Resolutions in Model Categories

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Abstract

Let $\mathcal{C}$ be a closed model category. Following work of Dwyer, Kan, and Stover, we develop a technique for building simplicial resolutions of objects in $\mathcal{C}$ which are free, or homotopically free, in a precise sense. Specifically, we specify in advance a set $\mathcal{A}$ of small, cofibrant, co-H objects in $\mathcal{C}$ and the simplicial resolutions X will have the property that at each level n, the object X_n is homotopy equivalent to a coproduct of objects of $\mathcal{A}$. These Stover-type resolutions are modeled on the free resolutions developed by Quillen, and are meant to be a replacement for projective resolutions in this context. In the end we develop some computational tools for spectra and structured ring spectra.

The purpose of this paper is to revisit and expand on two papers by Dwyer, Kan, and Stover [5, 6] which, among other things, systematically built simplicial resolutions of pointed topological spaces out of spheres. We wish to carry the same program over to spectra and, especially, to A_∞ or E_∞ ring spectra. The latter case—which we will call the case of structured ring spectra—falls somewhat outside the Dwyer-Kan-Stover program as these categories are not pointed: the initial object is not isomorphic to the terminal object.

This paper is part of a larger project intended to come to terms with the homotopy category of simplicial structured ring spectra and the spaces of maps between them. An overview of our goals and results can be found in [11].

The simplicial resolutions we have in mind are cofibrant objects in some model category. Historically, the resolutions – known as Stover resolutions – predated that model category (see [18]), however, the model category structure is a convenient way to manipulate and compare resolutions, and we choose to emphasize that approach here. In fact, we will prove the existence of the appropriate model category structure on simplicial objects before producing the resolutions. By taking the approach, we are returning more to the tactics deployed by Quillen ([16]), who first produced model category structures in simplicial objects, and then, latterly, identified the simplicial cofibrant objects as ones built from projective objects in the base category. Thus, a secondary object of this paper is to rework the existence of the model category structures developed by Dwyer, Kan, and Stover. To be honest, many of the main technical lemmas are the same.

Once we have the resolutions, we'd like to compute with them. This is accomplished by means of a spectral sequence (hence the "E_2" of E_2 model categories in the parlance of Dwyer, Kan, and Stover). The construction of this spectral sequence has two parts. The first is to create the "spiral exact sequence" of [6] which compares two different ways of defining weak equivalence in the categories of simplicial objects. Then, by splicing together various of these exact sequences, one obtains an exact couple, and, thus, a spectral sequence. This much goes through verbatim in a large number of contexts. Much more subtle, however, is showing that the resulting spectral sequence computes the homotopy type of the geometric realization of the simplicial object under discussion. For simplicial pointed connected simplicial sets this was done by Bousfield and Friedlander [2]; for simplicial pointed connected spaces this was done in [6]. However, neither method applies to spectra or structured ring spectra. Thus, the third point of this paper is to prove that in these two cases one is, indeed, computing the homotopy type of the geometric realization.

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In the case of spectra, this is easy. There is a standard spectral sequence for computing the generalized homology of the geometric realization of a simplicial spectrum, and we show that this spectral sequence and the one above coincide. For structured ring spectra, the question is much more difficult and highly technical. The key point is that a Reedy cofibrant simplicial structured ring spectra need not be Reedy cofibrant as a spectrum. Much of the technical maneuvering of sections 4, 5, and 6 is dedicated to coming to grips with this point. In the end (Proposition 6.13), however, we show that indeed, we can calculate as expected, at least in the A_∞ case. The E_∞ case is more delicate, as, from our point of view, one must resolve the operad as well as the spectru. This is developed further in [10].

We now explain what our resolutions will look like. In Quillen's case, we have the following. There is a fixed category $\mathcal{C}$ and a set of projective objects $\{Pi\}$ in $\mathcal{C}$. The the cofibrant object in the category $s\mathcal{C}$ are retracts of objects which Quillen calls "free". An object $X \in s\mathcal{C}$ is free in this sense if two things happened. First, the underlying degeneracy diagram (i.e., forget the face maps) is free as a diagram, meaning there there are objects Z_k in $\mathcal{C}$ so that

$$X_n \cong \bigsqcup_{\varphi:[n]\to[k]} Z_k$$

where φ runs over the surjections in the ordinal number category out of $[n]$ and the face maps respect this decomposition. Second, each of the Z_k must be a coproduct of the Pi. These are the resolutions, if you like.

In our case, we have a variation on this notion of freeness. This variation first arose explicitly with Stover resolutions, although it appears in a rudimentary form in Quillen's book [16]. We have a category $\mathcal{C}$ which is a topological model category (see Section 1) and a set of cofibrant objects $\mathcal{A} = \{Aj\}$ which is closed under suspension and with the property that each Aj is a homotopy co-group object; that is, the functor

$$X \mapsto [Aj, X]$$

defined on the homotopy category of $\mathcal{C}$ is a functor to groups. Out of this data we build a model category – which we will call the $\mathcal{A}$-resolution model category on $s\mathcal{C}$, which has among its cofibrant objects one which are homotopically $\mathcal{A}$-free. The exact definition is Definition 4.4; effectively, X is homotopically $\mathcal{A}$-free if X is, up to level-wise homotopically equivalence, free in Quillen's sense on the Aj. More exactly, there is a nice contractible subobect $Z \subseteq X$ so that collapsing off Z yields an object free as above. In any case, if one applies a homotopy invariant functor to a homotopically $\mathcal{A}$-free object it behaves as Quillen's free objects would.

As should be clear from the preceding paragraph, the resulting model category structure on $s\mathcal{C}$ depends heavily on the choice of $\mathcal{A}$. In the category of spectra, there is no obvious choice of such, and, indeed, the applications demand flexibility in this matter. For this reason we call this the $\mathcal{A}$-resolution model category structure.

Here is an outline of the paper. Section 1 is devoted to definitions and preliminaries. In section 2, we produce the $\mathcal{A}$-resolution model category. In section 3, we discuss the spiral exact sequence and show that the two ways of defining weak equivalence are the same. Section 4 is devoted to Stover resolutions and homotopically $\mathcal{A}$-free objects. Section 5 describes the cofibrations in the $\mathcal{A}$-resolution model category. Section 6 contains the hardest new result: that the resulting spectral sequence for simplicial structured ring spectra computes what it should. Finally, there is an appendix to which we consign various technical simplicial results used in the body of the paper, but whose proofs are tangential to everything else we do.

When is comes to spectra, we will work with the category of spectra developed by Lewis, May, and Stienberger in [14] with the closed model category structure that appears in [8] and [9]. This category of spectra has the advantage that every object is fibrant. This means, among other things, that our basic category of topological space **Top** is the category of compactly generated weak Hausdorff spaces with the model category structure obtained from weak equivalences and Serre fibrations.

Contents

1 The Reedy model category structure.

This model category structure has level-wise weak equivalences, but is still useful for building resolutions, and is the foundation for what follows.

To begin we fix a category $\mathcal{C}$ that has all limits and colimits. Then the category $s\mathcal{C}$ of simplicial objects in $\mathcal{C}$ has a standard structure of a simplicial category in the sense of [16]. In particular, if $\mathbf{S}$ denotes simplicial sets, there are functors

(1.1)
$$-\otimes\ - : \mathbf{S}\times s\mathcal{C}\to s\mathcal{C} \qquad \hom(\cdot,\cdot) : \mathbf{S}^{\mathrm{op}}\times s\mathcal{C}\to s\mathcal{C}$$
$$\mathrm{map}_{s\mathcal{C}}(\cdot,\cdot) = \mathrm{map}(\cdot,\cdot) : (s\mathcal{C})^{\mathrm{op}}\times s\mathcal{C}\to \mathbf{S}$$

satisfying the adjointness properties

$$\begin{aligned}\mathrm{map}(K\otimes X, Y) &\cong \mathrm{map}_{\mathbf{S}}(K, \mathrm{map}(X,Y))\\ &\cong \mathrm{map}(X, \hom(K,Y)).\end{aligned}$$

One has $*\otimes X\cong X$, and

$$\mathrm{map}(X,Y)_0 \cong \mathrm{Hom}_{\mathcal{C}}(X,Y).$$

Thus, in particular, $s\mathcal{C}$ is enriched over simplicial sets.

If $A\in\mathcal{C}$ we may regard A as a constant object in $s\mathcal{C}$. The following lemma defines matching objects.

Lemma 1.2 *Let $K\in\mathbf{S}$. Then there is a functor $M_K : s\mathcal{C}\to\mathcal{C}$ so that there are natural isomorphisms*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(A, M_K X) &\cong \mathrm{Hom}_{s\mathcal{C}}(K\otimes A, X)\\ &\cong \mathrm{Hom}_{\mathbf{S}}(K, \mathrm{map}(A,X)).\end{aligned}$$

Furthermore, if $K\to L$ is a morphism of simplicial sets, there is a natural transformation $M_L\to M_K$.

Proof. Since $A\in s\mathcal{C}$ is constant, $\mathrm{Hom}_{s\mathcal{C}}(A,X)\cong\mathrm{Hom}_{\mathcal{C}}(A,X_0)$. Hence

$$\begin{aligned}\mathrm{Hom}_{s\mathcal{C}}(K\otimes A, X) &\cong \mathrm{Hom}_{s\mathcal{C}}(A, \hom(K,X))\\ &\cong \hom(K,X)_0.\end{aligned}$$

Set $M_K = \hom(K, X)_0$. ■

The object $M_K X$ is the (generalized) matching object. Following [16], $M_K X$ may be constructed as follows. If $K : I \to \mathbf{S}$ is any diagram and $L = \operatorname{colim}_I K$, then $\hom(L, X) \cong \lim_{I^{\mathrm{op}}} \hom(K, X)$, so

$$M_L X \cong \lim_{I^{\mathrm{op}}} M_K X. \tag{1.3}$$

In particular $K \cong \operatorname{colim}_I \Delta^n$, where I is the diagram of simplices of K, so

$$M_K X \cong \lim_{I^{\mathrm{op}}} M_{\Delta^n} X \cong \lim_{I^{\mathrm{op}}} X_n.$$

Of particular interest are

$$M_n X = M_{\partial\Delta^n} X$$

and

$$M_n^k X = M_{\Delta_k^n} X.$$

Here $\partial\Delta^n$ is the boundary of the n-simplex and $\Delta_k^n = \partial\Delta^n - d_k\Delta^{n-1}$. The object $M_n X$ is often called the matching object. We have already used implicitly that $M_{\Delta^n} X = X_n$; hence, the inclusion $\partial\Delta^n \subseteq \Delta^n$ induces a map $X_n \to M_n X$. Note that $M_0 X = M_\phi X = *$, where $*$ denotes the terminal object.

For the rest of the section, we assume that $\mathcal{C}$ is a closed model category.

Proposition 1.4 [The Reedy Structure, [17], [7]] *If $\mathcal{C}$ is a closed model category, then the category $s\mathcal{C}$ has the structure of a closed model category, where a morphism $X \to Y$ is*

1.) a weak equivalence if $X_n \to Y_n$ is a weak equivalence for all n; and

2.) a fibration if $X_n \to Y_n \times_{M_n Y} M_n X$ is a fibration. ■

Cofibrations can be described by "latching objects." Specifically if $X \in s\mathcal{C}$ define

$$L_n X = \operatorname*{colim}_{\psi:[n]\to[k]} X_k$$

where ψ runs over the surjections in the ordinal number category with $k < n$. This is the latching object, and there is a natural map $L_n X \to X_n$. Note $L_0 X = \phi$, the initial object. A morphism in the Reedy model category structure on $s\mathcal{C}$ is a cofibration if and only if the induced maps, $n \geq 0$,

$$X_n \sqcup_{L_n X} L_n Y \longrightarrow Y_n$$

are cofibations in $\mathcal{C}$.

Important for our applications is the following two results, which are now more or less immediate. We refer to the fibrations (and so on) of the Reedy structure as Reedy fibrations.

Lemma 1.5 *If $j : K \to L$ is a cofibration of simplicial sets and $f : A \to B$ is a Reedy cofibration, then*

$$j \otimes f : K \otimes B \sqcup_{K\otimes A} L \otimes A \to L \otimes B$$

is a Reedy cofibration which is trivial if f is trivial. ■

The reader will note that this is most of Quillen's axiom SM7. However, if $K \to L$ is a trivial cofibration in $\mathbf{S}$, it is not necessarily true that $j \otimes f$ is trivial.

The next result says that the Reedy model category is cofibrantly generated if $\mathcal{C}$ is cofibrantly generated. In fact, let $I = \{i_\alpha : A_\alpha \to B_\alpha\}$ and $J = \{j_\beta : C_\beta \to D_\beta\}$ generate, respectively, the trivial cofibrations and the cofibrations of $\mathcal{C}$. This means, for example, that the class of cofibrations in $\mathcal{C}$ is the smallest class of maps containing J and closed under isomorphisms, coproducts, push-outs, sequential colimits, and retracts.

Lemma 1.6 *If $\mathcal{C}$ is cofibrantly generated, the Reedy model category structure on $s\mathcal{C}$ is cofibrantly generated. The sets, with $n \geq 0$,*

$$\begin{aligned} I' &= \{\Delta^n \otimes A_\alpha \sqcup_{\partial\Delta^n \otimes A_\alpha} \partial\Delta^n \otimes B_\alpha \to \Delta^n \otimes B_\alpha\} \\ J' &= \{\Delta^n \otimes C_\beta \sqcup_{\partial\Delta^n \otimes C_\beta} \partial\Delta^n \otimes D_\beta \to \Delta^n \otimes D_\beta\} \end{aligned}$$

generate the trivial cofibrations and the cofibrations respectively. ∎

We also have the following useful fact about Reedy fibrations.

Lemma 1.7 *A morphism $X \to Y$ in $s\mathcal{C}$ is a Reedy fibration if and only if for all cofibrations $K \to L$ in $\mathbf{S}$, the induced map*

$$M_L X \to M_L Y \times_{M_K Y} M_K X$$

is a fibration.

Proof. First suppose $K \subseteq L$ is obtained by attaching n-simplices; that is, there is a push-out diagram

$$\begin{tikzcd} \coprod_U \partial\Delta^n \arrow[r] \arrow[d] & K \arrow[d] \\ \coprod_U \Delta^n \arrow[r] & L \end{tikzcd}$$

for some set U of map $\partial\Delta^n \to K$. Then there is a pull-back diagram

(1.7.1)
$$\begin{tikzcd} M_L X \arrow[r] \arrow[d] & \prod_U X_n \arrow[d] \\ M_L Y \times_{M_K Y} M_K X \arrow[r] & \prod_U Y_n \times_{M_n Y} M_n X. \end{tikzcd}$$

For the general case, use that a sequence $K_0 \subseteq K \subseteq L$ yields maps

$$M_L X \to M_L Y \times_{M_K Y} M_K X \to M_L Y \times_{M_K Y} M_K Y \times_{M_{K_0} Y} M_{K_0} X \cong M_L Y \times_{M_{K_0} X} M_{K_0} X. \quad ∎$$

We now add more structure to the model category $\mathcal{C}$ – we will require that it satisfy the analog of Quillen's axiom SM7 over **Top**. Specifically, in analogy with (1.1) for each topological space W and each object $X \in \mathcal{C}$ there are natural objects $W \square X$ and X^W in $\mathcal{C}$ and for any two objects X, Y in $\mathcal{C}$ there is a mapping space $\mathrm{map}_{\mathcal{C}}(X, Y) \in \mathbf{Top}$ subject to the adjoint relationships

$$\begin{aligned} \mathrm{map}(W \square X, Y) &\cong \mathrm{map}_{\mathbf{Top}}(W, \mathrm{map}_{\mathcal{C}}(X, Y)) \\ &\cong \mathrm{map}(X, Y^W)). \end{aligned}$$

Furthermore, this structure fits with the model category structure on $\mathcal{C}$ as follows: we require that if $A \to B$ is a cofibration in $\mathcal{C}$ and $X \to Y$ is a fibration, then

$$\mathrm{map}_{\mathcal{C}}(B, X) \to \mathrm{map}_{\mathcal{C}}(A, X) \times_{\mathrm{map}_{\mathcal{C}}(A,Y)} \mathrm{map}_{\mathcal{C}}(B, Y)$$

be a (Serre) fibration, which is trivial if either $A \to B$ or $X \to Y$ is. This last stipulation is often called the "corner axiom". The model category $\mathcal{C}$ with this additional structure will be called a topological model category.

If $\mathcal{C}$ is a topological model category, then so is $s\mathcal{C}$ in the Reedy model category strucure, with $W \square X$ and X^W defined level-wise. Thus $s\mathcal{C}$ has both a simplicial structure and a topological structure and one must take some care to keep them straight. We will emphasize the simplicial structure, but the internal structure is useful for constructions (see after Definition 1.9) and it does arise in the next argument and later on (see Lemma 2.5).

Lemma 1.8 *Let $\mathcal{C}$ be a cofibrantly generated topological model category. Suppose every object of $\mathcal{C}$ is fibrant. Consider a diagram in $s\mathcal{C}$*

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, "q"] & X' \arrow[d, "q'"'] \\ Y \arrow[r, "i_Y"] & Y' \end{tikzcd}$$

where q and q' are Reedy fibrations, and i_X and i_Y are Reedy weak equivalences. Then for any cofibrant A in $\mathcal{C}$ and any $K \subseteq \partial\Delta^n$, the natural map

$$\mathrm{map}_{\mathcal{C}}(A, Y_n \times_{M_K Y} M_K X) \to \mathrm{map}_{\mathcal{C}}(A, Y'_n \times_{M_K Y'} M_K X')$$

is a weak equivalence.

Proof. If $K = \phi$, this follows from the fact that every object of $\mathcal{C}$ is fibrant. If L is obtained from K by attaching a k-simplex, then there is a pull-back diagram, natural in q,

$$\begin{tikzcd} Y_n \times_{M_L Y} M_L X \arrow[r] \arrow[d] & X_k \arrow[d, "d"'] \\ Y_n \times_{M_K Y} M_K X \arrow[r] & Y_k \times_{M_k Y} M_k Y. \end{tikzcd}$$

Since d is a fibration, $\mathrm{map}_{\mathcal{C}}(A, d)$ is a fibration and the diagram remains a pull-back diagram after applying $\mathrm{map}_{\mathcal{C}}(A, \cdot)$. The result now follows by a double induction on n and the number of simplices in K. ■

We next introduce an important class of cofibrant objects. We assume $\mathcal{C}$ is a topological model category in which every object is fibrant. As above, let

$$-\square- : \mathbf{Top} \times \mathcal{C} \to \mathcal{C}$$

denote the left action of **Top** on $\mathcal{C}$. Let $\phi \in \mathcal{C}$ denote the initial object.

Definition 1.9 *A homotopy co-group object A in $\mathcal{C}$ is a cofibrant object A equipped with an augmentation $\epsilon : A \to \phi$, a coproduct*

$$\mu : A \to A \sqcup A$$

and a conjugation $\chi : A \to A$, so that both μ and χ preserve the augmentation and $\mathrm{map}_{\mathcal{C}}(A, X)$ *is with product*

$$\mu^* : \mathrm{map}_{\mathcal{C}}(A, X) \times \mathrm{map}_{\mathcal{C}}(A, X) \to \mathrm{map}_{\mathcal{C}}(A, X)$$

is a homotopy associative H-space, with homotopy inverse given by χ^* and homotopy unit given by $A \to \phi \to X$.

We will call the map $A \to \phi \to X$ the constant map.

In our examples, $\mathrm{map}_{\mathcal{C}}(A, X)$ will be naturally a loop space. Note that the assumptions imply $\pi_0 \,\mathrm{map}_{\mathcal{C}}(A, X) \cong [A, X]_{\mathcal{C}}$ and that $\pi_0 \,\mathrm{map}_{\mathcal{C}}(A, X)$ is a group. If $\mathrm{map}_{\mathcal{C}}(A, X)_0 \subseteq \mathrm{map}_{\mathcal{C}}(A, X)$ is the component of $A \to \phi \to X$, then a choice of representatives for the elements of $\pi_0 \,\mathrm{map}_{\mathcal{C}}(A, X)$ defines a weak equivalence

$$\pi_0 \,\mathrm{map}_{\mathcal{C}}(A, X) \times \mathrm{map}_{\mathcal{C}}(A, X)_0 \simeq \mathrm{map}_{\mathcal{C}}(A, X).$$

If $W \in \mathbf{Top}$ is cofibrant—that is, a retract of a cellular space—and $A \in \mathcal{C}$ is a homotopy co-group object, then $W \square A$ is also a homotopy cogroup object, with coproduct

$$W \square \mu : W \square A \to W \square (A \sqcup A) \cong W \square A \sqcup W \square A$$

and augmentation $W \square A \to W \square \phi \cong \phi$. If $W \in \mathbf{Top}$ is pointed and the map $* \to W$ is a cofibration, we may define $W \wedge A$ by the push-out diagram

$$\begin{tikzcd} S \cong * \square A \arrow[r] \arrow[d] & \phi \arrow[d] \\ W \square S \arrow[r] & W \wedge A. \end{tikzcd}$$

Note that $W \wedge A$ is also a homotopy cogroup object. In fact the map $A \cong * \square A \to W \square A$ is a morphism of homotopy co-group objects and $\mathrm{map}_{\mathcal{C}}(W \wedge A, X)$ is the fiber of the map

$$\mathrm{map}_{\mathcal{C}}(W \wedge A, X) \to \mathrm{map}_{\mathcal{C}}(A, X)$$

at the constant map $A \to \phi \to X$.

If $\mathcal{C} = \mathbf{Top}_*$, then $W \square A = W_+ \wedge A$ and $W \wedge A$ is the usual smash-product. Define $\Sigma A = S^1 \wedge A$ and $\Sigma^n A = \Sigma(\Sigma^{n-1} A)$. Note that we can define ΣA for any $A \in \mathcal{C}$ equipped with an augmentation $A \to \phi$ and then ΣA is a homotopy cogroup object.

Now suppose we have a pull-back diagram in $\mathcal{C}$

$$\begin{tikzcd} Y \times_B X \arrow[r] \arrow[d] & X \arrow[d, "q"'] \\ Y \arrow[r] & B \end{tikzcd}$$

with q a fibration. Then, by SM7, there is a pull-back diagram in $\mathbf{Top}$

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}}(A, Y \times_B X) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{C}}(A, X) \arrow[d, "q_*"'] \\ \mathrm{map}_{\mathcal{C}}(A, Y) \arrow[r] & \mathrm{map}_{\mathcal{C}}(A, B) \end{tikzcd}$$

and q_* is still a fibration. Hence, taking the constant map as basepoint in all cases, there is a Barratt-Puppe sequence, which is an exact sequence of groups

(1.10.1)
$$\begin{aligned} \cdots \;&\to\; \pi_1 \,\mathrm{map}_{\mathcal{C}}(A, B) \to \pi_0 \,\mathrm{map}_{\mathcal{C}}(A, Y \times_B X) \\ &\to\; \pi_0 \,\mathrm{map}_{\mathcal{C}}(A, X) \times \pi_0 \,\mathrm{map}_{\mathcal{C}}(A, Y) \to \pi_0 \,\mathrm{map}(A, B). \end{aligned}$$

and a natural isomorphism of groups

$$\pi_n \operatorname{map}_{\mathcal{C}}(A, X) \cong [\Sigma^n S, X]_{\mathcal{C}}. \tag{1.10.2}$$

By standard arguments, $\pi_n \operatorname{map}_{\mathcal{C}}(A, X)$ is abelian if $n \geq 1$.

Actually, more is true. The action of $\pi_1 \operatorname{map}_{\mathcal{C}}(A, X)$ on $\pi_n \operatorname{map}_{\mathcal{C}}(A, X)$, $n \geq 1$, is trivial, since $\operatorname{map}_{\mathcal{C}}(A, X)$ is an H-space with homotopy inverse. Hence one doesn't need based homotopy classes of maps on the right hand side of (1.10.2).

We finish this section with a technical result. Let $X \in s\mathcal{C}$ and A a homotopy co-group object. Define

$$[A, X] = [A, X]_{\mathcal{C}} = \pi_0 \operatorname{map}_{\mathcal{C}}(A, X).$$

Then $[A, X]$ is a simplicial group, hence fibrant as a simplicial set. This is equivalent to asserting that

$$[A, X_n] \to M_n^k[A, X]$$

is onto for all $n \geq 1$ and all $k, 0 \leq k \leq n$. The following considerations identify the target of this map with $[A, M_n^k X]$ if X is Reedy fibrant.

In the next result, we assume K can be built up from the one-point object $* = \Delta^0$ by a finite sequence of push-outs of the form

$$\begin{tikzcd} \Delta^n_k \arrow[r] \arrow[d] & K_i \arrow[d] \\ \Delta^n \arrow[r] & K_{i+1}. \end{tikzcd}$$

In particular, K is contractible. Note that among such K are the horns Δ^n_k themselves.

Lemma 1.11 *Suppose $X \in s\mathcal{C}$ is Reedy fibrant and suppose $K \in \mathbf{S}$ can be built as in the previous paragraph. Then the natural map*

$$[A, M_K X] \to M_K[A, X]$$

is an isomorphism.

Proof. We get an induction argument: if $K = \Delta^0$, we are asserting

$$[A, M_0 X] \cong [A, X_0] \cong M_0[A, X]$$

which is patently true. Now suppose the result holds for K and for all finite contractible simplicial sets with top non-degenerate simplex is degree $n-1$ or less. Suppose L is obtained from K by a push-out diagram

$$\begin{tikzcd} \Delta^n_k \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^n \arrow[r] & L. \end{tikzcd}$$

By the inductive hypothesis $[\Sigma A, M_n^k X] \cong M_n^k[A, X]$ and so

$$[\Sigma A, X_n] \to [\Sigma A, M_n^k X] \tag{1.12}$$

is onto since every simplicial group is fibrant. There is a pull-back square in $\mathcal{C}$

$$\begin{tikzcd} M_L X \arrow[r] \arrow[d] & X_n \arrow[d] \\ M_K X \arrow[r] & M_n^k X; \end{tikzcd}$$

therefore 1.12 and the Barratt-Puppe sequence of 1.10 yield a diagram of exact sequences of groups

$$\begin{tikzcd} * \arrow[r] & {[A, M_L X]} \arrow[r] \arrow[d] & {[A, M_K X] \times [A, X_n]} \arrow[r] \arrow[d, "\cong"'] & {[A, M_n^k X]} \arrow[d, "\cong"'] & \\ * \arrow[r] & M_L[A, X] \arrow[r] & M_K[A, X] \times [A, X]_n \arrow[r] & M_n^k[A, X] \arrow[r] & *. \end{tikzcd}$$

■

2 The resolution model category.

We now modify the Reedy model category structure on $s\mathcal{C}$ to produce a model category which has the correct cofibrant objects, at least from the point of view of our applications. We will actually not discuss cofibrations until the section 5, but instead follow Quillen's standard route [16] to a simplicial model category structure—a route that focuses on fibrations. Nonetheless, cofibrations will be built from homotopy co-group objects, which we now specify.

Fix a small category J and a J diagram $\mathcal{A} = \{j \mapsto Aj\}$ in $\mathcal{C}$ with the property that for all $j \in J$, Aj is a homotopy co-group object in $\mathcal{C}$. Most often here J will be discrete, but in some examples the collection of homotopy cogroups objects naturally arises as a diagram (see 2.1.2 below). If necessary we can always focus on the underlying discrete diagram.

Example 2.1 1) Let $\mathcal{C} = \mathbf{Top}_*$ and J be the set $\{1, 2, 3, \ldots\}$. Define $Aj = S^j$. This is the main example of Dwyer, Kan, and Stover [5].

2) Let $\mathcal{C} = \mathcal{S}$, the category of spectra, and let $E \in \mathcal{S}$ be a homotopy associative ring spectrum which satisfies the condition Adams imposed in order to get a universal coefficient spectral sequence [1, §III.13]. Thus there is an isomorphism $\operatorname{colim}_{\alpha \in I} E_\alpha \cong E$, where E_α runs over a filtered system of finite CW spectra with the properties that

i) $E_* DE_\alpha$ is projective over E_*, and

ii) for all E module spectra F, the natural map

$$F^* DE_\alpha \to \mathrm{Hom}^*_{E_*}(E_* DE_\alpha, F_*)$$

is an isomorphism.

Here $D(\cdot)$ denotes Spanier-Whitehead duality. Many spectra satisfy this hypothesis; for example, any spectrum representing a Landweber exact theory.

Let $J = I^{\mathrm{op}} \times \mathbb{Z}$, where $\mathbb{Z}$ is regarded as a discrete category, and let

$$A(\alpha, k) = \Sigma^k DE_\alpha.$$

This and the following are among are motivating examples.

3) Let $\mathcal{C} = \mathcal{S}^T$ be the algebras in $\mathcal{S}$ over some operad T over the linear isometries operad. Let J be as in the previous example and

$$A(\alpha, k) = T(\Sigma^k DE_\alpha).$$

We will write $T(X)$ for the free T-algebra on $X \in \mathcal{S}$. Thus

$$T(X) = \bigvee_{n \geq 0} T_n \ltimes_{\Sigma_n} \underbrace{X \wedge \cdots \wedge X}_{n}$$

with the 0 summand interpreted as $T(0)_+$. Note that $T(*) = T(0)_+$ is the initial object of $\mathcal{S}^T$ and $*$ is the terminal object.

We may regard $\mathcal{A}$ as a diagram of constant objects in $s\mathcal{C}$. Then, using the simplicial set of maps (as in 1.1), $\mathcal{A}$ defines a functor from $s\mathcal{C}$ to the category $\mathbf{S}^{J^{\rm op}}$ of $J^{\rm op}$ diagrams of simplicial sets, given by

$$X \mapsto \mathrm{map}(\mathcal{A}, X) = \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X) = \{\mathrm{map}(Aj, X)\}. \tag{2.2.1}$$

This functor has a left adjoint which we denote by

$$Y \mapsto Y \otimes_J \mathcal{A}. \tag{2.2.2}$$

If $Y \in \mathbf{S}^{J^{\rm op}}$, we are writing $X \otimes_J \mathcal{A}$ for the coend $\int^j Yj \otimes Aj$, in the notation of [15, IX].

The category $\mathbf{S}^{J^{\rm op}}$ has a simplicial model category structure [3]. If $K \in \mathbf{S}$ and $Y \in \mathbf{S}^{J^{\rm op}}$, $(K \otimes Y)j = K \otimes (Yj)$; furthermore, a map $Y \to Z$ of diagrams is a weak equivalence (or fibration) if $Yj \to Zj$ is a weak equivalence or fibration for all i. Note that the functors of 2.2 preserve the simplicial structure in the sense that

$$\begin{aligned} \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, \hom(K, X)) &\cong \mathrm{map}_{\mathbf{S}}(K, \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X)) \\ (K \otimes Y) \otimes_J \mathcal{A} &\cong K \otimes (Y \otimes_J \mathcal{A}) \\ \mathrm{map}_{\mathbf{S}^{J^{\rm op}}}(Y, \mathrm{map}(\mathcal{A}, X)) &\cong \mathrm{map}_{s\mathcal{C}}(Y \otimes_J \mathcal{A}, X). \end{aligned} \tag{2.3}$$

We now define what will be the weak equivalences and so on of the resolution model category. Later, in section 3, we will be able to interpret the weak equivalences as E_2 equivalence in the sense of Dwyer, Kan, and Stover.

A Reedy fibrant replacement of a map $f : X \to Y$ in $s\mathcal{C}$ is a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

where the horizontal maps are Reedy weak equivalences and X' and Y' are Reedy fibrant.

Definition 2.4 *Define a morphism $f : X \to Y$ in $s\mathcal{C}$ to be*

1) an $\mathcal{A}$-equivalence if for some Reedy fibrant replacement $f' : X' \to Y'$ of f, the induced map

$$f'_* : \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X') \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y')$$

is a weak equivalence in $\mathbf{S}^{J^{\rm op}}$;

2) an $\mathcal{A}$-fibration if f is a Reedy fibration and if

$$f_* : \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X) \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y)$$

is a fibration in $\mathbf{S}^{J^{\mathrm{op}}}$.

3) an $\mathcal{A}$-cofibration if it has the left lifting property with respect to all $\mathcal{A}$-fibrations which are $\mathcal{A}$-equivalences.

We will characterize $\mathcal{A}$-cofibrations in the case that J is discrete in section 5.

Before proving that this defines a model category structure on $s\mathcal{C}$, we give a few lemmas intended to strengthen our hold on the concepts defined in 2.4. The first result sharpens the notion of $\mathcal{A}$-equivalence.

Lemma 2.5 *Let $g : X \to Y$ be a Reedy weak equivalence between Reedy fibrant objects. Then $g_* : \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X) \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y)$ is a weak equivalence in $\mathbf{S}^{J^{\mathrm{op}}}$.*

Proof. First note that if $q : E \to B$ is any Reedy trivial fibration, then

$$q_* : \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, E) \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, B)$$

is a trivial fibration in $\mathbf{S}^{J^{\mathrm{op}}}$. This follows from Lemma 1.5 and an adjointness argument.

From this assertion it follows that we may assume g is a Reedy trivial cofibration. For, in general, we may form a diagram

$$\begin{tikzcd} X \arrow[r, "g'"] \arrow[d, "="'] & Y' \arrow[d, "q"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

where q is a Reedy trivial fibration, and g' is a Reedy cofibration. Then g_* is a weak equivalence if and only if g'_* is, and, because fibrations are closed under composition, Y' is fibrant.

Now, because g is a Reedy trivial cofibration and Y is Reedy fibrant, g has a retraction $r : Y \to X$. Thus $rg = 1_X : X \to X$. Since 1_Y and gr are both solutions to the lifting problem and Y is fibrant

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "g"'] & Y \\ Y \arrow[ur, dashed] & \end{tikzcd}$$

they are homotopic in the following sense. We use that $s\mathcal{C}$, in its Reedy structure, has an internal structure as a topological model category. The homotopy $h : Y \to Y^{[0,1]}$ is a solution to the lifting problem

(2.6)
$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] & Y^{[0,1]} \arrow[d] \\ Y \arrow[r, "{(1,gr)}"'] \arrow[ur, dashed] & Y \times Y = Y^{\{0,1\}} \end{tikzcd}$$

which exists since the right map is a Reedy fibration.

To finish the argument, we show, for any choice of basepoint

$$\pi_* g_* : \pi_* \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X) \to \pi_* \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y)$$

is an isomorphism. As always with simplicial sets, we mean the homotopy groups of the geometric realization. The two projections

$$d_i : Y^{[0,1]} \to Y = Y^{\{i\}}, \quad i = 0, 1$$

are Reedy trivial fibrations, so $(d_i)_* = \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, d_i)$ are both trivial fibrations in $\mathbf{S}^{J^{\mathrm{op}}}$. Since $\pi_*(d_i)_*$ is an isomorphism, for $i = 0, 1$, and these two maps have a common inverse induced by

$$\mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y) \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y^{[0,1]}),$$

we have $\pi_*(d_0)_* = \pi_*(d_1)_*$. The homotopy h, diagram (2.6), and the fact that $rg = 1_X$ now imply the result. ■

The previous result is crucial for the following lemma, which removes the ambiguity in the definition of $\mathcal{A}$-equivalence.

Lemma 2.7 *Let $f : X \to Y$ be a morphism in $s\mathcal{C}$. Then*

$$(f')_* : \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X') \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y')$$

is a weak equivalence for one Reedy fibrant replacement f' of f if and only if it is a weak equivalence for all Reedy fibrant replacements f' of f.

Proof. Fix a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_0 \arrow[d, "f_0"] \\ Y \arrow[r] & Y_0 \end{tikzcd}$$

where the horizontal maps are Reedy trivial cofibrations and X_0, Y_0 are Reedy fibrant. Let $f' : X' \to Y'$ be any Reedy fibrant replacement for f. We show $(f_0)_*$ is a weak equivalence in $\mathbf{S}^{J^{\mathrm{op}}}$ if and only if $(f')_*$ is.

Factor f' as $X' \xrightarrow{j} Z \xrightarrow{q} Y'$ where j is a Reedy weak equivalence and q is a Reedy fibration. Then $(j)_*$ is a weak equivalence, by Lemma 2.5, so we need to show $(f_0)_*$ is a weak equivalence if and only if q_* is. Since $X \to X_0$ and $Y \to Y_0$ are Reedy trivial cofibrations, there is a diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "f_0"'] & Z \arrow[d, "g"] \\ Y_0 \arrow[r] & Y' \end{tikzcd}$$

where the horizontal maps are Reedy weak equivalences. The result now follows from Lemma 2.5.■

The following is immediate and reassuring.

Corollary 2.8 *Any Reedy weak equivalence is an $\mathcal{A}$-equivalence.*

Proof. If $f : X \to Y$ is a Reedy weak equivalence and $Y \to Y'$ is a Reedy weak equivalence with Y' Reedy fibrant, then the following is a Reedy fibrant replacement for f:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y' \arrow[d, "="] \\ Y \arrow[r] & Y'. \end{tikzcd}$$

■

The next two results complete the lemmas necessary for us to use Quillen's arguments.

Lemma 2.9 *Any Reedy fibrant object is $\mathcal{A}$-fibrant.*

Proof. We need to show that for all $j \in J$, the simplicial set

$$\mathrm{map}_{s\mathcal{C}}(Aj, X) = \{\mathrm{Hom}_{\mathcal{C}}(Aj, X_n)\}_{n\geq 0}$$

is fibrant; in other words, that

$$\mathrm{Hom}_{\mathcal{C}}(Aj, X_n) \to M_n^k \mathrm{Hom}_{\mathcal{C}}(Aj, X) \cong \mathrm{Hom}_{\mathcal{C}}(Ai, M_n^k X)$$

is onto. Since Aj is cofibrant and $X_n \to M_n^k X$ is a fibration, it is sufficient to show

$$[Aj, X_n]_{\mathcal{C}} \to [Aj, M_n^k X]_{\mathcal{C}}$$

is onto. Now Lemma 1.11 implies $[Aj, M_n^k X]_{\mathcal{C}} \cong M_n^k [Aj, X]_{\mathcal{C}}$, and the result follows from the fact that $[Aj, X]_{\mathcal{C}}$ is a simplicial group. ■

Lemma 2.10 *A Reedy fibration $f : X \to Y$ is at once an $\mathcal{A}$-fibration and an $\mathcal{A}$-equivalence if and only if*

$$f_* : \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X) \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y)$$

is a trivial fibration in $\mathbf{S}^{J^{\mathrm{op}}}$.

Proof. The proof is similar in spirit to the argument used in Lemma 2.7. Choose a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X_0 \arrow[d, "f_0"] \\ Y \arrow[r] & Y_0 \end{tikzcd}$$

where the horizontal maps are Reedy weak equivalences, Y_0 is Reedy fibrant, and f_0 is a Reedy fibration. Then X_0 is Reedy fibrant. We need to show

$$\mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X)_n \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y)_n \times_{M_n \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y)} M_n \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X) \tag{2.11.1}$$

is surjective if and only if

$$\mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X_0)_n \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y_0)_n \times_{M_n \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y_0)} M_n \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X_0)$$

is surjective. Since each Aj is constant as a simplicial object (2.11.1) is isomorphic to

$$\mathrm{Hom}_{\mathcal{C}}(\mathcal{A}, X_n) \to \mathrm{Hom}_{\mathcal{C}}(\mathcal{A}, Y_n \times_{M_n Y} M_n X). \tag{2.11.2}$$

Since f is a Reedy fibration $X_n \to Y_n \times_{M_n Y} M_n X$ is a fibration. Thus (2.11.2) is surjective if and only if

$$[\mathcal{A}, X_n]_{\mathcal{C}} \to [\mathcal{A}, Y_n \times_{M_n Y} M_n X]_{\mathcal{C}}$$

is onto. Now apply Lemma 1.8. ■

Remark 2.12 The reader sensitive to generalization may wonder where we used that Aj was a homotopy co-group object. This hypothesis was used in Lemma 2.9, which ultimately turned on the fact that every simplicial is fibrant and that the Barratt-Puppe sequence of (1.10.1) was a sequence of groups. The latter would be less necessary if Lemma 2.9 could be proved some other way. One possibility is to force the Reedy model structure to be a simplicial structure. By this we mean that under appropriate hypotheses one can localize (in the sense of [4], [12]) with respect to set of Reedy cofibrations

$$\Delta^n \otimes A_\alpha \cup_{\Delta^n_k \otimes A_\alpha} \Delta^n_k \otimes B_\alpha \to \Delta^n \otimes B_\alpha$$

where $A_\alpha \to B_\alpha$ runs over the generators for the cofibrations of $\mathcal{C}$. This yields a localized Reedy structure and the analog of Lemma 2.9 for this new structure is formal. Furthermore, the analog of Theorem 2.14 below is true if one is careful about the generators for trivial cofibrations. Furthermore, geometric realization passes to a functor on the localized homotopy category. We had difficulty with the spiral exact sequence (section 3), however, and, thus, trouble producing methods for analyzing the homotopy type of the realization.

The resolution model category we are constructing will be cofibrantly generated. We now describe the generating maps. For all $j \in J$, there is a functor $\mathbf{S}^{J^{\mathrm{op}}} \to \mathbf{S}$ sending Y to Yj. This functor has a left adjoint Fi given by left Kan extension; concretely

$$(F_j K)_i = \coprod_{j \to i} K$$

where the coproduct over morphisms in J^{op}. A morphism $X \to Y$ in $\mathbf{S}^{J^{\mathrm{op}}}$ is a fibration (or trivial fibration) if and only if it has the right lifting property with respect to $F_j \Delta^n_k \to F_j \Delta^n$ (or $F_j \partial\Delta^n \to F_j \Delta^n$) for all $j \in J$. Since the composition of left adjoints is a left adjoint, one has, for all $K \in \mathbf{S}$ and $j \in J$, that

$$F_j K \otimes_J \mathcal{A} \cong K \otimes Aj.$$

Thus a morphism $X \to Y$ in $s\mathcal{C}$ is an $\mathcal{A}$-fibration if and only if it has the right lifting property with respect to the cofibrations

$$\text{(2.13.1)} \qquad \begin{array}{rcl} \Delta^n_k \otimes Aj \to \Delta^n \otimes Aj, & & n > 0, 0 \le k \le n, j \in J, \\ \Delta^n \otimes A_\alpha \sqcup_{A_\alpha \otimes \partial\Delta^n} B_\alpha \otimes \partial\Delta^n & \to & B_\alpha \otimes \Delta^n, n \ge 0. \end{array}$$

Here $A_\alpha \to B_\alpha$ run over the generators for the trivial cofibrations. See Lemma 1.6. Let $\mathcal{E}$ denote this set of cofibrations. Similarly, a morphism $X \to Y$ in $s\mathcal{C}$ is an $\mathcal{A}$-trivial fibration if and only if it has the right lifting property with respect to the set of cofibrations

$$\text{(2.13.2)} \qquad \begin{array}{rcl} \partial\Delta^n \otimes Aj \to \Delta^n \otimes Aj, & & n > 0, 0 \le k \le n, j \in J \\ \Delta^n \otimes A_\alpha \sqcup_{A_\alpha \otimes \partial\Delta^n} B_\alpha \otimes \partial\Delta^n & \to & B_\alpha \otimes \Delta^n \end{array}$$

where $A_\alpha \to B_\alpha$ is as above. Note that we are asserting that a morphism is an $\mathcal{A}$-trivial fibration if and only if it is a Reedy fibration and $\mathrm{map}_{s\mathcal{C}}(Aj, X) \to \mathrm{map}_{s\mathcal{C}}(Aj, Y)$ is a trivial fibration for all j. This is the content of Lemma 2.10. Call this set of morphisms $\mathcal{F}$.

Theorem 2.14 *With the notion of $\mathcal{A}$-equivalence, $\mathcal{A}$-fibration, and $\mathcal{A}$-cofibration defined above, and with its standard simplicial structure, $s\mathcal{C}$ is a cofibrantly generated simplicial model category. The trivial cofibrations and cofibrations are generated, respectively, by the sets $\mathcal{E}$ and $\mathcal{F}$ above.*

Proof. Let U be the source of any of the morphisms in $\mathcal{E}$ or $\mathcal{F}$. Because each of the $Ai \in \mathcal{C}$ is cofibrant, and because $\mathcal{C}$ is cofibrantly generated, there is an ordinal number κ, so that $\mathrm{Hom}_{s\mathcal{C}}(U, \cdot)$ commutes with colimits over κ over a particular nice sort: specifically, if $X : \kappa \to s\mathcal{C}$ is a functor so that

(1) for all limit ordinals $s < \kappa$, $\operatorname{colim}_{t<s} X_t \cong X_s$; and

(2) for all $s < \kappa$, $X_s \to X_{s+1}$ is a level-wise cofibration, then

$$\operatorname{colim}_{s<\kappa} \mathrm{Hom}_{s\mathcal{C}}(U, X_s) \to \mathrm{Hom}_{s\mathcal{C}}(U, \operatorname{colim}_{s<\kappa} X_s)$$

is an isomorphism.

Because of this, the small object argument [16, §II.3–4] verifies all of the model category axioms except one: we can, at this stage, only factor an arbitrary morphism $X \to Y$ as $X \xrightarrow{j} Z \xrightarrow{q} Y$ where q is an $\mathcal{A}$-fibration and j is an $\mathcal{A}$-cofibration with the left lifting property with respect to all $\mathcal{A}$-fibrations. However, any such $\mathcal{A}$-cofibration will be an $\mathcal{A}$-trivial cofibration provided we can find, for all $X \in s\mathcal{C}$, a natural $\mathcal{A}$-equivalence $X \to X'$ so that X' is $\mathcal{A}$-fibrant. This is accomplished by Quillen's argument [16, §II.4]. Since $s\mathcal{C}$ in the Reedy structure is cofibrantly generated, there is a natural Reedy weak equivalence $X \to X'$ so that X' is Reedy fibrant. Now apply Corollary 2.8 and Lemma 2.9.

To verify SM7, we need to show that if $j : K \to L$ is a cofibration in $\mathbf{S}$ and $q : X \to Y$ is an $\mathcal{A}$-fibration in $s\mathcal{C}$, then

$$X^L \to X^K \times_{Y^K} Y^L$$

is an $\mathcal{A}$-fibration, which is an $\mathcal{A}$-equivalence if either j or q is a weak equivalence. However, this follows from the top formula of (2.3) and the fact that SM7 hold for $S^{J^{\mathrm{op}}}$. ■

3 The Spiral Exact Sequence.

This section imports the ideas of [6] to our more general setting. The arguments are nearly the same.

We have defined a morphism $f : X \to Y$ in $s\mathcal{C}$ to be an $\mathcal{A}$-equivalence if the simplicial set of maps $\mathrm{map}_{s\mathcal{C}}(\mathcal{A}, X') \to \mathrm{map}_{s\mathcal{C}}(\mathcal{A}, Y')$ is a weak equivalence in $\mathbf{S}^{I^{\mathrm{op}}}$, where $X' \to fY'$ is a Reedy fibrant model for f. Since each Aj is a homotopy co-group object, this is equivalent to the assertion that $\pi_n \operatorname{map} s\mathcal{C}(Aj, X') \to \pi_n \mathrm{map}_{s\mathcal{C}}(Aj, Y')$ be an isomorphism for all $n \geq 0$ and all $j \in J$, where we take as basepoint the constant map $Aj \to \phi \to X'$. By adjointness, this is equivalent to demanding that

$$[\Delta^n/\partial\Delta^n \wedge Aj, X'] \to [\Delta^n/\partial\Delta^n \wedge Aj, Y']$$

be an isomorphism, where the brackets denote homotopy classes of maps in the $\mathcal{A}$-resolution model category structure. Here, we define for $K \in \mathbf{S}$ pointed and $B \in s\mathcal{C}$ equipped with an augmentation $B \to \phi$, the object $K \wedge B$ by the push-out diagram

$$\begin{tikzcd} * \otimes B \arrow[r] \arrow[d] & K \otimes B \arrow[d] \\ \phi \arrow[r] & K \wedge B. \end{tikzcd}$$

Compare the construction after 1.9.

We define, for $n \geq 0$, the *natural homotopy groups* of $X \in s\mathcal{C}$ with coefficients in Aj by the equation

$$\pi_n^\natural(X, Aj) = [\Delta^n/\partial\Delta^n \wedge Aj, X'] \tag{3.1}$$

where $X \to X'$ is a Reedy fibrant replacement of X. Note that

$$\pi_n^\natural(X, \mathcal{A}) = \{\pi_n^\natural(X, Aj)\}_{j \in J}$$

is a J^{op} diagram of sets functorial in X, and independent of the choices. It is a diagram of groups if for each morphism in J, the map $Ai \to Aj$ respects the homotopy cogroup structure. This happens in all the examples of 2.2 or if J is discrete. It is because of this nuance that we phrase the results below in terms of Aj rather than in terms of the diagram $\mathcal{A}$.

Notice also that $\partial\Delta^0 = \phi$, so $\Delta^0/\partial\Delta^0 = \Delta^0_+$.

There is another collection of homotopy groups associated to X. If $[Aj, X]_\mathcal{C}$ denotes the homotopy class of maps in $\mathcal{C}$; this is a simplicial group. Define

$$\pi_n(X, Aj) = \pi_n[Aj, X]_\mathcal{C}. \tag{3.2}$$

Thus $\pi_n(X, Aj)$ is a homotopy group of a simplicial group and can be computed by the normalized Moore complex, which is defined as follows. Let

$$C_n[Aj, X]_\mathcal{C} = \bigcap_{i=1}^{n} \ker\{d_i : [Aj, X_n] \to [Aj, X_{n-1}]\}.$$

Then d_0 defines a homomorphism $d_0 : C_n[Aj, X]_\mathcal{C} \to C_{n-1}[Aj, X]_\mathcal{C}$ and

$$\pi_n(X; Aj) \cong \ker(d_0)/\operatorname{Im}(d_0). \tag{3.3}$$

The purpose of this section is to prove

Proposition 3.4 *For $X \in s\mathcal{C}$ and each $j \in J$ there is a natural isomorphism of groups*

$$\pi_0^\natural(X, Aj) \cong \pi_0(X; Aj)$$

and a natural long exact of abelian groups

$$\begin{aligned} \cdots \to \pi_{n-1}^\natural(X; \Sigma Aj) \to \pi_n^\natural(X; Aj) \to \pi_n(X, Aj) &\to \pi_{n-2}^\natural(X; \Sigma Aj) \to \\ \cdots \to \pi_1^\natural(X; Aj) &\to \pi_1(X, Aj) \to 0. \end{aligned}$$

Naturality in the result means in the X variable. Naturality in the Aj variable is subject to the conditions above.

We begin by identifying $C_n[Aj, X]_\mathcal{C}$ with another object. The following is a variant on Lemma 1.3. Let $\mathcal{C}_*$ be the category of objects augmented objects in $\mathcal{C}$; that is, objects equipped with a map $\epsilon : Z \to \phi$. Morphisms in $\mathcal{C}_*$ respect the augmentation.

Lemma 3.5 *Let $K \in \mathbf{S}$ be a pointed simplicial set. Then there is a functor $C_K : s\mathcal{C} \to \mathcal{C}_*$ so that for all $Z \in \mathcal{C}_*$ there is a natural isomorphism*

$$\operatorname{Hom}_{s\mathcal{C}}(K \wedge Z, X) \cong \operatorname{Hom}_{\mathcal{C}_*}(Z, C_K X).$$

Proof. Define $\hom_*(K, X)$ by the pull-back diagram

$$\begin{tikzcd} \hom_*(K,X) \arrow[r] \arrow[d] & \hom(K,X) \arrow[d] \\ \phi \arrow[r] & \hom(*,X) \simeq X. \end{tikzcd}$$

Then for $Y \in s\mathcal{C}_*$, we have

$$\mathrm{Hom}_{s\mathcal{C}}(Y \wedge K, X) \cong \mathrm{Hom}_{s\mathcal{C}_*}(Y, \hom_*(K, X)).$$

Thus we may set $C_K X = \hom_*(K, X)_0$. ■

Note there is a pull-back diagram (using $\Delta^0 \cong *$)

$$\begin{tikzcd} C_K X \arrow[r] \arrow[d] & M_K X \arrow[d] \\ \phi \arrow[r] & M_0 X \cong X. \end{tikzcd}$$

In particular, if $K \subseteq L$, then there is a pull-back diagram

(3.6)
$$\begin{tikzcd} C_{L/K} X \arrow[r] \arrow[d] & M_L X \arrow[d] \\ \phi \arrow[r] & M_K X. \end{tikzcd}$$

If X is Reedy fibrant, Lemma 1.7 implies $M_L X \to M_K X$ is a fibration, and this implies $C_{L/K} X \to \phi$ is a fibration. We are particularly interested in two examples: we define $C_n X = C_{\Delta^n/\Delta^n_0} X$ and $Z_n X = C_{\Delta^n/\partial\Delta^n} X$.

The category $\mathcal{C}_*$ of augmented objects in $\mathcal{C}$ inherits a topological model category structure for $\mathcal{C}$. Indeed, the weak equivalences, fibrations, and cofibrations are the same as in $\mathcal{C}$ (although the notion of fibrant object changes) and the topological structure is determined by the formulas

$$K \square_{\mathcal{C}_*} Z = K \square_{\mathcal{C}} Z$$

a pull-back diagram in $\mathcal{C}$

$$\begin{tikzcd} \hom_{\mathcal{C}_*}(K,Z) \arrow[r] \arrow[d] & \hom_{\mathcal{C}}(K,Z) \arrow[d] \\ \phi \arrow[r] & \hom_{\mathcal{C}}(K,\phi) \end{tikzcd}$$

and a pull-back diagram in **Top**

$$\begin{tikzcd} \mathrm{map}_{\mathcal{C}_*}(Y,Z) \arrow[r] \arrow[d] & \mathrm{map}_{\mathcal{C}}(Y,Z) \arrow[d] \\ * \arrow[r, "\epsilon"] & \mathrm{map}_{\mathcal{C}}(Y,\phi) \end{tikzcd}$$

where $\epsilon(*)$ is the augmentation of Y. If $Y \in \mathcal{C}_*$ is cofibrant (in either $\mathcal{C}$ or $\mathcal{C}_*$—the notions are equivalent) and $Z \in \mathcal{C}_*$ is fibrant (i.e., $\epsilon : Z \to \phi$ is a fibration in $\mathcal{C}$), then

$$[Y, Z]_{\mathcal{C}_*} \cong \pi_0 \, \mathrm{map}_{\mathcal{C}_*}(Y, Z).$$

Lemma 3.7 *Let $A \in \mathcal{C}$ be a homotopy co-group object, and $Z \in \mathcal{C}_*$ fibrant. Then for each $i \in I$ there is a split short exact sequence of groups*

$$\{1\} \to [A, Z]_{\mathcal{C}_*} \to [A, Z]_{\mathcal{C}} \to [A, \phi]_{\mathcal{C}} \to \{1\}.$$

The splitting is natural in Z and in the A-variable maps between co-group objects that respect the co-group structure.

Proof. The fiber sequence

$$\operatorname{map}_{\mathcal{C}_*}(A, Z) \to \operatorname{map}_{\mathcal{C}}(A, Z) \to \operatorname{map}_{\mathcal{C}}(A, \phi)$$

has a section induced by $\phi \to A$. ■

The crucial calculation is this:

Lemma 3.8 *Let $X \in s\mathcal{C}$ be Reedy fibrant and $A \in \mathcal{C}$ a homotopy cogroup object. Then there is a natural isomorphism*

$$C_n[A, X]_{\mathcal{C}} \cong [A, C_nX]_{\mathcal{C}*}.$$

Proof. By Lemma 1.7, the Barratt-Puppe sequence 1.10.1 and (3.6) we have an exact sequence of groups

$$\begin{aligned} \cdots \to [\Sigma A, \phi]_{\mathcal{C}} \times [\Sigma A, M_nX]_{\mathcal{C}} &\to [\Sigma A M_n^0 X]_{\mathcal{C}} \to \\ [A, C_nX]_{\mathcal{C}} &\to [A, \phi]_{\mathcal{C}} \times [A, M_nX]_{\mathcal{C}} \to [A, M_0^nX]_{\mathcal{C}}. \end{aligned}$$

The result now follows from Lemma 1.11 and the fact that the maps

$$M_n[\Sigma A, X]_{\mathcal{C}} \to M_n^0[\Sigma A, X]_{\mathcal{C}} \qquad \text{and} \qquad M_n[A, X]_{\mathcal{C}} \to M_n^0[A, X]_{\mathcal{C}}$$

are onto – remember, $[\Sigma A, X]_{\mathcal{C}}$ is a simplicial group. ■

Next notice that there is a push-out diagram of pointed simplicial sets

$$\begin{tikzcd} \Delta^n/\partial\Delta^n \arrow[r, "d^0"] \arrow[d] & \Delta^{n+1}/\Delta_0^{n+1} \arrow[d] \\ * \arrow[r] & \Delta^{n+1}/\partial\Delta^{n+1}. \end{tikzcd}$$

Hence, for $X \in s\mathcal{C}$, there is a pull-back diagram in $\mathcal{C}_*$

(3.9)
$$\begin{tikzcd} Z_{n+1}X \arrow[r, "i"] \arrow[d] & C_{n+1}X \arrow[d, "d_0"] \\ \phi \arrow[r] & Z_nX. \end{tikzcd}$$

If X is Reedy fibrant, then d_0 is a fibration in C. Let, for $Z \in \mathcal{C}_*$ fibrant and Aj one our distinguished homtopy cogroup objects,

$$[\Sigma^n Aj, Z]_* \cong \pi_n \operatorname{map}_{\mathcal{C}_*}(Aj, Z)$$

with $Aj \to \phi \to Z$ as the basepoint. Then $[\Sigma^n Aj, Z]_* \cong [|\Delta^n/\partial\Delta^n| \wedge Aj, Z]_*$. Since $\mathrm{map}_{\mathcal{C}_*}(Aj, \phi) = *$, there is a long exact sequence

$$\cdots \to [\Sigma^1 Aj, Z_n X]_* \longrightarrow [Aj, Z_{n+1}X]_* \longrightarrow [Aj, C_{n+1}X]_* \xrightarrow{(d_0)_*} [Aj, Z_n X]_*.$$

We combine these sequences into an exact couple

(3.10)
$$\begin{tikzcd}[column sep=small]
{[\Sigma^* Aj, Z_0 X]_*} \arrow[rr, dashed, "g_*"] & & {[\Sigma^* Aj, Z_1 X]_*} \arrow[rr, dashed] \arrow[dl, "i_*"'] & & {[\Sigma^* Aj, Z_2 X]_*} \arrow[rr, dashed] \arrow[dl, "i_*"'] & & {} \arrow[dl] \\
& {[\Sigma^* Aj, C_1 X]_*} \arrow[ul] & & {[\Sigma^* Aj, C_2 X]_*} \arrow[ul, "(d_0)_*"] & & {[\Sigma^* Aj, C_3 X]_*} \arrow[ul] & .
\end{tikzcd}$$

The dotted arrows indicate a map that shifts degree (in $\Sigma^* Aj$) by -1. Note that $Z_0 X = C_{\Delta^0/\partial\Delta^0} X = X_0$. The last follows for the usual reason: $\partial\Delta^0 = \phi$, so the push-out diagram

$$\begin{tikzcd}
\partial\Delta^0 \arrow[r] \arrow[d] & \Delta^0 \arrow[d] \\
* \arrow[r] & \Delta^0/\partial\Delta^0
\end{tikzcd}$$

identifies $\Delta^0/\partial\Delta^0$ with Δ^0_+. Note $C_0 X = \phi$.

3.11: Proof of the existence of the spiral exact sequence The composite

$$C_{n+1}X \xrightarrow{d_0} Z_n X \xrightarrow{i} C_n X$$

is induced by the map $d_0 : \Delta^n/\Delta^n_0 \to \Delta^{n+1}/\Delta^{n+1}_0$. Thus, using Lemma 3.8

$$i_*(d_0)_* = d_0 : C_{n+1}[\Sigma^* Aj, X] \to C_n[\Sigma^* Aj, X].$$

Thus the E_2-term of the spectral sequence associated to (3.10) is given by

(3.12.1)
$$E^2_{n,*} = H_n(C_*[\Sigma^* Aj, X], d_0) \cong \pi_n(X, \Sigma^* Aj).$$

If we define

$$D^2_{n,*} = [\Sigma^* Aj, Z_n X]_*/(d_0)_*[\Sigma^* Aj, C_{n+1}X]_*$$

then there is a long exact sequence

$$\cdots \to D^2_{n-1,*+1} \xrightarrow{g_*} D^2_{n,*} \xrightarrow{i_*} E^{2_{n,*}} \xrightarrow{i_* \circ g_*^{-1} \circ (d_0)_*} D^2_{n-2,*+1} \xrightarrow{S_*} D^2_{n-1,*} \to \cdots .$$

We claim that there is a natural exact sequence

(3.12.2)
$$[\Sigma^* Aj, C_{n+1}X]_* \xrightarrow{(d_0)_*} [\Sigma^* Aj, Z_n X]_* \to \pi^\natural_n(X, \Sigma^* Aj) \to 0.$$

This is proved exactly as in [6], Proposition 5.8. We will give the argument in the next paragraph. Note that (3.12.2) at the case $n = 0$ and the equation $Z_0 X = X_0$ supplies the isomorphism $\pi_0(X, Aj) \cong \pi^\natural_0(X, Aj)$. The spiral exact sequence is obtained from the derived sequence by setting $* = 0$.

To see (3.12.2) is exact, note that by Lemma 3.5,

$$\mathrm{Hom}_{\mathcal{C}_*}(\Sigma^q Aj, Z_n X) \cong \mathrm{Hom}_{s\mathcal{C}}(\Sigma^q Aj \wedge \Delta^n/\partial\Delta^n, X).$$

Similarly, one has a correspondence for homotopies in $\mathcal{C}_*$:

$$\mathrm{Hom}_{\mathcal{C}_*}(\Sigma^q Aj \wedge [0,1]_{+1} Z_n X) \cong \mathrm{Hom}_{s\mathcal{C}}((\Sigma^q Aj \wedge [0,1]_+) \wedge \Delta^n/\partial\Delta^n, X).$$

Since the endpoint inclusions $d_i : \Sigma^q Aj \to \Sigma^q Aj \wedge [0,1]_+$ $i = 0,1$ are weak equivalences in $\mathcal{C}_*$, the induced maps

$$d_i : \Sigma^q Aj \wedge \Delta^n/\partial\Delta^n \to (\Sigma^q Aj \wedge [0,1]_+) \wedge \Delta^n/\partial\Delta^n$$

are Reedy weak equivalences, hence $\mathcal{A}$-equivalences (by Corollary 2.8). Thus we get a map

$$[\Sigma^q Aj, Z_n X]_{\mathcal{C}_*} \to \pi_n^\natural(X, \Sigma^q Aj)$$

which is onto since both source and target are quotients of $\mathrm{Hom}_{s\mathcal{C}}(\Sigma^q Aj \wedge \Delta^n/\partial\Delta^n, X)$. Finally, $\Sigma^q Aj \wedge \Delta^n/\partial\Delta^n \to X$ is the zero map in the $\mathcal{A}$-resolution homotopy category if and only if it can be factored

$$\begin{tikzcd} \Sigma^q Aj \wedge \Delta^n/\partial\Delta^n \arrow[r] \arrow[d, "d^0"] & X \\ \Sigma^q Aj \wedge \Delta^{n+1}/\Delta_0^{n+1}. \arrow[ur] & \end{tikzcd}$$

This provides exactness at $[\Sigma^q Aj, Z_n X]_{\mathcal{C}_*}$. ■

The reader will have noted that the exact couple (3.10), the short exact sequence (3.12.2) and the identification of (3.12.1) supply a spectral sequence

$$\pi_p(X; \Sigma^q Aj) \Rightarrow \pi_{p+q}^\natural(X; Aj). \qquad (3.13)$$

Under favorable circumstances, there is an isomorphism

$$\pi_n^\natural(X; Aj) \cong [\Sigma^n Aj, |X|]_{\mathcal{C}}.$$

An example of this is supplied in section 6. We now record the obvious corollary to the spiral exact sequence.

Proposition 3.14 *A morphism $f : X \to Y$ in $s\mathcal{C}$ is an $\mathcal{A}$-equivalence if and only if the induced maps*

$$f_* : \pi_p(X; \Sigma^q Aj) \to \pi_p(Y; \Sigma^q Aj)$$

are isomorphisms for $p \geq 0$, $q \geq 0$, and $j \in J$.

Proof. If $X \to Y$ is an $\mathcal{A}$-equivalence, then (since $\mathcal{A}$ is closed under suspension), the homomorphism $\pi_n^\natural(f; \Sigma^n \mathcal{A})$ is an isomorphism for all n. Now use the spiral exact sequence. ■

4 Stover Resolutions.

If $\mathcal{C}$ is a model category and $X \in \mathcal{C}$, functor

$$\mathrm{Hom}_{\mathcal{C}}(X, \cdot) : \mathcal{C} \to \mathbf{Sets}$$

has left adjoint $U \mapsto \bigsqcup_U X$. If $F^X : \mathcal{C} \to \mathcal{C}$ is the resulting composite functor, one gets, for $Y \in \mathcal{C}$, an augmented simplicial object $F^X_\cdot(Y) \to Y$ with

$$F^X_n(Y) = (F^X)^{n+1} Y.$$

However, this functor doesn't have good homotopical properties, and neither do some of the more obvious modifications. The Stover resolution is a modification of this idea that works well if X is a homotopy cogroup object.

Let $\mathcal{A} : J \to \mathcal{C}$ be a diagram of homotopy cogroup objects. The diagram should be closed under suspension. We will most often take J to be discrete. For each $j \in J$, let $CAj = [0,1] \wedge (Aj)$ $1 \in [0,1]$ as basepoint be the cone on Aj and let $Aj \to CAj$ be the inclusion induced by $\{0\} \to [0,1]$. Note that the map $[0,1] \to *$ induces a weak equivalence

$$CAj \longrightarrow A \wedge * = \phi.$$

This is all natural in j, and we get a new diagram $\bar{A} = \{\mathcal{A} \to C\mathcal{A}\}$ in $\mathcal{C}$. $\bar{A}$ is a functor on $I = J \times \{0 \to 1\}$. There is a functor $\mathcal{C} \to \mathbf{Sets}^{I^{\mathrm{op}}}$ given by

$$Y \mapsto \mathrm{Hom}_{\mathcal{C}}(\bar{A}, Y);$$

this functor has left adjoint $F \mapsto \bar{A} \otimes_I F$. For example, if J is discrete, there is a push-out diagram

$$\begin{tikzcd} \coprod_{F(CAj)} Aj \arrow[r] \arrow[d] & \coprod_{F(Aj)} Aj \arrow[d] \\ \coprod_{F(CAj)} CAj \arrow[r] & \bar{A} \otimes_I F. \end{tikzcd}$$

In general $\bar{A} \otimes_I F$ can be expressed as a coend. Let $V(\cdot)$ be the composite functor

$$V(\cdot) = \bar{A} \otimes_I \mathrm{Hom}_{\mathcal{C}}(\bar{A}, \cdot) : \mathcal{C} \to \mathcal{C}.$$

This is a composition of adjoints, and, as such, yields a natural augmented simplicial object in $\mathcal{C}$

$$\epsilon : V_\cdot(Y) \to Y$$

where $V_n Y = V^{n+1} Y$. This construction can be extended from $\mathcal{C}$ to $s\mathcal{C}$ by setting, for $Y \in s\mathcal{C}$,

$$V(Y) = \operatorname{diag} V_\cdot(Y_\cdot).$$

The augmentation of above yields a natural map $\epsilon : V(Y) \to Y$.

More generally, if $X \in \mathcal{C}$ is a fixed object and $X/\mathcal{C}$ the category of arrows $X \to Y$ in $\mathcal{C}$, then the functor that assigns to $X \to Y$

$$\mathrm{Hom}_{\mathcal{C}}(\bar{A}, Y) \in \mathbf{Sets}^{I^{\mathrm{op}}}$$

has left adjoint $F \mapsto X \sqcup \bar{A} \otimes_I F$. This induces a composite functor

$$V^X = X \sqcup \bar{A} \otimes_I \mathrm{Hom}_{\mathcal{C}}(\bar{A}, \cdot) : X/\mathcal{C} \to X/\mathcal{C}$$

and an augmented simplicial object

(4.1) $$\epsilon : V_{\cdot}^{X} Y \to Y.$$

If $X \in s\mathcal{C}$, this construction can be extended to $X/s\mathcal{C}$ by

(4.2) $$V^{X}(Y) = \mathrm{diag}(V_{\cdot}^{X_{\cdot}}(Y_{\cdot})).$$

There is an augmentation $\epsilon : V^X(Y) \to Y$ and a map $X \xrightarrow{i} V^X(Y)$ so that the composite $\epsilon \circ i : X \to Y$ is the original object in $X/s\mathcal{C}$. This is the *Stover resolution* of Y under X.

Lemma 4.3 *The natural map of simplicial objects $\epsilon : V^X(Y) \to Y$ induces an isomorphism*

$$\epsilon_* \pi_p(\Sigma^q Aj, V^A(Y)) \to \pi_p(\Sigma^q Aj, Y)$$

for all $p \geq 0$ and $q \geq 0$.

Proof. If Y is constant, as in 4.1, this is proved exactly as in Stover's paper [18]. The key points are that $[\Sigma^q Aj, V^X(Y)] \to [\Sigma^q Aj, Y]$ is a morphism of simplicial groups, so that one can calculate with the Moore complex, and if $a \in [\Sigma^q Aj, V_n^X(Y)]$ has $(d_i)_* a = 0$ for $0 \leq i \leq n$, then there is a $b \in [\Sigma^q Aj, V_{n+1}^X(Y)]$ so that $(d_0)_* b = a$ and $(d_i)_* b = 0$, $1 \leq i \leq n+1$. If $f : \Sigma^q Aj \to V_n^X(Y)$ represents a, then b is the class of

$$\begin{aligned} \Sigma^q Aj = \Sigma^q Aj \Box \{f\} \to \Sigma^q Aj \Box \mathrm{Hom}(\Sigma^q Aj, V_n Y) \quad &\to \quad A \sqcup \int^I \bar{\mathcal{A}} \otimes_I \mathrm{Hom}(\bar{\mathcal{A}}, V_n Y) \\ &= \quad V_{n+1} Y. \end{aligned}$$

The general case follows from the bicomplex argument indicated by the definition 4.2. ■

We now undertake a detailed assessment of the homotopy type of $V^X(Y)$. We assume J is discrete, so that $I = \coprod_J \{(j,0) \to (j,1)\}$. In the classical case, as studied by Quillen [16, II§4], constructions analogous to the Stover resolution tend to provide s-free morphisms of a particular sort. (See the next paragraph for s-free.) This is not the case here, but it serves as a useful guide.

Let $\Delta_+ \subseteq \Delta$ be the subcategory of the ordinal number category with the same objects but only those morphisms the epimorphisms. Let $s_+\mathcal{C}$ be the category of functors $(\Delta_+)^{op} \to \mathcal{C}$. Thus an object in $s_+\mathcal{C}$ is a "degeneracy diagram". A free object in $s_+\mathcal{C}$ is one obtained by left Kan extension from a discrete diagram over the ordinal numbers. An object in $s\mathcal{C}$ is s-free if it is free when regarded as an object in $s_+\mathcal{C}$.

We note that if $\mathcal{C}$ is a closed model category, then $s_+\mathcal{C}$ acquires a closed model category structure with fibrations and weak equivalences defined level-wise and cofibrations exactly the Reedy cofibrations.

Define an augmented object $Z \in \mathcal{C}$ to be *contractible* if Z is cofibrant and if the augmentation $Z \to \phi$ is a weak equivalence. We have deliberately made the augmentation part of the data of contractibility. Note that the unique map $\phi \to Z$ is a homotopy equivalence; in effect, we have fixed the homotopy inverse.

Definition 4.4 *An object $X \in s\mathcal{C}$ is $\mathcal{A}$-free if there is a cofibrant, level-wise contractible object $Z \in s_+\mathcal{C}$ and a cofibration in $s_+\mathcal{C}$*

$$Z \to X$$

so that if $\bar{X}$ is defined by the push-out diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ \phi \arrow[r] & \bar{X} \end{tikzcd}$$

then $\bar{X}$ is s-free on

$$\{U_n\}_n = \{\sqcup_j(\sqcup_{I_{n,j}} A_j)\}_n$$

for some indexing sets $I_{n,j}$.

This definition has the following implications.

4.5.1) If X is homotopically $\mathcal{A}$-free then X and $\bar{X}$ are Reedy cofibrant.

4.5.2) The map $X_n \to \bar{X}_n$ is a homotopy equivalence for all $n \geq 0$.

4.5.3) Let $U_n \to X_n$ be any lift of the inclusion $U_n \to \bar{X}_n$ up to homotopy. Then

$$L_n X \sqcup U_n \to X_n$$

is a homotopy equivalence. Indeed, there is a homotopy commutative diagram

$$\begin{tikzcd} L_n X \sqcup U_n \arrow[r] \arrow[d] & X_n \arrow[d] \\ L_n \bar{X} \sqcup U_n \arrow[r, "\cong"] & \bar{X}_n \end{tikzcd}$$

and the vertical maps are homotopy equivalences between cofibrant objects.

We can extend this definition to maps:

Definition 4.6 *A morphism $X \to Y$ in $s\mathcal{C}$ is homotopically $\mathcal{A}$-free if, as a morphism in $s_+\mathcal{C}$, it is isomorphic to the inclusion of a summand*

$$X \longrightarrow X \sqcup Y_0$$

with Y_0 homotopically $\mathcal{A}$-free.

We shall prove that Stover resolutions are homotopically $\mathcal{A}$-free. This turns on a technical result, which we now state. Let $\Delta_0 \subset \Delta$ be the sub-category of the ordinal number category with the same objects, but the morphisms $\psi : [n] \to [m]$ must have $\psi(0) = 0$. Thus Δ_0 does not have d^0. Let $s_0\mathcal{C}$ be the functors from Δ_0^{op} to $\mathcal{C}$. The notions of homotopically $\mathcal{A}$-free objects and morphisms certainly apply to $s_0\mathcal{C}$. Let $I = J \times \{0 \to 1\}$ with J discrete.

Proposition 4.7 *Let $X \in s_0\mathbf{Sets}^{I^{\mathrm{op}}}$. Then $\overline{S} \otimes_I X \in s_0\mathcal{C}$ is homotopically $\mathcal{A}$-free.*

We postpone the proof to record:

Proposition 4.8 *Let $A \to Y$ be a morphism in $s\mathcal{C}$. Then $A \to V^A Y$ is homotopically $\mathcal{A}$-free.*

Proof. We recall that $V^X : X/\mathcal{C} \to X/\mathcal{C}$ is the composite functor

$$V^X(Y) = X \sqcup \bar{A} \otimes_I \mathrm{Hom}_{\mathcal{C}}(\bar{A}, Y).$$

Then

$$V_n^X(Y) = (V^X)^{n+1}Y = X \sqcup \bar{A} \otimes_I \mathrm{Hom}_{\mathcal{C}}(\bar{A}, (V^X)^n Y).$$

Thus, $X \to V^X(Y)$, regarded as an object in $X/s_0\mathcal{C}$ can be written

$$V^X(Y) = X \sqcup \bar{A} \otimes_I \mathrm{Hom}(\bar{A}, (V^X)^{\cdot} Y).$$

For general $X \to Y$ in $s\mathcal{C}$, there is an isomorphism in $X/s_0\mathcal{C}$ between $X \to V^X(Y)$ and

$$V^X(Y) \cong X \sqcup \bar{A} \otimes_I \mathrm{diag}\ \mathrm{Hom}(\bar{A}, (V^X)^{\cdot}(Y)_{\cdot})).$$

The result now follows from the previous Proposition. ■

We begin the proof of Proposition 4.7 with the following preliminary lemma.

Lemma 4.9 *1.) Let $X \to Y$ be any level-wise inclusion in s_0***Sets***. Then*

$$X_n \sqcup_{L_n X} L_n Y \to Y_n$$

is an injection for all $n \geq 0$.

2.) Any lifting problem

$$\begin{tikzcd} & X \arrow[d, "q"] \\ Z \arrow[r, "f"] & Y \end{tikzcd}$$

*in s_0***Sets** *with the image of f contained in the image of q has a solution in s_+***Sets**.

Proof. The first statement is an exercise in simplicial identities; indeed, we take the corresponding result for simplicial sets for granted and, upon checking the details, one sees that one uses only d_i, $i \geq 1$. For the second part, one inductively defines lifts of $f : Z_n \to Y_n$ to maps $g : Z_n \to X_n$ using the fact, implied by part 1, that $L_n Z \to Z_n$ is an injection. ■

4.10. Proof of Proposition 4.7. Since J is a discrete category

$$I \cong \bigsqcup_{j \in J} \{(j,0) \to (j,1)\}.$$

Thus $X \in \mathbf{Sets}^{I^{op}}$ is naturally isomorphic to $\sqcup_{j \in J} X_j$ where

$$X_j(i,t) = \begin{cases} X(j,t), & j = i; \\ \phi, & j \neq i. \end{cases}$$

This decomposition yields a natural isomorphism

$$\bar{A} \otimes_I X \cong \bigsqcup_{j \in J} \{A \to CA\} \otimes_{\{0 \to 1\}} X_j.$$

Hence we may assume that J has a single object (and morphism) and that $I \cong \{0 \to 1\}$.

Let $X \in s_0\mathbf{Sets}^{I^{op}}$; thus X is a morphism $X(1) \to X(0)$ is $s_0\mathbf{Sets}$. Let $Y(0) \subseteq X(0)$ be the subobject in $s_0\mathbf{Sets}$ of elements x so that $p^{-1}(x)$ is not empty. By the previous lemma there is a lift

$$\begin{tikzcd} & X(1) \arrow[d] \\ Y(0) \arrow[ur] \arrow[r] & X(0) \end{tikzcd}$$

in the category $s_+\mathbf{Sets}$. Define $Y \in s_0\mathbf{Sets}^{I^{op}}$ to be

$$Y = \{Y(0) \xrightarrow{1} Y(0)\}.$$

These considerations then give a morphism $Y \to X$ in $s_+\mathbf{Sets}^{I^{op}}$. This gives a morphism in $s_+\mathcal{C}$

$$Z = \bar{A} \otimes_I Y \to \bar{A} \otimes_I X.$$

For each n,

$$Z_n = [\bar{A} \otimes_I Y]_n \cong \bar{A} \otimes_I (Y_n) \cong \bigsqcup_{Y(0)_n} CA,$$

and the canonical augmentation $CA \to \phi$ induces an augmentation $Z_n \to \phi$ which is a weak equiavalence. Since, by the previous lemma $L_nY(0) \to Y(0)_n$ is an injection,

$$L_nZ = L_n(\bar{A} \otimes_I Y) \cong \bar{A} \otimes_I L_nY \to [\bar{A} \otimes_I Y]_n$$

is the inclusion of a coproduct summand with the other summand cofibrant. Hence Z is Reedy cofibrant. Similarly, by the lemma

$$Y_n \sqcup_{L_nY} \to X_n$$

is an inclusion, so $Z \to \bar{A} \otimes_I X$ is a Reedy cofibration. Thus it remains to discuss what happens when we collapse out Z.

For all n and all $y \in X(0)_n$, we have an I^{op} diagram

$$X_y = p^{-1}(y) \to \{y\}.$$

This yields a decomposition

$$\bar{A} \otimes_I X_n \cong \bigsqcup_{y \in X(0)_n} \bar{A} \otimes_I X_y.$$

There is pushout-diagram

$$\begin{tikzcd} \sqcup_{p^{-1}(y)} A \arrow[r] \arrow[d] & A \arrow[d] \\ \sqcup_{p^{-1}(y)} CA \arrow[r] & \bar{A} \otimes_I X_y. \end{tikzcd}$$

Define a pointed space K_y by the push-out diagram

$$\begin{tikzcd} \bigvee_{p^{-1}(y)} S^0 \arrow[r] \arrow[d] & S^0 \arrow[d] \\ \bigvee_{p^{-1}(y)} [0,1] \arrow[r] & K_y \end{tikzcd}$$

Recall that $1 \in [0,1]$ is the basepoint. Then $K_y \wedge A \cong \bar{\mathcal{A}} \otimes_I X_y$. If $p^{-1}(y) = \phi$, then $K \cong S^0$ and $\bar{\mathcal{A}} \otimes_I X_y = A$. If $p^{-1}(y) \neq \phi$, then $Y_y = Y \cap X_y = \{\{y\} \to \{y\}\}$ defines a distinguised element in $p^{-1}(y)$ and hence and inclusion $[0,1] \to K_y$.This yields a push-out diagram

$$\begin{tikzcd} {[0,1]} \arrow[r] \arrow[d] & K_y \arrow[d] \\ S^1 \arrow[r] & \bigvee_{p^{-1}(y)} S^1 \end{tikzcd}$$

Collapsing out this distinguished S^1 gives a push-out diagram

$$\begin{tikzcd} {[0,1]} \arrow[r] \arrow[d] & K_y \arrow[d] \\ * \arrow[r] & {[\bigvee_{p^{-1}(y)} S^1]/S^1} \end{tikzcd}$$

Applying $(\cdot) \wedge A$ gives a push-out square

$$\begin{tikzcd} \bar{\mathcal{A}} \otimes_I Y_y \arrow[r] \arrow[d] & \bar{\mathcal{A}} \otimes_I X_y \arrow[d] \\ \phi \arrow[r] & {[(\sqcup_{p^{-1}(y)} \Sigma A)/\Sigma A].} \end{tikzcd}$$

This is all that is required, since the original $\mathcal{A}$ (as in the statement of the proposition) was closed under suspension. ∎

Remark 4.11 Note that the analysis at the end of the previous proof shows that $(\bar{\mathcal{A}} \otimes_I X)_n$ is a coproduct of objects of the form $K \wedge Aj$ where $K = S^0$ or K fits into a push-out diagram of the form

$$\begin{tikzcd} {[0,1]} \arrow[r] \arrow[d] & K \arrow[d] \\ * \arrow[r] & \bigvee_D S^1 \end{tikzcd}$$

for some index set D.

5 $\mathcal{A}$-cofibrations and $\mathcal{A}$-cofibrant objects.

We use the Stover resolution of the previous section and the simplicial technology of the appendix on resolutions to give working models for cofibrations and cofibrant objects in the $\mathcal{A}$-resolution model category structure. In this section we assume that J is discrete – so that $\mathcal{A}$ is a set of objects $\{Aj\}$ – and that $\{Aj\}$ is closed under suspension.

We begin with the following two lemmas which will be used to test for $\mathcal{A}$-fibrations and $\mathcal{A}$-trivial fibrations.

Lemma 5.1 *Let $q : X \to Y$ be a Reedy fibration in $s\mathcal{C}$. Then q is an $\mathcal{A}$-fibration if and only if for all $j \in J$ the induced map of simplicial groups.*

$$q_* : [Aj, X] \to [Aj, Y]$$

is a fibration of simplicial groups.

Proof. We need to show that $\mathrm{map}(Aj, X) \to \mathrm{map}(Aj, Y)$ is a fibration of simplicial sets if and only if $[Aj, X] \to [Aj, Y]$ is a fibration of simplicial groups; or, by definition, that

$$\mathrm{Hom}_{\mathcal{C}}(Aj, X_n) \to \mathrm{Hom}_{\mathcal{C}}(Aj, Y_n \times_{M_n^K Y} M_n X) \tag{5.2.1}$$

is onto for all n if and only if

$$[Aj, X_n] \to [Aj, Y_n] \times_{M_n^k[Aj,Y]} M_n^k[Aj, X] \tag{5.2.2}$$

is onto. Since $X_n \to Y_n \times_{M_n Y} M_n X$ is a fibration the map of (5.2.1) is onto if and only if

$$[Aj, X_n] \to [Aj, Y_n \times_{M_n^k Y} M_n^k X] \tag{5.2.3}$$

is onto. In light of this observation and Lemma 1.8, we may assume X and Y are Reedy fibrant. Lemma 1.11 implies that for all $j \in J$, $[Aj, M_n^k X] \cong M_n^k[Aj, X]$. The set $\{Aj\}$ is closed under suspension and every simplicial group is fibrant, $[\Sigma Aj, X_n] \to [\Sigma Aj, M_n^k X]$ is onto. So the Barratt-Puppe sequence (1.10) and Lemma 1.11 supply isomorphisms

$$\begin{aligned}
[Aj, Y_n \times_{M_n^k Y} M_n^k X] &\cong [Aj, Y_n] \times_{[Aj, M_n^k Y]} [Aj, M_n^k X] \\
&\cong [Aj, Y_n] \times_{M_n^k[Aj,Y]} M_n^k[Aj, X].
\end{aligned}$$

■

Lemma 5.3 *Let $q : X \to Y$ be an $\mathcal{A}$-trivial fibration. Then the natural map*

$$[Aj, Y_n \times_{M_n Y} M_n X] \to [Aj, Y_n] \times_{M_n[Aj,Y]} M_n[Aj, X]$$

is an isomorphism.

Proof. This is proved by induction on n. If $n = 0$, we are asserting that $[Aj, Y_0] \to [Aj, Y_0]$ is an isomorphism, which is a tautology. For $n \geq 1$, we use that there is a push-out diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "d^0"] \arrow[d] & \Delta_0^{n+1} \arrow[d] \\
\Delta^n \arrow[r, "d^0"'] & \partial\Delta^{n+1}
\end{tikzcd}$$

and, hence, a natural pull-back diagram

$$\begin{tikzcd}
M_{n+1}X \arrow[r] \arrow[d] & X_n \arrow[d] \\
M_{n+1}^0 X \arrow[r] & M_n X.
\end{tikzcd}$$

This leads to a pull-back diagram

$$\begin{tikzcd}
Y_{n+1} \times_{M_{n+1}Y} M_{n+1}X \arrow[r] \arrow[d] & X_n \arrow[d] \\
Y_{n+1} \times_{M_{n+1}^0 Y} M_{n+1}^0 X \arrow[r] & Y_n \times_{M_n X} M_n Y.
\end{tikzcd}$$

If $X \to Y$ is an $\mathcal{A}$-trivial fibration, then

$$[\Sigma Aj, X_n] \to [\Sigma Aj, Y_n \times_{M_n X} M_n Y]$$

is surjective for all i, since $X \to Y$ has the right lifting property with respect to $Aj \otimes \partial\Delta^n \to Aj \otimes \Delta^n$ and the set $\{Aj\}$ is closed under suspension. Thus the Puppe sequence (1.10), Lemma 1.11 and the inductive hypotheses show that if the result holds for n it holds for $n+1$. ■

Proposition 5.4 *Any homotopically $\mathcal{A}$-free map is an $\mathcal{A}$-cofibration.*

Proof. Let us write $L_n(f) = L_n B \sqcup_{L_n A} A_n$. Then the inclusion of relative skeleta fits into a push-out diagram

(5.5)
$$\begin{tikzcd}
L_n(f) \otimes \Delta^n \sqcup_{L_n(f)\otimes\partial\Delta^n} B_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & sk^A_{n-1}B \arrow[d] \\
B_n \otimes \Delta^n \arrow[r] & sk^A_n B.
\end{tikzcd}$$

Given a lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "q"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

with q an $\mathcal{A}$-trivial fibration, one wants to use that $sk^A_{-1}B = A$ and extend liftings inductively over the skeleta.. Having lifted $sk^A_{n-1}B$, the push-out diagram and (5.5) and adjointness shows that one must solve the lifting problem

$$\begin{tikzcd}
L_n(f) \arrow[r] \arrow[d, "s"'] & X_n \arrow[d, "d"] & & \\
B_n \arrow[r] \arrow[ur, dashed] & Y_n \times_{M_n Y} M_n X & = & M_n(q).
\end{tikzcd}$$

Now s and d are, respectively, a fibration and cofibration in $\mathcal{C}$, whence

$$\mathrm{map}_{\mathcal{C}}(B_n X_n) \to \mathrm{map}_{\mathcal{C}}(B, M_n(q)) \times_{\mathrm{map}_{\mathcal{C}}(L_n(f), M_n(q))} \mathrm{map}_{\mathcal{C}}(L_n(f), X_n)$$

is a fibration. Thus we need only show that this morphism of mapping space is surjective on π_0. This is equivalent, by (4.5.3), to showing every lifting problem of the form

$$\begin{tikzcd}
L_n(f) \arrow[r] \arrow[d] & X_n \arrow[d, "d"] \\
L_n(f) \sqcup U_n \arrow[r] \arrow[ur, dashed] & M_n(q)
\end{tikzcd}$$

has a solution. Because $X_n \to M_n(q)$ is a fibration, this is equivalent to the assertion that

$$[Aj, X_n] \to [Aj, M_n(q)]$$

is onto for all i. By the previous lemma, this is equivalent to asserting

$$[Aj, X_n] \to [Aj, Y_n] \times_{M_n^k[Aj,Y]} M_n^k[Aj, X]$$

is onto. This amounts to asserting that $[Aj, X] \to [Aj, Y]$ is a trivial fibration of simplicial groups. But by Lemma 5.1, $[Aj, X] \to [Aj, Y]$ is a fibration of simplicial groups and Proposition 3.14, $[Aj, X] \to [Aj, Y]$ is a weak equivalence of simplicial groups. ■

An immediate corollary of this result and of the existence of the Stover resolution is the following. For the proof combine Proposition 5.4, Lemma 4.3, and Proposition 4.8.

Proposition 5.6 *Any morphism $A \to B$ in $s\mathcal{C}$ can be naturally factored as*

$$A \xrightarrow{j} X \xrightarrow{q} B$$

where q is an $\mathcal{A}$-equivalence and j is an $\mathcal{A}$-cofibration and homotopically $\mathcal{A}$-free. ■

Proposition 5.7 *Any morphism $A \to B$ in $s\mathcal{C}$ can be naturally factored*

$$A \xrightarrow{i} Y \xrightarrow{q} B$$

where q is a Reedy trivial fibration and i can be factored

$$A \xrightarrow{j} X \xrightarrow{f} Y$$

where j is homotopically $\mathcal{A}$-free and f is a Reedy trivial cofibration. ■

Note that Reedy trivial cofibrations are $\mathcal{A}$-trivial cofibrations since every $\mathcal{A}$-fibration is a Reedy fibration and Corollary 2.8. Also any Reedy trivial fibration is an $\mathcal{A}$-trivial fibration by Lemma 1.5 and an adjointness argument. Thus Proposition 5.7 implies

Proposition 5.8 *Any $\mathcal{A}$-cofibration $A \to B$ is a retract of a composition*

$$A \xrightarrow{j} X \xrightarrow{f} Y$$

where j is a homotopically $\mathcal{A}$-free map and f is a Reedy trivial cofibration. ■

Remark 5.9 The map $i : A \to Y$ constructed in Proposition 5.7 has the following properties: a) it is a Reedy cofibration and b) for each $n \geq 0$ there is an object $W_n \in \mathcal{C}$ isomorphic to the coproduct of the Aj so that the cofibration $A_n \sqcup_{L_n A} L_n Y \to Y_n$ can be extended to a homotopy equivalence

$$A_n \sqcup_{L_n A} L_n Y \sqcup W_n \to Y_n.$$

This is a consequence of Remark 4.5.3 and the fact that, since every object of $\mathcal{C}$ is fibrant, any trivial cofibration is a homotopy equivalence. Conversely, any morphism satisfying a) and b) is an $\mathcal{A}$-cofibration, by the same argument that proved Proposition 5.4.

Remark 5.10 If the applications require it, we can refine the notion of homotopically $\mathcal{A}$-free. Consider $X \in s\mathcal{C}$ and let $\epsilon : VX \to X$ be the Stover resolution. Then VX is homotopically $\mathcal{A}$-free and ϵ is a weak equivalence. Now let $W(X) = sd'X$, where sd' is the "big sub-division operator" of Corollary A.8. Then Corollary A.8 and Lemma A.10 imply there is a weak equivalence

$$W(X) \to V(X)$$

and Proposition A.7 says that $W(X)$ is $\mathcal{A}$-cofibrant. Finally, Proposition A.6 says $W(X)$ is free as a Δ_+^{op} diagram. In fact, using Proposition A.6 and Remark 4.11, $W(X)$ is s-free on $\{V_n\}$ where V_n

is a coproduct of objects of the form $Aj \wedge K$ where $K \in \mathbf{Top}_*$ fits into a push-out diagram

$$\begin{tikzcd} {[0,1]} \arrow[r, "y"] \arrow[d] & K \arrow[d] \\ * \arrow[r] & \bigvee_A S^1 \end{tikzcd}$$

with y a cofibration and A some, potentially very large, indexing set.

6 Simplicial structured ring spectra.

This section is devoted to examining the special case of spectra: we are interested in $\mathcal{A}$-resolutions in $s\mathcal{C}$ where $\mathcal{C} = \mathcal{S}$, the category of spectra and $\mathcal{C} = \mathcal{S}^T$, where T is an operad over the linear isometries operad.

We begin with the case of spectra and $\mathcal{A} = \{\Sigma^i DE_\alpha\}$ where $E = \operatorname{colim}_\alpha E_\alpha$ is some ring spectrum satisfying Adams's condition. See Example 2.1.2; however, we regard $\mathcal{A}$ as a discrete diagram. For ease of notation we relabel the spectra of $\mathcal{A}$ by Aj and suspress, as much as possible the internal suspension.

Note that $\mathcal{S}$ is a pointed category: the terminal object is isomorphic to the initial object.

Proposition 6.1 *Let $X \in s\mathcal{S}$ be Reedy cofibrant. Then for all $Aj \in \mathcal{A}$,*

$$\pi_n^\natural(X, Aj) \cong [\Sigma^n Aj, |X|].$$

We will prove more. If F_* is any homology theory and $X \in s\mathcal{S}$ is Reedy cofibrant, then there is a well-known spectral sequence for X Reedy cofibrant.

(6.2) $$\pi_p F_q X \Rightarrow F_{p+q}|X|.$$

If we set $F_* = (E_\alpha)_*$, then

$$(E_\alpha)_q X = [\Sigma^q DE_\alpha, X]$$

and this spectral sequence has the same E_2 term as the one of (4.13). We wish to show that they are isomorphic spectral sequences. In particular, this will imply 6.1.

For $X \in s\mathcal{S}$ Reedy cofibrant, let $N_n X$ be defined by the cofiber sequence

$$L_n X \to X_n \to N_n X.$$

Because there is a push-out diagram

$$\begin{tikzcd} \Delta^n_+ \wedge L_n X \sqcup_{\partial\Delta^n_+ \wedge L_n X} \partial\Delta^n_+ \wedge S_n \arrow[r] \arrow[d] & sk_{n-1}|X| \arrow[d] \\ \Delta^n_+ \wedge X_n \arrow[r] & sk_n|X| \end{tikzcd}$$

there is a cofiber sequence

(6.3) $$sk_{n-1} X \to sk_n X \to \Sigma^n N_n X.$$

Then one proves that if F_* is a homology theory,

$$F_* N_n X \cong N_n F_* X$$

where $N_* F_* X$ is the normalization of the simplicial abelian group $F_* X$ and, further, that

$$N_n F_q X \cong F_{q+n} \Sigma^n N_n X \to F_{q+n-1} sk_{n-1} X \to F_{q+n-1} \Sigma^{n-1} N_{n-1} X \cong N_{n-1} F_q X$$

is the boundary map, up to sign. The exact couple obtained by applying F_* to (6.2) now yields the spectral sequence (6.2). To relate this to the spectral sequence of section3, we need the next comparison result. The objects $C_n X$ and $Z_n X$ were defined in section 3.

We need some notation. Since we have functorial factorizations in spectra, any cofiber sequence $A \xrightarrow{i} B \to B/C$ yields a new functorial cofiber sequence

$$B \to B \sqcup_A C(A) \to B \sqcup_A C(A)/B,$$

where $C(A)$ is the cone on A. Furthermore, $B \sqcup_A C(A)$ is weakly equivalent to B/A and, if A is cofibrant, $B \sqcup_A C(A)/B$ is naturally weakly equivalent to ΣA. We abbreviate this statement by saying $B \to B/C \to \Sigma A$ is a homotopy cofiber sequence.

Proposition 6.4 *Let $\mathcal{C}$ be pointed. Let $X \in s\mathcal{C}$ be Reedy fibrant and Reedy cofibrant. Then there are natural maps*

$$\begin{array}{lcl} \Sigma^n Z_n X & \to & |sk_{n-1} X| = sk_n |X| \text{ and} \\ \Sigma^n C_n X & \to & \Sigma^n N_n X \end{array}$$

so that there is a commutative diagram

$$\begin{tikzcd} \Sigma^n Z_n X \arrow[r] \arrow[d] & \Sigma^n C_n X \arrow[r] \arrow[d] & \Sigma^n Z_{n-1} X \arrow[d] \\ sk_n |X| \arrow[r] & \Sigma^n N_n X \arrow[r] & \Sigma sk_n |X|. \end{tikzcd}$$

The bottom row is the homotopy cofiber sequence obtained by $sk_{n-1}|X| \to sk_n |X|$.

Proof. Let $K \in \mathbf{S}$ be pointed and $Z \in \mathcal{C}_* = \mathcal{C}$. Then $C_K X \in \mathcal{C}_*$ is defined by the formula

$$\mathrm{Hom}_{s\mathcal{C}}(Z \wedge K, X) \cong \mathrm{Hom}_{\mathcal{C}_*}(Z, C_K X).$$

Setting $Z = C_K X$ we obtain an evaluation map

$$C_K X \wedge K \to X$$

which is as natural as possible in K: if $f : K \to L$ is a pointed map, the following diagram commutes

(6.5)
$$\begin{tikzcd} C_L X \wedge K \arrow[r] \arrow[d] & C_L X \wedge L \arrow[d] \\ C_K X \wedge K \arrow[r] & X. \end{tikzcd}$$

Notice that $sk_n(Z \wedge K) = Z \wedge sk_n K$. Setting $K = \Delta^n/\partial\Delta^n$ yields a map

$$Z_n X \wedge \Delta^n/\partial\Delta^n \cong sk_n(Z_n X \wedge \Delta^n/\partial\Delta^n) \to sk_n X.$$

Taking the realization yields $\Sigma^n Z_n X \to |sk_n X|$. Now $K = \Delta^n/\Delta^n_0$ yields a diagram

$$\text{(6.6)} \qquad \begin{tikzcd} C_n X \wedge \Delta^{n-1}/\partial\Delta^{n-1} \arrow[r] \arrow[d, "\psi"] & C_n X \wedge \Delta^n/\Delta^n_0 \arrow[r] \arrow[d] & C_n X \wedge \Delta^n/\partial\Delta^n \arrow[d, "\varphi"] \\ sk_{n-1}X \arrow[r] & sk_n X \arrow[r] & sk_n X/sk_{n-1}X \end{tikzcd}$$

Taking the realization of φ yields the map $\Sigma^n C_n X \to \Sigma^n N_n X$. There is also a map of homotopy cofiber sequence

$$\begin{tikzcd} C_n \wedge |\Delta^n/\Delta^0| \arrow[r] \arrow[d] & \Sigma^n C_n X \arrow[r] \arrow[d] & \Sigma\Sigma^{n-1} C_n X \arrow[d] \\ |sk_n X| \arrow[r] & \Sigma^n N_n X \arrow[r] & \Sigma|sk_{n-1}X|. \end{tikzcd}$$

The diagram 6.5 where $f = d^0 : \Delta^{n-1}/\partial\Delta^{n-1} \to \Delta^n/\Delta^n_0$ shows that the map ψ of (6.6) factors

$$C_n X \wedge \Delta^{n-1}/\partial\Delta^{n-1} \xrightarrow{d_0 \wedge 1} Z_{n-1}X \wedge \Delta^{n-1}/\partial\Delta^n \to sk_{n-1}X.$$

The result follows. ■

Proposition 6.7 *Let $X \in s\mathcal{S}$ be Reedy cofibrant and Reedy fibrant. Then the maps $\Sigma^n Z_n X \to |sk_n X|$ and $\Sigma^n C_n X \to \Sigma^n N_n X$ are stable weak equivalences.*

Proof. A standard calculation (see [3] for the result is the cosimplicial setting) implies that for any homology theory F_* (including stable homotopy π_*), the natural map $L_n F_* X \to F_* L_n X$ is an isomorphism. It follows that $F_* N_n X \cong N_n F_* X$. Similarly, Lemma 3.8 implies that $(E_\alpha)_* C_n X \cong C_n(E_\alpha)_* X$. The assertion that $\Sigma^n C_n X \to \Sigma^n N_n X$ is a weak equivalence follows from the standard calculation that

$$C_n(E_\alpha)_* X \cong \bigcap_{i=1}^{n} \ker\{d_i\} \to (E_\alpha)_* X_n / \operatorname{Im}(s_0)_* + \cdots + \operatorname{Im}(s_{n-1})_* \cong N_n(E_\alpha)_* X$$

is an isomorphism. Since $Z_0 X = X_0 = |sk_0 X|$, the assertion that $\Sigma^n Z_n X \to |sk_n X|$ is a stable equivalence follows by induction and Proposition 6.4. ■

Corollary 6.8 *The maps of Proposition 6.4 induce an isomorphism of spectral sequences*

$$\begin{tikzcd} \pi_p(X, \Sigma^q DE_\alpha) \arrow[r, Rightarrow] \arrow[d] & \pi^\natural_{p+q}(X, DE_\alpha) \arrow[d] \\ \pi_p(E_\alpha)_q X \arrow[r, Rightarrow] & (E_\alpha)_{p+q}|X| \cong \pi_{p+q}(|X|, DE_\alpha). \end{tikzcd}$$

Proof. Proposition 6.7 supplies an isomorphism of E_1 terms. ■

We are next interested in the case $\mathcal{C} = \mathcal{S}^T$ where T is an operad over the linear isometries operad. Our goal is to understand the homotopy type of $|X|$ where $X \in s\mathcal{S}^T$ is Reedy cofibrant. The geometric realization $|X|$ can either be computed as a coequalizer in $\mathcal{S}^T$ or in $\mathcal{S}$. See [9]. Since $|X|$ can be computed in $\mathcal{S}^T$, a weak equivalence between Reedy cofibrant objects in $\mathcal{S}^T$ passes to a weak equivalence in $\mathcal{S}^T$. However, even if X is Reedy cofibrant in $s\mathcal{S}^T$, X might not be Reedy cofibrant in $s\mathcal{S}$, because cofibrations in $\mathcal{S}^T$ need not be cofibrations in $\mathcal{S}$. Thus Proposition 6.1 does not immediately help in determining the homotopy type of $|X|$. This leads to the technical Lemmas 6.9 and 6.10 below.

We chose to build our resolution model category for $s\mathcal{S}^T$ on the diagram $T(\mathcal{A}) : I \to s\mathcal{S}^T$ where $S : I \to \mathcal{S}$ is the diagram $\{\Sigma^i DE_\alpha\}$ associated to the Adams-type ring spectrum E.

Lemma 6.9 *Let $X \in s\mathcal{S}^T$. Then there is a natural $T(\mathcal{A})$-equivalence $W(X) \to X$ so that $W(X)$ is $T(\mathcal{A})$-cofibrant and so that, as a diagram in $s_+\mathcal{S}^T$, there is a natural isomorphism*

$$W(X) \cong T(Z_X)$$

where $Z_X \in s_+\mathcal{S}$ is free as a $\Delta_+^{\rm op}$ diagram on a set of cofibrant spectra.

Proof. Let $W(X) \to X$ be the $T(\mathcal{A})$-equivalence supplied by Remark 5.10. Then $W(X)$, as a $\Delta_+^{\rm op}$ diagram, is free on V_n, where V_n is a coproduct in $\mathcal{S}^T$; in fact, there are sets $A(n, \alpha, k)$ so that

$$V_n \cong \bigsqcup_{(\alpha,k)} \bigsqcup_{X \in A(n,\alpha,k)} T(\Sigma^k DE_\alpha) \wedge K_x$$

where $K_x \in \mathbf{Top}_*$ is a CW complex homotopy equivalent to a wedge of circles. There is a push-out in $\mathcal{S}^T$

$$\begin{tikzcd}
T(\Sigma^k DE_\alpha)\Box * \arrow[r] \arrow[d] & S^0 \arrow[d] \\
T(\Sigma^k DE_\alpha)\Box K_x \arrow[r] & T(\Sigma^k DE_\alpha) \wedge K_x
\end{tikzcd}$$

since S^0 is the initial object; hence

$$T(\Sigma^k DE_\alpha) \wedge K_x \cong T(\Sigma^k DE_\alpha \wedge K_x)$$

and

$$V_n \cong T(\bigsqcup_{(\alpha,k)} \bigsqcup_{A(n,\alpha,k)} \Sigma^k DE_\alpha \wedge K_x). \qquad \blacksquare$$

The adjunct to this result is the following.

Lemma 6.10 *Let $X \in s_+\mathcal{S}$ be free on $\{Z_n\}$. If each Z_n is a cofibrant and T is cofibrant as an operad, then $T(X) \in s_+\mathcal{S}$ is free on a set of cofibrant spectra.*

Proof. We first describe how to decompose $T(Y)$ when Y is a wedge. Suppose $Y = \bigvee_I Y_i$ where I is some indexing set. Let I^n/Σ_n be the orbits and $q : I^n \to I^n/\Sigma_n$ the quotient map. Write $\vec{i} = (i_1, \ldots, i_n)$ for a vector in I^n. Then if

$$T(Y) \cong \bigvee_n \bigvee_{x \in I^n/\Sigma_n} (T_n)_+ \wedge_{\Sigma_n} (\bigvee_{\vec{i} \in q^{-1}(x)} Yi_1 \wedge \cdots \wedge Yi_n).$$

If $X \in s_+\mathcal{S}$ is free on $\{Z_j\}$, then

$$X_k \cong \bigvee_{[k] \to [j]} Z_k \cong \bigvee_{I_k} Z_k$$

where I_k is the objects of $[k]/\Delta_+$. One now calculates that $T(X)$ is free on

$$\overline{Z}_k = \bigvee_n \bigvee_{x\in B_n} (T_n)_+ \wedge_{\Sigma_n} (\bigvee_{\vec{i}\in q^{-1}(x)} Zi_1 \wedge \cdots \wedge Zi_n)$$

where

$$B_n \subseteq I_k^n/\Sigma_n$$

is the subset of orbits $x = [([k] \to [i_1], \ldots , [k] \to [i_n])]$ where $i_t = k$ for at least one t. To finish the argument, use that if $Y \in S_{\Sigma^n}$ is cofibrant, and T is cofibrant, then $(T_n)_+ \wedge_{\Sigma_n} Y$ is cofibrant. ■

We now supply a result stipulating how geometric realization behaves with respect to $T(\mathcal{A})$-equivalences.

In preparation, we make the following preliminary remark. We have regarded the category $\mathcal{S}$ (or $\mathcal{S}^T$) as enriched as a topological model category; that is, $\mathcal{S}$ is enriched over topological spaces in a strong way and the model category structure satisfies the "corner axiom" – the analog of SM7 – over spaces. But any such category is equally a simplicial model category: If $K \in \mathbf{S}$ define $X \otimes_{\mathcal{S}} K = X\Box|K|$ and $\hom_{\mathcal{S}}(K, X) = X^{|K|}$. These are adjoint and define an action of $\mathbf{S}$ on $\mathcal{S}$; indeed,

$$(X \otimes_{\mathcal{S}} K) \otimes_{\mathcal{S}} L \cong X \otimes_{\mathcal{S}} (K \times L) \text{ and } X \otimes_{\mathcal{S}} * \cong X.$$

There is also the simplicial set of maps $\mathrm{map}_{\mathcal{S}}(X, Y)$ with

$$\text{(6.11)} \qquad \begin{aligned} \mathrm{map}_{\mathcal{S}}(X, Y)_n &\cong \mathrm{Hom}_{\mathcal{S}}(X\Box|\Delta^n|, Y) \\ &\cong \mathrm{Hom}_{\mathcal{S}}(X \otimes_{\mathcal{S}} \Delta^n, Y). \end{aligned}$$

Note that $\mathrm{map}_{\mathcal{S}}(X, Y)$ is the singular set of $\mathcal{S}(X, Y)$, and the functor $Y \mapsto \mathrm{map}_{\mathcal{S}}(X, Y)$ is right adjoint to $K \to X \otimes_{\mathcal{S}} K$. Finally, with these definitions, $\mathcal{S}$ satisfies SM7, because if $f : K \to L$ is a cofibration of simplicial sets $|f| : |K| \to |L|$ is a cofibration of topological spaces which is trivial if f is. Note that we are using, almost without saying it, the standard adjoint equivalence between the homotopy categories of spaces and simplicial sets.

Proposition 6.12 *The geometric realization functor*

$$|\cdot| : s\mathcal{S}^T \to \mathcal{S}^T$$

sends $T(\mathcal{A})$ equivalences between $T(\mathcal{A})$-cofibrant objects to weak equivalences.

Proof. We use the adjoint pair

$$\mathrm{Hom}_{s\mathcal{S}^T}(X, Y^{\Delta}) \cong \mathrm{Hom}_{\mathcal{S}^T}(|X|, Y).$$

We begin by investigating the behavior of $Y \mapsto Y^{\Delta}$ in the model category structure. First notice that if $X, Y \in \mathcal{S}$, then there is an isomorphism of simplicial sets

$$\mathrm{map}_{s\mathcal{S}}(X, Y^{\Delta}) \cong \mathrm{map}_{\mathcal{S}}(|X|, Y).$$

This is because $|X \otimes \Delta^n| \cong |X|\Box|\Delta^n| \cong |X| \otimes_{\mathcal{S}} \Delta^n$. Thus if $f : Y \to Z$ is a map in $s\mathcal{S}^T$ nd $T(Aj) \in \mathcal{S}^T$ is one of the spectra in $\mathcal{A}$, we have that

$$\mathrm{map}_{s\mathcal{S}^T}(T(Aj), Y^{\Delta}) \to \mathrm{map}_{s\mathcal{S}^T}(T(Aj), Z^{\Delta})$$

is isomorphic to

$$\mathrm{map}_{\mathcal{S}^T}(T(Aj), Y) \to \mathrm{map}_{\mathcal{S}^T}(T(Aj), Z),$$

since $|TAj| \cong T(Aj)$. The last isomorphism is because $T(Aj) \in s\mathcal{S}^T$ is constant. Thus f^Δ is a $T(\mathcal{A})$-fibration if f is a fibration and a $T(\mathcal{A})$-trivial fibration if f is a trivial fibration, by SM7, 2.4.2 and Lemma 2.10. It follows immediately by adjointness that $|\cdot| : (s\mathcal{S})^T \to \mathcal{S}^{|T|}$ preserves $T(\mathcal{A})$-cofibrations and $T(\mathcal{A})$-trivial cofibrations.

Let $g : A \to B$ be a $T(\mathcal{A})$-equivalence between $T(\mathcal{A})$-cofibrant objects. Factor g as $A \xrightarrow{j} C \xrightarrow{q} B$ where j is a $T(\mathcal{A})$-trivial cofibration and q is a $T(\mathcal{A})$-trivial fibration. Then we need only show $|q|$ is a weak equivalence. Since j is a cofibration and A is cofibrant, C is cofibrant and, hence, q is a $T(\mathcal{A})$-trivial fibration between $T(\mathcal{A})$-cofibrant objects. Thus q has a section $S : B \to C$ and there is a homotopy in $(s\mathcal{S})^T$

$$C \otimes \Delta^1 \to C$$

between the identity 1_C and sq. Taking realizations, and using that

$$|C \otimes \Delta^1| \cong |C| \otimes_{\mathcal{S}} \Delta^1 = |C|\Box|\Delta^1|$$

shows that $|q|$ is a homotopy equivalence. Thus q is a weak equivalence as required. ■

In the following result we write $\pi_*(\cdot,\cdot)_T$ to mean homotopy classes of maps in the category $\mathcal{S}^T$. Then there is a formula

$$\pi_*(X, T(Z))_T \cong \pi_*(X, Z)$$

where the unadorned homotopy groups are calculated in $\mathcal{S}$. Recall that the suspension operator in the category $\mathcal{S}^T$ has the property that is X is a spectrum, then $\Sigma T(X) \cong T(\Sigma X)$, so that for $q \geq 0$

$$\pi_*(X, \Sigma^q T(Z))_T \cong \pi_*(X, \Sigma^q Z).$$

The following result implies, by setting $Aj = S^0$, that for an object $X \in s\mathcal{S}^T$ which is $T(\mathcal{A})$ cofibrant, there is a spectral sequence

$$\pi_p \pi_q(X) \Longrightarrow \pi_{p+q}|X|.$$

Proposition 6.13 *Let T be any operad over the linear isometries operad and let $X \in s\mathcal{S}^T$ be $T(\mathcal{A})$-cofibrant. Then there is a spectral sequence*

$$\pi_p(X, \Sigma^q T(Aj))_T \Rightarrow \pi_{p+q}(|X|, T(Aj))_T.$$

Proof. One can equally assert there is a spectral sequence

$$\pi_p(X, \Sigma^q Aj) \Rightarrow \pi_{p+q}(|X|, Aj)$$

for $X \in s\mathcal{S}^T$ which is $T(\mathcal{A})$-cofibrant. First assume T is cofibrant as an operad, and let $W(X) \to X$ be the $T(\mathcal{A})$-equivalence supplied by Lemma 6.9. By the previous result $|W(X)| \to |X|$ is a weak equivalence. By Proposition 4.14,

$$\pi_q(W(X), \Sigma^q T(Aj))_T \cong \pi_q(X, \Sigma^q T(Aj))_T.$$

By Lemma 6.10, $W(X)$ is Reedy cofibrant in $\mathcal{S}$. Now apply Proposition 6.1.

For the general case, we will use the result of [GH], which claims that if $T_0 \to T$ is a weak equivalence of operads over the linear isometries operad and if X is cofibrant as a T_0 spectrum, then the canonical map $X \to T \otimes_{T_0} X$ is a weak equivalence of T_0 spectra. To use this, choose a weak equivalence of operads $T_0 \to T$ with T_0 cofibrant. The restriction function $\mathcal{S}^T \to \mathcal{S}^{T_0}$ has a left adjoint $X \mapsto T \otimes_{T_0} X$ and if $X \in \mathcal{S}^{T_0}$ is cofibrant $X \to T \otimes_{T_0} X$ is a weak equivalence.

By assumpiton, $X \in \mathcal{S}^T$ be $T(S)$-cofibrant. Regard X as an object in $\mathcal{S}^{T_0}$ and if $W(X) = W_{T_0}(X) \to X$ be the $T_0(\mathcal{A})$ equivalence supplied by Lemma 6.9. Then

$$\begin{aligned} \pi_q(W(X), \Sigma^q T_0(\mathcal{A}))_{T_0} &\cong \pi_q(X, \Sigma^q T_0(\mathcal{A}))_{T_0} \\ &\cong \pi_q(X, \Sigma^q T(\mathcal{A}))_T \end{aligned} \tag{6.14}$$

so we need only show $|W(X)| \to |X|$ is a weak equivalence. Since $|W(X)| \in \mathcal{S}^{T_0}$ is cofibrant $|W(X)| \to T \otimes_{T_0} |W(X)|$ is a weak equivalence, so we need only show

$$T \otimes_{T_0} |W(X)| \cong |T \otimes_{T_0} W(X)| \to |X|$$

is a weak equivalence. By Lemma 6.9, $T \otimes_{T_0} W(X)$ is $T(\mathcal{A})$-cofibrant. Furthermore $W(X) \to T \otimes_{T_0} W(X)$ is a level-wise weak equivalence. Thus 6.14 implies $T \otimes_{T_0} W(X) \to X$ is a $T(\mathcal{A})$-equivalence between cofibrant objects and Lemma 6.10 applies. ■

A Appendix on Resolutions.

The purpose of this section is to spell out the simplicial technology we need to build and modify various resolutions. We begin with the notion of s-free objects and morphisms, then show how to subdivide an object or morphism so that it becomes s-free. The principal results of the section (Propositions A.7 and Corollary A.8) states when this subdivision preserves homotopy type. Lemmas A.9 and A.10 apply the result to the $\mathcal{A}$-resolution model category.

Fix a category $\mathcal{C}$, which has all limits and colimits, and let $s\mathcal{C}$ be the simplicial objects in $\mathcal{C}$; thus $s\mathcal{C} = \mathcal{C}^{\Delta^{\mathrm{op}}}$ where Δ is the ordinal number category. Let $\Delta_+ \subseteq \Delta$ be the sub-category with the same objects and morphisms the surjective maps $[n] \to [m]$, and let $\Delta^\delta \subseteq \Delta$ be the sub-category with no non-identity morphisms. Let $s_+\mathcal{C} = \mathcal{C}^{(\Delta_+)^{\mathrm{op}}}$. Then $X \in s_+\mathcal{C}$ is a simplicial object with only degeneracies. Let $n\mathcal{C} = \mathcal{C}^{\Delta^\delta}$. It is the category of graded objects in $\mathcal{C}$. There are restriction functors

$$s\mathcal{C} \xrightarrow{i^*} s_+\mathcal{C} \xrightarrow{j^*} n\mathcal{C}$$

and these have left adjoints, given by left Kan extensions. For example the left adjoint of j^* is

$$(j_* Z)_n = \coprod_{\phi:[n]\to[m]} Z_m$$

where ϕ runs over surjective maps.

Definition A.1 *An object $X \in s\mathcal{C}$ is s-free if there is an object $Z \in n\mathcal{C}$ so that $i^*X \cong j_*Z$ in $s_+\mathcal{C}$. More generally, a morphism $A \to X$ is s-free if there is an isomorphism in $s_+\mathcal{C}\backslash i^*A$*

$$\{i^*A \to i^*X\} \cong \{i^*A \to i^*A \sqcup j^*Z\}$$

for some $Z \in n\mathcal{C}$.

The last statement says that $A \to X$ is almost free if

$$X_n \cong A_n \sqcup \coprod_{\phi:[n]\to[m]} Z_m$$

and this isomorphism respects the degeneracies. The importance of this notion lies in the following decomposition result. If $X \in s\mathcal{C}$, define the n^{θ} skeleton $sk_n X \in s\mathcal{C}$

$$(sk_n X)_t \cong \operatorname*{colim}_{\phi:[t]\to[m]} X_m$$

where $\phi : [t] \to [m]$ runs over the surjections with $m \leq n$. If $\Delta[n] \subseteq \Delta$ denotes the full subcategory on objects $[m]$, $m \leq n$, then $sk_n : s\mathcal{C} \to s\mathcal{C}$ is the composite of the restriction $s\mathcal{C} \to \mathcal{C}^{\Delta[n]^{\mathrm{op}}}$ followed by the left adjoint to this restriction. There are natural maps $sk_n X \to sk_{n+1} X$, $sk_{-1} X = \phi$, the initial object, and colim $sk_n X \cong X$. If $A \to X$ is a morphism in $s\mathcal{C}$, define $sk_n^A X$ by the push-out diagram

$$\begin{tikzcd}
sk_n A \arrow[r] \arrow[d] & sk_n X \arrow[d] \\
A \arrow[r] & sk_n^A X
\end{tikzcd}$$

Note that $sk_{-1}^A X \cong A$.

Lemma A.2 *Let $A \to X$ be s-free in $s\mathcal{C}$. Then there are push-out diagrams, $n \geq 0$,*

$$\begin{tikzcd}
Z_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & sk_{n-1}^A X \arrow[d] \\
Z_n \otimes \Delta^n \arrow[r] & sk_n^A X.
\end{tikzcd}$$

■

Notice this includes the case of s-free objects: X is s-free if and only $\phi \to X$ is an s-free morphism. This result asserts, for example, that if X is s-free on Z, then $sk_0 X \cong Z_0 \otimes \Delta^0 = Z_0$ regarded as a constant simplicial object.

We also have the following closure properties:

Lemma A.3 *The class of s-free morphisms is closed under coproducts, cobase change, and colimits over ordinal numbers.* ■

The last means that if κ is an ordinal number regarded as a category with one morphism $s \to t$ for each $s \leq t < \kappa$, and $X : \kappa \to s\mathcal{C}$ is a functor so that $X_s \to X_t$ is s-free for each $s \leq t < \kappa$, then

$$X_0 \to \operatorname*{colim}_{s<\kappa} X_s$$

is s-free.

We next introduce a procedure that automatically produces an s-free object from a simplicial object. We will worry later what this procedure does to homotopy type.

For each $[n] \in \Delta$, let Δ/n be the category of objects over $[n]$. Thus an object in Δ/n is a morphism $[m] \to [n]$ in Δ and a morphism is a triangle over $[n]$. For ease of notation we may drop the brackets on $[n]$ and write n instead. Let $B(\Delta/n)$ denote the classifying space, or nerve, of the category. If $\phi : n \to m$ is a morphism in Δ, there is an obvious functor $\phi : \Delta/n \to \Delta/m$ given by composition with ϕ; therefore $B(\Delta/\cdot)$ is a cosimplicial simplicial set. We abbreviate this by writing $B(\Delta/\cdot) \in c\mathbf{S}$. Notice that $B(\Delta/n)$ is contractible since Δ/n has a terminal object.

Now let us consider $\mathcal{D}$ to be a category equipped with an $\mathbf{S}$-action $-\otimes - : \mathcal{D} \times \mathbf{S} \to \mathcal{D}$. Suppose $\mathcal{D}$ has all colimits. Then the coend formula

$$(X, K) \mapsto \int^{\Delta} X \otimes K \tag{A.3}$$

defines a functor $s\mathcal{D} \times c\mathbf{S} \to \mathcal{D}$. To be brutally specific $\int^{\Delta} X \otimes K$ is defined by a coequalizer diagram

$$\coprod_{\phi: n \to m} X_m \otimes K^n \rightrightarrows \coprod_n X_n \otimes K^n \to \int^{\Delta} X \otimes K. \tag{A.4}$$

If $K = \Delta = \{\Delta^n\} \in c\mathbf{S}$, this is the geometric realization of X.

We now consider the case where $\mathcal{D} = s\mathcal{C}$ in its standard simplicial structure. Then $X = \{X_n\} \in s\mathcal{D} = s(s\mathcal{C})$ has an external simplicial degree n and an internal simplicial degree k: for each n, $X_n = \{X_{n,k}\} \in s\mathcal{C}$.

Lemma A.4 *Let $X \in s(s\mathcal{C})$. Then*

$$[\int^{\Delta} X \otimes B(\Delta/\cdot)]_k \cong \coprod_{n_0 \to n_1 \to \cdots \to n_k} X_{n_k, k} \tag{A.5.1}$$

where the coproduct is over composable k-tuples of morphisms in Δ. The face and degeneracy maps are given by summing together the formulas

$$d_0 \Big|_{\vec{n}} = d_0 : (X_{n_k,k})_{n_0 \to n_1 \to \cdots \to n_k} \to (X_{n_k,k-1})_{n_1 \to \cdots \to n_k}$$

$$d_i \Big|_{\vec{n}} = d_i : (X_{n_k,k})_{n_0 \to n_1 \to \cdots \to n_k} \to (X_{n_k,k-1})_{n_0 \to \cdots \to n_{i-1} \to n_{i+1} \to \cdots \to n_k}$$

$1 \le i \le k-1$

$$d_k \Big|_{\vec{n}} = d_k \circ \phi^* = \phi^* \circ d_k : (X_{n_k,k})_{n_0 \to \cdots \to n_k} \to (X_{n_{k-1},k-1})_{n_0 \to \cdots \to n_{k-1}}$$

where $\phi : n_{k-1} \to n_k$ is the last map, and

$$s_i \Big|_{\vec{n}} = s_i : (X_{n_k,k})_{n_0 \to \cdots \to n_k} \to (X_{n_k,k+1})_{n_1 \to \cdots \to n_i \xrightarrow{1} n_i \to n_{i+1} \to \cdots \to n_k}.$$

Proof. We use the formula (A.4). Then

$$[\coprod_n X_n \otimes B(\Delta/n)]_k \cong \coprod_{n_0 \to n_1 \to \cdots \to n_k \to n} X_{n,k}$$

where the coproduct is now over composable $(k+1)$ tuples in Δ and the face and degeneracy maps operate only on the k-variable. Define a map ϵ from this coproduct to the coproduct of (A.5.1) by summing together the maps

$$\psi^* : (X_{n,k})_{n_0 \to \cdots \to n_k \xrightarrow{\psi} n} \to (X_{n_k,k})_{n_0 \to n_1 \to \cdots \to n_k}.$$

One checks that this equalizes the diagram of (A.4). The formulas for the face and degeneracy maps follow from those of $\coprod_n X_n \otimes B(\Delta/n)$. ■

The reader will have noticed that this proof is essentially the same as the observation that for $X \in s(s\mathcal{C})$

$$|X| = \int^{\Delta} X \otimes \Delta \cong \mathrm{diag}(X) = \{X_{n,n}\}.$$

Proposition A.6 *If $X \in s(s\mathcal{C})$, then $\int^{\Delta} X \otimes B(\Delta/\cdot)$ is s-free on Z where $Z_0 = \coprod_n X_{n,0}$ and for $k \geq 1$*

$$Z_k = \coprod_{n_0 \to n_1 \to \cdots \to n_k} X_{n_k,k}$$

and the coproduct is over composable k-tuples of morphisms in Δ where none of the morphisms $n_i \to n_{i+1}$ are the identity. ■

More generally, if $A \to X$ is a morphism in $s(s\mathcal{C})$ so that for all pairs (n,k), $A_{n,k} \to X_{n,k}$ is isomorphic to the inclusion of a coproduct summand, then $\int^{\Delta} A \otimes B(\Delta/\cdot) \to \int^{\Delta} X \otimes B(\Delta/\cdot)$ is an s-free morphism.

There is a canonical map of cosimplicial objects $B(\Delta/\cdot) \to \Delta$, and hence a natural map

$$\int^{\Delta} X \otimes B(\Delta/\cdot) \to \int^{\Delta} X \otimes \Delta = |X|$$

defined as follows. If $\Delta_- \subseteq \Delta$ is the subcategory with same objects and only monomorphisms for morphisms, then the subdivision of the standard n-simplex is given by the formula

$$sd\Delta^n = B(\Delta_-/n).$$

The epi-monic factorization in Δ yields functor $\Delta/n \to \Delta_-/n$ and, for each $\phi : n \to m$ in Δ, a functor $\phi : \Delta_-/n \to \Delta_-/m$. From this one gets a morphism of cosimplicial objects

$$B(\Delta_-/\cdot) \to sd\Delta.$$

There is now (see [13]) a standard surjection $sd\Delta \to \Delta$. To exploit these maps we have the next result. We give $c\mathbf{S}$ the Bousfield-Kan simplicial category structure: it is the Reedy structure on $s(\mathbf{S}^{\mathrm{op}})$ with the internal simplicial structure inherited from $\mathbf{S}$. We assume $\mathcal{D}$ is a simplicial model category. We give $s\mathcal{D}$ the resulting Reedy model category structure. If $X \in s\mathcal{D}$ and $K \in \mathbf{S}$, then $X_n \otimes K \in \mathcal{D}$ for all n. The following result is implicit in Bousfield and Kan [3], and we quote from that work freely.

Proposition A.7 *Let $X \in s\mathcal{D}$ be Reedy cofibrant and $K \to L$ a weak equivalence between cofibrant objects in $c\mathbf{S}$. Then*

$$\int^{\Delta} X \otimes K \to \int^{\Delta} X \otimes L$$

is a weak equivalence between cofibrant objects in $\mathcal{D}$.

Proof. For fixed $X \in s\mathcal{D}$ the functor $K \mapsto \int^{\Delta} X \otimes K$ from $c\mathbf{S}$ to $\mathcal{D}$ has right adjoint

$$Y \mapsto \mathrm{map}_{\mathcal{D}}(X,Y) = \{\mathrm{map}_{\mathcal{D}}(X^n, Y)\}.$$

By construction, there is an isomorphism

$$\mathrm{map}_{\mathcal{D}}(L_nX,Y) \cong M_n\,\mathrm{map}_{\mathcal{D}}(X,Y)$$

where L_nX is the latching object of X and $M^n(\cdot)$ is the matching space functor of Bousfield-Kan. Put another way, if $K \in c\mathbf{S}$, then M^nK is the latching object of $K \in s(\mathbf{S}^{\mathrm{op}})$. If $Y \to Z$ is a map in $s\mathcal{C}$, then

$$\mathrm{map}_{\mathcal{D}}(X_n,Y) \to M^n\,\mathrm{map}_{\mathcal{D}}(X,Y) \times_{M^n\,\mathrm{map}_{\mathcal{D}}(X,Z)} \mathrm{map}_{\mathcal{D}}(X_n,Z)$$

is isomorphic to

$$\mathrm{map}_{\mathcal{D}}(X_n,Y) \to \mathrm{map}_{\mathcal{D}}(L_nX,Y) \times_{\mathrm{map}_{\mathcal{D}}(L_nX,Z)} \mathrm{map}_{\mathcal{D}}(X_n,Z).$$

Since $L_nX \to X$ is a cofibration, this map is a fibration if $Y \to Z$ is, and it is a trivial fibration if $Y \to Z$ is. Thus $\mathrm{map}_{\mathcal{D}}(X,\cdot)$ preserves fibrations and trivial fibrations. Hence $\int^{\Delta} X \otimes (\cdot)$ preserves cofibrations and trivial cofibrations. In particular if K is cofibrant, $\int^{\Delta} X \otimes K$ is cofibrant. Now suppose $f : K \to L$ is a weak equivalence between cofibrant objects. We may factor f as $K \xrightarrow{j} K' \xrightarrow{q} L$ where j is a trivial cofibration and q is a trivial fibration. Then $\int^{\Delta} X \otimes K \to \int^{\Delta} X \otimes K'$ is a weak equivalence and K' is cofibrant, so we may assume f is a trivial fibration in $c\mathbf{S}$. Since L is cofibrant f has a section $s : L \to K$; since K is cofibrant, there is a homotopy $K \times I \to K$ from 1_K to sf. The result now follows from the observation that

$$\int^{\Delta} X \otimes (K \times I) = (\int^{\Delta} X \otimes K) \otimes I. \qquad \blacksquare$$

Again we consider the case $\mathcal{D} = s\mathcal{C}$. Now suppose $X \in s\mathcal{C}$. We may regard X as an object in $s(s\mathcal{C})$ by setting $X_{n,k} = X_n$ for all k. Thus we run X out in the external simplicial direction. Define a "big subdivision" by the formula

$$sd'X = \int^{\Delta} X \otimes B(\Delta/\cdot).$$

The traditional subdivision of X is

$$sdX = \int^{\Delta} X \otimes sd\Delta[\cdot].$$

Corollary A.8 *Suppose $X \in s\mathcal{C}$ has the property that for all $n \geq 0$,*

$$L_nX \to X_n$$

is a cofibration of constant objects in $s\mathcal{C}$. Then X is Reedy cofibrant in $s(s\mathcal{C})$ and the natural maps

$$sd'X \to sdX \to X$$

are all weak equivalences. $\blacksquare$

It is convenient to have a converse, and for this we need more generality. We fix a model category structure on $s\mathcal{C}$ and say $X \to Y$ is an external Reedy cofibration if it is a cofibration in $s(s\mathcal{C})$ when put in as above. The class of external Reedy cofibrations is closed under cobase change, coproduct, colimits over natural numbers, and retracts. So one has

Lemma A.9 *If $s\mathcal{C}$ is cofibrantly generated and the generating cofibrations are external Reedy cofibrations, then every cofibration is an external Reedy cofibration. In particular, every cofibrant object satisfies the hypotheses of Corollary A.8.* ■

Thus, for example, any of the $\mathcal{A}$-resolution model categories structures of sections 2–5 satisfy hypotheses of Lemma A.9:

Lemma A.10 *Let $s\mathcal{C}$ be given an $\mathcal{A}$-resolution simplicial model category structure. Let $f : A \to B$ be an $\mathcal{A}$-cofibration. Then f is an external Reedy cofibration in $s(s\mathcal{C})$.*

Proof. First observe that if $A \to B$ is a cofibration in $\mathcal{C}$, then $f : A \to B$ is a Reedy cofibration of constant objects in $s\mathcal{C}$. This is because

$$L_k B \sqcup_{L_k A} A_k \to B_k$$

is an isomorphism if $k \geq 1$ and isomorphic to $A \to B$ if $k = 0$. From this we conclude that if $f : A \to B$ is a trivial cofibration in $\mathcal{C}$, then $f : A \to B$ is a trivial Reedy cofibration (hence an $\mathcal{A}$-cofibration) of constant objects in $s\mathcal{C}$.

Next observe, that if $A \to B$ in $\mathcal{C}$ is isomorphic to $A \to A \sqcup (\bigsqcup_\alpha (Aj)_\alpha)$ then $A \to B$ is an $\mathcal{A}$-cofibration of constant objects in $s\mathcal{C}$. This is because there is a push-out diagram in $s\mathcal{C}$

$$\begin{tikzcd} \bigsqcup_\alpha Aj \otimes \partial\Delta^0 \arrow[r] \arrow[d] & A \arrow[d] \\ \bigsqcup_\alpha Aj \otimes \Delta^0 \arrow[r] & B. \end{tikzcd}$$

Of course, $\partial\Delta^0 = \phi$.

Finally let $f : C \to D$ be any of the generating cofibrations of $s\mathcal{C}$ (See 2.13.2). Then we must check that

$$L_k f : L_k D \sqcup_{L_k C} C_k \to D_k$$

is an $\mathcal{A}$-cofibration of constant objects. If f is a trivial Reedy cofibration, then $L_k f$ is a trivial cofibration in $\mathcal{C}$, hence an $\mathcal{A}$-cofibration in $s\mathcal{C}$. If f is of the form $Aj \otimes \partial\Delta^n \to Aj \otimes \Delta^n$, then $L_n f$ is an isomorphism unless $n = k$; in that case

$$D_n \cong [L_n D \sqcup_{L_n C} C_n] \sqcup Aj.$$ ■

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