Simplicial Structured Ring Spectra

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Abstract

We examine the foundations of simplicial algebras in spectra over a simplicial operad. We are led to simplicial operads and simplicial algebras over simplicial operads because certain operads which are notoriously hard to work with – mainly the E_∞ operad – can be simplicially resolved by simpler pieces. Our main goals are to a build spectral sequence for computing spaces of maps between structured ring spectra, and to develop a Dwyer-Kan-Stover style obstruction theory for deciding when a spectrum actually can be a structured ring spectrum.

In this paper we work out some of the foundations of the homotopy theory of simplicial ring spectra over a simplicial operad. This is not a gratuitous act of generalization. Simplicial objects in any category are a standard mechanism for building the resolutions necessary for computations; this is how simplicial spectra arise. The simplicial operads arise as an answer to an immediate practical problem. If T is an E_∞ operad over the linear isometries operad and X is a spectrum, one can form the free E_∞ ring spectrum $T(X)$. Then, if E_* is a generalized homology theory, one can ask for a computation of $E_*T(X)$. This is extremely difficult; at the very least one needs to know $E_*B\Sigma_n$ for all $n \geq 0$ and, practically, one also needs a firm hold on the E_* Dyer-Lashof algebra. Our response is resolve T by a simplicial operad $\{T_n\}$ for which is is possible to compute $E_*T_n(X)$. The application is to develop tools for deciding when a spectrum is an E_∞ ring spectrum, and for computing the homotopy type of the space of E_∞ maps between E_∞ ring spectra. Actually, there's nothing special about the E_∞ operad. Our results will apply to any operad over the linear isometries operad.

This is the third of a series of papers, following and building upon [11] and [12]. An overview of our entire project has been supplied in [13]. This paper and [11] are both informed by the work of Dwyer, Kan, and Stover [5],[6]; the obstruction theory these authors allude to was worked out by Blanc [4]. The paper [12], which is necessary for some of our computations, should be regarded as an algebraic intermezzo.

Here is a brief synopsis of the results of this paper. As the paragraphs above indicate, we are working with the Lewis-May-Steinberger category of spectra over some fixed universe $\mathcal{U}$ [16]. Since $\mathcal{U}$ is a countably infinite dimensional inner product space, we can form the linear isometries operad $\mathcal{L}$ with

$$\mathcal{L}(0) = \mathcal{L}(\mathcal{U}^{\oplus n}, \mathcal{U})$$

the space of linear isometries from $\mathcal{U}^{\oplus n}$ to $\mathcal{U}$. We are interested in operads T equipped with a map of operads $T \to \mathcal{L}$; that is, in operads over the linear isometries operads. It is such operads that supply structured ring spectra. If $\mathcal{S} = \mathcal{S}_\mathcal{U}$ is the category of spectra over $\mathcal{U}$, let $\mathcal{S}^T$ denote the category of algebras over T. To specify an object in $X \in \mathcal{S}^T$ amounts to specifying a map of spectra $T(X) \to X$ from the free T-algebra on X to X so that certain diagrams commute. This is natural enough that the definition can be extended to simplicial objects: if T is a simplicial operad and X is a simplicial spectrum, then X is a simplicial T-algebra if there is a map of simplicial spectra $T(X) \to X$ so that the appropriate diagrams commute. We call the category of such simplicial algebras $s\mathcal{S}^T$.

The main thrust of the first five sections of this paper is to define and study a geometric realization functor of $s\mathcal{S}^T$. The geometric realization of simplicial operad T is an operad $|T|$ and we construct

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a geometric realization functor

$$|\cdot|: s\mathcal{S}^T \longrightarrow \mathcal{S}^{|T|}$$

and discuss its homotopy invariance and computability. This a generalization of a result of [10] (see also [8]) and depends on that result. This is done in the third section; the first two sections are spent building a Reedy model category structure on $s\mathcal{S}^T$ so that we can claim that this geometric realization functor preserves weak equivalences between Reedy cofibrant objects. See Proposition 3.4 and 3.5.

Sections 4 and 5 are devoted to importing the "E_2" model category structure of Dwyer, Kan, and Stover [5, 6] into the setting of simplicial spectra over a simplicial operad. The essential point here is that this allows one to build simplicial resolutions $Y \to X$ of objects $X \in s\mathcal{S}^T$ by objects $Y \in s\mathcal{S}^T$ so that each of the objects Y_n of Y is of the form $T_n(Z_n)$ with $E_*T_n(Z_n)$ computable, where E_* is some appropriate homology theory. With this in hand, we can build spectral sequences for computing space of algebra maps between structured ring spectra. The E_2 term is an appropriate André-Quillen cohomology. This is done in section 6. In section 8, we develop the obstruction theory for deciding when a spectrum is a structured ring spectrum; again the obstruction groups are André-Quillen cohomology groups. Section 7 is devoted to construction of the Eilenberg-Mac Lane objects necessary to the Postnikov tower approach to the obstruction theory.

The immediate applications involve studying the spectra E_n realizing the Lubin-Tate universal deformation of the height n formal group law. In [13] we showed how the Hopkins-Miller result showing that E_n is an A_∞ ring spectrum can be adapted to show that E_n is, in fact, an E_∞ ring spectrum. With the obstruction theory of this paper, and with the algebra of [12] at hand, the ideas of [20] can be adapted quite cleanly.

Contents

1 Preliminaries on operads and diagrams of algebras

We will be dealing with diagrams of operads – for example, simplicial operads – and diagrams of algebras over such diagrams of operads. In this setting one naturally confronts the notion of limits and colimits, or even enriched limits and colimits – such as a geometric realization. This section is devoted to working out the categorical preliminaries necessary for this discussion.

Let $\mathcal{L}$ be the linear isometries operad and $\mathcal{O}$ the category of operads over $\mathcal{L}$. The category $\mathcal{O}$ has all limits and colimits. Let I be a small category and $\mathcal{S}^I$ the category of I-diagrams in spectra. Then if $T: I \to \mathcal{O}$ is a diagram of operads and $X \in \mathcal{S}^I$ we can form $T(X) = T \circ X \in \mathcal{S}^I$. The functor

$T(\cdot) : \mathcal{S}^I \to \mathcal{S}^I$ is the functor of a triple on $\mathcal{S}^I$, and we write $\mathcal{S}^T$ for the category of algebras over this triple. This is an abbreviation of the more usual $(\mathcal{S}^I)^T$, which is cumbersome. The exception is this: if $I = \Delta^{\mathrm{op}}$ we write $(s\mathcal{S})^T$ for simplicial algebras over the simplicial operad T.

Fix $T : I \to \mathcal{O}$ and consider $\lim\limits_I T$ and $\operatorname{colim}\limits_I T$; these are again operads in $\mathcal{O}$. Abbreviate $\lim\limits_I$ as $\lim T$, and so on.

Lemma 1.1 *1) The constant diagram functor $\mathcal{S}^{\lim T} \to \mathcal{S}^T$ has a right adjoint $\lim\limits_I{}^T : \mathcal{S}^T \to \mathcal{S}^{\lim T}$.*

2) The constant diagram functor $\mathcal{S}^{\operatorname{colim} T} \to \mathcal{S}^T$ has a left adjoint

$$\operatorname*{colim}_I{}^T : \mathcal{S}^T \to \mathcal{S}^{\operatorname{colim} T}.$$

Proof. For 1), let $X \in \mathcal{S}^T$. Then we may regard $X \in \mathcal{S}^I$ and $\lim\limits_I X \in \mathcal{S}$ has a natural structure as a $\lim T$ algebra; indeed, the natural maps $(\lim\limits_I T) \circ (\lim\limits_I X) \to T_i \circ X_i$ coupled with the evaluation maps $\eta_i : T_i \circ X_i \to X_i$ yield a map

$$(\lim_I T) \circ (\lim_I X) \to \lim_I (T_i \circ X_i) \to \lim X,$$

which defines the $\lim T$ structure on $\lim\limits_I X$. The object $\lim\limits_I X \in \mathcal{S}^{\lim T}$ so defined is $\lim\limits_I{}^T X$.

For 2), let $X \in \mathcal{S}^T$. The maps of operads $T_i \to \operatorname{colim} T$ defines a functor

$$\operatorname{colim} T \otimes_T (\cdot) : \mathcal{S}^T \to (\mathcal{S}^{\operatorname{colim} T})^I$$

with $(\operatorname{colim} T \otimes_T X)_i = \operatorname{colim} T \otimes_{T_i} X_i$. Then

$$\operatorname*{colim}_I{}^T X = \operatorname*{colim}_I{}^{\operatorname{colim} T} (\operatorname{colim} T \otimes_T X). \tag{1.1}$$

This completes the argument.

Also in the vein of categorical preliminaries we record the following. Let I be a category and $J \subseteq I$ a subcategory. If $T : I \to \mathcal{O}$ is an I-diagram of operads we have a restricted diagram of operads $r_* T : J \to \mathcal{O}$, and we get a restriction functor

$$r_* : \mathcal{S}^T \to \mathcal{S}^{r_* T}.$$

Lemma 1.2 *The restriction functor r_* has both a left adjoint r^* and a right adjoint $r^!$.*

Proof. The restriction $r_* : \mathcal{O}^I \to \mathcal{O}^J$ has a left and right adjoint r^* and $r^!$ given by the left and right Kan extension functors respectively. For example

$$(r^* T)_i = \operatorname*{colim}_{j \to i} T_j$$

where the colimit occurs in $\mathcal{O}$, and is over all maps $j \to i$ in I with $j \in J$. Let us define $r^* : \mathcal{S}^{r_* T} \to \mathcal{S}^T$. First note that Lemma 1.1 supplies a functor $F : \mathcal{S}^{r_* T} \to \mathcal{S}^{r^* r_* T}$. The counit of the adjunction between r^* and r_* supplies a morphism in $\mathcal{O}^I$

$$r^* r_* T \to T.$$

Then $r^* X = T \otimes_{r^* r_* T} FX$.

The right adjoint is easier: if $X \in \mathcal{S}^{r_* T}$, then $X \in \mathcal{S}^J$ and the right Kan extension $r^! X \in \mathcal{S}^I$ is already an $r^! r_* T$ algebra and hence a T-algebra via the unit $T \to r^! r_* T$.

We now focus specifically on the case $(s\mathcal{S})^T$ where T is a simplicial operad. The first observation is that $(s\mathcal{S})^T$ is a simplicial category in the sense of Quillen [17, Chap. II]. To see this, give $s\mathcal{S}$ the standard simplicial structure of [17, II §2]. Thus if $K \in \mathbf{S}$ is a simplicial set and $X \in s\mathcal{S}$ is a simplicial spectrum, then

$$(X \otimes_{\mathcal{S}} K)_n = \bigvee_{K_n} X_n.$$

The subscript on the tensor product is used here only to make the next few paragraphs clear. It will be dropped when the category is understood. If $X \in s\mathcal{S}$, define $T(X) \otimes K \in (s\mathcal{S})^T$ by the formula

$$T(X) \otimes K = T(X \otimes_{\mathcal{S}} K).$$

If $f : T(X) \to T(Y)$ is any map in $(s\mathcal{S})^T$, there is a unique map $f \otimes 1 : T(X) \otimes K \to T(Y) \otimes K$ defined by unique T-algebra filler of the diagram

$$\begin{tikzcd}
X \otimes_{\mathcal{S}} K \arrow[r, "g\otimes 1"] \arrow[d] & T(Y) \otimes_{\mathcal{S}} K \arrow[d, "\theta"] \\
T(X \otimes_{\mathcal{S}} K) \arrow[r, dashed] & T(Y \otimes_{\mathcal{S}} K).
\end{tikzcd}$$

The map g of spectra is adjoint to f and the map θ in degree n is the map of spectra

$$\bigvee_{K_n} T_n(Y_n) \to \coprod_{K_n} T_n(Y_n)$$

induced by the inclusion of the summands. The coproduct in the target is formed in the category of T_n algebras. For general $X \in (s\mathcal{S})^T$ define $X \otimes K$ by the coequalizer diagram in $(s\mathcal{S})^T$

(1.2) $$T^2(X) \otimes K \rightrightarrows T(X) \otimes K \to X \otimes K$$

where the two maps are $T\eta_X \otimes 1$ and $\eta_{T_X} \otimes 1$. Since colimits are formed level-wise,

(1.3) $$(X \otimes K)_n \cong \coprod_{K_n}{}^{T_n} X_n.$$

For fixed K, the functor $X \mapsto X \otimes K$ from $(s\mathcal{S})^T$ to $(s\mathcal{S})^T$ has a right adjoint $X \mapsto \mathbf{hom}(K, X)$. (We reserve X^K for another purpose.) In fact, the object $\mathbf{hom}(K, X) \in s\mathcal{S}$ obtained by Quillen's structure on $s\mathcal{S}$ has a canonical T-algebra structure adjoint to

$$T\mathbf{hom}(K, X) \otimes_{\mathcal{S}} K \to T(\mathbf{hom}(K, X) \otimes K) \to T(X) \to X.$$

Finally there is a simplicial set of maps for $X, Y \in (s\mathcal{S})^T$ specified by

$$\mathbf{map}_{(s\mathcal{S})^T}(X, Y)_n = \mathrm{Hom}_{(s\mathcal{S})^T}(X \otimes \Delta^n, Y).$$

We will write $\mathbf{map}(X, Y)$ to ease notation if $(s\mathcal{S})^T$ is understood.

Lemma 1.3 *With these definitions $(s\mathcal{S})^T$ becomes a simplicial category.*

We next address the decomposition of objects in $(s\mathcal{S})^T$ via skeleta and coskeleta. Let $\Delta[n] \subseteq \Delta$ be the full subcategory of the ordinal number with objects $[k]$, $k < n$. For any category $\mathcal{C}$, let $s_n\mathcal{C} = \mathcal{C}^{\Delta[n]^{\mathrm{op}}}$ and $r_* : s\mathcal{C} \to s_n\mathcal{C}$ the restriction. If $T \in s_n\mathcal{O}$, let us write $\mathcal{S}^T$ as $(s_n\mathcal{S})^T$. As in Lemma 1.2 there is a restriction functor $r_* : (s\mathcal{S})^T \to (s_n\mathcal{S})^{r_*T}$. If $X \in (s\mathcal{S})^T$ define the n^{th} skeleton of X by, for $n \geq -1$.

$$sk_n X = r^* r_* X.$$

Note that $sk_{-1}X = T(0)$ is the initial object in $(s\mathcal{S})^T$.

More generally, if $A \to X$ is a map in $(s\mathcal{S})^T$ define $sk_n^A X$ by the push-out diagram in $(s\mathcal{S})^T$

$$\begin{tikzcd} sk_n A \arrow[r] \arrow[d] & A \arrow[d] \\ sk_n X \arrow[r] & sk_n^A X. \end{tikzcd}$$

Then $sk_{-1}^A X = A$ and the map $A \to \operatorname{colim}_n sk_n^A X \cong X$ is isomorphic to the original map $A \to X$. If A is the initial object, and $A \to X$ is the unique map, then $sk_n^A X = sk_n X$.

To describe how $sk_n^A X$ is built from $sk_{n-1}^A X$ we must slightly generalize the idea of $X \otimes K$ defined above. The restriction functor $(s\mathcal{S})^T \to \mathcal{S}^{T_n}$ given by $X \mapsto X_n$ has a left adjoint by Lemma 1.2. Following the prescription supplied by that result we have, for $Y \in \mathcal{S}^{T_n}$ an object $FY \in (s\mathcal{S})^{T_n \otimes \Delta^n}$ with

$$(FY)_k = \coprod_{[k] \to [n]} Y$$

where $[k] \to [n]$ runs the morphisms in Δ to $[n]$. Then the left adjoint is given by $Y \mapsto T \otimes_{T_n \otimes \Delta_n} FY$, where $T_n \otimes \Delta^n \to T$ is the natural map. We will write

$$Y \mapsto Y \otimes \Delta^n \tag{1.4}$$

for this functor. If T is constant, as a simplicial object, then this notation agrees with 1.2 if we regard $Y \in (s\mathcal{S})^T$ as a constant object. Otherwise, it is different, as Y does not live in $(s\mathcal{S})^T$ in any sense.

This can be slightly generalized. If $K \in \mathbf{S}$ we let $\mathcal{K}$ be its simplex category: the objects in $\mathcal{K}$ are morphisms $\Delta^n \to K$ and morphisms are triangles. There is a natural isomorphism $\operatorname{colim}_{\mathcal{K}} \Delta^n \cong K$. If $\mathcal{C}$ is any category with limits and colimits and $s\mathcal{C}$ is the category of simplicial objects with its standard simplicial structure, then the functor $\mathcal{C} \to s\mathcal{C}$ sending Z to $Z \otimes K$ has right adjoint M_K with

$$M_K X \cong \lim_{\mathcal{K}^{\mathrm{op}}} X_n.$$

If $X \in (s\mathcal{S})^T$ we may define a functor $M_K : (s\mathcal{S})^T \to \mathcal{S}^{M_K T}$ by the same formula. This is the generalized matching object formula.

Lemma 1.4 *1) For fixed K, the functor M_K has a left adjoint $Z \to Z \otimes K$.*

2) For fixed X, the functor $\mathbf{S}^{\mathrm{op}} \to \mathcal{S}$ sending $X \mapsto M_K X$ has a left adjoint.

Proof. If $Z \in \mathcal{S}^{M_K T}$ define $FZ \in (s\mathcal{S})^{M_K T \otimes K}$ by the coproduct formula 1.4.2; then $Z \otimes K = T \otimes_{M_K T \otimes K} FZ$ where $M_K T \otimes K \to T$ is the counit of the adjunction. The second statement follows from the fact that we have an isomorphism

$$\operatorname{Hom}_{\mathcal{S}}(Z, M_K X) \cong \operatorname{Hom}_{\mathbf{S}}(K, \mathbf{hom}_{s\mathcal{S}}(Z,X)) = \operatorname{Hom}_{\mathbf{S}^{\mathrm{op}}}(\mathbf{map}_{s\mathcal{S}}(Z,X), K).$$

A consequence of Lemma 1.4.2 is the following: if $K : I \to \mathbf{S}$ is a diagram and $L = \operatorname{colim}_i K_i$, then there is a natural isomorphism in $\mathcal{S}^{M_L T}$

$$M_L X \cong \lim_{I^{\mathrm{op}}} M_{K_i} X. \tag{1.5}$$

We may now describe how $sk_n^A X$ is built from $sk_{n-1}^A X$. For $X \in (s\mathcal{S})^T$ define $L_n X = (sk_{n-1}X)_n \in \mathcal{S}^{T_n}$. If $f : A \to X$ is a morphism in $(s\mathcal{S})^T$, let $L_n(f) = L_n X \sqcup_{L_n A} A_n$, the coproduct as T_n algebras. By definition of $sk_{n-1}^A X$ there is a natural isomorphism

$$L_n(f) \cong (sk_{n-1}^A X)_n,$$

and, hence, by adjointness, natural maps

$$L_n(f) \otimes \Delta^n \longrightarrow sk_{n-1}^A X$$

$$X_n \otimes \Delta^n \longrightarrow sk_n^A X$$

An easy calculation shows $sk_{n-1}(X_n \otimes \Delta^n) = X_n \otimes \partial\Delta^n$; thus, there is a diagram

$$\begin{tikzcd} L_n(f) \otimes \Delta^n \sqcup_{L_n(f)\otimes\partial\Delta^n} X_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & sk_{n-1}^A X \arrow[d] \\ X_n \otimes \Delta^n \arrow[r] & sk_n^A X. \end{tikzcd} \tag{1.6}$$

Proposition 1.5 *This diagram is a push-out diagram in $(s\mathcal{S})^T$.*

Proof. The proof is the usual one. It is sufficient to prove it is a push-out in $(s_n\mathcal{S})^{r_* r^* T}$ where $r_* T \in s_n\mathcal{O}$ is the restriction of T. Since push-out are computed level-wise, one does level-wise calculations. Since $(\partial\Delta^n)_k = (\Delta^n)_k$ for $k < n$, the left vertical map is an isomorphism for $k < n$, as required. In degree n, $(\Delta^n)_n = (\partial\Delta^n)_k \cup \{1 : [n] \to [n]\}$, so the right vertical map is isomorphic to

$$(X_n \otimes \partial\Delta^n)_n \sqcup L_n(f)_n \xrightarrow{k} (X_n \otimes \partial\Delta^n)_n \sqcup X_n.$$

There is a dual theory of coskeleta which we won't belabor. The standard theory of coskeleta for $s\mathcal{S}$ (See [14], if nowhere else) passes to a theory of coskeleta for $(s\mathcal{S})^T$ as the usual diagrams and constructions in $s\mathcal{S}$ lie canonically in $(s\mathcal{S})^T$. The upshot is that if $f : X \to B$ is a morphism in $(s\mathcal{S})^T$ then $X = \lim_n \mathrm{cosk}_n^B X$ and $\mathrm{cosk}_n^B X$ is the pull-back of a diagram built from $\mathrm{cosk}_{n-1}^B X$, X_n, and the matching objects

$$M_n(f) = (\mathrm{cosk}_{n-1}^B X)_n.$$

These last can be described as follows: $M_n X = M_{\partial\Delta^n} X$ and

$$M_n(f) = B_n \times_{M_n B} M_n X. \tag{1.7}$$

More important for what follows is the generalization and decomposition of the latching objects $L_n X$. If $F : J \to \Delta$ is any functor from a small category and $X \in s\mathcal{C}$, define the generalized latching object to be

$$L_J X = \operatorname*{colim}_J (X \circ F).$$

The functor F is usually suppressed from the notation, as it is implicit in the category J. If $X \in (s\mathcal{S})^T$, then the functor $F : J \to \Delta$, determines functors $T \circ F : J^{\mathrm{op}} \to \mathcal{O}$ and $X \circ F \in \mathcal{S}^{T \circ F}$

$$L_J X = \operatorname*{colim}_J (X \circ F) \in \mathcal{S}^{L_J T}$$

should be computed via Lemma 1.1. For example, let $J = [n] \downarrow \Delta$ be the category morphisms $[n] \to [m]$ in Δ, and $F : J \to \Delta^{\mathrm{op}}$ sending $[n] \to [m]$ to $[m]$. Then $L_n X \cong T_n \otimes_{L_J T} L_J X$. If $\mathcal{J}_n \subseteq J$ is the full subcategory with morphisms $[n] \to [m]$, $m < n$, which are surjections, then $\mathcal{J}_n$ is final in J (since every morphism in Δ factors uniquely as a surjection followed by an injection); hence $L_n T \cong L_{\mathcal{J}_n} T$ and

$$L_n X \cong T_n \otimes_{L_n T} L_{\mathcal{J}_n} X. \tag{1.8}$$

To decompose $L_n X$ we specify some subcategories of $\mathcal{J}_n$. Let $\mathcal{J}_{n,k} \subseteq \mathcal{J}_n$ be the full subcategory of surjections $\phi : [n] \to [m]$, $m < n$, with $\phi(k) < k$. Then $\mathcal{J}_{n,0} = \phi$ and $\mathcal{J}_{n,n} = \mathcal{J}_n$. Define

$$L_{n,k} X = L_{\mathcal{J}_{n,k}} X.$$

Thus $L_n X \cong T_n \otimes_{L_n T} L_{n,n} X$. Let $\mathcal{J}(k) \subseteq \mathcal{J}_n$ be the full subcategory of surjections $\phi : [n] \to [m]$ with $\phi(k) = \phi(k+1)$. Note $s^k : [n] \to [n-1]$ is initial in $\mathcal{J}(k)$; therefore,

$$L_{\mathcal{J}(k)} X \cong X_{n-1}.$$

There are inclusions $\mathcal{J}_{n,k} \subseteq \mathcal{J}_{n,k+1}$, $\mathcal{J}(k) \subseteq \mathcal{J}_{n,k+1}$ and an isomorphism

$$- \circ s^k : \mathcal{J}_{n-1,k} \to \mathcal{J}(k) \cap \mathcal{J}_{n,k}$$

sending ϕ to $\phi \circ s^i$.

The following is our decomposition result.

Proposition 1.6 *Let T be a simplicial operad and $X \in (s\mathcal{S})^T$. Then there is a push-out diagram of operads*

$$\begin{tikzcd} L_{n-1,k} T \arrow[r] \arrow[d] & T_{n-1} \arrow[d] \\ L_{n,k} T \arrow[r] & L_{n,k+1} T \end{tikzcd}$$

and a push-out diagram of $L_{n,k+1}T$ algebras

$$\begin{tikzcd} L_{n,k+1} T \otimes_{L_{n-1,k} T} L_{n-1,k} X \arrow[r] \arrow[d] & L_{n,k+1} T \otimes_{T_{n-1}} X_{n-1} \arrow[d] \\ L_{n,k+1} T \otimes_{L_{n,k} T} L_{n,k} X \arrow[r] & L_{n,k+1} X. \end{tikzcd}$$

Proof. The diagram

$$\begin{tikzcd} \mathcal{J}_{n-1,k} \arrow[r] \arrow[d] & \mathcal{J}(k) \arrow[d] \\ \mathcal{J}_{n,k} \arrow[r] & \mathcal{J}_{n,k+1} \end{tikzcd}$$

is a push-out diagram in the category of small categories.

2 The Reedy model category structure

The purpose of this section and the next is to define a geometric realization functor

$$|\cdot|:(s\mathcal{S})^T\to\mathcal{S}^{|T|}$$

and to investigate its homotopical properties. This section is devoted to the Reedy model category structure on $(s\mathcal{S})^T$, which is a model category structure suitable for discussing realization. This is a minor generalization of Reedy's point of view [18] in that the objects of $(s\mathcal{S})^T$ are not actually simplicial diagrams in any underlying category.

Recall the a morphism $f:X\to Y$ of spectra over some universe $\mathcal{U}$ is a weak equivalence (or fibration) if and only if $f:X_V\to Y_V$ is a weak equivalence (or fibration) of spaces for all finite dimensionsal subspaces $V\subseteq\mathcal{U}$ closed under the inner product. This structure lifts to a model category structure of $\mathcal{F}$ algebras in spectra, where $\mathcal{F}$ is any operad over the linear isometries operad. See [10], or [8].

Definition 2.1 A morphism $f:X\to Y$ in $(s\mathcal{S})^T$ is

(1) a Reedy weak equivalence if $f:X_n\to Y_n$ is a weak equivalence of T_n-algebras or, equivalently, of spectra;

(2) a Reedy fibration if $X_n\to Y_n\times_{M_nY}M_nX$ is a fibration of T_n-algebras or, equivalently, of spectra, for $n\geq 0$;

(3) a Reedy cofibration if $X_n\sqcup_{L_nX}L_nY\to X_n$ is a cofibration of T_n-algebras for $n\geq 0$.

Note that L_0X is the initial object, and $M_0X=*$, the terminal object. The result is that with these definitions $(s\mathcal{S})^T$ becomes a model category. The standard preliminary result is this:

Lemma 2.2 *A morphism $f:X\to Y$ in $(s\mathcal{S})^T$ is*

(1) a Reedy trivial fibration if and only if

$$X_n\to X_n\times_{M_nY}M_nX$$

is a trivial fibration of T_n-algebras or, equivalently, of spectra, for $n\geq 0$;

(2) a Reedy trivial cofibration if and only if

$$X_n\sqcup_{L_nX}L_nY\to Y_n$$

is a trivial cofibration of T_n-algebras for all $n\geq 0$.

Proof. The push-out diagram of Proposition 1.6 leads to a push-out diagram of T_n-algebras

$$\begin{tikzcd} T_n\otimes_{T_{n-1}}(X_{n-1}\sqcup_{L_{n-1,k-1}X}L_{n-1,k-1}Y) \arrow[r]\arrow[d] & X_n\sqcup_{L_{n,k}X}L_{n,k}Y \arrow[d]\\ T_n\otimes_{T_{n-1}}Y_{n-1} \arrow[r] & X_n\sqcup_{L_{n,k+1}X}L_{n,k+1}Y \end{tikzcd}\tag{2.1}$$

since $Y_{n-1}\cong X_{n-1}\sqcup_{X_{n-1}}Y_{n-1}$.

We use this to prove part 2. Assume $f:X\to Y$ is a Reedy cofibration in $(s\mathcal{S})^T$. Then $f:X_0\to Y_0$ is a cofibration of T_0 algebras and this begins an induction argument where the inductive hypothesis is that

$$X_{n-1}\sqcup_{L_{n-1,k}X}L_{n-1,k}Y\to Y_{n-1}\tag{2.2}$$

is a cofibration of T_{n-1} algebras for all k, $-1 \leq k \leq n-1$. To complete the inductive step, use (2.1) to show

$$X_n \sqcup_{L_{n,k}X} L_{n,k}Y \to X_n \sqcup_{L_{n,k+1}X} L_{n,k+1}Y \tag{2.3}$$

is a cofibration of T_n algebras. Here one uses that $T_n \otimes_{T_{n-1}} (\cdot)$ preserves cofibrations (see [10]). Since $X_n \sqcup_{L_n X} L_nY \to Y_n$ is a cofibration of T_n algebras and compositions of cofibrations are cofibrations we have completed in the inductive step.

Now suppose $f : X \to Y$ is a Reedy trivial cofibration. Then, inductively, each of the maps of (2.2) is a trivial cofibration of T_n-algebras. In particular, we can set $n = k$ and get the desired result. The inductive step is slightly different: one uses (2.1) to show each of the maps of (2.3) in a trivial cofibration; then since $X_n \to Y_n$ is a weak equivalence, the "two-out-of-three" axiom for weak equivalences finishes the argument.

Conversely, suppose $X_n \sqcup_{L_n X} L_nY \to Y_n$ is a trivial cofibration of T_n-algebras for all n. Then the same sort of argument gives that each of the maps of (2.2) is a trivial cofibration of T_n-algebras. Then one sets $k = 0$.

To prove part 1, we use a similar argument after decomposing the matching objects M_nX. Let $\Delta^{n,k} = d^0\Delta^{n-1} \cup \cdots \cup d^k\Delta^{n-1} \subseteq \partial\Delta^n$, $-1 \leq k \leq n$. Then $\Delta^{n,-1} = \phi$ and $\Delta^{n,n} = \partial\Delta^n$. There are push-out diagrams

$$\begin{tikzcd}
\Delta^{n-1,k} \arrow[r] \arrow[d] & \Delta^{n-1} \arrow[d] \\
\Delta^{n,k} \arrow[r] & \Delta^{n,k+1}
\end{tikzcd}$$

for $-1 \leq k \leq n-1$. Define $M_{n,k}X = M_{\Delta^{n,k}}X$. Then Lemma 1.4.2 implies there is a pull-back diagram

$$\begin{tikzcd}
M_{n,k+1}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
M_{n,k}X \arrow[r] & M_{n-1,k}X.
\end{tikzcd}$$

We now prove that the lifting axiom of a model category holds for $(s\mathcal{S})^T$.

Lemma 2.3 *The Reedy structure on $(s\mathcal{S})^T$ satisfies Axiom CM 4.*

Proof. Suppose we are given a lifting problem in $(s\mathcal{S})^T$

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "q"] \\
B \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}$$

where j is a Reedy cofibration, q is a Reedy cofibration, and either j or q is a Reedy weak equivalence. We inductively solve the lifting problems

$$\begin{tikzcd}
sk^A_{n-1}B \arrow[r] \arrow[d] & X \arrow[d] \\
sk^A_nB \arrow[r] \arrow[ur, dashed] & Y,
\end{tikzcd}$$

for $n \geq 0$. By Proposition 1.5 it suffices to solve a lifting problem

$$\begin{tikzcd} L_n(j) \otimes \Delta^n \sqcup_{L_m(j)\otimes\partial\Delta^n} B_m \otimes \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ B_n \otimes \Delta^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

This is adjoint (by Lemma 1.4.1) to a lifting problem

$$\begin{tikzcd} A_n \sqcup_{L_n A} L_n B \arrow[r] \arrow[d] & X_n \arrow[d] \\ B_n \arrow[r] \arrow[ur, dashed] & X_n \times_{M_n Y} M_n X \end{tikzcd}$$

in S^{T_n}. This is solvable by Lemma 2.2.

We next show that the Reedy structure on $(s\mathcal{S})^T$ satisfies the factorization axiom.

Lemma 2.4 *The Reedy structure on $(s\mathcal{S})^T$ satisfies CM 5.*

Proof. We prove the "trivial cofibration-fibration" factorization. The other is identical. Fix $f : X \to Y$. To do this we inductively factor $sk_n X \to Y$ as

$$sk_n X \xrightarrow{j_n} Z(n) \xrightarrow{q_n} Y$$

so that $sk_{n-1}Z(n) \cong Z(n-1)$, there are commutative diagrams

$$\begin{tikzcd} sk_{n-1}X \arrow[r, "j_{n-1}"] \arrow[d] & Z(n-1) \arrow[r, "q_{n-1}"] \arrow[d] & Y \arrow[d] \\ sk_n X \arrow[r] & Z(n) \arrow[r, "q_n"] & Y \end{tikzcd}$$

and so that $X_n \sqcup_{L_n X} L_n Z(n) \to Z(n)_n$ is a trivial cofibration of T_n algebras and $Z(n)_n \to Y_n \times_{M_n Y} M_n Z(n)$ is a fibration. Then we may set $j = \operatorname{colim} j_n$, $q = \operatorname{colim} q_n$ and $Z = \operatorname{colim} Z(n)$ to obtain the needed factorization.

To define $Z(0)$, factor $X_0 \to Y_0$ as

$$X_0 \to Z_0 \to Y_0$$

where the first map is a trivial cofibration of T_0 algebras and the second a fibration. Let $Z(0) = r^* Z_0$, where $r^* : \mathcal{S}^{T_0} \to (s\mathcal{S})^T$ is left adjoint to the functor $X \mapsto X_0$.

Now suppose $Z(n-1)$ is defined. For any $X \in (s\mathcal{S})^T$ we have natural maps $L_n X \to X_n \to M_n X$ of T_n-algebras. In addition $L_n sk_{n-1} X = L_n X$ by (1.8). Hence we get a map of T_n algebras

$$X_n \sqcup_{L_n X} L_n Z(n-1) \to Y_n \times_{M_n Y} M_n Z(n-1).$$

Factor this map as

$$X_n \sqcup_{L_n X} L_n Z(n) \to Z_n \to Y_n \times_{M_n Y} M_n Z(n-1)$$

where the first map is a trivial cofibration of T_n algebras and the second a fibration. Define $Z(n)$ by the push-out diagram in $(s\mathcal{S})^T$

(2.4)
$$\begin{tikzcd} L_n Z(n-1) \otimes \Delta^n \sqcup_{L_n Z(n-1)\otimes\partial\Delta^n} Z_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & Z(n-1) \arrow[d] \\ Z_n \otimes \Delta^n \arrow[r] & Z(n). \end{tikzcd}$$

The map $Z_n \otimes \partial\Delta^n \to Z(n-1)$ is adjoint to $Z_n \to M_n Z(n-1)$. See Lemma 1.4. The map $X_n \to Z_n$ and the push-out diagram 1.6 supply $j_n : sk_n X \to Z(n)$. To define $q_n : Z(n) \to Y$ we must complete

$$\begin{tikzcd} Z(n-1) \arrow[r, "q_{n-1}"] \arrow[d] & Y \\ Z(n). \arrow[ur, dashed] & \end{tikzcd}$$

Using the push-out diagram 2.4 and an adjointness argument this is equivalent to finding a lift

$$\begin{tikzcd} L_n Z(n-1) \arrow[r] \arrow[d] & Y_n \arrow[d] \\ Z_n \arrow[r] \arrow[ur, dashed] & M_n Y. \end{tikzcd}$$

This lift was supplied by the factorization that produced Z_n. The inductive step is completed by noting that $L_n Z(n) = L_n Z(n-1)$ and $M_n Z(n) = M_n Z(n-1)$.

These two lemmas yield immediately:

Theorem 2.5 *The Reedy structure on $(s\mathcal{S})^T$ is the structure of a closed model category.*

This is in fact a topological model category. The categories $\mathcal{S}^{T_n}$ are topological model categories; that is, there are functors

$$(X, K) \mapsto X \square K \qquad \text{and} \qquad (X, K) \to X^K$$

from $\mathcal{S}^{T_n} \times \mathbf{Top}$ and $\mathcal{S}^{T_n} \times \mathbf{Top}^{\mathrm{op}}$ to $\mathcal{S}^{T_n}$ respectively and mapping space functors

$$\mathcal{S}^{T_n}(\cdot, \cdot) : \mathcal{S}^{T_n} \times (\mathcal{S}^{T_n})^{\mathrm{op}} \to \mathbf{Top}$$

which satisfy a number of adjoint relationships and are connected to the model category structure on $\mathcal{S}^{T_n}$ by the "corner axiom" SM 7. This structure can be extended to $(s\mathcal{S})^T$ is by the following formulas: if $X \in (s\mathcal{S})^T$ and $K \in \mathbf{Top}$, let

$$(X \square K)_n = X_n \square K \tag{2.5}$$

and

$$(X^K)_n = X_n^K. \tag{2.6}$$

If $X, Y \in (s\mathcal{S})^T$ define $(s\mathcal{S})^T(X, Y)$ by the equalizer diagram

$$(s\mathcal{S})^T(X, Y) \to \prod_n \mathcal{S}^{T_n}(X_n, Y_n) \overset{a}{\underset{b}{\rightrightarrows}} \prod_{\phi:[n]\to[m]} \mathcal{S}^{T_m}(X_m, Y_n) \tag{2.7}$$

where a and b are defined respectively by the diagrams

$$\begin{tikzcd} \prod_n \mathcal{S}^{T_n}(X_n, Y_n) \arrow[r, "a"] \arrow[d] & \prod_\phi \mathcal{S}^{T_m}(X_m, Y_n) \arrow[d] \\ \mathcal{S}^{T_n}(X_n, Y_n) \arrow[r, "\phi^*"] & \mathcal{S}^{T_m}(X_m, Y_n) \end{tikzcd}$$

and

$$
\begin{tikzcd}
\prod_n \mathcal{S}^{T_n}(X_n, Y_n) \arrow[r] \arrow[d] & \prod_\phi \mathcal{S}^{T_m}(X_m, Y_n) \arrow[d] \\
\mathcal{S}^{T_m}(X_m, Y_m) \arrow[r] & \mathcal{S}^{T_m}(X_m, Y_n).
\end{tikzcd}
$$

Note that $\phi^* : T_m \to T_n$ gives Y_n the structure of a T_m algebra.

We call this structure on $(s\mathcal{S})^T$ the internal topological structure.

Proposition 2.6 *With the internal topological structure, the Reedy structure on $(s\mathcal{S})^T$ becomes a topological model category.*

Proof. One need only check the analog of Quillen's SM7a, which need only be checked in $s\mathcal{S}$, where it is an easy (and classical) exercise.

One can ask to what extent the external simplicial structure on $(s\mathcal{S})^T$ defined in the previous section interacts with the Reedy structure. It is not a simplicial model category; however, we do have:

Proposition 2.7 *Let $f : X \to Y$ be a Reedy cofibration and $j : K \to L$ a cofibration of simplicial sets. Then*

$$X \otimes L \sqcup_{X \otimes K} Y \otimes K \to Y \otimes L$$

is a Reedy cofibration which is trivial if f is trivial.

Proof. The adjoint statement is: if $X \to Y$ is a Reedy fibration and $j : K \to L$ is a cofibration then

$$\mathbf{hom}(L, X) \to \mathbf{hom}(K, X) \times_{\mathbf{hom}(K,Y)} \mathbf{hom}(L, Y) \tag{2.8}$$

is a Reedy fibration which is trivial if f is trivial. This can be checked in $s\mathcal{S}$, where it is known to be true: indeed, in $s\mathcal{S}$ the adjoint statement to (2.8) is easy to check directly.

Finally, we note that the Reedy structure on $(s\mathcal{S})^T$ is cofibrantly generated. This follows from the fact that $\mathcal{S}^{T_n}$ is cofibrantly generated and Lemma 2.2. A simple way to see this is as follows. Give the category $s\mathcal{S}$ its Reedy model category structure. Then since $\mathcal{S}$ is cofibrantly generated, $s\mathcal{S}$ is cofibrantly generated. Specifically the morphisms

$$S^{-V} \wedge |\partial \Delta^n|_+ \to S^{-V} \wedge |\Delta^n|_+ \tag{2.9}$$

with $V \subseteq \mathcal{U}$ a finite dimensional sub-inner product spaces and $n \geq 0$ generate the cofibrations of $\mathcal{S} = \mathcal{S}_{\mathcal{U}}$ and

$$S^{-V} \wedge |\Delta^n_k|_+ \to S^{-V} \wedge |\Delta^n|_+ \tag{2.10}$$

with $V \subseteq \mathcal{U}$ finite dimensional and $0 \leq k \leq n$, $n \geq 1$, generate the trivial cofibrations. Then set of maps

$$A_\alpha \otimes \Delta^k \sqcup_{A_\alpha \otimes \partial \Delta^k} B_\alpha \otimes \partial \Delta^k \to B_\alpha \otimes \Delta^k, \qquad k \geq 0, \tag{2.11}$$

with $A_\alpha \to B_\alpha$ among the maps of (2.10) generate the cofibrations of $s\mathcal{S}$ and

$$C_\beta \otimes \Delta^k \sqcup_{C_\beta \otimes \partial \Delta^k} D_\beta \otimes \partial \Delta^k \to D_\beta \otimes \Delta^k, k \geq 0 \tag{2.12}$$

with $C_\beta \to D_\beta$ among the maps of (2.10) generate the trivial cofibrations of $s\mathcal{S}$. We regard any object in $\mathcal{S}$ as a constant object in $s\mathcal{S}$.

Proposition 2.8 *The Reedy model category structure on $(s\mathcal{S})^T$ is cofibrantly generated. The cofibrations are generated by*

$$T(A_\alpha) \otimes \Delta^k \sqcup_{T(A_\alpha)\otimes\partial\Delta^k} T(B_\alpha) \otimes \partial\Delta^k \to T(B_\alpha) \otimes \Delta^k \tag{2.13}$$

where $A_\alpha \to B_\alpha$ runs over the generators for the cofibrations of $\mathcal{S}$. The trivial cofibrations are generated by

$$T(C_\beta) \otimes \Delta^k \sqcup_{T(C_\beta)\otimes\partial\Delta^k} T(D_\beta) \otimes \partial\Delta^k \to T(D_\beta) \otimes \Delta^k. \tag{2.14}$$

Proof. The forgetful functor $U^T : (s\mathcal{S})^T \to s\mathcal{S}$ has left adjoint $F^T X = T(X)$. Since $F^T(X \otimes_{\mathcal{S}} K) = T(X) \otimes K = F^T(X) \otimes K$, by construction, the maps of (2.13) and (2.14) are obtained by applying F^T to those of (2.11) and (2.12) respectively.

Now $f : X \to Y$ in $(s\mathcal{S})^T$ is a Reedy fibration or Reedy weak equivalence if and only if $U^T f$ is such. We use F^T to lift the Reedy model category structure on $s\mathcal{S}$ to $(s\mathcal{S})^T$. The result will follow from this.

Since U^T commutes with filtered colimits two points remain. First we must show the objects $T(B_\alpha) \otimes \Delta^k$ and $T(D_\beta) \otimes \Delta^k$ are small; that is

$$\mathrm{Hom}_{(s\mathcal{S})^T}(T(B_\alpha) \otimes \Delta^k, \operatorname*{colim}_n X_n) \cong \operatorname*{colim}_n \mathrm{Hom}_{(s\mathcal{S})^T}(T(B_\alpha) \otimes \Delta^k, \operatorname*{colim}_n X_n)$$

where $X_0 \to X_1 \to \cdots$ is a directed system over the natural numbers and the morphisms $X_{i-1} \to X_i$ fit into push-out diagrams in $(s\mathcal{S})^T$

$$\begin{tikzcd} T(W) \arrow[r] \arrow[d] & X_{i-1} \arrow[d] \\ T(Z) \arrow[r] & X_i \end{tikzcd}$$

where $W \to Z$ is a Reedy cofibration in $s\mathcal{S}$. Now

$$\mathrm{Hom}_{(s\mathcal{S})^T}(T(B_\alpha) \otimes \Delta^k, Y) = \mathrm{Hom}_{\mathcal{S}}(B_\alpha, Y_k)$$

so it is sufficient to know $(X_{i-1})_k \to (X_i)_k$ is a closed inclusion of spectra. For this (see [10]) it suffices to know $(X_{i-1})_k \to (X_i)_k$ is a cofibration of T_k-algebras. This follows from the fact that $T(W) \to T(Z)$ is a Reedy cofibration in $(s\mathcal{S})^T$, which implies $T(W)_k \to T(Z)_k$ is a cofibration of T_k-algebras.

The second point is to show every cofibration with the left lifting property with respect to all Reedy fibrations is a weak equivalence. But this follows from the fact that we already have a model category structure.

3 Geometric realization

The main result of this section is that there is a geometric realization functor $(s\mathcal{S})^T \to \mathcal{S}^{|T|}$ with good homotopical properties.

We begin by noting that since the category of operads $\mathcal{O}$ over the linear isometries operad is a topological model category, and that the category $s\mathcal{O}$ inherits a Reedy model category structure, which has an inherited "level-wise" topological model category structure. (Compare Proposition 2.6). There is a geometric realization functor

$$|\cdot| : s\mathcal{O} \to \mathcal{O}$$

which sends level-wise weak equivalences between Reedy cofibrant simplicial operads to weak equivalences. Also we note that every object in $s\mathcal{O}$ is a simplicial symmetric space, meaning a set of spaces $\{A_n\}_{n\geq 0}$ such that each A_n has an action on the symmetric group on n letters. Call the category of such $\mathbf{Top}^{\Sigma}$. For later use we record

Lemma 3.1 *Let $T \in s\mathcal{O}$, and let $|\cdot|_{\mathcal{O}} : s\mathcal{O} \to \mathcal{O}$ and $|\cdot| : s\mathbf{Top}^{\Sigma} \to \mathbf{Top}^{\Sigma}$ be the geometric realization functors. Then there is a natural isomorphism in $\mathbf{Top}^{\Sigma}$,*

$$|T|_{\mathcal{O}} \cong |T|.$$

Proof. See [19] for the simplicial case. The argument goes through verbatim.

Now let $T \in s\mathcal{O}$ and $K \in \mathbf{Top}$. Then the set of symmetric spaces $\{T_n^K\}$ is also a simplicial operad and $T^K \in \mathcal{O}$ is defined by the pull-back diagram

$$\begin{tikzcd} T^K \arrow[r] \arrow[d] & \{T_n^K\} \arrow[d] \\ \mathcal{L} \arrow[r] & \{\mathcal{L}_n^K\} \end{tikzcd}$$

Here $\mathcal{L}$ is the linear isometries operad and $\mathcal{L} \to \{\mathcal{L}_n^K\}$ is induced by $K \to *$.

Lemma 3.2 *Let $T \in \mathcal{O}$ and $X \in \mathcal{S}^T$. If $K \in \mathbf{Top}$, then X^K is naturally a T^K algebra.*

Proof. Let $A \to \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $X \in \mathcal{S}_{\mathcal{U}}$. If $K \in \mathbf{Top}$, then there is a natural map of spectra over $\mathcal{V}$

$$A^K \ltimes X^K \to (A \ltimes X)^K$$

adjoint to the composite

$$\begin{aligned} (A^K \ltimes X^K)\Box K &\overset{1\Box\Delta}{\to} (A^K \ltimes X^K)\Box(K \times K) \\ &\cong (A^K \times K) \ltimes (X^K \Box K) \to A \ltimes X. \end{aligned}$$

Furthermore, if $X \in \mathcal{S}_{\mathcal{U}}$ and $K \in \mathbf{Top}$, then there is a map in $\mathcal{S}_{\mathcal{U}^n}^{\Sigma_n}$

$$\underbrace{X^K \wedge X^K \wedge \cdots \wedge X^K}_{n} \to \underbrace{(X \wedge \cdots \wedge X)}_{n}{}^K$$

adjoint to

$$(\bigwedge_n X^K)\Box K \xrightarrow{1\Box\Delta} (\bigwedge_n X^K)\Box K^n \cong \bigwedge_n (X^K \Box K) \to \bigwedge_n X.$$

Combining these two observations yields a map

$$T_n^K \ltimes_{\Sigma_n} (\bigwedge_n X^K) \to (T_n \ltimes \bigwedge_n X)^K.$$

Following by the structure map of the T algebra structure on X yields the T^K structure on X^K.

Now let T be a simplicial operad over the linear isometries operad. Then we may form the geometric realization $|T|$ and this is again an operad over the linear isometries operad. The functor

$$|\cdot| : s\mathcal{O} \to \mathcal{O}$$

has right adjoint $S \to S^{\Delta} = \{S^{\Delta^n}\}$; hence there is a natural map $T \to |T|^{\Delta}$ of simplicial operads. If $Y \in \mathcal{S}^{|T|}$, then, by Lemma 3.2, $Y^{\Delta} \in (s\mathcal{S})^T$, and we claim the functor

$$(\cdot)^{\Delta} : \mathcal{S}^{|T|} \to (s\mathcal{S})^T$$

has a left adjoint, which we will call the geometric realization. In fact, we will prove that the geometric realization of $X \in (s\mathcal{S})^T$, regarded simply as a simplicial spectrum, has a canonical $|T|$ algebra structure and, with this structure, the functor $X \mapsto |X|$ is the required adjoint.

We start with:

Lemma 3.3 *Let $Z \in s\mathcal{S}$. Then there is a natural isomorphism in $\mathcal{S}$*

$$|TZ| \cong |T|\,|Z|.$$

Proof. Notice that TZ is the diagonal of a bisimplicial object:

$$TZ = \operatorname{diag} T.Z.$$

where $(T.Z.)_{p,q} = T_pZ_q$. Let p be the horizontal direction and q the vertical direction. Then

$$|TZ| = |\,|T.Z.|_v|$$

where the v denotes realization in the vertical direction. Now for fixed p, $(|T.Z.|_v)_p = |T_pZ| \cong T_p|Z|$. The last isomorphism uses [10]. Thus we need to show

$$|T.|Z|\,| \cong |T|\,|Z|$$

or, put another way, we may assume Z is constant.

For fixed Z, the functor $\mathbf{Top}^\Sigma \downarrow \mathcal{L} \to \mathcal{S}$ given by $A \mapsto AZ$ has a continuous right adjoint given by

$$Y \mapsto \{\mathcal{S}(\bigwedge_n Z, Y)\}_n.$$

Thus the functor $A \mapsto AZ$ commutes with geometric realization: if $A \in s\mathbf{Top}^\Sigma$, then $|A|Z \cong |AZ|$. But if $T \in s\mathcal{O}$, the realization of T as a simplicial operad is isomorphic to the realization of T as an object in $s\mathbf{Top}^\Sigma \downarrow \mathcal{L}$, by Lemma 3.1.

Proposition 3.4 *Let $Z \in (s\mathcal{S})^T$. Then the geometric realization $|Z|$ of Z, regarded as an object in $s\mathcal{S}$, has a natural structure as a $|T|$ algebra and the resulting functor*

$$|\cdot| : (s\mathcal{S})^T \to \mathcal{S}^{|T|}$$

is left adjoint to $Y \mapsto Y^\Delta$.

Proof. Let $\epsilon : TZ \to Z$ be the T-algebra structure map. Then

$$|T|\,|Z| \cong |TZ| \xrightarrow{|\epsilon|} |Z|$$

gives $|Z|$ the structure of a $|T|$ algebra. To get the adjointness we first show for $Y \in \mathcal{S}^{|T|}$ that the map of spectra

$$|Y^\Delta| \to Y$$

is a map of $|T|$ algebras; this will yield a map

$$(s\mathcal{S})^T(Z, Y^\Delta) \to \mathcal{S}^{|T|}(|Z|, Y)$$

which we will then show is an isomorphism.

If $Y \in \mathcal{S}^{|T|}$, the T-algebra structure on Y^Δ is specified by the diagram

$$\begin{tikzcd} T(Y^\Delta) \arrow[r] \arrow[d] & |T|^\Delta Y^\Delta \arrow[r] \arrow[d] & (|T|Y)^\Delta \arrow[d] \\ Y^\Delta \arrow[r, "="] & Y^\Delta \arrow[r, "="] & Y^\Delta. \end{tikzcd}$$

Taking geometric realizations yields, by Lemma 3.3,

$$\begin{tikzcd} |T|\,|Y^\Delta| \arrow[r] \arrow[d] & |\,|T|^\Delta|\,|Y^\Delta| \arrow[r] \arrow[d] & |(|T|Y)^\Delta| \arrow[r] \arrow[d] & |T|Y \arrow[d] \\ |Y^\Delta| \arrow[r, "="] & |Y^\Delta| \arrow[r, "="] & |Y^\Delta| \arrow[r] & Y. \end{tikzcd}$$

This proves the first claim. For the second, let $Z \in (s\mathcal{S})^T$. Then there is a coequalizer diagram in $s\mathcal{S}$

$$T^2 Z \rightrightarrows TZ \to Z; \tag{3.1}$$

hence a coequalizer diagram in $\mathcal{S}$

$$|T|^2|Z| \rightrightarrows |T|\,|Z| \to |Z|. \tag{3.2}$$

Again we use Lemma 3.3. Since these coequalizers are split coequalizers, they are in fact coequalizers in $(s\mathcal{S}^T)$ and $\mathcal{S}^{|T|}$ respectively. Now consider the induced maps of equalizers

$$\begin{tikzcd} (s\mathcal{S})^T(T^2Z, Y^\Delta) \arrow[d] & (s\mathcal{S})^T(TZ, Y^\Delta) \arrow[l, shift left] \arrow[l, shift right] \arrow[d] & (s\mathcal{S})^T(Z, Y^\Delta) \arrow[l] \arrow[d] \\ \mathcal{S}^{|T|}(|T|^2|Z|, Y) & \mathcal{S}^{|T|}(|T|\,|Z|, Y) \arrow[l, shift left] \arrow[l, shift right] & \mathcal{S}^{|T|}(|Z|, Y). \arrow[l] \end{tikzcd}$$

If we can show the map $(s\mathcal{S})^T(TZ, Y^\Delta) \to \mathcal{S}^{|T|}(|T|\,|Z|, Y)$ is an isomorphism for all Z, then the two left-most vertical arrows in this diagram will be isomorphisms and, then, so will the third—which is what is required. The following calculation finishes the proof:

$$\begin{aligned} (s\mathcal{S})^T(TZ, Y^\Delta) &\cong (s\mathcal{S})(Z, Y^\Delta) \\ &\cong \mathcal{S}(|Z|, Y) \\ &\cong \mathcal{S}^{|T|}(|T|\,|Z|, Y). \end{aligned}$$

We now come to the homotopical properties of realization. We give $(s\mathcal{S})^T$ the Reedy model category structure.

Proposition 3.5 *The geometric realization functor* $|\cdot| : (s\mathcal{S})^T \to \mathcal{S}^{|T|}$ *preserves cofibrations, trivial cofibrations, and weak equivalences between Reedy cofibrant objects.*

Proof. First note that if $K \in \mathbf{Top}$, then $|Z \square K| \cong |Z| \square K$. This is because if $Y \in \mathcal{S}^{|T|}$, then $(Y^\Delta)^K \cong (Y^K)^\Delta$ in $(s\mathcal{S})^T$. The result will now follow once we show $(\cdot)^\Delta : \mathcal{S}^{|T|} \to s\mathcal{S}^T$ preserves fibrations and trivial fibrations.

To do this we calculate that $M_n(Y^\Delta) \cong Y^{\partial\Delta^n}$. This is because if $\mathcal{K}$ is the simplex category of $\partial\Delta^n$, then

$$
\begin{aligned}
M_n Y &\cong \lim_{\mathcal{K}^{\mathrm{op}}} (Y^\Delta)_k \\
&\cong \lim_{\mathcal{K}^{\mathrm{op}}} Y^{\Delta^k} \cong Y^{\operatorname{colim}_{\mathcal{K}} \Delta^k} \cong Y^{\partial\Delta^n}.
\end{aligned}
\tag{3.3}
$$

Thus if $f : X \to Y$ is a morphism in $\mathcal{S}^{|T|}$, then

$$(Y^\Delta)_n \to (Y^\Delta)_n \times_{M_n Y^\Delta} M_n X^\Delta$$

is isomorphic to

$$X^{\Delta_n} \to Y^{\Delta_n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n}.$$

If f is a fibration in $\mathcal{S}^{|T|}$, this is a fibration in $\mathcal{S}^{|T|}$, hence in $\mathcal{S}^{T_n}$. The same applies if f is a trivial fibration.

4 The resolution model category structure

We refine the Reedy model category structure on $(s\mathcal{S})^T$ to give cofibrant objects which are much more free with respect to the operad structure. The principle reference for this section is [11], which depends heavily on the work of Dwyer, Kan, and Stover [5, 6].

We choose a set of $\mathcal{A} = \{Ai\}$, $i \in I$, of finite CW spectra with the property if $Aj \in \mathcal{A}$, then $\Sigma Aj \in \mathcal{A}$. Our main (in fact, only) example is the following. Let E be a homotopy associative ring spectrum which satisfies the condition Adams imposed in order to get a universal coefficient spectral sequence [1, §III.13]. Thus there is an isomorphism $\operatorname{colim}_{\alpha \in I} E_\alpha \cong E$, where E_α runs over a filtered system of finite CW spectra with the properties that

1) $E_* DE_\alpha$ is projective over E_*, and
2) for all E module spectra F, the natural map

$$F^* DE_\alpha \to \operatorname{Hom}^*_{E_*}(E_* DE_\alpha, F_*)$$

is an isomorphism.

3) $E_\alpha = S^0$ for some α.

Here $D(\cdot)$ denotes Spanier-Whitehead duality. The final condition was not listed by Adams, but can be included without loss of generality, and it helps us later. Many spectra satisfy this hypothesis; for example, any spectrum representing a Landweber exact homology theory. Then $\mathcal{A} = \{\Sigma^n DE_\alpha\}$ where α is as above and n runs over the integers.

As a bit of terminology, we say that a *Reedy fibrant replacement* of an arrow $X \to Y$ in $s\mathcal{S}^T$ is a diagram

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & X' \arrow[d] \\
Y \arrow[r] & Y'
\end{tikzcd}
$$

where the horizontal maps are weak equivalences and the the objects X' and Y' are Reedy fibrant.

Definition 4.1 A morphism $f : X \to Y$ is

1) a $T(\mathcal{A})$-equivalence if for some Reedy fibrant replacement $f' : X' \to Y'$ of f, the simplicial set of maps

$$\mathbf{map}_{(s\mathcal{S})^T}(T(Ai), X') \to \mathbf{map}_{(s\mathcal{S})^T}(T(Ai), Y')$$

is a weak equivalence of simplicial sets for all i;

2) a $T(\mathcal{A})$-fibration if

$$\mathbf{map}_{(s\mathcal{S})^T}(T(Ai), X) \to \mathbf{map}_{(s\mathcal{S})^T}(T(Ai), Y)$$

is a fibration for all i;

3) a $T(\mathcal{A})$-cofibration if it has the left lifting property with respect to all $T(\mathcal{A})$-trivial fibrations.

We make some remarks on this definition. The category $s\mathcal{S}$ with the analogous definitions (using Ai instead of $T(Ai)$) is a simplicial model category using the standard external simplicial structure on $s\mathcal{S}$. This is the point of section 2 of [11]. In particular if $X \in s\mathcal{S}$ is Reedy fibrant it is $\mathcal{A}$-fibrant, by Lemma 2.9 of [11]. Since $\mathbf{map}_{(s\mathcal{S})^T}(T(Ai), X) \cong \mathbf{map}_{s\mathcal{S}}(Ai, X)$, the simplicial sets in 4.1.1 are, in fact, fibrant. In addition, Lemma 2.7 of [11] implies a morphism $f : X \to Y$ is an $\mathcal{A}$-equivalence independent of the choice of Reedy fibrant replacement. More is true: if we equip $s\mathcal{S}$ with this $\mathcal{A}$-resolution model category, we are saying $f : X \to Y$ in $s\mathcal{S}^T$ is an $\mathcal{A}$-equivalence or $\mathcal{A}$ fibration in $s\mathcal{S}^T$ if and only if it is in $s\mathcal{S}$. Finally, the $\mathcal{A}$-resolution model category is cofibrantly generated. The generators for the cofibrations are

$$\begin{gathered} Ai \otimes \partial\Delta^n \to Ai \otimes \Delta^n, i \in I, n \geq 0 \\ C_\beta \otimes \Delta^n \sqcup_{C_\beta \otimes \partial\Delta^n} D_\beta \otimes \partial\Delta^n \to D_\beta \otimes \partial\Delta^n, n \geq 0 \end{gathered} \tag{4.1}$$

where $C_\beta \to D_\beta$ runs over the generators for the trivial cofibrations of $\mathcal{S}$ (see Equations 2.9 – 2.12 of this paper). The generators for the trivial cofibrations are

$$\begin{gathered} Ai \otimes \Delta^n_k \to Ai \otimes \Delta^n, i \in I, 0 \leq k \leq n, n \geq 1 \\ C_\beta \otimes \Delta^n \sqcup_{C_\beta \otimes \partial\Delta^n} D_\beta \otimes \partial\Delta^n \to D_\beta \otimes \Delta^n, n \geq 0. \end{gathered} \tag{4.2}$$

Theorem 4.2 *With the definition of $T(\mathcal{A})$-resolution equivalence, fibration, and cofibration and in its standard external simplicial structure, $(s\mathcal{S})^T$ becomes a simplicial model category. It is cofibrantly generated, with generators obtained by applying $T(\cdot)$ to the morphisms of Equations 4.1 and 4.2.*

Proof. As in the proof of Proposition 2.8 we lift the model category structure from $s\mathcal{S}$ to $(s\mathcal{S})^T$. The argument is the same except we must find some other way to prove that $T(\mathcal{A})$-cofibration with the left lifting property with respect to all fibrations is a weak equivalence. By Quillen [17, §II.4] it is sufficient to find, for all $X \in (s\mathcal{S})^T$ a natural $T(\mathcal{A})$-equivalence $X \to Y$ where Y is $T(\mathcal{A})$-fibrant. Since Reedy weak equivalences are $\mathcal{A}$-equivalences (Corollary 2.8 of [11]) and any Reedy fibrant object in $\mathcal{A}$-fibrant (Lemma II2.9) we take a Reedy equivalence $X \to Y$ in $(s\mathcal{S})^T$ with Y Reedy fibrant. This can be done naturally since $(s\mathcal{S})^T$ in Reedy structure, is cofibrantly generated.

To check SM7 it is sufficient to check SM7a, which can be done in $s\mathcal{S}$.

The weak equivalences in the resolution model category structure can be characterized in another manner. If $X \in (s\mathcal{S})^T$,

$$\begin{aligned} \pi_n^\natural(X; Ai) &= \pi_n \mathbf{map}_{(s\mathcal{S})^T}(T(Ai), X') \\ &\cong \pi_n \mathbf{map}_{s\mathcal{S}}(Ai, X') \end{aligned} \tag{4.3}$$

where $X \to X'$ is some weak equivalence with X' Reedy fibrant, and with $T(Ai) \to T(*) \to X'$ as the basepoint. Then $f : X \to Y$ in $(s\mathcal{S})^T$ is a $T(\mathcal{A})$-equivalence if and only if $\pi_n^\natural(f, Ai)$ is an isomorphism for all n. This is because $\mathbf{map}_{s\mathcal{S}}(Ai, X')$ is a fibrant homotopy associative H-space with unit. There are other homotopy groups associated to X and Ai. Let $[\cdot,\cdot]_{T_n}$ denote homotopy classes of maps of T_n-algebras and $[\cdot,\cdot]$ homotopy classes of maps of spectra. If $X \in s\mathcal{S}^T$, then,

$$[T_n(Ai), X_n]_{T_n} \cong [Ai, X_n]$$

is an abelian group, and these groups assemble into a simplicial abelian group $[T(Ai), X]_T \cong [Ai, X]$. Define the groups

$$\pi_n(X; Ai) = \pi_n[Ai, X]. \tag{4.4}$$

Then, because an object $X \in (s\mathcal{S})^T$ is Reedy fibrant if and only if X is Reedy fibrant in $s\mathcal{S}$, Proposition 4.4 of [11] implies

$$\pi_0^\natural(X; Ai) \cong \pi_0(X; Ai) \tag{4.5}$$

for all i and that there is a natural long exact sequence

$$\cdots \to \pi_{n-1}^\natural(X; \Sigma Ai) \to \pi_n^\natural(X; Ai) \to \pi_n(X; Ai) \to \pi_{n-2}^\natural(X; Ai) \to \cdots \to \pi_1(X; Ai) \to 0. \tag{4.6}$$

This is the *spiral exact sequence* and we have

Proposition 4.3 *A morphism $f : X \to Y$ is an $T(\mathcal{A})$-equivalence if and only if*

$$\pi_n(f; Ai) : \pi_n(X; Ai) \to \pi_n(Y; Ai)$$

is an isomorphism for all n and i.

We now want to talk about particular $T(\mathcal{A})$-cofibrant models for objects in $(s\mathcal{S})^T$. We begin with some terminology. Let $\Delta_+ \subseteq \Delta$ be the subcategory with the same objects, but only surjections as morphism. An object $X \in s_+\mathcal{C} = \mathcal{C}^{\Delta_+^{\mathrm{op}}}$ is a "simplicial object with degeneracies only." Such an object is *free* if X is obtained by left Kan extension of a trivial diagram $\{Z_n\}_{n\geq 0}$. Thus

$$X_n = \coprod_{\phi:[n]\to[k]} Z_k$$

with $\phi \in \Delta_+$. An object $X \in s\mathcal{C}$ is s-free if it is free in $s_+\mathcal{C}$.

We we also need another related category. Let $\Delta_0 \subseteq \Delta$ be the subcategory with the same objects, but we require that morphism $\phi; [m] \to [n]$ sends 0 to 0. An object $X \in s_0\mathcal{C} = \mathcal{C}^{\Delta_0}$ is a "simplicial object without d_0". The importance of these objects is that if $X \in s_0C$, then the natural maps $L_nX \to X_n$ from the latching objects are injections. Note that we can speak of an object in $s_0\mathcal{C}$ as being s-free.

An object $Z \in \mathcal{S}^{T_n}$ is contractible if the map $T_n(*) \to Z$ from the initial object is a trivial cofibration.

Definition 4.4 An object $X \in (s\mathcal{S})^T$ is homotopically $T(\mathcal{A})$-free if there is a morphism $Z \to X$ in $(s_+\mathcal{S})^T$ so that

1) $Z_n \sqcup_{L_nZ} L_nX \to X_n$ and $L_nZ \to Z_n$ are cofibrations of T_n-algebras;

2) $Z_n \in \mathcal{S}^{T_n}$ is contractible;

3) there is a push-out diagram in $(s_+\mathcal{S})^T$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ T(*) \arrow[r] & \overline{X} \end{tikzcd}$$

where $\overline{X}$ is free on $\{Y_n\}$ where each Y_n is isomorphic to a coproduct of $T_n(Ai)s$.

Note that the definition of homotopically $T(\mathcal{A})$-free can be extended to objects of $(s_+\mathcal{S})^T$. More generally we have

Definition 4.5 A morphism $A \to X$ in $(s\mathcal{S})^T$ is homotopically $T(\mathcal{A})$-free if

1) as a morphism in $(s_+\mathcal{S})^T$, $A \to X$ is isomorphic to $A \to A \sqcup Y$ for some $Y \in (s_+\mathcal{S})^T$

2) Y is homotopically $T(\mathcal{A})$-free.

The following gives a recognition principal for cofibrations.

Proposition 4.6 *Any homotopically $T(\mathcal{A})$-free map is an $T(\mathcal{A})$-cofibration in $(s\mathcal{S})^T$.*

Proof. The argument is the same as in Proposition 5.4 of [11]. The crucial observation is that if D is a cofibrant spectrum

$$[T_n(D), Z]_{T_n} \cong [D, Z].$$

This is necessary so that Proposition 5.1 of [11] applies.

We can now give, for each $X \in (s\mathcal{S})^T$, a particular model for the $T(\mathcal{A})$-cofibrant replacement of X. We use the device of the Stover resolution; let us recall how this is defined.

Let $\{Ai\}_{i\in I}$, be the set of objects used to define the $\mathcal{A}$-resolution model category structure. For each $i \in I$, let $CAi = Ai \wedge [0,1]$ with $\{1\} \in [0,1]$ the basepoint; CAi is the cone on Ai. Let J be the category $I \times \{0 \to 1\}$ where I is regarded as a discrete category. The inclusions $Ai \to CAi$ define a diagram $\bar{\mathcal{A}} : J \to \mathcal{S}$ with $\bar{\mathcal{A}}[(i,0) \to (i,1)] = Ai \to CAi$.

Fix an integer $n \geq 0$. Then there is a functor

$$\mathrm{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), \cdot) : \mathcal{S}^{T_n} \to \mathbf{Sets}^{J^{\mathrm{op}}}$$

with left adjoint $F \mapsto T_n(\bar{\mathcal{A}}) \otimes_J F$. There is a push-out diagram in $\mathcal{S}^{T_n}$

$$\begin{tikzcd}
\coprod_i \coprod_{F(i,1)} T_n(Ai) \arrow[r] \arrow[d] & \coprod_i \coprod_{F(i,0)} T_n(Ai) \arrow[d] \\
\coprod_i \coprod_{F(i,1)} T_n(CAi) \arrow[r] & T_n(\bar{\mathcal{A}}) \otimes_J F.
\end{tikzcd}$$

In particular we have $T_n(\bar{\mathcal{A}}) \otimes_J F \cong T_n(\bar{\mathcal{A}} \otimes_J F)$, where the tensor product must be interpreted in the relevant category.

Let $\overline{V}^{T_n} : \mathcal{S}^{T_n} \to \mathcal{S}^{T_n}$ be the composite functor

$$\overline{V}^{T_n} = T_n(\bar{\mathcal{A}}) \otimes_J \mathrm{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), \cdot).$$

Then for each $Z \in \mathcal{S}^{T_n}$ there is an augmented simplicial object

$$\epsilon : V^{T_n}(Z) \to Z$$

with $(V^{T_n}Z)_k = (\overline{V}^{T_n})^{k+1}Z$ with the following properties

4.6.1) for all i, the augmentation ϵ induces an isomorphism

$$\pi_*\pi_q(V^{T_n}(Z); T_n(Ai)) \cong \pi_q(Z; T_n(A_i))$$

concentrated in degree 0, and

4.6.2) as objects in $s_0\mathcal{S}^{T_n}$, there is an isomorphism

$$\begin{aligned} V^{T_n}(X) &\cong T_n(\bar{\mathcal{A}}) \otimes_J \mathrm{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), (\overline{V}^{T_n})^{\cdot} X) \\ &\cong T_n[\bar{\mathcal{A}} \otimes_J \mathrm{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), (\overline{V}^{T_n})^{\cdot} X)]. \end{aligned}$$

See [11, §4] for the results.

We extend this construction to the simplicial setting. Let $X \in (s\mathcal{S})^T$, where $T = \{T_n\}$ is a simplicial operad. If $\phi^* : T_n \to T_m$ is induced by an ordinal number map, then ϕ^* gives X_m the structure of a T_n-algebra and $\phi^* : X_n \to X_m$ is a morphism of T_n-algebras. Thus we get maps

$$\mathrm{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), X_n) \to \mathrm{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), X_m) \cong \mathrm{Hom}_{T_m}(T_m(\bar{\mathcal{A}}), X_m)$$

and, hence a functor

$$\mathrm{Hom}_T(\mathcal{A}, \cdot) : (s\mathcal{S})^T \to (s\,\mathbf{Sets})^{J^{\mathrm{op}}}.$$

This functor has a left adjoint: if $F \in (s\,\mathbf{Sets})^{J^{\mathrm{op}}}$, the adjoint is given by $T(\bar{\mathcal{A}}) \otimes_J F = \{T_n(\bar{\mathcal{A}}) \otimes_J F_n\}$. Note that if $\phi : [m] \to [n]$ is an ordinal number map, then we have

$$T_n(\bar{\mathcal{A}}) \otimes_J F_n \xrightarrow{\phi^* \otimes \phi^*} T_m(\bar{\mathcal{A}}) \otimes_J F_m$$

is a morphism of T_n algebras; hence $T(\bar{\mathcal{A}}) \otimes_J F \in (s\mathcal{S})^T$.

Let $\overline{V}^T = T(\bar{\mathcal{A}}) \otimes_J \mathrm{Hom}_T(T(\bar{\mathcal{A}}), \cdot) : (s\mathcal{S})^T \to (s\mathcal{S})^T$ be the composite. Then $(\overline{V}^{TX})_n = \overline{V}^{T_n} X_n$. If $X \in (s\mathcal{S})^T$, define $\epsilon : \overline{V}^T_{\cdot}(X) \to X$ to be the augmented simplicial object in $(s\mathcal{S})^T$ with $(\overline{V}^T_{\cdot}(X))_k = (\overline{V}^T)^{k+1} X$. Finally, let

$$V(X) = \mathrm{diag}\ \overline{V}^T_{\cdot}(X)$$

be the diagonal. This is an object in $(s\mathcal{S})^T$ equipped with a natural map

$$\epsilon : V(X) \to X$$

induced from the augmentation of $\overline{V}^T_{\cdot}(X)$.

Proposition 4.7 *For $X \in (s\mathcal{S})^T$, the natural map*

$$\epsilon : V(X) \to X$$

is a $T(\mathcal{A})$-equivalence. Furthermore, $V(X)$ is homotopically $T(\mathcal{A})$-free, and for each $n \geq 0$, there is a cofibrant spectrum Z_n so that

$$V(X)_n \cong T_n(Z_n).$$

Proof. That the morphism ϵ is an $\mathcal{A}$-equivalence follows from 4.6.1) and a bicomplex argument. And, as objects in $(s_+\mathcal{S})^T$, there is a natural isomorphism, by 4.6.2,

$$\begin{aligned} V(X) &\cong T(\bar{\mathcal{A}}) \otimes_J \mathrm{diag}\ \mathrm{Hom}_T(T(\bar{\mathcal{A}}), (\overline{V}^T)^{\cdot} X) \\ &\cong T(\bar{\mathcal{A}} \otimes_J \mathrm{diag}\ \mathrm{Hom}_T(T(\bar{\mathcal{A}}), (\overline{V}^T)^{\cdot} X). \end{aligned}$$

The rest of the result now follows from Proposition 4.7 of [11].

If necessary, we can further refine this result. To do this we use the technique of subdivision reviewed in the appendix to [11]. To recapitulate, let $K \in c\mathbf{S}$ be a cosimplicial simplicial set and let $X \in (s\mathcal{S})^T$. Then we can define a coend $\int^\Delta X \otimes K$; in fact, this coend is given explicitly by the coequalizer diagram

$$\coprod_{\phi:[n]\to[m]} X_m \otimes K^n \rightrightarrows \coprod_n X_n \otimes K^n \to \int^\Delta X \otimes K \tag{4.7}$$

where ϕ runs over the morphisms of Δ, the coproduct is in $(s\mathcal{S})^T$, and the two maps to be equalized evaluate ϕ in the X variable and the K variable respectively. For example, if $K = \Delta[\cdot] = \{\Delta^n\}$, we note that if k-simplices of Δ^n are all maps in Δ

$$\phi : [k] \to [n].$$

There is a map of T_k algebras

$$(\coprod_n X_n \otimes \Delta^n)_k \cong \coprod_n \coprod_{\phi:[k]\to[n]} X_{n,k} \to X_{k,k}$$

evaluating ϕ in the first variable. This map coequalizes the two maps of Equation 4.7 and defines an isomorphism

$$\int^\Delta X \otimes \Delta[\cdot] \cong \operatorname{diag} X. \tag{4.8}$$

In short, the generalized Eilenberg-Zilber Theorem holds in $(s\mathcal{S})^T$.

It is a standard result that these constructions preserve certain types of weak equivalences. We give cosimplicial spaces the Bousfield-Kan model category structure [2]; this is the Reedy structure on $(s(\mathbf{S}^{\mathrm{op}}))^{\mathrm{op}}$. Cofibrant objects have a simple characterization: $K \in c\mathbf{S}$ is cofibrant if and only if the equalizer of d^0, $d^1 : K^0 \to K^1$ is empty. We give $(s\mathcal{S})^T$ its Reedy model category structure of Theorem 2.5. The topological model category structure of Proposition 2.6 gives the Reedy model category structure on $(s\mathcal{S})^T$ an induced simplicial model category structure. Give $s(s\mathcal{S})^T$ the inherited Reedy model category structure. Then Proposition A.7 of [11] implies that if $K \to L$ is a weak equivalence between cofibrant objects in $c\mathbf{S}$ and $X \in s(s\mathcal{S})^T$ is Reedy cofibrant, then

$$\int^\Delta X \otimes K \to \int^\Delta X \otimes L \tag{4.9}$$

is a weak equivalence in $(s\mathcal{S})^T$. Thus the following result allows us an input for this theory.

Lemma 4.8 *The simplicial object $\overline{V}^T(X) \in s(s\mathcal{S})^T$ of the bisimplicial Stover resolution is Reedy cofibrant.*

Proof. As an object in $s_0(s\mathcal{S})^T$, there is an isomorphism

$$\overline{V}^T(X) \cong T(\bar{\mathcal{A}} \otimes_J \operatorname{Hom}_T(T(\bar{\mathcal{A}}), (\overline{V}^T)^{\cdot} X)).$$

Thus we need only show that for any $X \in s_0(s\,\mathbf{Sets})^{J^{\mathrm{op}}}$, $T(\bar{\mathcal{A}} \otimes_J X)$ is Reedy cofibrant; that, if we let n vary in the external, or s_0, direction, we need

$$L_n T(\bar{\mathcal{A}} \otimes_J X) \to T(\bar{\mathcal{A}} \otimes_J X)_n$$

to be a Reedy cofibration in $(s\mathcal{S})^T$. Since $T(\cdot)$ is a left adjoint, it is sufficient to show

$$L_n(\bar{\mathcal{A}} \otimes_J X) \to (\bar{\mathcal{A}} \otimes_J X)_n$$

is a Reedy cofibration in $s\mathcal{S}$. Thus, letting k vary in the internal, or s, direction, we need,

$$L_n(\bar{\mathcal{A}} \otimes_J X)_k \sqcup_{L_k L_n(\bar{\mathcal{A}} \otimes_J X)} L_k(\bar{\mathcal{A}} \otimes_J X)_n \to (\bar{\mathcal{A}} \otimes_J X)_{n,k}$$

to be a cofibration in $\mathcal{S}$. Since $\bar{\mathcal{A}} \otimes_J \cdot$ is a left adjoint, this map may be rewritten

$$\bar{\mathcal{A}} \otimes_J (L_n X_k \sqcup_{L_k L_n X} L_k X_n) \to \bar{\mathcal{A}} \otimes_J X_{n,k}.$$

The functor $\bar{\mathcal{A}} \otimes_J (\cdot)$ sends injections in $\mathbf{Sets}^{J^{\mathrm{op}}}$ to cofibrations in $\mathcal{S}$; indeed, if $X \to Y$, then

$$\bar{\mathcal{A}} \otimes_J X \to \bar{\mathcal{A}} \otimes_J Y$$

is the inclusion of a wedge summand where the other factor is cofibrant. This can be seen directly from the definition of by examining the beginning of the proof of Proposition 4.7 in [11]. Thus we need only assert that for all $j \in J$

$$L_n X_k(j) \sqcup_{L_k L_n X(j)} L_k X_n(j) \to X_{n,k}(j)$$

is an injection. This amounts to the assertion that for every $Z \in s_0(s\,\mathbf{Sets})$,

$$L_n Z \to Z_n$$

is a degree-wise injection of simplicial sets. But this is clear, since $(L_n Z)_k \subseteq (Z_n)_k$ is the inclusion of the degenerate k-simplices.

We now record:

Proposition 4.9 *Let $X \in (s\mathcal{S})^T$ then there is a natural $T(\mathcal{A})$-equivalence*

$$\eta : W(X) \to X$$

so that $W(X)$ is $T(\mathcal{A})$-cofibrant and as an object in $(s_0\mathcal{S})^T$, there is an isomorphism

$$W(X) \cong T(Z)$$

with $Z \in s_0\mathcal{S}$ s-free on a set of cofibrant spectra.

Proof. For $n \geq 0$, let Δ/n be the category of arrows in Δ over n and let $B(\Delta/\cdot)$ be the resulting cosimplicial classifying space. Define

$$W(X) = \int^{\Delta} \overline{V}^T(X) \otimes B(\Delta/\cdot).$$

Then $B(\Delta/\cdot)$ is cofibrant and there is a weak equivalence $B(\Delta/\cdot) \to \Delta[\cdot]$. Let η be the composite

$$W(X) \to \int^{\Delta} \overline{V}^T(X) \otimes \Delta[\cdot] = V(X) \xrightarrow{\epsilon} X.$$

This is a $T(\mathcal{A})$ equivalence.

By Lemma A.5 of [11], there is an isomorphism of T_k algebras

$$W(X)_k \cong \coprod_{n_0 \to n_1 \to \cdots \to n_k} \overline{V}^{T_k}_{n_k}(X_k)$$

where the coproduct is over composable morphisms in Δ. A degeneracy morphism $s_i : W(X)_k \to W(X)_{k+1}$ inserts an identity in the i^{th} slot. Let us write

$$Z_{n_k,k} = \bar{\mathcal{A}} \otimes_J \operatorname{Hom}(\bar{\mathcal{A}}, (\overline{V}^{T_k})^{n_k} X).$$

Then in the category $\mathcal{S}^{T_k}$, there is an isomorphism

$$W(X)_k \cong T(Z_k)$$

where

$$Z_k = \bigvee_{n_0 \to n_1 \to \cdots \to n_k} Z_{n_k k}.$$

Again by Lemma A.5 of [11] this yields an isomorphism $W(X) \cong T(Z)$ in $(s_0\mathcal{S})^T$. Furthermore Z is s-free on $\{\overline{Z}_k\}$ where

$$\overline{Z}_k = \bigvee_{n_0 \to n_1 \to \cdots \to n_k} Z_{n_k,k}$$

with none of the morphisms the identity.

Finally, we note that each pair (n,k), Remark 4.11 of [11] implies there is a cofibration $Y_{n,k} \to Z_{n,k}$ so that $Y_{n,k}$ is contractible and there is a push-out diagram

$$\begin{tikzcd}
Y_{n,k} \arrow[r] \arrow[d] & Z_{n,k} \arrow[d] \\
* \arrow[r] & \bigvee_A (Ai)_\alpha
\end{tikzcd}$$

for α in some index set A. It follows that $W(X)$ is homotopically $T(\mathcal{A})$-free.

5 The geometric realization of $T(\mathcal{A})$-cofibrant objects

We will need to be able to analyze the homotopy type of geometric realization of various objects in $(s\mathcal{S})^T$. This will be done by means of a spectral sequence we develop in the next section. In this section we determine what that spectral sequence will converge to.

We begin with a lemma. If $X \in (s\mathcal{S})^T$ and K a simplicial set we write $X \otimes K$ for the object in $(s\mathcal{S})^T$ constructed in Lemma 1.4 and $|X|\Box|K|$ for the extended object in $\mathcal{S}^{|T|}$.

Lemma 5.1 *Let $X \in (s\mathcal{S})^T$ and K a simplicial set. Then there is an isomorphism in $\mathcal{S}^{|T|}$, natural in X and K,*

$$|X|\Box|K| \cong |X \otimes K|.$$

Proof. In this argument we will write $X \otimes_{\mathcal{S}} K$ for $X \otimes K \in s\mathcal{S}$ to distinguish it from an object of the form $X \otimes K \in (s\mathcal{S})^T$.

First suppose $X = T(Z)$ for some $Z \in s\mathcal{S}$. Then

$$X \otimes_K \cong T(Z \otimes_{\mathcal{S}} K).$$

Since $\mathbf{hom}_T(K,Y) \cong \mathbf{hom}(K,Y)$ for all $Y \in (s\mathcal{S})^T$, there are maps $Z \otimes_{\mathcal{S}} K \to T(Z) \otimes_{\mathcal{S}} K \to T(Z \otimes_{\mathcal{S}} K) \cong T(Z) \otimes K$ and if $f : T(Z) \to T(Z')$ is and T-algebra map, then

$$f \otimes_T K : T(Z) \otimes K \to T(Z) \otimes K'$$

is the unique T-algebra map completing the following diagram of spectra

(5.1)
$$\begin{tikzcd}
Z \otimes_{\mathcal{S}} K \arrow[r] & T(Z) \otimes_{\mathcal{S}} K \arrow[r] \arrow[d, "f \otimes K"'] & T(Z) \otimes K \arrow[d, "f \otimes_T K"'] \\
& T(Z') \otimes_{\mathcal{S}} K \arrow[r] & T(Z') \otimes K.
\end{tikzcd}$$

In spectra, $|Z \otimes_{\mathcal{S}} K| \cong |Z| \Box_{\mathcal{S}} |K|$. This yields a map

$$|T|\,|Z|\Box|K| \cong |T|(|Z|\Box_{\mathcal{S}}|K|) \to |T(Z) \otimes K|$$

which is natural is all T-algebra maps $T(Z) \to T(Z')$, using (5.1). It is also an isomorphism since $|T(Z \otimes_{\mathcal{S}} K)| \cong |T|\,|Z \otimes_{\mathcal{S}} K|$. For general X examine the diagram of coequalizer diagrams

$$\begin{tikzcd}
|T|^2|X|\Box|K| \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & |T|\,|X|\Box|K| \arrow[r] \arrow[d] & |X|\Box|K| \arrow[d, dashed] \\
|T^2(X) \otimes K| \arrow[r, shift left] \arrow[r, shift right] & |T(X) \otimes K| \arrow[r] & |X \otimes K|
\end{tikzcd}$$

Then the dotted arrow exists and is an isomorphism.

This past lemma is crucial in the following result.

Proposition 5.2 *The geometric realization functor*

$$|\cdot| : (s\mathcal{S})^T \to \mathcal{S}^{|T|}$$

sends $T(\mathcal{A})$-equivalences between $T(\mathcal{A})$-cofibrant objects to weak equivalences.

Proof. While we have regarded $\mathcal{S}^{|T|}$ as a topological model category, any such is automatically a simplicial model category. (See the remarks between Proposition 6.10 of [11]). Then Lemma 5.1 implies that the adjoint isomorphism

$$\mathrm{Hom}_{(s\mathcal{S})^T}(X, Y^{\Delta}) \cong \mathrm{Hom}_{\mathcal{S}^{|T|}}(|X|, Y)$$

extends to an isomorphism of simplicial mapping spaces

$$\mathbf{map}_{(s\mathcal{S})^T}(X, Y^{\Delta}) \cong \mathbf{map}_{\mathcal{S}^{|T|}}(|X|, Y).$$

Since $|T(Ai)| \cong |T|(Ai)$, it follows from Definition 4.1 and the subsequent remarks that Y^{Δ} sends fibrations (or trivial fibrations)in $\mathcal{S}^{|T|}$ to $T(\mathcal{A})$-fibrations (or $T(\mathcal{A})$-trivial fibrations) in $(s\mathcal{S})^T$. Then $|\cdot|$ sends $T(\mathcal{A})$-cofibrations (and $T(\mathcal{A})$-trivial cofibrations) to cofibrations (and trivial cofibrations). The standard argument and Lemma 5.1 now finish the proof.

The next result, used with Proposition 4.9, will provide the key tool for our main result. Recall that $s_+\mathcal{C}$ denotes the category of "simplicial objects with only degeneracies" in $\mathcal{C}$.

Lemma 5.3 *Suppose $X \in (s\mathcal{S})^T$, regarded as an object in $(s_+\mathcal{S})^T$, is isomorphic to $T(Y)$ where $Y \in s_+\mathcal{S}$ is free on some set of cofibrant spectra. Suppose T is Reedy cofibrant as a simplicial operad. Then X is Reedy cofibrant as a simplicial spectrum.*

Proof. First suppose T is constant and Y is free on Z_k. Then in Lemma 6.10 of [11] we proved that $T(Y)$, regarded as an object in $s_+\mathcal{S}$, is free on

$$\vec{Z}_k = \bigvee_n \bigvee_{x \in B_n} T(n)_+ \wedge_{\Sigma_n} \Big(\bigvee_{\vec{i} \in q^{-1}(x)} Z_{i_1} \wedge \cdots \wedge Z_{i_n} \Big) \qquad (5.2)$$

where B_n is an indexing set we now define. Let I_k be the objects of $[k]/\Delta_+$, thus I_k consists of surjections $[k] \to [j]$ in Δ. Let I_k^n be the Cartesian product of I_k with itself n-times. The symmetric group acts by permuting coordinates. Let $q : I_k^n \to K_k^n/\Sigma_n$ be the quotient map and $B_n \subseteq I_k^n/\Sigma_n$ the subset of orbits

$$x = [([k] \to [i_1], \ldots , [k] \to [i_n])]$$

where $i_t = k$ for at least one t.

To prove the result we use Equation 5.2 and a diagonal argument:

$$T(Y) = \mathrm{diag}(T_\bullet(Y))$$

where $T_\bullet(Y)_{n,m} = T_n(Y_m)$. Let n be the horizontal variable and m the vertical variable and L_n^h, L_m^v the horizontal and vertical latching objects (as spectra) respectively. Then

$$L_n(T(Y)) = L_n^v L_n^h(T_\bullet(Y)).$$

It follows from Equation 5.2 that

$$L_n^h(T_\bullet(Y))_k = \bigvee_m \bigvee_{C_m} T_k(m)_+ \wedge_{\Sigma_m} (\bigvee_{\vec{i}\in q^{-1}(x)} Z_{i_1} \wedge \cdots \wedge Z_{i_m})$$

with $C_m = I_n^m/\Sigma_m - B_m$. Note that C_m does not depend on k. Hence

$$L_n(T(Y)) = \bigvee_m \bigvee_{v\in C_m} L_n T(m)_+ \wedge_{\Sigma_m} (\bigvee_{\vec{i}\in q^{-1}(x)} Z_{i_1} \wedge \cdots \wedge Z_{i_m}).$$

This is a wedge summand of the standard coproduct, with C_m replaced by I_n^m/Σ_m. Since $L_n T(m) \to T_n(m)$ is a cofibration of spaces over $\mathcal{L}(\mathcal{U}^m, \mathcal{U})$, $L_n(T(Y)) \to T(Y)$ is a cofibration, because

$$T(Y)_n = \bigvee_m \bigvee_{I_n^m/\Sigma_m} T_n(m)_+ \wedge_{\Sigma_m} (\bigvee_{\vec{i}\in q^{-1}(x)} Z_{i_1} \wedge \cdots \wedge Z_{i_m}).$$

We now come to the main result.

Theorem 5.4 *Let $X \in (s\mathcal{S})^T$ be $T(\mathcal{A})$-cofibrant and suppose T is a Reedy cofibrant simplicial operad over the linear isometries operads. Then there is a homotopy spectral sequence*

$$\pi_p(X, \Sigma^q Ai) \Rightarrow \pi_{p+q}(|X|, Ai).$$

Remark. The simplicial homotopy groups of the E_2 term and the homotopy groups of the abutment are to be computed in $s\mathcal{S}$ and $\mathcal{S}$ respectively. Also, for any T, note that

$$\pi_{p+q}(|X|, Ai) \cong [|T|(\Sigma^{p+q}Ai), |X|]_{\mathcal{S}^{|T|}}$$

and the latter expression simplifies if T is constant, since $|T| \cong T$.

Proof. Since geometric realization preserves weak equivalences between $T(\mathcal{A})$-cofibrant objects (by Proposition 5.2) we may use Proposition 4.9 to replace X by $W(X)$. By Lemma 5.3, $W(X)$ is Reedy cofibrant as a simplicial spectrum. Now apply Corollary 6.8 of [11].

This spectral sequence can be manipulated to obtain a much-more familiar spectral sequence. Let E be an Adams-type homology theory, and let $\mathcal{A} = \{\Sigma^k DE_\alpha\}$ be our standard collection of spectra. Then, for any spectrum Z, we have

$$\pi_q(Z, DE_\alpha) = [\Sigma^q DE_\alpha\, Z] = \pi_q(E_\alpha \wedge Z);$$

hence, we have that

$$\operatorname{colim}_\alpha \pi_q(Z, DE_\alpha) \cong E_q Z \tag{5.3}$$

and the spectral sequences of Theorem 5.4 yield a spectral sequence

$$\pi_p E_q(X) \Longrightarrow E_{p+q}(|X|). \tag{5.4}$$

This is, in fact, the standard homology spectrals sequence of a Reedy cofibrant simplicial spectrum. (See §6 of [11]). Note that if X is a simplicial T spectrum, this spectral sequence should be more than a spectral sequence of E_*E comodules; this we now explore.

We will suppose that E is a homotopy associative ring spectrum and of Adams-type. If T is a simplicial operad over the linear isometries operad, then E_*T is a simplicial operad in E_*E comodules and π_*E_*T is a graded operad in E_*E comodules. If each of the simplicial spaces $T(n)$ is Reedy cofibrant as a simplicial spaces, then we have spectral sequence of operads

$$E^2_{p,q}T = \pi_p E_q T \Longrightarrow E_{p+q}|T|.$$

If X is a simplicial T algebra in spectra which is Reedy cofibrant as as a simplicial spectrum, then we have that π_*E_*X is a π_*E_*X algebra and that spectral sequence of Equation 5.3 abuts to $E_*|T|$ algebra structure on $E_*|X|$ in the sense that

5.5.1) $E^r_{*,*}X$ is an $E^r_{*,*}T$ algebra;

5.5.2) if $\alpha \in E^r_{*,*}T(n)$ and $x_i \in E^r_{*,*}X$, for $1 \leq i \leq n$, then

$$d_r(\alpha(x_1 \otimes \ldots \otimes x_n)) = d_r(\alpha)(x_1 \otimes \ldots \otimes x_n) + \sum_i \pm\alpha(x_1 \otimes \ldots \otimes d_r(x_i) \otimes \ldots \otimes x_n);$$

5.5.3) $E^{r+1}_{*,*}X \cong H_*(E^r_{*,*}X)$ as $E^{r+1}_{*,*}T$ algebras;

5.5.4) the $E^\infty_{*,*}T$ algebra structure on $E^\infty_{*,*}X$ obtained from the spectal sequence agrees with the structure obtained by regarding $|X|$ as filtered algebra over $|T|$ and taking the asssociated graded object.

These spectral sequences are obtained by taking the skeletal filtration of the simplicial object and applying the geometric realization functor and then E_* homology to obtain an exact couple

$$\begin{tikzcd}
E_*|sk_0X| \arrow[rr, "i_*"] & & E_*|sk_1X| \arrow[rr, "i_*"] \arrow[dl, "\pi_*"] & & E_*|sk_2X| \arrow[rr, "i_*"] \arrow[dl, "\pi_*"] & & E_*|sk_3X| \cdots \arrow[dl, "\pi_*"] \\
& E_*\Sigma N_1X \arrow[ul, dashed, "\partial"] & & E_*\Sigma^2 N_3X \arrow[ul, dashed, "\partial"] & & E_*\Sigma^3 N_3X \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

Here $\Sigma^p N_p X$ is the cofiber of $|sk_{p-1}X| \to |sk_pX|$ and a dotted arrow means a morphism of degree -1. If we define

$$D^2_{p,*}X = E_*|sk_pX|/\partial[E_*\Sigma^{p+1}N_{p+1}X]$$

then we obtain exact sequences – the derived sequences –

$$\cdots D^2_{p-2,q} \xrightarrow{i_*\circ i_*} D^2_{p,q}X \xrightarrow{\pi} E^2_{p,q} = \pi_p E_q X \xrightarrow{i_*^{-1}\circ\partial} D^2_{p-2,q-1} \cdots \tag{5.5}$$

and it follows immediately from the above disussion that this is an exact sequence of π_0E_*X algebras over π_0E_*T. However, we can rename the terms in this exact sequence. Recall that there is a "spiral exact sequence" (§3 of [11])

$$\ldots \to \pi^\natural_{p-1}(X, \Sigma^{q+1}DE_\alpha) \to \pi^\natural_p(X, \Sigma^q DE_\alpha) \to \pi_p\pi_q(X, DE_\alpha) \to \pi^\natural_{p-2}(X, \Sigma^{q+1}DE_\alpha) \to \ldots$$

Taking the colimit over α, using Equation 5.3 and the results of §6 of [11] (specifically Proposition 6.7 and Corollary 6.8) we obtain an exact sequence which is isomorphic to the exact sequence of Equation 5.5. Putting this all together we obtain:

Proposition 5.5 *Suppose that T is an operad over the linear isometries operad which is level-wise Reedy cofibrant as a space, and suppose that X is a T-algebra which is Reedy cofibrant as a simplicial spectrum. Then the spiral exact sequence*

$$\ldots \to \operatorname*{colim}_{\alpha} \pi^{\natural}_{p-1}(X, \Sigma^{q+1} DE_\alpha) \to \operatorname*{colim}_{\alpha} \pi^{\natural}_{p}(X, \Sigma^{q} DE_\alpha) \to \pi_p E_q X$$
$$\to \operatorname*{colim}_{\alpha} \pi^{\natural}_{p-2}(X, \Sigma^{q+1} DE_\alpha) \to \ldots \to \operatorname*{colim}_{\alpha} \pi^{\natural}_{1}(X, \Sigma^{q} DE_\alpha) \to \pi_1 E_q X \to 0$$

is an exact sequence of $\pi_0 E_* X$ modules over the operad $\pi_0 E_* T$.

6 A spectral sequence for computing the homotopy groups of mapping spaces

Fix an an Adams-type commutative ring spectrum E and an operad $\mathcal{F}$ over the linear isometries operad. Suppose X is a $\mathcal{F}$ algebra in spectra and F is a $\mathcal{F}$ algebra which is also an E module spectrum. For example, F might be E with some $\mathcal{F}$ algebra structure. We develop a spectral sequence to compute the homotopy type of $\mathbf{map}_{\mathcal{F}}(X, F)$.

To begin with, we resolve the operad $\mathcal{F}$. Let $n\mathbf{Top}$ be the catgegory of non-negatively graded spaces $\{X_n\}_{n\geq 0}$. The forgetful functor from the category of operads (in spaces) to $n\mathbf{Top}$ has a left adjoint G; hence the forgetful functor from the category $\mathcal{O}$ of operads over the linear isometries operad to the category $n\mathbf{Top}/\mathcal{L}$ has a left adjoint, also given by G. We also write G for the induced triple on $\mathcal{O}$. Then, given $\mathcal{F} \in \mathcal{O}$ we can form the augmented simplicial object in $\mathcal{O}$

$$G_\bullet \mathcal{F} = \{G^{n+1}\mathcal{F}\} \longrightarrow \mathcal{F}.$$

This construction has the following properties:

6.1.1.) If each of the spaces $\mathcal{F}(n)$ of $\mathcal{F}$ is cofibrant as a space, then $G_\bullet \mathcal{F}$ is Reedy cofibrant as a simplicial operad.

6.1.2.) If each of the spaces $\mathcal{F}(n)$ of $\mathcal{F}$ is cofibrant as a space, then the natural map of operads $|G_\bullet \mathcal{F}| \to \mathcal{F}$ is a weak equivalence of operads.

6.1.3.) Suppose $E_*\mathcal{F}(n)$ is projective as an $E_*\mathcal{F}(n)$ is projective as an E_* module. Then, because E_* is a homology theory with a Künneth spectral sequence, $E_* G_\bullet \mathcal{F}$ is a cofibrant simplicial operad in E_* modules or E_*E comodules.

The first and the third of these properties follow from the description of the free operad functor given in [14] and the description of cofibrant operads as retracts of ones that are free in the sense of [17], §II.4. The second follows from the fact that that, regarded as an object in $n\mathbf{Top}$, the simplicial object $G.\mathcal{F} \to \mathcal{F}$ has a natural retraction. Since the forgetful functor $\mathcal{O} \to n\mathbf{Top}/\mathcal{L}$ makes the geometric realization, we have that claim.

The hypothesis that $\mathcal{F}$ be level-wise cofibrant, as in 6.1.1 and 6.1.2, is innocuous. If $\mathcal{F}$ is an operad in $\mathcal{O}$ we may choose a morphism of operads $\mathcal{F}' \to \mathcal{F}$ which is a level-wise weak equivalence and so that $\mathcal{F}'$ is cofibrant as an operad. Since cofibrant operads are level-wise cofibrant as graded topological spaces, we can form the simplicial object $G_\bullet \mathcal{F}'$ and there will be an augmentation

$$G_\bullet \mathcal{F}' \longrightarrow \mathcal{F}$$

so that the analogs of 6.1.1 and 6.1.2 hold.

Thus we have proved:

Proposition 6.1 *Let $\mathcal{F}$ be an operad over the linear isometries operad. Then there is an augmented simplicial operad $T \to \mathcal{F}$ over the linear isometies operad so that T is Reedy cofibrant as an operad and so that $|T| \to \mathcal{F}$ is a weak equivalence of operads. If, in addition, each of the spaces $\mathcal{F}(n)$ is projective as E_* module, then E_*T is a cofibrant simplicial operad in E_* modules or E_*E comodules.*

We now begin the construction of the promised spectral sequence. Let E, $\mathcal{F}$, X, and F be as at the beginning of the section.

First, we may assume $\mathcal{F}$ is cofibrant. If not, we choose a cofibrant replacement $\mathcal{F}_0 \to \mathcal{F}$. This does not affect the homotopy theory: if $X_0 \to X$ is a cofibrant replacement for X as an $\mathcal{F}_0$ algebra then by [10], $\mathcal{F} \otimes_{\mathcal{F}_0} X_0 \to X$ is a weak equivalence and $\mathcal{F} \otimes_{\mathcal{F}_0} X_0$ is cofibrant. Thus

$$\mathbf{map}_{\mathcal{F}_0}(X_0, E) \cong \mathbf{map}_{\mathcal{F}}(\mathcal{F} \otimes_{\mathcal{F}_0} X_0, E)$$

is weakly equivalent to $\mathbf{map}_{\mathcal{F}}(X, E)$.

Next, let $T \to \mathcal{F}$ be the canonical resolution of $\mathcal{F}$ provided by Proposition 6.1. Then $X \in (s\mathcal{S})^T$ as a constant object. We give $s\mathcal{S}^T$ the $T(\mathcal{A})$-resolution model category structure, where $\mathcal{A} = \{\Sigma^n DE_\alpha\}$ is the set of finite CW spectra built from E. (See the beginning of section 4). Choose a $T(\mathcal{A})$-cofibrant object $Y \to X$ weakly equivalent to X and define a cosimplicial space

$$\mathbf{map}_T(Y, F) = \{\mathbf{map}_T(Y_n, F)\} \cong \{\mathbf{map}_{\mathcal{F}}(\mathcal{F} \otimes_{T_n} Y_n, F)\}_{n \geq 0}. \tag{6.1}$$

A morphism $f : X \to F$ of $\mathcal{F}$-algebras defines a basepoint for this cosimplicial space, and we get a Bousfield-Kan spectral sequence

$$\pi^s \pi_t \mathbf{map}_T(Y, F) \Rightarrow \pi_{t-s} \mathrm{T{\small OT}}\, \mathbf{map}_T(Y, F). \tag{6.2}$$

We next identify the E_2 terms and target of this spectral sequence. This is done in Lemmas 6.4 and 6.2 respectively.

Lemma 6.2 *There is a natural weak equivalence*

$$\mathrm{T{\small OT}}\, \mathbf{map}_T(Y, F) \cong \mathbf{map}_{\mathcal{F}}(X, F).$$

Proof. First, since $\mathbf{map}_T(Y, F) \cong \mathbf{map}_{\mathcal{F}}(\mathcal{F} \otimes_T Y, F)$, we have

$$\begin{aligned} \mathrm{T{\small OT}}\, \mathbf{map}_T(Y, F) &\cong \mathrm{T{\small OT}}\, \mathbf{map}_{\mathcal{F}}(\mathcal{F} \otimes_T Y, F) \\ &\cong \mathbf{map}_{\mathcal{F}}(|\mathcal{F} \otimes_T Y|, F). \end{aligned}$$

Next, we have that $|\mathcal{F} \otimes_T Y| \cong \mathcal{F} \otimes_{|T|} |Y|$. This follows from the string of natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{F}}(|\mathcal{F} \otimes_T Y|, F) &\cong \mathrm{Hom}_{\mathcal{F}}(\mathcal{F} \otimes_T Y, F^{\Delta}) \\ &\cong \mathrm{Hom}_T(Y, F^{\Delta}) \\ &\cong \mathrm{Hom}_{|T|}(|Y|, F) \\ &\cong \mathrm{Hom}_{\mathcal{F}}(\mathcal{F} \otimes_{|T|} |Y|, F). \end{aligned}$$

Since $|T| \to \mathcal{F}$ is a weak equivalence between cofibrant operads,

$$|Y| \to \mathcal{F} \otimes_{|T|} |Y|$$

is a weak equivalence. Thus we need only show $|Y| \to X$ is a weak equivalence. Since $Y \to X$ is an $T(\mathcal{A})$-equivalence, and T is a Reedy cofibrant this follows from Theorem 5.4.

To identify the E_2 term, we review the basics of André Quillen homology of simplicial algebras over a simplicial operad. A more exhaustive and systematic treatment is in [12].

First we define derivations. Let R be a commutative ring, $\mathcal{C}$ is an operad in R modules, and A a $\mathcal{C}$ algebra. Then A-module M is an R module equipped with an action map

$$\mathcal{C}(A,M) = \underset{n}{\oplus}\, \mathcal{C}(n) \otimes_{R[\Sigma_{n-1}]} A^{\otimes(n-1)} \otimes M \to M$$

satisfying certain associativity and unital conditions. We can use this data to define a new $\mathcal{C}$ algebra over A called $A \ltimes M$ as follows: as an R-module $A \ltimes M$ is simply $A \oplus M$, but the $\mathcal{C}$ algebra structure is defined by noting that there is a natural decomposition

$$\mathcal{C}(A \oplus M) \cong \mathcal{C}(A) \oplus \mathcal{C}(A,M) \oplus E(A,M)$$

where $E(A,M)$ consists of those summands of $\mathcal{C}(A \oplus M)$ with more that one M term. Since M is an A-module we get a composition

$$\mathcal{C}(A \oplus M) \to \mathcal{C}(A) \oplus \mathcal{C}(A,M) \to A \oplus M$$

which defines the $\mathcal{C}$ algebra structure on $A \ltimes M$. The algebra $A \ltimes M$ is a "square-zero extension" and an abelian object in the category of algebras over A; indeed, all abelian group objects have this form. The object $A \ltimes M$ in the category of $\mathcal{C}$ algebras over A represents an abelian group valued functor which we might as well call derivations. If X is a $\mathcal{C}$ algebra over A and M is an A-module, a *derivation* $d : X \to M$ is an R-module homomorphism such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(A) \arrow[r, "\Delta"] \arrow[d, "\epsilon_A"'] & \mathcal{C}(A,A) \arrow[r, "\mathcal{C}(A,d)"] & \mathcal{C}(A,M) \arrow[d, "\eta"] \\
A \arrow[rr, "d"] & & M.
\end{tikzcd}$$

where $\Delta : \mathcal{C}(A) \to \mathcal{C}(A,A)$ is induced by the diagonal maps $A^{\otimes n} \to \oplus_n A^{\otimes n}$. If we call the group of such derivations $\mathrm{Der}_{\mathcal{C}}(X,M)$, then the functor

$$X \mapsto \mathrm{Der}_{\mathcal{C}}(X,M)$$

is represented by $A \ltimes M$.

Now let $s\mathcal{M}_R$, the category of simplicial R-modules. Let $\mathcal{C}$ be a simplicial operad in the category of R modules. In our application, R will be E_* and $\mathcal{C} = E_*T$. Let $s\mathcal{M}_R^{\mathcal{C}}$ be the category of $\mathcal{C}$ algebras in $s\mathcal{M}_R$. There is a simplicial closed model category structure on $s\mathcal{M}_R^{\mathcal{C}}$ where one uses the standard simplicial structure [17, §II.2] and defines $f : X \to Y$ to be a fibration or weak equivalence if it is so as simplicial R-modules. A cofibrant object is a retract of a "free object" X [17, §II.4]—which is to say as an object in $(s_+\mathcal{M}_R)^{\mathcal{C}}$, there is an isomorphism $X \cong \mathcal{C}(M)$ where $M \in s_+\mathcal{M}_R$ is projective as an R-module in each degree.

Now suppose $\mathcal{C}$ has an augmented simplicial operad $\epsilon : \mathcal{C} \to \mathcal{C}_{-1}$ and $X \in s\mathcal{M}_R^{\mathcal{C}}$ has an augmentation $\epsilon : X \to X_{-1}$ to some $\mathcal{C}_{-1}$-algebra X_{-1}. If M is an $\mathcal{C}_{-1}$ module over X_{-1}, then M inherits a unique structure as a $\mathcal{C}_n$ module over X_n and we can form the cosimplicial abelian group

$$\mathrm{Der}_{\mathcal{C}}(X,M) \cong \{\mathrm{Der}_{\mathcal{C}_n}(X_n,M)\}. \tag{6.3}$$

This is a cosimplicial R-module or, if R is graded, a cosimplicial R_0 module. It does not have good homotopical properties; however, if we choose a weak equivalence $Y \to X$ is $s\mathcal{M}_R^{\mathcal{C}}$ with Y cofibrant, then $\mathrm{Der}_C(Y,M)$ is well-defined and natural up to homotopy. We define the André-Quillen cohomology of X with coefficients in M by

$$D^s_{\mathcal{C}}(X,M) \cong \pi^s\, \mathrm{Der}_{\mathcal{C}}(Y,M). \tag{6.4}$$

If (R, Γ) is a Hopf algebroid, there is a variant of this construction which uses Γ-comodules as the ground category, rather than R-modules. If $\mathcal{C}$ is a simplicial operad in Γ comodules, we will write

$$D^*_{\Gamma(\mathcal{C})}(X, m)$$

for the relevant cohomology. The notation is supposed to suggest the role of the Hopf algebroid.

We now return to the spectral sequence of (6.2). Thus we have an operad $\mathcal{F}$ over the linear isometries operad, E an Adams-type commutative ring spectra, and $f : X \to F$ a morphism of $\mathcal{F}$-algebras. We note that E_*X and E_*F inherit an $E_*\mathcal{F}$ algebra structure. Because F is an E module spectrum, F_* is also an $E_*\mathcal{F}$ algebra and the evaluation $E_*F \to F_*$ is a morphism of $E_*\mathcal{F}$ algebras. The morphism $f : X \to E$ of $\mathcal{F}$-algebras gives F_{*+t} an $E_*\mathcal{F}$ module structure over E_*X. (Here and elsewhere M_{*+n} means the graded module with $(M_{*+n})_k = M_{n+k}$.) Let $T \to \mathcal{F}$ be the canonical simplicial resolution of $\mathcal{F}$ and $Y \to X$ and $T(\mathcal{A})$ cofibrant model for X regarded as a constant object in $(s\mathcal{S})^T$. We may assume that Y is homotopically s-free by Propositions 4.6 and 4.7.

Lemma 6.3 *The simplicial E_*T algebra E_*Y is cofibrant, and the augmentation $Y \to X$ induces an isomorphism*

$$\pi_*E_*Y \cong E_*X.$$

Proof. Since Y is homotopically $T(\mathcal{A})$-free, there is a homotopy equivalence, as T_n algebras,

$$Y_n \simeq T_n(Z)$$

where Z is a wedge of spectra of the form $\Sigma^j DE_\alpha$. Now $E_*(\Sigma^j DE_\alpha)$ is a projective E_* module. Also $\pi_0 T_n$ is a free Σ_n set, hence E_*T_n is an extended $E_*\Sigma_n$ module. Since E_* has a Künneth spectral sequence, there is an isomorphism of E_*T_n algebras

$$(E_*T_n)(E_*Z) \cong E_*Y_n. \tag{6.5}$$

A further consequence of being homotopically s-free is that E_*Y is free, hence cofibrant, as an E_*T algebra.

To calculate π_*E_*Y, we use the spectral sequence of Theorem 5.4, which implies that

$$\pi_p[\Sigma^n DE_\alpha, Y] = \begin{cases} [\Sigma^n DE_\alpha, X], & p = 0; \\ 0 & p \neq 0. \end{cases}$$

Now use the fact that

$$\pi_p E_n Y \cong \operatorname{colim}_\alpha \pi_p (E_\alpha)_n Y \cong \operatorname{colim}_\alpha \pi_p[\Sigma^n DE_\alpha, Y]$$

to finish the argument.

The following identifies the E_2 term of the spectral sequence of Equation (6.3).

Lemma 6.4 *There is a natural isomorphism*

$$\pi^0\pi_0 \mathbf{map}_T(Y, F) \cong \operatorname{Hom}_{E_*\mathcal{F}}(E_*X, F_*)$$

and for all $t \geq 1$ and all $s \geq 0$, and all choices of $\mathcal{F}$-algebra maps $f : X \to E$, a natural isomorphism

$$\pi^s\pi_t \mathbf{map}_T(Y, F) \cong D^s_{E_*T}(E_*X, F_{*+t}).$$

Proof. First notice that the map that evaluates on E_*

$$\pi_0 \mathbf{map}_{T_n}(Y_n, F) \to \mathrm{Hom}_{E_*T_n}(E_*Y_n, F_*) \tag{6.6}$$

is an isomorphism; indeed, using that $Y_n \simeq T_n(Z)$ and Equation 6.5, we have that

$$\begin{aligned} \pi_0 \mathbf{map}_{T_n}(Y_n, F) &\cong \pi_0 \mathbf{map}_{\mathcal{S}}(Z_n, F) \\ &\cong \mathrm{Hom}_{E_*}(E_*Z_n, F_*) \\ &\cong \mathrm{Hom}_{E_*T_n}(E_*Y_n, F_*). \end{aligned}$$

For the calculation of $\pi^0\pi_0\mathbf{map}(Y, F)$, use that there is a split coequalizer diagram

$$E_*Y_1 \rightrightarrows E_*Y_0 \to E_*X$$

as E_* modules Hence, there is a coequalizer diagram of $E_*\mathcal{F}$ algebras

$$E_*\mathcal{F} \otimes_{E_*T_1} E_*Y_1 \rightrightarrows E_*\mathcal{F} \otimes_{E_*T_0} E_*Y_0 \to E_*X. \tag{6.7}$$

There is, by Equation 6.6, an equalizer diagram

$$\pi^0\pi_0\mathbf{map}_T(Y, F) \to \mathrm{Hom}_{E_*T_0}(E_*Y_0, F_*) \rightrightarrows \mathrm{Hom}_{E_*T_1}(E_*Y_1, F_*)$$

hence an equalizer diagram

$$\pi^0\pi_0\mathbf{map}_T(Y, F) \to \mathrm{Hom}_{E_*\mathcal{F}}(E_*\mathcal{F} \otimes_{E_*T_0} E_*Y_0, F_*) \rightrightarrows \mathrm{Hom}(E_*\mathcal{F} \otimes_{E_*T_1} E_*Y_1, F_*)$$

and the result follows from Equation 6.7.

To calculate $\pi^s\pi_t\mathbf{map}(Y, F)$ with $t \geq 1$, let $\mathbf{hom}(S^t, F)$ be the spectrum of maps from S^t to F. Then $\mathbf{hom}(S^t, F)$ is an E-module spectrum and a $\mathcal{F}$ algebra; furthermore

$$\pi_*\mathbf{hom}(S^t, E) \cong F_* \ltimes F_{*+t}$$

as an $E_*\mathcal{F}$ algebra. Here F_* is a F_* module in the canonical way. Then, arguing as above, we have, for a choice of basepoint in $\mathbf{map}_{T_n}(Y_n, E)$, a natural isomorphism

$$\begin{aligned} \pi_t\mathbf{map}_{T_n}(Y, F) &\cong \mathrm{Hom}_{E_*T_n}(E_*Y_n, F_* \ltimes F_{*+t}) \\ &\cong \mathrm{Der}_{E_*T_n}(E_*Y, F_{*+t}). \end{aligned}$$

The result now follows from Lemma 6.3 and the definition of André-Quillen cohomology.

7 Eilenberg-Mac Lane objects

Let T be a simplicial operad over the linear isometries operad and let E be an Adams type homology theory. If $X \in s\mathcal{S}^T$, then E_*X is a simplicial algebra over E_*T in E_*E comodules. If M is a π_0E_*T module over π_0E_*X (all in E_*E comodules) we can form the André-Quillen cohomology groups

$$D^n(E_*X, M)$$

in E_*E comodules. To be consistent we should write

$$D^n_{E_*E(E_*T)}(E_*X, M)$$

for these groups, but we forebear, for the obvious notational reasons. In this section we discuss to what extent this functor is representable.

We will use the standard resolution model category structure inherent in our choice of homology theory E_*. See Definition 4.1. Write $\mathrm{Hom}_{E_*T}(\cdot,\cdot)$ for maps of E_*T algebras in E_*E comodules.

We will assume throughout this section that T satisfies 6.1.1 – 6.1.3. This, in particular, implies that if X is a $T(\mathcal{A})$ cofibrant simplicial T algebra, then E_*X is a cofibrant E_*T algebra.

The main result of this section is the following. Fix a $\pi_0 E_*T$ algebra Λ and a $\pi_0 E_*T$ module M over Λ in E_*E comodules.

Proposition 7.1 *1.) There is a fibrant simplicial object $B\Lambda \in s\mathcal{S}^T$ and natural isomorphisms for all $X \in s\mathcal{S}^T$*

$$\begin{aligned}[X, B\Lambda]_{s\mathcal{S}^T} &\overset{\cong}{\to} \mathrm{Hom}_{E_*T}(E_*X, A) \\ &\cong \mathrm{Hom}_{\pi_0 E_*T}(\pi_0 E_*X, \Lambda).\end{aligned}$$

2.) Let $X \in s\mathcal{C}^T$ be $T(\mathcal{A})$-cofibrant and let $f_* \in \mathrm{Hom}_{\pi_0 E_*T}(\pi_0 E_*X, \Lambda)$ be represented by $f : X \to B\Lambda$. Then there is an object $q : E_\Lambda(M,n) \to B\Lambda$ in $s\mathcal{S}^T/B\Lambda$ and a natural isomorphism

$$[X, E_\Lambda(M,n)]_{s\mathcal{S}^T/B\Lambda} \cong D^n(E_*X, M).$$

The argument in both cases is essentially Spanier's proof of Brown Representability [21]. We would also appeal to Heller [15] or Baues [2]. The first observation is that the functors of Proposition 7.1 satisfy a variant of Brown's axioms for a homotopy functor.

Lemma 7.2 *Let $\{X_\alpha\}_{\alpha\in A}$ be a set of objects in $s\mathcal{S}^T$. Then*

$$\mathrm{Hom}_{\pi_0 E_*T}(\pi_0 E_*(\coprod_\alpha X_\alpha), \Lambda) \cong \prod_\alpha \mathrm{Hom}_{\pi_0 E_*T}(\pi_0 E_* X_\alpha, \Lambda).$$

Furthermore, fixing $E_*f_\alpha : E_*X_\alpha \to \Lambda$,

$$D^n(E_*(\coprod_\alpha X_\alpha), M) \cong \prod_\alpha D^n(A_*X_\alpha, M).$$

Proof. We may assume each X_α is homotopically $T(\mathcal{A})$-free. We will show that

$$E_*(\coprod_\alpha X_\alpha) \cong \coprod_\alpha E_*X_\alpha,$$

with the coproduct on the right in the category of E_*T algebras in E_*E comodules. The result will then follow.

At level n, $(X_\alpha)_n \cong T_n(V_n^\alpha)$ where $V_n^\alpha \simeq \bigvee_w \Sigma^{n_w} DE_w$. Since $\pi_0 T_n(k)$ is a free Σ_k set and because $E_*V_n^\alpha$ is projective as an E_* module, we have

$$E_*(X_\alpha)_n \cong E_*T_n(E_*V_n).$$

Also, because $(\coprod_\alpha X_\alpha)_n \cong \coprod_\alpha (X_\alpha)_n \cong T_\alpha(\vee V_\alpha^n)$, the result follows.

The second half of Brown's axioms is the following.

Lemma 7.3 *Suppose we have a push-out diagram in $s\mathcal{A}^T$*

$$\begin{tikzcd} C \arrow[r] \arrow[d] & X \arrow[d] \\ D \arrow[r] & Y \end{tikzcd}$$

where C and $X \in s\mathcal{A}^T$ are $T(\mathcal{A})$-cofibrant and $C \to D$ is a $T(\mathcal{A})$-cofibration. Then

$$\begin{tikzcd}
\mathrm{Hom}_{E_*T}(E_*Y,\Lambda) \arrow[r] \arrow[d] & \mathrm{Hom}_{E_*T}(E_*D,\Lambda) \arrow[d] \\
\mathrm{Hom}_{E_*T}(E_*X,\Lambda) \arrow[r] & \mathrm{Hom}_{E_*T}(E_*C,\Lambda)
\end{tikzcd}$$

is a pull-back diagram. Furthermore, given $f \in \mathrm{Hom}_{E_*T}(E_*Y,\Lambda)$, we have that for all $n \geq 0$; the sequence

$$D^n(E_*Y,M) \to D^n(E_*X,M) \times D^n(E_*D,M) \to D^n(E_*C,M)$$

is exact.

Proof. It is sufficient to show that

(7.1)
$$\begin{tikzcd}
E_*C \arrow[r] \arrow[d] & E_*X \arrow[d] \\
E_*D \arrow[r] & E_*Y
\end{tikzcd}$$

is a push-out in E_*T algebras of a particular type: namely, that E_*C and E_*X are cofibrant and $E_*C \to E_*D$ is a cofibration of E_*T algebras. If this is the case, the result follows immediately; indeed, the first statement only requires that the diagram 7.1 be a push-out and the second statement follows from the Meyer-Vietoris sequence for André-Quillen cohomology: Propositions 2.13 and Lemma 4.8 of [12].

To see that diagram 7.1 is a push-out it is sufficient to work level-wise. We may assume that C is homotopically $T(\mathcal{A})$ free and the $C \to D$ is a homotopically $T(\mathcal{A})$ free morphism. At level n, $C_n \simeq T(\bigvee_z \Sigma^{n_z} DE_z)$ for z in some indexing set z, and

$$D_n \simeq C_n \coprod T(\bigvee_y \Sigma^{n_y} DE_y)$$

for y in some indexing set y. Thus

$$E_*D_n \cong E_*C_n \coprod E_*(T(\bigvee_y \Sigma^{n_y} DE_y).$$

Thus we need only check

$$\begin{aligned}
E_*Y_n &\cong E_*(X_n \textstyle\coprod T(\bigvee_y \Sigma^{n_y} DE_y)) \\
&\cong E_*X_n \textstyle\coprod E_*T(\bigvee_y \Sigma^{n_y} DE_y).
\end{aligned}$$

But $X_n \cong T(V_n)$ where $V_n \simeq \bigvee_x \Sigma^{n_x} DE_x$ for some indexing set. The computation follows. Actually, this analysis proves that $E_*C \to E_*D$ is a cofibration of E_*T algebras. Finally, E_*X is a cofibrant E_*T algebra because of the hypothesis on X and the assumptions on T. This completes the argument.

A final bit of analysis is the following, which describes how our functors behave under certain types of directed colimits.

Lemma 7.4 *Suppose we have a directed system of $T(\mathcal{A})$-cofibrations in $s\mathcal{S}^T$ and suppose that X_0 is $T(\mathcal{A})$-cofibrant:*

$$X_0 \to X_1 \to X_2 \to \cdots .$$

Then the natural map

$$\mathrm{Hom}_{E_*T}(E_* \operatorname*{colim}_m X_m, \Lambda) \longrightarrow \lim_m \mathrm{Hom}_{E_*T}(E_*X_n, \Lambda)$$

is an isomorphism. Furthermore, given $f \in \mathrm{Hom}_{E_*T}(E_* \operatorname*{colim}_m X_m, \Lambda)$, the natural map

$$D^n(E_* \operatorname*{colim}_m X_m, M) \to \lim_m D^n(E_*X_m, M)$$

is a surjection.

Proof. We may assume that X_0 is homotopically $T(\mathcal{A})$ free and each of the maps is also homotopically $T(\mathcal{A})$ free. Then in any given simplicial degree k, there are spectra V_m, $m \geq 0$, and a set I_n so that there is a weak equivalence

$$V_m \simeq \bigvee_{\alpha \in I_n} \Sigma^{s_\alpha} DE_\alpha$$

and a homotopy equivalence of T_k algebras

$$(X_m)_k \to (X_{m-1})_k \sqcup T_k(V_m) \cong T(\bigvee_{k=0}^{m} V_m).$$

We set X_{-1} to be the initial object. Hence

$$(\operatorname{colim} X_m)_k \simeq T_k(\bigvee_{m=0}^{\infty} V_m)$$

and a direct calculation, using our assumptions of T, implies

$$E_* \operatorname{colim}(X_m)_k \cong \operatorname{colim} E_*(X_m)_k \tag{7.2}$$

or, simply $E_* \operatorname{colim} X_m \cong \operatorname{colim} E_*X_m$ and the statement about the functor $\mathrm{Hom}_{E_*T}(?, \Lambda)$ now follows.

Also each of the maps $E_*X_{m-1} \to E_*X_m$ is a cofibration of E_*T algebras; hence,

$$D^n(E_* \operatorname*{colim}_\alpha X_m, M) = \pi^n \operatorname{Der}(E_* \operatorname*{colim}_\alpha X_m, M).$$

The isomorphism of 7.2 implies there is an isomorphism of E_* module

$$\operatorname{Der}(E_* \operatorname*{colim}_\alpha X_m), M) \cong \lim_m \operatorname{Der}(E_*X_m, M)$$

and, hence, we have a Milnor exact sequence

$$0 \to \lim{}^1 D^{n-1}(E_*X_m, M) \to D^n(E_* \operatorname*{colim}_\alpha X_m, M) \to \lim_m D^n(E_*X_m, M) \to 0.$$

This completes the argument.

These results suggest the following definition.

Definition 7.5 Define a functor $H : \mathbf{Ho}(s\mathcal{S}^T)^{\mathrm{op}} \to \mathbf{Sets}$ to be a *homotopy functor* if

1. $H(\coprod_\alpha X_\alpha) \cong \prod H(X_\alpha)$;
2. given a two source
$$D \leftarrow C \to X$$
with C and X $T(\mathcal{A})$-cofibrant and $C \to D$ a $T(\mathcal{A})$-cofibration, then
$$H(D \sqcup_C X) \to H(D) \times_{H(C)} H(X)$$
is a surjection; and
3. given a directed system
$$X_0 \to X_1 \to X - 2 \to \cdots$$
with X_0 a $T(\mathcal{A})$-cofibrant object and each of the maps a $T(\mathcal{A})$ cofibration, then the natural map $H(\operatorname{hocolim}_m X_m) \to \lim_m H(X_m)$ is a surjection.

Example 7.6 1.) The functor
$$X \mapsto \mathrm{Hom}_{E_*T}(E_*X, \Lambda) = \mathrm{Hom}_{\pi_0 E_*T}(\pi_0 E_*X, \Lambda)$$
is a homotopy functor, by Lemmas 7.2, 7.3, and 7.4.

2.) Define $H^n(?; M)$ to be the functor that assigns to $X \in \mathbf{Ho}(s\mathcal{S}^T)^{\mathrm{op}}$ the set of all pairs (f, x) where
$$f \in \mathrm{Hom}_{\pi_0 E_*T}(\pi_0 E_*X, \Lambda)$$
and
$$x \in D^n(E_*X; M),$$
where the E_*T module structure on M over E_*X is determined by f. Then lemmas 7.2, 7.3, and 7.4 imply $H^n(?; M)$ is a homotopy functor.

A Brown Representability Theorem would assert that any homotopy functor is representable. This is proved below – see Proposition 7.9. Then Proposition 7.1 is proved by appealing to the previous example.

The argument for Brown Representability goes as in Spanier [[21], §7.7]. If $H : [\mathbf{Ho}(s\mathcal{S}^T)]^{\mathrm{op}} \to$ **Sets** is any homotopy functor define a pair (Y, u), $u \in H(Y)$, to be *n-universal* for H if Y is fibrant and for all k, and α the natural map
$$\phi_u : [T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta^q), Y]_{s\mathcal{S}^T} \to H(T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta^q))$$
given by
$$\phi_\mu(f) = H(f)(u)$$
is an isomorphism for $q < n$ and onto for $q = n$. The pair (Y, μ) is *universal* if it is n-universal for all n. Note that since $T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta^q)$ is a homotopy cogroup object, ϕ_u is automatically a group homomorphism. If H is a group-valued functor, the two group operations on $H(T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta^q)$ agree (cf. [[21], Lemma 7.7.6]).

Lemma 7.7 *Let H be a homotopy functor. Let $X \in s\mathcal{S}^T$ be $T(\mathcal{A})$-cofibrant and $y \in H(X)$. Then there exists a universal pair (Y, u) and a $T(\mathcal{A})$-cofibration $f : X \to Y$ so that $H(f)(u) = y$.*

Proof. We define a sequence of maps over Z

$$X = Y_{-1} \xrightarrow{f_0} Y_0 \xrightarrow{f_1} Y_1 \xrightarrow{f_2} Y_2 \to \cdots$$

and elements $\mu_i \in H(Y_i)$ so that

1. $\mu_{-1} = y$ and $H(f_n)u_n = u_{n-1}$;
2. (Y_n, u_n) is n-universal;
3. for each n there is a push-out diagram

$$\begin{tikzcd}
\coprod_{I_n} T(\Sigma^k DE_\alpha \wedge \Delta^n/\partial\Delta^n) \arrow[r] \arrow[d] & Y_n \arrow[d] \\
\coprod_{I_n} T(\Sigma^k DE_\alpha \wedge \Delta^{n+1}/\Delta_0^{n+1}) \arrow[r] & \bar{Y}_{n+1}
\end{tikzcd}$$

   for some index set I_n and a Reedy trivial cofibration $\bar{Y}_{n+1} \to Y_{n+1}$ which is an isomorphism in simplicial degrees less than $n+1$.

Assuming this to be the case for the moment, let $Y = \operatorname{colim} Y_n$ and $f : X \to Y$ the evident map. Let $\mu \in H(Y)$ be any element restricting to $\lim u_i$ under the surjective map $H(Y) \to \lim H(Y_i)$. Since, writing q for simplicial degree,

$$(Y_n)_q \cong (Y_{n+1})_q$$

for $q \leq n$, we have for $q < n$

$$[T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta, Y] \cong [T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta^q, Y_n]$$

and, hence, (Y, u) is universal.

To construct (Y_n, u_n), we work recursively. To construct Y_0, let I_0 be the set of triples (k, α, z) where $z \in H(T(\Sigma^k DE_\alpha))$. Let

$$\bar{Y}_0 = X \sqcup (\bigsqcup_{I_0} T(\Sigma^k D_\alpha E))$$

and $\mu_0 \in H(Y_0)$ the evident class determined by the isomorphism

$$H(Y_0) \cong H(X) \times \prod_{I_0} H(T(\Sigma^k DE_\alpha)).$$

Now let $\bar{Y}_0 \to Y_0$ be a Reedy trivial cofibration with Y_0 Reedy fibrant. Now suppose Y_n has been constructed.

Define $\bar{Y}_{n+1}$ as follows. Let $I_n = J_n \sqcup K_n$ where J_n is the set of triples (k, α, z) where

$$z \in H(T(\Sigma^k DE_\alpha \wedge \Delta^{n+1}/\partial\Delta^{n+1})$$

and K_n is the set of triples (k, α, f) where

$$f \in [T(\Sigma^k DE_\alpha \wedge \Delta^n/\partial\Delta^n), Y_n]$$

so that $H(f)(u_n) = 0$. Let W be defined by the pushout diagram

$$\begin{tikzcd}
\coprod_{K_n} T(\Sigma^k DE_\alpha \wedge \Delta^n/\partial\Delta^n) \arrow[r] \arrow[d] & Y_n \sqcup (\coprod_{J_n} T(\Sigma^k DE_\alpha) \wedge \Delta^{n+1}/\partial\Delta^{n+1}) \arrow[d] \\
\coprod_{K_n} T(\Sigma^k DE_\alpha \wedge \Delta^{n+1}/\Delta^{n+1}_0) \arrow[r] & W
\end{tikzcd}$$

and let

$$\bar{Y}_{n+1} = W \sqcup (\coprod_{J_n} T(\Sigma^k DE_\alpha) \wedge \Delta^{n+1}/\partial\Delta^{n+1}).$$

Now choose a Reedy trivial cofibration $\bar{Y}_{n+1} \to Y_{n+1}$ with Y_{n+1} fibrant. Here we mean $T(\mathcal{A})$-fibrant, but, by [11], this is the same as Reedy fibrant. Since $(Y_n)_k \cong (\bar{Y}_{n+1})_k$ in simplicial degrees $k \leq n$, and since the Reedy "trivial cofibration-fibration" factorization is done recursively in simplicial degree (see §2), we may assume $(\bar{Y}_{n+1})_k \cong (Y_{n+1})_k$ in degrees $k \leq n$.

By Definition 7.5, we have an element $\mu_{n+1} \in H(Y_{n+1}) \cong H(\bar{Y}_{n+1})$ so that 1) and 3) above hold. We need to show (Y_{n+1}, μ_{n+1}) is $(n+1)$-universal.

Write φ_n for φ_{u_n}, and consider the diagram

$$\begin{tikzcd}
{[T(\Sigma^k DE_\alpha \wedge \Delta^q/\Delta^q), Y_n)]} \arrow[rr, "(f_{n+1})_*"] \arrow[dr, "\varphi_n"'] & & {[T(\Sigma^k DE_\alpha \wedge \Delta^q/\Delta^q), Y_{n+1}]} \arrow[dl, "\varphi_{n+1}"] \\
& H(T(\Sigma^k DE_\alpha \wedge \Delta^q/\Delta^q). &
\end{tikzcd}$$

If $q = n+1$, φ_{n+1} is onto by construction. If $q = n$, φ_n is onto, so φ_{n+1} is onto. Also, since $(Y_n)_k \cong (Y_{n+1})_k$ in simplicial degrees $k \leq n$, $(f_{n+1})_*$ is onto. Since all the maps are group maps and every element in the kernel of φ_n is in the kernel of $(f_{n+1})_*$, φ_{n+1} is one-to-one. Finally, if $q < n$, the fact that $(Y_n)_k \cong (Y_{n+1})_k$ for $k \leq n$ implies $(f_{n+1})_*$ is an isomorphism. Since φ_n is an isomorphism, so is φ_{n+1}. This completes the argument.

Lemma 7.8 *Let $j : A \to X$ be an $T(\mathcal{A})$-cofibration in $(s\mathcal{S})^T$ with A $T(\mathcal{A})$-cofibrant and suppose (Y, μ) be universal for H. Suppose there is an element $v \in X$ and a morphism $f : A \to Y$ so that $H(f)\mu = H(j)v$. Then there is a map $g : X \to Y$ so that $H(g)u = v$ and $g \circ j = f$ in $\mathbf{Ho}(s\mathcal{S}^T)$.*

Proof. Consider the push-out diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d, "i"] \\
X \arrow[r, "q"'] & Z.
\end{tikzcd}$$

There is an element $w \in H(Z)$ so that $H(i)w = \mu$ and $H(q)w = v$. Apply the previous lemma to produce a universal pair (Y', u') and a map $h : Z \to Y'$ so that $H(h)'_u = w$. Then $h \circ i : Y \to Y'$ is a weak equivalence, and the lifting problem

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d] \\
X \arrow[r, "h \circ q"'] \arrow[ur, dotted] & Y'
\end{tikzcd}$$

can be solved up to homotopy.

Proposition 7.9 *If H is a homotopy functor, it is representable: there is an object $Y \in s\mathcal{S}^T$ and a class $\mu \in H(Y)$ so that*

$$[X,Y]_{s\mathcal{S}^T} \to H(X)$$

given by $f \mapsto H(f)\mu$ is bijection.

Proof. See [[21], Thm 7.7.14]. We have arranged that this argument proceeds verbatim.

In particular Proposition 7.1 follows:

Proof of Proposition 7.1: The existence of $B\Lambda$ follows immediately from example 7.6 and Proposition 7.9. For the object $E_\Lambda(M,n) \to B\Lambda)$ we proceed as follows. Consider the functor $H^n(?,M)$ of Example 7.6. By Proposition 7.9 $H^n(?;M)$ is represented by a cofibrant-fibrant object $\bar{E}_\Lambda(M,n)$. There is a natural transformation

$$\Phi : H^n(X,M) \to \mathrm{Hom}_{\pi_0 E_* T}(\pi_0 E_* X, \Lambda) \tag{7.3}$$

sending a pair (f,x) to f. This yields a map in $s\mathcal{S}^T$

$$\bar{E}_\Lambda(M,n) \to B\Lambda.$$

Factor this map as $\bar{E}_\Lambda(M,n) \xrightarrow{j} E_\Lambda(M,n) \xrightarrow{q} B\Lambda$ where j is a trivial cofibration and q is a fibration.

Let $f_* \in \mathrm{Hom}_{\pi_0 E_* T}(\pi_0 E_* X, \Lambda)$ be represented by $f : X \to B\Lambda$. Consider the pull-back square

$$\begin{tikzcd}
\mathbf{map}_{s\mathcal{S}^T/B\Lambda}(X, E_\Lambda(M,n)) \arrow[r] \arrow[d] & \mathbf{map}_{s\mathcal{S}^T}(X, E_\Lambda(M,n)) \arrow[d, "q_*"] \\
\{f\} \arrow[r] & \mathbf{map}_{s\mathcal{S}^T}(X, B\Lambda).
\end{tikzcd}$$

Then Axiom SM7 for the simplicial model category $s\mathcal{S}^T$ shows that q_* is a fibration. Also q_* has a section, since the natural transformation (7.3) has a section sending f to $(f,0)$. Finally

$$\pi_0 \mathbf{map}_{s\mathcal{S}^T/BA}(X, E_\Lambda(M,n)) = [X, E_\Lambda(M,n)]_{s\mathcal{S}^T/B\Lambda},$$

so the result follows from the fact that there is a pull-back square

$$\begin{tikzcd}
D^n(X,M) \arrow[r] \arrow[d] & H^n(X,M) \arrow[d, "\Phi"] \\
\{f\} \arrow[r] & \mathrm{Hom}_{\pi_0 E_* T}(\pi_0 E_* X, \Lambda).
\end{tikzcd}$$

Remark 7.10 We could forget that E_*T and E_*X and the rest of the objects under consideration are in E_*E comodules, and simply take the base category to be E_* modules. Then we could ask, for $\pi_0 E_* X$ module N over $\pi_0 E_* T$, whether the functor

$$X \longmapsto D^n_{E_*T}(E_*X, N)$$

is representable. However, we know (by [12] §4) that

$$D^n_{E_*T}(E_*X, N) \cong D^n_{E_*E(E_*T)}(E_*X, E_*E \otimes_{E_*} N)$$

and we obtain an affirmative answer from Proposition 7.1.

8 The obstruction theory

In this section we develop an obstruction theory for realizing structured ring spectra. There are many possible variants on this; for this first time through we choose a simple, direct example.

We are given an operad $\mathcal{F}$ over the linear isometries operad and an $E_*\mathcal{F}$ algebra Λ in the category of E_*E comodules. Here E_* is a fixed Adams type homology theory. We ask whether there is an $X \in \mathcal{S}^{\mathcal{F}}$ so that $E_*X \cong \Lambda$ as $E_*\mathcal{F}$ modules. This is the realization problem.

To simplify the process we will assume that as an E_*E comodule, $\Lambda \cong E_*E \otimes_{E_*} N$ where N is a projective E_* module. This eliminates the problem of existence and uniqueness of a spectrum (not necessarily a structured ring spectrum) with $E_*Z \cong Z$ as E_*E comodules.

Before stating the precise result, we make a reduction in the problem. We suppose we may choose an augmented simplicial operad

$$\epsilon : T \to \mathcal{F}$$

so that

8.1.1) E_*T is a cofibrant simplicial E_* operad,

8.1.2) the morphism $\epsilon : |T| \to \mathcal{F}$ is a weak equivalence of operads over the linear isometries operad, and

8.1.3) $\pi_*E_*T \cong E_*\mathcal{F}$ concentrated in degree 0.

We saw earlier that this is possible if $E_*\mathcal{F}(n)$ is a projective E_* module for all $n \geq 0$. (See 6.1.1 – 6.1.3). Note that 8.1.2 and one of the main results of [10] now would imply that if X is a cofibrant $|T|$ spectrum, then the natural map

$$X \to \mathcal{F} \otimes_{|T|} X$$

is a weak equivalence.

The program is to define a simplicial object $Y \in s\mathcal{S}^T$ so that

$$\pi_k E_* Y \cong \begin{cases} \Lambda, & k = 0, \\ 0 & k \neq 0. \end{cases}$$

Then, applying Theorem 5.4 and the result of [10] cited above, we will have that

$$X = \mathcal{F} \otimes_{|T|} |Y| \in \mathcal{S}^{\mathcal{F}}$$

is the desired structured ring spectra.

We give $s\mathcal{S}^T$ the $T(\mathcal{A})$-resolution model category structure, where $\mathcal{A} = \{\Sigma^j DE_\alpha\}$. We will write Aj for an element of $\mathcal{A}$.

If X actually existed, the fact that $\Lambda \cong E_*E \otimes_{E_*} N$ would force

$$\begin{aligned} \pi(X, Aj) &= [Aj, X] \\ &\cong \mathrm{Hom}_{E_*E}(E_*Aj, \Lambda). \end{aligned}$$

On the other hand, suppose we had $Y \in s\mathcal{S}^T$ with

$$\pi_k \pi(Y, Aj) \cong \begin{cases} \mathrm{Hom}_{E_*E}(E_*Aj, \Lambda) & k = 0 \\ 0 & k > 0. \end{cases} \tag{8.2}$$

Then

$$\pi_k E_* Y \cong \mathrm{colim}_\alpha\, \pi(Y, \Sigma^* DE_\alpha) \cong \begin{cases} \Lambda & k = 0 \\ 0 & k > 0. \end{cases}$$

Thus, we will work to produce Y satisfying (8.2). The method for producing Y is, in effect, to produce its Postnikov tower. If Y is to exist, the spiral exact sequence of section 4 (Equation 4.6) implies that for all $Aj \in \mathcal{A}$,

$$\begin{aligned}\pi_k^\natural(Y, Aj) &\cong \pi_0^\natural(Y, \Sigma^k Aj)\\ &\cong \pi_0\pi(Y, \Sigma^k Aj)\\ &\cong \mathrm{Hom}_{E_*E}(E_*\Sigma^k Aj, \Lambda)\end{aligned}$$

for all $k \geq 0$. Indeed, the connecting maps

$$\pi_{k-1}^\natural(Y, \Sigma Aj) \to \pi_k^\natural(Y, Aj)$$

must be isomorphisms for $n \geq 1$. Thus we attempt to produce $Y(n) \in s\mathcal{S}^T$ and fibrations $q_n : Y(n) \to Y(n-1)$, $n \geq 0$, so that

(8.3.1) $Y(-1) = *$ and

$$\pi_k^\natural(Y(n), Aj) = \begin{cases} \mathrm{Hom}_{E_*E}(E_*\Sigma^k Aj, \Lambda) & 0 \leq k \leq n \\ 0 & k > n; \end{cases}$$

(8.3.2) $\pi_k^\natural q_n : \pi_k^\natural(Y(n), Aj) \to \pi_k^\natural(Y(n-1), Aj)$ is an isomorphism for $k \leq n-1$; and

(8.3.3) the natural morphism $\pi_{k-1}^\natural(Y, \Sigma Aj) \to \pi_k^\natural(Y, Aj)$ are isomorphisms for $k \leq n$.

Note that E_*T, E_*X, etc., are objects in the category of E_*E comodules. If M is an $E_*\mathcal{F}$ module over E_*X in the category of E_*E comodules we can define the André-Quillen cohomology

$$D^*_{E_*E(E_*T)}(E_*X, M).$$

The following is the main result. Note that Λ is canonically a Λ module over $E_*\mathcal{F} = \pi_0 E_*T$. Let Λ_{*+n} denote the shifted Λ module over $pi_0 E_*T$ with $(\Lambda_{*+n})_k = \Lambda_{k+n}$.

Proposition 8.1 *The object $Y(0) \in s\mathcal{S}^T$ always exist and, having $Y(n-1)$, $n \geq 1$, there is an obstruction*

$$\theta_n \in D^{n+2}_{E_*E(E_*T)}(\Lambda, \Lambda_{*+n})$$

to producing $Y(n)$.

This will be proved in successive steps below. This result then gives a sequence of recursively defined obstructions to producing an $\mathcal{F}$ structure on X.

We begin by producing $Y(0)$. Since Λ is a $\pi_0 E_*T$ algebra via the augmentation, we may define

$$Y(0) = B(\Lambda)$$

as in Proposition 7.1. Then

$$\begin{aligned}\pi_0^\natural(Y(0), Aj) &\cong [Aj, Y(0)]_{s\mathcal{S}}\\ &\cong [T(Aj), Y(0)]_{s\mathcal{S}^T}\\ &\cong \mathrm{Hom}_{E_*T}(E_*T(E_*Aj), \Lambda)\\ &\cong \mathrm{Hom}_{E_*E}(E_*Aj, \Lambda)\end{aligned}$$

Furthermore, if $k \geq 1$,

$$\begin{aligned}
\pi_k^\natural(Y(0), Aj) &\cong \mathrm{Hom}_{E_*T}(E_*T(E_*Aj \wedge \Delta^k/\partial\Delta^k), \Lambda) \\
&\cong \mathrm{Hom}_{\pi_0 E_* T(0)\mathcal{F}}(E_*, \Lambda) \\
&= 0
\end{aligned}$$

since

$$\pi_0 E_* T(E_*Aj \wedge \Delta^k/\partial\Delta^k) \cong \pi_0 E_* T(0) = E_*\mathcal{F}(0)$$

is the initial object in $\pi_0 E_* T = E_*\mathcal{F}$ algebras.

Now suppose $Y(n-1)$ exists. Then the spiral exact sequence for $Y(n-1)$ implies that

$$\pi_k \pi_*(Y(n-1), Aj) \cong \begin{cases} \mathrm{Hom}_{E_*E}(E_*Aj, \Lambda) & k = 0 \\ \mathrm{Hom}_{E_*E}(E_*\Sigma^n Aj, \Lambda) & k = n+1 \\ 0 & k \neq 0, n+1. \end{cases}$$

From this it follows that

$$\pi_k E_* Y(n-1) \cong \begin{cases} \Lambda & k = 0 \\ \Lambda_{*+n} & k = n+1 \\ 0 & k \neq 0, n+1. \end{cases}$$

Thus the simplicial E_*T algebra (in E_*E comodules) $E_*Y(n-1)$ has two non-vanishing homotopy groups. Furthermore, from Proposition 5.5, we have that, as a Λ module over $\pi_0 E_* T(0) = E_*\mathcal{F}$, Λ_{*+n} has its canonical structure. It follows immediately that the n^θ Postnikov section $P_n E_* Y(n-1)$ is weakly equivalent to $P_0 E_* Y(n-1) \cong \Lambda$. Let

(8.3) $$\theta_n \in H^{n+2}_{E_*T}(\Lambda, \Lambda_{*+n})$$

be the k-invariant. This is the obstruction; this is verified by the following result.

Lemma 8.2 *There is a simplicial T-algebra $Y(n)$ over $Y(n-1)$ satisfying (8.2.1–8.2.3) if and only if $\theta_n = 0$.*

Proof. The k-invariant θ_n is zero if and only if there is a weak equivalence of E_*T algebras over Λ

(8.4) $$E_*Y(n-1) \to K_\Lambda(\Lambda_{*+n}, n+1).$$

This, in turn, happens if and only if the projection

$$E_*Y(n-1) \to P_0 E_* Y(n-1) \simeq \Lambda$$

has a homotopy section as E_*T algebras. Here the object $K_\Lambda(\Lambda_{*+n}, n+1)$ is the simplicial E_*T algebra in E_*E comodules representing André-Quillen cohomology. In particular, if $Y(n)$ exists, then

$$\Lambda \simeq P_{n+1} E_* Y(n) \to P_{n+1} E_* Y(n-1) \cong E_* Y(n-1)$$

provides the desired section and, hence $\theta_n = 0$. This proves one implication.

So suppose $\theta_n = 0$. Then the weak equivalence (8.4) produces a cohomology class

$$\iota_{n+1} \in D^{n+1}_{E_*E(E_*T)}(E_*Y(n-1), \Lambda_{*+n})$$

and, hence, a morphism of simplicial T-algebras

$$Y(n-1) \to E_\Lambda(\Lambda_{*+n}, n+1).$$

Here $E_\Lambda(\Lambda_{*+n}, n+1)$ is the simplicial spectrum produced in Proposition 7.1; it should not be confused with the algebraic object $K_\Lambda(\Lambda_{*+n}, n+1)$.

Define $Y(n)$ by the homotopy pull-back

(8.5)
$$\begin{array}{ccc} Y(n) & \longrightarrow & B\Lambda \\ {\scriptstyle q_n}\downarrow & & \downarrow \\ Y(n-1) & \xrightarrow{\iota_{n+1}} & E_\Lambda(\Lambda_{*+n}, n+1). \end{array}$$

The evident long exact sequence in $\pi_*^\natural(\cdot, Aj)$ implies that $Y(n)$ satisfies 8.3.1 and 8.3.2. To prove that 8.3.3 holds we argue as follows.

First note that the natural Hurewicz map

$$\begin{aligned} \pi_{n+1}\pi(Y(n-1), Aj) &\to \pi_{n+1}\mathrm{Hom}_{E_*E}(E_*Aj, E_*Y(n-1)) \\ &\to \mathrm{Hom}_{E_*E}(E_*Aj, \pi_{n+1}E_*Y(n-1)) \end{aligned}$$

is an isomorphism since $\pi_{n+1}E_*Y(n-1) \cong \Lambda_{*+n}$.

Second, note that the map

$$E_*\iota_{n+1} : E_*Y(n-1) \to E_*E_\Lambda(\Lambda_{*+n}, n+1)$$

is split, up to homotopy, in simplicial E_*T algebras by classifying the fundamental class in the cohomology group $D^{n+1}_{E_*E(E_*T)}E_*E_\Lambda(\Lambda_{*+n}, n+1)$:

$$E_*E_\Lambda(\Lambda_{*+n}, n+1) \to K_\Lambda(\Lambda_{*+n}, n+1).$$

Third, notice that the

$$\pi_{n+1}\pi(E_\Lambda(\Lambda_{*+n}, n+1), Aj) \to \mathrm{Hom}_{E_*E}(E_*Aj, \pi_{n+1}E_*E_\Lambda(\Lambda_{*+n}, n+1))$$

is an isomorphism by the spiral exact sequence. Thus we get a diagram

$$\begin{array}{ccc} \pi_{n+1}\pi(Y(n-1), Aj) & \longrightarrow & \pi_{n+1}\pi(E_\Lambda(\Lambda_{*+n}, n+1), Aj) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{Hom}_{E_*E}(E_*Aj, \pi_{n+1}E_*Y(n-1)) & \xrightarrow{\cong} & \mathrm{Hom}_{E_*E}(E_*Aj, \pi_{n+1}E_*E_\Lambda(\Lambda_{*+n}, n+1)). \end{array}$$

The top horizontal map is now an isomorphism. There is a short exact sequence in $\pi_*\pi(\cdot, Aj)$ for the homotopy pull-back square of (8.5) since

$$\pi(K\Lambda, Aj) \to \pi(E_\Lambda(\Lambda_{*+n}, n+1))$$

is split surjective. From this and the spiral exact sequence (8.3.3) follows.

Example 8.3 Suppose we fixed an Adams-type ring spectrum E and we are trying to decided whether E supports and A_∞ (or E_∞) ring structure compatible with the homotopy ring structure on E. In this case $E_*\mathcal{F}$ would be the associative (or commutative) algebra operad in E_*E comodules, and the $E_*\mathcal{F}$ algebra structure on $E_*E = E_*E \otimes_{E_*} E_*$ in E_*E comodules would be induced up from the $E_*\mathcal{F}$ structure on E_* in E_* modules. Then Proposition 4.11 of [12] supplies an isomorphism

$$D^{n+2}_{E_*E(E_*T)}(E_*E, E_{*+n}E) \cong D^{n+2}_{E_*(T)}(E_*E, E_{*+n}),$$

and, in this case, the obstruction groups are an extension of the type of group encountered in spectral sequence of section 6. Compare Lemma 6.4.

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