Simplicial Structured Ring Spectra

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In this paper we work out some of the foundations of the homotopy theory of simplicial ring spectra over a simplicial operad. This is not a gratuitous act of generalization. Simplicial objects in any category are a standard mechanism for building the resolutions necessary for computations; this is how simplicial spectra arise. The simplicial operads arise as an answer to an immediate practical problem. If T is an E_∞ operad over the linear isometries operad and X is a spectrum, one can form the free E_∞ ring spectrum $T(X)$. Then, if E_* is a generalized homology theory, one can ask for a computation of $E_*T(X)$. This is extremely difficult; at the very least one needs to know $E_*B\Sigma_n$ for all $n \geq 0$ and, practically, one also needs a firm hold on the E_* Dyer-Lashof algebra. Our response is resolve T by a simplicial operad $\{T_n\}$ for which is is possible to compute $E_*T_n(X)$. The application is to develop tools for deciding when a spectrum is an E_∞ ring spectrum, and for computing the homotopy type of the space of E_∞ maps between E_∞ ring spectra. Actually, there's nothing special about the E_∞ operad. Our results will apply to any operad over the linear isometries operad.

This is the third of a series of papers, following and building upon [10] and [11]. This paper and [10] are both informed by the work of Dwyer, Kan, and Stover [4],[5]; the obstruction theory these authors alude to was worked out by Blanc [3]. The paper [11], which is necessary for some of our computations, should be regarded as an algebraic intermezzo.

Here is a brief synopsis of the results of this paper. As the paragraphs above indicate, we are working with the Lewis-May-Steinberger category of spectra over some fixed universe $\mathcal{U}$ [14]. Since $\mathcal{U}$ is a countably infinite dimensional inner product space, we can form the linear isometries operad $\mathcal{L}$ with

$$\mathcal{L}(0) = \mathcal{L}(\mathcal{U}^{\oplus n}, \mathcal{U})$$

the space of linear isometries from $\mathcal{U}^{\oplus n}$ to $\mathcal{U}$. We are interested in operads T equipped with a map of operads $T \to \mathcal{L}$ and so that $T(0) = L(0) = *$ is a point. We will call such operads *unital* operads over the linear isometries operad. If $\mathcal{S} = \mathcal{S}_{\mathcal{U}}$ is the category of spectra over $\mathcal{U}$, let $\mathcal{S}^T$ denote the category of algebras over T. To specify an object in $X \in \mathcal{S}^T$ amounts to specifying a map of spectra $T(X) \to X$ from the free T-algebra on X to X so that certain diagrams commute. This is natural enough that the definition can be extended to simplicial objects: if T is a simplicial operad and X is a simplicial spectrum, then X is a simplicial T-algebra if there is a map of simplicial spectra $T(X) \to X$ so that the appropriate diagrams commute. We call the category of such simplicial algebras $s\mathcal{S}^T$.

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The main thrust of the first five sections of this paper is to define and study a geometric realization functor of $s\mathcal{S}^T$. The geometric realization of simplicial operad T is an operad $|T|$ and we construct a geometric realization functor

$$|\cdot| : s\mathcal{S}^T \longrightarrow \mathcal{S}^{|T|}$$

and discuss its homotopy invariance and computability. This a generalization of a result of [9] (see also [7]) and depends on that result. This is done in the third section; the first two sections are spent building a Reedy model category structure on $s\mathcal{S}^T$ so that we can claim that this geometric realization functor preserves weak equivalences between Reedy cofibrant objects. See Proposition 3.4 and 3.6.

Sections 4 and 5 are devoted to importing the "E_2" model category structure of Dwyer, Kan, and Stover [4, 5] into the setting of simplicial spectra over a simplicial operad. The essential point here is that this allows one to build simplicial resolutions $Y \to X$ of objects $X \in s\mathcal{S}^T$ by objects $Y \in s\mathcal{S}^T$ so that each of the objects Y_n of Y is of the form $T_n(Z_n)$ with $E_*T_n(Z_n)$ computable, where E_* is some appropriate homology theory. With this in hand, we can build spectral sequences for computing space of algebra maps between structured ring spectra. The E_2 term is an appropriate André-Quillen cohomology. This is done in section 6. In section 8, we develop the obstruction theory for deciding when a spectrum is a structured ring spectrum; again the obstruction groups are André-Quillen cohomology groups. Section 7 is devoted to construction of the Eilenberg-Mac Lane objects necessary to the Postnikov tower approach to the obstruction theory.

The immediate applications involve studying the spectra E_n realizing the Lubin-Tate universal deformation of the height n formal group law. In an appendix we show how the Miller-Hopkins result showing that E_n is an A_∞ ring spectrum can be adapted to show that E_n is, in fact, and E_∞ ring spectrum. With the obstruction theory of this paper, and with the algebra of [11] at hand, the ideas of [18] can be adapted quite cleanly.

1 Some preliminaries on operads and diagrams of algebras

Let $\mathcal{L}$ be the linear isometries operad and $\mathcal{O}$ the category of unital operads over $\mathcal{L}$. Recall that an operad $T \to \mathcal{L}$ is unital if $T(0) = \mathcal{L}(0) = *$ is a point. The category $\mathcal{O}$ has all limits and colimits. Let I be a small category and $\mathcal{S}^I$ the category of I-diagrams in spectra. Then if $T : I \to \mathcal{O}$ is a diagram of operads and $X \in \mathcal{S}^I$ we can form $T(X) = T \circ X \in \mathcal{S}^I$. The functor $T(\cdot) : \mathcal{S}^I \to \mathcal{S}^I$ is the functor of a triple on $\mathcal{S}^I$, and we write $\mathcal{S}^T$ for the category of algebras over this triple. This is an abbreviation of the more usual $(\mathcal{S}^I)^T$, which is cumbersome. The exception is this: if $I = \Delta^{\mathrm{op}}$ we write $(s\mathcal{S})^T$ for simplicial algebras over the simplicial operad T.

Fix $T : I \to \mathcal{O}$ and consider $\lim_I T$ and $\operatorname{colim}_I T$, again operads in $\mathcal{O}$. Abbreviate $\lim_I$ as $\lim T$, and so on.

Lemma 1.1 *1) The constant diagram functor $\mathcal{S}^{\lim T} \to \mathcal{S}^T$ has a right adjoint $\lim\limits_I{}^T : \mathcal{S}^T \to \mathcal{S}^{\lim T}$.*

2) The constant diagram functor $\mathcal{S}^{\operatorname{colim} T} \to \mathcal{S}^T$ has a left adjoint

$$\operatorname*{colim}_I{}^T : \mathcal{S}^T \to \mathcal{S}^{\operatorname{colim} T}.$$

Proof. For 1), let $X \in \mathcal{S}^T$. Then we may regard $X \in \mathcal{S}^I$ and $\lim\limits_I X \in \mathcal{S}$ has a natural structure as a $\lim T$ algebra; indeed, the natural maps $(\lim\limits_I T) \circ (\lim\limits_I X) \to T_i \circ X_i$ coupled with the evaluation maps $\eta_i : T_i \circ X_i \to X_i$ yield a map

$$(\lim_I T) \circ (\lim_I X) \to \lim_I (T_i \circ X_i) \to \lim X,$$

which defines the $\lim T$ structure on $\lim\limits_I X$. The object $\lim\limits_I X \in \mathcal{S}^{\lim T}$ so defined is $\lim\limits_I{}^T X$.

For 2), let $X \in \mathcal{S}^T$. The maps of operads $T_i \to \operatorname{colim} T$ defined a functor

$$\operatorname{colim} T \otimes_T (\cdot) : \mathcal{S}^T \to (\mathcal{S}^{\operatorname{colim} T})^I$$

with $(\operatorname{colim} T \otimes_T X)_i = \operatorname{colim} T \otimes_{T_i} X_i$. Then

$$\operatorname*{colim}_I{}^T X = \operatorname*{colim}_I{}^{\operatorname{colim} T} (\operatorname{colim} T \otimes_T X). \tag{1.2}$$

■

Also in the vein of categorical preliminaries we record the following. Let I be a category and $J \subseteq I$ a subcategory. If $T : I \to \mathcal{O}$ is an I-diagram of operads we have a restricted diagram of operads $r^*T : J \to \mathcal{O}$, and we get a restriction functor

$$r^* : \mathcal{S}^T \to \mathcal{S}^{r^*T}.$$

Lemma 1.3 *The restriction functor r^* has both a left adjoint r_* and a right adjoint $r^!$.*

Proof. The restriction $r^* : \mathcal{O}^I \to \mathcal{O}^J$ has a left and right adjoint r_* and $r^!$ given by the left and right Kan extension functors respectively. For example

$$(r_*T)_i = \operatorname*{colim}_{j \to i} T_j$$

where the colimit occurs in $\mathcal{O}$, and is over all maps $j \to i$ in I with $j \in J$. Let us define $r_* : \mathcal{S}^{r^*T} \to \mathcal{S}^T$. First note that Lemma 1.1 supplies a functor $F : \mathcal{S}^{r^*T} \to \mathcal{S}^{r_*r^*T}$. The counit of the adjunction between r_* and r^* supplies a morphism in $\mathcal{O}^I$

$$r_*r^*T \to T.$$

Then $r_*X = T \otimes_{r_*r^*T} FX$.

The right adjoint is easier: if $X \in \mathcal{S}^{r^*T}$, then $X \in \mathcal{S}^J$ and the right Kan extension $r^!X \in \mathcal{S}^I$ is already an $r^!r^*T$ algebra and hence a T-algebra via the unit $T \to r^!r^*T$. ∎

We now focus specifically on the case $(s\mathcal{S})^T$ where T is a simplicial operad. The first observation is that $(s\mathcal{S})^T$ is a simplicial category in the sense of Quillen [15, Chap. II]. To see this, give $s\mathcal{S}$ the standard simplicial structure of [15, II §2]. Thus if $K \in \mathbf{S}$ and $X \in s\mathcal{S}$,

$$(X \otimes K)_n = \bigvee_{K_n} X_n.$$

If $X \in s\mathcal{S}$, define $T(X) \otimes K \in (s\mathcal{S})^T$ by the formula

$$T(X) \otimes K = T(X \otimes K).$$

If $f : T(X) \to T(Y)$ is any map in $(s\mathcal{S})^T$, there is a unique map $f \otimes 1 : T(X) \otimes K \to T(Y) \otimes K$ defined by unique T-algebra filler of the diagram

$$\begin{tikzcd} X \otimes K \arrow[r, "g\otimes 1"] \arrow[d] & T(Y) \otimes K \arrow[d, "\theta"] \\ T(X \otimes K) \arrow[r, dashed] & T(Y \otimes K). \end{tikzcd}$$

The map g of spectra is adjoint to f and the map θ in degree n is the map of spectra

$$\bigvee_{K_n} T_n(Y_n) \to \coprod_{K_n} T_n(Y_n)$$

induced by the inclusion of the summands. The coproduct in the target is formed in the category of T_n algebras. For general $X \in (s\mathcal{S})^T$ define $X \otimes K$ by the coequalizer diagram in $(s\mathcal{S})^T$

(1.4.1) $$T^2(X) \otimes K \rightrightarrows T(X) \otimes K \to X \otimes K$$

where the two maps are $T\eta_X \otimes 1$ and $\eta_{T_X} \otimes 1$. Since colimits are formed level-wise,

(1.4.2) $$(X \otimes K)_n \cong \coprod_{K_n}{}^{T_n} X_n.$$

For fixed K, the functor $X \mapsto X \otimes K$ from $(s\mathcal{S})^T$ to $(s\mathcal{S})^T$ has a right adjoint $X \mapsto \mathbf{hom}(K, X)$. (We reserve X^K for another purpose.) In fact, the object $\mathbf{hom}(K, X) \in s\mathcal{S}$ obtained by Quillen's structure on $s\mathcal{S}$ has a canonical T-algebra structure adjoint to

$$T\mathbf{hom}(K, X) \otimes K \to T(\mathbf{hom}(K, X) \otimes K) \to T(X) \to X.$$

Finally there is a simplicial set of maps for $X, Y \in (s\mathcal{S})^T$ specified by

$$\mathrm{map}_{(s\mathcal{S})^T}(X, Y)_n = \mathbf{Hom}_{(s\mathcal{S})^T}(X \otimes \Delta^n, Y).$$

We will write $\mathrm{map}(X, Y)$ to ease notation if $(s\mathcal{S})^T$ is understood.

Lemma 1.5 *With these definitions $(s\mathcal{S})^T$ becomes a simplicial category.*

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We next address the decomposition of objects in $(s\mathcal{S})^T$ via skeleta and coskeleta. Let $\Delta[n] \subseteq \Delta$ be the full subcategory of the ordinal number with objects $[k]$, $k < n$. For any category $\mathcal{C}$, let $s_n\mathcal{C} = \mathcal{C}^{\Delta[n]^{\mathrm{op}}}$ and $r_* \cdot s\mathcal{C} \to s_n\mathcal{C}$ the restriction. If $T \in s_n\mathcal{O}$, let us write $\mathcal{S}^T$ as $(s_n\mathcal{S})^T$. As in Lemma 1.3 there is a restriction functor $r^* : (s\mathcal{S})^T \to (s_n\mathcal{S})^{r^*T}$. If $X \in (s\mathcal{S})^T$ define the n^{th} skeleton of X by, for $n \geq -1$.

$$sk_n X = r_* r^* X.$$

Note $sk_{-1}X = S^0$, the initial object. We have restricted attention to unital operads so that S^0 is the initial object.

More generally, if $A \to X$ is a map in $(s\mathcal{S})^T$ define $sk_n^A X$ by the push-out diagram

$$\begin{array}{ccc} sk_n A & \longrightarrow & A \\ \downarrow & & \downarrow \\ sk_n X & \longrightarrow & sk_n^A X. \end{array}$$

Then $sk_{-1}^A X = A$ and the map $A \to \operatorname*{colim}_n sk_n^A X \cong X$ is isomorphic to the original map $A \to X$. If $A = S^0$, and $S^0 \to X$ is the unique map, then $sk_n^{S^0} X = sk_n X$.

To describe how $sk_n^A X$ is built from $sk_{n-1}^A X$ we must slightly generalize the idea of $X \otimes K$ defined above. The functor $(s\mathcal{S})^T \to \mathcal{S}^{T_n}$ given by $X \mapsto X_n$ has a left adjoint by 1.1 Following the prescription of that result we have, for $Y \in \mathcal{S}^{T_n}$ an object $FY \in (s\mathcal{S})^{T_n \otimes \Delta^n}$ with

$$(FY)_k = \coprod_{[k] \to [n]} Y$$

where $[k] \to [n]$ runs the morphisms in Δ to $[n]$. Then the left adjoint is given by $Y \mapsto T \otimes_{T_n \otimes \Delta_n} FY$, where $T_n \otimes \Delta^n \to T$ is the natural map. We will write

(1.6)
$$Y \mapsto Y \otimes \Delta^n$$

for this functor. If T is constant, as a simplicial object, then this notation agrees with 1.4.1 if we regard $Y \in (s\mathcal{S})^T$ as a constant object. Otherwise, it is different, as Y does not live in $(s\mathcal{S})^T$ in any sense.

This can be slightly generalized. If $K \in \mathbf{S}$ we let $\mathcal{K}$ be its simplex category: the objects in $\mathcal{K}$ are morphisms $\Delta^n \to K$ and morphisms are triangles. There is a natural isomorphism $\operatorname*{colim}_{\mathcal{K}} \Delta^n \cong K$. If $\mathcal{C}$ is any category with limits and colimits and $s\mathcal{C}$ is the category of simplicial objects with its standard simplicial structure, then the functor $\mathcal{C} \to s\mathcal{C}$ sending Z to $Z \otimes K$ has right adjoint M_K with

$$M_K X \cong \lim_{\mathcal{K}^{\mathrm{op}}} X_n.$$

If $X \in (s\mathcal{S})^T$ we may define a functor $M_K : (s\mathcal{S})^T \to \mathcal{S}^{M_K T}$ by the same formula. This is the generalized matching object formula.

Lemma 1.7 *1) For fixed K, the functor M_K has a left adjoint $Z \to Z \otimes K$.*
2) For fixed X, the functor $\mathbf{S}^{\mathrm{op}} \to \mathcal{S}$ sending $X \mapsto M_K X$ has a left adjoint.

Proof. If $Z \in \mathcal{S}^{M_K T}$ define $FZ \in (s\mathcal{S})^{M_K T \otimes K}$ by the coproduct formula 1.4.2; then $Z \otimes K = T \otimes_{M_K T \otimes K} FZ$ where $M_K T \otimes K \to T$ is the counit of the adjunction. The second statement follows from the fact that in $s\mathcal{S}$

$$\mathbf{Hom}_{\mathcal{S}}(Z, M_K X) \cong \mathbf{Hom}_{\mathbf{S}}(K, \mathbf{Hom}_{s\mathcal{S}}(Z, X)). \qquad \blacksquare$$

A consequence of 1.7.2 is the following: if $K : I \to \mathbf{S}$ is a diagram and $L = \operatorname{colim}_i K_i$, then there is a natural isomorphism in $\mathcal{S}^{M_L T}$

$$M_L X \cong \lim_{I^{\mathrm{op}}} M_{K_i} X. \tag{1.8}$$

We may now describe how $sk_n^A X$ is built from $sk_{n-1}^A X$. For $X \in (s\mathcal{S})^T$ define $L_n X = (sk_{n-1} X)_n \in \mathcal{S}^{T_n}$. If $f : A \to X$ is a morphism in $(s\mathcal{S})^T$, let $L_n(f) = L_n X \sqcup_{L_n A} A_n$, the coproduct as T_n algebras. By definition of $sk_{n-1}^A X$ there is a natural isomorphism

$$L_n(f) \cong (sk_{n-1}^A X)_n,$$

and, hence, by adjointness, natural maps

$$L_n(f) \otimes \Delta^n \longrightarrow sk_{n-1}^A X$$

$$X_n \otimes \Delta^n \longrightarrow sk_n^A X$$

An easy calculation shows $sk_{n-1}(X_n \otimes \Delta^n) = X_n \otimes \partial\Delta^n$; thus, there is a diagram

$$\begin{tikzcd} L_n(f) \otimes \Delta^n \sqcup_{L_n(f) \otimes \partial\Delta^n} X_n \otimes \partial\Delta^n \arrow[r] \arrow[d] & sk_{n-1}^A X \arrow[d] \\ X_n \otimes \Delta^n \arrow[r] & sk_n^A X. \end{tikzcd} \tag{1.9}$$

Proposition 1.10 *This diagram is a push-out diagram in $(s\mathcal{S})^T$.*

Proof. The proof is the usual one. It is sufficient to prove it is a push-out in $(s_n\mathcal{S})^{r^*_* T}$ where $r^*T \in s_n\mathcal{O}$ is the restriction of T. Since push-out are computed level-wise, one does level-wise calculations. Since $(\partial\Delta^n)_k = (\Delta^n)_k$ for $k < n$, the left vertical map is an isomorphism for $k < n$, as required. In degree n, $(\Delta^n)_n = (\partial\Delta^n)_k \cup \{1 : [n] \to [n]\}$, so the right vertical map is isomorphic to

$$(X_n \otimes \partial\Delta^n)_n \sqcup L_n(f)_n \xrightarrow{k} (X_n \otimes \partial\Delta^n)_n \sqcup X_n. \qquad \blacksquare$$

There is a dual theory of coskeleta which we won't belabor. The standard theory of coskeleta for $s\mathcal{S}$ (See [13], if nowhere else) passes to a theory of coskeleta for $(s\mathcal{S})^T$ as the usual diagrams and constructions in $s\mathcal{S}$ lie canonically in $(s\mathcal{S})^T$. The upshot is that if $f : X \to B$ is a morphism in $(s\mathcal{S})^T$ then $X = \lim_n \operatorname{cosk}^B_n X$ and $\operatorname{cosk}^B_n X$ is the pull-back of a diagram built from $\operatorname{cosk}^B_{n-1} X$, X_n, and the matching objects

$$M_n(f) = (\operatorname{cosk}^B_{n-1} X)_n.$$

These last can be described as follows: $M_n X = M_{\partial\Delta^n} X$ and

$$M_n(f) = B_n \times_{M_n B} M_n X. \tag{1.11}$$

More important for what follows is the generalization and decomposition of the latching objects $L_n X$. If $F : J \to \Delta$ is any functor from a small category and $X \in s\mathcal{C}$, define the generalized latching object to be

$$L_J X = \operatorname*{colim}_J X \circ F.$$

The functor F is usually suppressed from the notation, as it is implicit in the category J. If $X \in (s\mathcal{S})^T$, then the functor $F : J \to \Delta$, determines functors $T \circ F : J^{\mathrm{op}} \to \mathcal{O}$ and $X \circ F \in \mathcal{S}^{T \circ F}$

$$L_J X = \operatorname*{colim}_J X \circ F \in \mathcal{S}^{L_J T}$$

should be computed via Lemma 1.1. For example, let $J = [n] \downarrow \Delta$ be the category morphisms $[n] \to [m]$ in Δ, and $F : J \to \Delta^{\mathrm{op}}$ sending $[n] \to [m]$ to $[m]$. Then $L_n X \cong T_n \otimes_{L_J T} L_J X$. If $\mathcal{J}_n \subseteq J$ is the full subcategory with morphisms $[n] \to [m]$, $m < n$, which are surjections, then $\mathcal{J}_n$ is final in J (since every morphism in Δ factors uniquely as a surjection followed by an injection); hence $L_n T \cong L_{\mathcal{J}_n} T$ and

$$L_n X \cong T_n \otimes_{L_n T} L_{\mathcal{J}_n} X. \tag{1.12}$$

To decompose $L_n X$ we specify some subcategories of $\mathcal{J}_n$. Let $\mathcal{J}_{n,k} \subseteq \mathcal{J}_n$ be the full subcategory of surjections $\phi : [n] \to [m]$, $m < n$, with $\phi(k) < k$. Then $\mathcal{J}_{n,0} = \phi$ and $\mathcal{J}_{n,n} = \mathcal{J}_n$. Define

$$L_{n,k} X = L_{\mathcal{J}_{n,k}} X.$$

Thus $L_n X \cong T_n \otimes_{L_n T} L_{n,n} X$. Let $\mathcal{J}(k) \subseteq \mathcal{J}_n$ be the full subcategory of surjections $\phi : [n] \to [m]$ with $\phi(k) = \phi(k+1)$. Note $s^k : [n] \to [n-1]$ is initial in $\mathcal{J}(k)$; therefore,

$$L_{\mathcal{J}(k)} X \cong X_{n-1}.$$

There are inclusions $\mathcal{J}_{n,k} \subseteq \mathcal{J}_{n,k+1}$, $\mathcal{J}(k) \subseteq \mathcal{J}_{n,k+1}$ and an isomorphism

$$- \circ s^k : \mathcal{J}_{n-1,k} \to \mathcal{J}(k) \cap \mathcal{J}_{n,k}$$

sending ϕ to $\phi \circ s^i$.

The following is our decomposition result.

Proposition 1.13 *Let T be a simplicial operad and $X \in (s\mathcal{S})^T$. Then there is a push-out diagram of operads*

$$\begin{tikzcd} L_{n-1,k}T \arrow[r] \arrow[d] & T_{n-1} \arrow[d] \\ L_{n,k}T \arrow[r] & L_{n,k+1}T \end{tikzcd}$$

and a push-out diagram of $L_{n,k+1}T$ algebras

$$\begin{tikzcd} L_{n,k+1}T \otimes_{L_{n-1,k}T} L_{n-1,k}X \arrow[r] \arrow[d] & L_{n,k+1}T \otimes_{T_{n-1}} X_{n-1} \arrow[d] \\ L_{n,k+1}T \otimes_{L_{n,k}T} L_{n,k}X \arrow[r] & L_{n,k+1}X. \end{tikzcd}$$

Proof. The diagram

$$\begin{tikzcd} \mathcal{J}_{n-1,k} \arrow[r] \arrow[d] & \mathcal{J}(k) \arrow[d] \\ \mathcal{J}_{n,k} \arrow[r] & \mathcal{J}_{n,k+1} \end{tikzcd}$$

is a push-out diagram in the category of small categories. ∎

2 The Reedy Model Category Structure.

The purpose of this section and the next is to define a geometric realization functor

$$|\cdot| : (s\mathcal{S})^T \to \mathcal{S}^{|T|}$$

and to investigate its homotopical properties. This section is devoted to the Reedy model category structure on $(s\mathcal{S})^T$, which is a model category structure suitable for discussing realization. This is a minor generalization of Reedy's point of view [16]; for example, the objects of $(s\mathcal{S})^T$ are not actually simplicial diagrams in any underlying category.

Definition 2.1 *A morphism $f : X \to Y$ in $(s\mathcal{S})^T$ is a*

(1) Reedy weak equivalence if $f : X_n \to Y_n$ is a weak equivalence of T_n-algebras or, equivalently, of spectra;

(2) Reedy fibration if $X_n \to Y_n \times_{M_nY} M_nX$ is a fibration of T_n-algebras or, equivalently, of spectra, for $n \geq 0$;

(3) Reedy cofibration if $X_n \sqcup_{L_nX} L_nY \to X_n$ is a cofibration of T_n-algebras for $n \geq 0$.

Note that $L_0X = S^0$, the initial object, and $M_0X = *$, the terminal object. The result is that with these definitions $(s\mathcal{S})^T$ becomes a model category. The standard preliminary result is this:

Lemma 2.2 *A morphism $f : X \to Y$ in $(s\mathcal{S})^T$ is*

(1) A Reedy trivial fibration if and only if

$$X_n \to X_n \times_{M_nY} M_nX$$

is a trivial fibration of T_n-algebras or, equivalently, of spectra, for $n \geq 0$;

(2) Reedy trivial cofibration if and only if

$$X_n \sqcup_{L_nX} L_nY \to Y_n$$

is a trivial cofibration of T_n-algebras for all $n \geq 0$.

Proof. The push-out diagram of Proposition 1.13 leads to a push-out diagram of T_n-algebras

$$\begin{tikzcd} T_n \otimes_{T_{n-1}} (X_{n-1} \sqcup_{L_{n-1,k-1}X} L_{n-1,k-1}Y) \arrow[r] \arrow[d] & X_n \sqcup_{L_{n,k}X} L_{n,k}Y \arrow[d] \\ T_n \otimes_{T_{n-1}} Y_{n-1} \arrow[r] & X_n \sqcup_{L_{n,k+1}X} L_{n,k+1}Y \end{tikzcd} \tag{2.3.1}$$

since $Y_{n-1} \cong X_{n-1} \sqcup_{X_{n-1}} Y_{n-1}$.

We use this to prove part 2. Assume $f : X \to Y$ is a Reedy cofibration in $(s\mathcal{S})^T$. Then $f : X_0 \to Y_0$ is a cofibration of T_0 algebras and this begins an induction argument where the inductive hypothesis is that

$$X_{n-1} \sqcup_{L_{n-1,k}X} L_{n-1,k}Y \to Y_{n-1} \tag{2.3.2}$$

is a cofibration of T_{n-1} algebras for all k, $-1 \leq k \leq n-1$. To complete the inductive step, use (2.3.1) to show

$$X_n \sqcup_{L_{n,k}X} L_{n,k}Y \to X_n \sqcup_{L_{n,k+1}X} L_{n,k+1}Y \tag{2.3.3}$$

is a cofibration of T_n algebras. Here one uses that $T_n \otimes_{T_{n-1}} (\cdot)$ preserves cofibrations (see [9]). Since $X_n \sqcup_{L_nX} L_nY \to Y_n$ is a cofibration of T_n algebras and compositions of cofibrations are cofibrations we have completed in the inductive step.

Now suppose $f : X \to Y$ is a Reedy trivial cofibration. Then, inductively, each of the maps of (2.3.2) is a trivial cofibration of T_n-algebras. In particular, we can set $n = k$ and get the desired result. The inductive step is slightly different: one uses (2.3.1) to show each of the maps of (2.3.3) in a trivial cofibration; then since $X_n \to Y_n$ is a weak equivalence, the "two-out-of-three" axiom for weak equivalences finishes the argument.

Conversely, suppose $X_n \sqcup_{L_n X} L_n Y \to Y_n$ is a trivial cofibration of T_n-algebras for all n. Then the same sort of argument gives that each of the maps of (2.3.2) is a trivial cofibration of T_n-algebras. Then one sets $k = 0$.

To prove part 1, we use a similar argument after decomposing the matching objects $M_n X$. Let $\Delta^{n,k} = d^0\Delta^{n-1} \cup \cdots \cup d^k\Delta^{n-1} \subseteq \partial\Delta^n$, $-1 \leq k \leq n$. Then $\Delta^{n,-1} = \phi$ and $\Delta^{n,n} = \partial\Delta^n$. There are push-out diagrams

$$\begin{tikzcd} \Delta^{n-1,k} \arrow[r] \arrow[d] & \Delta^{n-1} \arrow[d] \\ \Delta^{n,k} \arrow[r] & \Delta^{n,k+1} \end{tikzcd}$$

for $-1 \leq k \leq n-1$. Define $M_{n,k}X = M_{\Delta^{n,k}}X$. Then Lemma 1.7.2 implies there is a pull-back diagram

$$\begin{tikzcd} M_{n,k+1}X \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ M_{n,k}X \arrow[r] & M_{n-1,k}X. \end{tikzcd}$$

■

We now prove that the lifting axiom of a model category holds for $(s\mathcal{S})^T$.

Lemma 2.3 *The Reedy structure on $(s\mathcal{S})^T$ satisfies Axiom CM 4.*

Proof. Suppose we are given a lifting problem in $(s\mathcal{S})^T$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "q"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where j is a Reedy cofibration, q is a Reedy cofibration, and either j or q is a Reedy weak equivalence. We inductively solve the lifting problems

$$\begin{tikzcd} sk^A_{n-1}B \arrow[r] \arrow[d] & X \arrow[d] \\ sk^A_n B \arrow[r] \arrow[ur, dashed] & Y, \end{tikzcd}$$

for $n \geq 0$. By Proposition 1.10 it suffices to solve a lifting problem

$$\begin{tikzcd} L_n(j) \otimes \Delta^n \sqcup_{L_m(j)\otimes\partial\Delta^n} B_m \otimes \partial\Delta^n \arrow[r] \arrow[d] & X \arrow[d] \\ B_n \otimes \Delta^n \arrow[r] \arrow[ur, dashed] & Y. \end{tikzcd}$$

This is adjoint (by Lemma 1.7.1) to a lifting problem

$$\begin{tikzcd} A_n \sqcup_{L_n A} L_n B \arrow[r] \arrow[d] & X_n \arrow[d] \\ B_n \arrow[r] \arrow[ur, dashed] & X_n \times_{M_n Y} M_n X \end{tikzcd}$$

in S^{T_n}. This is solvable by Lemma 2.2. ■

We next show that the Reedy structure on $(s\mathcal{S})^T$ satisfies the factorization axiom.

Lemma 2.5 *The Reedy structure on $(s\mathcal{S})^T$ satisfies CM 5.*

Proof. We prove the "trivial cofibration-fibration" factorization. The other is identical. Fix $f : X \to Y$. To do this we inductively factor $sk_n X \to Y$ and

$$sk_n X \xrightarrow{j_n} Z(n) \xrightarrow{q_n} Y$$

so that $sk_{n-1}Z(n) \cong Z(n-1)$, there are commutative diagrams

$$\begin{tikzcd} sk_{n-1}X \arrow[r, "j_{n-1}"] \arrow[d] & Z(n-1) \arrow[r, "q_{n-1}"] \arrow[d] & Y \arrow[d] \\ sk_n X \arrow[r] & Z(n) \arrow[r, "q_n"] & Y \end{tikzcd}$$

and so that $X_n \sqcup_{L_n X} L_n Z(n) \to Z(n)_n$ is a trivial cofibration of T_n algebras and $Z(n)_n \to Y_n \times_{M_n Y} M_n Z(n)$ is a fibration. Then we may set $j = \operatorname{colim} j_n$, $q = \operatorname{colim} q_n$ and $Z = \operatorname{colim} Z(n)$ to obtain the needed factorization.

To define $Z(0)$, factor $X_0 \to Y_0$ as

$$X_0 \to Z_0 \to Y_0$$

where the first map is a trivial cofibration of T_0 algebras and the second a fibration. Let $Z(0) = r_* Z_0$, where $r_* : \mathcal{S}^{T_0} \to (s\mathcal{S})^T$ is left adjoint to the functor $X \mapsto X_0$.

Now suppose $Z(n-1)$ is defined. For any $X \in (s\mathcal{S})^T$ we have natural maps $L_n X \to X_n \to M_n X$ of T_n-algebras. In addition $L_n sk_{n-1} X = L_n X$ by (1.12). Hence we get a map of T_n algebras

$$X_n \sqcup_{L_n X} L_n Z(n-1) \to Y_n \times_{M_n Y} M_n Z(n-1).$$

Factor this map as

$$X_n \sqcup_{L_n X} L_n Z(n) \to Z_n \to Y_n \times_{M_n Y} M_n Z(n-1)$$

where the first map is a trivial cofibration of T_n algebras and the second a fibration. Define $Z(n)$ by the push-out diagram in $(s\mathcal{S})^T$

(2.6)
$$\begin{tikzcd} L_nZ(n-1)\otimes\Delta^n \sqcup_{L_nZ(n-1)\otimes\partial\Delta^n} Z_n\otimes\partial\Delta^n \arrow[r] \arrow[d] & Z(n-1) \arrow[d] \\ Z_n\otimes\Delta^n \arrow[r] & Z(n). \end{tikzcd}$$

The map $Z_n\otimes\partial\Delta^n\to Z(n-1)$ is adjoint to $Z_n\to M_nZ(n-1)$. See 1.7. The map $X_n\to Z_n$ and the push-out diagram 1.9 supply $j_n : sk_nX\to Z(n)$. To define $q_n : Z(n)\to Y$ we must complete

$$\begin{tikzcd} Z(n-1) \arrow[r, "q_{n-1}"] \arrow[d] & Y \\ Z(n). \arrow[ur, dashed] & \end{tikzcd}$$

Using the push-out diagram 2.6 and an adjointness argument this is equivalent to finding a lift

$$\begin{tikzcd} L_nZ(n-1) \arrow[r] \arrow[d] & Y_n \arrow[d] \\ Z_n \arrow[r] \arrow[ur, dashed] & M_nY. \end{tikzcd}$$

This lift was supplied by the factorization that produced Z_n. The inductive step is completed by noting that $L_nZ(n)=L_nZ(n-1)$ and $M_nZ(n)=M_nZ(n-1)$. ∎

These two lemmas yield immediately:

Theorem 2.7 *The Reedy structure on $(s\mathcal{S})^T$ is the structure of a closed model category.*

This is in fact a topological model category. The category $\mathcal{S}^{T_n}$ are topological model categories meaning there are functors

$$(X,K)\mapsto X\square K \text{ and } (X,K)\to X^K$$

from $\mathcal{S}^{T_n}\times\mathbf{Top}$ and $\mathcal{S}^{T_n}\times\mathbf{Top}^{\mathrm{op}}$ to $\mathcal{S}^{T_n}$ respectively and mapping space functors

$$\mathcal{S}^{T_n}(\cdot,\cdot):\mathcal{S}^{T_n}\times(\mathcal{S}^{T_n})^{\mathrm{op}}\to\mathbf{Top}$$

which satisfy a number of adjoint relationships and are connected to the model category structure on $\mathcal{S}^{T_n}$ by the “corner axiom” SM 7. This structure can be extended to $(s\mathcal{S})^T$ is by the following formulas: if $X\in(s\mathcal{S})^T$ and $K\in\mathbf{Top}$, let

(2.8.1)
$$(X\square K)_n = X_n\square K$$

and

$$(X^K)_n = X_n^K. \tag{2.8.2}$$

If $X, Y \in (s\mathcal{S})^T$ define $(s\mathcal{S})^T(X,Y)$ by the equalizer diagram

$$(s\mathcal{S})^T(X,Y) \to \prod_n \mathcal{S}^{T_n}(X_n, Y_n) \overset{a}{\underset{b}{\rightrightarrows}} \prod_{\phi:[n]\to[m]} \mathcal{S}^{T_m}(X_m, Y_n) \tag{2.8.3}$$

where a and b are defined respectively by the diagrams

$$\begin{tikzcd}
\prod_n \mathcal{S}^{T_n}(X_n, Y_n) \arrow[r, "a"] \arrow[d] & \prod_\phi \mathcal{S}^{T_m}(X_m, Y_n) \arrow[d] \\
\mathcal{S}^{T_n}(X_n, Y_n) \arrow[r, "\phi^*"] & \mathcal{S}^{T_m}(X_m, Y_n)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\prod_n \mathcal{S}^{T_n}(X_n, Y_n) \arrow[r] \arrow[d] & \prod_\phi \mathcal{S}^{T_m}(X_m, Y_n) \arrow[d] \\
\mathcal{S}^{T_m}(X_m, Y_m) \arrow[r] & \mathcal{S}^{T_m}(X_m, Y_n).
\end{tikzcd}$$

Note that $\phi^* : T_m \to T_n$ gives Y_n the structure of a T_n algebra.

We call this structure on $(s\mathcal{S})^T$ the internal topological structure.

Proposition 2.9 *With the internal topological structure, the Reedy structure on $(s\mathcal{S})^T$ becomes a topological model category.*

Proof. One need only check the analog of Quillen's SM7a, which need only be checked in $s\mathcal{S}$, where it is an easy (and classical) exercise. ■

One can ask to what extent the external simplicial structure on $(s\mathcal{S})^T$ defined in the previous section interacts with the Reedy structure. It is not a simplicial model category, however, we do have:

Proposition 2.10 *Let $f : X \to Y$ be a Reedy cofibration and $j : K \to L$ a cofibration of simplicial sets. Then*

$$X \otimes L \sqcup_{X\otimes K} Y \otimes K \to Y \otimes L$$

is a Reedy cofibration which is trivial if f is trivial.

Proof. The adjoint statement is: if $X \to Y$ is a Reedy fibration and $j : K \to L$ is a cofibration then

$$\mathbf{hom}(L, X) \to \mathbf{hom}(K, X) \times_{\mathbf{hom}(K,Y)} \mathbf{hom}(L, Y) \tag{2.10}$$

is a Reedy fibration which is trivial if f is trivial. This can be checked in $s\mathcal{S}$, where it is known to be true: indeed, in $(s\mathcal{S})$ the adjoint statement to (2.10) is easy to check directly. ■

Finally, we note that the Reedy structure on $(s\mathcal{S})^T$ is cofibrantly generated. This follows from the fact that $\mathcal{S}^{T_n}$ is cofibrantly generated and Lemma 2.2. A simple way to see this is as follows. Give the category $s\mathcal{S}$ its Reedy model category structure. Then since $\mathcal{S}$ is cofibrantly generated, $s\mathcal{S}$ is cofibrantly generated. Specifically the morphisms

$$S^{-V} \wedge |\partial\Delta^n|_+ \to S^{-V} \wedge |\Delta^n|_+ \tag{2.11.1}$$

with $V \subseteq \mathcal{U}$ finite dimension and $n \geq 0$ generate the cofibration of $\mathcal{S} = \mathcal{S}_{\mathcal{U}}$ and

$$S^{-V} \wedge |\Delta^n_k|_+ \to S^{-V} \wedge |\Delta^n|_+ \tag{2.11.2}$$

with $V \subseteq \mathcal{U}$ finite dimensional and $0 \leq k \leq n$, $n \geq 1$, generate the trivial cofibrations. Then set of maps

$$A_\alpha \otimes \Delta^k \sqcup_{A_\alpha \otimes \partial\Delta^k} B_\alpha \otimes \partial\Delta^k \to B_\alpha \otimes \Delta^k, \qquad k \geq 0, \tag{2.12.1}$$

with $A_\alpha \to B_\alpha$ among the maps of (2.11.1) generate the cofibrations of $s\mathcal{S}$ and

$$C_\beta \otimes \Delta^k \sqcup_{C_\beta \otimes \partial\Delta^k} D_\beta \otimes \partial\Delta^k \to D_\beta \otimes \Delta^k, k \geq 0 \tag{2.12.2}$$

with $C_\beta \to D_\beta$ among the maps of (2.11.2) generate the trivial cofibrations of $s\mathcal{S}$. We regard any object in $\mathcal{S}$ as a constant object in $s\mathcal{S}$.

Proposition 2.12 *The Reedy model category structure on $(s\mathcal{S})^T$ is cofibrantly generated. The cofibrations are generated by*

$$T(A_\alpha) \otimes \Delta^k \sqcup_{T(A_\alpha) \otimes \partial\Delta^k} T(B_\alpha) \otimes \partial\Delta^k \to T(B_\alpha) \otimes \Delta^k \tag{2.13.1}$$

where $A_\alpha \to B_\alpha$ runs over the generators for the cofibrations of $\mathcal{S}$. The trivial cofibrations are generated by

$$T(C_\beta) \otimes \Delta^k \sqcup_{T(C_\beta) \otimes \partial\Delta^k} T(D_\beta) \otimes \partial\Delta^k \to T(D_\beta) \otimes \Delta^k. \tag{2.13.2}$$

Proof. The forgetful functor $U^T : (s\mathcal{S})^T \to s\mathcal{S}$ has left adjoint $F^T X = T(X)$. Since $F^T(X \otimes K) = T(X) \otimes K = F^T(X) \otimes K$, by construction, the maps of (2.13.1) and (2.13.2) are obtained by applying F^T to those of (2.12.1) and (2.12.2) respectively.

Now $f : X \to Y$ in $(s\mathcal{S})^T$ is a Reedy fibration or Reedy weak equivalence if and only if $U^T f$ is such. We use F^T to lift the Reedy model category structure on $s\mathcal{S}$ to $(s\mathcal{S})^T$. The result follows.

Since U^T commutes with filtered colimits two points remain. First we must show the objects $T(B_\alpha) \otimes \Delta^k$ and $T(D_\beta) \otimes \Delta^k$ are small; that is

$$\mathbf{Hom}_{(s\mathcal{S})^T}(T(B_\alpha) \otimes \Delta^k, \operatorname*{colim}_n X_n) \cong \operatorname*{colim}_n \mathbf{Hom}_{(s\mathcal{S})^T}(T(B_\alpha) \otimes \Delta^k, \operatorname*{colim}_n X_n)$$

where $X_0 \to X_1 \to \cdots$ is a directed system over the natural numbers and the morphisms $X_{i-1} \to X_i$ fit into push-out diagrams in $(s\mathcal{S})^T$

$$\begin{tikzcd} T(W) \arrow[r] \arrow[d] & X_{i-1} \arrow[d] \\ T(Z) \arrow[r] & X_i \end{tikzcd}$$

where $W \to Z$ is a Reedy cofibration in $s\mathcal{S}$. Now

$$\mathbf{Hom}_{(s\mathcal{S})^T}(T(B_\alpha) \otimes \Delta^k, Y) = \mathbf{Hom}_{\mathcal{S}}(B_\alpha, Y_k)$$

so it is sufficient to know $(X_{i-1})_k \to (X_i)_k$ is a closed inclusion of spectra. For this (see [9]) it suffices to know $(X_{i-1})_k \to (X_i)_k$ is a cofibration of T_k-algebras. This follows from the fact that $T(W) \to T(Z)$ is a Reedy cofibration in $(s\mathcal{S})^T$, which implies $T(W)_k \to T(Z)_k$ is a cofibration of T_k-algebras.

The second point is to show every cofibration with the left lifting property with respect to all Reedy fibrations is a weak equivalence. But this follows from the fact that we already have a model category structure. ■

3 Geometric Realization.

The main result of this section is that there is a geometric realization functor $(s\mathcal{S})^T \to \mathcal{S}^{|T|}$ with good homotopical properties.

We begin by noting that since the category of operads $\mathcal{O}$ over the linear isometries operad is a topological model category, there is a geometric realization functor

$$|\cdot| : s\mathcal{O} \to \mathcal{O}$$

which sends level-wise weak equivalences between Reedy cofibrant simplicial operads to weak equivalences. Also we note that every object in $s\mathcal{O}$ is a simplicial symmetric space, meaning a set of spaces $\{A_n\}_{n\geq 0}$ such that each A_n has an action on the symmetric group on n letters. Call the category of such $\mathbf{Top}^\Sigma$. For later use we record

Lemma 3.1 *Let $T \in s\mathcal{O}$, and let $|\cdot|_{\mathcal{O}} : s\mathcal{O} \to \mathcal{O}$ and $|\cdot| : s\mathbf{Top}^{\Sigma} \to \mathbf{Top}^{\Sigma}$ be the geometric realization functors. Then there is a natural isomorphism in $\mathbf{Top}^{\Sigma}$,*

$$|T|_{\mathcal{O}} \cong |T|.$$

Proof. See [17] for the simplicial case. ∎

Now let $T \in \mathcal{O}$ and $K \in \mathbf{Top}$. Then the set of symmetric spaces $\{T_n^K\}$ is also a simplicial operad and $T^K \in \mathcal{O}$ is defined by the pull-back diagram

$$\begin{array}{ccc} T^K & \longrightarrow & \{T_n^K\} \\ \downarrow & & \downarrow \\ \mathcal{L} & \longrightarrow & \{\mathcal{L}_n^K\} \end{array}$$

Here $\mathcal{L}$ is the linear isometries operad and $\mathcal{L} \to \{\mathcal{L}_n^K\}$ is induced by $K \to *$.

Lemma 3.2 *Let $T \in \mathcal{O}$ and $X \in \mathcal{S}^T$. If $K \in \mathbf{Top}$, then X^K is naturally a T^K algebra.*

Proof. Let $A \to \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $X \in \mathcal{S}_{\mathcal{U}}$. If $K \in \mathbf{Top}$, then there is a natural map of spectra over $\mathcal{V}$

$$A^K \ltimes X^K \to (A \ltimes X)^K$$

adjoint to the composite

$$\begin{aligned} (A^K \ltimes X^K)\Box K &\stackrel{1\Box\Delta}{\to} (A^K \ltimes X^K)\Box(K \times K) \\ &\cong (A^K \times K) \ltimes (X^K \Box K) \to A \ltimes X. \end{aligned}$$

Furthermore, if $X \in \mathcal{S}_{\mathcal{U}}$ and $K \in \mathbf{Top}$, then there is a map in $\mathcal{S}_{\mathcal{U}^n}^{\Sigma_n}$

$$\underbrace{X^K \wedge X^K \wedge \cdots \wedge X^K}_{n} \to (\underbrace{X \wedge \cdots \wedge X}_{n})^K$$

adjoint to

$$(\bigwedge_n X^K)\Box K \xrightarrow{1\Box\Delta} (\bigwedge_n X^K)\Box K^n \cong \bigwedge_n (X^K \Box K) \to \bigwedge_n X.$$

Combining these two observations yields a map

$$T_n^K \ltimes_{\Sigma_n} (\bigwedge_n X^K) \to (T_n \ltimes \bigwedge_n X)^K.$$

Following by the structure map of the T algebra structure on X yields the T^K structure on X^K. ∎

Now let T be a simplicial operad over the linear isometries operad. Then we may form the geometric realization $|T|$ and this is again an operad over the linear isometries operad. The functor

$$|\cdot| : s\mathcal{O} \to \mathcal{O}$$

has right adjoint $S \to S^\Delta = \{S^{\Delta^n}\}$; hence there is a natural map $T \to |T|^\Delta$ of simplicial operads. If $Y \in \mathcal{S}^{|T|}$, then, by Lemma 3.2, $Y^\Delta \in (s\mathcal{S})^T$, and we claim the functor

$$(\cdot)^\Delta : \mathcal{S}^{|T|} \to (s\mathcal{S})^T$$

has a left adjoint, which we will call the geometric realization. In fact, we will prove that the geometric realization of $X \in (s\mathcal{S})^T$, regarded simply as a simplicial spectrum, has a canonical $|T|$ algebra structure and, with this structure, the functor $X \mapsto |X|$ is the required adjoint.

We start with:

Lemma 3.3 *Let $Z \in s\mathcal{S}$. Then there is a natural isomorphism in $\mathcal{S}$*

$$|TZ| \cong |T|\,|Z|.$$

Proof. Notice that TZ is the diagonal of a bisimplicial object:

$$TZ = \operatorname{diag} T.Z.$$

where $(T.Z.)_{p,q} = T_pZ_q$. Let p be the horizontal direction and q the vertical direction. Then

$$|TZ| = |\,|T.Z.|_v|$$

where the v denotes realization in the vertical direction. Now for fixed p, $(|T.Z.|_v)_p = |T_pZ| \cong T_p|Z|$. The last isomorphism uses [9]. Thus we need to show

$$|T.|Z|\,| \cong |T|\,|Z|$$

or, put another way, we may assume Z is constant.

For fixed Z, the functor $\mathbf{Top}^\Sigma \downarrow \mathcal{L} \to \mathcal{S}$ given by $A \mapsto AZ$ has a continuous right adjoint given by

$$Y \mapsto \{\mathcal{S}(\bigwedge_n Z, Y)\}_n.$$

Thus the functor $A \mapsto AZ$ commutes with geometric realization: if $A \in s\mathbf{Top}^\Sigma$, then $|A|Z \cong |AZ|$. But if $T \in s\mathcal{O}$, the realization of T as a simplicial operad is isomorphic to the realization of T as an object in $s\mathbf{Top}^\Sigma \downarrow \mathcal{L}$, by Lemma 3.1. ∎

Proposition 3.4 *Let $Z \in (s\mathcal{S})^T$. Then the geometric realization $|Z|$ of Z, regarded as an object in $s\mathcal{S}$, has a natural structure as a $|T|$ algebra and the resulting functor*

$$|\cdot| : (s\mathcal{S})^T \to \mathcal{S}^{|T|}$$

is left adjoint to $Y \mapsto Y^\Delta$.

Proof. Let $\epsilon : TZ \to Z$ be the T-algebra structure map. Then

$$|T|\,|Z| \cong |TZ| \xrightarrow{|\epsilon|} |Z|$$

gives $|Z|$ the structure of a $|T|$ algebra. To get the adjointness we first show for $Y \in \mathcal{S}^{|T|}$ that the map of spectra

$$|Y^\Delta| \to Y$$

is a map of $|T|$ algebras; this will yield a map

$$(s\mathcal{S})^T(Z, Y^\Delta) \to \mathcal{S}^{|T|}(|Z|, Y)$$

which we will then show is an isomorphism.

If $Y \in \mathcal{S}^{|T|}$, the T-algebra structure on Y^Δ is specified by the diagram

$$\begin{tikzcd}
T(Y^\Delta) \arrow[r] \arrow[d] & |T|^\Delta Y^\Delta \arrow[r] \arrow[d] & (|T|Y)^\Delta \arrow[d] \\
Y^\Delta \arrow[r, "="] & Y^\Delta \arrow[r, "="] & Y^\Delta .
\end{tikzcd}$$

Taking geometric realizations yields, by Lemma 3.3,

$$\begin{tikzcd}
|T|\,|Y^\Delta| \arrow[r] \arrow[d] & |\,|T|^\Delta|\,|Y^\Delta| \arrow[r] \arrow[d] & |(|T|Y)^\Delta| \arrow[r] \arrow[d] & |T|Y \arrow[d] \\
|Y^\Delta| \arrow[r, "="] & |Y^\Delta| \arrow[r, "="] & |Y^\Delta| \arrow[r] & Y.
\end{tikzcd}$$

This proves the first claim. For the second, let $Z \in (s\mathcal{S})^T$. Then there is a coequalizer diagram in $s\mathcal{S}$

(3.5.1) $$T^2Z \rightrightarrows TZ \to Z;$$

hence a coequalizer diagram in $\mathcal{S}$

(3.5.2) $$|T|^2|Z| \rightrightarrows |T|\,|Z| \to |Z|.$$

Again we use Lemma 3.3. Since these coequalizers are split coequalizers, they are in fact coequalizers in $(s\mathcal{S}^T)$ and $\mathcal{S}^{|T|}$ respectively. Now consider the induced maps of equalizers

$$\begin{tikzcd}
(s\mathcal{S})^T(T^2Z, Y^\Delta) \arrow[d, shift left] \arrow[d, shift right] & (s\mathcal{S})^T(TZ, Y^\Delta) \arrow[l, shift left] \arrow[l, shift right] & (s\mathcal{S})^T(Z, Y^\Delta) \arrow[l] \arrow[d] \\
\mathcal{S}^{|T|}(|T|^2|Z|, Y) & \mathcal{S}^{|T|}(|T|\,|Z|, Y) \arrow[l, shift left] \arrow[l, shift right] & \mathcal{S}^{|T|}(|Z|, Y). \arrow[l]
\end{tikzcd}$$

If we can show the map $(s\mathcal{S})^T(TZ, Y^\Delta) \to \mathcal{S}^{|T|}(|T|\,|Z|, Y)$ is an isomorphism for all Z, then the two left-most vertical arrows in this diagram will be isomorphisms and, then, so will the third—which is what is required. The following calculation finishes the proof:

$$\begin{aligned}
(s\mathcal{S})^T(TZ, Y^\Delta) &\cong (s\mathcal{S})(Z, Y^\Delta) \\
&\cong \mathcal{S}(|Z|, Y) \\
&\cong \mathcal{S}^{|T|}(|T|\,|Z|, Y).
\end{aligned}$$

■

We now come to the homotopical properties of realization. We give $(s\mathcal{S})^T$ the Reedy model category structure.

Proposition 3.6 *The geometric realization functor $|\cdot| : (s\mathcal{S})^T \to \mathcal{S}^{|T|}$ preserves cofibrations, trivial cofibrations, and weak equivalences between Reedy cofibrant objects.*

Proof. First note that if $K \in \mathbf{Top}$, then $|Z \square K| \cong |Z| \square K$. This is because if $Y \in \mathcal{S}^{|T|}$, then $(Y^\Delta)^K \cong (Y^K)^\Delta$ in $(s\mathcal{S})^T$. The result will now follow once we show $(\cdot)^\Delta : \mathcal{S}^{|T|} \to s\mathcal{S}^T$ preserves fibrations and trivial fibrations.

To do this we calculate that $M_n(Y^\Delta) \cong Y^{\partial\Delta^n}$. This is because if $\mathcal{K}$ is the simplex category of $\partial\Delta^n$, then

$$\begin{aligned}
M_n Y &\cong \lim_{\mathcal{K}^{\mathrm{op}}} (Y^\Delta)_k \\
&\cong \lim_{\mathcal{K}^{\mathrm{op}}} Y^{\Delta^k} \cong Y^{\operatorname{colim}_{\mathcal{K}} \Delta^k} \cong Y^{\partial\Delta^n}.
\end{aligned} \tag{3.7}$$

Thus if $f : X \to Y$ is a morphism in $\mathcal{S}^{|T|}$, then

$$(Y^\Delta)_n \to (Y^\Delta)_n \times_{M_n Y^\Delta} M_n X^\Delta$$

is isomorphic to

$$X^{\Delta_n} \to Y^{\Delta_n} \times_{Y^{\partial\Delta^n}} X^{\partial\Delta^n}.$$

If f is a fibration in $\mathcal{S}^{|T|}$, this is a fibration in $\mathcal{S}^{|T|}$, hence in $\mathcal{S}^{T_n}$. The same applies if f is a trivial fibration. ■

4 The resolution model category structure.

We refine the Reedy model category structure on $(s\mathcal{S})^T$ to give cofibrant object which are much more free with respect to the operad structure. The principle reference for this section is [10], which depends heavily on the work of Dwyer, Kan, and Stover [4, 5].

We choose a set of $\mathcal{A} = \{Ai\}$, $i \in I$, of finite cofibrant spectra with the property if $Aj \in \mathcal{A}$, then $\Sigma Aj \in \mathcal{A}$. Our main (in fact, only) example is the following. Let $\mathcal{C} = \mathcal{S}$

and let $E \in \mathcal{S}$ be a homotopy associative ring spectrum which satisfies the condition Adams imposed in order to get a universal coefficient spectral sequence [1, §III.13]. Thus there is an isomorphism $\operatorname{colim}_{\alpha \in I} E_\alpha \cong E$, where E_α runs over a filtered system of finite CW spectra with the properties that

i) $E_* DE_\alpha$ is projective over E_*, and

ii) for all E module spectra F, the natural map

$$F^* DE_\alpha \to \mathbf{Hom}^*_{E_*}(E_* DE_\alpha, F_*)$$

is an isomorphism.

iii) $E_\alpha = S^0$ for some α.

Here $D(\cdot)$ denotes Spanier-Whitehead duality. The final condition is was not listed by Adams, but can be included without loss of generality, and it helps us later. Many spectra satisfy this hypothesis; for example, any spectrum representing a Landweber exact theory. The $\mathcal{A} = \{\Sigma^n DE_\alpha\}$ where α is as above and n runs over the integers.

A Reedy fibrant replacement of an arrow $X \to Y$ in $s\mathcal{S}^T$ is a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

where the horizontal maps are weak equivalences and the the objects X' and Y' are Reedy fibrant.

Definition 4.1 *A morphism $f : X \to Y$ is*

(1) a $T(\mathcal{A})$-equivalence if for some Reedy fibrant replacement $f' : X' \to Y'$ of f, the simplicial set of maps

$$\operatorname{map}_{(s\mathcal{S})^T}(T(Ai), X') \to \operatorname{map}_{(s\mathcal{S})^T}(T(Ai), Y')$$

is a weak equivalence of simplicial sets for all i;

(2) a $T(\mathcal{A})$-fibration if

$$\operatorname{map}_{(s\mathcal{S})^T}(T(Ai), X) \to \operatorname{map}_{(s\mathcal{S})^T}(T(Ai), Y)$$

is a fibration for all i;

(3) a $T(\mathcal{A})$-cofibration if it has the left lifting property with respect to all $T(\mathcal{A})$-trivial fibrations.

We make some remarks on this definition. The category $s\mathcal{S}$ with the analogous definitions (using Ai instead of $T(Ai)$) is a simplicial model category using the standard external simplicial structure on $s\mathcal{S}$. This is the point of section 2 of [10]. In particular if $X \in s\mathcal{S}$ is Reedy fibrant it is $\mathcal{A}$-fibrant, by Lemma 2.9 of [10]. Since $\mathrm{map}_{(s\mathcal{S})^T}(T(Ai), X) \cong \mathrm{map}_{s\mathcal{S}}(Ai, X)$, the simplicial sets in 3.1.1 are, in fact, fibrant. In addition, Lemma 2.7 of [10] implies a morphism $f : X \to Y$ is an $\mathcal{A}$-equivalence independent of the choice of Reedy fibrant replacement. More is true: if we equip $s\mathcal{S}$ with this $\mathcal{A}$-resolution model category, we are saying $f : X \to Y$ in $s\mathcal{S}^T$ is an $\mathcal{A}$-equivalence or $\mathcal{A}$ fibration in $s\mathcal{S}^T$ if and only if it is in $s\mathcal{S}$. Finally, the $\mathcal{A}$-resolution model category is cofibrantly generated. The generators for the cofibrations are

(4.2.1)
$$\begin{gathered} Ai \otimes \partial\Delta^n \to Ai \otimes \Delta^n, i \in I, n \geq 0 \\ C_\beta \otimes \Delta^n \sqcup_{C_\beta \otimes \partial\Delta^n} D_\beta \otimes \partial\Delta^n \to D_\beta \otimes \partial\Delta^n, n \geq 0 \end{gathered}$$

where $C_\beta \to D_\beta$ runs over the generators for the trivial cofibrations of $\mathcal{S}$ (see 2.11 and 2.12). The generators for the trivial cofibrations are

(4.2.2)
$$\begin{gathered} Ai \otimes \Delta^n_k \to Ai \otimes \Delta^n, i \in I, 0 \leq k \leq n, n \geq 1 \\ C_\beta \otimes \Delta^n \sqcup_{C_\beta \otimes \partial\Delta^n} D_\beta \otimes \partial\Delta^n \to D_\beta \otimes \Delta^n, n \geq 0. \end{gathered}$$

Theorem 4.3 *With the definition of $T(\mathcal{A})$-resolution equivalence, fibration, and cofibration and in its standard external simplicial structure, $(s\mathcal{S})^T$ becomes a simplicial model category. It is cofibrantly generated, with generators obtained by applying $T(\cdot)$ to the morphisms of (3.2.1) and (3.2.2).*

Proof. As in the proof of Proposition 2.12 we lift the model category structure from $s\mathcal{S}$ to $(s\mathcal{S})^T$. The argument is the same except we must find some other way to prove that $T(\mathcal{A})$-cofibration with the left lifting property with respect to all fibrations is a weak equivalence. By Quillen [15, §II.4] it is sufficient to find, for all $X \in (s\mathcal{S})^T$ a natural $T(\mathcal{A})$-equivalence $X \to Y$ where Y is $T(\mathcal{A})$-fibrant. Since Reedy weak equivalences are $\mathcal{A}$-equivalences (Corollary 2.8 of [10]) and any Reedy fibrant object in $\mathcal{A}$-fibrant (Lemma II2.9) we take a Reedy equivalence $X \to Y$ in $(s\mathcal{S})^T$ with Y Reedy fibrant. This can be done naturally since $(s\mathcal{S})^T$ in Reedy structure, is cofibrantly generated.

To check SM7 it is sufficient to check SM7a, which can be done in $s\mathcal{S}$. ■

The weak equivalences in the resolution model category structure can be characterized in another manner. If $X \in (s\mathcal{S})^T$,

(4.4)
$$\begin{aligned} \pi_n^\natural(X; Ai) &= \pi_n \,\mathrm{map}_{(s\mathcal{S})^T}(T(Ai), X') \\ &\cong \pi_n \,\mathrm{map}_{s\mathcal{S}}(Ai, X') \end{aligned}$$

where $X \to X'$ is some weak equivalence with X' Reedy fibrant, and with $T(Ai) \to T(*) \to X'$ as the basepoint. Then $f : X \to Y$ in $(s\mathcal{S})^T$ is an $T(\mathcal{A})$-equivalence if and only if $\pi_n^\natural(f, Ai)$ is an isomorphism for all n. This is because $\mathrm{map}_{s\mathcal{S}}(Ai, X')$ is a fibrant homotopy associative H-space with unit. There are other homotopy groups associated to X and Ai. Let $[\cdot,\cdot]_{T_n}$ denote homotopy classes of maps of T_n-algebras and $[\cdot,\cdot]$ homotopy classes of maps of spectra. If $X \in s\mathcal{S}^T$, then,

$$[T_n(Ai), X_n]_{T_n} \cong [Ai, X_n]$$

is an abelian group, and these groups assemble into a simplicial abelian group

$$[T.(Ai), X]_{T.} \cong [Ai, X].$$

Define the groups

$$\pi_n(X; Ai) = \pi_n[T.(Ai), X]_{T.} \cong \pi_n[Ai, X]. \tag{4.5}$$

Then, because an object $X \in (s\mathcal{S})^T$ is Reedy fibrant if and only if X is Reedy fibrant in $s\mathcal{S}$, Proposition 4.4 of [10] implies

$$\pi_0^\natural(X; Ai) \cong \pi_0(X; Ai) \tag{4.6}$$

for all i and that there is a natural long exact sequence

$$\cdots \to \pi_{n-1}^\natural(X; \Sigma Ai) \to \pi_n^\natural(X; Ai) \to \pi_n(X; Ai) \to \pi_{n-2}^\natural(X; Ai) \to \cdots \to \pi_1(X; Ai) \to 0.$$

This is the spiral exact sequence and we have

Proposition 4.7 *A morphism $f : X \to Y$ is an $T(\mathcal{A})$-equivalence if and only if*

$$\pi_n(f; Ai) : \pi_n(X; Ai) \to \pi_n(Y; Ai)$$

is an isomorphism for all n.

∎

We now want to talk about particular $T(\mathcal{A})$-cofibrant models for objects in $(s\mathcal{S})^T$. We begin by giving a particular model for cofibrant objects. Let $\Delta_+ \subseteq \Delta$ be the subcategory with the same objects, but only surjections as morphism. An object $X \in s_+\mathcal{C} = \mathcal{C}^{\Delta_+^{\mathrm{op}}}$ is a "simplicial object with degeneracies only." Such an object is *free* if X is obtained by left Kan extension of a trivial diagram $\{Z_n\}_{n\geq 0}$. Thus

$$X_n = \coprod_{\phi:[n]\to[k]} Z_k$$

with $\phi \in \Delta_+$. An object $X \in s\mathcal{C}$ is s-free if it is free in $s_+\mathcal{C}$.

An object $Z \in \mathcal{S}^{T_n}$ is contractible if the unit map $S^0 \to Z$ is a trivial cofibration.

Definition 4.8 *An object $X \in (s\mathcal{S})^T$ is* homotopically $T(\mathcal{A})$-free *if there is a morphism $Z \to X$ is $(s_+\mathcal{S})^T$ so that*

(1) $Z_n \sqcup_{L_n Z} L_n X \to X_n$ *and* $L_n Z \to Z_n$ *are cofibrations of* T_n*-algebras;*

(2) $Z_n \in \mathcal{S}^{T_n}$ *is contractible;*

(3) *there is a push-out diagram in* $(s_+\mathcal{S})^T$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ S^0 \arrow[r] & \overline{X} \end{tikzcd}$$

where $\overline{X}$ *is free on* $\{Y_n\}$ *where each* Y_n *is isomorphic to a coproduct of* $T_n(Ai)$*s.*

Note that the definition of homotopically $T(\mathcal{A})$-free can be extended to objects of $(s_+\mathcal{S})^T$. More generally we have

Definition 4.9 *A morphism* $A \to X$ *in* $(s\mathcal{S})^T$ *is homotopically* $T(\mathcal{A})$*-free if*

(1) *as a morphism in* $(s_+\mathcal{S})^T$, $A \to X$ *is isomorphic to* $A \to A \sqcup Y$ *for some* $Y \in (s_+\mathcal{S})^T$

(2) Y *is homotopically* $T(\mathcal{A})$*-free.*

The following gives a recognition principal for cofibrations.

Proposition 4.10 *Any homotopically* $T(\mathcal{A})$*-free map is an* $T(\mathcal{A})$*-cofibration in* $(s\mathcal{S})^T$*.*

Proof. The argument is the same as in Proposition 5.4 of [10]. The crucial observation is that if D is a cofibrant spectrum

$$[T_n(D), Z]_{T_n} \cong [D, Z].$$

This is necessary so that Proposition 5.1 of [9] applies. ∎

We can now give, for each $X \in (s\mathcal{S})^T$, a particular model for the cofibration replacement of X. We use the device of the Stover resolution; let us recall how this is defined.

Let $\{Ai\}_{i \in I}$, be the set of objects used to define the $\mathcal{A}$-resolution model category structure. For each $i \in I$, let $CAi = Ai \wedge [0,1]$ with $\{1\} \in [0,1]$ the basepoint; CAi is the cone on Ai. Let J be the category $I \times \{0 \to 1\}$ where I is regarded as a discrete category. The inclusions $Ai \to CAi$ define a diagram $\bar{\mathcal{A}} : J \to \mathcal{S}$ with $\bar{\mathcal{A}}[(i,0) \to (i,1)] = Ai \to CAi$.

Fix an integer $n \geq 0$. Then there is a functor

$$\mathbf{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), \cdot) : \mathcal{S}^{T_n} \to \mathbf{Sets}^{J^{\mathrm{op}}}$$

with left adjoint $F \mapsto T_n(\bar{\mathcal{A}}) \otimes_J F$. There is a push-out diagram in $\mathcal{S}^{T_n}$

$$\begin{tikzcd} \coprod_i \coprod_{F(i,1)} T_n(Ai) \arrow[r] \arrow[d] & \coprod_i \coprod_{F(i,0)} T_n(Ai) \arrow[d] \\ \coprod_i \coprod_{F(i,1)} T_n(CAi) \arrow[r] & T_n(\bar{\mathcal{A}}) \otimes_J F. \end{tikzcd}$$

In particular we have $T_n(\bar{\mathcal{A}}) \otimes_J F \cong T_n(\bar{\mathcal{A}} \otimes_J F)$.

Let $\overline{V}^{T_n} : \mathcal{S}^{T_n} \to \mathcal{S}^{T_n}$ be the composite functor

$$\overline{V}^{T_n} = T_n(\bar{\mathcal{A}}) \otimes_J \mathbf{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), \cdot).$$

Then for each $Z \in \mathcal{S}^{T_n}$ there is an augmented simplicial object

$$\epsilon : V^{T_n}(Z) \to Z$$

with $(V^{T_n}Z)_k = (\overline{V}^{T_n})^{k+1}Z$ with the following properties

4.11.1) the augmentation ϵ induces an isomorphism

$$\pi_*\pi_q(\overline{V}^{T_n}(Z); T_n(Ai)) \cong \pi_q(Z; T_n(A_i))$$

concentrated in degree 0, and

4.11.2) as objects in $s_0\mathcal{S}^{T_n}$, there is an isomorphism

$$\begin{aligned} V^{T_n}(X) &\cong T_n(\bar{\mathcal{A}}) \otimes_J \mathbf{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), (\overline{V}^{T_n})^{\cdot} X) \\ &\cong T_n[\bar{\mathcal{A}} \otimes_J \mathbf{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), (\overline{V}^{T_n})^{\cdot} X)]. \end{aligned}$$

See [10, §4] for the results.

We extend this construction to the simplicial setting. Let $X \in (s\mathcal{S})^T$, where $T = \{T_n\}$ is a simplicial operad. If $\phi^* : T_n \to T_m$ is induced by an ordinal number map, then ϕ^* gives X_m the structure of a T_n-algebra and $\phi^* : X_n \to X_m$ is a morphism of T_n-algebras. Thus we get maps

$$\mathbf{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), X_n) \to \mathbf{Hom}_{T_n}(T_n(\bar{\mathcal{A}}), X_m) \cong \mathbf{Hom}_{T_m}(T_m(\bar{\mathcal{A}}), X_m)$$

and, hence a functor

$$\mathbf{Hom}_T(\mathcal{A}, \cdot) : (s\mathcal{S})^T \to (s\,\mathbf{Sets})^{J^{\mathrm{op}}}.$$

This functor has a left adjoint: if $F \in (s\,\mathbf{Sets})^{J^{\mathrm{op}}}$, the adjoint is given by $T(\bar{\mathcal{A}}) \otimes_J F = \{T_n(\bar{\mathcal{A}}) \otimes_J F_n\}$. Note that if $\phi : [m] \to [n]$ is an ordinal number map, then we have

$$T_n(\bar{\mathcal{A}}) \otimes_J F_n \xrightarrow{\phi^* \otimes \phi^*} T_m(\bar{\mathcal{A}}) \otimes_J F_m$$

is a morphism of T_n algebras; hence $T(\bar{\mathcal{A}}) \otimes_J F \in (s\mathcal{S})^T$.

Let $\overline{V}^T = T(\mathcal{A}) \otimes_J \mathbf{Hom}_T(T(\bar{\mathcal{A}}), \cdot) : (s\mathcal{S})^T \to (s\mathcal{S})^T$ be the composite. Then $(\overline{V}^{TX})_n = \overline{V}^{T_n}X_n$. If $X \in (s\mathcal{S})^T$, define $\epsilon : \overline{V}^T_{\cdot}(X) \to X$ to be the augmented simplicial object in $(s\mathcal{S})^T$ with $(\overline{V}^T_{\cdot}(X))_k = (\overline{V}^T)^{k+1}X$. Finally, let

$$V(X) = \operatorname{diag} \overline{V}^T_{\cdot}(X)$$

be the diagonal. This is an object in $(s\mathcal{S})^T$ equipped with a natural map

$$\epsilon : V(X) \to X$$

induced from the augmentation of $\overline{V}^T_{\cdot}(X)$.

Proposition 4.12 *For $X \in (s\mathcal{S})^T$, the natural map*

$$\epsilon : V(X) \to X$$

is an $\mathcal{A}$-equivalence. Furthermore, $V(X)$ is homotopically $\mathcal{A}$-free, and for each $n \geq 0$, there is a cofibrant spectrum Z_n so that

$$V(X)_n \cong T_n(Z_n).$$

Proof. That the morphism ϵ is an $\mathcal{A}$-equivalence follows from 4.11.1) and a bicomplex argument. And, as objects in $(s_0\mathcal{S})^T$, there is a natural isomorphism, by 4.11.2,

$$\begin{aligned} V(X) &\cong T(\bar{\mathcal{A}}) \otimes_J \operatorname{diag} \mathbf{Hom}_T(T(\bar{\mathcal{A}}), (\overline{V}^T)^{\cdot} X) \\ &\cong T(\bar{\mathcal{A}} \otimes_J \operatorname{diag} \mathbf{Hom}_T(T(\bar{\mathcal{A}}), (\overline{V}^T)^{\cdot} X). \end{aligned}$$

The rest of the result now follows from Proposition 4.7 of [10]. ∎

If necessary, we can further refine this result. To do this we use the technique of subdivision reviewed in the appendix to [10]. To recapitulate, let $K \in c\mathbf{S}$ be a cosimplicial simplicial set and let $X \in (s\mathcal{S})^T$, then we can define a coend $\int^\Delta X \otimes K$; in fact, this coend is given explicitly by the coequalizer diagram

$$\coprod_{\phi:[n]\to[m]} X_m \otimes K^n \rightrightarrows \coprod_n X_n \otimes K^n \to \int^\Delta X \otimes K \tag{4.13}$$

where ϕ runs over the morphisms of Δ, the coproduct is in $(s\mathcal{S})^T$, and the two maps to be equalized evaluate ϕ in the X variable and the K variable respectively. For example, if $K = \Delta[\cdot] = \{\Delta^n\}$, we note that if k-simplices of Δ^n are all maps in Δ

$$\phi : [k] \to [n].$$

There is a map (using 1.4) of T_k algebras

$$(\coprod_n X_n \otimes \Delta^n)_k \cong \coprod_n \coprod_{\phi:[k]\to[n]} X_{n,k} \to X_{k,k}$$

evaluating ϕ in the n variable. This map coequalizes the two maps of 4.13 and defines an isomorphism

$$\int^\Delta X \otimes \Delta[\cdot] \cong \operatorname{diag} X. \tag{4.14}$$

In short, the generalized Eilenberg-Zilber Theorem holds in $(s\mathcal{S})^T$.

It is a standard result that these constructions preserve certain types of weak equivalences. We give cosimplicial spaces the Bousfield-Kan model category structure [2]; this is the Reedy structure on $(s(\mathbf{S}^{\mathrm{op}}))^{\mathrm{op}}$. Cofibrant objects have a simple characterization: $K \in c\mathbf{S}$ is cofibrant if and only if the equalizer of d^0, $d^1 : K^0 \to K^1$ is empty. We give $(s\mathcal{S})^T$ its Reedy model category structure of Theorem 2.7. The topological model category structure of Proposition 2.9 gives the Reedy model category structure on $(s\mathcal{S})^T$ an induced simplicial model category structure. Give $s(s\mathcal{S})^T$ the inherited Reedy model category structure. Then Proposition A.7 of [10] implies that if $K \to L$ is a weak equivalence between cofibrant objects in $c\mathbf{S}$ and $X \in s(s\mathcal{S})^T$ is Reedy cofibrant, then

$$\int^{\Delta} X \otimes K \to \int^{\Delta} X \otimes L \tag{4.15}$$

is a weak equivalence in $(s\mathcal{S})^T$.

Lemma 4.16 *The simplicial object $\overline{V}^T(X) \in s(s\mathcal{S})^T$ of the bisimplicial Stover resolution is Reedy cofibrant.*

Proof. As an object in $s_0(s\mathcal{S})^T$, there is an isomorphism

$$\overline{V}^T(X) \cong T(\bar{\mathcal{A}} \otimes_J \mathbf{Hom}_T(T(\bar{\mathcal{A}}), (\overline{V}^T)^{\cdot} X)).$$

Thus we need only show that for any $F \in s_0(s\,\mathbf{Sets})^{J^{\mathrm{op}}}$, $T(\bar{\mathcal{A}} \otimes_J X)$ is Reedy cofibrant; that, if we let n vary in the external, or s_0, direction, we need

$$L_n T(\bar{\mathcal{A}} \otimes_J X) \to T(\bar{\mathcal{A}} \otimes_J X)_n$$

to be a Reedy cofibration in $(s\mathcal{S})^T$. Since $T(\cdot)$ is a left adjoint, it is sufficient to show

$$L_n(\bar{\mathcal{A}} \otimes_J X) \to (\bar{\mathcal{A}} \otimes_J X)_n$$

is a Reedy cofibration in $s\mathcal{S}$. Thus, letting k vary in the internal, or s, direction, we need,

$$L_n(\bar{\mathcal{A}} \otimes_J X)_k \sqcup_{L_k L_n(\bar{\mathcal{A}} \otimes_J X)} L_k(\bar{\mathcal{A}} \otimes_J X)_n \to (\bar{\mathcal{A}} \otimes_J X)_{n,k}$$

to be a cofibration in $\mathcal{S}$. Since $\bar{\mathcal{A}} \otimes_J \cdot$ is a left adjoint, this map may be rewritten

$$\bar{\mathcal{A}} \otimes_J (L_n X_k \sqcup_{L_k L_n X} L_k X_n) \to \bar{\mathcal{A}} \otimes_J X_{n,k}.$$

The functor $\bar{\mathcal{A}} \otimes_J (\cdot)$ sends injections in $\mathbf{Sets}^{J^{\mathrm{op}}}$ to cofibrations in $\mathcal{S}$; indeed, if $X\ toY$, then

$$\bar{\mathcal{A}} \otimes_J X \to \bar{\mathcal{A}} \otimes_J Y$$

is the inclusion of a wedge summand where the other factor is cofibrant. This can be seen directly from the definition of by examining the beginning of the proof of Proposition 4.7 in [10]. Thus we need only assert that for all $j \in J$

$$L_n X_k(j) \sqcup_{L_k L_n X(j)} L_k X_n(j) \to X_{n,k}(j)$$

is an injection. This amounts to the assertion that for every $Z \in s_0(s\,\mathbf{Sets})$,

$$L_n Z \to Z_n$$

is a degree-wise injection of simplicial sets. But this is clear, since $(L_n Z)_k \subseteq (Z_n)_k$ is the inclusion of the degenerate k-simplices. ■

We now record:

Proposition 4.17 *Let $X \in (s\mathcal{S})^T$ then there is a natural $T(\mathcal{A})$-equivalence*

$$\eta : W(X) \to X$$

so that $W(X)$ is $T(\mathcal{A})$-cofibrant and as an object in $(s_0\mathcal{S})^T$, there is an isomorphism

$$W(X) \cong T(Z)$$

with $Z \in s_0\mathcal{S}$ s-free on a set of cofibrant spectra.

Proof. For $n \geq 0$, let Δ/n be the category of arrows in Δ over n and let $B(\Delta/\cdot)$ be the resulting cosimplicial classifying space. Define

$$W(X) = \int^{\Delta} \overline{V}^T(X) \otimes B(\Delta/\cdot).$$

Then $B(\Delta/\cdot)$ is cofibrant and there is a weak equivalence $B(\Delta/\cdot) \to \Delta[\cdot]$. Let η be the composite

$$W(X) \to \int^{\Delta} \overline{V}^T(X) \otimes \Delta[\cdot] = V(X) \xrightarrow{\epsilon} X.$$

This is a $T(\mathcal{A})$ equivalence.

By Lemma A.5 of [10], there is an isomorphism of T_k algebras

$$W(X)_k \cong \coprod_{n_0 \to n_1 \to \cdots \to n_k} \overline{V}^{T_k}_{n_k}(X_k)$$

where the coproduct is over composable morphisms in Δ. A degeneracy morphism $s_i : W(X)_k \to W(X)_{k+1}$ inserts an identity in the i^{th} slot. Write $Z_{n_k,k} = \bar{\mathcal{A}} \otimes_J \mathbf{Hom}(\bar{\mathcal{A}}, (\overline{V}^{T_k})^{n_k} X)$. Then in $\mathcal{S}^{T_k}$

$$W(X)_k \cong T(Z_k)$$

where

$$Z_k = \bigvee_{n_0 \to n_1 \to \cdots \to n_k} Z_{n_k k}.$$

Again by Lemma A.5 of [10] this yields an isomorphism $W(X) \cong T(Z)$ in $(s_0\mathcal{S})^T$. Furthermore Z is s-free on $\{\overline{Z}_k\}$ where

$$\overline{Z}_k = \bigvee_{n_0 \to n_1 \to \cdots \to n_k} Z_{n_k k}$$

with none of the morphism the identity.

Finally, we note that each pair (n,k), Remark 4.11 of [10] implies there is a cofibration $Y_{n,k} \to Z_{n,k}$ so that $Y_{n,k}$ is contractible and there is a push-out diagram

$$\begin{tikzcd} Y_{n,k} \arrow[r] \arrow[d] & Z_{n,k} \arrow[d] \\ * \arrow[r] & \bigvee_A (Ai)_\alpha \end{tikzcd}$$

for α in some index set A. It follows that $W(X)$ is homotopically $T(\mathcal{A})$-free. ■

5 The geometric realization of $T(\mathcal{A})$-cofibrant objects.

We will need to be able to analyze the homotopy type of geometric realization of various objects in $(s\mathcal{S})^T$. This will be done by means of a spectral sequence we develop in the next section. In this section we determine what that spectral sequence will converge to.

We begin with a lemma. If $X \in (s\mathcal{S})^T$ and K a simplicial set we write $X \otimes K$ for the object in $(s\mathcal{S})^T$ constructed in (1.4) and $|X|\Box|K|$ for the extended object in $\mathcal{S}^{|T|}$.

Lemma 5.1 *Let $X \in (s\mathcal{S})^T$ and K a simplicial set. Then there is an isomorphism in $\mathcal{S}^{|T|}$, natural in X and K,*

$$|X|\Box|K| \cong |X \otimes K|.$$

Proof. In this argument (and only in this argument) we will write $X \otimes_T K$ for $X \otimes K \in (s\mathcal{S})^T$ to distinguish it from $X \otimes K \in s\mathcal{S}$.

First suppose $X = T(Z)$ for some $Z \in s\mathcal{S}$. Then

$$X \otimes_T K \cong T(Z \otimes K).$$

Since $\mathbf{hom}_T(K,Y) \cong \mathbf{hom}(K,Y)$ for all $Y \in (s\mathcal{S})^T$, there are maps $Z \otimes K \to T(Z) \otimes K \to T(Z \otimes K) \cong T(Z) \otimes_T K$ and if $f : T(Z) \to T(Z')$ is and T-algebra map, then

$$f \otimes_T K : T(Z) \otimes_T K \to T(Z) \otimes_T K'$$

is the unique T-algebra map completing the following diagram of spectra

(5.2)
$$\begin{tikzcd} Z \otimes K \arrow[r] & T(Z) \otimes K \arrow[r] \arrow[d, "f \otimes K"'] & T(Z) \otimes_T K \arrow[d, "f \otimes_T K"'] \\ & T(Z') \otimes K \arrow[r] & T(Z') \otimes_T K. \end{tikzcd}$$

In spectra, $|Z \otimes K| \cong |Z|\Box|K|$. This yields a map

$$|T|\,|Z|\Box_{|T|}|K| \cong |T|(|Z|\Box|K|) \to |T(Z) \otimes_T K|$$

which is natural is all T-algebra maps $T(Z) \to T(Z')$, using (5.2). It is also an isomorphism since $|T(Z \otimes K)| \cong |T|\,|Z \otimes K|$. For general X examine the diagram of coequalizer diagrams

$$\begin{tikzcd}
|T|^2|X|\Box_{|T|}|K| \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & |T|\,|X|\Box_{|T|}|K| \arrow[r] \arrow[d] & |X|\Box_{|T|}|K| \arrow[d, dashed] \\
|T^2(X) \otimes_T K| \arrow[r, shift left] \arrow[r, shift right] & |T(X) \otimes_T K| \arrow[r] & |X \otimes_T K|
\end{tikzcd}$$

Then the dotted arrow exists and is an isomorphism. ■

This past lemma is crucial in the following result.

Proposition 5.3 *The geometric realization functor*

$$|\cdot| : (s\mathcal{S})^T \to \mathcal{S}^{|T|}$$

sends $T(\mathcal{A})$-equivalences between $T(\mathcal{A})$-cofibrant objects to weak equivalences.

Proof. While we have regarded $\mathcal{S}^{|T|}$ as a topological model category, any such is automatically a simplicial model category. (See the remarks between Proposition 6.10 of [10]). Then Lemma 5.1 implies that the adjoint isomorphism

$$\mathbf{Hom}_{(s\mathcal{S})^T}(X, Y^\Delta) \cong \mathbf{Hom}_{\mathcal{S}^{|T|}}(|X|, Y)$$

extends to an isomorphism of simplicial mapping spaces

$$\mathrm{map}_{(s\mathcal{S})^T}(X, Y^\Delta) \cong \mathrm{map}_{\mathcal{S}^{|T|}}(|X|, Y).$$

Since $|T(Ai)| \cong |T|(Ai)$, it follows from Definition 4.2 and the subsequent remarks, that Y^Δ sends fibrations (or trivial fibrations)in $\mathcal{S}^{|T|}$ to $T(\mathcal{A})$-fibrations (or $T(\mathcal{A})$-trivial fibrations) in $(s\mathcal{S})^T$. Then $|\cdot|$ sends $T(\mathcal{A})$-cofibrations (and $T(\mathcal{A})$-trivial cofibrations) to cofibrations (and trivial cofibrations). The standard argument and Lemma 5.1 now finish the proof. ■

The next result is very technical, but used with Proposition 4.17, it will provide the key tools for our main result.

Lemma 5.4 *Suppose $X \in (s\mathcal{S})^T$, regarded as an object in $(s_+\mathcal{S})^T$, is isomorphic to $T(Y)$ where $Y \in s_+\mathcal{S}$ is free on some set of cofibrant spectra. Suppose T is Reedy cofibrant as a simplicial operad. Then X is Reedy cofibrant as a simplicial spectrum.*

Proof. First suppose T is constant and Y is free on Z_k. Then in Lemma 6.10 of [10] we proved that $T(Y)$, regarded as an object in $s_+\mathcal{S}$, is free on

$$\vec{Z}_k = \bigvee_n \bigvee_{x\in B_n} T(n)_+ \wedge_{\Sigma_n} (\bigvee_{\vec{i}\in q^{-1}(x)} Z_{i_1}\wedge\cdots\wedge Z_{i_n}) \tag{5.5}$$

where B_n is an indexing set we now define. Let I_k be the objects of $[k]/\Delta_+$, thus I_k consists of surjections $[k]\to[j]$ in Δ. Let I_k^n be the Cartesian product of I_k with itself n-times. The symmetric group acts by permuting coordinates. Let $q: I_k^n \to K_k^n/\Sigma_n$ be the quotient map and $B_n \subseteq I_k^n/\Sigma_n$ the subset of orbits

$$x = [([k]\to[i_1],\ldots,[k]\to[i_n])]$$

where $i_t = k$ for at least one t.

To prove the result we use (5.5) and a diagonal argument:

$$T(Y) = \mathrm{diag}(T.(Y))$$

where $T.(Y)_{n,m} = T_n(Y_m)$. Let n be the horizontal variable and m the vertical variable and L_n^h, L_m^v the horizontal and vertical latching objects (as spectra) respectively. Then

$$L_n(T(Y)) = L_n^v L_n^h(T.(Y)).$$

It follows from (5.5) that

$$L_n^h(T.(Y))_k = \bigvee_m \bigvee_{C_m} T_k(m)_+ \wedge_{\Sigma_m} (\bigvee_{\vec{i}\in q^{-1}(x)} Z_{i_1}\wedge\cdots\wedge Z_{i_m})$$

with $C_m = I_n^m/\Sigma_m - B_m$. Note that C_m does not depend on k. Hence

$$L_n(T(Y)) = \bigvee_m \bigvee_{v\in C_m} L_nT(m)_+ \wedge_{\Sigma_m} (\bigvee_{\vec{i}\in q^{-1}(x)} Z_{i_1}\wedge\cdots\wedge Z_{i_m}).$$

This is a wedge summand of the standard coproduct, with C_m replaced by I_n^m/Σ_m. Since $L_nT(m)\to T_n(m)$ is a cofibration of spaces over $\mathcal{L}(\mathcal{U}^m,\mathcal{U})$, $L_n(T(Y))\to T(Y)$ is a cofibration, because

$$T(Y)_n = \bigvee_m \bigvee_{I_n^m/\Sigma_m} T_n(m)_+ \wedge_{\Sigma_m} (\bigvee_{\vec{i}\in q^{-1}(x)} Z_{i_1}\wedge\cdots\wedge Z_{i_m}). \qquad \blacksquare$$

We now come to the main result.

Theorem 5.6 *Let $X\in(s\mathcal{S})^T$ be $T(\mathcal{A})$-cofibrant and suppose T is a Reedy cofibrant simplicial operad over the linear isometries operad. Then there is a homotopy spectral sequence*

$$\pi_p(X,\Sigma^q Ai)\Rightarrow \pi_{p+q}(|X|,Ai).$$

Remark. The simplicial homotopy groups of the E_2 term and the homotopy groups of the abutment are to be computed in $s\mathcal{S}$ and $\mathcal{S}$ respectively. Also, for any T, note that

$$\pi_{p+q}(|X|, Ai) \cong [|T|(Ai), |X|]_{\mathcal{S}^{|T|}}$$

and the latter expression simplifies if T is constant, since $|T| \cong T$.

Proof. Since geometric realization preserves weak equivalences between $T(\mathcal{A})$-cofibrant objects (by Proposition 5.3) we may use Proposition 4.17 to replace X by $W(X)$. By Lemma 5.4, $Y(X)$ is Reedy cofibrant as a simplicial spectrum. Now apply Corollary 6.8 of [10]. ∎

6 A spectral sequence for computing the homotopy groups of mapping spaces.

Suppose E is an Adams-type commutative ring spectrum and an $\mathcal{F}$-algebra where $\mathcal{F}$ is an operad over the linear isometries operad. For any $\mathcal{F}$-algebra X, we wish to discuss the homotopy type of $\mathrm{map}_{\mathcal{F}}(X, E)$.

To begin with, we resolve the operad $\mathcal{F}$. Let $n\mathbf{Top}$ be the catgegory of non-negatively graded spaces $\{X_n\}_{n\geq 0}$. The forgetful functor from the category of operads (in spaces) to $n\mathbf{Top}$ has a left adjoint G; hence the forgetful functor from the category $\mathcal{O}$ of operads over the linear isometries operad to the category $n\mathbf{Top}/\mathcal{L}$ has a left adjoint, also given by G. We also write G for the induced triple on $\mathcal{O}$. Then, given $\mathcal{F} \in \mathcal{O}$ we can form the augmented simplicial object in $\mathcal{O}$

$$G.\mathcal{F} = \{G^{n+1}\mathcal{F}\} \longrightarrow \mathcal{F}.$$

This construction has the following properties:

6.1.1.) If each of the spaces $\mathcal{F}(n)$ of $\mathcal{F}$ is cofibrant as a space, then $G.\mathcal{F}$ is Reedy cofibrant as a simplicial operad.

6.1.2.) If each of the spaces $\mathcal{F}(n)$ of $\mathcal{F}$ is cofibrant as a space, then the natural map of operads $|G.|\mathcal{F} \to \mathcal{F}$ is a weak equivalence of operads.

6.1.3.) Suppose E_* is a homology theory with a Künneth spectral sequence, and suppose $E_*\mathcal{F}(n)$ is projective as an $E_*\mathcal{F}(n)$ is projective as an E_* module. Then $E_*G.\mathcal{F}$ is a cofibrant simplicial operad in E_* modules or E_*E comodules.

The first and the third of these properties follow from the description of the free operad functor given in [13] and the discussion of cofibrant simplicial operads given in [11]. The second follows from the fact that that, regarded as an object in $n\mathbf{Top}$, the simplicial object $G.\mathcal{F} \to \mathcal{F}$ has a natural retraction. Since the forgetful functor $\mathcal{O} \to n\mathbf{Top}/cL$ makes the gemoetric realization, we have that claim.

The hypothesis that $\mathcal{F}$ be level-wise cofibrant, as in 6.1.1 and 6.1.2, is innocuous. If $\mathcal{F}$ is an operad in $\mathcal{O}$ we may choose a morphism of operads $\mathcal{F}' \to \mathcal{F}$ which is a level-wise weak equivalence and so that $\mathcal{F}'$ is cofibrant as an operad. Since cofibrant operads are level-wise coibrant as graded topological spaces (again by [11]), we can form the simplicial object $G.\mathcal{F}'$ and there will be an augmentation

$$G.\mathcal{F}' \longrightarrow \mathcal{F}$$

so that the analogs of 6.1.1 and 6.1.2 hold.

Thus we have proved:

Proposition 6.2 *Let $\mathcal{F}$ be an operad over the linear isometries operad. Then there is an augmented simplicial operad $T \to \mathcal{F}$ over the linear isometies operad so that T is Reedy cofibrant as an operad and so that $|T| \to \mathcal{F}$ is a weak equivalence of operad. If, in addition, each of the spaces $\mathcal{F}(n)$ is projective as E_* module, then E_*T is a cofibrant operad in E_* modules or E_*E comodules.*

∎

We now fix an Adams style ring spectrum E. We assume E is an $\mathcal{F}$-algebra for some operad $\mathcal{F} \in \mathcal{O}$. Let X also be a cofibrant $\mathcal{F}$-algebra. We attempt to compute $\mathrm{map}_{\mathcal{F}}(X, E)$.

First, we may assume $\mathcal{F}$ is cofibrant. If not, we choose a cofibrant replacement $\mathcal{F}_0 \to \mathcal{F}$. This does not affect the homotopy theory: if $X_0 \to X$ is a cofibrant replacement for X as an $\mathcal{F}_0$ algebra then by [9], $\mathcal{F} \otimes_{\mathcal{F}_0} X_0 \to X$ is a weak equivalence and $\mathcal{F} \otimes_{\mathcal{F}_0} X_0$ is cofibrant. Thus

$$\mathrm{map}_{\mathcal{F}_0}(X_0, E) \cong \mathrm{map}_{\mathcal{F}}(\mathcal{F} \otimes_{\mathcal{F}_0} X_0, E)$$

is weakly equivalent to $\mathrm{map}_{\mathcal{F}}(X, E)$.

Next, let $T = \bar{F}.(\mathcal{F}) \to A$ be the canonical resolution of $\mathcal{F}$. Then $X \in (s\mathcal{S})^T$ as a constant object. We give $s\mathcal{S}^T$ the $T(\mathcal{A})$-resolution model category structure, where $\mathcal{A} = \{\Sigma^n DE_\alpha\}$ is the set of finite CW spectra built from E. (See the beginning of section 4). Choose a $T(\mathcal{A})$-cofibrant object $Y \to X$ weakly equivalent to X and define a cosimplicial space

$$\mathrm{map}_T(Y, E) = \{\mathrm{map}_T(Y_n, E)\} \cong \{\mathrm{map}_{\mathcal{F}}(\mathcal{F} \otimes_{T_n} Y_n, E)\}_{n \geq 0}. \tag{6.3}$$

A morphism $f : X \to E$ of $\mathcal{F}$-algebras defines a basepoint for this cosimplicial space, and we get a Bousfield-Kan spectral sequence

$$\pi^s \pi_t \,\mathrm{map}_T(Y, E) \Rightarrow \pi_{t-s} \mathrm{T{\scriptsize OT}}\ \mathrm{map}_T(Y, E). \tag{6.4}$$

We next identify the E_2 terms and target of this spectral sequence. This is done in Lemmas 6.8 and 6.5 respectively.

Lemma 6.5 *There is a natural weak equivalence*

$$\mathrm{T{\scriptsize OT}}\ \mathrm{map}_T(Y, E) \cong \mathrm{map}_{\mathcal{F}}(X, E).$$

Proof. First, since $\mathrm{map}_T(Y,E) \cong \mathrm{map}_{\mathcal{F}}(\mathcal{F} \otimes_T Y, E)$, we have

$$\begin{aligned} \mathrm{Tot}\ \mathrm{map}_T(Y,E) &\cong \mathrm{Tot}\ \mathrm{map}_{\mathcal{F}}(\mathcal{F} \otimes_T Y, E) \\ &\cong \mathrm{map}_{\mathcal{F}}(|\mathcal{F} \otimes_T Y|, E). \end{aligned}$$

Next, we have that $|\mathcal{F} \otimes_T Y| \cong \mathcal{F} \otimes_{|T|} |Y|$. This follows from the string of natural isomorphisms

$$\begin{aligned} \mathbf{Hom}_{\mathcal{F}}(|\mathcal{F} \otimes_T Y|, E) &\cong \mathbf{Hom}_{\mathcal{F}}(\mathcal{F} \otimes_T Y, E^{\Delta}) \\ &\cong \mathbf{Hom}_T(Y, E^{\Delta}) \\ &\cong \mathbf{Hom}_{|T|}(|Y|, E) \\ &\cong \mathbf{Hom}_{\mathcal{F}}(\mathcal{F} \otimes_{|T|} |Y|, E). \end{aligned}$$

Since $|T| \to \mathcal{F}$ is a weak equivalence between cofibrant operads,

$$|Y| \to \mathcal{F} \otimes_{|T|} |Y|$$

is a weak equivalence. Thus we need only show $|Y| \to X$ is a weak equivalence. Since $Y \to X$ is an $T(\mathcal{A})$-equivalence, and $T = \mathfrak{S}.(\mathcal{F})$ is a Reedy cofibrant this follows from Theorem 5.6. ∎

To identify the E_2 term, we review the basics of André Quillen homology of simplicial algebras over a simplicial ring. A more exhaustive and systematic treatment is in [11]. Let R be a commutative ring, $s\mathcal{M}_R$, the category of simplicial R-modules. Let C be a simplicial operad in the category of R modules. In our application, R will be E_* and $C = E_*T$. Let $s\mathcal{M}_R^C$ be the category of C algebras in $s\mathcal{M}_R$. There is a simplicial closed model category structure on $s\mathcal{M}_R^C$ where one uses the standard simplicial structure [15, §II.2] and defines $f : X \to Y$ to be a fibration or weak equivalence if it is so as simplicial R-modules. A cofibrant object is a retract of a "free object" X [15, §II.4]—which is to say as an object in $(s_+\mathcal{M}_R)^C$, there is an isomorphism $X \cong C(M)$ where $M \in s_+\mathcal{M}_R$ is projective as an R-module in each degree.

Now suppose C has an augmented operad $\epsilon : C \to C_{-1}$ and $X \in s\mathcal{M}_R^C$ has an augmentation $\epsilon : X \to X_{-1}$ to some C_{-1}-algebra X_{-1}. If M is an C_{-1} module over X_{-1}, the M inherits a unique structure as a C_n module over X_n and we can form the cosimplicial abelian group

$$\mathrm{Der}_C(X,M) \cong \{\mathrm{Der}_{C_n}(X_n, M)\}. \tag{6.6}$$

This is a cosimplicial R-module or, if R is graded, a cosimplicial R_0 module. It does not have good homotopical properties; however, if we choose a weak equivalence $Y \to X$ is $s\mathcal{M}_R^C$ with Y cofibrant, then $\mathrm{Der}_C(Y,M)$ is well-defined and natural up to homotopy. We define the André-Quillen cohomology of X with coefficients in M by

$$D^s_C(X,M) \cong \pi^s\,\mathrm{Der}_C(Y,M). \tag{6.7}$$

We now return to the spectral sequence of (6.4). Thus we have an operad $\mathcal{F}$ over the linear isometries operad, E an Adams-type commutative ring spectra, and $f : X \to E$ a morphism of $\mathcal{F}$-algebras. We note that E_*E inherits an $E_*\mathcal{F}$ algebra structure. Also there is a morphism $E_*\mathcal{F} \to E_*\mathcal{L}$. Since each of the spaces of $\mathcal{L}$ is contractible, $E_*\mathcal{L}$ is the commutative algebra operad, so E_*E, as a commutative E_* algebra gets a potentially different $E_*\mathcal{F}$ structure. We assume these two structures are the same. Then since the multiplication $u_* : E_*E \to E_*$ is a morphism of E_* algebras, E_* is an $E_*\mathcal{F}$ module over E_*E, hence so is

$$E_{*+t} = \{E_{n+t}\}_n.$$

The morphism $f : X \to E$ of $\mathcal{F}$-algebras gives E_{*+t} an $E_*\mathcal{F}$ module structure over E_*X. Let $T = \bar{F}_\cdot(\mathcal{F})$ be the canonical resolution of $\mathcal{F}$ and $Y \to X$ and $T(\mathcal{A})$ cofibrant model for X regarded as a constant object in $(s\mathcal{S})^T$. We may assume that Y is homotopically s-free by Propositions 4.10 and 4.12.

Lemma 6.8 *The simplicial E_*T algebra E_*Y is cofibrant, and the augmentation $Y \to X$ induces an isomorphism*

$$\pi_*E_*Y \cong E_*X.$$

Proof. Since Y is homotopically $T(\mathcal{A})$-free, there is a homotopy equivalence, as T_n algebras

$$Y_n \cong T_n(Z)$$

where Z is a wedge of spectra of the form $\Sigma^j DE_\alpha$. Now $E_*(\Sigma^j DE_\alpha)$ is a projective E_* module. Also $\pi_0 T_n$ is a free Σ_n set, hence E_*T_n is a free $E_*\Sigma_n$ module. Since E_* has a Künneth spectral sequence, there is an isomorphism of E_*T_n algebras

$$(E_*T_n)E_*Z \cong E_*Y_n.$$

A further consequence of being homotopically s-free is that E_*Y is free, hence cofibrant, as an E_*T algebra.

To calculate π_*E_*Y, we use the spectral sequence of Theorem 5.6, which implies that

$$\pi_p[\Sigma^n DE_\alpha, Y] = \begin{cases} [\Sigma^n DE_\alpha, X], & p = 0; \\ 0 & p \neq 0. \end{cases}$$

Now use the fact that

$$\pi_p E_n Y \cong \operatorname{colim}_\alpha \pi_p (E_\alpha)_n Y \cong \operatorname{colim}_\alpha \pi_p[\Sigma^n DE_\alpha, Y]$$

to finish the argument. ∎

The following identifies the E_2 term of the spectral sequence (6.4).

Lemma 6.9 *There is a natural isomorphism*

$$\pi^0\pi_0 \operatorname{map}_T(Y, E) \cong \mathbf{Hom}_{E_*\mathcal{F}}(E_*X, E_*)$$

and for all $t \geq 1$ and all $s \geq 0$, and all choices of $\mathcal{F}$-algebra maps $f : X \to E$, a natural isomorphism

$$\pi^s\pi_t \operatorname{map}_T(Y, E) \cong D^s_{E_*T}(E_*X, E_{*+t}).$$

Proof. First notice that the map that evaluates on E_*

$$\pi_0 \operatorname{map}_{T_n}(Y_n, E) \to \mathbf{Hom}_{E_*T_n}(E_*Y_n, E_*) \tag{6.10}$$

is an isomorphism; indeed,

$$\begin{aligned} \pi_0 \operatorname{map}_{T_n}(Y_n, E) &\cong \pi_0 \operatorname{map}_{\mathcal{S}}(Z_n, E) \\ &\cong \mathbf{Hom}_{E_*}(E_*Z_n, E_*) \\ &\cong \mathbf{Hom}_{E_*T_n}(E_*Y_n, E_*). \end{aligned}$$

For the calculation of $\pi^0\pi_0 \operatorname{map}(Y, E)$, use that there is a coequalizer diagram

$$E_*Y_1 \rightrightarrows E_*Y_0 \to E_*X$$

as E_* modules Hence, because it is a split coequalizer there is a coequalizer of $E_*\mathcal{F}$ algebras

$$E_*\mathcal{F} \otimes_{E_*T_1} E_*Y_1 \rightrightarrows E_*\mathcal{F} \otimes_{E_*T_0} E_*Y_0 \to E_*X. \tag{6.11}$$

There is, by 6.10, an equalizer diagram

$$\pi^0\pi_0 \operatorname{map}_T(Y, E) \to \mathbf{Hom}_{E_*T_0}(E_*Y_0, E_*) \rightrightarrows \mathbf{Hom}_{E_*T_1}(E_*Y_1, E_*)$$

hence an equalizer diagram

$$\pi^0\pi_0 \operatorname{map}_T(Y, E) \to \mathbf{Hom}_{E_*\mathcal{F}}(E_*\mathcal{F} \otimes_{E_*T_0} E_*Y_0, E_*) \rightrightarrows \mathbf{Hom}(E_*\mathcal{F} \otimes_{E_*T_1} E_*Y_1, E_*)$$

and the result follows from (6.11).

To calculate $\pi^s\pi_t \operatorname{map}(Y, E)$ with $t \geq 1$, let $\mathbf{hom}(S^t, E)$ be the spectrum of maps from S^t to E. Then $E_*\mathbf{hom}(S^t, E) \cong \Lambda_{E_*}(x_t)$, the exterior algebra over E_* one a generator in degree $-t$. Then, arguing as above, we have for a choice of basepoint in $\operatorname{map}_{T_n}(Y_n, E)$, a natural isomorphism

$$\begin{aligned} \pi_t \operatorname{map}_{T_n}(Y, E) &\cong \mathbf{Hom}_{E_*T_n}(E_*Y_n, \Lambda_{E_*}(x_t) \\ &\cong \operatorname{Der}_{E_*T_n}(E_*Y, E_{*+t}). \end{aligned}$$

The result now follows from Lemma 6.8 and the definition of André-Quillen cohomology.■

7 Eilenberg-MacLane Objects.

Let T be a simplicial operad over the linear isometries operad and let E be an Adams type homology theory. If $X \in s\mathcal{S}^T$, then E_*X is a simplicial algebra over E_*T; in short,

$$E_*X \in s\mathcal{A}^{E_*T}.$$

If M is a $\pi_0 E_*T$ module over $\pi_0 E_*X$ we can form the André-Quillen cohomology groups

$$D^n(E_*X, M),$$

as in section 3. In this section we discuss to what extent this functor is representable.

To begin with, we widen our scope a bit. Fix a $\pi_0 E_*T$ algebra A and a $\pi_0 E_*T$ module M over A. The main result of this section is the following; we are using the standard resolution model category structure.

Proposition 7.1 *1) There is a fibrant simplicial object $KA \in s\mathcal{S}^T$ and a natural isomorphism for all $X \in s\mathcal{S}^T$*

$$\begin{aligned} [X, KA]_{s\mathcal{S}^T} &\overset{\cong}{\to} \mathbf{Hom}_{E_*T}(E_*X, A) \\ &\cong \mathbf{Hom}_{\pi_0 E_*T}(\pi_0 E_*X, A). \end{aligned}$$

2) Let $X \in s\mathcal{C}^T$ be s-cofibrant $f_* \in \mathbf{Hom}_{\pi_0 E_*T}(\pi_0 E_*X, A)$ be represented by $f : X \to KA$. Then there is an object $q : K(M, n) \to KA$ in $s\mathcal{S}^T/KA$ and a natural isomorphism

$$[X, K(M, n)]_{s\mathcal{S}^T/KA} \cong D^n(E_*X, M).$$

The argument in both cases is essentially Spanier's proof of Brown Representability [Spa]. We would also appeal to Heller [H]. The first observation is that the functors of Proposition 7.1 satisfy a variant of Brown's axioms for a homotopy functor.

Lemma 7.2 *Let $\{X_\alpha\}_{\alpha \in A}$ be a set of objects in $s\mathcal{S}^T$. Then*

$$\mathbf{Hom}_{\pi_0 E_*T}(\pi_0 E_*(\coprod_\alpha X_\alpha), A) \cong \prod_\alpha \mathbf{Hom}_{\pi_0 E_*T}(\pi_0 E_* X_\alpha, A).$$

Furthermore, fixing $E_* f_\alpha : E_* X_\alpha \to A$,

$$D^n(E_*(\coprod_\alpha X_\alpha), M) \cong \prod_\alpha D^n(A_* X_\alpha, M).$$

Proof. We may assume each X_α is S-free. We will show that

$$E_*(\coprod_\alpha X_\alpha) \cong \coprod_\alpha E_* X_\alpha,$$

the coproduct on the right being in $s\mathcal{A}^{E_*T}$. The result will then follow. At level n, $(X_\alpha)_n \cong T_n(V_n^\alpha)$ where $V_n^\alpha \simeq \bigvee_w \Sigma^{n_w} DE_w$. Since T is cofibrant, T_n is a Σ-free operad. Also $E_*V_n^\alpha$ is projective as an E_* module. Hence

$$E_*(X_\alpha)_n \cong E_*T_n(E_*V_n).$$

Since $(\coprod_\alpha X_\alpha)_n \cong \coprod_\alpha (X_\alpha)_n \cong T_\alpha(\vee V_\alpha^n)$, the result follows. ■

The second half of Brown's axioms is the following.

Lemma 7.3 *Suppose we have a push-out diagram in* $s\mathcal{A}^T$

$$\begin{tikzcd} C = \coprod_\beta T(\Sigma^{n_\beta} DE_\beta \otimes \Delta^{k_\beta}/\partial\Delta^{k_p}) \arrow[r] \arrow[d] & X \arrow[d] \\ D = \coprod_\beta T(\Sigma^{n_\beta} DE_\beta \otimes \Delta^{k_\beta+1}/\Delta_0^{k_\beta+1}) \arrow[r] & Y \end{tikzcd}$$

where $X \in s\mathcal{A}^T$ *is* S*-cofibrant. Then*

$$\begin{tikzcd} \mathbf{Hom}_{E_*T}(E_*Y, A) \arrow[r] \arrow[d] & \mathbf{Hom}_{E_*T}(E_*D, A) \arrow[d] \\ \mathbf{Hom}_{E_*T}(E_*X, A) \arrow[r] & \mathbf{Hom}_{E_*T}(E_*C, A) \end{tikzcd}$$

is a pull-back diagram. Furthermore, given $f \in \mathbf{Hom}_{E_*T}(E_*Y, A)$*, we have that for all* $n \geq 0$*; the sequence*

$$D^n(E_*Y, M) \to D^n(E_*X, M) \times D^n(E_*B, M) \to D^n(E_*A, M)$$

is exact.

Proof. It is sufficient to show that

$$\begin{tikzcd} E_*C \arrow[r] \arrow[d] & E_*X \arrow[d] \\ E_*D \arrow[r] & E_*Y \end{tikzcd} \tag{7.4}$$

is a push-out in $s\mathcal{A}^{E_*T}$ and that E_*X and E_*Y are cofibrant as simplicial E_* modules. Then the first statement—which only requires that 7.4 be a push-out—follows immediately. The second statement follows from the Meyer-Vietoris sequence for André-Quillen cohomology (0.2.0) and (0.HC.0) which says 7.4 is, in fact, weakly equivalent to the homotopy push-out.

To see that 7.4 is a push-out it is sufficient to work level-wise. At level n, $C_n \cong T(\bigvee_z \Sigma^{n_z} DE_z)$ for z in some indexing set z, and

$$D_n \cong C_n \coprod T(\bigvee_y \Sigma^{n_y} DE_y)$$

for y in some indexing set y. Thus

$$E_* D_n \cong E_* C_n \coprod E_*(T(\bigvee_y \Sigma^{n_y} DE_y)$$

as in the proof of 7.2. Thus we need only check

$$\begin{aligned} E_* Y_n &\cong E_*(X_n \coprod T(\bigvee_y \Sigma^{n_y} DE_y)) \\ &\cong E_* X_n \coprod E_* T(\bigvee_y \Sigma^{n_y} DE_y). \end{aligned}$$

But $X_n \cong T(\bigvee_n)$ where $V_n \simeq \bigvee_x \Sigma^{n_x} DE_x$ for some indexing set. The computation follows.

To see $E_* X$ and $E_* Y$ are cofibrant as simplicial E_* modules, we use the following observation: Since T is S-cofibrant, $E_* T$ is cofibrant as a simplicial operad in $s\mathcal{M}_{E_*}$. Now apply HC.? ■

Lemma 7.5 *Suppose we have a directed system of S-cofibrations in $s\mathcal{S}^T$, with X_0 cofibrant*

$$X_0 \to X_1 \to X_2 \to \cdots .$$

Then the natural map

$$\mathbf{Hom}_{E_* T}(E_* \operatorname{colim}_m X_m, A) \overset{\rightarrow}{\cong} \lim_m \mathbf{Hom}_{E_* T}(E_* X_n, M)$$

is an isomorphism. Furthermore, given $f \in \mathbf{Hom}_{E_* T}(E_* \operatorname{colim}_m X_m, A)$, the natural map

$$D^n(E_* \operatorname{colim}_m X_m, M) \to \lim_m D^n(E_* X_m, M)$$

is a surjection.

Proof. In any given simplicial degree k, there are spectra V_m, $m \geq 0$, and a set I_n so that there is a weak equivalence

$$\vee_m \simeq \bigvee_{\alpha \in I_n} \Sigma^{s_\alpha} DE_\alpha$$

and a homotopy equivalence of T_k algebras

$$(X_m)_k \to (X_{m-1})_k \sqcup T_k(V_m) \cong T(\bigvee_{k=0}^{m} V_m).$$

We set X_{-1} to be the initial object. Hence

$$(\operatorname{colim} X_m)_k \simeq T_k(\bigvee_{m=0}^{\infty} V_m)$$

and a direct calculation, using our assumptions of T, implies

$$E_* \operatorname{colim}(X_m)_k \cong \operatorname{colim} E_*(X_m)_k \tag{7.6}$$

or, simply $E_* \operatorname{colim} X_m \cong \operatorname{colim} E_* X_m$ and the statement about $\mathbf{Hom}_{E_*T}(?, A)$ now follows.

Further analysis, using 0.0.0, implies each $E_*X_{m-1} \to E_*X_m$ is a cofibration of E_*T algebras. Hence

$$D^n(E_* \operatorname*{colim}_{\alpha} X_m, M) = \pi^n \operatorname{Der}(E_* \operatorname*{colim}_{\alpha} X_m, M).$$

The isomorphism of 7.6 implies there is an isomorphism of E_* module

$$\operatorname{Der}(E_* \operatorname*{colim}_{\alpha} X_m), M) \cong \lim_m \operatorname{Der}(E_* X_m, M)$$

and, hence, we have a Milnor exact sequence

$$0 \to \lim{}^1 D^{n-1}(E_* X_m, M) \to D^n(E_* \operatorname*{colim}_{\alpha} X_m, M) \to \lim_m D^n(E_* X_m, M) \to 0. \qquad \blacksquare$$

Definition 7.6 *Define a functor* $H : \mathbf{Ho}(s\mathcal{S}^T) \stackrel{\text{op}}{\to} \mathbf{Sets}$ *to be a homotopy functor if*

1) $H(\coprod_{\alpha} X_\alpha) \cong \prod H(X_\alpha)$

2) *Given a two source of the type*

$$D = \coprod_{\beta} T(\Sigma^{n_\beta} DE_\beta \otimes \Delta^{k_\alpha+1}/\Delta_0^{k_\alpha+1}) \leftarrow C = \coprod_{\alpha} T(\Sigma^{n_\alpha} DE_\beta \otimes) \cdots \mathit{etc.}$$

then

$$H(\operatorname{hocolim}(D \leftarrow C \to X)) \to H(D) \times_{H(C)} H(X)$$

is a surjection.

3) *Given a directed system*

$$X_0 \to X_1 \to X - 2 \to \cdots$$

the natural map $H(\operatorname*{hocolim}_m X_m) \to \lim_m H(X_m)$ *is a surjection.*

Example 7.7 1) The functor

$$X \mapsto \mathbf{Hom}_{E_*T}(E_*X, A) = \mathbf{Hom}_{\pi_0 E_*T}(\pi_0 E_*X, A)$$

is a homotopy functor, by Lemmas 7.2, 7.3, and 7.5.

2) Define $H^n(?; M)$ to be the functor that assigns to $X \in \mathbf{Ho}(s\mathcal{S}^T)^{\mathrm{op}}$ the set of all pairs (f, x) where

$$f \in \mathbf{Hom}_{\pi_0 E_*T}(\pi_0 E_*X, M)$$

and

$$x \in D^n(E_*X; M),$$

where the E_*T module structure on M over E_*X is determined by f. Then lemmas 7.2, 7.3, and 7.5 imply $H^n(?; M)$ is a homotopy functor.

We now prove the following Brown-Representability Theorem.

The argument goes as in Spanier [Spa, §7.7]. If $H : [\mathbf{Ho}(s\mathcal{S}^T)/z]^{\mathrm{op}} \to \mathbf{Sets}$ is any homotopy functor define a pair (Y, u), $u \in H(Y)$, to be *n-universal* for H if Y is fibrant and for all k, and α the natural map

$$\phi_u : [T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta^q), Y]_{s\mathcal{S}^T} \to H(T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta^q))$$

given by

$$\phi_\mu(f) = H(f)(u)$$

is an isomorphism for $q < n$ and onto for $q = n$. The pair (Y, μ) is *universal* if it is n-universal for all n. Note that since $T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta^n)$ is a homotopy cogroup object, ϕ_u is automatically a group homomorphism. If H is a group-valued functor, the two group operations on $H(T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta^q)$ agree (cf. [Spa, Lemma 7.7.6]).

Lemma 7.8 *Let $X \in s\mathcal{S}^T/Z$ be S-cofibrant and $y \in H(X)$. Then there exists a universal pair (Y, u) and a skeletal map $f : X \to Y$ so that $H(f)(u) = y$.*

Proof. We define a sequence of maps over Z

$$X = Y_{-1} \xrightarrow{f_0} Y_0 \xrightarrow{f_1} Y_1 \xrightarrow{f_2} Y_2 \to \cdots$$

and elements $\mu_i \in H(Y_i)$ so that

1) $\mu_{-1} = y$ and $H(f_n)u_n = u_{n-1}$;

2) (Y_n, u_n) is n-universal

3) for each n there is a push-out diagram

$$\begin{tikzcd} \coprod_{I_n} T(\Sigma^k DE_\alpha \wedge \Delta^n/\partial\Delta^n) \arrow[r] \arrow[d] & Y_n \arrow[d] \\ \coprod_{I_n} T(\Sigma^k DE_\alpha \wedge \Delta^{n+1}/\Delta_0^{n+1}) \arrow[r] & \bar{Y}_{n+1} \end{tikzcd}$$

for some index set I_n and a Reedy trivial cofibration $\bar{Y}_{n+1} \to Y_{n+1}$ which is an isomorphism in simplicial degrees less than $n+1$.

Assuming this to be the case for the moment, let $Y = \operatorname{colim} Y_n$ and $f : X \to Y$ the evident map over Z. Let $\mu \in H(Y)$ be any element restricting to $\lim u_i$ under the map $H(Y) \to \lim H(Y_i)$. Since, in any simplicial degree

$$(Y_n)_{k_q} \cong (Y_{n+1})_{k_q}$$

for $q \leq n$, we have for $q < n$

$$[T(\Sigma^k DE_\alpha \wedge \Delta^q/\partial\Delta, Y] \cong [T(\Sigma^l DE_\alpha \wedge \Delta^q/\partial\Delta^q, Y_n]$$

and, hence, (Y, u) is universal.

To construct (Y_n, u_n), we work recursively. To construct Y_0, let I_0 be the set of triples (k, α, z) where $z \in H(T(\Sigma^k D_\alpha))$. Let

$$\bar{Y}_0 = X \sqcup (\bigsqcup_{I_0} T(\Sigma^k D_\alpha E))$$

and $\mu_1 \in H(Y_0)$ the evident class determined by the isomorphism

$$H(Y_0) \cong H(X) \times \prod_{I_0} H(T(\Sigma^k DE_\alpha)).$$

Now let $\bar{Y}_0 \to Y_0$ be a Reedy trivial cofibration with Y_0 Reedy fibrant. Now suppose Y_n has been constructed.

Define $\bar{Y}_{n+1}$ as follows. Let $I_n = J_n \sqcup K_n$ where J_n is the set of triples (k, α, z) where

$$Z \in H(T(\Sigma^k DE_\alpha \wedge \Delta^{n+1}/\partial\Delta^{n+1})$$

and K_n is the set of triples (k, α, f) where

$$f \in [T(\Sigma^k DE_\alpha \wedge \Delta^n/\partial\Delta^n), Y_n]$$

so that $H(f)(u_n) = 0$. Let Y_{n+1} be defined by the pushout diagram

$$\begin{tikzcd} \coprod_{n} T(\Sigma^k DE_\alpha \wedge \Delta^n/\partial\Delta^n) \arrow[r] \arrow[d] & Y_n \sqcup (\coprod_{J_n} T(\Sigma^k DE_\alpha) \wedge \Delta^{n+1}/\partial\Delta^{n+1}) \arrow[d] \\ \coprod_{K_n} T(\Sigma^k DE_\alpha \wedge \Delta^{n+1}/\Delta_0^{n+1}) \arrow[r] & \bar{Y}_{n+1}. \end{tikzcd}$$

Now choose a Reedy trivial cofibration $\bar{Y}_{n+1} \to Y_{n+1}$ with Y_{n+1} fibrant. Here we mean S-fibrant, but, by 0.0.0, this is the same as Reedy fibrant. Since $(Y_n)_k \cong (\bar{Y}_{n+1})_k$ in simplicial degrees $k \leq n$, and since the Reedy "trivial cofibration-fibration" factorization is done recursively in simplicial degree (see 0.0.0), we may assume $(\bar{Y}_{n+1})_k \cong (Y_{n+1})_k$ in degrees $k \leq n$.

By definition 7.6, we have an element $\mu_{n+1} \in H(Y_{n+1}) \cong H(\bar{Y}_{n+1})$ so that 1) and 3) above hold. We need to show (Y_{n+1}, μ_{n+1}) is $(n+1)$-universal.

Write φ_n for φ_{u_n}, and consider the diagram

$$\begin{tikzcd}
{[T(\Sigma^k DE_\alpha \wedge \Delta^q/\Delta^q), Y_n)]} \arrow[rr, "(f_{n+1})_*"] \arrow[dr, "\varphi_n"'] & & {[T(\Sigma^k DE_\alpha \wedge \Delta^q/\Delta^q), Y_{n+1}]} \arrow[dl, "\varphi_{n+1}"] \\
& H(T(\Sigma^k DE_\alpha \wedge \Delta^q/\Delta^q). &
\end{tikzcd}$$

If $q = n+1$, φ_{n+1} is onto by construction. If $q = n$, φ_n is onto, so φ_{n+1} is onto. Also, since $(Y_n)_k \cong (Y_{n+1})_k$ in simplicial degrees $k \leq n$, $(f_{n+1})_*$ is onto. Since all the maps are group map and every element in the kernel of φ_n is in the kernel of $(f_{n+1})_*$, φ_{n+1} is one-to-one. .Finally, if $q < n$, the fact that $(Y_n)_k \cong (Y_{n+1}) + k$ for $k \leq n$ implies $(f_{n+1})_*$ is an isomorphism. Since φ_n is an isomorphism, so is φ_{n+1}. ■

The next lemma sets the stage for proving Proposition 7.1.

Lemma 7.9 *Let $j : A \to X$ be an S-cofibration in $(s\mathcal{S})^T$ with A cofibrant and let (Y, μ) be universal for H. Suppose there is an element $v \in X$ and a morphism $f : A \to Y$ so that $H(f)\mu = H(j)v$. Then there is a map $g : X \to Y$ so that $H(g)u = v$ and $g \circ j = f$ in* $\mathbf{Ho}(s\mathcal{S}^T)$.

Proof. Consider the push-out diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d, "i"] \\
X \arrow[r, "q"'] & Z.
\end{tikzcd}$$

By Lemma 7.3 there is an element $w \in H(Z)$ so that $H(i)w = \mu$ and $H(q)w = v$. Apply the previous lemma to produce a universal part (Y', u') and a map $h : Z \to Y'$ so that $H(n)'_u = w$. Then $h \circ i : Y \to Y'$ is a weak equivalence, and the lifting problem

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & Y \arrow[d] \\
X \arrow[r, "h\circ q"'] \arrow[ur, dotted] & Y'
\end{tikzcd}$$

can be solved up to homotopy. ■

Proposition 7.10 *If H is a homotopy functor, it is representable: there is an object $Y \in s\mathcal{S}^T$ and a class $\mu \in H(Y)$ so that*

$$[X, Y]_{s\mathcal{S}^T} \to H(x)$$

given by $f \mapsto H(f)\mu$ is bijection.

Proof. See [Sp, Thm 7.7.14]. ■

In particular Proposition 7.1 follows:

7.12: Proof of Proposition 7.1: The existence of KA follows immediately from example 7.7 and Proposition 7.10. For the object $K(M,n) \to KA)$ we proceed as follows. Consider the functor $H^n(?, M)$ of Example 7.7.2. By Proposition 7.10 there is $H^n(?; M)$ is represented by a cofibrant-fibrant object $\bar{K}(M,n)$. There is a natural transformation

$$\Phi : H^n(X, M) \to \mathbf{Hom}_{\pi_0 E_* T}(\pi_0 E_* X, A) \tag{7.13}$$

sending a pair (f, x) to f. This yields a map in $s\mathcal{S}^T$

$$\bar{K}(M,n) \to KA.$$

Factor this map as $\bar{K}(M,n) \xrightarrow{j} K(M,n) \xrightarrow{q} KA$ where j is a trivial cofibration and q is a fibration.

Let $f_* \in \mathbf{Hom}_{\pi_0 E_* T}(\pi_0 E_* X, A)$ be represented by $f : X \to KA$. Consider the pull-back square

$$\begin{tikzcd}
\mathrm{map}_{s\mathcal{S}^T/KA}(X, K(M,n)) \arrow[r] \arrow[d] & \mathrm{map}_{s\mathcal{S}^T}(X, K(M,n)) \arrow[d, "q_*"] \\
\{f\} \arrow[r] & \mathrm{map}_{s\mathcal{S}^T}(X, KA).
\end{tikzcd}$$

Then SM7 for $s\mathcal{S}^T$ shows that q_* is a fibration. Also q_* has a section, since the natural transformation (7.13) has a section sending f to $(f, 0)$. Finally

$$\pi_0 \,\mathrm{map}_{s\mathcal{S}^T/KA}(X, K(M,n)) = [X, K(M,n)]_{s\mathcal{S}^T},$$

so the result follows from the fact that there is a pull-back square

$$\begin{tikzcd}
D^n(X, M) \arrow[r] \arrow[d] & H^n(X, M) \arrow[d, "\Phi"] \\
\{f\} \arrow[r] & \mathbf{Hom}_{\pi_0 E_* T}(\pi_0 E_* X, A)
\end{tikzcd}$$

■

8 The Obstruction Theory

In this section we develop an obstruction theory for realizing structured ring spectra. There are many possible variants on this; for this first time through we choose a simple, direct example.

We are given an operad $\mathcal{F}$ over the linear isometries operad and an $E_*\mathcal{F}$ algebra Λ in the category of E_*E comodules. Here E_* is a fixed Adams type homology theory. We ask whether there is an $X \in \mathcal{S}^{\mathcal{F}}$ so that $E_*X \cong \Lambda$ as $E_*\mathcal{F}$ modules. This is the realization problem.

To simplify the process we will assume that as an E_*E comodule, $\Lambda \cong E_*E \otimes_{E_*} N$ where N is a projective E_* module. This eliminates the problem of existence and uniqueness of a spectrum (not necessarily a structured ring spectrum) with $E_*Z \cong Z$ as E_*E comodules.

Before stating the precise result, we make a reduction in the problem. We suppose we may choose an augmented simplicial operad

$$\epsilon : T \to \mathcal{F}$$

so that

8.1.1) E_*T is a cofibrant simplicial E_* operad,

8.1.2) the morphism $\epsilon : |T| \to \mathcal{F}$ is a weak equivalence of operads over the linear isometries operad, and

8.1.3) $\pi_*E_*T \cong E_*\mathcal{F}$ concentrated in degree 0.

We saw earlier that this is possible if $E_*\mathcal{F}(n)$ is a projective E_* module for all $n \geq 0$. (See Proposition 6.2). Note that 8.1.2 and one of the main results of [9] now would imply that if X is a cofibrant $|T|$ spectrum, then the natural map

$$X \to \mathcal{F} \otimes_{|T|} X$$

is a weak equivalence.

The program is to define a simplicial object $Y \in s\mathcal{S}^T$ so that

$$\pi_k E_* Y \cong \begin{cases} \Lambda, & k = 0, \\ 0 & k \neq 0. \end{cases}$$

Then, applying Theorem 5.6 and the result of [9] cited above, we will have that

$$X = \mathcal{F} \otimes_{|T|} |Y| \in \mathcal{S}^{\mathcal{F}}$$

is the desired structured ring spectra.

We give $s\mathcal{S}^T$ the $\mathcal{A}$-resolution model category structure, where $\mathcal{A} = \{\Sigma^j DE_\alpha\}$. We will write Aj for an element of $\mathcal{A}$.

If X actually existed, the fact that $\Lambda \cong E_*E \otimes_{E_*} N$ would force

$$\begin{aligned} \pi(X, Aj) &= [Aj, X] \\ &\cong \mathbf{Hom}_{E_*E}(E_*Aj, \Lambda). \end{aligned}$$

On the other hand, suppose we had $Y \in s\mathcal{S}^T$ with

$$\pi_k \pi(Y, Aj) \cong \begin{cases} \mathbf{Hom}_{E_*E}(E_*Aj, \Lambda) & k = 0 \\ 0 & k > 0. \end{cases} \tag{8.2}$$

Then

$$\pi_k E_*Y \cong \operatorname{colim}_\alpha \pi(Y, \Sigma^* DE_\alpha) \cong \begin{cases} \Lambda & k = 0 \\ 0 & k > 0. \end{cases}$$

Thus, we will work to produce Y satisfying (8.2). The method for producing Y is, in effect, to produce its Postnikov tower. If Y is to exist, the spiral exact sequence of section 4 (see before Proposition 4.7) implies that for all $Aj \in \mathcal{A}$,

$$\begin{aligned} \pi_k^\natural(Y, Aj) &\cong \pi_0^\natural(Y, \Sigma^k Aj) \\ &\cong \pi_0 \pi(Y, \Sigma^k Aj) \\ &\cong \mathbf{Hom}_{E_*E}(E_*\Sigma^k Aj, \Lambda) \end{aligned}$$

for all $k \geq 0$. Indeed, the connecting maps

$$\pi_{k-1}^\natural(Y, \Sigma Aj) \to \pi_k^\natural(Y, Aj)$$

must be isomorphisms for $n \geq 1$. Thus we attempt to produce $Y(n) \in s\mathcal{S}^T$ and fibrations $q_n : Y(n) \to Y(n-1)$, $n \geq 0$, so that

(8.3.1) $Y(-1) = *$ and

$$\pi_k^\natural(Y(n), Aj) = \begin{cases} \mathbf{Hom}_{E_*E}(E_*\Sigma^k Aj, \Lambda) & 0 \leq k \leq n \\ 0 & k > n; \end{cases}$$

(8.3.2) $\pi_k^\natural q_n : \pi_k^\natural(Y(u), Aj) \to \pi_k^\natural(Y(n-1), Aj)$ is an isomorphism for $k \leq n-1$; and

(8.3.3) the natural morphism $\pi_{k-1}^\natural(Y, \Sigma Aj) \to \pi_k^\natural(Y, Aj)$ are isomorphisms for $k \leq n$.

Note that E_*T, E_*X, etc., are objects in the category of E_*E comodules. If M is an $E_*\mathcal{F}$ module over E_*X in the category of E_*E comodules we can define the André-Quillen cohomology

$$H^*_{E_*T}(E_*X, M).$$

The following is the main result. Let Λ_{*+n} denote the shifted module over Λ with $(\Lambda_{*+n})_k = \Lambda_{k-n}$.

Proposition 8.4 *The object $Y(0) \in s\mathcal{S}^T$ always exist and, having $Y(n-1)$, $n \geq 1$, there is an obstruction*

$$\theta_n \in H^{n+2}_{E_*T}(\Lambda, \Lambda_{*+n})$$

to producing $Y(n)$.

This will be proved in successive steps below. This result then gives a sequence of recursively defined obstructions to producing an $\mathcal{F}$ structure on X.

We begin by producing $Y(0)$. Since Λ is a $\pi_0 E_*T$ algebra via the augmentation, we may define

$$Y(0) = K(\Lambda)$$

as in Proposition 7.1. Then

$$\begin{aligned} \pi_0^\natural(Y(0), Aj) &\cong [Aj, Y(0)]_{s\mathcal{S}} \\ &\cong [T(Aj), Y(0)]_{s\mathcal{S}^T} \\ &\cong \mathbf{Hom}_{E_*T}(E_*T(E_*Aj), \Lambda) \\ &\cong \mathbf{Hom}_{E_*E}(E_*Aj, \Lambda) \end{aligned}$$

Furthermore, if $k \geq 1$,

$$\begin{aligned} \pi_k^\natural(Y(0), Aj) &\cong \mathbf{Hom}_{E_*T}(E_*T(E_*Aj \wedge \Delta^k/\partial\Delta^k), \Lambda) \\ &\cong \mathbf{Hom}_{E_*\mathcal{F}}(E_*, \Lambda) \\ &= 0 \end{aligned}$$

since

$$\pi_0 E_*T(E_*Aj \wedge \Delta^k/\partial\Delta^k) \cong \pi_0 E_*T(0) = E_*$$

is the initial object in $\pi_0 E_*T = E_*\mathcal{F}$ algebras. (Recall that our operads are unital.)

Now suppose $Y(n-1)$ exists. Then the spiral exact sequence for $Y(n-1)$ implies that

$$\pi_k\pi(Y(n-1), Aj) \cong \begin{cases} \mathbf{Hom}_{E_*E}(E_*Aj, \Lambda) & k = 0 \\ \mathbf{Hom}_{E_*E}(E_*\Sigma^n Aj, \Lambda) & k = n+1 \\ 0 & k \neq 0, n+1. \end{cases}$$

From this it follows that

$$\pi_k E_*Y(n-1) \cong \begin{cases} \Lambda & k = 0 \\ \Lambda_{*+n} & k = n+1 \\ 0 & k \neq 0, n+1. \end{cases}$$

Thus the simplicial E_*T algebra (in E_*E comodules) $E_*Y(n-1)$ has two non-vanishing homotopy groups. It follows immediately that the n^{th} Postnikov section $P_n E_*Y(n-1)$ is weakly equivalent to $P_0 E_*Y(n-1) \cong \Lambda$. Let

(8.5) $$\theta_n \in H^{n+2}_{E_*T}(\Lambda, \Lambda_{*+n})$$

be the k-invariant. This is the obstruction; this is verified by the following result.

Lemma 8.6 *There is a simplicial T-algebra $Y(n)$ over $Y(n-1)$ satisfying (8.2.1–8.2.3) if and only if $\theta_n = 0$.*

Proof. The k-invariant θ_n is zero if and only if there is a weak equivalence of E_*T algebras over Λ

$$E_*Y(n-1) \to K_\Lambda(\Lambda_{*+n}, n+1). \tag{8.7}$$

This, in turn, happens if and only if the projection

$$E_*Y(n-1) \to P_0E_*Y(n-1) \simeq \Lambda$$

has a homotopy section as E_*T algebras. In particular, if $Y(n)$ exists, then

$$\Lambda \simeq P_{n+1}E_*Y(n) \to P_{n+1}E_*Y(n-1) \cong E_*Y(n-1)$$

provides the desired section and, hence $\theta_n = 0$. This proves one implication.

So suppose $\theta_n = 0$. Then the weak equivalence (8.7) produces a cohomology class

$$\iota_{n+1} \in H^{n+1}(E_*Y(n-1), \Lambda_{*+n})$$

and, hence, a morphism of simplicial T-algebras

$$Y(n-1) \to H_\Lambda(\Lambda_{*+n}, n+1).$$

Here $H_\Lambda(\Lambda_{*+n}, n+1)$ is the simplicial spectrum produced in Proposition 7.1; by constrast, $K_\Lambda(\Lambda_{*+n}, n+1)$ is the simplicial E_*T algebra representing André-Quillen cohomology.

Define $Y(n)$ by the homotopy pull-back

$$\begin{tikzcd} Y(n) \arrow[r] \arrow[d, "q_n"'] & K(\Lambda) \arrow[d] \\ Y(n-1) \arrow[r, "\iota_{n+1}"] & H_\Lambda(\Lambda_{*+n}, n+1). \end{tikzcd} \tag{8.8}$$

The evident long exact sequence in $\pi_*^\natural(\cdot, Aj)$ implies that $Y(n)$ satisfies 8.3.1 and 8.3.2. To prove that 8.3.3 holds we argue as follows.

First note that the natural Hurewicz map

$$\begin{aligned} \pi_{n+1}\pi(Y(n-1), Aj) &\to \pi_{n+1}\mathbf{Hom}_{E_*E}(E_*Aj, E_*Y(n-1)) \\ &\to \mathbf{Hom}_{E_*E}(E_*Aj, \pi_{n+1}E_*Y(n-1)) \end{aligned}$$

is an isomorphism since $\pi_{n+1}E_*Y(n-1) \cong \Lambda_{*+n}$.

Second, note that the map

$$E_*\iota_{n+1} : E_*Y(n-1) \to E_*H_\Lambda(\Lambda_{*+n}, n+1)$$

is split, up to homotopy, in simplicial E_*T algebras by classifying the fundamental class in $D^{n+1}_{E_*T}E_*H_\Lambda(\Lambda_{*+n}, n+1)$:

$$E_*H_\Lambda(\Lambda_{*+n}, n+1) \to K_\Lambda(\Lambda_{*+n}, n+1).$$

Third, notice that the

$$\pi_{n+1}\pi(H_\Lambda(\Lambda_{*+n}, n+1), Aj) \to \mathbf{Hom}_{E_*E}(E_*Aj, \pi_{n+1}, E_*H_\Lambda(\Lambda_{*+n}, n+1)$$

is an isomorphism by the spiral exact sequence. Thus we get a diagram

$$\begin{tikzcd}
\pi_{n+1}\pi(Y(n-1), Aj) \arrow[r] \arrow[d, "\cong"'] & \pi_{n+1}\pi(H_\Lambda(\Lambda_{*+n}, n+1), Aj) \arrow[d, "\cong"] \\
\mathbf{Hom}_{E_*E}(E_*Aj, \pi_{n+1}E_*Y(n-1)) \arrow[r, "\cong"] & \mathbf{Hom}_{E_*E}(E_*Aj, \pi_{n+1}E_*H_\Lambda(\Lambda_{*+n}, n+1)).
\end{tikzcd}$$

The top horizontal map is now an isomorphism. There is a long exact sequence in $\pi_*\pi(\cdot, Aj)$ for (8.8) since

$$\pi(K\Lambda, Aj) \to \pi(H_\Lambda(\Lambda_{*+n}, n+1))$$

is split surjective. From this and the spiral exact sequence (8.3.3) follows. ■

A Appendix: The Miller-Hopkins Result

In this section, we show that the argument used by Haynes Miller and the second author to show that the Lubin-Tate spectrum $E = E_n$, $n \geq 1$, is an A_∞ ring spectrum actually shows that E_n is an E_∞ ring spctrum. We adapt the conventions and notation of [18], which the reader is assumed to have on hand.

Fix a prime p and spectrum $E = E_{k,\Gamma}$. Here k is a perfect field of characteristic p and Γ is a formal group of height $n \geq 1$ over k. Let $\mathcal{L}$ be the linear isometries operad. As in Proposition 6.2, choose an augmented simplicial operad

$$\epsilon : T \longrightarrow \mathcal{L}$$

so that T is Reedy cofibrant as an operad, $|T| \to \mathcal{L}$ is a weak equivalence, and E_*T is a cofibrant simplicial operad. Note that we may assume $\pi_*E_*T \to E_*\mathcal{L}$ is an isomorphism. Of course, $E_*\mathcal{L}$ is the commutative algebra operad. Thus, E_*T is a simplicial E_∞ operad in E_* modules, in the sense of [11]. The crucial calculation is the following. $F = F_{k',\Gamma'}$ be another spectrum, this one associated to a formal group over k'.

Proposition A.1 *For all $s \geq 0$ and all integers t,*

$$D^s_{E_*T}(E_*F, E_{*+t}) = 0.$$

The proof proceeds, as in [18], by a sequence of reductions. Let $m \subset E_0$ be the maxiaml ideal and let $\mathbb{K} = E_*/M \cong k[u, u^{-1}]$, where the degree of u is 2. There is a short exact sequence of E_* modules

$$0 \to \mathbb{K} \otimes_k m^n/m^{n+1} \to E_*/m^n \to E_*/m^{n+1} \to 0. \tag{A.2}$$

Lemma A.3 *Suppose that*

$$D^s_{E_*T}(E_*F, \mathbb{K}_{*+t}) = 0$$

for all $s \geq 0$ and all integers t, then

$$D^s_{E_*T}(E_*F, E_{*+t}) = 0.$$

for all $s \geq 0$ and all integers t.

Proof. Compare Corollary 19.3 of [18]. Since E_* is complete

$$E_{*+t} = \lim E_{*+t}/m^n.$$

Thus there is a short exact sequence

$$0 \to \lim{}^1 D^{s-1}_{E_*T}(E_*F, E_{*+t}/m^n) \to D^s_{E_*T}(E_*F, E_{*+t}) \to \lim D^s_{E_*T}(E_*F, E_{*+t}/m^n) \to 0.$$

There is also a long exact sequence in $D^*_{E_*T}(E_*F, \cdot)$ resulting from the short exact seqeunce of (A.2). Since

$$D^*_{E_*T}(E_*F, \mathbb{K} \otimes_k m^n/m^{n+1}) \cong D^*_{E_*T}(E_*F, \mathbb{K}) \otimes_k m^n/m^{n+1},$$

the result follows. ∎

Remark A.4 : The hypotheses that E_* be complete is crucuial to the previous result and unavoidable for this approach, The question of whether the Johnson-Wolson spectrum $E(n)$ is E_∞ or A_∞ cannot be addressed by these methods.

To reduce further, define $\mathbb{K}_*T$ by the equation

$$\mathbb{K}_*T = \mathbb{K} \otimes_{E_*} E_*T$$

and, similarly, let

$$\mathbb{K}_*F = \mathbb{K} \otimes_{E_*} E_*F = E_*F/m.$$

Then $\mathbb{K}_*T$ is an $\mathbb{K}$ operad, $\mathbb{K}_*F$ is a $\mathbb{K}_*T$ algebra and $\mathbb{K}_{*+t}$ is a $\mathbb{K}_*T$ module over $\mathbb{K}_*F$.

Lemma A.5 *: There is an isomorphism of E_* modules*

$$D^*_{E_*T}(E_*F, \mathbb{K}_{*+t}) \cong D^*_{\mathbb{K}_*T}(\mathbb{K}_*F, \mathbb{K}_{*+t}).$$

Proof. Let $X \to E_*F$ be a cofibrant model for E_*F as a simplicial E_*T module. We may assume that X is free in the sense of Quillen. Then, sence E_*T is cofibrant as an operad [11] implies that X is cofibrant as a simplicial E_* module. Since E_*F is flat over E_* (by Landweber exactness),

$$\mathbb{K} \otimes_{E_*} X \longrightarrow \mathbb{K}_* F$$

is a weak equivalence of simplicial $\mathbb{K}$ modules. More is true, since E_*T_n is a free operad,

$$X_n \cong [E_*T_n](V_n)$$

for some projective E_* module V_n. Then, by the description of free operads given in [13], we have that

$$\mathbb{K} \otimes_{E_*} X_n \cong [\mathbb{K}_*T_n](\mathbb{K} \otimes_{E_*} V_n);$$

hence, $\mathbb{K} \otimes_{E_*} X \to \mathbb{K}_*F$ is a cofibrant resolution of $\mathbb{K}_*F$ as a simplicial $\mathbb{K}_*T$ module. Since

$$\begin{aligned} \mathrm{Der}_{E_*T_n}(X_n, \mathbb{K}_{*+t}) &\cong \mathbf{Hom}_{E_*}(V_n, \mathbb{K}_{*+t}) \\ &\cong \mathbf{Hom}_{\mathbb{K}}(\mathbb{K} \otimes_{E_*} V_n, \mathbb{K}_{*+t}) \\ &\cong \mathrm{Der}_{\mathbb{K}_*T_n}(\mathbb{K} \otimes_{E_*} X_n, \mathbb{K}_{*+t}) \end{aligned}$$

the result follows. ∎

The final resuction is as follows. Let $k[T] = [\mathbb{K}_*T]_0$ be the operad over k obtained from the degree 0 part of $\mathbb{K}_*T$. The $k[T]$ is the E_∞ operad over the field k.

Lemma A.6 *There is an isomorphism*

$$D^*_{\mathbb{K}_*T}(\mathbb{K}_*F, \mathbb{K}_{*+t}) \cong D^*_{k[T]}(\mathbb{K}_0F, \mathbb{K}_t).$$

Proof. Arguing as in the previous result, if $X \to \mathbb{K}_0F$ is a free resolution of $\mathbb{K}_0F$ as a simplicial $k[T]$ algebra, then

$$\mathbb{K} \otimes_k X \longrightarrow \mathbb{K}_*F$$

is a free resolution of $\mathbb{K}_*F$ as a simplicial $\mathbb{K}_*T$ algebra. ∎

Proposition A.1 now follows from Lemmas A.3, A.5 and A.6, the spectral sequences of Propositions 8.10 and 8.11 of [11] and the following result, which is the sum of Corollary 21.3 and Corollary 21.6 of [18]. Let **Comm** be the commutative algebra operad.

Lemma A.7 $D_*^{\mathbf{Comm}}(\mathbb{K}_0F) = 0.$

∎

An immediate corollary of Proposition A.1 and Lemma 4.9 of [11] is the following, which is the necessary input for the obstruction theory of section 8.

Proposition A.8 $H^*_{E_*T}(E_*E, E_{*+t}E) = 0.$

∎

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