Slides for ICM talk

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1 Introduction

My goals in this talk are:

- To describe the current state of the calculations of $\pi_* S^n$.
- To describe some classical global results.
- To describe the current effort in getting more global results and the nature of such results.
- In particular, we will describe what we know of the current calculation in so far as it fits into some infinite families.
- To describe some new invariants which seem to help get additional structure.

Detailed calculations • EHP calculations for unstable spheres were carried out to $\pi_{j+n}(S^n)$ for all $j \leq 30$. (Oda, Toda, Mimura)

- The calculations for S^3 were done for $j \leq 52$ assisted by computer calculations.
- At 3, detailed stable calculations have been done through dimension 100.
- At 5, detailed stable calculations have been done though dimension 1000 (Ravenel).
- At 2, detailed stable calculations have been done through dimension 64 (Kochman, Mahowald, Tangora).

Global Results • (Serre) The groups, $\pi_j S^n$ are finite except if $j = n$ or if $n = 2k$ and $j = 4k - 1$.

- (Nisihida) The stable ring, $\pi_*(S^0)$, is a nil ring in positive dimensions.
- (Toda, Cohen, Moore, Neisendorfer) If $j \neq 2n + 1$ then $p^n \pi_j(S^{2n+1}) = 0$ for p an odd prime.
- (Gray) There are classes of order p^n in $\pi_*(S^{2n+1})$.
- (James) At the prime two $2^{2n}\pi_j(S^{2n+1}) = 0$ if $j \neq 2n + 1$.

Basically, the EHP sequence is a consequence of the result that

$$S^n \to \Omega S^{n+1} \to \Omega S^{2n+1}$$

is a 2 local fibration.

At odd primes there is a similar result with some twists. The spectral sequence results from the filtration:

$$S^1 \to \Omega S^2 \to \cdots \to \Omega^{n-2} S^{n-1} \to \Omega^{n-1} S^n.$$

Applying homotopy we get:

$$E_1^{s,t} = \pi_{t+1}(\Omega^{s-1} S^{2s-1}) = \pi_{t+s}(S^{2s-1}).$$

The key feature of this spectral sequence is:

Input is output of earlier calculation

In particular, once $\pi_1(S^1)$ is determined, no other outside calculation is necessary.

Here is a general result which leads to the rough exponent results mentioned above.

Theorem 1.1 *In the EHP spectral sequence for S^{2n+1} at all primes, the E_2-term is a F_p vector space for $t > 0$.*

The filtration induced on $\pi_t(S^0)$ refers to the sphere of origin of the class. That is the smallest integer s such that the homotopy is in the image of the suspension map

$$\Omega^{s-1}S^s \to \Omega^{\infty-1}S^\infty.$$

The name an element $\alpha \in \pi_t(S^n)$ projects to in this spectral sequence is called the Hopf Invariant of α.

The exponent question at the prime two needs expansion.

James implies $2^{2n}\pi_j(S^{2n+1}) = 0$.

Comparison with stunted projective spaces suggests something like

$$2^{n+1}\pi_j(S^{2n+1}) = 0.$$

Let $P(k)$ be the power map $\Omega^n S^n \to \Omega^n S^n$ of order 2^k.

The usual way to prove exponent results is to show that for some k, $P(k)$ is null. This implies $2^k\pi_*\Omega^n S^n = 0$. This approach gave the Cohen, Moore Neisendorfer result and the results of James and Toda.

Conjecture 1.2 *At the prime 2, where $P(k)$ refers to the 2^k power map acting on $\Omega^{2n+1}S^{2n+1}$ we expect:*

- *If $n \equiv -1, 0$ mod 4, then $P(n)$ is null and $P(n-1)$ is essential.*
- *If $n \equiv 1, 2$ mod 4 and $n > 1$, then $P(n+1)$ is null and $P(n)$ is essential.*
- *Among the torsion classes in $\pi_*(S^{2n+1})$, the element of maximal order is detected by K-theory.*

As opposed to the result of Gray mentioned above we have:

Theorem 1.3 *If $n \equiv -1, 0$ mod 4, then there is a homotopy class of order 2^n detected by K-theory. If $n \equiv 1, 2$ mod 4 and $n > 1$, then the maximum order among the classes detected by K-theory is 2^{n-1}.*

Since $E_1^{s,t} = \pi_{t+s}(S^{2s-1})$ it is clear that if $t < 3s - 3$ then $E_1^{s,t}$ depends only on the value of $t - s$.

In general,

$$E_r^{s,t} = E_r^{s+2^{r/2+1},t+2^{r/2+1}}$$

provided that

$$2^{r/2+1} + t < 3(s + 2^{r/2+1}) - 3.$$

This follows from the fact that

$$(\Omega^k S^{n+k}, S^n) \simeq \Sigma^n(P^{n+k-1}/P^{n-1}) := \Sigma^n P_n^{n+k-1}$$

through the stable range of the right hand side.

This allows one to describe a stable EHP spectral sequence in which

$$SE_1^{s,t} = \pi_{t-s-1}(S^0).$$

$s \in \mathbb{Z}$. It is like the Atiyah-Hirzebruch spectral sequence for the stable gadget $P_{-\infty}^{\infty}$.

It is a consequence of the theorem of Lin, which started the Segal conjecture flurry of 15 years ago, that this spectral sequence converges to $\pi_t(S_2^{-1})$, the 2 completed stable -1 sphere.

The Root Invariant of a class $\alpha \in \pi_t(S_2^{-1})$ is the representative of that class in $SE_1^{s,t}$.

Here are some examples:

- $RI(16^n\iota)$ is a class of order 2 in the $8n-1$-stem.
- $RI(\eta) = \nu$.
- For each homotopy class, α, all iterates of the Root Invariant are non-zero.
- The stem of the Root Invariant of α is at least twice the stem of α.

2 Periodic phenomenon

Let $M^k := S^{k-1} \cup_{2\iota} e^k$

Theorem 2.1 *There is a map $A : M^{15} \to M^7$ such that all composites*

$$M^{15+8k} \to M^{7+8k} \to \cdots \to M^7$$

are essential.

The proof follows immediately from the observation that $K^*(A)$ is an isomorphism.

This constructs homotopy classes by the following compositions

$$S^{14+8k} \to M^{15+8k} \to M^7 \to S^7$$

where the first map is the obvious inclusion and the last map is the pinch map.

The result of Adams shows that the composite

$$M^{15+8k} \to M^7 \to S^7 \to O(N)$$

is essential for large N. Then it is clear that the above composition is essential.

Work of Devinatz, Hopkins and Smith have clarified which complexes have what kind of self maps. The result is:

Theorem 2.2 *Let F be a finite complex and $v : \Sigma^k F \to F$. The composite*

$$\Sigma^{k \cdot j} F \to \Sigma^{k(j-1)} F \to \cdots \to F$$

is essential for all j if and only if $MU_*(v) \neq 0$ where MU_* is complex bordism theory.

At each prime MU splits into smaller spectra, BP. The homotopy of BP is $\pi_* BP = Z_{(2)}[v_i; i = 1, \ldots]$ where the dimension of v_i is $2(p^i - 1)$.

Hopkins and Smith have shown that for each i there is a finite complex F_i and self map

$$v : \Sigma^k F_i \to F_i$$

which induces multiplication by v_i^j for some j. The Adams map is "detected" by v_1^4.

We can tie this up with the root invariant constructed from the EHP spectral sequence. The result and discussion is easier at large primes.

The root invariant of $p^k \iota$ is the element of order p in the image of J in the $k \cdot q - 1$. This is the class "detected" by v_1^k in BP. Call this class α_k.

The root invariant of α_k is a class in dimension

$$2(p^2 - 1) - 2(p - 1) - 2$$

It is detected by v_2^k in BP.

Miller, Ravenel and Wilson have proposed a filtration of homotopy theory called the chromatic filtration. The Z in the zero stem is the first filtration. The elements detected by K-theory is the second filtration. One labels these filtrations by L_i.

- Let $E(n) = v_n^{-1}BP$. The chromatic tower refers to Bousfield localization with respect to $E(n)$. That is, $L_i = L_{E(i)}$.
- For any finite spectrum we have the tower

  $$\cdots \to L_n(X) \to L_{n-1}(X) \to \cdots \to L_1(X) \to L_0(X)..$$

- The inverse limit of this tower is the p-completed spectrum X.
- The example above suggests that the root invariant of a class of filtration i has filtration at least $i+1$.

By adjointing the Adams map we get

$$M^7 \to \Omega^8 M^7 \to \Omega^{18} M^7 \cdots .$$

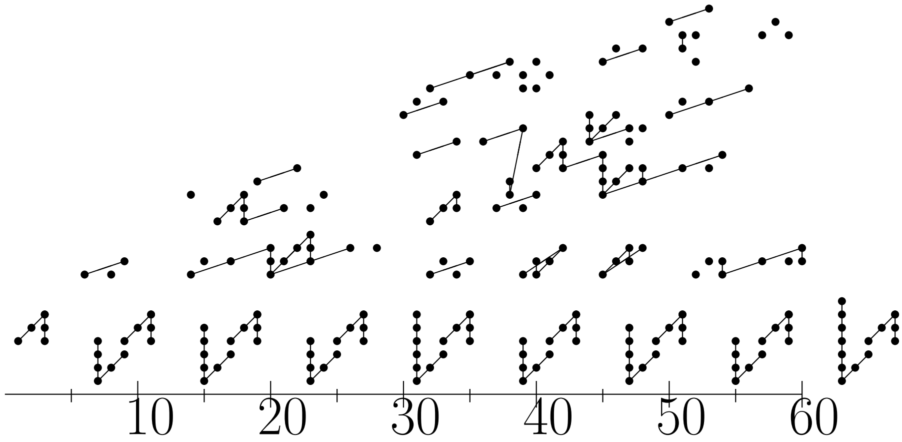
We call the periodic Ω spectrum which results out of this direct limit the telescope.

If we look at the cofiber of A, $M^7 \cup_A CM^{15}$, then the Hopkins Smith theorem says that this space admits a v_2 to some power self map. This way the process continues.

The telescope conjecture suggests that localizations with respect to these telescopes should be the same as the appropriate stage of the chromatic tower.

The telescope conjecture is correct for L_1. It seems false for other L_i. Indeed there is a conjectured proof that it is false by Mahowald, Ravenel and Shick.

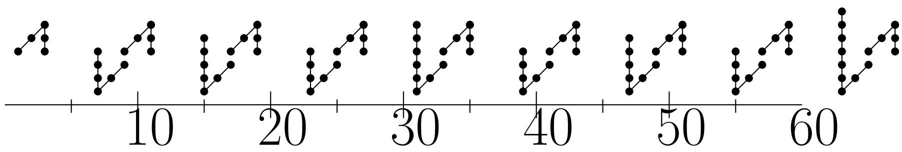
3 2-primary homotopy



The organization scheme for this arrangement is the following:

- Each dot represents a $Z/2$ in an associated graded group. Vertical lines represent group extensions. Slanting lines to the right represent compositions with η and ν.
- The image of the J homomorphism is on the bottom row. These represent the classes which can be detected by K theory.
- The Hopf invariant of a class lies to the left or below a given class.
- The Root Invariant (defined from Lin's Theorem) lies to the right or above a given class.

The image of J family



The image of J, as an infinite periodic family, is understood both stably and unstably. The pattern continues as indicated. We have the following theorem.

Theorem 3.1 • *The classes of order two in the 1 and 2 stems and the class of order 8 in the 3 stem are continued periodically with period 8.*

- *The classes of order 2 in the 8 and 9 stems are continued periodically with period 8.*
- *In the 8k-1 stem there is a class of order $2^{\nu(k)+4}$ where $\nu(k)$ is the power of 2 which divides k.*

It is easier to state the result for odd primes.

Theorem 3.2 *Let $q = 2(p-1)$. Then in the $k \cdot q - 1$ stem there is an element of order $p^{|k|_p+1}$ where $|k|_p$ is the power of p which divides k.*

This illustrates the phenomenon that a lot of information about homotopy groups is given by number theoretic statements allowing the answer for an infinite number of primes to given in a simple statement.

Unstable v_1 periodic classes

The following two results give the unstable v_1 periodic classes.

Theorem 3.3 *The Snaith map*

$$\Omega^{2n+1}S^{2n+1} \to \Omega^{\infty}S^{\infty}RP^{2n}$$

induces an isomorphism in v_1 periodic classes at the prime 2.

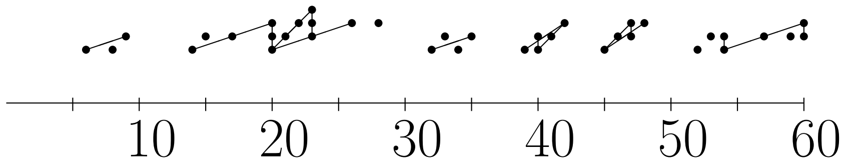
Theorem 3.4 (Thompson) *The Snaith map*

$$\Omega^{2n+1}S^{2n+1} \to \Omega^{\infty}S^{\infty}B\Sigma_p^{\langle n\cdot q\rangle}$$

induces an isomorphism in v_1 periodic homotopy at odd primes.

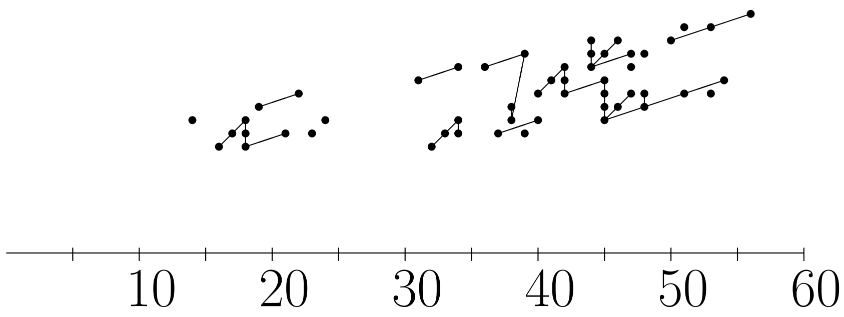
The key point is that the unstable calculation is the same as a different stable calculation. Stable calculations of this sort are comparatively easy.

The topological modular forms classes



- These classes are understood by use of the spectrum constructed by Hopkins and Miller which they label EO_2. In particular, they are essentially in the Hurewicz image.
- Hopkins and Mahowald have computed EO_{2*}.
- The v_1-torsion classes in this theory are periodic of period 192 (v_2^{32} periodic), have order 8 or less and all are killed by v_1^4.
- There is a gap in this family from dimension 162 to 198.
- There is an almost isomorphism among classes in this family between the groups in dimension k and dimension $168 - k$. This is a manifestation of Brown-Comenetz duality.

Secondary EO_2 classes



These classes fit into v_2 periodic classes and are detected by secondary EO_2 considerations.

The complete family is not known.

This much is like the e-invariant of Adams.

The classes left are not immediately a part of any infinite family.

The majority of what is known in detail is known to fit into infinite periodic families.

v_2 periodic phenomenon at other primes

Shimomura and Yabe have calculated the L_2 localization of the spheres at all primes larger than 3.

The important point is that the answer can be given for all primes simultaneously in terms of number theoretic functions.

This is analogous to what could be done for the image of J for primes greater than 2.

Unstable situation

Let X be some space (or spectrum) and let F be some functor. Then Goodwillie constructs a tower of functors

$$\begin{array}{ccccccccc} P_1F(X) & \leftarrow & P_2F(X) & \leftarrow & \cdots & \leftarrow & P_nF(X) & \leftarrow & \cdots \\ \downarrow & & \downarrow & & & & \downarrow & & \\ D_2F(X) & & D_3F(X) & & & & D_{n+1}F(X) & & \end{array}$$

Apply this to $X = S^{2n+1}$ and F be the identity functor. Arone and Mahowald show:

Theorem 3.5 *For each prime and each n, the L_n localization of S^{2k+1} can be represented by a tower of n fibrations. Each of the fibers is an infinite loop space.*

In particular, in the Goodwillie tower for this case, only the fibers, D_{p^k} are not p-locally a point. Furthermore, $L_{k-1}D_{p^k} = pt$.

This reduces the calculation of the L_i-localization of an odd sphere to $i+1$ stable questions and i fibrations.

For example, we have a fibration:

$$L_1 S^{2k+1} \to L_1 \Omega^\infty \Sigma^\infty S^{2k+1} \to L_1 \Omega^\infty \Sigma^\infty (\Sigma^{2k+1} P_{2k+1})$$

where $P_{2k+1} = RP^\infty / RP^{2k}$.

This allows an easy calculation of K-theory detected homotopy of unstable spheres.

$L_2 S^{2k+1}$ is the fiber of a two stage system. The implied homotopy is only partially worked out. The families discussed above will be a part of the story.