André-Quillen (Co-)homology for Simplicial Algebras over Simplicial Operads

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Abstract

We develop a Quillen-type homology and cohomology theory for simplicial algebras over simplicial operads. In form, this theory follows closely that of the classical André-Quillen (co-)homology for commutative algebras: there are differentials, derivations, square-zero extensions and so on in this setting. For the needs of stable homotopy theory, we are led to consider operads, algebras, and so on in the category of Γ-comodules for some Hopf algebroid (R, Γ), and here we pick up the classical thread developed by Illusie. We close with a discussion of Postnikov towers in this setting. The observation here is that the stages in a Postnikov tower for a simplicial algebra are homotopically square-zero extensions.

In his work on homotopical algebra [21], Quillen was led to consider the homotopy theory and homology of simplicial modules over simplicial algebras. These problems arise naturally in the context of the homotopical algebra of R-algebras over some commutative ring R, as a cofibrant model (or resolution) of any R-algebra is a non-constant simplicial R-algebra. Since operads are to algebras what algebras are to modules, one might expect to encounter non-constant simplicial operads and, then, one must consider the homotopy theory of simplicial algebras over simplicial operads. We are especially interested in the André-Quillen homology of such simplicial algebras, as these are the groups that arise in the context of stable homotopy theory.

In fact, we were led to consider the simplicial operads because of the following computational difficulty that arises in homotopy theory. If $\mathcal{F}$ is an E_∞ operad over the linear isometries operad $\mathcal{L}$ – for example, $\mathcal{F}$ could be $\mathcal{L}$ itself, which is the case in [7] – and X is a spectrum, one would like to calculate with the free $\mathcal{F}$-algebra $\mathcal{F}(X)$. In particular, if E is some generalized homology theory, one would like to compute $E_*\mathcal{F}(X)$. In general this is very difficult, as it requires, at the very least, a thorough knowledge of the E_* Dyer-Lashof algebra. However, it is possible to resolve $\mathcal{F}$ by a simplicial operad T over $\mathcal{L}$ so that if E_* has a Künneth spectral sequence and E_*X is projective as an E_* module, then the simplicial object $E_*T(X)$ is a free algebra over the simplicial operad E_*T on E_*X. This makes computations accessible. This application is discussed in detail in [14].

As this last paragraph indicates, we will not only be interested in simplicial operads in the category of R modules for some commutative ring R, but also in simplicial operads in the category of Γ comodules where (R, Γ) is some Hopf algebroid over R – for example, (E_*, E_*E).

This paper is the second in a series of three papers designed to come to terms with existence of multiplicative structures on spectra and with the homotopy types of spaces of multiplicative maps between multiplicative spectra. The other two ([12],[13]) are more homotopy theoretic in nature; this paper is something of an algebraic intermezzo. This paper, and, indeed, all of the work of these papers should be compared with the work of Robinson and Whitehouse; see [26] and [27]

The theory of André and Quillen adapts in a fairly straightforward manner to define the homology and cohomology of simplicial algebras over a simplicial operad, at least when working in the base

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category of R-modules over some commutative ring R. This is mostly because most of the necessary algebraic notions of commutative ring theory – modules, derivations, Kähler differentials and the like – have close generalizations in the operadic setting. The point of sections 1 and 2 is to explain this, and we suspect that there is very little new here; indeed, much of this seems to have been more or less understood by people working with operads – see, for example, [9] and, to our knowledge, Benoit Fresse and Muriel Livernet have thought about such things – but the prevailing emphasis on the differential graded category is slightly different than our emphasis on simplicial objects, and the latter is much more suitable for our purposes. The simplicial point of view has arisen, however; see [24].

Our purposes, in fact, take us into a slightly deeper waters. As mentioned above, we are interested in operads, algebras over operads, and so on in the category of Γ-comodules for some Hopf algebroid (R, Γ). The underlying abelian category will be simplicial Γ-comodules, and this category has the difficulty that in its natural closed model category structure, not every object is fibrant – making this category different than simplicial R-modules. (See §3.) As a result, many of the standard arguments need care and modification, and we drift very close to the situation encountered by Illusie [16] when he worked out the definitions and applications of André-Quillen homology and cohomology for commutative algebras in some category of sheaves or, more generally, in a topos. There one mixes locally-free sheaves and injective sheaves, and we adapt these techniques to our situation; however, we do add some model category underpinnings to our work. The reader aware of the connection between Γ-comodules and sheaves on an associated algebraic stack (c.f. the appendix to [28]) will realize that the connection of our work to Illusie's is quite strong. We could have worked out the theory of simplicial algebras over simplicial operads and the associated cohomology theories in a topos, but it would require translation for our applications; thus, it seemed appropriate to leave that for another day. Also, in some sense, the homology of algebras in Γ-comodules can be computed as homology in algebras in R-modules (see Lemma 4.8, Proposition 4.9, and Proposition 4.11) – a fact that is important for our applications.

The fifth section is devoted to working out the theory of square-zero (or "infinitesimal") extensions in both our categories of interest. The theory goes more or less as expected, although we work more generally than usual: we allow square-zero extensions of simplicial algebras by simplicial modules. This allows for a simple classification result and an easy connection to André-Quillen cohomology in all degrees, not simply in degree 1. It also makes our discussion of Postnikov towers for simplicial algebras presented in section 6 to be quite simple: in effect, the stages of a Postnikov tower are obviously weakly equivalent to a square-zero extensions, and the k-invariants are the cohomology classes corresponding to these extensions

Contents

1 Preliminaries on operads

Here we review the basic definitions of operads, algebras over operads, modules, and the like. Our primary purpose is to establish language; none of this material is new, and can certainly be found in standard sources (e.g, [10], [17]). There is also an elegant exposition at the beginning of [8].

We fix an abelian category $\mathcal{C}$ with all limits and colimits equipped with a symmetric monoidal product

$$\cdot \otimes \cdot : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

We will assume the symmetric monoidal structure is closed in the sense there is an exponential functor $\hom : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$ so that

$$\mathrm{Hom}_{\mathcal{C}}(M \otimes N, K) \cong \mathrm{Hom}_{\mathcal{C}}(M, \hom(N, K)).$$

The two examples we have in mind are the category of R modules, where R is some commutative ring, and the category of Γ-comodules, where (R, Γ) is a Hopf algebroid ([23] §A.1). In the latter case, the symmetric monoidal structure is defined by

$$(M, N) \mapsto M \otimes_R N.$$

We now let $\mathcal{C}_\Sigma$ be the category of *symmetric objects* in C. This is the category with objects $T = \{T_n\}_{n\geq 0}$ where $T_n \in \mathcal{C}$ is equipped with a Σ_n action in $\mathcal{C}$. The morphisms in $\mathcal{C}_\Sigma$ are the obvious ones. This is the category of Σ objects in $\mathcal{C}$. There is a pairing

$$\cdot \otimes_\Sigma \cdot : \mathcal{C}_\Sigma \times \mathcal{C}_\Sigma \to \mathcal{C}$$

induced by the symmetric monoidal structure in $\mathcal{C}$:

$$S \otimes_\Sigma T = \bigoplus_{n\geq 0} S_n \otimes_{\Sigma_n} T_n. \tag{1.1}$$

Here $S_n \otimes_{\Sigma_n} T_n$ denotes the colimit of the diagonal Σ_n action on $S_n \otimes T_n$. Note that for fixed $T \in \mathcal{C}_\Sigma$, the functor $\cdot \otimes_\Sigma T : \mathcal{C}_\Sigma \to \mathcal{C}$ has right adjoint

$$M \to \{\hom(T_n, M)\}_{n\geq 0} = \hom_*(T, M)$$

where Σ_n acts on the source of $\hom(T_n, M)$.

Here are some common uses of these constructions.

Example 1.1 Let $M \in \mathcal{C}$. Define $\Phi(M) \in \mathcal{C}_\Sigma$ by the formula

$$\Phi(M)_n = M \underset{\leftarrow n \rightarrow}{\otimes \cdots \otimes} M = M^{\otimes n}$$

with the permutation action. If $T \in \mathcal{C}_\Sigma$ we write

$$T(M) = T \otimes_\Sigma \Phi(M) = \bigoplus_n T_n \otimes_{\Sigma_n} M^{\otimes n}.$$

Example 1.2 Let X and M be in $\mathcal{C}$. Define $\Psi(X, M) \in \mathcal{C}_\Sigma$ by

$$\Psi(X, M)_n = \bigoplus_{i=1}^{n} X \otimes \cdots \otimes \underset{i}{M} \otimes \cdots \otimes X$$

again with the permutation action. If $T \in \mathcal{C}_\Sigma$ we write

$$\begin{aligned} T(X,M) &= T \otimes_\Sigma \Psi(X,M) \\ &\cong \bigoplus_n T_n \otimes_{\Sigma_{n-1}} X^{\otimes n-1} \otimes M. \end{aligned}$$

This last formula suggests we introduce an operator on $\mathcal{C}_\Sigma$

$$D_* : \mathcal{C}_\Sigma \to \mathcal{C}_\Sigma \tag{1.2}$$

with $(D_*T)_n = T_{n+1}$ with Σ_n action induced by $\Sigma_n \subseteq \Sigma_{n+1}$ fixing $n+1 \in \{1, 2, \ldots, n, n+1\}$. Then

$$T(X,M) = D_*T(X) \otimes M.$$

Example 1.3 The category $\mathcal{C}_\Sigma$ itself inherits a symmetric monoidal structure from $\mathcal{C}$. If S and T are objects in $\mathcal{C}_\Sigma$ define $S \otimes T \in \mathcal{C}_\Sigma$ by

$$(S \otimes T)_n = \bigoplus_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} S_p \otimes T_q.$$

In particular, if $T \in \mathcal{C}_\Sigma$ is some object, we can form $T^{\otimes n} \in \mathcal{C}_\Sigma$. This object has an external action of the symmetric group Σ_n in addition to the internal action of the symmetric group Σ_q on the object $(T^{\otimes n})_q$. If $S \in \mathcal{C}_\Sigma$ is another object, define $S \circ T \in \mathcal{C}_\Sigma$ by

$$(S \circ T)_q = S \otimes_\Sigma (T^{\otimes n})_{q\,n} = \bigoplus_n S_n \otimes_{\Sigma_n} (T^{\otimes n})_q.$$

This is the *circle product*

$$\circ : \mathcal{C}_\Sigma \times \mathcal{C}_\Sigma \to \mathcal{C}_\Sigma.$$

Proposition 1.4 *With the circle product, the category $\mathcal{C}_\Sigma$ becomes a closed monoidal category. The unit is given by $\mathbf{1} \in \mathcal{C}_\Sigma$ where*

$$\mathbf{1}_n = \begin{cases} e & n = 1 \\ 0 & n \neq 1 \end{cases}$$

with $e \in \mathcal{C}$ the unit of the symmetric monoidal structure on $\mathcal{C}$. The internal function object of $\mathcal{C}_\Sigma$ is given by the formula

$$\hom_{\mathcal{C}_\Sigma}(S,T)_n = \prod_{q \geq 0} \hom_{\Sigma_m}((S_q^{\otimes n}, T_q).$$

Note that this monoidal structure is not a symmetric monoidal structure; in particular $S \circ T$ need not be isomorphic to $T \circ S$. The next result is obtained by unwinding the definitions and regrouping the parentheses.

Proposition 1.5 *Let $S, T \in \mathcal{C}_\Sigma$ and $M, N \in \mathcal{C}$. Then there are natural isomorphisms*

1) *$S(T(M)) \cong (S \circ T)M$, and*
2) *$S(T(M), T(M,N)) \cong (S \circ T)(M,N)$.*

Definition 1.6 An *operad* in $\mathcal{C}$ is a monoid object $T \in \mathcal{C}_\Sigma$ in the monoidal structure given by the circle product.

Thus if T is an operad, there are morphisms

$$\mu : T \circ T \to T$$

and $\eta : \mathbf{1} \to T$ in $\mathcal{C}_\Sigma$ so that the evident associativity and unitary diagrams commute. Let $\mathcal{O}$ be the category of operads on $\mathcal{C}$.

If T is an operad, then the functor

$$M \mapsto T(M)$$

becomes the functor of a triple (i.e., monad) on $\mathcal{C}$ and a T-*algebra* is an algebra over this triple. In general, we will write $\mathcal{A}^T$ for the category of T-algebras in $\mathcal{C}$.

Example 1.7 If $M \in \mathcal{C}$, then the object

$$\mathrm{End}(M) = \{\hom(M^{\otimes n}, M)\}_{n \geq 0}$$

becomes an operad in $\mathcal{C}$, called the endomorphism operad of M. If T is any operad, then the set of T-algebra structures on M is isomorphic to

$$\mathrm{Hom}_{\mathcal{O}}(T, \mathrm{End}(M)).$$

Definition 1.8 Let T be an operad and A a T-algebra. An object $M \in \mathcal{C}$ is an A-module over T if there is a morphism in $\mathcal{C}$

$$\epsilon_M : T(A, M) \to M$$

so that the following diagrams commute:

$$\begin{tikzcd}
T(TA, T(A,M)) \arrow[r, "T(\epsilon_A, \epsilon_M)"] \arrow[dd, "\cong"] & T(A,M) \arrow[d, "\epsilon_M"] \\
 & M \\
(T \circ T)(A,M) \arrow[r, "\mu(A,M)"] & T(A,M) \arrow[u, "\epsilon_M"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
M = \mathbf{1}(A,M) \arrow[r] \arrow[dr, "="'] & T(A,M) \arrow[d] \\
 & M.
\end{tikzcd}$$

The first diagram is an associativity condition, and uses Proposition 1.5. The second is a unit condition. If the operad is understood, we may simply refer to M as an A-module.

If we fix T and A, let $\mathcal{M}_A^T$ denote the A-modules over T. The summation map $\Psi(A, A) \to \Phi(A)$ defines a composite

$$T(A, A) \to T(A) \xrightarrow{\epsilon_A} A$$

making $A \in \mathcal{M}_A^T$. Also, if $f : A \to B$ is a morphism of T-algebras, and M is a B-module over T, then M is a A-module over T via the composite

$$T(A, M) \xrightarrow{T(f,M)} T(B, M) \to M.$$

Before going further, it will be useful to lay out the concepts of various kinds of special coequalizers. A *reflexive coequalizer* is a coequalizer diagram

$$M_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} M_0$$

that can be equipped with a morphism $s_0 : M_0 \to M_1$ so that $d_0 s_0 = d_1 s_0 = 1$. We will write d_0 for the map

$$d_0 : M_0 \to \operatorname{colim}(M_1 \rightrightarrows M_0).$$

For example, If $A \in \mathcal{A}^T$ is a T-algebra, the diagram,

$$(T \circ T)A \underset{T_{\epsilon_A}}{\overset{\epsilon_{T_A}}{\rightrightarrows}} T(A) \overset{\epsilon_A}{\to} A \tag{1.3}$$

is a reflexive coequalizer diagram with $s_0 = T_{\eta_A} : TA \to (T \circ T)(A)$. Write $d_0 = \epsilon_{TA}$, $d_1 = T\epsilon_A$ and $d_0 = \epsilon_A$. Notice that in $\mathcal{C}$, this diagram comes equipped with $s_{-1} = \eta : A \to TA$ and $s_{-1} : \eta : TA \to T \circ T(A)$ so that $d_0 s_{-1} = 1$ and $d_1 s_{-1} = s_{-1} d_0$. It follows that (1.3) is an equalizer diagram in $\mathcal{C}$, hence in $\mathcal{A}^T$ – as we will see below.

In the closed symmetric monoidal category $\mathcal{C}$, the product $\otimes$ preserves reflexive coequalizers in the strongest possible sense: if

$$M_{1i} \rightrightarrows M_{0i} \to M_i, \qquad i = 1, 2$$

are reflexive coequalizers then so is

$$M_{11} \otimes M_{12} \rightrightarrows M_{01} \otimes M_{02} \to M_1 \otimes M_2.$$

To see this note that the functors $M_1 \otimes \cdot$ and $\cdot \otimes M_2$ both preserve all coequalizers, being left adjoints, so one need only check the coequalizer of $M_{11} \otimes M_{12} \rightrightarrows M_{01} \otimes M_{02}$ is the colimit of the diagram

$$\begin{tikzcd}
M_{11} \otimes M_{12} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & M_{11} \otimes M_{02} \arrow[d, shift left] \arrow[d, shift right] \\
M_{01} \otimes M_{12} \arrow[r, shift left] \arrow[r, shift right] & M_{01} \otimes M_{02}.
\end{tikzcd}$$

To do this, we argue as follows: any map $f : M_{01} \otimes M_{02} \to N$ which equalizes $d_0 \otimes d_0$, $d_1 \otimes d_1 : M_{11} \otimes M_{12} \rightrightarrows M_{01} \otimes M_{02}$ must also equalize

$$1 \otimes d_0, 1 \otimes d_1 : M_{01} \otimes M_{12} \rightrightarrows M_{01} \otimes M_{12}$$

and

$$d_0 \otimes 1, d_1 \otimes 1 : M_{11} \otimes M_{02} \rightrightarrows M_{01} \otimes M_{12},$$

since, for example,

$$f(1 \otimes d_0) = f(d_0 \otimes d_0)(s_0 \otimes 1) = f(d_1 \otimes d_1)(s_0 \otimes 1) = f(1 \otimes d_1),$$

where $s_0 : M_{01} \to M_{11}$ is any splitting.

Because $\otimes$ preserves reflexive coequalizers, so does

$$T(\cdot) : \mathcal{C} \to \mathcal{C}$$

where $T \in \mathcal{C}_\Sigma$ is any object. From this comes the following observation. Let T is an operad and let

$$d_0, d_1 : A_1 \rightrightarrows A_0$$

be two maps in $\mathcal{A}^T$ that can be split by $s_0 : A_0 \to A_1$ in $\mathcal{C}$. Then if $A \in \mathcal{C}$ is the coequalizer d_0 and d_1 in $\mathcal{C}$, there is a unique map $T(A) \to A$ filling in the following diagram

$$\begin{tikzcd}
T(A_1) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & T(A)_0 \arrow[r] \arrow[d] & T(A) \arrow[d] \\
A_1 \arrow[r, shift left] \arrow[r, shift right] & A_0 \arrow[r] & A.
\end{tikzcd}$$

This gives A the structure of a T-algebra and with this structure, A is the coequalizer of d_0, d_1 in $\mathcal{A}^T$. In shorthand: the forgetful functor $\mathcal{A}^T \to \mathcal{C}$ makes reflexive coequalizers. It follows from this that $\mathcal{A}^T$ has all finite colimits. In our two main examples – R-modules and (R, Γ) comodules – the forgetful functor also makes filtered colimits, so the category $\mathcal{A}^T$ has all colimits. Finally, the forgetful functor also makes limits, $\mathcal{A}^T$ has all limits and colimits.

We now list a technical result needed in the results that follow.

Lemma 1.9 *Let $A_1 \rightrightarrows A_0 \to A$ be a reflexive coequalizer of T-algebras, and let $M_1 \rightrightarrows M_0 \to M$ be a reflexive coequalizer in $\mathcal{C}$. Suppose M_i, $i = 1, 2$, is A_i-module over T, $i = 1, 2$, $d_0, d_1 : M_1 \to M_0$ are A_1 modules maps and $s_0 : M_0 \to M_1$ is an A_0-module map. Then M has a unique structure as a A-module over T so that $d_0 : M_0 \to M$ is an A_0 module map.*

Proof. The functor $T(\cdot, \cdot)$ preserves reflexive coequalizers in the strong sense of both variables at once; hence there is a unique map $T(A, M) \to M$ in $\mathcal{C}$ making the following diagram commute.

$$\begin{tikzcd}
T(A_1, M_1) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & T(A_0, M_0) \arrow[r] \arrow[d] & T(A, M) \arrow[d] \\
M_1 \arrow[r, shift left] \arrow[r, shift right] & M_0 \arrow[r] & M.
\end{tikzcd}$$

Proposition 1.10 *The forgetful functor $\mathcal{M}_A^T \to \mathcal{C}$ has a left adjoint*

$$M \mapsto A \otimes^T M.$$

Proof. First assume $A = T(N)$ for some N. Then

$$A \otimes^T M \cong T(N, M)$$

with A-module structure over T defined by

$$T(A, A \otimes^T M) \cong (T \circ T)(N, M) \to T(N, M) = A \otimes^T M.$$

The natural map, with $K \in \mathcal{M}_A^T$

$$\mathrm{Hom}_{\mathcal{C}}(M, K) \to \mathrm{Hom}_{\mathcal{M}_A^T}(A \otimes^T M, K)$$

sending $f : M \to K$ to the composite

$$A \otimes^T M = T(N, M) \to T(T(N), M) = T(A, M) \xrightarrow{T(A,f)} T(A, K) \to K$$

is an isomorphism with inverse sending $g : A \otimes^T M \to K$ to

$$M = \mathbf{1}(N, M) \to A \otimes^T M \xrightarrow{g} K.$$

Note that if $f : A_1 = T(N_1) \to A_2 = T(N_2)$ is any morphism of T-algebras, the natural map $M \to A_2 \otimes^T M$ and the fact that f gives $A_2 \otimes^T M$ the structure of an A_1 module yields, by adjointness, a map of A_1-modules over T

$$f \otimes^T M : A_1 \otimes^T M \to A_2 \otimes^T M.$$

To define $A \otimes^T M$ for more general M, we use the reflexive coequalizer diagram of 1.3 of T-algebras

$$T^2 A \underset{d_1}{\overset{d_0}{\rightrightarrows}} T(A) \xrightarrow{d_0} A$$

and define $A \otimes^T M$ by the coequalizer diagram in $\mathcal{C}$

$$T^2(A) \otimes^T M \rightrightarrows T(A) \otimes^T M \to A \otimes^T M. \tag{1.4}$$

Lemma 1.9 supplies the A-module structure on $A \otimes^T M$.

To get the adjointness property note that under the situation described by the hypotheses of Lemma 1.9, there is a reflexive equalizer diagram of abelian groups

$$\mathrm{Hom}_{\mathcal{M}_A^T}(M, N) \to \mathrm{Hom}_{\mathcal{M}_{A_0}^T}(M_0, N) \rightrightarrows \mathrm{Hom}_{\mathcal{M}_{A_1}^T}(M_1, N)$$

for any $N \in \mathcal{M}_A^T$. Apply this to the sequence of Definition 1.4.

Lemma 1.11 *The forgetful functor $\mathcal{M}_A^T \to \mathcal{C}$ makes all limits and colimits, and $\mathcal{M}_A^T$ is an abelian subcategory of $\mathcal{C}$.*

Proof. Because of the equation $T(A, M) \cong D_* T(A) \otimes M$ (see 1.2), the functor $T(A, \cdot)$ preserves all colimits $\mathcal{C}$. Hence, if $M : I \to \mathcal{M}_A^T$ is any diagram, the colimit in $\mathcal{C}$, $\operatorname{colim}_I M$, acquires a module structure via

$$T(A, \operatorname{colim}_I M) = \operatorname{colim}_I T(A, M) \to \operatorname{colim}_I M$$

with their structure, $\operatorname{colim}_I M \in \mathcal{M}_T^A$ becomes the colimit in $\mathcal{M}_T^A$. The case of limits is easier.

Since the direct sum in $\mathcal{M}_T^A$ of two objects $M_1, M_2 \in \mathcal{M}_T^A$ is made in $\mathcal{C}$, $\mathcal{M}_T^A$ is an additive subcategory. Since cokernels and kernels are, respectively, a colimit and limit, the result follows from the fact that colimits and limits are made in $\mathcal{C}$.

Lemma 1.12 *The category $\mathcal{M}_A^T$ is equivalent to the category of algebras over a triple with underlying functor*

$$A \otimes^T (\cdot) : \mathcal{C} \to \mathcal{C}.$$

Proof. This follows from Lemma 1.11 and Beck's Theorem. ([19], VI §7).

It is standard to note that $\mathcal{M}_A^T$ is, in fact, the category of some left modules over some algebra in $\mathcal{C}$. This is proved by the following sequence of ideas. First note that if $K \in \mathcal{M}_A^T$ and $M \in \mathcal{C}$, then the mapping object $\hom(M,K) \in \mathcal{C}$ has a natural structure as an object of $\mathcal{M}_A^T$; the evaluation formula

$$T(A, \hom(M,K)) \to \hom(M,K)$$

is adjoint to the composite

$$T(A, \hom(M,K)) \otimes M \cong D_*T \otimes \hom(M,K) \otimes M \to D_*T \otimes K \cong T(A,K) \to K.$$

Similarly, $K \otimes M$ is an object in $\mathcal{M}_A^T$ with evaluation map

$$T(A, K \otimes M) \cong T(A,K) \otimes M \to K \otimes M.$$

Lemma 1.13 *There is a natural adjunction isomorphism, $M, K \in \mathcal{M}_A^T$, $N \in \mathcal{C}$*

$$\mathrm{Hom}_{\mathcal{M}_A^T}(M \otimes N, K) \cong \mathrm{Hom}_{\mathcal{M}_A^T}(M, \hom(N,K)).$$

Proof. This is a result of the comparison of equalizers

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{M}_A^T}(M \otimes N, K) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{C}}(M \otimes N, K) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathcal{C}}(T(A, M \otimes N), K) \arrow[d, "\cong"] \\
\mathrm{Hom}_{\mathcal{M}_A^T}(M, \hom(N,K) \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(M, \hom(N,K)) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}_{\mathcal{C}}(T(M), \hom_{\mathcal{C}}(N,K)).
\end{tikzcd}$$

An immediate consequence of this result is that the natural map of A-modules over T

$$A \otimes^T (M \otimes N) \to (A \otimes^T M) \otimes N \tag{1.5}$$

is an isomorphism, as

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{M}_A^T}((A \otimes^T M) \otimes N, K) &\cong \mathrm{Hom}_{\mathcal{M}_A^T}(A \otimes^T M, \hom(N,K)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(M, \hom(N,K)) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(M \otimes N, K) \\
&\cong \mathrm{Hom}_{\mathcal{M}_A^T}(A \otimes^T (M \otimes N), K).
\end{aligned}$$

Thus. if $e \in \mathcal{C}$ is the unit of the monoidal structure, the object $A \otimes^T e$ gets a multiplication in $\mathcal{C}$

$$(A \otimes^T e) \otimes (A \otimes^T e) \cong A \otimes^T A \otimes^T e \to A \otimes^T e,$$

using the fact that A is an A module over itself. This multiplication is associative and has a unit $e \to A \otimes^T e$. This object is the enveloping algebra of A. If M is a A-module over T, then M is naturally a left $A \otimes^T e$ module via

$$(A \otimes^T e) \otimes M \cong A \otimes^T (e \otimes M) \cong A \otimes^T M \to M.$$

With this assignment of left $A \otimes^T e$ module structure we have

Proposition 1.14 *The category $\mathcal{M}_A^T$ of A-modules over T is equivalent to the category of left $A \otimes^T e$ modules.*

Proof. If M is $A \otimes^T e$ module, then M becomes an algebra over the triple $M \to A \otimes^T M$ via $A \otimes^T M \cong (A \otimes^T e) \otimes M \to M$. The result follows from Lemma 1.11.

Example 1.15 Suppose the underlying abelian category $\mathcal{C}$ is the category of R-modules over a commutative ring R with symmetric monoidal structure given by tensor product. Thus R itself is the unit of the symmetric monoidal structure and $A \otimes_T R$ is the enveloping algebra of A. Then the forgetful functor $\mathcal{M}_A^T \to \mathcal{C}$ has a right adjoint. By the previous result, it is sufficient to argue that the forgetful functor from left $A \otimes^T R$ modules to $\mathcal{C}$ has a right adjoint. This adjoint is given by the functor

$$M \longmapsto \mathrm{Hom}_R(A \otimes^T R, M)$$

where the target gets a left $A \otimes^T R$ module structure via the formula

$$(a\psi)(x) = \psi(ax).$$

The counit of adjunction $M \to \mathrm{Hom}_R(A \otimes^T R, M)$ sends an element x of the left $A \otimes^T R$ module to the homomorphism ψ_x with $\psi_x(a) = xa$.

Notice that the existence of this adjoint immediately implies that the category $\mathcal{M}_A^T$ has enough injectives, because the category of R-modules has enough injectives.

If R is a graded ring and C is the category of graded R-modules, similar observations apply. The graded adjoint is

$$M \longmapsto \mathrm{Hom}_R^*(A \otimes^T R, M)$$

with

$$\mathrm{Hom}_R^n(A \otimes^T R, M) = \mathrm{Hom}_R^n(A \otimes^T R, M_{*+n})$$

where $(M_{*+n})_k = M_{k+n}$. Again M_A^T will have enough injectives.

We next give a result allowing extension of scalars and change of operads. We have already noted that if $A \to B$ is a morphism of T-algebras we get a functor $\mathcal{M}_B^T \to \mathcal{M}_A^T$ which might be called restriction of scalars. Similarly if $S \to T$ is a morphism of operads, any T-algebra A becomes an S-algebra via

$$S(A) \to T(A) \xrightarrow{\epsilon} A.$$

Furthermore if M is a A-module over T, then M is an A-module over S via

$$S(A, M) \to T(A, M) \to M.$$

Thus we obtain functors $\mathcal{A}^T \to \mathcal{A}^S$ and $\mathcal{M}_A^T \to \mathcal{M}_A^S$.

Lemma 1.16 *1) If $A \to B$ is a morphism of T-algebras, the restriction of scalars functor $\mathcal{M}_B^T \to \mathcal{M}_A^T$ has a left adjoint $B \otimes_A (\cdot) : \mathcal{M}_A^T \to \mathcal{M}_B^T$ characterized by the natural isomorphism*

$$B \otimes_A (A \otimes^T M) \cong B \otimes^T M.$$

2) If $S \to T$ is a morphism of operads the forgetful functor $\mathcal{A}^T \to \mathcal{A}^S$ has a left adjoint $T \otimes_S (\cdot)$ characterized by

$$T \otimes_S S(M) \cong T(M).$$

3) If $S \to T$ is a morphism of operads and A is a T-algebra, the functor $\mathcal{M}_A^T \to \mathcal{M}_A^S$ has left adjoint $T \otimes_S (\cdot)$ characterized by

$$T \otimes_S (A \otimes^S M) \cong A \otimes^T M.$$

Proof. All three follow from [2]. The argument is a modification of the argument given for Proposition 1.10

2 Derivations, differentials, and (co-)homology

André-Quillen cohomology is obtained as the right derived functors of derivations. We introduce and explore the relevant concepts.

If A is a commutative algebra over a commutative ring R and M is an A-module, then the abelian group of R-derivations $d : A \to M$ is naturally isomorphic to the abelian group of R-algebra maps over A

$$A \to A \ltimes M$$

where $A \ltimes M$ is the "square-zero" extension of A by M. As abelian groups $A \ltimes M \cong A \oplus M$, and the multiplication is defined by

$$(a, x)(b, y) = (ab, ax + yb).$$

For algebras over an operad, we will simply defined derivations to be maps into a square-zero extension; then we will give an alternative description more in line with the usual definition of derivation.

If T is an operad, A a T-algebra, and M a A-module over T, we need to define the square-zero extension of A by M. Note that if $\Phi(A)$ and $\Psi(A, M)$ are the functors of Examples 1.1 and 1.2, then, in $\mathcal{C}_\Sigma$, there is a decomposition,

$$\Phi(A \oplus M) \cong \Phi(A) \oplus \Psi(A, M) \oplus L(A, M)$$

hence a projection $T(A \oplus M) \to T(A) \oplus T(A, M)$.

The T-algebra $A \ltimes M$ is the object $A \oplus M$ with algebra structure

$$T(A \oplus M) \to T(A) \oplus T(A, M) \xrightarrow{\epsilon_A \oplus \epsilon_M} A \oplus M$$

where ϵ denotes the evaluation map.

If we let $\mathcal{A}^T/A$ denote the category of T algebras over A, then $A \ltimes M$ is an object in $\mathcal{A}^T/A$; indeed, $A \ltimes M$ is an abelian group object in the category of T-algebras over A, and all abelian group objects in $\mathcal{A}^T/A$ are of this form.

Definition 2.1 If T is an operad, A a T-algebra, and M a A-module over T, and $B \in \mathcal{A}^T/A$, define the abelian group of T-*derivations* by the equation

$$\mathrm{Der}_T(B, M) \cong \mathrm{Hom}_{\mathcal{A}^T/A}(B, A \ltimes M).$$

We can give another characterization of this group. If we have a T-algebra map $f : B \to A \ltimes M$, then as a morphism is $\mathcal{C}$, we may write

$$f = \epsilon + d : B \to A \ltimes M = A \oplus M$$

where $\epsilon : B \to A$ is the augmentation. One traditionally thinks of d as the derivation and we'd like to know what properties this morphism has.

For fixed $M \in \mathcal{C}$, there is a natural diagonal map

$$M^{\otimes n} = \Phi(M)_n \to \Psi(M, M)_n = \bigoplus_n M^{\otimes n}$$

and hence, for any operad T, a diagonal map

$$\Delta : T(M) \to T(M, M).$$

The following result, which is an exercise in definitions, gives an equivalent definition of derivations.

Proposition 2.2 *Let A be a T-algebra and M a A-module over T. Then a T-derivation is determined by a morphism $d : B \to M$ is $\mathcal{C}$ so that the following diagram commutes*

$$\begin{tikzcd} T(B) \arrow[r, "\Delta"] \arrow[d, "\epsilon_B"'] & T(B,B) \arrow[r, "T(B,d)"] & T(B,M) \arrow[d, "\epsilon_M"] \\ B \arrow[rr, "d"] & & M. \end{tikzcd}$$

We next produce Kähler differentials. For fixed $A \in \mathcal{A}^T$, the Kähler differentials on A is a module $\Omega_T A \in \mathcal{M}_A^T$ representing the functor $M \mapsto \mathrm{Der}_T(A, M)$. The next result shows this object exists.

Lemma 2.3 *Let A be a T-algebra. Then there is a module $\Omega_T A \in \mathcal{M}_A^T$ and a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{M}_A^T}(\Omega_T A, M) \cong \mathrm{Der}_T(A, M).$$

Proof. First assume $A = T(N)$ for some $N \in \mathcal{C}$. Then the map, for $M \in \mathcal{M}_A^T$

$$\mathrm{Hom}_{\mathcal{C}}(N, M) \to \mathrm{Der}_T(A, M)$$

sending $f : N \to M$ to the derivation induced (via Proposition 2.2) by the composite

$$A = T(N) \xrightarrow{\Delta} T(N,N) \xrightarrow{T(\eta,f)} T(T(N), M) \xrightarrow{\epsilon} M.$$

is an isomorphism. Hence we have

$$\Omega_T(N) \cong T(N) \overset{T}{\otimes} N \cong T(N,N). \tag{2.1}$$

If $T(N_1) \to T(N_2)$ is any morphism of T-algebras, we get a natural homomorphism for $M \in \mathcal{M}_{T(N_2)}^T$

$$\mathrm{Der}_T(T(N_2), M) \to \mathrm{Der}_T(T(N_1), M)$$

hence a morphism of $T(N_2)$ modules

$$T(N_2) \otimes_{T(N_1)} \Omega_T T(N_1) \to \Omega_T T(N_2).$$

For general $A \in \mathcal{A}^T$, there is a coequalizer diagram of T-algebras

$$T^2(A) \rightrightarrows T(A) \to A$$

and we define $\Omega_T(A)$ by the coequalizer diagram

$$A \otimes_{T^2 A} \Omega_T T^2 A \rightrightarrows A \otimes_{TA} \Omega_T TA \to \Omega_T A.$$

This completes the argument.

Implicit in the preceding proof is the following result.

Lemma 2.4 *Let $A \to B$ be a morphism of T-algebras. Then there is an induced map of B-modules over T*

$$B \otimes_A \Omega_T A \to \Omega_T B$$

inducing the natural map

$$\mathrm{Der}_T(B, M) \to \mathrm{Der}_T(A, M).$$

Also, if $S \to T$ is a morphism of operads we get, for any T-algebra A, and $M \in \mathcal{M}_A^T$, a map

$$\mathrm{Der}_T(A, M) \to \mathrm{Der}_S(A, M)$$

which is, in turn, induced by a map A-modules over T

$$T \otimes_S \Omega_S A \to \Omega_T A. \tag{2.2}$$

Note that if $S(A) \to T(A)$ is surjective, $\mathrm{Der}_T(A, M) \to \mathrm{Der}_S(A, M)$ is an isomorphism; hence $T \otimes_S \Omega_S A \cong \Omega_T A$.

We now come to simplicial T-algebras. If T is an operad in $\mathcal{C}$, we can form the category $s\mathcal{A}^T$ of simplicial T-algebras; however, for our applications, we need something more general. If $T = \{T_n\}$ is a simplicial operad and $X \in s\mathcal{C}$, then one can form the object

$$T(X) = \{T_n(X_n)\} \in s\mathcal{C}$$

and a simplicial operad over T is an object $X \in s\mathcal{C}$ equipped with a multiplication $\epsilon : T(X) \to X$ so that the usual diagrams commute. Write $s\mathcal{A}^T$ for the category of simplicial T-algebras. If T is any operad $\mathcal{C}$, we may regard T as a constant simplicial operad and the two definitions of $s\mathcal{A}^T$ agree.

In a similar vein, if $X \in s\mathcal{A}^T$ a simplicial X-module over T is an object $M \in s\mathcal{C}$ equipped with a multiplication in $s\mathcal{C}$

$$\epsilon : T(X, M) \to M$$

subject to the usual diagrams.

Finally, as a bit of notation, an augmented simplicial object Z is a simplicial object $Z = \{Z_n\}$ equipped with a morphism $d_0 : Z_0 \to Z_{-1}$ so that $d_0 d_1 = d_0 d_0 : Z_1 \to Z_{-1}$. With this condition, all possible compositions of face maps

$$d_{i_0} d_{i_1} \cdots d_{i_n} : Z_n \to Z_{-1}$$

are equal. The coequalizer of the two maps $d_0, d_1 : Z_1 \to Z_0$, if it exists, is denoted $\pi_0 Z$ and the obvious map $d_0 : Z_0 \to \pi_0 Z$ is initial among all augmentations.

In particular, if $T \to T_{-1}$ is an augmented simplicial operad, $X \to X_{-1}$ are algebra over over $T \to T_{-1}$ and M a T_{-1} module over X_{-1}, then M has a unique induced X_n module structure over T_n.

For the rest of this section we will assume that our base category of $\mathcal{C}$ is the category $\mathcal{M}_R$ of R-modules for some commutative ring R. This is because in this case, the category of simplicial T-algebras, for a simplicial operad T, has a good simplicial model category structure with every object fibrant. We will explore the case of Γ-comodules, where (R, Γ) is a Hopf algebroid in the next two sections.

Before stating the precise result, we introduce some terminology, If $\mathcal{C}$ is any category and $s\mathcal{C}$ is the category of simplicial objects in $\mathcal{C}$, then an object $X \in s\mathcal{C}$ is called *s-free* if the underlying degeneracy diagram is free; that, is, there are objects $Z_n \in \mathcal{C}$ and isomorphisms

$$X_n \cong \bigsqcup_{\phi:[n]\to[m]} Z_m$$

where ϕ runs over the surjections in the ordinal number category, and these isomorphisms respect the degeneracy maps. If R is a commutative ring and T is a simplicial operad over R, then an object in $X \in s\mathcal{A}^T$ will be called *free* (following [21] §II.4) if the underlying degeneracy diagram is of the form $T(M)$ where M is a degeneracy diagram of R-modules which is free on a set $\{P_n\}$ of projective R-modules. A morphism $X \to Y$ is called free if the underlying morphism of degeneracy diagrams is isomorphic to one of the form $X \to X \sqcup Y'$ where Y' is free.

Proposition 2.5 *The category $s\mathcal{A}^T$, with the standard simplicial structure, has a cofibrantly generated simplicial model category structure so that $f : X \to Y$ is*

1. *a weak equivalence if and only if $\pi_* X \to \pi_* Y$ is an isomorphism of graded objects R-modules;*
2. *a fibration is, after normalization, it is surjection in positive degrees; and*
3. *a cofibration if it is a retract of a free map.*

This follows from the general result of Quillen's; see [21] §II.4. Similar remarks apply to $s\mathcal{M}_A^T$. Indeed, since $\mathcal{M}_A^T$ is an abelian category with enough projectives, it, too has a simplicial model category structure with homotopy isomorphisms as weak equivalences and fibrations those maps which are, after normalization, a surjection in positive degrees. (See [21]). Cofibrations are retracts of those maps $A \to B$ which are appropriately free; that, is the underlying morphism of degeneracy diagrams is of the form

$$A \longrightarrow A \oplus B'$$

where B' is s-free on a set $A_n \otimes^{T_n} P_n$ with P_n a projective R-module.

Definition 2.6 Let $A \in s\mathcal{A}^T$ be a simplicial T-algebra. The cotangent complex of A is defined by the equation

$$\mathbb{L}\Omega_T A \cong A \otimes_X \Omega_T X$$

where $X \to A$ is a trivial fibration in $s\mathcal{A}^T$ with X cofibrant. The operad will be dropped from the notation when understood.

The André-Quillen homology of A is defined by

$$D_*^T(A) = D_*(A) \cong \pi_* \mathbb{L}\Omega_T A.$$

If $T \to T_{-1}$ is an augmented simplicial operad, $A \to A_{-1}$ is an augmented $T \to T_{-1}$ algebra and M a A_{-1} module over T_{-1}, then the *André-Quillen cohomology* of A with coefficients in M is defined by

$$D_T^*(A, M) = D^*(A, M) \cong \pi^* \operatorname{Der}_T(X, M) \cong \pi^* \operatorname{Hom}_{\mathcal{M}_A^T}(\mathbb{L}\Omega_T A, M).$$

We make some remarks on the definition. If we assume X is free, then we may write the underlying degeneracy diagram of X as $T(Z^0)$ where Z^0 is some free degeneracy diagram in $\mathcal{C}$, and then the underlying degeneracy diagram of $A \otimes_X \Omega_T X = \mathbb{L}\Omega_T A$ is

$$A \otimes_X \Omega_T X \cong A \otimes^T Z^0$$

which is, in turn, free in $s\mathcal{M}_A^T$.

Since every cofibrant object X in $s\mathcal{A}^T$ is a retract of a free object, $\Omega_T X$ is a retract of a free object in $s\mathcal{M}_X^T$; hence $A \otimes_X \Omega_T X$ is cofibrant in $s\mathcal{M}_A^T$. It is also well-defined up to homotopy equivalence. Thus $D_*(A, M)$ and $D^*(A, M)$ are well-defined.

The next point to make is that André-Quillen cohomology is representable. Let A be a simplicial algebra over a simplicial operad T. If M is a simplicial A-module over T one can form the simplicial algebra $A \ltimes M$. If M is a $\pi_0 A$-module over $\pi_0 T$ we may form a simplicial A-module $K(M, n)$. As a simplicial object in $\mathcal{C}$, $NK(M, n) \cong M$ concentrated in degree n (where N is the normalization functor) and $K(M, n)_k$ acquires a A_k module over T_k structure via the augmentation. Write $K_A(M, n)$ for $A \ltimes K(M, n)$.

Proposition 2.7 *Let $Y \in s\mathcal{A}^T/A$. Then there is a natural isomorphism*

$$[Y, K_A(M, n)]_{s\mathcal{A}^T/A} \cong D^n(Y, M).$$

Proof. Definition 2.1 and the existence of Kähler differentials (Lemma 2.3), the functor $A \ltimes (\cdot) : s\mathcal{M}^T_A \to s\mathcal{A}^T/A$ has left adjoint $Y \mapsto A \otimes_Y \Omega_T Y$. Note $A \ltimes (\cdot)$ preserves all weak equivalences and fibrations. Hence $A \otimes_Y \Omega_T(\cdot)$ preserves cofibrations and weak equivalences between cofibrant objects and

$$\begin{aligned}[][Y, K_A(M,n)]_{s\mathcal{A}^T/A} &\cong [\mathbb{L}\Omega_T Y, K(M,n)]_{s\mathcal{M}^T_A}\\ &\cong D^n(Y,M).\end{aligned}$$

This finishes the proof.

For computing one has an attenuated form of the transitivity sequence and, perhaps more usefully, a Meyer-Vietoris sequence. For both of these, we need a relative version of the above concepts. Suppose $A \to X$ is a morphism of simplicial T-algebras and M is a $\pi_0 X$-module over $\pi_0 T$.

Definition 2.8 The abelian group of T-derivations from X to M under A is defined by the exact sequence

$$0 \to \mathrm{Der}_A(X,M) \to \mathrm{Der}_T(X,M) \to \mathrm{Der}_T(A,M).$$

The functor $M \mapsto \mathrm{Der}_A(X,M)$ is representable; indeed, if we define $\Omega_T(X/A) \in s\mathcal{M}^T_A$ by the exact sequence in $s\mathcal{M}^T_A$

$$X \otimes_A \Omega_T A \to \Omega_T X \to \Omega_T(X/A) \to 0$$

then there is a natural isomorphism

$$\mathrm{Hom}_{s\mathcal{M}^T_A}(\Omega_T(X/A), M) \cong \mathrm{Der}_A(X,M). \tag{2.3}$$

Then $\mathbb{L}\Omega_T(X/A)$ is defined as follows. Choose a cofibrant model $A_0 \to A$ for A in $s\mathcal{A}^T$ and factor the composite $A_0 \to A \to X$ as $A_0 \xrightarrow{j} X_0 \xrightarrow{q} X$ where j is a cofibration and q is a weak equivalence. Then

$$\mathbb{L}\Omega_T(X/A) = X \otimes_{X_0} \Omega(X_0/A_0). \tag{2.4}$$

The arrow $A_0 \to X_0$ is a cofibrant model for the arrow $A \to X$ is the arrow category of $s\mathcal{A}^T$ with the Bousfield-Kan model category structure: level-wise weak equivalences and fibrations. (See [4]). Since every object in this arrow category is fibrant, $\mathbb{L}\Omega_T(X/A)$ is well-defined in the homotopy category of simplicial X-modules. In fact, we have:

Lemma 2.9 *Suppose we have a diagram of T-algebras*

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

with the horizontal maps weak equivalences. Then there is a natural weak equivalence of simplicial X-modules

$$X \otimes_Y \mathbb{L}\Omega_T(Y/B) \xrightarrow{\simeq} \mathbb{L}\Omega_T(X/A).$$

Proof. Let $A_0 \to X_0$ be a cofibrant model for the arrow $B \to Y$. Then $A_0 \to X_0$ is also a cofibrant model for the arrow $A \to X$. Hence,

$$X \otimes_Y \mathbb{L}\Omega_T(Y/B) \simeq X \otimes_Y Y \otimes_{X_0} \Omega_T(X_0/A_0) \cong X \otimes_{X_0} \Omega_T(X_0/A_0) \simeq \mathbb{L}\Omega_T(X/A).$$

Note that, in particular, this last result implies that if $A_0 \to X_0$ is cofibrant model for $A \to X$, then

$$\mathbb{L}\Omega_T(X_0/A_0) = \Omega_T(X_0/A_0) \to X \otimes \Omega_T(X_0/A_0) = \mathbb{L}\Omega_T(X/A)$$

is a weak equivalence.

We define the relative André-Quillen homology by

$$D_*(X/A) = D_*^T(X/A) = \pi_*\mathbb{L}\Omega_T(X/A).$$

One could also attempt to define $\mathbb{L}\Omega_T(X/A)$ without recourse to resolving A; that is, factor $A \to X$ as $A \xrightarrow{j'} Y \xrightarrow{q} X$ where q is a weak equivalence and j is cofibration, and consider $\mathbb{L}'\Omega_T(X/A) = X \otimes_Y \Omega_T(Y/A)$. This is again well-defined in $\mathbf{Ho}(s\mathcal{M}_X^T)$; however, this object may not be weakly equivalent to $\mathbb{L}\Omega_T(X/A)$. We have the following result. As far as we know, they most important example where the "left proper" hypothesis of this Proposition is valid is the case of the commutative algebra operad.

Proposition 2.10 *There is a natural map in* $\mathbf{Ho}(s\mathcal{M}_X^T)$

$$\mathbb{L}\Omega_T(X/A) \to \mathbb{L}'\Omega_T(X/A)$$

and this map is an isomorphism in $\mathbf{Ho}(s\mathcal{M}_X^T)$ *if the simplicial model category* $s\mathcal{A}^T$ *is left proper.*

Proof. Let $A_0 \to X_0$ be a cofibrant model for $A \to X$. There is a sequence of arrows

$$A \xrightarrow{i} A \sqcup_{A_0} X_0 \xrightarrow{p} X$$

and i is a cofibration. If $s\mathcal{A}^T$ is left proper, then p is a weak equivalence. In any case, factor p as

$$A \sqcup_{A_0} X_0 \xrightarrow{f} Y \xrightarrow{g} X$$

where f is a cofibration and g is a weak equivalence. Then we have morphisms

$$\begin{aligned} \mathbb{L}\Omega_T(X/A) = X \otimes_{X_0} \Omega_T(X_0/A_0) &\to X \otimes_{(X_0 \sqcup_{A_0} A)} \Omega_T(X_0 \sqcup_{A_0} A/A) \\ &\to X \otimes_Y \Omega_T(Y/A) \simeq \mathbb{L}'\Omega_T(X/A). \end{aligned}$$

If $s\mathcal{A}^T$ is left proper, we may choose f to be the identity; hence, the second map is an isomorphism in $\mathbf{Ho}(s\mathcal{M}_X^T)$. Also

$$\Omega_T(X_0 \sqcup_{A_0} A/A) \cong (X_0 \sqcup_{A_0} A) \otimes_{X_0} \Omega_T(X_0/A_0),$$

so the first map may be rewritten

$$X \otimes_{X_0} \Omega_T(X_0/A_0) \to X \otimes_{(X_0 \sqcup_{A_0} A)} (X_0 \sqcup_A A) \otimes_{X_0} \Omega_1(X_0/A_0),$$

which is an isomorphism, completing the proof.

There is also a definition of relative André-Quillen cohomology. Given a morphism of simplicial T-algebras $A \to X$ and a $\pi_0 X$ module M over $\pi_0 X$, we choose cofibrant model $A_0 \to X_0$ for the morphism $A \to X$ and set

$$D_T^*(X/A, M) = \pi^* \operatorname{Der}_A(X, M).$$

This can be rewritten in various ways. The module M has an induced structure as an X_q module over T_q – where q denotes the simplicial degree – and we obtain a cosimplicial object

$$\mathrm{Hom}_{s\mathcal{M}^T_X}(\mathbb{L}\Omega_T(X/A), M) = \{\mathrm{Hom}_{\mathcal{M}^{T_q}_{X_q}}(\mathbb{L}\Omega_T(X/A)_q, M)\}$$

and Equation 2.3 immediately implies

$$D^*_T(X/A, M) \cong \pi^*\mathrm{Hom}_{s\mathcal{M}^T_X}(\mathbb{L}\Omega_T(X/A), M). \tag{2.5}$$

Also, relative André-Quillen cohomology is representable. If B is a $\pi_0 T$ algebra and M is a B module over π_0, then one way to make M into a $\pi_0 X$ module over $\pi_0 T$ is by specifying an augmentation $X \to B$; indeed, if $B = \pi_0 X$, this is the only way. If we form the simplicial T-algebra $B \ltimes K(M,n) = K_B(M,n)$, the projection map $K_B(M,n) \to B$ has a natural section $B \to K_B(M,n)$ corresponding to the zero derivation. Let Let $I = \{0 \to 1\}$ be the category with two objects and one non-trivial morphism. The diagram category $(s\mathcal{A}^T)^I$ is the "arrow category" with objects the morphisms $A \to X$ in $s\mathcal{A}^T$. As mentioned above, the category $(s\mathcal{A}^T)^I$ has a simplicial model category structure with level-wise weak equivalences and fibrations. Let $(s\mathcal{A}^T)^I/B$ denote the category of arrows over $1 : B \to B$. Then we have a natural isomorphism

$$\left[\begin{array}{ccc} A & & B \\ \downarrow & , & \downarrow \\ X & & K_B(M,n) \end{array}\right]_{(s\mathcal{A}^T)^I/B} \cong D^n_T(X/A, M) \tag{2.6}$$

We now come to a version of the transitivity sequence. Compare the results of [22]. See also Remark 2.12 below.

Proposition 2.11 *Let $A \to B \to C$ be a sequence of morphisms in $s\mathcal{A}^T$. Then there is a homotopy cofiber sequence in $s\mathcal{M}^T_C$.*

$$C \otimes_B \mathbb{L}\Omega_T(B/A) \to \mathbb{L}\Omega_T(C/A) \to \mathbb{L}\Omega_T(C/B).$$

Proof. Choose a free model $A_0 \to A$ of A in $s\mathcal{A}^T$, factor the composition $A_0 \to A \to B$ as $A_0 \to B_0 \to B$ where the first map is free and the second a weak equivalence, then factor the composition $B_0 \to B \to C$ as $B_0 \to C_0 \to C$ where the first map is free and the second a weak equivalence. We thus obtain a diagram

$$\begin{tikzcd} A_0 \arrow[r, tail] \arrow[d] & B_0 \arrow[r, tail] \arrow[d] & C_0 \arrow[d] \\ A \arrow[r] & B \arrow[r] & C \end{tikzcd}$$

where the vertical maps a weak equivalence, A_0 is free and the arrows labeled $\rightarrowtail$ are free. We will show that the resulting composition of simplicial C-modules

$$C \otimes_{B_0} \Omega_T(B_0/A_0) \to C \otimes_{C_0} \Omega_T(C_0/A_0) \to C \otimes_{C_0} \Omega_T(C_0/B_0) \tag{2.7}$$

is level-wise split short exact. The result will follow.

For a fixed integer $n \geq 0$, $(A_0)_n \cong T_n(V_n)$ for some $V_n \in \mathcal{C}$,

$$(B_0)_n \cong (A_0)_n \sqcup T_n(W_n)$$

for some $W_n \in \mathcal{C}$ and

$$(C_0)_n \cong (B_0)_n \sqcup T_n(Z_n)$$

for some $Z_n \in \mathcal{C}$. It follows that

$$\Omega_T(B_0/A_0)_n \cong B_0 \otimes^{T_n} W_n$$
$$\Omega_T(C_0/A_0)_n \cong C_0 \otimes^{T_n} (W_n \oplus Z_n)$$

and

$$\Omega_T(C_0/B_0)_n \cong C_0 \otimes^{T_n} (Z_n).$$

Thus, at level n, (2.7) becomes

$$0 \to C \otimes^{T_n} W_n \to C \otimes^{T_n} (W_n \oplus Z_n) \to C \otimes^{T_n} (Z_n) \to 0.$$

Remark 2.12 If $s\mathcal{A}^T$ is proper, one can replace $\mathbb{L}\Omega_T(X/A)$ by $\mathbb{L}'_T\Omega(X/A)$. This is the case if T is the commutative algebra operad, as in Quillen's original argument. Note that the spectral sequence Quillen invokes at the end of his proof implies that that category of simplicial commutative algebras is proper. This is also implied by the spectral sequences of [21] §II.6.

A similar proof leads to the following result.

Proposition 2.13 (Meyer-Vietoris) *Let A be a fixed object in $s\mathcal{A}^T$ and let*

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "j"] & X \arrow[d] \\ C \arrow[r] & Y. \end{tikzcd}$$

be a push-out diagram of objects under A. If A is cofibrant, if B and X are cofibrant under A, and if $j : B \to C$ is a cofibration, then there is a homotopy cofiber sequence in $s\mathcal{M}_Y^T$

$$Y \otimes_B \mathbb{L}\Omega_T(B/A) \to Y \otimes_C \mathbb{L}\Omega_T(C/A) \oplus Y \otimes_X \mathbb{L}\Omega_T(X/A) \to \mathbb{L}\Omega_T(Y/A).$$

Proof. If j and f are both free maps, then an argument very similar to that given for Proposition 2.11 shows that there is a short exact sequence of simplicial Y modules over T

$$0 \to Y \otimes_B \Omega_T(B/A) \to Y \otimes_C \Omega_T(C/A) \oplus Y \otimes_X \Omega_T(X/A) \to \Omega_T(Y/A) \to 0.$$

The hypotheses imply that at each entry of the short exact sequence the simplicial module of differentials is in fact equivalent to the derived module of differentials and we have the result in this case. We show that other cases reduce to this one.

In general, factor $j : B \to C$ as a $B \to C' \to C$ where the first map is free and the second map is a weak equivalence. Similarly factor $f : B \to X$ as $B \to X' \to X$ where the first map is free and the second is a weak equivalence. Because j is a cofibration, and B, C, and X are cofibrant, Proposition 3.5 of [5] implies that the induced map

$$C' \sqcup_B X' \to Y$$

is a weak equivalence. We can now apply Lemma 2.9 to complete the argument.

The next result follows immediately from Proportion 2.13 and Equation 2.5

Corollary 2.14 *Suppose given a push-out square as in Proposition 2.13, and suppose M is a $\pi_0 Y$ module over $\pi_0 T$. Then there is a long exact sequence in André-Quillen cohomology*

$$0 \to D_T^0(Y/A, M) \to D_T^0(X/A, M) \times D_T^0(C/A, M) \to D_T^0(B/A, M) \to D_T^1(Y/A, M) \to \cdots.$$

3 Comodules over a Hopf algebroid.

This section and the next are devoted to proving that under a standard hypotheses (Definition 3.1 below) one has good homotopy theoretic properties for algebras over operads and related categories in the category of left comodules over some operad.

One begins with a $\mathbb{Z}$-graded commutative ring R, which is typically the coefficient ring of some homology theory. If (R, Γ) is a Hopf algebroid and Γ is a flat R module, then the category $\mathcal{C}_\Gamma$ of left comodules over Γ is abelian. However, it doesn't have enough projectives, so one must think a bit to get a good closed model category structure on $s\mathcal{C}_\Gamma$.

It is extremely useful to place the following restriction on (R, Γ).

Definition 3.1 A Hopf algebroid (R, Γ) is an *Adams Hopf algebroid* if Γ is flat over R and there is a filtered system $\{\Gamma_\alpha\}$ of left Γ comodules so that each Γ_α is a finitely generated projective R-module and there is an isomorphism in $\mathcal{C}_\Gamma$:

$$\operatorname*{colim}_\alpha \Gamma_\alpha \cong \Gamma.$$

The terminology is justified by the following observation. Let $D(\cdot)$ denote the Spanier-Whitehead duality functor.

Example 3.2 In [1] Adams identified a condition under which a homology theory E_* had a universal coefficient spectral sequence; namely, if E is the representing spectrum for E_*, then there is a weak equivalence

$$\operatorname*{colim}_\alpha E_\alpha \to E$$

so that

1. the E_α form a diagram of finite CW spectra,
2. for all α, E_*DE_α is projective as an E_* module, and
3. for all E module spectra F, the Künneth map

$$F^*DE_\alpha \to \operatorname{Hom}^*_{E_*}(E_*DE_\alpha, F_*)$$

   is an isomorphism.

Given such a spectrum E, the pair (E_*, E_*E) might not form a Hopf algebroid. However, if E_*E is flat over E_*, then (E_*, E_*E) is a Hopf algebroid and it is an Adams Hopf algebroid. The only point to argue here is that the statement that if E_*DE_α is a projective E_* module, then so is E_*E_α. However, since E_α has the homotopy type of a finite CW spectrum, E_*DE_α is necessarily a finitely generated E_* module. Then

$$E_*E_\alpha \cong E^{-*}DE_\alpha \cong \operatorname{Hom}^*_{E_*}(E_*DE_\alpha, E_{-*})$$

is the dual of a finitely generated projective module, hence a finitely generated projective module

If E_* is a Landweber exact homology theory, then E_*E is flat over E_* by Landweber's theorem [18] and it satisfies the condition of Adams. This last statement is implicit in [1] and made explicit in [25]

For any Hopf algebroid, there is also a notion of a right comodule; that is, a graded R-module M with coproduct

$$\psi : M \to M \otimes_R \Gamma.$$

The categories of left and right comodules are equivalent: if M is a right Γ-comodule, we define a left comodule structure of Γ via the composition, called ψ_M^*,

$$M \xrightarrow{\psi_M} M \otimes_R \Gamma \xrightarrow{t} \Gamma \otimes_R M \xrightarrow{c\otimes 1} \Gamma \otimes M$$

where t is the graded twist map $t(a\otimes b) = (-1)^{|a|\,|b|} b\otimes a$ and c is the conjugation of Γ. The resulting functor from right comodules to left comodules has an obvious inverse.

It is worth pointing out that Γ is both a left and right Γ-comodule via $\Delta : \Gamma \to \Gamma \otimes_R \Gamma$ and these two structures are different in the sense that the above functor from right comodules to left comodules doesn't send one to the other; indeed, the diagram

$$\begin{tikzcd}
\Gamma \arrow[r, "\Delta"] \arrow[d, "c"'] & \Gamma \otimes_R \Gamma \arrow[r, "t"] & \Gamma \otimes_R \Gamma \arrow[d, "c\otimes c"] \\
\Gamma \arrow[rr, "\Delta"] & & \Gamma \otimes_R \Gamma
\end{tikzcd}$$

shows that $c : \Gamma \to \Gamma$ defines an isomorphism between Γ with the right comodule structure obtained from its left comodule structure and Γ with the right comodule structure obtained from Δ.

Also note that if we have an Adams Hopf algebroid and if $i_\alpha : \Gamma_\alpha \to \Gamma$ are the maps of Definition 3.1, then $c \circ i_\alpha : \Gamma_\alpha \to \Gamma$ define a system of right comodule maps so that $\operatorname{colim}_\alpha \Gamma_\alpha \cong \Gamma$ as right comodules.

For the next construction we suppose $M \in \mathcal{C}_\Gamma$ is a left comodule and finitely generated and projective as an R-module. We define a comodule DM, called the dual of $M \in \mathcal{C}_\Gamma$, as follows. As an R-module

$$(DM)_k = \operatorname{Hom}_R(M_{-k}, R)$$

or, succinctly, $DM = \operatorname{Hom}_R^*(M, R)$. This R-module has a right Γ-comodule structure via

$$\operatorname{Hom}_R^*(M,R) \to \operatorname{Hom}_R^*(M,\Gamma) \xleftarrow{\cong} \operatorname{Hom}_R^*(M,R) \otimes_R \Gamma$$

where the first map sends ϕ to $(\phi \otimes 1) \circ \psi_M$ and the second sends $\phi \otimes x$ to $\phi(\cdot)x$. This last map is an isomorphism because M is finitely generated and projective. Let $DM \in \mathcal{C}_\Gamma$ be the associated left comodule. Note that since M is a retract of $\bigoplus_i R_{*+n_i}$ for some finite set $\{i\}$, DM is a retract of $\bigoplus_i R_{*-n_i}$; hence, DM is finitely generated and projective as an R-module. The following results lists the properties of this construction.

Lemma 3.3 *Let $M \in \mathcal{C}_\Gamma$ be finitely generated and projective of R.*

1) The natural map $M \to D(DM)$ is an isomorphism of left comodules.

2) Let $N_1, N_2 \in \mathcal{C}_\Gamma$. Then the adjoint isomorphism

$$\operatorname{Hom}_R(N_1 \otimes M, N_2) \cong \operatorname{Hom}_R(N_1, \operatorname{Hom}_R^*(M,R) \otimes N_2)$$

restricts to an isomorphism

$$\operatorname{Hom}_\Gamma(N_1 \otimes M, N_2) \cong \operatorname{Hom}_\Gamma(N_1, DM \otimes N_2).$$

We belabored these ideas so that we could prove the following result. The proof is essentially the same as in [20].

Lemma 3.4 *Let (R,Γ) be an Adams Hopf algebroid. The left comodules $\Sigma^n D\Gamma_\alpha$, $n \in \mathbb{Z}$, generate the category $\mathcal{C}_\Gamma$. In particular, every left comodule is the filtered colimit of its sub-comodules which are finitely generated over R.*

Proof. Let $M \in \mathcal{C}_\Gamma$ and let $x \in M$ have degree n. We show there is a map $g : \Sigma^n D\Gamma_\alpha \to M$ in $\mathcal{C}_\Gamma$ for some n and α so that x is in the image. Both claims follow.

Consider $\Gamma \otimes_R M$ with the diagonal comodule action

$$\Gamma \otimes_R M \xrightarrow{\Delta\otimes\psi_M} \Gamma \otimes_R \Gamma \otimes_R \Gamma \otimes_R M \xrightarrow{1\otimes t\otimes 1} \Gamma \otimes_R \Gamma \otimes_R \Gamma \otimes_R M \xrightarrow{m\otimes 1} \Gamma \otimes_R \Gamma \otimes_R M.$$

This is isomorphic to the extended Γ comodule $\Gamma \otimes_R M$ with action

$$\Gamma \otimes_R M \xrightarrow{\Delta\otimes 1} \Gamma \otimes_R \Gamma \otimes_R M;$$

in fact, the isomorphism is given by

$$\Gamma \otimes_R M \xrightarrow{1\otimes\psi_M} \Gamma \otimes_R \Gamma \otimes_R M \xrightarrow{m\otimes 1} \Gamma \otimes_R M.$$

Hence (with the diagonal action)

$$M_n \cong \mathrm{Hom}_\Gamma(\Sigma^n R, \Gamma \otimes_R M) \cong \operatorname*{colim}_\alpha \mathrm{Hom}_\Gamma(\Sigma^n R, \Gamma_\alpha \otimes M).$$

Let $f : \Sigma^n R \to \Gamma_\alpha \otimes M$ map to x under this isomorphism and let

$$g : \Sigma^n R \otimes D\Gamma_\alpha = \Sigma^n D\Gamma_\alpha \to M$$

be the adjoint guaranteed by Lemma 3.3. This is the map needed to complete the proof.

We use the generators $\Sigma^n D\Gamma_\alpha \in \mathcal{C}_\Gamma$ to build a simplicial model category on the category of simplicial objects $s\mathcal{C}_\Gamma$. This category is naturally a simplicial category by ([21] §II.2). Temporarily, we consider the following sort of weak equivalence, fibration, and cofibration.

Definition 3.5 Let (R,Γ) be an Adams Hopf algebroid. Define a morphism $f : X \to Y$ in $s\mathcal{C}_\Gamma$ to be a Quillen weak equivalence (or Quillen fibration) if and only if

$$f_* : \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X) \to \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, Y)$$

is a weak equivalence (or fibration) of simplicial sets for all (n,α). Define $f : X \to Y$ to be a Quillen cofibration if and only if it has the left lifting property with respect to all acyclic Quillen fibrations.

Remark 3.6 Since the object $\Sigma^n D\Gamma_\alpha$ is constant

$$\mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X)_m \cong \mathrm{Hom}_{\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X_m).$$

In particular, $\mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X)$ is a simplicial abelian group. Thus every object in $s\mathcal{C}_\Gamma$ is Quillen fibrant. Also note that

$$\begin{aligned} \operatorname*{colim}_\alpha \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n D\Gamma_\alpha, X) &\cong \operatorname*{colim}_\alpha \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n R, \Gamma_\alpha \otimes_R X) \\ &\cong \mathrm{map}_{s\mathcal{C}_\Gamma}(\Sigma^n R, \Gamma \otimes_R X) \cong X_n. \end{aligned}$$

Hence, if $f : X \to Y$ is a Quillen weak equivalence, then

$$\pi_* f : \pi_* X \to \pi_* Y$$

is an isomorphism of graded objects in $\mathcal{C}_\Gamma$. The converse is not true however. We will repair this defect below in Proposition 3.11.

Proposition 3.7 *With the definitions of 3.5, the category $s\mathcal{C}_\Gamma$ becomes a simplicial model category.*

Proof. This follows from the arguments of ([21] §II.4) using the fact that every object is fibrant.

Remark 3.8 One can also characterize these cofibrations. If $X \in s\mathcal{C}_\Gamma$, the normalization NX is a chain complex of left Γ-comodules. A morphism $f : X \to Y$ is a cofibration if it is a retract of a *free map* and a free map if $Nf : NX_m \to NY_m$ is the inclusion of a summand in $\mathcal{C}_\Gamma$ and there is an isomorphism in $\mathcal{C}_\Gamma$

$$NY_m/NX_m \cong \bigoplus_A \Sigma^{ni} D\Gamma_{\alpha_i}$$

where A runs over some indexing set of pairs (n_i, α_i). We refer to a map of chain complexes as "free", also.

Because of the type of weak equivalences that arise in this model category structure (see Remark 3.6), these results are somewhat unsatisfactory. Hence:

Definition 3.9 Define a morphism $f : X \to Y$ to be

1. a weak equivalence if $\pi_* f : \pi_* X \to \pi_* Y$ is an isomorphism;
2. a cofibration if f is a Quillen cofibration;
3. a fibration if f has the right lifting property with respect to all morphisms which are at once cofibrations and π_* weak equivalences.

We will argue that, with these definitions, we have a simplicial model category structure. The argument goes back to [3], §11, and is now standard: it is a kind of localization. The key input is the next result; it proves there are enough generating acyclic cofibrations.

Let λ be an infinite cardinal greater than the cardinality of $\bigoplus_n R_n$. If N is a simplicial R-module, let $\#N$ be the cardinality of $\oplus(N_n)_m$, with n the internal degree and m the simplicial degree.

Lemma 3.10 *Let $f : A \to B$ be a acyclic cofibration in $s\mathcal{C}_\Gamma$, and let $x \in B$. Then there is a sub-simplicial comodule $B' \subseteq B$ so that $A \cap B' \to B'$ is a acyclic cofibration, $x \in B'$, and $\#B' \leq \lambda$.*

Proof. We may, by Remark 3.8, assume f is free. We will then prove the following stronger assertion about chain complexes. Suppose $C \to D$ is an $\mathbb{N}$-graded chain complex in $\mathcal{C}_\Gamma$ so that $H_*C \cong H_*D$ and for all $m \geq 0$ there is an indexing set I_m so that $C_m \to D_m$ is isomorphic to

$$C_m \to C_m \oplus \bigoplus_{i \in I_m} \Sigma^{n_i} D\Gamma_{\alpha_i}.$$

Let $x \in D_m$ for some m. Then there is a subchain complex $D' \subseteq D$ so that $\#D' \leq \lambda$, $x \in D'$, $H_*(C \cap D') \cong H_*D'$, and for all $m \geq 0$ there is a sub-indexing set $I'_m \subseteq I_m$ so that $C_m \cap D'_m \to D'_m$ is isomorphic to

$$C_m \cap D'_m \to C_m \cap D'_m \oplus \bigoplus_{I_m} \Sigma^{n_i} D\Gamma_{\alpha_i}.$$

We begin with the following observation.

Let $M \subseteq D_m$ be a finitely generated sub-comodule. Then there is a sub-chain complex $E(M) \subseteq D$ so that $\#E(M) \leq \lambda$, $M \subseteq E(M)_m$ and $C \cap E(M) \to E(M)$ is free. To see this, let $\{y_j\} \subseteq M$ be a set of generators. Then there is a unique expression

$$y_j = y_{0j} + \Sigma y_{ij}$$

where $y_{0j} \in C_m$ and $y_{ij} \in \Sigma^{n_i} D\Gamma_{\alpha_i}$. Then $y_{ij} \neq 0$ for only finitely many pairs (i,j), hence for only finitely many i. Let $N \subseteq C_m$ be a finitely generated sub-comodule containing $\{y_{0j}\}$. Set

$$E(M)_m = N \oplus \bigoplus_{I(M)} \Sigma^{n_i} D\Gamma_{\alpha_i}$$

where $I(M) = \{i \in I_m | y_{ij} \neq 0 \text{ for some } (i,j)\}$. Since $E(M)_m$ is finitely generated, so is $\partial D(M)_m \subseteq D_{m-1}$. Choose $D(M)_{m-1}$ in the same manner. A downward recursion finishes the argument.

To finish the proof of the lemma, we define sub-complexes

$$D_0 \subseteq D_1 \subseteq D_2 \subseteq \cdots$$

so that $\#D_k \leq \lambda$, $C \cap D_k \to D_k$ is free, $x \in D_0$ and $H_*(D_k/C \cap D_k) \to H_*(D_{k+1}/C \cap D_{k+1})$ is zero.

To define D_0, let $M \subseteq D_m$ be any finitely generated sub-comodule with $x \in M$ and let $D_0 = E(M)$. Having defined D_k, let

$$a_t \in H_*(D_k/C \cap D_k)$$

be a set of generators and let $x_t \in D_k$ pass to a_t. Since $H_*(D/C) = 0$ there exists $y_t \in D$ so that $\partial y_t = x_t \mod C$. Let $M_t \subseteq D_k$ be any finitely generated sub-module containing y_t and let

$$D_{k+1} = D_k + \sum_t E(M_t).$$

This finishes the proof.

Proposition 3.11 *With these definitions, and its usual simplicial structure, $s\mathcal{C}_\Gamma$ becomes a simplicial model category.*

Proof. Lemma 3.10 and the small object argument supply the "trivial cofibration – fibration" factorization. To obtain the other factorization, we examine the Quillen model category structure on $s\mathcal{C}_\Gamma$. There one obtains a factorization of any morphism as a cofibration (in either sense) followed by a morphism which has the right lifting property with respect to all cofibrations. Thus it is a fibration in the sense of Definition 3.9 and a weak equivalence in the Quillen model category structure. By Remark 3.6 it is a π_* isomorphism, as required. To prove the lifting axiom, we note that half of this axiom is the definition of fibration. To obtain the lifting of cofibrations against trivial fibrations, note that we have shown that one can factor any trivial fibration $p : X \to Y$ as a composite

$$X \xrightarrow{j} Z \xrightarrow{q} Y$$

where j is a cofibration and q is a trivial cofibration with right lifting property with respect to all cofibrations. Then j is a trivial cofibration and we use this to show that p is a retract of q. Since the morphism q has the requisite lifting property, so does p.

To prove Quillen's corner axiom SM7 that supplies the simplicial part of the simplicial model category structure, we prove the following equivalent statement: if $i : A \to B$ is a cofibration in $s\mathcal{C}_\Gamma$ and $j : K \to L$ is a cofibration of simplicial sets, then

$$i \otimes j : B \otimes K \sqcup_{A \otimes K} A \otimes L \to B \otimes L$$

is a cofibration which is a weak equivalence if i or j is. It is certainly a cofibration, as SM7 holds in the Quillen structure on $s\mathcal{C}_\Gamma$ in the Quillen structure. To see that it is a weak equivalence, note that Remark 3.8 implies that i is a cofibration of simplicial R-modules. Thus if i or j is a weak equivalence, $i \otimes j$ is a weak equivalence of simplicial R modules, because SM7 holds for that category. Thus $i \otimes j$ is a π_* isomorphism, as required. This completes the proof.

4 Simplicial algebras over a simplicial operad in comodules

In this section we promote the model category structure on $s\mathcal{C}_\Gamma$ to a model category structure on $s\mathcal{C}_\Gamma^T$, where T is a simplicial operad in $\mathcal{C}_\Gamma$ and a simplicial T modules over a simplicial T-algebra A. We concentrate on the homotopy weak equivalence structure of Proposition 3.11.

Fix a simplicial operad T in $\mathcal{C}_\Gamma$ and let $s\mathcal{C}_\Gamma^T$ denote the algebras in $s\mathcal{C}_\Gamma$ over T.

Definition 4.1 Let $f : X \to Y$ be a morphism in $s\mathcal{C}_\Gamma^T$. Then

1. f is a weak equivalence if $\pi_* f : \pi_* X \to \pi_* Y$ is a weak equivalence;

2. f is a fibration if it is a fibration in $s\mathcal{C}_\Gamma$;

3. a cofibration if it has the left lifting property with respect to all trivial fibrations.

The category $s\mathcal{C}_\Gamma^T$ has a canonical simplicial structure (see [13], §1), and the first result is the following.

Proposition 4.2 *With these definitions $s\mathcal{C}_\Gamma^T$ becomes a simplicial model category.*

Since the model category on $s\mathcal{C}_\Gamma$ is cofibrantly generated and the forgetful functor $U_T : s\mathcal{C}_\Gamma^T \to s\mathcal{C}_\Gamma$ preserves filtered colimits and is continuous in the sense that for all $K \in \mathbf{S}$

$$U_T(X^K) \cong U_T(X)^K,$$

this falls into the standard situation where one is trying to lift a model category structure ([6], [15], §II.6). There are two crucial lemmas which we now state and prove.

Let $I = \{\Sigma^n D\Gamma_\alpha \otimes \partial\Delta^k \to \Sigma^n D\Gamma_\alpha \otimes \Delta^k\}$ be the generating cofibrations of $s\mathcal{C}_\Gamma$ and let $J = \{i_\beta : C_\beta \to D_\beta\}$ be the generating trivial cofibrations of $s\mathcal{C}_\Gamma$. Let $\mathcal{M}_{T(I)}$ be the class of morphisms in $s\mathcal{C}_\Gamma^T$ generated by the morphisms $\{T(\Sigma^n D\Gamma_\alpha) \otimes \partial\Delta^k \to T(\Sigma^n D\Gamma_\alpha) \otimes \Delta^k\}$; thus $\mathcal{M}_{T(I)}$ is the smallest class of morphisms containing the listed morphisms and closed under isomorphism, coproducts, co-base change, sequential colimits, and retracts. Similarly, we may define $\mathcal{M}_{T(J)}$. If $i_\beta : C_\beta \to D_\beta$ is in J, then $T(i_\beta) \in \mathcal{M}_{T(I)}$; hence $\mathcal{M}_{T(J)} \subseteq \mathcal{M}_{T(I)}$. Here are the crucial lemmas.

Lemma 4.3 *Every morphism in $\mathcal{M}_{T(J)}$ is a weak equivalence in $s\mathcal{C}_\Gamma^T$.*

Proof. Since $i_\beta : C_\beta \to D_\beta$ is a trivial cofibration of simplicial R-modules, and every simplicial R-module is fibrant, i_β is a strong deformation retraction of simplicial R-modules. Hence i_β has a retraction $r_\beta : D_\beta \to C_\beta$ and there is a homotopy $D_\beta \to D_\beta^{\Delta^1}$ from 1_{D_β} to $i_\beta r_\beta$. It follows from the natural map $T(D_\beta^{\Delta^1}) \cong T(D_\beta)^{\Delta^1}$ that $Ti_\beta : TC_\beta \to TD_\beta$ is a strong deformation retraction of simplicial algebras over T, regarded as a simplicial operad in R-modules. Thus $TC_\beta \to TD_\beta$ is a weak equivalence of simplicial R-modules, which is all that is required.

Lemma 4.4 *Let $M \in s\mathcal{C}_\Gamma$. Then $T(M) \in s\mathcal{C}_\Gamma^T$ is small with respect to the class of morphisms $\mathcal{M}_{T(I)}$.*

Proof. The comodule M is small with respect to the class of morphisms $X \to Y$ in $s\mathcal{C}_\Gamma$ which are level-wise inclusions. Since $\mathrm{Hom}_{s\mathcal{C}_\Gamma^T}(T(M), X) = \mathrm{Hom}_{s\mathcal{C}_\Gamma}(M, X)$, we need only show that the morphisms in $\mathcal{M}_{T(I)}$ are level-wise inclusions.

Since the generating morphisms $T(\Sigma^n D\Gamma_\alpha \otimes \partial\Delta^k) \to T(\Sigma^n D\Gamma_\alpha \otimes \Delta^k)$ are free (see [11]), every morphism in $\mathcal{M}_{T(I)}$ is a retract of a free morphism. Since retracts of inclusions are inclusions, we need only show s-free morphisms in $s\mathcal{C}_\Gamma^T$ are level-wise inclusions. If $A \to B$ in $s\mathcal{C}_\Gamma^T$ is free there is, for each $k \geq 0$, a comodule $M_k \in \mathcal{C}_\Gamma$ and an isomorphism of $A_k \to B_k$ with

$$i_1 : A_k \to A_k \sqcup T_k(M_k)$$

where the coproduct is as T_k-algebras. The unique map of comodules $M_k \to 0$ defines a map

$$r : A_k \sqcup T_k(M_k) \to A_k \sqcup T_k(0) \cong A_k$$

which is a retraction of i_1. The result follows.

Proof of Proposition 4.2: A morphism $f : X \to Y$ in $(s\mathcal{C}_\Gamma)^T$ is a trivial fibration if and only if it has the left lifting property with respect to the morphisms $T(\Sigma^n D\Gamma_\alpha \otimes \partial\Delta^k) \to T(\Sigma^n D\Gamma_\alpha \otimes \Delta^k)$, and f is a fibration if and only if it has the left lifting property with respect to the morphisms $T(C_\beta) \to T(D_\beta)$. The small object argument now applies, because of Lemma 4.4. That the small object argument to produce the "trivial cofibration-fibration" factorization works follows from Lemma 4.3. The Axiom SM7 holds because SM7a ([21], §II.2) holds in $(s\mathcal{C}_\Gamma^T)$, which follows from the facts that it holds in $s\mathcal{C}_\Gamma$ and that the forgetful functor preserves exponential objects.

Remark 4.5 It is possible to characterize the cofibrations in $(s\mathcal{C}_\Gamma)^T$ as retracts of the *free* morphisms. If we write $(n\mathcal{C}_\Gamma)^T$ as the non-negatively graded T-algebras in $\mathcal{C}_\Gamma$, where T is regarded as a discrete operad, and $(s_+\mathcal{C}_\Gamma)^T$ for the T-algebras in $s_+\mathcal{C}_\Gamma$ (degeneracy diagrams) with T regarded as an operad with only its degeneracy morphisms, then the restriction functor

$$r_* : (s_+\mathcal{C})^T \to (n\mathcal{C})^T$$

has left adjoint r^* with

$$r^*\{Z_*\}_k = T_k \otimes_{r^*T} \Big(\coprod_{\phi:[k]\to[j]} Z_j \Big)$$

with ϕ running over the morphisms of Δ_+ and

$$r^*_A T = \coprod_{\phi:[k]\to[j]} T_j \to T_k$$

the coproduct of operads.

A morphism $A \to B$ in $s\mathcal{C}_\Gamma^T$ is s-free if the underlying degeneracy diagram is isomorphic to

$$i_1 : A \to A \sqcup r^*Z$$

for some $Z \in (n\mathcal{C})^T$. It is *free* if $Z_j = T_j(M_j)$ where M_j is a sum of the comodules $\Sigma^n D\Gamma_\alpha$. Certainly, every cofibration created by the small object argument is free; conversely, if a morphism is free a skeletal decomposition (as in [13],§1) shows the morphism is a cofibration; if in fact $A \to \operatorname{colim} sk_n^A B \cong B$ where $sk_{n-1}^A B = A$ and there are push-out diagrams

$$\begin{array}{ccc} T(M_n) \otimes \partial\Delta^n & \longrightarrow & sk_{n-1}^A B \\ \downarrow & & \downarrow \\ T(M_n) \otimes \Delta^n & \longrightarrow & sk_n^A B. \end{array}$$

Note that every cofibration in $(s\mathcal{C}_\Gamma)^T$ is a cofibration in $(s\mathcal{M}_R)^T$.

If we fix a T-algebra A, the same sort of discussion puts a model category structure on the category of simplicial A-modules in $\mathcal{C}_\Gamma$. Recall that T is a simplicial operad, and let $(s\mathcal{C}_\Gamma)^T_A$ be the category of A-modules over T. Then $(s\mathcal{C}_\Gamma)^T_A$ acquires a simplicial structure and we have:

Proposition 4.6 *There is a simplicial model category structure on $(s\mathcal{C}_\Gamma)^T_A$ where $f: M \to N$ is a weak equivalence or fibration if and only if it is a weak equivalence or fibration in $s\mathcal{C}_\Gamma$.*

Proof. The forgetful functor $(s\mathcal{C}_\Gamma)^T_A \to s\mathcal{C}_\Gamma$ has a left adjoint, so again the small object argument will work, provided one proves the requisite analogs of Lemmas 4.3 and 4.4. The proofs remain virtually unchanged.

Remark 4.7 Again, the cofibrations are retracts of free morphisms, which we now define. Let $(n\mathcal{C}_\Gamma)^T_A$ and $(s_+\mathcal{C}_\Gamma)^T_A$ be the evident restricted categories. Then the restriction functor

$$r_*: (s_+\mathcal{C}_\Gamma)^T_A \to (n\mathcal{C}_\Gamma)^T_A$$

has a left adjoint. If $M = \{M_k\} \in (n\mathcal{C}_\Gamma)^T_A$, then each M_k is a A_k module over T_k. If $\phi: [k] \to [j]$ is any surjection in Δ, then $\phi^*: A_j \to A_k$ is a morphism of T_j algebras, and induces a morphism of T_k algebras

$$T_k \otimes_{T_j} A_j \to A_k.$$

Thus we may form a A_k module over T_k.

$$T_k \otimes_{T_j} (A_k \otimes_{A_j} M_j) \cong A_k \otimes_{(T_k \otimes_{Tj} A_k)} (T_k \otimes_{T_j} M_j).$$

Then the left adjoint to r_* is defined by

$$(r^*M)_k = \bigoplus_{[k]\to[j]} T_k \otimes_{T_j} (A_k \otimes_{A_j} M_j).$$

In particular, if $M_j = A_j \overset{T_j}{\otimes} N_j$, then

$$\begin{aligned}(r^*M)_k &\cong \bigoplus_{[k]\to[j]} A_k \overset{T_k}{\otimes} N_j \\ &\cong A_k \overset{T_k}{\otimes}(\bigoplus_{[k]\to[j]} N_j).\end{aligned}$$

A morphism $X \to Y$ in $(s\mathcal{C})^A_\Gamma$ is free if there exist $N_k \in \mathcal{C}_\Gamma$, $k \geq 1$, so that each N_k is a direct sum of comodules of the form $\Sigma^n D\Gamma_\alpha$, and so that in $(s_+\mathcal{C}_\Gamma)^T_A$, the map $X \to Y$ is isomorphic to

$$X \to X \oplus r^*(\{A_k \overset{T_k}{\bigotimes} N_k\}_{k\geq 0}).$$

In particular, every cofibration in $(s\mathcal{C}_\Gamma)^T_A$ is a cofibration in $(s\mathcal{M})^T_A$.

We finish the section by writing down a comparison result for André-Quillen cohomology.

The forgetful functor from Γ-comodules to R-modules has a right adjoint which sends an R-module to the extended Γ-comodule:

$$M \to \Gamma \otimes_R M.$$

If T is a simplicial operad in Γ-comodules, then T is an operad in R-modules. If A is a simplicial algebra over T in Γ comodules and M is a simplicial A-module over T in R modules, then $\Gamma \otimes_R M$ is a simplicial A-module over T in Γ comodules: the structure map

$$T(A, \Gamma \otimes_R M) \to \Gamma \otimes_R M$$

is adjoint to the map

$$T(A, \Gamma \otimes_R M) \to T(A, M) \to M$$

induced by the unit of the adjunction $\Gamma \otimes_R M \to M$; in fact, the extended comodule functor is the right adjoint to the forgetful functor

$$(s\mathcal{C}_\Gamma)^T_A \to (s\mathcal{M}_R)^T_A.$$

This has implications for derivations and differentials. For operads T in Γ-comodules and A over T in Γ-comodules, the formalism of section 2 applies: if $A \to X$ is a morphism of T-algebras in Γ-comodules, we have an R-module of derivations

$$\mathrm{Der}^\Gamma_A(X, M)$$

of T-derivations in Γ-comodules. These are morphisms of Γ-comodules $d : X \to M$ so that the diagram of Proposition 2.2 commutes and so that $d = 0$ on A. This functor is representable; see Equation 2.3 – the same argument apply. Let $\Omega_{\Gamma(T)}(X/A)$ be the representing object; it is an A-module over T in Γ-comodules. (The Γ in the notation is to indicate that the underlying abelian category is $\mathcal{C}_\Gamma$.) Let $\Omega_T(X/A)$ be the module of Kähler differentials of A in R-modules.

Lemma 4.8 *The A-module over T in R-modules $\Omega_T(X/A)$ has a natural structure of a Γ-comodule and, with this structure,*

$$\Omega_T(X/A) \cong \Omega_{\Gamma(T)}(X/A).$$

Proof. This could be argued directly by examining the construction of Lemma 2.3 and Equation 2.3 or as follows. If we let Der^Γ_A denote the T-derivations in the category of Γ-comodules Der_A denote derivations in the category of R-modules, then using Proposition 2.2, we see that there is an isomorphism of R-modules

$$\mathrm{Der}_A(X, M) \cong \mathrm{Der}^\Gamma_A(X, \Gamma \otimes_R M).$$

and, hence, a natural isomorphism

$$\mathrm{Hom}_{\mathcal{M}^T_X}(\Omega_T(X/A), M) \cong \mathrm{Hom}_{(\mathcal{C}_\Gamma)^T_X}(\Omega^\Gamma_T(X/A), \Gamma \otimes_R M) \cong \mathrm{Hom}_{\mathcal{M}^T_X}(\Omega^\Gamma_T(X/A), M).$$

The result follows.

We explore the ramification of Lemma 4.8 in André-Quillen homology and cohomology.

For example, if T is a simplicial operad in Γ-comodules and $A \to X$ is a morphism of T-algebras in Γ-comodules, then we define

$$D^{\Gamma(T)}_*(X/A) = \pi_*\Omega_{\Gamma(T)}(X/A) \cong \pi_*[X \otimes \Omega_{\Gamma(T)}(X_0/A_0)] \tag{4.1}$$

where $A_0 \to X_0$ is a cofibrant model for the map $A \to X$.

Note that if X is a simplicial T algebra and M is a simplicial X-module over T, then π_*M is a π_*X module over π_*T.

Proposition 4.9 *There is a natural isomorphism of π_*X modules over π_*T*

$$D^{\Gamma(T)}_*(X/A) \cong D^T_*(X/A).$$

Proof. Choose a weak equivalence $A_0 \to A$ in $(s\mathcal{C}_\Gamma)^T$ with A_0 free, and then factor the composite $A_0 \to A \to X$ as

$$A_0 \to X_0 \to X$$

where $A_0 \to X_0$ is free. Then, by Lemma 4.8

$$D_*^{\Gamma(T)}(X/A) = \pi_*[X \otimes \Omega_T^\Gamma(X_0/A_0)] \cong \pi_*[X \otimes \Omega_T(X_0/A_0)]$$

Thus we need only check that $A_0 \to X_0$ is cofibration with cofibrant source in $(s\mathcal{M}_R)^T$. Since each of the generating cofibrations of $(s\mathcal{C}_\Gamma)^T$

$$T(\Sigma^n D\Gamma_\alpha \otimes \partial\Delta^k) \to T(\Sigma^n D\Gamma_\alpha \otimes \Delta^k)$$

is a cofibration in $(s\mathcal{M}_R)^T$ – since $\Sigma^n D\Gamma_\alpha$ is a projective R-module – we have the result.

Defining André-Quillen cohomology of T-algebras in comodules is a bit harder, because an arbitrary simplicial Γ-comodule need not be cofibrant. We return to [21] §5 for the definition, and then give an interpretation.

Let $I = \{0 \to 1\}$ be the category with two objects and one non-trivial morphism. If $\mathcal{C}$ is any category, the diagram category $\mathcal{C}^I$ is the "arrow category" with objects the morphisms $A \to X$ in $\mathcal{C}$. In Γ-comodules, let $T \to T_{-1}$ be an augmented simplicial operad, B a T_{-1} algebra, and M a B-module over T_{-1}. Then we may form the augmented algebra $B \ltimes K(M,q)$ over B and we define, for any arrow $A \to X$ of T-algebras over B,

$$D^q_{\Gamma(T)}(X/A, M) = \mathbf{Ho}_{(s\mathcal{C}_\Gamma^T/B)^I}\left(\begin{array}{c} A \\ \downarrow \\ X \end{array}, \begin{array}{c} B \\ \downarrow \\ B \ltimes K(M,q) \end{array} \right) \tag{4.2}$$

where $s\mathcal{C}_\Gamma^T/B$ denotes the category of simplicial algebras over B. The map $B \to B \ltimes K(M,n)$ is the zero section. Since this map need not be fibrant and since $A \to X$ need not be cofibrant in $(s\mathcal{C}_\Gamma^T/B)^I$, this definition needs an explication.

We first produce a fibrant model for $B \ltimes K(M,n)$. The category $(sC_\Gamma)_B^{T_{-1}}$ has a model category structure, by Proposition 4.6. Let $K_f(M,n)$ be a fibrant model for $K(M,n)$ in this category. Then we may form a weak equivalence of augmented T algebras over B:

$$B \ltimes K(M,n) \to B \ltimes K_f(M,n).$$

Since $K_f(M,n)$ is fibrant in $(sC_\Gamma)_B^{T_{-1}}$ it is fibrant in sC_Γ; thus the projection

$$B \ltimes K_f(M,n) \to B$$

is a fibration in sC_Γ. It follows that this projection is a fibrant object in $(sC_\Gamma)^T/B$ and, therefore, the zero section

$$B \to K_f(M,n)$$

is a fibrant object in $(s\mathcal{C}_\Gamma^T/B)^I$. This is the desired fibrant model. Using this we can prove the following result. The symbol $[\cdot,\cdot]$ means the left homotopy classes of maps; that is, the set of maps modulo the standard homotopy relation defined using a cylinder object.

Lemma 4.10 *Let $A \to X$ be a morphism of T-algebras over B. Then for any $k \geq 0$*

$$D^q_{\Gamma(T)}(X/A, M) \cong \pi_k \operatorname{map}_{(sC^T_\Gamma/B)^I}\left(\begin{array}{c} A_0 \\ \downarrow \\ X_0 \end{array}, \begin{array}{c} B \\ \downarrow \\ B \ltimes K_f(M, q+k) \end{array} \right)$$

$$\cong \pi_k \operatorname{map}_{(sC_\Gamma)^T_X}(X \otimes_{X_0} \Omega_{\Gamma(T)}(X_0/A_0), K_f(M, q+k)).$$
$$\cong [\mathbb{L}\Omega_{\Gamma(T)}(X/A), K_f(M,q)]_{(sC_\Gamma)^T_X}$$

where $A_0 \to X_0$ is a cofibrant model for $A \to X$ and we take the zero map as the basepoint of both mapping spaces.

Proof. The second isomorphism simply follows from the fact that differential represent derivation. Compare Equation 2.3. In the first isomorphism, the case $k = 0$ is the definition. Thus we need only show that for any cofibrant object N in $(sC_\Gamma)^T_X$, there is a natural isomorphism

$$\pi_k \operatorname{map}_{(sC_\Gamma)^T_X}(N, K_f(M, q+k)) \cong \pi_0 \operatorname{map}_{(sC_\Gamma)^T_X}(N, K_f(M, q)).$$

But this is standard. The third isomorphism is similar.

Lemma 4.10 has the following immediate consequence.

Proposition 4.11 *Let $T \to T_{-1}$ be an augmented operad in Γ comodules and $A \to X$ morphism of T algebras over a T_1 algebra B, again in Γ-comodules. Let M be a B-module over T_{-1} in R-modules. Then there is a natural isomorphism*

$$D^*_{\Gamma(T)}(X/A, \Gamma \otimes_R M) \cong D^*_T(X/A, M).$$

Proof. Combine Lemma 4.10, Lemma 4.8, the representability of cohomology of T-algebras in R-modules and the fact that $K(\Gamma \otimes_R M, n) \cong \Gamma \otimes_R K(M, n)$ is automatically fibrant.

5 Square-zero extensions

One of the important features of André-Quillen cohomology in the classical case is the identification of the first cohomology group with a group of square-zero (or "infinitesimal") algebra extensions – much in the way group cohomology is used to compute central extensions. We extend this theory here in two ways: first, to simplicial algebras over a simplicial operad, and, second, we allow extensions by a simplicial module. In form, our discussion parallels that of Illusie [16]. We apply this theory in the next section to study Postnikov towers.

We begin with operads, and so on, in R modules for some commutative ring R.

Let T be an operad in R modules and $f : E \to B$ a morphism of T-algebras. The kernel of f, as a morphism of R-modules, is a sub E-module over T of E, regarded as an E-module over itself. Any such sub-module is an *ideal*; furthermore, given an ideal $I \subseteq E$, the quotient map of R-modules $E \to E/I$ is a morphism of T-algebras in a unique way. An ideal $I \subseteq E$ is a *square-zero ideal* if the module structure map $\epsilon : T(E, I) \to E$ factors through the quotient $T(E, I) \to T(B, I)$. Such a factoring is unique if it exists, and makes I into B-module over T. Note that the factorization condition is equivalent to asserting that

$$\epsilon(a \otimes x_1 \otimes \cdots x_{n-1} \otimes y) = 0$$

whenever $a \in T(n)$, $x_i \in E$, $y \in I$ and at least one $x_i \in I$. This justifies the term "square-zero". If $q : E \to B$ has a square-zero kernel I, we say that E is a *square-zero extension* of B by I, and we may write

$$X = \{I \longrightarrow E \xrightarrow{q} B\}$$

for such an extension. As usual, one defines an equivalence relation on square-zero extensions by setting

$$X = \{I \longrightarrow E \xrightarrow{q} B\} \sim Y = \{I \longrightarrow E' \xrightarrow{q'} B\}$$

if there is a morphism of T-algebras $g : E \to E'$ making the following diagram commute

$$\begin{tikzcd} I \arrow[r] \arrow[d, "="'] & E \arrow[r, "q"] \arrow[d, "g"] & B \arrow[d, "="] \\ I \arrow[r] & E' \arrow[r, "q'"'] & B. \end{tikzcd}$$

Note that g is necessarily an isomorphism. We write $\mathrm{Exal}(B, I)$ for the set of equivalence classes. Note that the operad does not appear in the notation; we hope never to need it.

Square-zero extensions are preserved under-base change; that is, if $X = \{I \longrightarrow E \longrightarrow B\}$ is square-zero extension $f : A \to B$ is a morphism of T-algebras, then

$$f^*X = \{I \to f^*E = A \times_B E \to A\}$$

is also a square-zero extension. Similarly, square-zero extensions are preserved under cobase change in the fiber: if $X = \{I \longrightarrow E \longrightarrow B\}$ is a square-zero extension and $\phi : I \to J$ is a morphism of B modules, then

$$f_*X = \{J \to J \oplus_I E \to B\}$$

is a square-zero extension. This last assertion requires some checking, and is implied by the next lemma. Granting that, we have that the set $\mathrm{Exal}(B, I)$ acquires the structure of an abelian group in the usual way. The neutral object is the abelian extension $B \ltimes I \to B$ of section 2; it is the unique square-zero extension $X = \{I \longrightarrow E \longrightarrow B\}$ so that $E \to B$ has a T-algebra section.

Lemma 5.1 *Let $q : E \to B$ be a morphism of T-algebras with kernel I, J a B-module over T, and $\phi : I \to J$ a morphism of E-modules over T. Then there is a T-algebra structure on $J \oplus_I E$ so that the following diagram is a commutative diagram of T-algebras:*

$$\begin{tikzcd} E \arrow[r] \arrow[d, "q"'] & J \oplus_I E \arrow[d, "p"] \\ B \arrow[r, "="'] & B. \end{tikzcd}$$

Furthermore $\{J \to J \oplus_I E \to B\}$ is a square-zero extension.

Proof. Because J is an E-module, we can form the abelian extension $E \ltimes J$; this is a T-algebra over E. Define

$$f : I \longrightarrow E \ltimes J$$

by $f(a) = (a, -\phi(a))$. This is an injection, and one checks, using the fact that J is actually a B-module, then f identifies I with an ideal in $E \ltimes J$. The desired algebra structure on $J \oplus_I E$ is then $E \ltimes J/f(I)$. The other assertions of of this result are easily checked.

All of the above has a relative version: if $A \to B$ is a morphism of T-algebras, a square-zero extension $X = \{I \longrightarrow E \longrightarrow B\}$ is a square-zero extension under A if $E \to B$ is a morphism of T-algebras under A. Similarly, the equivalence relation on such extensions must respect this structure. We let $\mathrm{Exal}_A(B, I)$ be the resulting abelian group.

Finally, all of these definitions make sense simplicially: if T is a simplicial operad in R-modules, $A \to B$ a morphism of simplicial T-algebras and I a simplicial B-module over T, then we can form the abelian group $\mathrm{Exal}_A(B, I)$ by copying the above definitions.

There is an abelian version of the forgoing. Let T be a simplicial operad and B a simplicial T-algebra. Denote by $s\mathcal{M}_B^T$ the category of simplicial B-modules over T. If N and I are objects in $s\mathcal{M}_B^T$, we define $\mathrm{Ext}_B^1(N, I)$ to be equivalence classes of short exact sequences in $s\mathcal{M}_B^T$

$$0 \to I \to M \to N \to 0;$$

the equivalence relation is similar to that above. Again, note that the operad is missing from the notation of the Ext group.

The group $\mathrm{Ext}_B^1(N, I)$ can be computed using the homotopy theory of simplicial B-modules over T. As noted in section 2 (see after Proposition 2.5), the category $s\mathcal{M}_B^T$ is a simplicial model category. As a result, we have a functorial suspension object defined as follows. If K is any pointed simplicial set and $M \in s\mathcal{M}_B^T$, we can define $M \wedge K$ by the push-out diagram in $s\mathcal{M}_B^T$

$$\begin{tikzcd} M \otimes * \arrow[r] \arrow[d] & M \otimes K \arrow[d] \\ 0 \arrow[r] & M \wedge K. \end{tikzcd}$$

The suspension of M is then defined by $\Sigma M = M \wedge [\Delta^1/\partial\Delta^1]$. If we define the cone of M by $C(M) = M \wedge \Delta^1$ with $d^1(\Delta^0)$ as the basepoint, then there is short exact sequence in $s\mathcal{M}_B^T$

$$0 \to M \xrightarrow{M \otimes d^0} C(M) \longrightarrow \Sigma M \to 0.$$

Since $\pi_* C(M) = 0$, we have a natural isomorphism $\pi_n M \cong \pi_{n+1}\Sigma M$. There is a natural map

$$\alpha : [N, \Sigma I]_{s\mathcal{M}_B^T} \to \mathrm{Ext}_B^1(N, I)$$

defined as follows: if $f : N \to \Sigma I$ is a morphism in $s\mathcal{M}_B^T$, define an extension by pulling back $C(I) \to I$ along f. The symbol $[\cdot,\cdot]$ stands for the left-homotopy classes of maps; two morphisms $N \to \Sigma I$ are left homotopic if they extend over $N \otimes \Delta^1$. If N is not cofibrant, then this is not necessarily compute the morphisms from N to ΣI in the homotopy category. Indeed, we compute

$$\mathbf{Ho}(N, \Sigma I) = [N', \Sigma I]_{s\mathcal{M}_B^T}$$

where $N' \to N$ is a weak equivalence with N' cofibrant. If we wish to avoid making a specific choice of a such cofibrant model for N, then we may write, since every object in $s\mathcal{M}_B^T$ is fibrant.

$$\mathbf{Ho}(N, \Sigma I) \cong \operatorname*{colim}_{N_0 \xrightarrow[\simeq]{} N} [N, \Sigma I]_{s\mathcal{M}_B^T}$$

where the colimit runs over all weak equivalences $N_0 \to N$.[1]

Note that the definition of the homomorphism α uses that fact that left-homotopic maps induce isomorphic pull-backs.

Here is the standard isomorphism result; the colimit is required if N is not cofibrant.

[1]The reader may object that we are taking a colimit over a category which is not small. Nonetheless, the colimit exists, as the displayed equation shows.

Proposition 5.2 *Let $N \in \mathcal{M}_B^T$ be a cofibrant simplicial B-module over T and let $I \in \mathcal{M}_B^T$ be any object. Then*

$$\alpha : [N, \Sigma I]_{s\mathcal{M}_B^T} \to \mathrm{Ext}_B^1(N, I)$$

is an isomorphism. More generally, for an arbitrary $N \in s\mathcal{M}_B^T$ there is an isomorphism

$$\mathbf{Ho}(N, \Sigma I) \cong \operatorname*{colim}_{N_0 \xrightarrow[\simeq]{} N} [N, \Sigma I]_{s\mathcal{M}_B^T} \xrightarrow{\alpha} \operatorname*{colim}_{N_0 \xrightarrow[\simeq]{} N} \mathrm{Ext}_B^1(N, I)$$

where the colimit runs over all weak equivalences $N_0 \to N$.

Proof. The statement about the colimit follows from the first, for if $N' \to N$ is a weak equivalence with N' cofibrant, then the canonical map

$$[N' \Sigma I]_{s\mathcal{M}_B^T} \to \operatorname*{colim}_{N_0 \xrightarrow[\simeq]{} N} [N, \Sigma I]_{s\mathcal{M}_B^T}$$

is an isomorphism.

For the first statement, the usual argument adapts, so we only give a sketch. We must give an inverse to α.

If $I \to M \to N$ is a representative for an element in $\mathrm{Ext}_B^1(N, I)$, then we can form the diagram (c.f., [16] §I.3.2.2)

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r, "d^0"] & C(I) \arrow[r] & \Sigma I \arrow[r] & 0 \\
0 \arrow[r] & I \arrow[r, "j"] \arrow[u, "="] \arrow[d, "="'] & C(I) \oplus M \arrow[r] \arrow[u, "pr_1"'] \arrow[d, "pr_2"'] & K \arrow[r] \arrow[u, "p"'] \arrow[d, "s"'] & 0 \\
0 \arrow[r] & I \arrow[r] & M \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

where j is forced by the fact the diagram commutes. The map s is an acyclic fibration; therefore, because N is cofibrant, s has a section, unique up to homotopy, and determines a class in $[N, \Sigma I]_{\mathcal{M}_B^T}$. One checks that isomorphic extensions give homotopic maps. This is defines a map

$$\mathrm{Ext}_B^1(N, I) \longrightarrow [N, \Sigma I]_{s\mathcal{M}_B^T}$$

and one now checks this map is the desired inverse.

This result has a technical consequence, which we record for later use.

Corollary 5.3 *Let $C \to B$ be a morphism of simplicial T-algebras, N a cofibrant simplicial C-module, and I a simplicial B-module, all over T. Then the natural map of C-modules over T*

$$\eta : N \longrightarrow B \otimes_C N$$

induces an isomorphism

$$\alpha^* : \mathrm{Ext}_B^1(B \otimes_C N, I) \xrightarrow{\cong} \mathrm{Ext}_C^1(N, I).$$

Proof. We use the previous result and the fact that

$$[B \otimes_C N, \Sigma I]_{s\mathcal{M}_B^T} \cong [N, \Sigma I]_{s\mathcal{M}_C^T}.$$

Now let $A \to B$ be a morphism of simplicial T-algebra and I a simplicial B-module over T. We define a homomorphism

$$\gamma : \mathrm{Ext}^1_B(\Omega_T(B/A), I) \to \mathrm{Exal}_A(B, I).$$

There is a canonical morphism of T-algebras under B

$$f : B \to B \ltimes \Omega_T(B/A)$$

adjoint to the universal derivation under the isomorphism

$$\mathrm{Der}_A(B, \Omega_T(B/A)) \cong \mathrm{Hom}_{T-alg/B}(B, B \ltimes \Omega_T(B/A)).$$

Then, if

$$0 \to I \to M \to \Omega_T(B/A) \to 0$$

represents an element of $\mathrm{Ext}^1_B(\Omega_T(B/A), I)$, there is square-zero extension

$$X = \{I \to B \ltimes M \to B \ltimes \Omega_T(B/A)\}.$$

Define $\gamma([X])$ to the be the class of f^*X. The main result of this section is the following. In order to get a general statement, we need an indexing category of weak equivalences. If $f : A \to B$ is a morphism of T-algebras, let $\mathcal{W}(f)$ denote the category with objects the diagrams of T-algebras

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "\simeq"'] & B_0 \arrow[d, "\simeq"] \\ Af \arrow[r] & B \end{tikzcd}$$

with the vertical maps weak equivalences.

Theorem 5.4 *Suppose that A is a cofibrant simplicial T-algebra, $A \to B$ is a cofibration of T-algebras and I is a simplicial B module over T. Then the homomorphism*

$$\gamma : \mathrm{Ext}^1_B(\Omega_T(B/A), I) \to \mathrm{Exal}_B(A, I)$$

is an isomorphism. More generally, if $f : A \to B$ is any morphism of T-algebras, we obtain an isomorphism

$$\mathrm{Ext}^1_B(\mathbb{L}\Omega_T(B/A), I) \to \operatorname*{colim}_{\mathcal{W}(f)} \mathrm{Exal}_{A_0}(B_0, I).$$

Proof. In outline, the argument is the same as that if Proposition 5.2. Note that the second statement follows from the first and Corollary 5.3; for if $f' : A' \to B'$ is a cofibrant model for f, then $B \otimes_{B'} \Omega_T(B'/A')$ is a model for $\mathbb{L}\Omega_T(B/A)$ and the canonical composite

$$\begin{aligned} \mathrm{Ext}^1_B(B \otimes_{B'} \Omega_T(B'/A'), I) &\to \mathrm{Ext}^1_{B'}(\Omega_T(B'/A', I) \\ &\to \operatorname*{colim}_{\mathcal{W}(f)} \mathrm{Ext}^1_{B_0}(\Omega_T(B_0/A_0, I) \end{aligned}$$

is a composition of isomorphisms.

We will give the proof of the first statement below. Our model is again [16], this time §§III.1.1 – III.1.3. It involves constructing a map backwards, and this depends on a lemma about differentials. See Lemma 5.10.

Before doing that, we first give a number of consequences. The first is formal: the module $\mathbb{L}\Omega_T(B/A)$ is a cofibrant object in $s\mathcal{M}^T_B$; thus Proposition 5.2 and Theorem 5.4 immediately imply

Corollary 5.5 *Suppose that A is a cofibrant simplicial T-algebra, $A \to B$ is a cofibration of T-algebras and I is a simplicial B module over T. Then the there is a natural isomorphism*

$$[\Omega_T(B/A), \Sigma I]_{s\mathcal{M}_R} \cong \mathrm{Exal}_B(A, I).$$

More generally, there is an isomorphism

$$[\mathbb{L}\Omega_T(B/A), \Sigma I]_{s\mathcal{M}_R} \cong \operatorname*{colim}_{\mathcal{W}(f)} \mathrm{Exal}_{A_0}(B_0, I).$$

In particular, we have the following situation. Suppose that T is actually an augmented simplicial operad $T \to T_{-1}$ and B is an augmented algebra $B \to B_{-1}$ over the augmented operad. Suppose that M is a B_{-1} module over T_1. Then we can form the simplicial B_{-1} module $K(M,n)$ over T_{-1}, and $K(N,n)$ can be regarded as an object in $s\mathcal{M}_B^T$ by restriction of structure. Then we have

Corollary 5.6 *Suppose that A is a cofibrant simplicial T-algebra, $A \to B$ is a cofibration of T-algebras. Then there is a natural isomorphism*

$$D_T^{n+1}(B/A, M) \cong \mathrm{Exal}_A(B, K(M,n)).$$

More generally, there is an isomorphism

$$D_T^{n+1}(B/A, M) \cong \operatorname*{colim}_{\mathcal{W}(f)} \mathrm{Exal}_{A_0}(B_0, K(M,n)).$$

Proof. This follows from the previous result, the fact that $\Sigma K(M,n) \simeq K(M, n+1)$, and the natural isomorphism

$$[\mathbb{L}\Omega_T(B/A), K(M,n+1)]_{s\mathcal{M}_R} \cong D_T^{n+1}(B/A, M)$$

of Proposition 2.7. (See the equation in the proof of that result.)

In a very special case, one can remove the cofibrancy hypotheses and obtain something very close to the classical results. Note that this result is essentially a constant result: we have an operad T, rather than a simplicial operad, and so on.

Corollary 5.7 *Suppose T is an operad in R-modules, B a T-algebra, and I a B-module over T. Then there is an isomorphism*

$$\mathrm{Exal}(B, I) \cong D_T^1(B; I).$$

Proof. Choose a weak equivalence $q : X \to B$ of simplicial T-algebras with X cofibrant. Then the previous result says that there is an isomorphism

$$\mathrm{Exal}(X, K(I,0)) \cong D^1(X; I) \cong D^1(B; I).$$

Cobase change along q defines a map $q^* : \mathrm{Exal}(B, I) \to \mathrm{Exal}(X, K(I,0))$, and we claim that this map is an isomorphism. Indeed, the inverse map sends a square-zero extension

$$K(I,0) \longrightarrow E \longrightarrow X$$

to

$$I \cong \pi_0 K(I,0) \to \pi_0 E \to \pi_0 X \cong B.$$

Note that none of $K(I,0)$, E, or X has any higher homotopy.

We now come to the proof of Theorem 5.4. We begin by examining the behavior of Kähler differentials under quotient maps of T-algebras under A. Suppose that $q : E \to B$ is a surjective map of T algebras and M is a B-module over T. Then Proposition 2.2 implies that if $d : E \to M$ is an A derivation over T, then the restricted A-module map $d : I \to M$ is actually a morphism of E-modules over T. Furthermore, if $d : E \to M$ is an A-derivation which restricts to the zero homomorphism on I, then the induced map $\bar{d} : B \to M$ of A-modules over A-modules over T is automatically a derivation. Thus we obtain an exact sequence

$$0 \to \mathrm{Der}_A(B, M) \to \mathrm{Der}_A(E, M) \to \mathrm{Hom}_{\mathcal{M}_E^T}(I, M) \cong \mathrm{Hom}_{\mathcal{M}_B^T}(B \otimes_E I, M). \tag{5.1}$$

By passing to representing objects we obtain an exact sequence, valid for all B-modules over T:

$$0 \to \mathrm{Hom}_{\mathcal{M}_B^T}(\Omega_T(B/A), M) \to \mathrm{Hom}_{\mathcal{M}_B^T}(B \otimes_E \Omega_T(E/A), M) \to \mathrm{Hom}_{\mathcal{M}_B^T}(B \otimes_E I, M) \tag{5.2}$$

Note that the canonical differential $d : E \to \Omega_T(E/A)$ passes to a morphism of B modules

$$d : B \otimes_E I \to B \otimes_E \Omega_T(E/A)$$

and that this map induces the surjection in Equation 5.2. Thus we come to the analog of the "second exact sequence" for differentials. In the example of the commutative algebra operad, we have that $B \otimes_E I \cong I/I^2$.

Lemma 5.8 *Let T be an operad in R-modules and $q : E \to B$ a surjection of T-algebras under a T-algebra A. Then there is an exact sequence of B-modules over T*

$$B \otimes_E I \xrightarrow{d} B \otimes_E \Omega_T(E/A), M) \longrightarrow \Omega_T(B/A) \to 0.$$

If the map $q : E \to B$ has splitting as T-algebras under A, then this sequence is also exact on the left; indeed, a choice of splitting yields a splitting of the resulting short exact sequence.

Proof. Only the statement about exactness on the left doesn't follow from Equation 5.2. This remaining claim can be proved as follows. Let $\phi : I \to B \otimes_E I$ be the canonical map of E-modules given by the unit of the adjunction. Use Lemma 5.1 to form a diagram of T-algebras under A

$$\begin{tikzcd} E \arrow[r] \arrow[d, "q"'] & (B \otimes_E I) \oplus_I E \arrow[d, "p"] \\ B \arrow[r, "="'] & B. \end{tikzcd}$$

The morphism $p : (B \otimes_E I) \oplus_I E \to B$ is a square-zero extension. A choice of splitting of $q : E \to B$ gives a choice of splitting of p, and an isomorphism of square-zero extensions

$$\begin{tikzcd} (B \otimes_E I) \arrow[r] \arrow[d, "="'] & (B \otimes_E I) \oplus_I E \arrow[r] \arrow[d, "\cong"] & B \arrow[d, "="] \\ (B \otimes_E I) \arrow[r] & B \ltimes (B \otimes_E I) \arrow[r] & B. \end{tikzcd}$$

The resulting map $E \to B \ltimes (B \otimes_E I)$ of algebras under A and over B gives an A-derivation $d : E \to B \otimes_E I$. Composition with d yields a homomorphism

$$d^* : \mathrm{Hom}_{\mathcal{M}_B^T}(B \otimes_E I, M) \to \mathrm{Der}_A(E, M)$$

that splits Equation 5.1 and, hence, Equation 5.2. The result follows.

Remark 5.9 An example where there is always the type of splitting required by this past lemma is supplied by the case where $B \cong A \sqcup T(M)$ where M is a projective R-module. Also notice that if $E \to B$ is a square-zero extension, then I is a B-module, so $I \cong B \otimes_E I$.

We can now supply the inverse to the homomorphism

$$\gamma : \mathrm{E{\scriptstyle XT}}^1_B(\Omega_T(B/A), I) \to \mathrm{Exal}_A(B, I)$$

of Theorem 5.4. Suppose that T is a simplicial operad, that $A \to B$ is a cofibration of T algebras, and that $X = \{I \to E \to B\}$ is square-zero extension of T-algebras under A. Then, because of the characterization of cofibrations of T-algebras given in Proposition 2.5, and because of Remark 5.9, Lemma 5.8 supplies a short exact sequence of B modules

$$0 \to I \longrightarrow B \otimes \Omega^T(E/A) \longrightarrow \Omega^T(B/A) \to 0. \tag{5.3}$$

Define $\delta : \mathrm{Exal}_A(B, I) \to \mathrm{E{\scriptstyle XT}}^1_B(\Omega^T(B/A), I)$ by sending the class of X to the class of the extension Equation 5.3. Notice that δ is defined only when $A \to B$ is a cofibration.

Theorem 5.4 now follows from the next result, which is an exercise in definitions.

Lemma 5.10 *The homomorphisms*

$$\gamma : \mathrm{E{\scriptstyle XT}}^1_B(\Omega^T(B/A), I) \rightleftarrows \mathrm{Exal}_A(B, I) : \delta$$

are inverse to each other.

We would also like to address the relationship between André-Quillen cohomology and square-zero extensions for algebras over operads in the category of (R, Γ) comodules. The analog of Theorem 5.4 holds; indeed, in light of Lemma 4.8, the proof goes through with only a change of notation, and we have

Theorem 5.11 *Let T be a simplicial operad in Γ-comodules, let A be a cofibrant simplicial T-algebra, and $A \to B$ a cofibration of T-algebras. If I is a simplicial B-module over T, we have an isomorphism*

$$\gamma : \mathrm{E{\scriptstyle XT}}^1_{\Gamma(B)}(\Omega^T(B/A), I) \xrightarrow{\cong} \mathrm{Exal}_{\Gamma(A)}(B, I).$$

More generally, there is a natural isomorphism

$$\mathrm{E{\scriptstyle XT}}^1_{\Gamma(B)}(\mathbb{L}\Omega^T(B/A), I) \cong \operatorname*{colim}_{\mathcal{W}(f)} \mathrm{Exal}_{\Gamma(A_0)}(B_0, I).$$

The appearance of the Γ in the subscript is to indicate that all the relevant definitions and constructions occur in the ground category of Γ-comodules.

To interpret this result in terms of the homotopy theory of simplicial B-modules over T, all in the category of Γ-comodules, we must come to terms with the fact that not every B-module in this category is fibrant.

If N and M are arbitrary simplicial modules in $s\mathcal{M}^T_{\Gamma(B)}$, then

$$\mathbf{Ho}_{s\mathcal{M}^T_{\Gamma(B)}}(N, M) \cong [N', M']_{s\mathcal{M}^T_{\Gamma(B)}}$$

where $N' \to N$ is a weak equivalence with N' cofibrant and $M \to M'$ is a weak equivalence with M' cofibrant. The choice-free way to write this is as

$$\mathbf{Ho}_{s\mathcal{M}^T_{\Gamma(B)}}(N, M) \cong \operatorname*{colim}_{N' \xrightarrow[\simeq]{} N} \ \operatorname*{colim}_{M \xrightarrow[\simeq]{} M'} [N', M']_{s\mathcal{M}^T_{\Gamma(B)}}.$$

Here $N' \to N$ and $M \to M'$ run through all maps that are weak equivalences.

We would now like to define a map

$$\alpha : \mathbf{Ho}_{s\mathcal{M}^T_{\Gamma(B)}}(N, \Sigma I) \to \operatorname*{colim}_{N' \xrightarrow[\simeq]{} N} \operatorname*{colim}_{I \xrightarrow[\simeq]{} J} \mathrm{EXT}^1_{\Gamma(B)}(N', J). \tag{5.4}$$

This will again be done by a pull-back.

First, since $s\mathcal{M}^T_{\Gamma(B)}$ is a simplicial model category, we have internal mapping objects M^K, where $M \in s\mathcal{M}^T_{\Gamma(B)}$ and K is a simplicial set. If K is a pointed simplicial set, we can define a pointed mapping object $\hom_*(K, M)$ by the pull-back square

$$\begin{tikzcd}
\hom_*(K,M) \arrow[r] \arrow[d] & M^K \arrow[d] \\
0 \arrow[r] & M \cong M^*.
\end{tikzcd}$$

Thus we have a path-loops sequence

$$\Omega M = \hom_*(\Delta^1/\partial\Delta^1, M) \to \hom_*(\Delta^1, M) \to M.$$

Note that $\pi_n \Omega M \cong \pi_{n+1} M$ and, for this category at least, the unit of the adjunction $N \to \Omega\Sigma N$ is a weak equivalence. In particular, suppose we are given a morphism $f : N \to J$ in $s\mathcal{M}^T_{\Gamma(B)}$ and a weak equivalence $\Sigma I \to K$. Then we may pull back that path-loops fibration for J via f and obtain an extension

$$0 \to \Omega K \longrightarrow M \longrightarrow N \to 0$$

equipped with a weak equivalence $I \to \Omega K$. This defines the morphism α of Equation 5.4

Here, then, is out substitute for Proposition 5.2 The proof is now nearly identical to that of Illusie.

Proposition 5.12 *The induced homomorphism*

$$\alpha : \mathbf{Ho}_{s\mathcal{M}^T_{\Gamma(B)}}(N, \Sigma I) \to \operatorname*{colim}_{N' \xrightarrow[\simeq]{} N} \operatorname*{colim}_{I \xrightarrow[\simeq]{} J} \mathrm{EXT}^1_{B}(N', J)$$

is an isomorphism.

Proof. We must construct the inverse map. We choose a representative for an element of the colimit

$$0 \to J \to M \to N' \to 0.$$

We may assume that N' is cofibrant. As in the proof of Proposition 5.2 we investigate the diagram

$$\begin{tikzcd}
0 \arrow[r] & J \arrow[r, "d^0"] & C(J) \arrow[r] & \Sigma J \arrow[r] & 0 \\
0 \arrow[r] & J \arrow[r, "j"] \arrow[u, "="] \arrow[d, "="'] & C(J) \oplus M \arrow[r] \arrow[u, "pr_1"] \arrow[d, "pr_2"] & X \arrow[r] \arrow[u, "p"] \arrow[d, "s"] & 0 \\
0 \arrow[r] & J \arrow[r] & M \arrow[r] & N' \arrow[r] & 0
\end{tikzcd}$$

The morphism s remains a weak equivalence, but it is not an acyclic fibration, in general. If we factor s as

$$X \xrightarrow{i} Y \xrightarrow{q} N'$$

with i an acyclic cofibration and q a fibration, then q is an acyclic fibration and, since N' is cofibrant, has a section $N' \to Y$, unique up to homotopy. If we form the push-out diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "i"'] & \Sigma J \arrow[d] \\ Y \arrow[r] & K \end{tikzcd}$$

Then $\Sigma J \to K$ is a weak equivalence and we obtain a composition $N' \to Y \to K$, which induces a morphism in $\mathbf{Ho}_{s\mathcal{M}^T_{\Gamma(B)}}(N, \Sigma I)$. This is the inverse to α.

Each of the Corollaries 5.5 – 5.7 has an analog in this setting. In particular, we record the analog of Corollary 5.6, which has an application in the next section.

Suppose we are working in the base category of Γ-comodules. Suppose that T is an augmented simplicial operad $T \to T_{-1}$ and B is an augmented algebra $B \to B_{-1}$ over the augmented operad. Suppose that I is a B_{-1}-module over T_1. Then we can form the simplicial B_{-1} module $K(I,n)$ over T_{-1}, and $K(I,n)$ can be regarded as an object in $s\mathcal{M}^T_{\Gamma(B)}$ by restriction of structure.

Corollary 5.13 *There is a natural isomorphism*

$$D^{n+1}_{\Gamma(T)}(B/A, M) \cong \operatorname*{colim}_{W(f)} \operatorname*{colim}_{I \xrightarrow[\simeq]{} J} \mathrm{Exal}_{\Gamma(A_0)}(B_0, J).$$

6 Postnikov towers for simplicial T-algebras

If $X \in s\mathcal{A}^T$ is a simplicial T-algebra, the standard Postnikov tower of X, regarded as a simplicial R-module, refines to a Postnikov tower of simplicial T-algebras. In this section we say what that means, and spell out the details. We then close with some remarks on the case of algebras in comodules over some Hopf algebroid.

Let $X \in s\mathcal{M}_R$ be a simplicial R-module. A Postnikov tower for X is a tower in $s\mathcal{M}_R$

$$\cdots \to P_nX \xrightarrow{q_n} P_{n-1}X \to \cdots \to P_2X \xrightarrow{q_2} P_1X \xrightarrow{q_1} P_0X$$

equipped with maps $f_n : X \to P_nX$ so that $q_nf_n = f_{n-1}$ and so that $\pi_kP_nX = 0$ for $k > n$ and

$$\pi_kf_n : \pi_kX \to \pi_kP_nX$$

is an isomorphism for $k \leq n$.

More generally, if $f : X \to B$ is a fibration of simplicial R-modules, [2] let $F = f^{-1}(0)$ be the fiber at zero. Then a Postnikov tower for the morphism f is a tower in $s\mathcal{M}_R$

$$\cdots \to P_n(f) \xrightarrow{q_n} P_{n-1}(f) \to \cdots \to P_2(f) \xrightarrow{q_2} P_1(f) \xrightarrow{q_1} P_0(f) = B$$

equipped with maps $f_n : X \to P_nX$ so that $q_nf_n = f_{n-1}$, $f_0 = f$, and so that

$$\pi_kP_n(f) = \begin{cases} \pi_kX & k \leq n; \\ coker\{\pi_{n+1}F \to \pi_{n+1}X\} \cong ker\{\pi_nB \to \pi_{n-1}F\} & k = n+1; \\ \pi_kB & k > n+1 \end{cases}$$

Note that the successive fibers of $P_n(f)$ form a Postnikov tower for F. The Postnikov tower of an object X, as defined above, is the Postnikov tower of the unique morphism $X \to 0$.

[2] Recall that a fibration of simplicial R-modules is a morphism for which the associated morphism of associated normalized chain complexes is surjective in positive degrees

Such towers are essentially unique and the Moore-Postnikov tower, which we now define, gives an example of such.

To define the Moore-Postnikov tower for a fibration $f : X \to B$ in $s\mathcal{M}_R$, again let F be the fiber, fix n, and let $s \geq 0$. Let

$$I_{s,n} = \bigcap_{\phi : k \to s} \ker\{\phi^* : X_s \to X_k\} \cap F_s \subseteq X_s \tag{6.1}$$

where ϕ runs over all injections with $k \leq n$. Define $P_n(f) \in s\mathcal{M}_R$ by the equation

$$(P_n(f))_s = X_s/I_{s,n}.$$

There are fibrations $f_n : X \to P_n(f)$ and $q_n : P_n(f) \to P_{n-1}(f)$. If we are considering the absolute case of the Postnikov tower of an object X (i.e., of the morphism $X \to 0$), then $X = F$ and it is a simple exercise to show that the normalized chain complex NP_nX of P_nX is given by

$$\cdots \to 0 \to B_nX \to NX_n \to \cdots \to NX_1 \to NX_0,$$

with $B_nX = \mathrm{Im}\{\partial : NX_{n+1} \to NX_n\} \cong NX_{n+1}/\ker(\partial)$, so that $\pi_* P_nX$ is as required. For a general fibration $f : X \to B$, the normalized chain complex $NP_n(f)$ of $P_n(f)$ fits into a diagram of chain complexes, with the columns short exact:

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & B_nF \arrow[r] \arrow[d] & NF_n \arrow[r] \arrow[d] & \cdots \arrow[r] & NF_1 \arrow[r] \arrow[d] & NF_0 \arrow[d] \\
\cdots \arrow[r] & NP_n(f)_{n+2} \arrow[r] \arrow[d] & NP_n(f)_{n+1} \arrow[r] \arrow[d] & NP_n(f)_n \arrow[r] \arrow[d] & \cdots \arrow[r] & NP_n(f)_1 \arrow[r] \arrow[d] & NP_n(f)_0 \arrow[d] \\
\cdots \arrow[r] & NB_{n+2} \arrow[r] & NB_{n+1} \arrow[r] & NB_n \arrow[r] & \cdots \arrow[r] & NB_1 \arrow[r] & NB_0.
\end{tikzcd}$$

Again, this implies that $\pi_* P_n(f)$ is as required. Also one immediately has that if $F_n(f)$ is the fiber of $P_n(f) \to P_{n-1}(f)$, then

$$\pi_k F_n(f) = \begin{cases} \pi_n F & k = n; \\ 0 & k \neq n \end{cases} \tag{6.2}$$

Now suppose that T is a simplicial operad in R modules, and that X is a simplicial T algebra, or, more generally, that $f : X \to B$ is a fibration of simplicial T-algebras. Recall that a morphism of simplicial T-algebras is a fibration if and only if it is a fibration of simplicial R-modules. Then the fiber F of f is a simplicial ideal in the T-algebra X. In fact, because the intersection of ideals is an ideal, the sub-R-modules $I_{s,n} \subseteq X_s$ of Equation 6.1 are ideals in the T_s algebra X_s, and we have:

Proposition 6.1 *Let $f : X \to B$ be a morphism of simplicial T-algebras. Then the Moore-Postnikov tower for f, regarded as a simplicial R-module, is a tower of simplicial T-algebras.*

In particular, simplicial T-algebras have Postnikov towers. The next point is to refine k-invariants to T-algebras. To begin this, we first note what properties a fiber sequence of T-algebras has. If $f : X \to B$ is a fibration of T-algebras, then $f : X \to B$ is a morphism of T-modules over X. If f is surjective (that is, $\pi_0 X \to \pi_0 B$ is onto), then we get a diagram of exact sequences

$$\begin{tikzcd}
 & T(X,F) \arrow[r] \arrow[d] & T(X,X) \arrow[r] \arrow[d] & T(X,B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] & X \arrow[r] & B \arrow[r] & 0.
\end{tikzcd}$$

So that $F \to X \to B$ is a fibration sequence of simplicial X-modules over T. This is simply a rephrasing of the remark that F is an ideal and f is surjective. If B happens to be flat over R, then the top row of this diagram is actually short exact, and if we apply homotopy, we get that a long exact sequence

$$\cdots \pi_* F \to \pi_* X \to \pi_* B \xrightarrow{\partial} \pi_{*-1} F \to \cdots$$

and each of the maps is a morphism of $\pi_* X$ modules over $\pi_* T$ except the map ∂, which is a $\pi_* X$ modules over $\pi_* T$ up to sign. This means that if $a \in \pi_* T(n)$, $x_i \in \pi_* X$, $y \in \pi_* B$ and $\epsilon_M : (\pi_* T)(\pi_* X, M) \to M$ is the module multiplication, then

$$\partial \epsilon_{\pi_* B}(a \otimes x_1 \otimes \cdots x_{n-1} \otimes y) = (-1)^k \epsilon_{\pi_* F}(a \otimes x_1 \otimes \cdots x_{n-1} \otimes \partial y)$$

where k is the degree of $a \otimes x_1 \otimes \cdots x_{n-1}$. We will simply say we have a long exact sequence of $\pi_* X$ modules over $\pi_* T$, and leave the sign understood.

In B happens not to be flat over R, we can recover this structure as follows. Choose a weak equivalence of simplicial operads $S \to T$ where S is cofibrant as a simplicial operad, and choose a weak equivalence $q : B' \to B$ where B' is a cofibrant S algebra. Then B' is flat over R and if we pull back the fibration $f : X \to B$ via q to a fibration $f' : X' \to B'$, then the long exact sequence in homotopy

$$\cdots \to \pi_* F \to \pi_* X' \cong \pi_* X \to \pi_* B' \cong \pi_* B \to \pi_{*-1} F \to \cdots$$

is a long exact sequence of $\pi_* X' \cong \pi_* X$ modules over $\pi_* S \cong \pi_* T$. Hence we have proved:

Proposition 6.2 *Let T be a simplicial operad and let $f : X \to B$ be a surjective fibration of simplicial T-algebras. Then the long exact sequence in homotopy of $F \to X \to B$ is a long exact sequence of $\pi_* X$ modules over $\pi_* T$.*

In particular, suppose we have a fibration of simplicial T-algebras $q : E \to B$ and suppose that fiber F has one non-zero homotopy group $\pi = \pi_n F$, $n \geq 1$. Then π is a $\pi_0 B \cong \pi_0 E$ module over $\pi_0 T$. Thus we can form the simplicial $\pi_0 B$ module $K(\pi, n)$. By restriction of scalars, this is a simplicial B module over T and there is a weak equivalence $F \to K(\pi, n)$ of simplicial E modules over T. If we then apply Lemma 5.1 we obtain a diagram of fibration sequences

(6.3)
$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\simeq"'] & E \arrow[r, "q"] \arrow[d, "\simeq"'] & B \arrow[d, "="] \\
K(\pi,n) \arrow[r] & K(\pi,n) \oplus_F E \arrow[r] & B
\end{tikzcd}$$

with the vertical maps all weak equivalences and the bottom row a square-zero extension. We define the k-invariant of $q : E \to B$ to be cohomology class $k(q) \in D^{n+1}(B, \pi)$ corresponding to this square-zero extension. Here we use Corollary 5.6. If q happens to be a morphism of T-algebras under some fixed T-algebra A, then the k-invariant refines to an element $k(q) \in D^{n+1}(B/A, \pi)$. Notice that $k(q)$ depends on the choice of weak equivalence $F \to K(\pi, n)$ and is, thus, well-defined up to automorphisms of π as a $\pi_0 B$ module over $\pi_0 T$.

One immediately notes that Corollary 5.6 has another implication. We may represent the cohomology class $k(q) \in D^{n+1}(B/A, \pi)$ as a morphism $\kappa(q) : B' \to \pi_0 B \ltimes K(\pi, n)$ of T-algebras over $\pi_0 B$ and under A, where $B' \to B$ is a weak equivalence of T-algebras with B' cofibrant. Then there is a diagram

(6.4)
$$\begin{tikzcd}
K(\pi,n) \oplus_F E \arrow[d] & E' \arrow[l, "\simeq"] \arrow[r] \arrow[d] & \pi_0 B \ltimes L(\pi, n+1) \arrow[d] \\
B & B' \arrow[l, "\simeq"] \arrow[r, "\kappa(q)"] & \pi_0 B \ltimes K(\pi, n+1)
\end{tikzcd}$$

where $L(\pi, n+1) \to K(\pi, n+1)$ is a fibration of $\pi_0 B$ modules over $\pi_0 T$ with $L(\pi, n+1)$ contractible, the right-hand square is a pull-back, and the left hand square has horizontal weak equivalences.

Combining Equation 6.3 and Equation 6.4 we obtain the following result.

Proposition 6.3 *Let $q : E \to B$ be a fibration of simplicial T-algebras under a T-algebras A so that the fiber F has one non-zero homotopy group $\pi = \pi_n F$, $n \geq 1$. Then the morphism q is weakly equivalent to a square-zero extension with fiber $K(\pi, n)$ classified by a cohomology class $k(q) \in D^{n+1}(B/A, \pi)$ and q is weakly equivalent to a morphism q' of T-algebras under A defined by a pull-back square*

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "q'"'] & \pi_0 B \ltimes L(\pi, n+1) \arrow[d] \\ B' \arrow[r, "\kappa(q)"'] & \pi_0 B \ltimes K(\pi, n+1) \end{tikzcd}$$

In particular, if $f : X \to B$ is a fibration of T-algebras under some fixed T-algebra A then there are k-invariants

$$k_n \in D^{n+1}(P_{n-1}(f)/A, \pi), \qquad n \geq 1,$$

and the morphisms $P_n(f) \to P_{n-1}(f)$, $n \geq 1$, of the Moore-Postnikov tower are weakly equivalent to morphisms $P'_n(f) \to P'_{n-1}(f)$ which fit into a pull-back diagram of the sort supplied by Proposition 6.3.

Now let us examine the case where we change the base category of Γ-comodules, where (R, Γ) is a Hopf algebroid of Adams type, as in Definition 3.1. Let T be a simplicial operad in Γ comodules, and let $f : X \to B$ is a fibration simplicial T-algebras in Γ-comodules, or, more generally, simply a fibration of simplicial R-modules. The the Moore-Postnikov tower for f is a tower in T-algebras in Γ-comodules. Thus Postnikov towers exist in this category.

Next suppose $E \to B$ is a morphism of simplicial T-algebras in Γ-comodules which is fibration of simplicial R-modules and suppose the fiber F has one non-zero vanishing homotopy group $\pi = \pi_n F$, $n \geq 1$. Then π is a $\pi_0 B$-module over $\pi_0 T$ in Γ-comodules; let $K_f(\pi, n)$ be a fibrant model for $K(\pi, n)$ in the category of simplicial $\pi_0 B$-modules over $\pi_0 T$, in Γ-comodules. If we then apply Lemma 5.1 we obtain a diagram of fibration sequences

(6.5)
$$\begin{tikzcd} F \arrow[r] \arrow[d, "\simeq"'] & E \arrow[r, "q"] \arrow[d, "\simeq"'] & B \arrow[d, "="] \\ K_f(\pi, n) \arrow[r] & K_f(\pi, n) \oplus_F E \arrow[r] & B \end{tikzcd}$$

with the vertical maps all weak equivalences and the bottom row a square-zero extension. If $q : E \to B$ is a morphism of T-algebras under A in Γ-comodules, then Corollary 5.13 now supplies the k-invariant $k(q) \in D^{n+1}_{\Gamma(T)}(B/A, \pi)$ and a diagram of T-algebras under A

(6.6)
$$\begin{tikzcd} K_f(\pi, n) \oplus_F E \arrow[d] & E' \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & \pi_0 B \ltimes L_f(\pi, n+1) \arrow[d] \\ B & B' \arrow[l, "\simeq"] \arrow[r, "\kappa(q)"] & \pi_0 B \ltimes K_f(\pi, n+1). \end{tikzcd}$$

The T-algebra B' will be a cofibrant model for B in the category of T-algebras in Γ-comodules. Combining Equations 6.5 and 6.6 yields the analog of Proposition 6.3 in this context:

Proposition 6.4 *Let $q : E \to B$ be a morphism of simplicial T-algebras under a T-algebras A in Γ-comodules. Suppose Q is a fibration of simplicial R-modules and suppose the fiber F has one non-zero vanishing homotopy group $\pi = \pi_n F$, $n \geq 1$. Then the morphism q is weakly equivalent to a square-zero extension of simplicial T-algebras in Γ-comodules with fiber $K_f(\pi, n)$ classified by a cohomology class $k(q) \in D^{n+1}(B/A, \pi)$ and q is weakly equivalent to a morphism q' of T-algebras under A defined by a pull-back square*

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "q'"'] & \pi_0 B \ltimes L_f(\pi, n+1) \arrow[d] \\ B' \arrow[r, "k(q)"'] & \pi_0 B \ltimes K_f(\pi, n+1) \end{tikzcd}$$

In particular, this result again applies to the successive stages of the Moore-Postnikov tower for a fibration; for example, if $f : X \to B$ is a fibration under A, there are k-invariants

$$k_n \in D^{n+1}_{\Gamma(T)}(P_{n-1}(f)/A, \pi), \qquad n \geq 1.$$

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