,n-1) & & A \\ \downarrow & , & \downarrow \\ WK(R,n-1) & & X \end{bmatrix}_{(s\mathcal{M}_R)^I} \cong \pi_n(X,A)$$

**Proof.** Write $\pi_n'(X,A)$ for the homotopy classes of maps in $(s\mathcal{M}_R)^I$ from $K(R,n-1) \to WK(R,n-1)$ to $A \to X$. Since $WK(R,n-1)/K(R,n-1) \cong K(R,n)$ there are natural maps $\pi_n X = [K(R,n), X]_{s\mathcal{M}_R} \to \pi_n'(X,A)$ and $\pi_n'(X,A) \to \pi_{n-1}(A)$; these fit into a long exact sequence

$$\pi_n A \to \pi_n X \to \pi_n'(X,A) \to \pi_{n-1} A \to \cdots \to \pi_0 A \to \pi_0 X. \tag{7.3}$$

We define a homomorphism $\pi_n'(X,A) \to \pi_n(X,A)$ as follows. We may assume $A \to X$ is cofibration and we can send a morphism in $(s\mathcal{M}_R)^I$

$$\begin{array}{ccc} K(A,n-1) & \longrightarrow & A \\ \downarrow & & \downarrow \\ WK(A,n-1) & \longrightarrow & X \end{array}$$

to the map on quotients

$$K(A,n) \cong WK(A,n-1)/K(A,n-1) \to X/A.$$

An easy exercise shows that we get a map of long exact sequences from (7.3) to that of (7.2) with the substitutions $A = 0$ and $B = A$. The result follows by the five lemma. ■

Now let $A \to X$ be a morphism in $s\mathcal{A}^T$, where $T$ is some simplicial operad. Define $\pi_n(X,A)$ by regarding $A \to X$ as a morphism in $s\mathcal{M}_R$. Then Proposition 7.2 implies there is a natural isomorphism, $n \geq 1$,

$$\begin{bmatrix} TK(A,n-1) & & A \\ \downarrow & , & \downarrow \\ TWK(A,n-1) & & X \end{bmatrix}_{(s\mathcal{A}^T)^I} \cong \pi_n(X,A). \tag{7.4}$$

**Definition 7.5** *A morphism $A \to X$ is* skeletal *if there is a sequence of morphisms*

$$A = X_{-1} \to X_0 \to X_1 \to X_2 \to \cdots$$

*so that*

1) $X_0 = A \sqcup \coprod_{C_0} TK(R,0)$ *for some set* $C_0$

2) *for* $s \geq 1$*, there is a push-out diagram*

$$\begin{tikzcd} \coprod_{C_s} TK(R,s-1) \arrow[r] \arrow[d] & X_{s-1} \arrow[d] \\ \coprod_{C_s} TWK(R,s-1) \arrow[r] & X_s \end{tikzcd}$$

*for some set* $C_s$*; and*

3) $A \to \operatorname{colim}_s X_s$ *is isomorphic to* $A \to X$.

Note that if we agree to write $K(R,-1) = 0$ and $WK(R,-1) = K(R,0)$, then 6.5.1 and 6.5.2 can be combined into a single statement.

The next result demonstrates that any morphism can be factored as a skeletal map followed by a weak equivalence.

**Proposition 7.6** *Let $f : A \to X$ be a morphism in $s\mathcal{A}^T$. Then there is a sequence of morphisms in $s\mathcal{A}^T$*

$$A = X_{-1} \xrightarrow{j_0} X_0 \xrightarrow{j_1} X_1 \to \cdots$$

*and morphisms $q_s : X_s \to X$ so that*

1) $q_{-1} = f$ *and* $q_s \circ j_s = q_{s-1}$ *for* $s \geq 0$*;*

2) $\pi_n(X, X_s) = 0$ *for* $n \leq s$*;*

3) *if* $\pi_n(X, A) = 0$ *for* $n \leq s$*, then* $A \to X_s$ *is the identity; and*

4) *for* $s \geq 0$*, there is a set* $C_s$ *and a pushout diagram*

$$\begin{tikzcd} \coprod_{C_s} TK(R,s-1) \arrow[r] \arrow[d] & X_{s-1} \arrow[d] \\ \coprod_{C_s} TWK(R,s-1) \arrow[r] & X_s. \end{tikzcd}$$

**Proof.** We define $X_s$ recursively. If $\pi_0(X,A)=0$, set $X_0=A$. If $\pi_0(X,A)\neq 0$, choose morphisms

$$f_\alpha : K(R,0)\to X$$

generating $\pi_0(X,A)$ and define $C_0=\{f_\alpha\}$ and

$$X_0 = A\sqcup \coprod_{C_0} TK(R,0).$$

Then $j_0$ and $q_0$ are the evident maps. Since $\pi_0(X_0,A)\to\pi_0(X,A)$ is onto, the sequence 6.2 for $A\to X_0\to X$ implies $\pi_0(X,X_0)=0$.

Now assume $X_{s-1}$ has been defined and 6.6.1–6.6.4 hold. Choose generators

$$\begin{tikzcd} K(R,s-1) \arrow[r, "f_\alpha^0"] \arrow[d] & X_{s-1} \arrow[d] \\ WK(R,s-1) \arrow[r, "f_\alpha^2"] & X \end{tikzcd}$$

for $\pi_s(X_s,X_{s-1})$. Note that $s\geq 1$. Let $C_s=\{f_\alpha\}$ and define $X_s$ by the pushout diagram of 6.6.4. Letting $k$ denote simplicial degree, we have that

$$TK(R,s-1)_k\to TWK(R,s-1)_k$$

is an isomorphism for $k\leq s-1$; hence, $(X_{s-1})_k\to(X_s)_k$ is an isomorphism for $k\leq s-1$. It follows that $\pi_n X_{s-1}\to\pi_n X_s$ is an isomorphism for $n<s-1$ and onto for $n=s-1$. Hence (7.2) for the sequence $0\to X_{s-1}\to X_s$ shows $\pi_n(X_s,X_{s-1})=0$ for $n\leq s-1$. Also

$$\pi_s(X_s,X_{s-1})\to\pi_s(X,X_{s-1})$$

is onto by construction. Thus the long exact sequence

$$\cdots\to\pi_s(X_s,X_{s-1})\to\pi_s(X,X_{s-1})\to\pi_s(X,X_s)\to\pi_{s-1}(X_s,X_{s-1})\to\cdots$$

shows $\pi_n(X,X_s)=0$ for $n\leq s$ as required. Finally, for 6.6.3, note that if $\pi_s(X,X_{s-1})=0$ we may take $X_s=X_{s-1}$. ∎

We can now define our relative Hurewicz homomorphism. Let $A\to X$ be a morphism in $s\mathcal{A}^T$. For our applications, we may assume this map to be an injection. Since, for all $T$ modules over $X$ there is a natural isomorphism

$$\mathrm{Der}_A(X,M)\cong\mathrm{Hom}_X(\Omega(X/A),M)$$

we may, by setting $M=\Omega(X/A)$, obtain a universal $A$-derivation

$$d : X\to\Omega(X/A).$$

Since

$$0 \to \mathrm{Der}_A(X, M) \to \mathrm{Der}_T(X, M) \to \mathrm{Der}_T(A, M)$$

is exact, the composition

$$A \to X \xrightarrow{d} \Omega(X/A)$$

is zero and we get a morphism in $s\mathcal{M}_R$

$$h : X/A \to \Omega(X/A).$$

If we choose a diagram

$$\begin{tikzcd} A_0 \arrow[r, "j"] \arrow[d] & X_0 \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

with the vertical maps weak equivalences, in $s\mathcal{A}^T$, $A_0$ cofibrant, and $j : A_0 \to X_0$ a cofibration, then we get a diagram

$$X/A \leftarrow X_0/A_0 \xrightarrow{h} X \otimes_{X_0} \Omega(X_0/A_0) = \mathbf{L}\Omega(X/A).$$

Since $\pi_*(X/A) \cong \pi_*(X_0/A_0) \cong \pi_*(X, A)$ we get a Hurewicz homomorphism

$$h_* : \pi_*(X, A) \to \pi_*\mathbf{L}\Omega(X/A) \cong D_*(X, A).$$

This leads to a Hurewicz Theorem:

**Theorem 7.7** *Let $A \to X$ be a morphism in $s\mathcal{A}^T$ and suppose $\pi_k(X, A) = 0$ for $k < n$, where $n \geq 1$. Then $D_k(X, A) = 0$ for $k < n$ and*

$$h_* : \pi_0 X \otimes_{\pi_0 A} \pi_n(X, A) \to D_n(X, A)$$

*is an isomorphism of $\pi_0 X$ modules.*

This argument is below in 7.13. The proof uses the skeletal decomposition of Proposition 7.6 and the following lemma.

**Lemma 7.8** *Let $A \in s\mathcal{A}^T$ be cofibrant and suppose there is a pushout diagram*

$$\begin{tikzcd} \coprod_C TK(R, n-1) \arrow[r] \arrow[d] & A \arrow[d] \\ \coprod_C TWK(R, n-1) \arrow[r] & X \end{tikzcd}$$

*for some set $C$ and some $n \geq 1$. Then the Hurewicz Theorem 7.7 holds for $A \to X$.*

The argument goes by calculating source and target. This is done in 9.9, 9.10, & 9.12. The first calculation is: let $V = \bigoplus_C R$. We claim

$$\mathbf{L}\Omega(X/A) \cong \Omega(X/A) \cong X \otimes^T K(V,n). \tag{7.9}$$

Since $A$ is cofibrant and $A \to X$ is a cofibration, the first isomorphism $\mathbf{L}\Omega(X/A) \cong \Omega(X/A)$ is the definition of $\mathbf{L}\Omega(X/A)$. For the second, the pushout diagram implies

$$\Omega(X/A) \cong X \otimes_{TWK(R,n-1)} \Omega(TWK(R,n-1)/TK(R,n-1))$$

and the right hand side is easily calculated; indeed, if $V \to W$ is an inclusion of simplicial $R$ modules, then

$$\mathrm{Der}_{T(V)}(T(W), M) \cong \mathrm{Hom}_R(W/V, M);$$

so we have $\Omega(T(W)/T(V)) \cong T(W) \otimes^T (W/V)$.

We next show

**Lemma 7.9** *There is a map $A \otimes^T K(V,n) \to X/A$ of $A$-modules which induces an isomorphism on $\pi_k$ for $k < 2n$.*

**Proof.** This uses the Rezk decomposition; see Proposition 5.10 and the material preceding that result. If $S$ is any operad and $A$ is any $S$-algebra, there are $R\Sigma_n$ modules $D^k_S A$ an isomorphism of $R$-modules

$$A \sqcup S(M) \cong \bigoplus_{k \geq 0} D^k_S A \otimes_{k\Sigma_n} M^{\otimes n}$$

for all $R$-modules $M$. This is natural in $A$, $S$, and $M$, hence extends to simplicial objects. Note that $D^0_S A = A$ and $D^1_S A \otimes M \cong A \otimes^S M$.

More is true: if $A_1 \sqcup T_1(M_1) \to A_2 \sqcup T_2(M_2)$ is induced by a map of $T_1$ algebras $A_1 \to A_2$ and a morphism of $R$-modules

$$M_1 \to A_2 \oplus M_2 \to A_2 \sqcup T_2(M_2)$$

then the composition

$$D^n_{T_1} A_1 \otimes_{R\Sigma_n} M_1^{\otimes n} \to A_2 \sqcup T_2(M_2)$$

factors through

$$\bigoplus_{k=0}^{n} D^k_{T_2} A_2 \otimes_{R\Sigma_n} M_2^{\otimes k}.$$

Since each of the face and degeneracy of $X$ are of this form we get a spectral sequence with $E_1^{p,0} = 0$ and

$$E_1^{p,q} = \pi_p(D_T^q A \otimes_{k\Sigma_n} K(V,n)^{\otimes n}) \Rightarrow \pi_p(X/A). \tag{7.11}$$

The commutative diagram

$$\begin{tikzcd} K(V,n-1) \arrow[r] \arrow[d] & WK(V,n-1) \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

defines a map of $R$-modules $K(V,n) \to X/A$ which extends to a map of $A$-modules

$$A \otimes^T K(V,n) \to X/A$$

and 7.11 now implies the result. ∎

Thus to prove Lemma 7.8, we need to calculate that the composite

$$A \otimes^T K(V,n) \to X/A \to X \otimes^T K(V,n)$$

induces an isomorphism on $\pi_n$. Checking the definitions shows that the map is the obvious one induced by $A \to X$ so Lemma 7.8 follows from the next result.

**Lemma 7.12** *Let $V$ be a free $R$-module and suppose $A \in s\mathcal{A}^T$ is cofibrant. Then there is a natural isomorphism*

$$\pi_0 A \otimes^{\pi_0 T} V \xrightarrow{\cong} \pi_n(A \otimes^T K(V,n)).$$

**Proof.** The morphism is the obvious one: using the unit of the adjunction there is a natural map $K(V,n) \to A \otimes^T K(V,n)$. Also if $M \in s\mathcal{M}_A^T$, $\pi_* M$ is a $\pi_* T$ module over $\pi_* A$. Hence there is a map

$$\varphi : \pi_* A \otimes_{\pi_* T} \pi_* K(V,n) \to \pi_*(A \otimes^T K(V,n)). \tag{7.13}$$

In degree $n$ this is map of the result. To prove the claim, first notice that since the source and target of 7.13 commute with filtered colimits and sums, we may assume $V = R$. Next since limits and colimits in $s\mathcal{M}_A^T$ are made in $s\mathcal{M}_R$, we have, for $M \in s\mathcal{M}_R$, that

$$W(A \otimes^T M) \cong A \otimes^T WM$$

and, hence, there are exact sequences in $s\mathcal{M}_A^T$

$$0 \to A \otimes^T K(R,n) \to A \otimes^T WK(R,n) \to A \otimes^T K(R,n+1) \to 0$$

and $\pi_*(A \otimes^T WK(R,n)) = 0$. Hence we are reduced to the case $n = 0$. Thus we are asking that $\pi_* A \otimes^{\pi_* T} R \to \pi_*(A \otimes^T R)$ is an isomorphism in degree 0.

To show this, first suppose $A = T(M)$ where $\pi_* M$ is a free $R$-module. Then $\pi_* A = \pi_* T(\pi_* M)$ and we are asking whether

$$\pi_* T(\pi_* M, R) \to \pi_* T(M, R)$$

is an isomorphism, and it is.

For general $A$, choose a Stover resolution

$$V.A \to A.$$

Then $|V.A| \to A$ is a homotopy equivalence; hence

$$|V.A| \otimes^T M \to A \otimes^T M$$

is a homotopy equivalence. Because colimits in $T$-modules over $A$ are made in the category of $R$-modules

$$|V.A \otimes^T M| \cong diag(V.A \otimes^T M) \cong |V.A| \otimes^T M.$$

Thus we get a spectral sequence

$$\pi_p \pi_q (V.(A) \otimes^T M) \Rightarrow \pi_{p+q}(A \otimes^T M)$$

and we get a morphism of coequalizer diagrams

$$\begin{tikzcd}
\pi_0(V_1(A)) \otimes^{\pi_0 T} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \pi_0(V_0(A)) \otimes^{\pi_0 T} R \arrow[r] \arrow[d, "\cong"] & \pi_0 A \otimes^{\pi_0 T} R \arrow[d] \\
\pi_0(V_1(A) \otimes^T R) \arrow[r, shift left] \arrow[r, shift right] & \pi_0 V_0(A) \otimes^T R \arrow[r] & \pi_0(A \otimes^T R)
\end{tikzcd}$$

and the result follows. ■

We can now prove the Hurewicz Theorem.

**(6.14) The proof of Theorem 7.7**: We may assume that $A$ is cofibrant and that $A \to X$ is a skeletal map of the type supplied by Proposition 7.6. Thus there is a sequence of cofibrations

$$A = X_{-1} \to X_0 \to X_1 \to \cdots$$

where $X = \operatorname{colim} X_s$ and $A = X_s$ for $s < n$. Lemma 7.8 implies that $D_r(X_s, X_{s-1}) = 0$ for $r < s$, so the transitivity sequence and a colimit argument shows that

$$D_n(X_{n+1}, A) \cong D_n(X, A).$$

Also by Lemma 7.8 $D_s(X_n, A) = 0$ for $s < n$ and we have an isomorphism

$$\pi_0 X_n \otimes_{\pi_0 A} \pi_n(X_n, A) \to D_n(X_n, A).$$

Finally, using the transitivity sequence, we get a diagram of $\pi_0 X_{n+1} = \pi_0 X$ modules

$$\begin{tikzcd}
\pi_0 X \otimes_{\pi_0 A} \pi_{n+1}(X_{n+1}, X_n) \arrow[r] \arrow[d, "h_{n+1}"] & \pi_0 X \otimes_{\pi_0 A} \pi_n(X_n, A) \arrow[r] \arrow[d, "h_*"] & \pi_0 X \otimes_{\pi_0 A} \pi_n(X_{n+1}, A) \arrow[r] \arrow[d, "h_*"] & 0 \\
D_{n+1}(X_{n+1}, X_n) \arrow[r] & \pi_n(X_{n+1} \otimes_{X_n} \mathbf{L}\Omega_T(X_n/A)) \arrow[r] & D_n(X_{n+1}, A) \arrow[r] & 0
\end{tikzcd}$$

The map labeled $h_{n+1}$ is an isomorphism by 9.8. Since

$$\pi_s \mathbf{L}\Omega_T(X_n/A) = 0$$

for $s < n$, we have

$$\begin{aligned}
\pi_n(X_{n+1} \otimes_{X_n} \mathbf{L}\Omega_T(X_n/A)) &\cong \pi_0 X_{n+1} \otimes_{\pi_0 X_n} \pi_n \mathbf{L}\Omega_T(X_n/A) \\
&\cong \pi_0 X \otimes_{\pi_0 X_n} D_n(X_n, A)
\end{aligned}$$

and the result follows from the five lemma. ■

# 8 Postnikov Towers for Simplicial $T$-algebras.

If $X \in s\mathcal{A}^T$ is a simplicial $T$-algebras, the standard Postnikov tower of $X$, regarded as a simplicial $R$-module, refines to a Postnikov tower of simplicial $T$-algebras. In this section we say what that means, and spell out the details.

Let $S$ be an operad and $A$ an $S$-algebra. Then a sub-$R$-module $I \subseteq A$ is an *ideal* if $I$ is a sub $S$-module over $A$ of $A$. That is, $I$ is an $S$-module over $A$ and $I \to A$ is a module map. If this is the case, there is a unique structure of an $S$-algebra on the quotient $A/I$ so that $A \to A/I$ is a morphism of $S$-algebras. If $f : A \to B$ is a morphism of $S$-algebras, and $I$ is the kernel of $A$, then $I$ is an ideal.

Now let $X \in s\mathcal{M}_R$ be a simplicial $R$-module. A Postnikov tower for $X$ is a tower in $s\mathcal{M}_R$

$$\cdots P_n X \xrightarrow{q_n} P_{n-1} X \cdots \to P_2 X \xrightarrow{q_2} P_1 X \xrightarrow{q_1} P_0 X$$

equipped with maps $f_n : X \to P_n X$ so that $q_n f_n = f_{n-1}$ and so that $\pi_k P_n X = 0$ for $k > n$ and

$$\pi_k f_n : \pi_k X \to \pi_k P_n X$$

is an isomorphism for $k \leq n$.

Such towers are essentially unique (see [13] if nowhere else) and the Moore-Postnikov tower, which we now define, gives an example of such.

To define the Moore-Postnikov tower, fix $n$, and let $s \geq 0$. Let

$$I_{s,n} = \bigcap_{\phi: k \to s} \ker\{\phi^* : X_s \to X_k\} \subseteq X_s \tag{8.1}$$

where $\phi$ runs over all injections with $k \leq n$. Define $P_n X \in s\mathcal{M}_R$

$$(P_n X)_s = X_s / I_{s,n}.$$

There are surjections $f_n : X \to P_n X$ and $q_n : P_n X \to P_{n-1} X$. Furthermore, it is a simple exercise (cf. [14],§5]) to show the normalized chain complex of $NP_n X$ is given by, with $B_n X = \mathrm{Im}\{\partial : NX_{n+1} \to NX_n\} \cong NX_{n+1}/\ker(\partial)$,

$$\cdots \to 0 \to B_n X \to NX_n \to \cdots \to NX_1 \to NX_0$$

so that $\pi_* P_n X$ is as required.

This description, plus the remarks on ideals made above, and the fact that the intersection of ideals is an ideal immediately imply:

**Proposition 8.2** *Let $X \in s\mathcal{A}^T$ be a simplicial $T$-algebra. Then the Moore-Postnikov tower for $X$, regarded as a simplicial $R$-module, is a tower of simplicial $T$-algebras.* ∎

In particular, simplicial $T$-algebras have Postnikov towers. The next point is to refine $k$-invariants. The shorthand for $k$-invariants is this: we say there is, for $X \in s\mathcal{M}_R$, a homotopy pull-back diagram

$$\begin{tikzcd} P_n X \arrow[r] \arrow[d] & WK(\pi_n X, n) \arrow[d] \\ P_{n-1} X \arrow[r, "k_n"] & K(\pi_n X, n+1). \end{tikzcd}$$

Since $P_{n-1}X$ need not be cofibrant, the map $k_n$ is only a morphism in $\mathbf{Ho}(s\mathcal{M}_R)$ and need not be represented by an actual map. For our purposes we need to be more explicit.

There are many ways to define $k_n$ and to spell out the details of the above. The one that generalizes well to the simplicial $T$-algebra case is the following.

Let $X \in s\mathcal{M}_R$ and $M$ an $R$-module. Define $H_R^*(X, M)$ by the equation

$$H_R^n(X, M) \cong \pi^n \mathrm{Hom}_R(Y, M)$$

where $Y \to X$ is a weak equivalence with $Y$ cofibrant. One knows

(7.3.1) $H_R^n(X, M)$ is representable: there is an isomorphism

$$[X, K(M, n)]_{s\mathcal{M}_R} \cong H^n(X, M);$$

(7.3.2) there is a spectral sequence

$$\mathrm{Ext}_R^p(\pi_q X, M) \Rightarrow H_R^{p+q}(X, M).$$

In particular, if $\pi_q X = 0$ for $q < n$, then

$$H^n(X, M) \cong \mathrm{Hom}_R(\pi_n X, M).$$

There is a relative version: if $A \to X$ is a morphism in $s\mathcal{M}_R$, define

$$H^*_R(X, A; M) = H^*_R(Y/A; M)$$

where $A \xrightarrow{j} Y \xrightarrow{q} X$ factors the original map as a cofibration followed by a weak equivalence. Then (7.3.2) becomes

$$\mathrm{E\small{XT}}^p_R(\pi_q(X, A), M) \Rightarrow H^{p+q}_R(X, A; M). \tag{7.3.3}$$

Finally, there is a long exact sequence

$$\cdots \to H^n_R(X, A; M) \xrightarrow{r^*} H^n_R(X; M) \to H^n_R(A; M) \to H^{n+1}_R(X, A; M) \to \cdots .$$

In particular, consider the morphism $P_n X \to P_{n-1} X$, where $X \in s\mathcal{M}_R$. Then

$$\pi_q(P_{n-1}X, P_n X) \cong \begin{cases} \pi_n X & q = n+1 \\ 0 & q \neq n+1. \end{cases}$$

Hence $H^{n+1}_R(P_{n-1}X, P_n X; \pi_n X) \cong \mathrm{Hom}_R(\pi_n X, \pi_n X)$. Let

$$k_n(X) = k_n = r^* 1_{\pi_n X} \in H^n(P_{n-1}X; \pi_n X).$$

Choosing cofibrant models $P'_n X \to P'_{n-1} X$ for $P_n X \to P'_{n-1} X$ we may represent $k_n$ by a map $f_n : P'_{n-1} X \to K(\pi_n X, n+1)$ and the composite

$$P'_n X \to P'_{n-1} X \xrightarrow{f_n} K(\pi_n X, n+1)$$

is null-homotopic. Abbreviate $\pi_n X$ as $\pi_n$. The standard assertion is that if $Y$ is defined by the pull-back square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & WK(\pi_n, n) \arrow[d] \\ P'_{n-1}X \arrow[r, "f_n"] & K(\pi_n, n+1) \end{tikzcd} \tag{7.3.4}$$

then there is a weak equivalence $g : P'_n X \to Y$ making

$$\begin{tikzcd} & Y \arrow[d] \\ P'_n X \arrow[ur, "g"] \arrow[r] & P'_{n-1} X \end{tikzcd}$$

commute. The morphism $g$ is not unique, but may be altered by any element of

$$H^n(P_{n-1}X;\pi_n) \cong H^n(X;\pi_n).$$

To further recapitulate, the morphism $g$ is constructed as follows. First one notes that $Y \cong P_n Y$ and $P_{n-1}Y \simeq P'_{n-1}X$ and that we have the following result.

**Lemma 8.4** *There is an automorphism $\alpha$ of $\pi_n X$ so that*

$$k_n(Y) = \alpha^* k_n(X) \in H^{n+1}(P_{n-1}Y;\pi_n X).$$

**Proof.** Factor $Y \to P'_{n-1}X$ as $H \xrightarrow{j} Z \xrightarrow{q} P'_{n-1}X$ where $j$ is a cofibration and $q$ is a weak equivalence. Consider the diagram

$$\begin{tikzcd}
 & Z/Y \\
Z \arrow[r, "i"] \arrow[d, "q"'] & Z \sqcup_Y WK(\pi_n, n) \arrow[u, "p"] \arrow[d, "\bar{q}"] \\
P'_{n-1}X \arrow[r, "f_n"'] & K(\pi_n, n)
\end{tikzcd}$$

The morphism $p$ is a weak equivalence since $s\mathcal{M}_R$ is a proper model category, $\bar{q}$ is a weak equivalence by calculation. Then

$$\begin{aligned}
k_n(Y) &= (q^*)^{-1} i^* p^* 1\pi_n \\
&= (q^*)^{-1} i^* \bar{q}^* \alpha^* 1\pi_n
\end{aligned}$$

for some $\alpha \in \mathrm{Aut}(A)$, or

$$k_n(y) = (q^*)^{-1} q^* f_n^* \alpha^* \pi_n = \alpha^* k_n(X). \qquad \blacksquare$$

To construct $g$, let $f : P'_n X \to Y$ be any lift of $q'_n : P'_n X \to P'_{n-1}X$. Then

$$f_* : \pi_n X \cong \pi_n P'_n X \to \pi_n Y \cong \pi_n X$$

need not be an isomorphism; however, one does obtain a diagram

$$\begin{tikzcd}
H^n(Y,\pi_n) \arrow[r] \arrow[d, "f^*"] & \mathrm{Hom}_R(\pi_n \pi_n) \arrow[r, "\bar{\partial}"] \arrow[d, "\theta"] & H^{n+1}(P'_{n-1}X,\pi_n) \arrow[d, "="] \\
H^n(P_n Y,\pi_n) \arrow[r] & \mathrm{Hom}_R(\pi_n,\pi_n) \arrow[r, "\partial"] & H^{n+1}(P'_{n-1}X,\pi_n)
\end{tikzcd}$$

By Lemma 8.4, $\bar{\partial} 1_{\pi_n} = k_n(Y) = \alpha^* k_n(X)$. Also

$$\partial\theta 1_{\pi_n} = \partial f_*$$

so $\partial f_* = \alpha^* k_n(X)$. Hence

$$\partial(\alpha^* 1_{\pi_n} - f_*) = 0$$

so there is a class $\psi \in H^n(P_n X, \pi_n)$ mapping to $\alpha^* 1\pi_n - f_*$. Define $g$ to be the composite

$$P_n X \xrightarrow{\psi \times f} K(\pi_n, n) \times Y \xrightarrow{\mu} Y$$

where $\mu$ is the action map. Then

$$g_k = \alpha_* : \pi_n P_n X \cong \pi_n X \to \pi_n X \cong Y$$

as $g$ is a weak equivalence.

Now suppose $X \in s\mathcal{A}^T$ is a simplicial $T$-algebra. We'd like to go through this essentially formal program for $X$. We will assume that every cofibrant $T$-algebra is cofibrant as a simplicial $R$-module.

First note that if $M$ is a $\pi_0 T$ module over $\pi_0 X$, if $A_0$ is cofibrant and $A_0 \to X_0$ is a cofibration in $s\mathcal{A}^T$, then there is a map

$$\mathrm{Der}_{A_0}(X_0, M) \to \mathrm{Hom}_R(X_0/A_0, M)$$

hence, by our assumptions, a homomorphism

$$D^*(X, A; M) \to H^*_R(X, A; M).$$

To calculate the source of this map, we use the following result. Note that $D_*(X/A)$ is a $\pi_* T$ module over $\pi_* X$. In order to simplify notation, write $\mathcal{M}_*$ for the category of $\pi_* T$ module over $\pi_* X$. If $M$ a $\pi_0 T$ module over $\pi_0 X$, then $M \in cM_*$, regarded as an object concentrated in degree 0. Let $M(q)$ denote the same module concentrated in degree $q$.

**Lemma 8.5** *Let $A \to X$ be a morphism in $s\mathcal{A}^T$ and $M$ a $\pi_0 T$ module over $\pi_0 X$. Then there is a spectral sequence*

$$\mathrm{E{\scriptstyle XT}}^p_{\mathcal{M}_*}(D_*(X/A), M(q)) \Rightarrow D^{p+q}(X, A; M).$$

**Proof.** Choose a cofibrant model $B \to A$ and factor $B \to A \to X$ as $B \xrightarrow{j} Y \xrightarrow{q} X$ where $j$ is a cofibration and $q$ is a weak equivalence. Then $D_q(X, A) \cong \pi_q \Omega(Y, B) \cong \pi_q(X \otimes_Y \Omega(Y, B))$. Let $V.(X \otimes_Y \Omega(Y, B))$ be the Stover resolution of $X \otimes_Y \Omega(Y, B)$ considered as a simplicial $T$ module over $X$ module. Then $\pi_* V_q(X \otimes_Y \Omega(Y, B))$ is a free $\pi_* X$ module on a graded set and

$$\pi_p \pi_* V.(X \otimes_Y \Omega(Y, B)) \cong \begin{cases} \pi_*(X \otimes_Y \Omega(Y, B)) & p = 0 \\ 0 & p > 0 \end{cases}$$

Thus, by filtering by degrees in $p$, we get a spectral sequence converging to $D^{p+q}(X, A; M)$ with

$$\begin{aligned} E_1^{p,q} &\cong \pi^q \mathrm{Hom}_{\mathcal{M}_X^T}(V_p(X \otimes_Y \Omega(Y, B)), M) \\ &\cong \mathrm{Hom}_{\mathcal{M}_*}(\pi_* V_p(X \otimes_Y \Omega(Y, B)), M(q)) \end{aligned}$$

so

$$E_2^{p,q} \cong \mathrm{Ext}^p_{\mathcal{M}_*}(D_q(Y, B), M(q)). \qquad \blacksquare$$

Now let $X \in s\mathcal{A}^T$ and $P_n X \to P_{n-1}X$ the map of Postnikov sections. According to the Hurewicz Theorem in the previous section

$$D_q(P_{n-1}X/P_nX) = \begin{cases} \pi_n X & q = n+1 \\ 0 & q > n, \end{cases}$$

If we let $\mathcal{M}_0$ denote the category of $\pi_0 T$ modules over $\pi_0 X$ we have, by the previous result

$$D^{n+1}(P_{n-1}, P_nX, \pi_nX) \cong \mathrm{Hom}_{\mathcal{M}_*}(D_*(P_{n-1}X/P_nX), \pi_nX(n+1)) \cong \mathrm{Hom}_{cM_0}(\pi_nX, \pi_nX).$$

There is a natural inclusion functor

$$\mathrm{Hom}_{\mathcal{M}_*}(\pi_nX, \pi_nX) \to \mathrm{Hom}_R(\pi_nX, \pi_nX)$$

and the identity is certainly in the image.

Let $k_n^T \in D^{n+1}(P_{n-1}X, \pi_nX)$ be the image of the identity under the map in the transitivity sequence

$$D^{n+1}(P_{n-1}X, P_nX)_{\pi_nX} \to D^{n+1}(P_{n-1}X, \pi_nX) \to 0.$$

Note that $k_n^T \in D^{n+1}(P_{n-1}X, \pi_nX)$ maps to $k_n \in H_R^{n+1}(P_{n-1}X, \pi_nX)$.

As we noticed in section 2, André-Quillen cohomology is representable; let $K_+(\pi_n, n+1) = \pi_0 X \rtimes K(\pi_n, n+1)$ and $W_+K(\pi_n, n+1) = \pi_0X \rtimes WK(\pi_n, n)$. Then there is a unique homotopy class of maps of $T$-algebras

$$f_n : P'_{n-1}X \to K_+(\pi_n, n+1)$$

(where $P'_{n-1}X$ is a cofibrant model for $P_{n-1}X$) and we may define a new simplicial $T$-algebra $Y$ by the pull-back diagram

(7.6)
$$\begin{tikzcd} Y \arrow[r] \arrow[d] & W_+K(\pi_n, n) \arrow[d] \\ P'_{n-1}X \arrow[r, "f_n"] & K_+(\pi_n, n+1). \end{tikzcd}$$

The claim is that there is a weak equivalence of simplicial $T$-algebras $P'_nX \to Y$. We note that there is an isomorphism in $s\mathcal{M}_R$ between the pull-back of (7.6) and that of (7.3.4);

hence there is a weak equivalence of simplicial $R$-modules $P'_{n-1}X \to Y$; we refine this to a weak equivalence of simplicial $T$-algebras.

First note that Lemma 8.4, and this observation imply there is an $\alpha \in \mathrm{Aut}(\pi_n)$ so that

$$k_n(Y) = \alpha^* k_n(X) \in H_R^{n+1}(P_{n-1}X, \pi_n).$$

The next result is the necessary refinement.

**Lemma 8.7** *There is an $\alpha \in \mathrm{Aut}(\pi_n)$ so that*

$$k_n^T(Y) = \alpha^* k_n^T(X).$$

**Proof.** The argument is only a minor variation on the one used for Lemma 8.4. Choose a cofibrant model $Y' \to Y$ and factor $Y' \to Y \to P'_{n-1}X$ as $Y' \xrightarrow{j} Z \xrightarrow{q} P'_{n-1}X$ where $j$ is a cofibration and $q$ is a trivial fibration. Then we get, from 10.6, a diagram

$$\begin{tikzcd} Y' \arrow[r] \arrow[d] & W_+K(\pi_n, n) \arrow[d] \\ Z \arrow[r] & K_+(\pi_n, n+1) \end{tikzcd}$$

which has adjoint

$$\begin{tikzcd} Z \otimes_{Y'} \Omega(Y') \arrow[r] \arrow[d] & WK(\pi_n, n) \arrow[d] \\ \Omega(Z) \arrow[r] & K(\pi_n, n+1) \end{tikzcd}$$

and, hence, there is a diagram

(7.8)
$$\begin{tikzcd} & \Omega(Z)/Z \otimes_{Y'} \Omega(Y') \arrow[r, "r"] & K(\pi_n, n+1) \\ \Omega(Z) \arrow[r, "i"] & \Omega(Z) \sqcup_{Z \otimes_{Y'} \Omega(Y')} WK(\pi_n, n) \arrow[u, "p"] \arrow[d] & \\ & K(\pi_n, n+1) & \end{tikzcd}$$

The map $p$ is a weak equivalence as $s\mathcal{M}_Z^T$ is proper. Also

$$\Omega(Z)/Z \otimes_{Y'} \Omega(Y) \cong \Omega(Z/Y');$$

hence

$$\pi_*[\Omega(Z)/Z \otimes_{Y'} \Omega(Y)] \cong D_*(P_{n-1}X/Y).$$

The diagram that appeared in the proof of 7.4

$$
\begin{tikzcd}
 & Z/Y' \\
Z \arrow[r] & Z \sqcup_{Y'} WK(\pi_n, n) \arrow[u] \arrow[d] \\
 & K(\pi_n, n+1)
\end{tikzcd}
\tag{7.9}
$$

maps to 7.8. Note $Z/Y' \cong K(\pi_n, n+1)$ and both vertical maps of 7.9 are weak equivalences; therefore, the element

$$
\begin{aligned}
r(p^{-1})i &\in [\Omega(Z), K(\pi_n, n+1)] \\
&\cong D^{n+1}(P_{n-1}X, \pi_n)
\end{aligned}
$$

determines $k_n(Y)$. Since (10.9) maps to (10.8), we have that

$$k_n(Y) = \alpha^* k_n(X)$$

where $\alpha$ is the automorphism determined by the weak equivalence

$$K(\pi_n, n+1) \leftarrow \Omega(Z, Y') \leftarrow Z/Y' \stackrel{\cong}{\to} K(\pi_n, n+1). \qquad \blacksquare$$

With this lemma in hand, the program for producing a weak equivalence of simplicial $T$-algebras

$$g : P'_n X \to Y$$

over $P'_{n-1}X$ goes through exactly as before. One chooses any lift

$$f : P'_n X \to Y$$

and use the induced sequence

$$
\begin{tikzcd}
D^n(P_n X; \pi_n) \arrow[r] & \mathrm{Hom}_{\mathcal{M}_0}(\pi_n, \pi_n) \arrow[r] & D^{n+1}(P_{n-1}X, \pi_n) \\
D^n(Y; \pi_n) \arrow[r] \arrow[u] & \mathrm{Hom}_{\mathcal{M}-0}(\pi_n, \pi_n) \arrow[r] \arrow[u, "\theta"'] & D^{n+1}(P_{n-1}X, \pi_n) \arrow[u, "="']
\end{tikzcd}
$$

to alter $f$ as necessary.

# 9 Some computational techniques for $E_\infty$ operads.

In this section we try to make some of the theory of the rest of this paper concrete by presenting a series of examples. In particular, we will give the reductions necessary to show how the input of the Hopkins-Miller theorem can be used to show the Lubin-Tate spectrum $E_n$ is an $E_\infty$ ring spectrum.

Thus our first example concerns the following situation. Let the fixed ground ring $R$ be a field $\mathbb{F}$, where $\mathrm{char}(\mathbb{F}) = p > 0$. Suppose $A$ is a simplicial commutative $\mathbb{F}$-algebra, and suppose $T$ is an $E_\infty$ operad (in the sense to be defined below). We would like to address the problem of computing $D^T_*(A)$ given knowledge of the ordinary André-Quillen cohomology $D_*A$. Note that $D_*A = D^{\mathbf{Comm}}_* A$ where **Comm** is the commutative algebra operad.

We will first say what we mean by an $E_\infty$ operad.

**Definition 9.1** *For any commutative ring $R$ (not necessarily a field) an $E_\infty$ operad $T$ is an operad in $R$-modules equipped with a weak equivalence $T \to$ **Comm** and so that for each $k \geq 0$, $T(k)$ is a cofibrant (i.e., level-wise projective) simplicial $R[\Sigma_k]$ module.*

There is a canonical such operad—namely a cofibrant model for **Comm**—but we don't need that much structure in this part of the discussion. If $V_*$ is a cofibrant simplicial $R$-module, the shuffle chain equivalence of normalized chain complexes

$$\underbrace{N(V) \otimes \cdots \otimes N(V)}_{n} \to \underbrace{N(V \otimes \cdots \otimes V)}_{n}$$

is $\Sigma_n$ equivariant; thus if $S$ is any simplicial $R$-module operad, the normalized object $NS = \{NS(k)\}_{k\geq 0}$ is an operad of $R$-chain complexes. In particular, if $T$ is an $E_\infty$ operad in the sense of 9.1, then $NT$ is an $E_\infty$ operad in the usual algebraic sense (citeKM, [8]). More is true, if $V$ is a simplicial $R$-module, and $S$ is a simplicial operad, then there is a natural map of chain complexes

$$NS(k) \otimes_{R\Sigma_k} NV^{\otimes k} \to N(S(k) \otimes_{R\Sigma_k} V^{\otimes k}) \tag{9.2.1}$$

and if $S(k)$ is cofibrant as an $R\Sigma_k$ module and $V$ is cofibrant as an $R$-module, this is a quasi-isomorphism. In shorthand,

$$NS(NV) \to NS(V) \tag{9.2.2}$$

is a quasi-isomorphism of NS algebras. Furthermore if $A$ is any simplicial $S$-algebra, $NA$ is an $NS$ algebra via

$$NS(NA) \to NS(A) \to A.$$

¿From these considerations, and from [17], it immediately follows that if $R = \mathbb{F}$, $\mathbb{F}$ a field of char $p > 0$, and $T$ is $E_\infty$, then the homotopy of any simplicial $T$-algebra is an "unstable" algebra over the Dyer-Lashof algebra. Thus, if $A \in s\mathcal{A}^T$, then $\pi_* A$ is a graded commutative algebra equipped with operations

$$Q^i : \pi_n A \to \pi_{n+i} A, \qquad i \geq 0, p = 2$$

or

$$\beta^\epsilon Q^i : \pi_n A \to \pi_{n+2i(p-1)-\epsilon} A \qquad i \geq 0, \epsilon = 0, 1, p > 2$$

subject to the Adem and Cartan formulas of [5], §I.1. Unstable in this context means, at $p = 2$,

$$Q^i(x) = \begin{cases} 0 & i < \deg(x) \\ x^2 & i = \deg(x) \end{cases}$$

and at $p > 2$

$$\beta^\epsilon Q^i(x) = \begin{cases} 0 & 2i - \epsilon < \deg(x) \\ x^p & 2i = \deg(x) \text{ and } \epsilon = 0. \end{cases}$$

Also if $\mathbb{F}$ is not the prime field $\mathbb{F}_P$, these operations are not $\mathbb{F}$-linear; if $\phi : \mathbb{F} \to \mathbb{F}$ is the Frobenius, $a \in \mathbb{F}$ and $x \in \pi_* A$, then

$$\beta^\epsilon Q^i(ax) = \phi(a) \beta^\epsilon Q^i(x).$$

There is an obvious category $\mathcal{UR}$ of unstable algebras over the Dyer-Lashof algebra. The forgetful functor $\mathcal{UR} \to n\mathbb{F}$ to graded vector spaces has a left adjoint $\mathcal{S}_R$, and it follows from the quasi-isomorphisms of 9.2.1 and 9.2.2 and the calculations of [17] that if $T$ is an $E_\infty$ operad and $V \in s\mathcal{M}_\mathbb{F}$, then the natural map

$$\mathcal{S}_R(\pi_* V) \to \pi_* T(V)$$

is an isomorphism in $\mathcal{UR}$.

Note that this isomorphism does not depend on $T$. From this it follows:

**Proposition 9.3** *Let $T \to T'$ be a morphism of $E_\infty$ operads over a field $\mathbb{F}$ of positive characteristic. Then the forgetful functor*

$$s\mathcal{A}^{T'} \to s\mathcal{A}^T$$

*and its adjoint $T' \otimes_T (\cdot)$ descend to a Quillen equivalence of homotopy categories*

$$\mathbf{Ho}(s\mathcal{A}^{T'}) \simeq \mathbf{Ho}(s\mathcal{A}^T).$$

■

The algebra $S_R(W)$ has a simple description. See [5] §I.1. The operations $\beta^\epsilon Q^i$ can be assembled into an algebra $\mathcal{R}$ over $\mathbb{F}_p$ using the Adam relations (see [5], §I.2). This is the Dyer-Lashof algebra. If $W \in n\mathcal{M}_{\mathbb{F}}$ is a graded $\mathbb{F}$ vector space, let

$$R(W) = \mathcal{R} \otimes_{\mathbb{F}_p} W/U \tag{9.4.1}$$

where $U$ is the sub-$\mathcal{R}$-module generated by elements of the form

$$\begin{array}{lll} Q^i \otimes x, & i < \deg(x) & (p=2) \\ \beta^\epsilon Q^i \otimes x, & 2i-\epsilon < \deg(x) & p > 2. \end{array}$$

Then

$$S_R(W) = S(R(W))/I \tag{9.4.2}$$

where $S$ is the graded symmetric algebra functor and $I$ is the ideal generated by the elements

$$\begin{array}{ll} Q^i(x) - x^2 & \deg(x) = i, \quad p = 2 \\ Q^i(x) - x^p & \deg(x) = 2i, \quad p > 2 \end{array}$$

In particular $S_R(W)$ is a free graded commutative algebra.

If $\Lambda \in \mathcal{UR}$ then $\Lambda$ is, among other things, a graded commutative algebra and, as such, we can form its André-Quillen homology $D_*\Lambda$ as a graded commutative algebra: $D_*\Lambda = \pi_*\mathbf{L}\Omega\Lambda$ where $\mathbf{L}\Omega\Lambda$ is the cotangent complex as a graded commutative algebra. As usual (cf. [9]), this object inherits structure from the Dyer-Lashof operations. We next outline this structure.

Let $\mathcal{U}$ be the category of $\mathbf{N}$-graded modules over the Dyer-Lashof algebras $\mathcal{R}$. This are graded $\mathbb{F}$ vector spaces and $\mathcal{R}$ acts with the Frobenius twist defined above, and unstable measns that

$$\begin{array}{lllr} Q^i(x) = 0 & \text{if} & i \leq \deg(x) & (p=2) \\ \beta^\epsilon Q^i(x) = 0 & \text{if} & 2i - \epsilon < \deg(x) \text{ or } 2i = \deg(x) & (p > 2) \end{array}$$

If $\Lambda \in \mathcal{UR}$ then $\mathcal{U}(\Lambda)$ is the category of objects $M$ which are at once in $\mathcal{U}$, and graded $\Lambda$-modules subject to the compatibility condition that the multiplication map

$$\Lambda \otimes M \to M$$

is a morphism in $\mathcal{U}$.

Such structures arise naturally as follows: If $M \in \mathcal{U}(\Lambda)$, let $\mathrm{Der}_{\mathcal{R}}(\Lambda, M)$ be the module of commutative algebra deviations that commute with the elements of $\mathcal{R}$.

**Lemma 9.5** *Let $\Lambda \in \mathcal{UR}$. The graded module $\Omega\Lambda$ of commutative algebra derivations is naturally an object in $\mathcal{U}(\Lambda)$ and there is a natural isomorphism*

$$\mathrm{Der}_{\mathcal{R}}(\Lambda, M) \cong \mathrm{Hom}_{\mathcal{U}(\Lambda)}(\Omega(\Lambda), M).$$

**Proof.** First suppose $\Lambda = \mathcal{S}_R(V)$ for some $V \in n\mathcal{M}_{\mathbb{F}}$. Then $\mathcal{S}_R(V)$ is a free commutative algebra (see [5], §I.2), and

$$\Omega S_R(V) \cong S_R(V) \otimes P(V)$$

where $P(V) = QS_R(V)$ is the $\mathcal{R}$-module of indecomposables. This has a structure as a $S_R(V)$ module and thus structure is natural with respect to all morphisms $S_R(V) \to S_R(W)$ in $\mathcal{UR}$. The general case now follows from the coequalizer diagram

$$\Lambda \otimes_{S^2_R(\Lambda)} \Omega(S^2_{\mathcal{R}}(\Lambda)) \rightrightarrows \Lambda \otimes_{S_{\mathcal{R}}(\Lambda)} \Omega(S_{\mathcal{R}}(V)) \to \Omega(V). \qquad \blacksquare$$

¿From this we can make the following observation. Choose a cofibrant $X$ model for $\Lambda \in \mathcal{UR}$ regarded as a constant object in $s\mathcal{UR}$. We may assume $X_q = S_{\mathcal{R}}(V_q)$ for some $V_q \in n\mathcal{M}_{\mathbb{F}}$; hence, as a simplicial graded commutative algebra, $X \to \Lambda$ is still a cofibrant model for $\Lambda$. Thus

$$\mathbf{L}\Omega(V) \simeq \Lambda \otimes_X \Omega X \tag{9.6}$$

acquires, by Lemma 9.5, the structure of a simplicial $s\mathcal{U}(\Lambda)$ module—in fact, $\Lambda \otimes_X \Omega X$ is a cofibrant object in $s\mathcal{U}(\Lambda)$. Thus

$$D_*\Lambda \cong \pi_*\mathbf{L}\Omega(\Lambda) \tag{9.7}$$

is a graded object in $\mathcal{U}(\Lambda)$, and this structure is independent of the choice of $X$.

One can go further. If $\Lambda \in \mathcal{UR}$, and $M$ is a $\Lambda$-module, then $M$ has a structure as an object as an $\mathcal{R}$-module simply defined by $\beta^\epsilon Q^i(x) = 0$ for all $i$ and $\epsilon$. Then $\Lambda \otimes M$ is also a trivial $\mathcal{R}$ module object in $\mathcal{U}(\Lambda)$ since

$$Q^i(a \otimes b) = \sum_{j+k=i} Q^j(a) \otimes Q^k(b) = 0;$$

hence $M \in \mathcal{U}(\Lambda)$. This functor from $\Lambda$-modules to $\mathcal{U}(\Lambda)$ has a left adjoint defined as follows.

Write $\Lambda \odot \mathcal{R}$ for the tensor product $\Lambda \otimes_{\mathbb{F}_p} \mathcal{R}$ with product

$$(a \otimes x)(b \otimes y) = \Sigma a x_i(b) \otimes \bar{x}_i y$$

where $\Delta x = \Sigma x_i \otimes \bar{x}_i$ in $\mathcal{R} \otimes \mathcal{R}$. An object in $\mathcal{U}(\Lambda)$ is an $\Lambda \odot \mathcal{R}$ module. There is an algebra homomorphism $\Lambda \odot \mathcal{R} \to \Lambda$ given by $a \otimes x \mapsto a\epsilon(x)$ where $\epsilon : \mathcal{R} \to \mathbb{F}_p$ is the augmentation. Then the left adjoint needed is the functor

$$M \mapsto \Lambda \otimes_{\Lambda \odot \mathcal{R}} M.$$

Notice this functor is characterized by its behavior on free $\Lambda$ modules:

$$\Lambda \otimes N \mapsto \Lambda \otimes (N/\bar{\mathcal{R}} \cdot N)$$

where $\bar{\mathcal{R}} \subseteq \mathcal{R}$ is the augmentation ideal. Thus if $\Lambda \in s\mathcal{UR}$ one can consider the "strong abelianization" functor $\Omega_{\mathcal{R}}(V)$

$$\Lambda \mapsto \Lambda \otimes_{\Lambda \odot \mathcal{R}} \Omega(V) := \Omega_{\mathcal{R}}(V). \tag{9.8}$$

This has left derived functors, a kind of André-Quillen homology given by

$$D_*^{\mathcal{R}}(\Lambda) = \pi_*[\Lambda \otimes_X \Omega_{\mathcal{R}}(X)]$$

where $X \to \Lambda$ is a cofibrant model for $\Lambda$ in $s\mathcal{UR}$. Note that $D_*^{\mathcal{R}}(\Lambda)$ is bigraded, the internal grading arising from that of $\mathcal{UR}$.

The groups $D_*^{\mathcal{R}}(\Lambda)$ can be computed by a composite functor spectral sequence. If $M \in \mathcal{U}(\Lambda)$ we write

$$\mathrm{Tor}_P^{\mathcal{U}(\Lambda)}(\Lambda_\epsilon, M) \tag{9.9}$$

for the left derived functors of $\Lambda \otimes_{\Lambda \odot \mathcal{R}} (\bullet)$ with respect to the projectives

$$P_\Lambda(V) = \Lambda \otimes P(V).$$

Here $P : n\mathcal{M}_{\mathbb{F}} \to \mathcal{U}$ is left adjoint to the forgetful functor. The notation $\Lambda_\epsilon$ is (9.9) refers to the fact that the augmentation $\Lambda \odot \mathcal{R} \to \Lambda$ induces a trivial $\mathcal{R}$-module structure on $\Lambda$.

**Proposition 9.10** *Let $\Lambda \in \mathcal{UR}$. Then there is a spectral sequence*

$$\mathrm{Tor}_p^{\mathcal{U}(\Lambda)}(\Lambda_\epsilon, D_q\Lambda) \Rightarrow D_{p+q}^{\mathcal{R}}(\Lambda).$$

**Proof.** If $\Lambda \in \mathcal{UA}$, let

$$(S_R)_\bullet(\Lambda) \to \Lambda$$

be the canonical cotriple resolution; similarly, if $M \in \mathcal{U}(\Lambda)$, let $(P_\Lambda)_\bullet(M) \to M$ be the cotriple resolution. Consider the bisimplicial object

$$\Lambda \otimes_{\Lambda \odot \mathcal{R}} (P_\Lambda)_\bullet[\Lambda \otimes_{(S_R)_\bullet \Lambda} \Omega(S_R)_\bullet(\Lambda)].$$

The result follows by considering the two different filtrations once one notices that (9.4.2) implies that for any $V \in n\mathcal{M}_{\mathbb{F}}$

$$\Lambda \otimes_{S_R(V)} \Omega(S_R(V)) \cong \Lambda \otimes P(V). \qquad \blacksquare$$

**Proposition 9.11** *Let $T$ be an $E_\infty$ operad over a field $\mathbb{F}$ of characteristic $p > 0$. Let $A \in s\mathcal{A}^T$. Then there is a spectral sequence*

$$D_p^{\mathcal{R}}(\pi_* A)_q \Rightarrow D_{p+q}^T A.$$

**Proof.** We may assume $A$ is cofibrant. Let $T_\bullet A \in s(s\mathcal{A}^T)$ be the standard resolution of $A$ obtained by iterating $T$:

$$T_p A = T^{p+1}(A).$$

Then, the underlying augmented bisimplicial vector space

$$T_\bullet A \to A$$

has a simplicial contraction; thus we have a weak equivalence of simplicial algebras

$$\mathrm{diag}(T_\bullet A) \cong |T_\bullet A| \to A.$$

Since both source and target are cofibrant, $D_*^T|T_\bullet A| \cong D_*^T A$; more specifically,

$$\begin{aligned} D_*^T A &\cong \pi_* A \otimes_{|T_\bullet A|} \Omega^T |T_\bullet A|| \\ &\cong \pi_* \,\mathrm{diag}\, A \otimes_{T_\bullet A} \Omega^T T_\bullet A. \end{aligned}$$

Thus filtering the simplicial object

$$A \otimes_{T_\bullet A} \Omega^T T_\bullet A$$

by degree in $p$ we get a spectral sequence converging to $D_*^T A$ with

$$E^1_{p,q} \cong \pi_q A \otimes_{T^{p+1}A} \Omega^T T^{p+1} A. \tag{9.12.1}$$

For any $T$-algebra map $T(V) \to A$, we have

$$A \otimes_{T(V)} \Omega^T T(V) \cong A \otimes^T V.$$

Hence, since $A$ is cofibrant, we have (by virtue of Lemma 9.13 below), natural isomorphism

$$\begin{aligned} \pi_* A \otimes_{T(V)} \Omega^T T(V) &\cong \pi_* A \otimes \pi_* V \\ &\cong \pi_* A \otimes_{\pi_* A \odot \mathcal{R}} \pi_* A \otimes P(\pi_* V). \end{aligned} \tag{9.12.2}$$

The functor $W \to \Lambda \otimes P(W)$ is the left adjoint to the forgetful functor $\mathcal{U}(\Lambda) \to n\mathcal{M}_{\mathbb{F}}$. Since $\pi_* T(V) \cong S_{\mathcal{R}}(\pi_* V)$, the isomorphism (9.4.2) yields

$$\Omega(\pi_* T(V)) = \pi_* T(V) \otimes P(\pi_* V). \tag{9.12.3}$$

Combining 9.12.1, 9.12.2, and 9.12.3, we have

$$E^1_{\bullet,*} \cong \pi_* A \otimes_{(S_{\mathcal{R}})_\bullet(\pi_* A)} \Omega^{\mathcal{R}} (S_{\mathcal{R}})_\bullet (\pi_* A)$$

where $(S_{\mathcal{R}})_\bullet \pi_* A \to \pi_* A$ is the canonical cotriple resolution of $\pi_* A$ in $s\mathcal{UR}$. Hence

$$E^2_{p,*} = D_p^{\mathcal{R}}(\pi_* A)$$

as required. ■

This argument used a property of $E_\infty$ operads which is not true in general; namely, there is a natural isomorphism, for cofibrant $A \in s\mathcal{A}^T$,

$$\pi_*(A \otimes V) \to \pi_*(A \otimes^T V).$$

We close this section by proving this.

For the moment, let $T$ be any simplicial operad and $V$ any simplicial $R$ module. Then there is a natural composition, for $A \in s\mathcal{A}^T$

$$\begin{aligned} T(2) \otimes A \otimes V &\to T(A, V) = \bigoplus_n T(n+1) \otimes_{\Sigma_n} A^{\otimes n} \otimes V \\ &\to T(A, A \otimes^T V) \\ &\to A \otimes^T V. \end{aligned}$$

Call the composition $\lambda$.

**Lemma 9.13** *Let $T$ be an $E_\infty$ operad over any commutative ring $R$, and suppose $A \in s\mathcal{A}^T$ is cofibrant and $V \in s\mathcal{M}_R$. Then*

$$\lambda : T(2) \otimes A \otimes V \to A \otimes^T V$$

*is a weak equivalence of simplicial $R$-modules and induces an isomorphism*

$$\pi_*(A \otimes V) \to \pi_*(A \otimes^T V).$$

**Proof.** Since $T(2)$ is cofibrant as a simplicial $R$-module and $\pi_* T(2) \cong R$, the statement on homotopy follows from the statement about $\lambda$. To prove that, first assume $A = T(W)$ for some cofibrant $W \in s\mathcal{M}_R$. Then, unraveling the definitions, $\lambda$ is the map induced by

$$\begin{aligned} T(2) \otimes T(n) \otimes_{R\Sigma_n} W^{\otimes n} \otimes V &\to T(2) \otimes T(n) \otimes_{R\Sigma_n} W^{\otimes n} \otimes T(1) \otimes V \\ &\to T(n+1) \otimes_{R\Sigma_n} W^{\otimes n} \otimes V. \end{aligned}$$

Since $T(n+1)$ is a cofibrant $R\Sigma_n$ module, $W^{\otimes n}$ is cofibrant as an $R$-module, and $T(2)$ is cofibrant and weakly equivalent to $R$, the result follows in this case.

More generally, take a cofibrant $A \in s\mathcal{A}^T$ and let

$$V_\bullet(A) \to A$$

be the Stover resolution of $A$ with respect to the homotopy cogroup object $T(R \wedge \Delta^n / \partial \Delta^n)$. Then $|V_\bullet(A)| \to A$ is a homotopy equivalence; hence $|V_\bullet(A)| \otimes^T V \to A \otimes^T V$ is a homotopy equivalence. Also $V_\bullet(A) \otimes V \to A \otimes V$ is a homotopy equivalence. Thus we need only show

$$T(2) \otimes |V_\bullet(A)| \otimes V \to |V_\bullet(A)| \otimes^T V$$

is a weak equivalence. Since $|V_\bullet(A)| = \operatorname{diag} V_\bullet(A)$ and $V_p(A) = T(W_q)$ for some cofibrant $W_q \in s\mathcal{M}_R$, the result follows by a diagonal argument. ■

Another useful property of $E_\infty$ operads is the following. let $A$ and $B$ be simplicial algebras over a simplicial $E_\infty$ operad $T$. Then we can define a map of simplicial $R$-modules

$$\eta : T(2) \otimes A \otimes B \to A \sqcup B$$

by the composite

$$T(2) \otimes A \otimes B \to T(2) \otimes A \sqcup B \otimes A \sqcup B \to A \sqcup B.$$

**Proposition 9.14** *If $A$ and $B$ are cofibrant $T$-algebras, then the morphism $\eta$ is a weak equivalence of simplicial $R$-modules.*

**Proof.** We begin with case $A = T(V)$ and $B = T(W)$, with $V$ and $W$ cofibrant simplicial $R$-modules. Then we are asserting that for $n, k \geq 0$,

$$\begin{aligned} T(2) \otimes T(k) \otimes_{R\Sigma_k} V^{\otimes k} \otimes T(n) \otimes_{R\Sigma_k} W^{\otimes k} &\cong T(2) \otimes T(k) \otimes T(n) \otimes_{R[\Sigma_k \times \Sigma_n]} V^{\otimes k} \otimes W^{\otimes k} \\ &\to T(k+n) \otimes_{R[\Sigma_k \times \Sigma_n]} V^{\otimes k} \otimes W^{\otimes k} \end{aligned}$$

is a weak equivalence. Since $T(2) \otimes T(k) \otimes T(n) \to T(k+n)$ is a weak equivalence of cofibrant $R[\Sigma_k \times \Sigma_n]$ modules, this is clear.

For the general case, let $V_\bullet A \to A$ and $V_\bullet B \to B$ be Stover resolutions in $s(s\mathcal{A}^T)$ with respect to the homotopy cogroup objects $T(R \wedge \Delta^n/\partial\Delta^n)$. Then there are homotopy equivalences

$$|V_\bullet A| \to A \text{ and } |V_\bullet B| \to B$$

hence a homotopy equivalence

$$\operatorname{diag}(V_\bullet A \sqcup V_\bullet B) = |V_\bullet A \sqcup V_\bullet B| \to A \sqcup B.$$

Thus we get a diagram of spectral sequences (filling in the external simplicial direction),

$$\begin{tikzcd} \pi_p\pi_q(T(2) \otimes V_\bullet(A) \otimes V_\bullet(B)) \arrow[r, Rightarrow] \arrow[d, "\cong"] & \pi_{p+1}(T(2) \otimes A \otimes B) \arrow[d] \\ \pi_p\pi_q(V_\bullet(A) \sqcup V_\bullet(B)) \arrow[r, Rightarrow] & \pi_{p+q}(A \sqcup B). \end{tikzcd}$$

The result follows. ■

¿From this result we can deduce the following.

**Proposition 9.15** *Suppose the ground $R$ is a field, and suppose we have a two source*

$$X \xleftarrow{j} A \xrightarrow{f} B$$

*in $s\mathcal{A}^T$, $T$ and $E_\infty$ operad, with $j$ a cofibration and $A$ and $B$ cofibrant in $s\mathcal{A}^T$. Then there is a spectral sequence*

$$\mathrm{Tor}_p^{\pi_* A}(\pi_* X, \pi_* A) \Rightarrow \pi_{p+q}(X \sqcup_A B).$$

**Proof.** Let $\mathbb{B}_\bullet(B, A, X) \in s(s\mathcal{A}^T)$ be the bar resolution; thus

$$\mathbb{B}_n(B, A, X) = B \sqcup \underleftrightarrow{A \sqcup A \cdots \sqcup A}_{n} \sqcup X.$$

If we take $f = 1 : A \to A$, then, since $A$ and $X$ are cofibrant, $\mathbb{B}_\bullet(A, A, X)$ is Reedy cofibrant and hence

$$A \to |\mathbb{B}(A, A, X)| = \mathrm{diag}\, \mathbb{B}_\bullet(A, A, X)$$

is a cofibration. Since $\mathbb{B}_\bullet(A, A, X)$ has a simplicial contraction

$$|\mathbb{B}_\bullet(A, A, X)| \to X$$

is a weak equivalence. Since $j$ is cofibration, it follows that this is a homotopy equivalence under $A$. Thus

$$B \sqcup_A |\mathbb{B}_\bullet(A, A, X)| = |\mathbb{B}_\bullet(B, A, X)| \to B \sqcup_A X$$

is a homotopy coequivalence. Filtering $\mathbb{B}_\bullet(B, A, X)$ in the external simplicial degree we have a spectral sequence converging to $\pi_*(B \sqcup_A X)$ with

$$\begin{aligned} E^1_{p,*} &= \pi_* \mathbb{B}_p(B, A, X) \\ &= \pi_* B \otimes \pi_* A^{\otimes p} \otimes \pi_* X \\ &= \mathbb{B}_p(\pi_* B, \pi_* A, \pi_* X). \end{aligned}$$

We use the previous result and the fact we are working over a field. Thus $E^2$ is as required. ■

More generally—dropping the requirement that $R$ be a field—we have:

**Proposition 9.16** *Suppose $T$ is a simplicial $E_\infty$ operad and $j : A \to X$ is a cofibration with $A$ cofibrant. Let*

$$A \xrightarrow{f} B \xrightarrow{g} B'$$

*be a sequence of morphisms in $s\mathcal{A}^T$ with $B$ and $B'$ cofibrant and $g$ a weak equivalence. Then*

$$X \sqcup_A B \to X \sqcup_A B'$$

*is a weak equivalence.*

**Proof.** Arguing as in the previous result, we get a morphism of spectral sequences

$$\begin{tikzcd}
\pi_q\mathbb{B}_p(B,A,X) \arrow[r, Rightarrow] \arrow[d] & \pi_{p+q}(X\sqcup_A B) \arrow[d] \\
\pi_q\mathbb{B}_p(B',A,X) \arrow[r, Rightarrow] & \pi_{p+q}(X\sqcup_A B)
\end{tikzcd}$$

9.14 and the fact that every cofibrant $T$-algebra is a cofibrant as a simplicial $R$-module (as in Corollary 6.13) give an isomorphism of $E^n$-torus.

In particular, we have the following:

**Corollary 9.17** *Let $T$ be a simplicial $E_\infty$-operad. Given a two source in $s\mathcal{A}^T$*

$$X \xleftarrow{j} A \xrightarrow{f} B$$

*with $j$ a cofibration and $A$ and $B$ cofibrant, the canonical map from the homotopy push-out to $X\sqcup_A B$ is a weak equivalence.* ∎

# A Appendix: Trees and free operads

The purpose of this section is to give a description of the free operad as a graded $R$-module using the language of trees. We will concentrate on non-$\Sigma$ operads, for if $F_V$ is the free non-$\Sigma$ operad on $V$, then $G_V = \{R[\Sigma_k]\otimes F_V(k)\}_{k\geq 0}$ is the free operad on $V$. The primary goal of this is to give a decomposition, in the category of $G_V$ algebras of

$$A\sqcup F(Z).$$

where $Z$ is an $R$-module. This was used in section 5 when discussing cofibrant operads (see Proposition 5.10, in particular).

This section is written in the spirit of [7], and we use the language of that paper.

A *totally ordered tree* $T$ is an oriented, contractible graph so that

1) each vertex has a (possibly empty) ordered set of input edges;

2) each vertex has at exactly one output edge;

3) there are only finitely many vertices and there is exactly one edge with no output vertex.

For each vertex $v\in T$, let $l(v)$ be the number of input edges. Note that $l(v)\geq 0$. If unique edge without an output vertex is called the output edge. There is an obvious notion of isomorphism of totally ordered trees.

The ordering an input edges places an ordering on the vertices. If $v$ is any vertex, there is a unique path $\sigma_v$ (i.e., chain of oriented edges) from $v$ to the output edge. If $v$ and $w$ are two vertices, let $u$ be the first vertex where $\sigma_v$ and $\sigma_w$ agree. Then $v > w$ if the input edge from $v$ to $u$ is greater than the input edge from $w$ to $u$ or if $u = w$. This in turn puts a total ordering on the incoming external edges of $T$: $e < f$ is the output vertex of $e$ is less than the output vertex of $f$, or if these output vertices are the same, if $e < f$ in the ordering already given.

If $T$ is a totally ordered tree with ordered incoming external edges $e_1, \dots, e_n$ and if $T_1, \dots, T_n$ is a set of totally ordered trees, then the tree $T[T_1, \dots, T_n]$ obtained by grafting $T_i$ on at $e_i$ is also a new totally ordered tree.

If $T$ is a totally ordered tree with exactly $n$ incoming external edges, the $T$ is a tree on the set $[n] = \{1, 2, \dots, n\}$ in the sense of [7]. Let $o\mathcal{T}(n)$ be the set of isomorphism classes of such totally ordered trees.

Fix a commutative ring $R$ and let $V = \{V(n)\}_{n \geq 0}$ be a graded $R$ module. We describe the free (non-$\Sigma$) operad on $V$. Let $T$ be totally ordered tree and define

$$V(T) = \bigotimes_{v \in T} V(l(v))$$

where we use the total ordering on the vertices to decide in which order to take the tensor product. If $T_0$ is the unique tree in $o\mathcal{T}(1)$ with on edge and no vertices, set $V(T_0) = R$. Define

$$F_V(n) = \bigoplus_{T \in o\mathcal{T}(n)} V(T).$$

Let $F_V = \{F_V(n)\}$ and let $\mathbf{1} \to F(V)$ be the map of graded $R$-modules induced by $R = V(T_0)$. Then the grafting operation on trees induces a multiplication

$$F_V(n) \otimes F_V(k_1) \otimes \cdots \otimes F_V(k_n) \to F_V(k_1 + \cdots + k_n);$$

indeed, there is a canonical isomorphism

$$V(T[T_1, \dots, T_n]) \cong V(T) \otimes V(T_1) \otimes \cdots \otimes V(T_n).$$

The following is now proved exactly as Proposition 1.10 of [7].

**Proposition A.1** *The graded $R$ module $F_V$ is a non-$\Sigma$ operad and the functor $V \to F_V$ is left adjoint to the forgetful functor from non-$\Sigma$ operads to graded $R$-modules.*

Note that

$$F_V(0) = \bigoplus_{T \in o\mathcal{T}(0)} V(T)$$

is the free $F_V$ algebra on $V(0)$. Also, if each $V(n)$ is projective and $Z$ is projective, then the free $F_V$ algebra $F_V(Z)$ is also projective as an $R$-module.

Now suppose $A$ is an $F_V$ algebra. We can use this same language to describe the category of $F_V$ algebras $B$ under $A$. In the set $o\mathcal{T}(n)$ of totally ordered trees we consider only those trees $T$ with the following property: if $v$ is a vertex of $T$ either the set of parents of $T$ is empty or there exists some parent $w$ of $v$ with $l(v) \neq 0$. Call this set of trees $o\mathcal{T}(n)_+$. Note that $o\mathcal{T}(0)_+$ has a single element.

We define a graded $R$ module $V_A$ by

$$V_A(n) = \begin{cases} V(n) & n \geq 1 \\ A & n = 0. \end{cases}$$

If $T \in o\mathcal{T}(n)$ there is a collapse map $T \to T'$ to an element $T' \in o\mathcal{T}(n)_+$ given by collapsing edges that which have an vertex $v$ with a non-empty set of parents $W$ so that $l(w) = 0$ for all $w \in W$. This collapse operation may need to be repeated. Since $A$ is a $F_V$ algebra, note that this collapse map is covered by a mulitplication map

$$V_A(T) \longrightarrow V_A(T').$$

Define an graded $R$-module $F_V[A]$ by

$$F_V[A](n) = \bigoplus_{T \in o\mathcal{T}(n)_+} V_A(T).$$

There is a multiplication

$$F_V[A](n) \otimes F_V[A](k_1) \otimes \cdots \otimes F_V[A](k_n) \to F_V[A](k_1 + \cdots + k_n)$$

given by the grafting of tress and the collapse operations described in the previous parapgrah. The following can now be proved as in [7], Lemma 1.18.

**Proposition A.2** *The graded $R$ module $F_V[A]$ is a non-$\Sigma$ operad and the category of $F_V[A]$ algebras is equivalent to the category of $F_V$ algebras under $A$.*

This result makes it simple to describe the Rezk decomposition of a free operad. Let $V$ be a graded $R$-moduel and $G_V = \{R[\Sigma_n] \otimes F_V(n)\}$ the free operad (as opposed to the free non-$\Sigma$ operad) on $V$. The a $G_V$ algebra is the same thing as an $F_V$ algebra and one would like describe the decomposition

$$A \sqcup G_V(Z) = \bigoplus_k D^{(k)}_{G_V} A \otimes_{R[\Sigma_k]} Z^{\otimes k}.$$

Here is $A$ is a $G_V$ algebra and the coproduct is in the category of $G_V$ algebras. The result is this:

**Proposition A.3** . *There is an isomorphism of $R[\Sigma_k]$ modules*

$$D^{(k)}_{G_V} A \cong R[\Sigma_k] \otimes F_V[A](k).$$

**Proof.** The essential observation is that the functor

$$Z \mapsto A \sqcup G_V(Z)$$

is left adjoint to the forgetful functor from $G_V$ algebras under $A$ to $R$ modules. Hence

$$\begin{aligned} A \sqcup G_V(Z) &\cong \bigoplus F_V[A](k) \otimes Z^{\otimes k} \\ &\cong \bigoplus [R[\Sigma_k] \otimes F_V[A](k)] \otimes_{R[\Sigma_k]} Z^{\otimes k} \end{aligned}$$

■

We remark that the arugument of this section apply to describe free operads on any decent symmetric monoidal category – for example the category of **Top** of compactly generated weak Hausdorff spaces. One gets some interesing formulas: if $E_*$ is a homology theory with a Küuneth spectrals sequence and $X = \{X(n)\}$ is a graded space such that $E_*X(n)$ is a projective $E_*$ module for all $n$, then there is an isomorphism of operads in $E_*$ modules

$$G_{E_*X} \cong E_*G_X,$$

where the free operad functor $G.$ has to be interpreted in the appropriate category.

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