Realizing Commutative Ring Spectra as E_∞ Ring Spectra

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Abstract

We outline an obstruction theory for deciding when a homotopy commutative and associative ring spectrum is actually an E_∞ ring spectrum. The obstruction groups are André-Quillen cohomology groups of an algebra over an E_∞ operad. The same cohomology theory is part of a spectral sequence for computing the homotopy type of mapping spaces between E_∞ ring spectrum. The obstruction theory arises out of techniques of Dwyer, Kan, and Stover, and the main application here is to prove an analog of a theorem of Haynes Miller and the second author: the Lubin-Tate spectra E_n are E_∞ and the space of E_∞ self-maps has weakly contractible components.

Some years ago, Haynes Miller and the second author proved that the Lubin-Tate spectra E_n, which are used to govern the chromatic layers of stable homotopy theory, are A_∞ ring spectra. (See [37] for an exposition.) Heuristically, this means that E_n is associative up to all higher homotopies and, specifically, this means that that there is a model for E_n which is an algebra over an A_∞ operad. They actually proved more as they showed, in fact, that the space of A_∞ self-maps of E_n has contractible components and they computed that the monoid of path-components was isomorphic to the Morava stabilizer group S_n. This implies that there is a model for E_n upon which S_n acts, and thus, for example, one can form homotopy fixed points of E_n with respect to finite subgroups of S_n. (This can be extended, in a sensible way, to closed subgroups after further work; see [10]). The applications were immediate and striking. The spectrum EO_2 obtained from E_2 at the prime 2 by taking homotopy fixed points with respect to a maximal finite subgroup of E_2 has received a lot of attention (see [22]), as has the homotopy fixed points of E_{p-1} at a prime $p > 0$, again with respect to a maximal finite subgroup (see [31] for a surprising computation). One of the purposes of this paper is to show how one might extend the Hopkins-Miller result to show that E_n is actually an E_∞ ring spectrum – commutative and associative up to all higher homotopy – and to show that the space of E_∞ self-maps is weakly equivalent to the space of A_∞ self-maps.

At about the same time, Dwyer, Kan, and Stover, in their investigation of Π-algebras, discovered an obstruction theory for deciding when there exists a space with specified homotopy groups. The obstruction groups are a type of André-Quillen cohomology group and, as such, have good categorical properties and there exists a battery of techniques – spectral sequences and so on – for computation. Dwyer, Kan, and Stover never actually published their ideas (they are alluded in the introduction of [12]), but they have been brought into print by Blanc [5] and, in another context, by Baues [4]. A second purpose of this paper is to import the ideas of these authors into stable homotopy and to develop an obstruction theory for deciding when a spectrum can be realized as an algebra over some operad; again, the obstruction groups are a type of André-Quillen

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cohomology. This should come as no surprise – Robinson's pioneering work on A_∞ ring spectra [38], used by Hopkins and Miller – places the obstructions in Hochschild cohomology, which is a shift of André-Quillen cohomology for associative algebras (see [32]). Indeed, we suspect that our obstructions, in the A_∞ case, reduce to Robinson's, but we've not checked the details.

The third point of this paper is to outline what we mean by André-Quillen cohomology for algebras over an E_∞ operad. The ideas are dictated by needs of homotopy theory. If X is a spectrum, the free A_∞ ring spectrum $\mathcal{A}(X)$ on X has the homotopy type of $\vee_{n\geq 0} X^{\wedge n}$, so that if E_* is a homology theory with a Künneth spectral sequence and E_*X is flat over E_*, then $E_*\mathcal{A}(X)$ is isomorphic to the tensor algebra over E_* on $E_*(X)$. This basic computation underlies much of the rest of the theory. However, the free E_∞ ring spectrum $\mathcal{E}(X)$ has the homotopy type of

$$\bigvee_{n\geq 0} (E\Sigma_n)_+ \wedge_{\Sigma_n} X^{\wedge n}$$

where $E\Sigma_n$ is free contractible Σ_n space. To compute $E_*\mathcal{E}(X)$ would require, at the very least, knowledge of $E_*B\Sigma_n$ and, practically, one would need to know a great deal about the E_* Dyer-Lashof algebra. While some things are known in the case $E = E_n$ (by work of the second author, Strickland and others), we propose a different solution; namely, to resolve the E_∞ operad $\mathcal{E}$ by a simplicial operad which allows for simpler computations, at least level-wise – one might phrase this as saying that we resolve the E_∞ operad by A_∞ operads. In any case, for us, André-Quillen cohomology will mean the cohomology of simplicial algebras over simplicial operads. This extra bit of structure would seem to be a technical thicket, but it is, in fact, quite liberating, and leads to effective computations in terms of familiar objects such as ordinary André-Quillen homology of commutative rings and the usual Dyer-Lashof algebra; see section 6 below.

The mixture of André-Quillen cohomology and structured ring spectra has a long history. We cited Alan Robinson's work in the A_∞ case above. In the latter half of the 1980s, Waldhausen began studying the André-Quillen cohomology of E_∞ ring spectra and he developed a homology theory of E_∞ algebras for this program; he lectured on these topics at the Northwestern homotopy theory conference in the spring of 1988. At the same time and independently, Robinson himself was studying these topics and outlined his ideas in a lecture at the Adams Memorial Symposium in 1990. This work has started to come into print. (See [39] and [40].) In particular, Robinson and Sarah Whitehouse have a homology theory for E_∞ algebras which looks very similar to ours, although they work with differential graded objects, rather than simplicial objects. (Most attention has been to the differential graded case; notably [14] and [15].) In the early 1990s Jim McClure and Tom Hunter began doing some computations along the lines suggested by Waldhausen; their work languished while awaiting the foundational underpinnings supplied by [13]. Later, Igor Kriz [23] and Maria Basterra [3] developed an André-Quillen cohomology of E_∞ ring spectra augmented over the Eilenberg-Mac Lane spectrum $H\mathbb{Z}/p\mathbb{Z}$. At some point, we would hope that all these efforts would coalesce.

Here is an outline of the paper. It is important for us to have complete freedom of choice with our operads. This means that we choose to work with the Lewis-May-Steinberger category of spectra [28]. An outline of this theory sufficient for our purposes is supplied in section 1. Sections 2 and 3 are devoted to simplicial algebras over simplicial operads, and developing the theory of "resolution model categories" (imported from [12]) in this setting. Section 4 defines André-Quillen cohomology and presents a spectral sequence for computing the homotopy type of the space of algebra maps between structured spectra; this is Bousfield-Kan spectral sequence of a standard type available when one has good resolutions. Hopkins and Miller used an analog in their A_∞ calculations. Techniques for computation are developed in section 6 after we present the obstruction theory in section 5. Finally, in section 7 we present a criterion (due to the second author and Miller) for showing that André-Quillen cohomology vanishes. Finally, we discuss the Lubin-Tate spectra at the end of that section. Our exposition of this last topic is concise; a more detailed exposition, which, in turn, is based partly on lectures of the second author, can be found in [37].

This paper is long, but we still don't give all the details. As always with spectra, there are foundational issues to be dealt with, and the reader can imagine that this would be especially true in the case of simplicial algebras over simplicial operads over the linear isometries operad. We try to be explicit about where these arise; details will appear in [17] and [20]. One can also imagine technicalities in dealing with André-Quillen cohomology in this setting. They main problem is proving a suitable general Meyer-Vietoris sequence; this is dealt with in [19]. Finally, we are a bit glib about the Dwyer-Kan-Stover technology. There are two problems here. First, the basic paper [11] is written in a pointed context, and algebras over an operad are not pointed, and, second, the canonical resolutions – the "Stover" resolutions and their subdivisions – require care and thought. This is the point of [18].

Contents

1 Spectra and structured ring spectra

This section reviews the basics of the Lewis-May category of spectra and the theory of algebras over operads in that category. The basic references are [28] and the appendix to [13]. This is the category of choice for us for two reason: first, in the standard model category structure for these type of spectra built from topological spaces, every object is fibrant – a fact which is important for applying the Dwyer-Kan-Stover theory of resolutions [11, 12] – and, second, we wish to be flexible about operads – in particular, we wish to be to able to vary an operad within its weak equivalence class. The result that allows us to do this is 1.11 below.

To begin, a topological space is, for us, a compactly generated weak Hausdorff spaces. The category of such will be called $\mathcal{T}$. The category of pointed objects in $\mathcal{T}$ will be written $\mathcal{T}_*$. This is, of course, the category of based spaces.

For spectra, one first fixes an infinite dimensional real inner product space $\mathcal{U}$ (a "universe"). For each finite dimensional sub-inner product space $V \subset \mathcal{U}$, let S^V denote the one-point compactification. A spectrum X is then a set of pointed spaces $\{X_V\}$ indexed by the finite dimensional V together with homeomorphisms

$$\tau_{W,V} : X_V \to \mathcal{T}_*(S^{W-V}, X_W) = \Omega^{W-V} X_W$$

whenever $V \subseteq W$. One requires that $\tau_{V,V} = 1$ and whenever $U \subseteq V \subseteq W$, then $\tau_{W,V} \circ \tau_{V,U} = \tau_{W,U}$. Here the space $W - V$ is the orthogonal complement to V in W. We write $\mathcal{S}_{\mathcal{U}}$, or simply $\mathcal{S}$ for the category of spectra.

A collection of spaces $\{X_V\}$ equipped with maps

$$\sigma_{W,V} : S^{W-V} \wedge X_V \longrightarrow X_W$$

satisfying the obvious associativity condition is called a *pre-spectrum.* The forgetful functor from spectra to pre-spectra has (non-obviously) a left adjoint L called spectrification. If each of the maps $\sigma_{W,V}$ is a closed inclusion then

$$L(X)_V \cong \mathrm{colim}_{V \subseteq W}\, \Omega^{W-V} X_W.$$

The category $\mathcal{S}$ is enriched over the cateory of topological spaces in a particular way. First, the set of morphisms between two spectra X and Y has a topology as a subspace of the product of the spaces $\mathcal{T}_*(X_V, Y_V)$; we write $\mathcal{S}(X,Y)$ for this space of maps. Furthermore, if K is any space the collection of spaces

$$X^K = \{\mathcal{T}(K, X_V)\}$$

is automatically a spectrum. The functor $X \to X^K$ has a left adjoint $X \to X \otimes K$ given by the spectrification of the pre-spectrum

$$\{X_V \wedge K_+\}.$$

where K_+ is K with a disjoint basepoint. These objects are all functors of both variables and there is an adjoint relationship in $\mathcal{T}$

$$\mathcal{T}(K, \mathcal{S}(X,Y)) \cong \mathcal{S}(X \otimes K, Y) \cong \mathcal{S}(X, Y^K). \tag{1.1}$$

The bifunctors $X \otimes K$, Y^K, and $\mathcal{S}(X,Y)$ and the adjunction formulas of Equation 1.1 are the data of a *topological category*; such are similar in spirit to the simplicial categories of Quillen ([33],§II.2).

The category $\mathcal{S}_{\mathcal{U}}$ inherits a closed model category structure from the $\mathcal{T}_*$ as follows: a morphism $X \to Y$ in $\mathcal{S}_{\mathcal{U}}$ is a weak equivalence (or a fibration) if $X_V \to Y_V$ is a weak equivalence (or Serre fibration) for all V. Because the functor $X \to X_V$ has a left adjoint

$$S^{-V} \wedge Z = L(\{S^{W-V} \wedge Z\})$$

Quillen's small object ([33] §II.3) argument allows one to prove:

Theorem 1.1 *With the notion of fibration and weak equivalence defined above the $\mathcal{S}_{\mathcal{U}}$ becomes a topological model category.*

The phrase *topological model category* means that $\mathcal{S}_\mathcal{U}$ is at once a closed model category and a topological category satisfying the addditional *corner axiom* analagous to Quillen's SM7 ([33] §II.2) as follows: if $j : A \to B$ is a fibration in $\mathcal{S}_\mathcal{U}$ and $q : X \to Y$ is a fibration in $\mathcal{S}_\mathcal{U}$, then

$$j \otimes q : \mathcal{S}(B,X) \to \mathcal{S}(B,Y) \times_{\mathcal{S}(A,Y)} \mathcal{S}(A,X) \tag{1.2}$$

is a fibration in $\mathcal{T}$ which is a weak equivalence if either j or q is.

The cofibrations in $\mathcal{S}_\mathcal{U}$ are retracts of cellular maps; and the class of cellular maps is the smallest collection of maps closed under coproducts, cobase change, sequential colimits, and containing the maps

$$S^{-V} \wedge S^{n-1} \to S^{-V} \wedge D^n$$

for finite V and $n \geq 1$. Here S^n and D^n are the standard sphere and ball in $\mathcal{T}_*$. One can conclude that $\mathcal{S}_U$ is cofibrantly generated.

A spectrum X so that the unique map $* \to X$ is a cellular map is a *cellular spectrum.* It is a finite cellular spectrum if it obtained by attaching only finitely many $S^{-V} \wedge D^n$. Every cellular spectrum is the filtered colimit of its finite cellular sub-spectra.

In citing Quillen's small object argument above we meant to imply that the proof of 1.1 was standard; nonetheless, it is not trivial. It is not immediately obvious, for example, that $\mathcal{S}_\mathcal{U}$ has colimits. And the proof that the small object argument works uses subleties of the spectrification functor. See the appendix to [28].

Also less than obvious is the following. We take as our standard stable homotopy category the one described by Adams [1].

Theorem 1.2 *The homotopy category obtained from the closed model category structure on $\mathcal{S}_\mathcal{U}$ is equivalent to the stable homotopy category.*

The next important item to discuss is the role of the universe $\mathcal{U}$. This leads naturally to the introduction of operads.

If $\mathcal{U}$ and $\mathcal{V}$ are two infinite dimensional real inner product spaces, let $\mathcal{L}(\mathcal{U},\mathcal{V})$ denote the space of linear isometries from $\mathcal{U}$ to $\mathcal{V}$. In an important insight in this theory is that this space is contractible, which can be easily seen by a dimension doubling trick. (See [28])

If $f : \mathcal{U} \to \mathcal{V}$ is a linear isometry (i.e., a point in $\mathcal{L}(\mathcal{U},\mathcal{V})$), the functor $f^* : \mathcal{S}_\mathcal{V} \to \mathcal{S}_\mathcal{U}$ given by

$$(f^*X)_V = X_{f^{-1}V}$$

has a left adjoint f_* and this adjoint pair induces a Quillen equivalence of homotopy categories

$$\mathbf{H}o(S_\mathcal{U}) \cong \mathbf{H}o(S_\mathcal{V})).$$

Because $\mathcal{L}(\mathcal{U},\mathcal{V})$ is contractible, this equivalence is independent of the choice of f. So the homotopy theory is independent of the universe in a canonical way. However, this is an avatar of a much richer structure which we now delineate.

The first observation, is that the functors f^* and f_* can be assembled continuously. This is one of the most important insights of [28] and worth separating out.

Proposition 1.3 *Let $q : A \to \mathcal{L}(\mathcal{U},\mathcal{V})$ be a space over the space of linear isometries. Then there is a functor*

$$A \otimes (\cdot) : \mathcal{S}_\mathcal{U} \to \mathcal{S}_\mathcal{V}$$

which has a right adjoint $X \mapsto X^A$ with the following properties

1. *These functors are continuous bifunctors of their respective arguments;*

2. *If $A = \{*\}$ is a point and $f = q(*)$, then $A \otimes X$ and X^A are naturally isomorphic to f_*X and f^*X respectively.*

The object $A \otimes X$ is the "half-smash product". One should really write $q \otimes X$ for $A \otimes X$, but, in practice, the map q is understood. The phrase *continuous functor* means that the map on morphism sets

$$\mathcal{S}_{\mathcal{U}}(X, Y) \longrightarrow \mathcal{S}_{\mathcal{V}}(A \otimes X, A \otimes Y)$$

is continuous.

One can ask if the functor $(\cdot) \otimes X : \mathcal{T}/\mathcal{L}(\mathcal{U}, \mathcal{V}) \to \mathcal{S}_{\mathcal{V}}$ has a right adjoint, and the answer is yes.

Proposition 1.4 *Given $X \in \mathcal{S}_{\mathcal{U}}$ and $Y \in \mathcal{S}_{\mathcal{V}}$ there is a space of maps $\mathcal{S}(X, Y)$ and a continuous map*

$$\mathcal{S}(X, Y) \to \mathcal{L}(\mathcal{U}, \mathcal{V})$$

so that

1. *$\mathcal{S}(X, Y)$ is a continuous bifunctor;*

2. *There are natural adjoint isomorphisms*

$$\mathcal{S}_{\mathcal{U}}(X, Y^A) \cong [\mathcal{T}/\mathcal{L}(\mathcal{U}, \mathcal{V})](A, \mathcal{S}(X, Y)) \cong \mathcal{S}_V(A \otimes X, Y);$$

3. *The corner axiom holds: if $j : X \to Y$ is a cofibration is cofibration in $\mathcal{S}_{\mathcal{U}}$ and $q : E \to B$ is a cofibration in $\mathcal{S}_{\mathcal{V}}$ then*

$$\mathcal{S}(Y, E) \longrightarrow \mathcal{S}(Y, B) \times_{\mathcal{S}(X,B)} \mathcal{S}(X, E)$$

is a a fibration of spaces over $\mathcal{L}(\mathcal{U}, \mathcal{V})$ which is a weak equivalence if either j or q is.

This extension of the model category structure is proved in [17]. The various adjoint relationships in this result lead to evaluation maps. One we will use often involves the half-smash product. If $A \to \mathcal{L}(\mathcal{U}, \mathcal{V})$, $X \in \mathcal{S}_{\mathcal{U}}$ and $Y \in \mathcal{S}_{\mathcal{W}}$ then there is an evaluation map

$$\mathcal{S}(A \otimes X, Y) \times A \to \mathcal{S}(X, Y) \tag{1.3}$$

covering the composition

$$\mathcal{L}(\mathcal{V}, \mathcal{W}) \times \mathcal{L}(\mathcal{U}, \mathcal{V}) \to \mathcal{L}(\mathcal{U}, \mathcal{W}).$$

There was one more important fact about this structure, which describes behavior with respect to composition of change of universes. If $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are all universes, and $A \to \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $B \to \mathcal{L}(\mathcal{V}, \mathcal{W})$ are given, then $B \times A$ is a space over $\mathcal{L}(\mathcal{U}, \mathcal{W})$ via the composition map:

$$B \times A \to \mathcal{L}(\mathcal{V}, \mathcal{W}) \times \mathcal{L}(\mathcal{U}, \mathcal{V}) \to \mathcal{L}(\mathcal{U}, \mathcal{W}).$$

We then have the following result.

Proposition 1.5 *If $X \in \mathcal{S}_{\mathcal{U}}$ there is a natural isomorphism of spectra in $\mathcal{S}_{\mathcal{W}}$*

$$(B \times A) \otimes X \cong B \otimes (A \otimes X).$$

Furthermore, this isomorphism is associative and has the identity maps $1_{\mathcal{U}} : * \to \mathcal{L}(\mathcal{U},\mathcal{U})$ as a two-sided unit.

This has the following consequence, as a result of the adjointness expressed in 1.4.

Proposition 1.6 *If $X \in \mathcal{S}_U$, $Y \in \mathcal{S}_V$, and $Z \in \mathcal{S}_{\mathcal{W}}$, then there is a composition law*

$$\mathcal{S}(Y,Z) \times \mathcal{S}(X,Y) \to \mathcal{S}(X,Z)$$

covering the composition

$$\mathcal{L}(\mathcal{V},\mathcal{W}) \times \mathcal{L}(\mathcal{U},\mathcal{V}) \to \mathcal{L}(\mathcal{U},\mathcal{W}).$$

Furthermore this compostition is associative and has the identity $1_X \in \mathcal{S}(X,X)$ as a two-sided inverse.

This brings us to smash products. If $X \in \mathcal{S}_{\mathcal{U}}$ and $Y \in \mathcal{S}_V$ then $X \wedge Y$ is spectrum in $\mathcal{S}_{\mathcal{U}\times\mathcal{V}}$ defined by taking the spectrum associated to the (partial) pre-spectrum $\{X_U \wedge Y_V\}$. The smash product is associative and and symmetric in the sense that if $\sigma : \mathcal{U} \times \mathcal{V} \to \mathcal{V} \times \mathcal{U}$ is the factor switching map, then there is a natural isomorphism

$$\sigma_*(X \wedge Y) \cong Y \wedge X.$$

In particular, if $X \in \mathcal{S}_{\mathcal{U}}$, then the n-fold iterate of the smash product $X^{(n)}$ is a spectrum in $\mathcal{S}_{\mathcal{U}^n}$ and it has an action of the symmetric group Σ_n in the following sense: if $\Sigma_n \to \mathcal{L}(\mathcal{U}^n,\mathcal{U}^n)$ is the embedding sending a permuation to the factor permutation, then there is a natural map

$$\Sigma_n \otimes X^{(n)} \to X^{(n)}.$$

This, in general is what is meant by an equivariant spectrum: if the universe $\mathcal{U}$ has an action by a group G through linear isometries (i.e., there is a monoid map $G \to \mathcal{L}(\mathcal{U},\mathcal{U})$, then a G-spectrum is an object $X \in \mathcal{S}_{\mathcal{U}}$ equipped with a map

$$G \otimes X \to X$$

such that the obvious unital and associative conditions are satisfied. We can write $\mathcal{S}^G_{\mathcal{U}}$ for the category of G spectra.

If $X \in \mathcal{S}^G_{\mathcal{U}}$ is a G spectrum and $Y \in \mathcal{S}_{\mathcal{V}}$ is a spectrum, we want to know what kind of structure the mapping object $\mathcal{S}(X,Y)$ has. First, note that $\mathcal{L}(\mathcal{U},\mathcal{V})$ has a G action by operating on $\mathcal{U}$. Then $\mathcal{S}(X,Y)$ is a G space over $\mathcal{L}(\mathcal{U},\mathcal{V})$: the action map

$$\mathcal{S}(X,Y) \times G \to \mathcal{S}(X,Y)$$

is composition

$$\mathcal{S}(X,Y) \times G \to \mathcal{S}(G \otimes X,Y) \times G \to \mathcal{S}(X,Y).$$

The first map is induced by the action $G \otimes X \to X$ and the last is the evaluation map of Equation 1.3.

As an example of the phenomenon of the previous paragraph, we note that if $X \in \mathcal{S}_{\mathcal{U}}$, then $\mathcal{S}(X^{(n)}, X)$ is a Σ_n space over $\mathcal{L}(\mathcal{U}^n,\mathcal{U})$.

Operads arise naturally at this point. For example, if we define $\mathcal{L}(\mathcal{U}^0,\mathcal{U})$ to be a point, then the collection $\mathcal{L} = \{\mathcal{L}(\mathcal{U}^n,\mathcal{U})\}_{n\geq 0}$ is an operad under composition. This is the *linear isometries operad*, and it depends in $\mathcal{U}$, although $\mathcal{U}$ is usually left understood. It is an E_∞ operad, as each of the spaces is the operad is contractible and has a free action by the appropriate symmetric group. It is also a *unital* operad, as the zeroth space is a point, the terminal object in the category of spaces. If $X \in \mathcal{S}_{\mathcal{U}}$ is a spectrum, we again define $\mathcal{S}(X^{(0)}, X)$ to be a point. Then the above considerations imply that the collection of spaces

$$\mathcal{E}_X = \{\mathcal{S}(X^{(n)}, X\}$$

is a unital operad over the linear isometries operad. This is the endomorphism operad of X.

Given operad T over the linear isometries operad, we'd like to define what it means for a spectrum X to be T algebra. This definition should yield that X is an $\mathcal{E}_X$ algebra in a natural way – every object should be an algebra over its endomorphism operad. For this we need two contructions.

First is G acts on $\mathcal{U}$ and $X \in \mathcal{S}^G_{\mathcal{U}}$ is a G-spectrum and $A \to \mathcal{L}(\mathcal{U},\mathcal{V})$ is G-space over the linear isometries we define $A \otimes_G X$ to the the coequalizer of the two maps

$$(A \times G) \otimes X \xrightarrow{\mu\times 1} A \otimes X$$

and

$$(A \times G) \otimes X \cong A \otimes (G \otimes X) \xrightarrow{1\otimes\mu} A \otimes X.$$

Second if A is a space we may regard A as a space over $\mathcal{L}(\mathcal{U}^0,\mathcal{U}) = *$. If X is pointed space we may (formally) regard X as a $\mathcal{U}^0 = 0$ spectrum. Then we define the spectrum $A \otimes X \in \mathcal{S}_{\mathcal{U}}$ as the suspension spectrum of $A_+ \wedge X$. One of the remarkable observations of [28] is that this works; that is, the appropriate extensions of 1.3 and 1.4 hold. In particular, if X is a pointed space and $A = *$, then $* \otimes X$ is the suspension spectrum of X and if B is a space over $\mathcal{L}(\mathcal{U},\mathcal{V})$ then the equation

$$B \otimes (* \otimes X) \cong (B \times *) \otimes X \cong B \otimes X,$$

or writing the suspension spectrum as $\Sigma^\infty X$,

$$B \otimes \Sigma^\infty X \cong \Sigma^\infty(B_+ \wedge X).$$

This has application. For example, in [9], one finds a very simple proof of the Snaith splittings using these ideas.

In any case, we can now form free algebras over an operad. Let $T = \{T(n)\}$ be an operad over the linear isometries operad and $X \in \mathcal{S}_{\mathcal{U}}$ a spectrum. Define

$$T(X) = \bigvee_{n\geq 0} T(n) \otimes_{\Sigma_n} X^{(n)}.$$

This is a spectrum in $\mathcal{S}_{\mathcal{U}}$. Because T is an operad, the functor $X \mapsto T(X)$ is a monad (or triple) on the category $\mathcal{S}_{\mathcal{U}}$. The algebras over this monad are the T-algebras, and the category of such is written $\mathcal{S}^T$ or $\mathcal{S}^T_{\mathcal{U}}$ as needed. An algebra of the form $T(X)$ is a free T-algebra. Note that $T(*)$ is the suspension spectrum of $T(0)_+$. It is the initial object in $\mathcal{S}^T$.

Two operads have gotten a lot of attention. The first is the linear isometries operad itself; the resulting category of algebras is the category of E_∞ ring spectra. The second is the operad $\mathcal{A}_\infty = \{\mathcal{L}(\mathcal{U}^n, \mathcal{U}) \times \Sigma_n\}$. The resulting category of algebras is the category of A_∞ ring spectra. Note that in the homotopy category of spectra one might write

$$\mathcal{A}_\infty(X) \cong \bigvee_{n \geq 0} \mathcal{L}(\mathcal{U}^n, \mathcal{U}) \otimes X^{(n)} \cong \bigvee_{n \geq 0} X^{(n)},$$

so that $\mathcal{A}_\infty(X)$ is the analog of the tensor algebra on X.

The category $\mathcal{S}^T$ inherits a topological model category structure from $\mathcal{S}_\mathcal{U}$. There are some technical difficulties in proving this fact. The first is that one must show that $\mathcal{S}^T$ actually has colimits. This actually a general fact about categories of algebras over a monad ([2]), but we'd like to highlight one point. It's enough to prove the existence of coproducts and reflexive coequalizers. Coproducts are simple. A reflexive pair of maps is one that can be made to look like the bottom of a simplicial object; that is, one has maps

$$f, g : X \to Y$$

and one can choose a map $s : Y \to X$ so that $fs = gs = 1$.

Lemma 1.7 *Suppose $f, g : X \to Y$ is a reflexive pair in $\mathcal{S}^T$. Then the coequalizer Z in $\mathcal{S}_\mathcal{U}$ has a unique and natural structure as a T-algebra so that $Y \to Z$ is a morphism of T-algebras and, with that stucture, Z is the coequalizer in $\mathcal{S}^T$.*

In brief, the forgetful functor makes reflexive coequalizers. The core of the argument is to show that the functor $T(\cdot) : \mathcal{S}_\mathcal{U} \to \mathcal{S}_\mathcal{U}$ preserves reflexive coequalizers. It's sufficient to show that the functor $X \mapsto X^{(n)}$ has this property, in turn, follows from the fact that the smash product on compactly generated weak Hausdorff spaces has the right property.

It follows from Lemma 1.7 that $\mathcal{S}^T$ has the structure of a topological category in the sense defined above. Indeed, this structure is stipulated by requiring that if $X \in \mathcal{S}_\mathcal{U}$ and $K \in \mathcal{T}$ then there is a natural isomorphism

$$T(X) \otimes K \cong T(X \otimes K).$$

Note that the forgetful functor $\mathcal{S}^T \to \mathcal{S}_\mathcal{U}$ and its left adjoint $X \mapsto T(X)$ are both continuous.

Then one can prove

Theorem 1.8 *Define a morphism $f : X \to Y$ in $\mathcal{S}^T$ to be a weak equivalence or fibration if the underlying morphism of spectra is a weak equivalence or fibration. Then $\mathcal{S}^T$ becomes a simplicial model category.*

The same techniques that apply to prove Lemma 1.7 can be applied to show the following result, which is of central importance. Note that geometric realization of simplicial objects exist in any topological category.

Proposition 1.9 *Let $X_\bullet$ be a simplicial object in $\mathcal{S}^T$. Then the geometric realization $|X_\bullet|$ of $X_\bullet$ in spectra has a natural structure as a T-algebra, and with that structure, it is isomorphic to the geometric realization of $X_\bullet$ in $\mathcal{S}^T$.*

There is a serious warning that accompanies this result. One needs some condition on a simplicial object in order to be able to calculate the homotopy type of the geometric realization. Typically one requires that $X_\bullet$ be "proper" or "Reedy cofibrant". The latter requirement, for example, means the following. Define

$$L_n X_\bullet = \operatorname*{colim}_{\phi:[n]\to[m]} X_m$$

where ϕ runs over the non-identity surjections in the ordinal number category. There is a natural map $L_n X \to X_n$ and $X_\bullet$ is Reedy cofibrant if these maps are cofibrations. The standard result [35] is that geometric realization preserves level-wise weak equivalences between Reedy cofibrant objects; therefore, in any category, one restricts attention to Reedy cofibrant simplicial objects. In spectra one can prove that for Reedy cofibrant objects $X_\bullet$ there is a spectral sequence

$$\pi_i \pi_j X_\bullet \Longrightarrow \pi_{i+j}|X_\bullet|.$$

However, a cofibration of T-algebras need not be a cofibration of spectra, so a Reedy cofibrant simplicial T-algebra need not be a Reedy cofibrant simplicial spectrum, and one cannot necessarily use the above spectral sequence to compute the homotopy type of the geometric realization of a simplicial T-algebra. Much of the work of [13], for example, is expended to come to terms with issues of this sort.

Our way out of this conundrum is to vary the operad. The category of operads over the linear isometries operad is itself a topological model category. A morphism of $T \to S$ of operads is a weak equivalence or fibration if each of the morphisms of spaces $T(n) \to S(n)$ is a weak equivalence or fibration. The underlying technical result we use is

Lemma 1.10 *Let T be a cofibrant operad over the linear isometries operad. Then any cofibration of T-algebras is a cofibration of spectra.*

The difficulty with this result is that the linear isometries operad is not cofibrant. (It's close, in the sense that it behaves much like a cofibrant operad; see the appendices to [13].) Thus, we are interested in studying the homotopy category of algebras over non-cofibrant operads. If we replace a non-cofibrant operad by a cofibrant model, we may change the homotopy category. The following result, in some sense the main theorem of [17], proves that this is not the case.

Theorem 1.11 *Let $T \to S$ be a morphism of operads over the linear isometries operad. Then the restriction of scalars functor $\mathcal{S}^S \to \mathcal{S}^T$ has a left ajoint*

$$S \otimes_T (\cdot) : \mathcal{S}^T \to \mathcal{S}^T.$$

These functors have the following properties:

1. *they are continuous;*
2. *they are a Quillen pair and induce adjoint functors on the homotopy categories; and*
3. *if $T \to S$ is a weak equivalence of operads, they induce an equivalence of homotopy categories $\mathbf{H}o(\mathcal{S}^T) \cong \mathbf{H}o(\mathcal{S}^S)$.*

This result is not formal, and fails in other categories. It turns on two facts. The first is that if $T \to \mathcal{L}$ is an operad over the linear isometries operad, then each of the spaces $T(n)$ has a free Σ_n action, simply because $\mathcal{L}(\mathcal{U}^n, \mathcal{U})$ does. The second turns on subtleties of the functor

$$X \mapsto T(n) \otimes_{\Sigma_n} X^{(n)}.$$

This functor does not necessarily preserve cofibrations, but it does send cofibrations to maps that a "flat" in a certain categorical sense. This fact, in turn, comes down to facts about the Thom space construction, which is at the heart of the construction of the half-smash product.

2 Simplicial spectra over simplicial operads

Simplicial objects have often been used to build resolutions. That is our main point here, also. However, given an algebra X in spectra over some operad, we will resolve not only X, but the operad as well. The main results of this section are that if X is a simplicial algebra over a simplicial operad T then the geometric realization $|X|$ is an algebra over the geometric realization $|T|$ and, furthermore, that geometric realization preserves level-wise weak equivalences between Reedy cofibrant objects, appropriately defined.

Most of the material in this section is in [20].

Let's begin by talking about the kind of simplicial operad we'll be interested in. As mentioned in the previous section, the category of operads is a topological model category, where $T \to S$ is a weak equivalence or fibration if each of the maps $T(n) \to S(n)$ is. The forgetful functor to graded spaces has a left adjoint, called the free operad functor (see [14] or [36]), which in turn is built from disjoint unions of finite products. Thus the small object argument applies. From this one gets the Reedy model category structure on simplicial operads ([35]). Weak equivalences are level-wise and cofibrations are defined using the latching objects of the previous section. This, again, has the property that geometric realization preserves weak equivalences between cofibrant objects. It also has a structure as a topological model category; for example if T is a simplicial operad and K is space, then

$$T^K = \{T_n^K\}.$$

Let us first spell out the kind of simplicial operads we want. If E_* is a generalized homology theory and $\mathcal{F}$ is an operad over the linear isometries operad, one might like to compute $E_*\mathcal{F}(X)$. As mentioned in the introduction, this is usually quite difficult, unless E_*X is projective as an E_* module and $\pi_0\mathcal{F}(q)$ is a free Σ_q set for all q. Thus we'd like to resolve $\mathcal{F}$ using operads of this sort.

If T is a simplicial operad over the linear isometries operad and E is a commutative ring spectrum in the homotopy category of spectra, then E_*T is an operad in the category of E_* modules. The category of simplicial operads in E_* modules has a simplicial model category structure in the sense of §II.4 of [33], precisely because there is a free operad functor. Cofibrant objects are retracts of diagrams which are "free" in the sense of [33]; meaning the underlying degeneracy diagram is a diagram of free operads.

Given an operad $\mathcal{F}$ over the linear isometries operad, we'd like to consider simplicial operads T of the following sort:

Theorem 2.1 *Let $\mathcal{F}$ be an operad over the linear isometries operad. Then there exists an augmented simplicial operad over the linear isometries operad*

$$T \longrightarrow \mathcal{F}$$

1. *T is Reedy cofibrant as a simplicial operad;*
2. *For each $n \geq 0$ and each $q \geq 0$, $\pi_0 T_n(q)$ is a free Σ_q set;*
3. *The map of operads $|T| \to \mathcal{F}$ induced by the augmentation is a weak equivalence;*
4. *If $E_*\mathcal{F}(q)$ is projective as an E_* module for all q, then E_*T is cofibrant as a simplicial operad in E_* modules and $E_*T \to E_*F$ is a weak equivalance of operads in that category.*

This theorem is not hard to prove, once one has the explicit construction of the free operad given in terms of trees as given in [14]. Indeed, if $F_{\mathcal{O}}$ is the free operad functor on graded spaces, one may take T to be the standard cotriple resolution, using $F_{\mathcal{O}}$ of a cofibrant model for $\mathcal{F}$.

Now we fix a simplicial operad $T = \{T_n\}$ over the linear isometries operad. Since the free algebra functor $X \mapsto S(X)$ is natural in X and the operad S, if X is a simplicial spectrum, so is $T(X)$. Hence a simplicial algebra in spectra over T is a simplicial spectrum X equipped with a multiplication map

$$T(X) \longrightarrow X$$

so that the usual associativity and unit diagrams commute. In particular, if $X = \{X_n\}$, then each X_n is a T_n algebra. Let $s\mathcal{S}^T$ be the category of simplicial T algebras.

The category $s\mathcal{S}^T$ is a topological model category. Recall that given a morphism of operads $\mathcal{E} \to \mathcal{F}$, then the restriction of operads functor $\mathcal{S}^{\mathcal{F}} \to \mathcal{S}^{\mathcal{E}}$ is continuous. This implies that if $K \in \mathcal{T}$ and $X \in s\mathcal{S}^T$, we may define $X \otimes K$ and X^K level-wise; for example,

$$X \otimes K = \{X_n \otimes K\}.$$

We could use this structure to define a geometric realization functor; however, we prefer to proceed as follows.

If $\mathcal{C}$ is a topological category, then the geometric realization functor $|\cdot| : s\mathcal{C} \to \mathcal{C}$ has a right adjoint

$$Y \mapsto Y^{\Delta} = \{Y^{|\Delta^n|}\}.$$

where $|\Delta^n|$ is the standard topological n-simplex.[1] In particular, this applies to simplical operads, and we are interested in the unit of the adjunction $T \to |T|^{\Delta}$. If S is any operad over the linear isomeries operad and Y is an S-algebra, then for all spaces $K \in \mathcal{T}$, the spectrum Y^K is an S^K algebra. From this it follows that Y^{Δ} is a simplicial S^{Δ} algebra. Setting $S = |T|$ and restricting scalars defines a functor

$$Y \mapsto Y^{\Delta} : \mathcal{S}^{|T|} \longrightarrow s\mathcal{S}^T.$$

The result we want is the following. It is an extension of, and depends on, Proposition 1.9.

[1] We need the notation Δ^n for the simplicial simplex later.

Theorem 2.2 *Let T be a simplicial operad and $X \in s\mathcal{S}^T$ a simplicial T-algebra. Then the geometric realization $|X|$ of X as a spectrum has a natural structure as a $|T|$ algebra and, with this structure, the functor*

$$X \mapsto |X|$$

is right adjoint to $Y \mapsto Y^\Delta$.

In light of Theorem 2.1 and Theorem 1.11, this gives a very useful tool for creating homotopy types of algebras over operads.

The next item to study is the homotopy invariance of the geometric realization functor. The usual result has been cited above: realization preserves level-wise weak equivalences between Reedy cofibrant objects. The same result holds in this case, but one must take some care to define "Reedy cofibrant". The difficulty is this: the definition of Reedy cofibrant involves the colimit

$$L_n X = \operatorname*{colim}_{\phi:[n]\to[m]} X_m$$

and one must define this colimit if each of the X_m is an algbera over a different operad. The observation needed is the following. Let $S : I \to \mathcal{O}$ be a diagram of operads. Then an I diagram of S algebras in an I diagram $X : I \to \mathcal{S}$ of spectra equipped with a natural tranformation of I diagram

$$S(X) \to S$$

satisfying the usual associativity and unital conditions. For example if $I = \Delta^{op}$ one recovers simplicial S algebras. Call the category of such $(\mathcal{S})^S$. (This is a slight varaition on the notation $s\mathcal{S}^T$,) Then one can form the colimit operad $\operatorname{colim} S = \operatorname{colim}_I S$ and there is a constant diagram functor

$$(\mathcal{S})^{\operatorname{colim} S} \longrightarrow (\mathcal{S})^S$$

sending X to the I diagram $i \mapsto X$ where X gets and S_i structure via restriction of scalars along

$$S_i \longrightarrow \operatorname*{colim}_I S.$$

Lemma 2.3 *This constant diagram functor has a left adjoint*

$$X \to \operatorname{colim}_I X.$$

Despite the notation, $\operatorname{colim}_I X$ is not the colimit of X as an I diagram of spectra; indeed, if $X = S(Y)$ where Y is an I-diagram of spectra

$$\operatorname{colim}_I X \cong (\operatorname{colim}_I S)(\operatorname{colim}_I Y).$$

If T is a simplicial operad we can form

$$L_n T = \operatorname*{colim}_{\phi:[n]\to[m]} T_m$$

where ϕ runs over the non-identity surjections in Δ. There is a natural maps $L_nT \to T_n$ of operads. If X is a simplicial T algebra we extend this definition slightly and define

$$L_n X = T_n \otimes_{L_n X} \operatorname*{colim}_{\phi:[n]\to[m]} X_m$$

where, again, ϕ runs over the non-identity surjections in Δ. In short we extend the scalars to make L_nX a T_n algebra and the natural map $L_nX \to X_n$ a morphism of T_n algebras.

With this construction at hand one can make the following definition. Let T be a simplicial operad and $f : X \to Y$ a morphism of simplicial T-algebras. Then f is a level-wise weak equivalence (or *Reedy weak equivalence*) if each of the maps $X_n \to Y_n$ is a weak equivalence of T_n algebras – or, by definition, a weak equivalence as spectra. The morphism f is a Reedy cofibration if the morphism of L_nT algebras

$$L_nY \sqcup_{L_nX} Y_n \longrightarrow Y_n$$

is a cofibration of T_n algebras. The coproduct here occurs in the category of T_n algebras. The main result is then:

Theorem 2.4 *With these definitions, and the topological structure defined above, the category $s\mathcal{S}^T$ becomes a topological model category. Furthermore*

1. *The geometric realization functor $|\cdot| : s\mathcal{S}^T \to \mathcal{S}^{|T|}$ sends level-wise weak equivalences between Reedy cofibrant objects to weak equivalences; and*
2. *if T is Reedy cofibrant as an operad, then any Reedy cofibration in $s\mathcal{S}^T$ is a Reedy cofibration of simplicial spectra.*

The importance of the second item in this result is that, in light of Theorem 2.2, one can calculate the homotopy type of the geometric realization of a T-algebra entirely in spectra.

3 Stover resolutions and cofibrant algebras over operads

Building on the results of the last section, we'd like to assert the following. Let X be a simplicial algebra over T, and suppose T satisfies all the conclusions of Theorem 2.1. Then there is a simplicial T-algbera Y and a morphism of T-algebras $Y \to X$ so that a.) $|Y| \to |X|$ is a weak equivalence and b.) E_*Y is cofibrant as an E_*T algebra. The device for this construction is an appropriate Stover resolution, [42],[11],[12]. We explain some of the details in this section. See also [18] and [20].

We begin by specifying the building blocks of our resolutions. We fix a spectrum E which is a commutative ring object in the homotopy category of spectra. Let $D(\cdot)$ denote the Spanier-Whitehead duality functor.

Definition 3.1 *A homotopy commutative and associative ring spectrum E satisfies Adams's condition if E can be written, up to homotopy, as a colimit of finite cellular spectra E_α with the properties that*

1. *E_*DE_α is projective as an E_* module; and*
2. *for every module spectrum F over E the Künneth map*

$$[DE_\alpha, F] \longrightarrow Hom_{E_*}(E_*DE_\alpha, F_*)$$

is an isomorphism.

This is the condition Adams wrote down in [1] to guarantee that the (co-)homology theory over E has Künneth spectral sequences. If F is a module spectrum over E, then so is every suspension or desuspension of E; therefore, one could replace the source and target of the map in this definition by the corresponding graded objects.

Many spectra of interest satisfy this condition; for example, if E is Landweber exact, it holds. (This is implicit in [1], and made explicit in [37].)

We want to use the spectra DE_α as detecting objects for a homotopy theory, but first we enlarge the scope a bit.

Definition 3.2 *Define $\mathcal{A}(E) = \mathcal{A}$ to be a set of finite cellular spectra so that*

1. *The spectrum $S^0 \in \mathcal{A}$ and E_*X is projective as an E_* module for all $X \in \mathcal{A}$;*
2. *For each α there is finite cellular spectrum homotopy equivalent to DE_α in $\mathcal{A}$;*
3. *$\mathcal{A}$ is closed under suspension and desuspension; and,*
4. *for all $X \in \mathcal{A}$ and all E module spectra F the Künneth map*

$$[X, F] \longrightarrow Hom_{E_*}(E_*X, F_*)$$

is an isomorphism.

In [12] it was pointed out that there are two ways to use the spectra of $\mathcal{A}$ to put a closed model category structure on simplicial spectra. We will phrase the results so that they are very close to the familiar examples obtained in the simplicial universal algebra case by Quillen [33], §II.4.

Because the category of spectra has all limits and colimits, the category of simplicial spectra is a simplicial category in the sense of Quillen. For example, if K is a simplicial set and X is a simplicial spectrum, then

$$(X \otimes K)_n = \vee_{K_n} X_n$$

and the simplicial set of maps $\mathrm{map}(X, Y)$ between two simplicial spectra has

$$\mathrm{map}(X, Y)_n = \mathrm{Hom}_{s\mathcal{S}}(X \otimes \Delta^n, Y),$$

by which we mean the set underlying the space $s\mathcal{S}(X, Y)$. However, the Reedy model category structure on simplicial spectra is not a simplicial model category; for example, if $i : X \to Y$ is a Reedy cofibration and $j : K \to L$ is a a cofibration of simplicial sets, then

$$i \otimes j : X \otimes L \sqcup_{X \otimes K} Y \otimes K \to Y \otimes L$$

is a Reedy cofibration, it is a level-wise weal equivalence if i is, but it is not a level-wise weak equivalence if j is. We use the objects of $\mathcal{A}$ to repair this.

Definition 3.3 *Let $f : X \to Y$ be a morphism of simplicial spectra. Then*

1. *the morphism f is an $\mathcal{A}$-fibration if it is a Reedy fibration and the morphism of simplicial mapping spaces*

$$\mathrm{map}(A, X) \to \mathrm{map}(A, Y)$$

is a fibration of simplicial sets for all $A \in \mathcal{A}$; and

2. the morphism f is an $\mathcal{A}$-equivalence if for some Reedy fibrant replacement $f' : X' \to Y'$ of f, the morphism of and the morphism of simplicial mapping spaces

$$\mathrm{map}(A, X') \to \mathrm{map}(A, Y')$$

is a weak equivalence of simplicial sets for all $A \in \mathcal{A}$.

The phrase "Reedy fibrant replacement" needs explanation. Not every object in $s\mathcal{S}$ is Reedy fibrant. A Reedy fibrant replacement for X is a level-wise weak equivalence $X \to X'$ where X' is Reedy fibrant. Since the Reedy model category structure is cofibrantly generated, this can be done functorially. A Reedy fibrant replacement for a morphism $X \to Y$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ Y \arrow[r] & Y' \end{tikzcd}$$

where the horizontal maps are Reedy fibrant replacements.

Theorem 3.4 *With these definitions, the category $s\mathcal{S}$ becomes a simplicial model category.*

The proof is given, essentially, in [11]. We call this the $\mathcal{A}$ resolution model category structure. It is cofibrantly generated and an object is $\mathcal{A}$-fibrant if and only if it is Reedy fibrant. We will discuss cofibrations below.

There are two ways to characterize $\mathcal{A}$-equivalences. Define "sphere objects" in $s\mathcal{S}$ as follows: let $A \in \mathcal{A}$, $n \geq 0$, and let $\Delta^n/\partial\Delta^n$ be the standard simplicial n-sphere. As always, $\Delta^0/\partial\Delta^0 = (\Delta^0)_+$ is the two-point simplicial set. Then the nth A sphere $A \wedge \Delta^n/\partial\Delta^n$ is defined by the push-out diagram

$$\begin{tikzcd} A \otimes * = A \arrow[r] \arrow[d] & A \otimes \Delta^n/\partial\Delta^n \arrow[d] \\ * \arrow[r] & A \wedge \Delta^n/\partial\Delta^n. \end{tikzcd}$$

If $X \in s\mathcal{S}$ is Reedy fibrant, then the mapping space $\mathrm{map}(A, X)$ that appears in 3.3 is a loop space, because A is a homotopy co-group object in spectra. As a result, a morphism between Reedy fibrant objects is an $\mathcal{A}$ equivalence if and only of

$$\pi_n \,\mathrm{map}(A, X) \longrightarrow \pi_n \,\mathrm{map}(A, Y)$$

is an isomorphism for all $n \geq 0$ with the constant map as basepoint. Put another way, if we define the "natural" homotopy groups of a Reedy fibrant simplicial spectrum X by the formula

$$\pi_n^\natural(X; A) = [A \wedge \Delta^n/\partial\Delta^n, X]_{\mathcal{A}} \cong \pi_n \,\mathrm{map}(A, X)$$

then we can say that $X \to Y$ is an $\mathcal{A}$ equivalence if and only if it induces an isomorphism on natural homotopy groups. The symbol $[\ ,\ ]_{\mathcal{A}}$ refers to morphisms in the homotopy category of obtained from the $\mathcal{A}$-resolution model category structure; therefore, the formula

$$\pi_n^\natural(X; A) = [A \wedge \Delta^n/\partial\Delta^n, X]_{\mathcal{A}}$$

makes sense even if X is not Reedy fibrant, and we take this as definition in general.

There are other homotopy groups in this setting. If $X \in s\mathcal{S}$ and $A \in \mathcal{A}$, then

$$[A, X] = \{[A, X_n]\}$$

is a simplicial abelian group, and we may define

$$\pi_i(X; A) = \pi_i[A, X].$$

This is often easier to compute, but both are important. They are related by the spiral exact sequence.

Proposition 3.5 *For all $A \in \mathcal{A}$ and all simplicial spectra A, there is a natural isomorphism*

$$\pi_0^\natural(X; A) \cong \pi_0(X; A)$$

and a long exact sequence

$$\cdots \to \pi_{n-1}^\natural(X; \Sigma A) \to \pi_n^\natural(X; A) \to \pi_n(X; A) \to \pi_{n-2}^\natural(X; \Sigma A) \to \cdots \to \pi_1(X; A) \to 0.$$

See [12]. These long exact sequences can be spliced together to give a spectral sequence

$$\pi_q(X, \Sigma^q A) \Longrightarrow \pi_0^\natural(X; \Sigma^{p+q} A).$$

Because A is constant as a simplicial object, geometric realization gives map

$$\pi_0^\natural(X; \Sigma^{p+q} A) \to [\Sigma^{p+q} A, |X|]$$

and, essentially because the functor $[\Sigma^* A, \cdot]$ is a homology theory, this map is an isomorphism for Reedy cofibrant spectra. See [18]. (The corresponding result for topological spaces is true, but harder; see [12].) Thus one gets a spectral sequence

$$\pi_q(X, \Sigma^q A) \Longrightarrow [\Sigma^{p+q} A, |X|]. \tag{3.1}$$

At this point we can explain one of the reasons for using the the models $\mathcal{A}$ to define the resolution model category. (Another, more important reason, is below in Proposition 3.6.) Suppose $X \to Y$ is a weak equivalence between cofibrant objects in the $\mathcal{A}$ resolution model category. Then for each of the spectra DE_α we have an isomorphism

$$\pi_p(X, \Sigma^q DE_\alpha) \cong \pi_p(Y, \Sigma^q DE_\alpha).$$

However

$$\pi_p E_q X \cong \operatorname{colim}_\alpha (E_\alpha)_q X \cong \operatorname{colim}_\alpha \pi_p[\Sigma^q DE_\alpha, X] = \operatorname{colim}_\alpha \pi_p(X, \Sigma^q DE_\alpha). \tag{3.2}$$

We note that the spectral sequence of Equation 3.1 is natural in A; thus, taking the colimit as in Equation 3.2, we obtain a spectral sequence

$$\pi_q E_q X \Longrightarrow E_{p+q}|X|. \tag{3.3}$$

This is, of course, the standard homology spectral sequence of a simplicial spectrum. In any case, of $X \to Y$ is an $\mathcal{A}$-equivalence, then they produce isomorphic E_* homology spectral sequences.

The true power of this model category structure arises when considering the cofibrant objects. It is possible to give a complete characterization of cofibrations in this category; however, we wish to focus on certain types of cofibrant models. The key result is below as Proposition 3.6. But before we can state that result, we need some terminology.

If $X : I \to \mathcal{C}$ is a diagram in any category, we say that X is *free* if it is the left Kan extension of a diagram $Z : I^\delta \to \mathcal{C}$ on the underlying discrete category. In formulas, this says that

$$X_i \cong \sqcup_{j \to i} Z_j \tag{3.4}$$

where the coproduct runs over morphisms in I into i. A simplicial object $X : \Delta^{op} \to \mathcal{C}$ is s-free if the underdyling degeneracy diagram is free; thus, there are objects Z_m in $\mathcal{C}$ so that

$$X_n \cong \sqcup_{\phi:[n]\to[m]} Z_m$$

where ϕ runs over the surjections in Δ. If $\mathcal{C}$ is a closed model category and each Z_m is cofibrant, an s-free object is Reedy cofibrant in $s\mathcal{C}$. In Quillen's terminology [33] §II.4, a simplicial algebra is *free* if it is s-free on free algberas.

Proposition 3.6 *Let $X \in s\mathcal{S}$ be a simplicial spectrum. Then there is a natural $\mathcal{A}$-equivalence*

$$P(X) \longrightarrow X$$

so that

1. *$P(X)$ is cofibrant in $\mathcal{A}$ resolution model category structure;*
2. *$P(X)$ is s-free on spectra Z_m, $m \geq 0$;*
3. *for each $m \geq 0$ there is a cofibrant spectrum $Z_m(0)$ and a cellular map $Z_m(0) \to Z_m$ so that $Z_m(0)$ is contractible and $Z_m/Z_m(0)$ is isomorphic to a wedge of elements of $\mathcal{A}$.*

The morphism $P(X) \to X$ is a mild variation on the Stover resolution [42], which is not actually s-free. This freeness is obtained by a sub-division argument in [18]. Note that part 3) of this result says that $E_*P(X)$ is a cofibrant (i.e., level-wise projective) simplicial E_* module.

It is worth remarking that one can now recover the universal coefficient theorem of Adams [1] §III.13 from this last result; indeed, the universal coefficient spectral sequence is the Bousfield-Kan spectral sequence of the cosimplicial space $\mathcal{S}(P(X), F)$ for an E module spectrum F:

$$\mathrm{Ext}^s_{E_*}(E_*X, F_{*+t}) \cong \pi^s\pi_{-t}\mathcal{S}(P(X), F) \Longrightarrow \pi_{-t-s}\mathcal{S}(|P(X)|, F) \cong F^{s+t}X.$$

The E_2 is identified using Definition 3.2.4. Here and throughout this paper the symbol F_{*+t} means the graded group with $(F_{*+t})_n = F_{n+t}$.

The $\mathcal{A}$-resolution model category structure can be promoted to a model category for simplicial algebras over a simplicial operad over the linear isometries operad. Fix a simplicial operad T and a let $s\mathcal{S}^T$ be the category of algebras over T. This category has a simplicial structure; indeed, if K is a simplicial set and $X \in s\mathcal{S}^T$, one has

$$(X \otimes K)_n = \coprod_{K_n}^{T_n} X_n. \tag{3.5}$$

The superscript T_n is indicates that the coproduct is taken in the category of T_n algebras. The simplicial set of maps is defined again by

$$\mathrm{map}_T(X, Y)_n = \mathrm{Hom}_{s\mathcal{S}^T}(X \otimes \Delta^n, Y).$$

We say that a morphism $X \to Y$ of simplicial T-algebras is an $\mathcal{A}$-fibration or $\mathcal{A}$-equivalence if the underlying morphism of simplicial spectra is. Then we have the *$\mathcal{A}$-resolution model category structure* on $s\mathcal{S}^T$:

Theorem 3.7 *With these definitions, the category $s\mathcal{S}^T$ becomes a simplicial model category. Furthermore, for each $X \in s\mathcal{S}^T$ there is a natural $\mathcal{A}$-equivalence*

$$P_T(X) \to X$$

so that

1. *$P_T(X)$ is cofibrant in the $\mathcal{A}$-resolution model category structure on $s\mathcal{S}^T$;*
2. *the underlying degeneracy diagram of $P_T(X)$ is of the form $T(Z)$ where Z is free as a degeneracy diagram;*
3. *for each $m \geq 0$ there is a cofibrant spectrum $Z_m(0)$ and a cellular map $Z_m(0) \to Z_m$ so that $Z_m(0)$ is contractible and $Z_m/Z_m(0)$ is isomorphic to a wedge of elements of $\mathcal{A}$.*

The existence of the model category structure in a standard lifting argument; the analysis of the cofibrant replacement $P_T(X)$ is again a sub-divided Stover resolution.

Here is how one might use this model category structure. We fix an operad $\mathcal{F}$ over the linear isometries operad and a simplicial resolution $T \to \mathcal{F}$ of $\mathcal{F}$ as in Theorem 2.1. If X is an $\mathcal{F}$ algebra, then X can be regarded as a constant object in $s\mathcal{S}^T$ and, hence, we have the resolution $P_T(X) \to X$ of the previous result. Then $P_T(X)$ is Reedy cofibrant in $s\mathcal{S}^T$ and, by Theorem 2.4.2, also Reedy cofibrant as a simplicial spectrum. Thus we can use the spectral sequence of Equation 3.1 with $A = S^0$ to show that the natural map

$$|P_T(X)| \to X$$

is a weak equivalence. But also, arguing as in Equation 3.2, we have that the augmentation $P_T(X)$ induces an isomorphism

$$\pi_* E_* P_T(X) \cong E_* X.$$

Finally, if $E_*\mathcal{F}(n)$ is projective as an E_* module for all n, then $E_*P_T(X)$ is a cofibrant E_*T algebra. In fact, since each of the spectra $X \in \mathcal{A}$ has the property that E_*X is projective as an E_* module, Theorems 2.1 and 3.7 imply that there is an isomorphism of underlying degeneracy objects

$$E_*(P_T(X)) \cong (E_*T)(E_*Z).$$

The role of cofibrant E_*T algebras will be explored in the next section.

4 André-Quillen cohomology and the homotopy type of mapping spaces

If A is a commutative algebra over a commutative ring k, M an A-module and $X \to A$ a morphism of commutative algebras, then the André-Quillen cohomology of X with coefficients in M is the non-abelian right derived functors of the functor

$$X \mapsto \mathrm{Der}_k(X, M)$$

which assigns to X the A-module of k-derivations from X to M. This cohomology has natural generalization to algebras over operads and their modules; indeed, much of the formalism of Quillen's paper [32] goes through without difficulty. This section outlines the details and explains the application to the computation of the homotopy type of the space of maps between structured ring spectra.

It should be said that the case when T was the A_∞ operad, regarded as constant operad, has been understood by many people. The extension here is to the simplicial operads.

The first part of this section is written algebraically. We fix a commutative ring R, possibly graded, and we consider R-modules (again possibly graded), operads in R-modules, and so on. In our applications R with be E_* for some homotopy commutative ring spectrum E. Any omitted details can be found in [19].

Let $\mathcal{C}$ be an operad in R-modules and suppose A is a $\mathcal{C}$ algebra. We define what it means for M to be an A-module. Let $\Phi(A, M)$ to be the graded R-module with

$$\Phi(A, M)_n = \oplus_i A \otimes \cdots \otimes M \otimes A \otimes \cdots \otimes A$$

with M appearing once in each summand and then in the ith slot. Here and below the tensor product is over R. Note that $\Phi(A, M)_n$ has an obvious action of the symmetric group Σ_n. Define

$$\mathcal{C}(A, M) = \oplus_n \mathcal{C}(n) \otimes_{R\Sigma_n} \Phi(A, M)_n = \oplus_n \mathcal{C}(n) \otimes_{R\Sigma_{n-1}} A^{\otimes(n-1)} \otimes M.$$

It is an exercise to show that there is a natural ismorphism of bifunctors

$$\mathcal{C}(\mathcal{C}(A), \mathcal{C}(A, M)) \cong (\mathcal{C} \circ \mathcal{C})(A, M))$$

where $(\cdot) \circ (\cdot)$ is the composition of operads. The R-module is an *A-module over $\mathcal{C}$* if there is a morphism of R-modules $\eta : \mathcal{C}(A, M) \to M$ which fits into a coequalizer diagram

$$\mathcal{C}(\mathcal{C}(A), \mathcal{C}(A, M)) \cong (\mathcal{C} \circ \mathcal{C})(A, M)) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathcal{C}(A, M) \xrightarrow{\eta} M$$

where the maps d_0 and d_1 are induced by the operad multiplication of $\mathcal{C}$, and by η and the algebra structure on A respectively. Furthermore, the unit $\mathbf{1} \to \mathcal{C}$ defines a morphism of R-modules $M = \mathbf{1}(A, M) \to \mathcal{C}(A, M)$ which is required to be a section of η.

We now come to derivations. If A is a commutative algebra over R, and M is an A-module, we can can form a new commutative algebra over A called $M \rtimes A$, which as an R-module is simply $M \oplus A$, but with algebra multiplication

$$(x, a)(y, b) = (ab + ay, ab).$$

The algebra $M \rtimes A$ is a *square-zero extension* and an abelian object in the category of algebras over A; all abelian group objects have this form. It also represents the functor that assigns to an algebra over A the A-module of R derivations from X to M:

$$\mathrm{Der}_R(X, M) \cong \mathrm{Hom}_{\mathbf{Comm}/A}(X, M \rtimes A)$$

where we write **Comm** for the commutative algebra operad in R modules.

These concepts easily generalize. If $\mathcal{C}$ is an operad, A a $\mathcal{C}$ algebra and M an A-module, define a new $\mathcal{C}$ algebra over A called $M \rtimes A$ as follows: as an R-module $M \rtimes A$ is simply $M \oplus A$, but the $\mathcal{C}$ algebra structure is defined by noting that there is a natural decomposition

$$\mathcal{C}(M \oplus A) \cong E(A, M) \oplus \mathcal{C}(A, M) \oplus \mathcal{C}(A)$$

where $E(A, M)$ consists of those summands of $\mathcal{C}(M \oplus A)$ with more that one M term. Since M is an A-module we get a composition

$$\mathcal{C}(M \oplus A) \to \mathcal{C}(A, M) \oplus \mathcal{C}(A) \to M \oplus A$$

which defines the $\mathcal{C}$ algebra structure on $M \rtimes A$. The algebra $M \rtimes A$ is again a square-zero extension and an abelian object in the category of algebras over A; again, all abelian group objects have this form. The object $M \rtimes A$ in the category of $\mathcal{C}$ algebras over A again represents an abelian group valued functor which we might as well call derivations. If X is a $\mathcal{C}$ algebra over A and M is an A-module, a *derivation* $d : X \to M$ is an R-module homomorphism such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}(A) \arrow[r, "\Delta"] \arrow[d, "\epsilon_A"'] & \mathcal{C}(A, A) \arrow[r, "\mathcal{C}(A,d)"] & \mathcal{C}(A, M) \arrow[d, "\eta"] \\
A \arrow[rr, "d"] & & M.
\end{tikzcd}$$

where $\Delta : \mathcal{C}(A) \to \mathcal{C}(A, A)$ is induced by the diagonal maps $A^{\otimes n} \to \oplus_n A^{\otimes n}$. If we call the group of such derivations $\mathrm{Der}_{\mathcal{C}}(X, M)$, then the functor

$$X \mapsto \mathrm{Der}_{\mathcal{C}}(X, M) \tag{4.1}$$

is represented by $M \rtimes A$.

Cohomology in this context should be derived functors of derivations; the lesson of Quillen's work, however, is that one carry out this program in a more general and more flexible context. Thus we work with simplicial algebras over a simplicial operad.

Thus we let $\mathcal{C} = \mathcal{C}_\bullet$ be simplicial operad in R-modules and $s\mathcal{A}^{\mathcal{C}}$ the category of simplicial algebras over $\mathcal{C}$. This is a simplicial category; for example, if K is a simplicial set and $X \in s\mathcal{A}^{\mathcal{C}}$ then

$$(A \otimes K)_n = \coprod_{K_n} A_n$$

with the coproduct in $\mathcal{C}_n$ algebras. (Compare Equation 3.5 and items following.) Also, among the morphisms of $s\mathcal{A}^{\mathcal{C}}$ we single out the *free* maps: a morphism $X \to Y$ is free if the underlying morphism of degeneracy diagrams is isomorphic to a map of the form

$$X \to X \sqcup \mathcal{C}(Z)$$

where Z is a free degeneracy diagram on a free R-module. Compare Equation 3.4.

The main theorem of [33] §II.4 immediately implies the following:

Proposition 4.1 *The $s\mathcal{A}^{\mathcal{C}}$ has the structure of a simplicial model category with a morphism $f : X \to Y$*

1. *a weak equivalence if $\pi_* f : \pi_* X \to \pi_* Y$ is an ismorphism;*
2. *a fibration if the induced map $Nf : NX \to NY$ of normalized chain complexes in R-modules is surjective in positive degrees;*
3. *a cofibration if it is a retract of a free map.*

If $A \in s\mathcal{A}^{\mathcal{C}}$ then $\pi_0 A$ is a $\pi_0\mathcal{C}$ algebra. If M is a $\pi_0 A$-module (over the operad $\pi_0\mathcal{C}$) then M is an A_n-module (over $\mathcal{C}_n$) for all $n \geq 0$. Then we can form the simplicial module $K(M,n)$ over A whose normalization $NK(M,n) \cong M$ concentrated in degree n. From this object we can form the simplicial $\mathcal{C}$-algebra $K_A(M,n) = K(M,n) \rtimes A$ over A and, for $X \in s\mathcal{A}^{\mathcal{C}}/A$ an algebra over A we will define the André-Quillen cohomology of X with coefficients in M by the formula

$$D^n_{\mathcal{C}}(X,M) == [X, K_A(M,n)]_{s\mathcal{A}^{\mathcal{C}}/A} \cong \pi_0 \operatorname{map}_{s\mathcal{A}^{\mathcal{C}}/A}(Y, K_A(M,n)) \tag{4.2}$$

where $Y \to X$ is a weak equivalence with Y cofibrant. (We will drop the operad in the subscript only if we are speaking of the commutative algebra operad; that is, of ordinary André-Quillen cohomology.) As usual, this cohomology can be written down as the cohomology of a chain complex; in light of Equation 4.1 there is a natural isomorphism

$$D^n_{\mathcal{C}}(X,M) \cong \pi^n \operatorname{Der}_{\mathcal{C}}(Y,M) \tag{4.3}$$

where Y a cofibrant model for X.

Defining such an object is one thing and computing it is another; we return to this point in a later section.

In our applications we will have a homology $E_*(\cdot)$ and $R = E_*$. We will also have a simplicial operad T over the linear isometries operad, so $\mathcal{C} = E_*T$ and a typical $\mathcal{C}$ algebra will be of the form E_*X where $X \in s\mathcal{S}^T$. If E_*E if flat over E_*, this will imply that we are actually working with operads, algebras and so forth in the category of E_*E comodules, rather than simply in the more basic category of E_*-modules. Under appropriate hypotheses – for example, if E satisfies the Adams condition of Definition 3.1 – the E_*E comodule version of Proposition 4.2 is true, and one can define André-Quillen cohomology in the category of E_*E comodules. Let us write this object as $D^*_{\mathcal{C}/E_*E}(X,M)$. However, with luck, one can reduce the calculation of this more complicated object to the first case. Here is the result we will use. Recall that if M is an E_*-module, one can form the extended E_*E comodule $E_*E \otimes_{E_*} M$.

Proposition 4.2 *Let $\mathcal{C}$ be a simplicial operad in E_*E comodules and A a simplicial $\mathcal{C}$ algebra in E_*E comodules. If M is a $\pi_0 A$-module in E_*-modules, then the extended comodule $E_*E \otimes_{E_*} M$ is $\pi_0 A$-module in E_*E comodules and there is a natural isomorphism for X over A in $\mathcal{C}$ algebras in E_*E comodules*

$$D^*_{\mathcal{C}/E_*E}(X, E_*E \otimes_{E_*} M) \cong D^*_{\mathcal{C}}(X,M).$$

With this technology at hand, we can know define a spectral sequence for computing the homotopy groups of the space of maps between two structured ring spectra. We fix a commutative ring spectrum E satisfying the Adams condition of Definition 3.1. Let $\mathcal{F}$ be an operad over the linear isometries operad and suppose X is a $\mathcal{F}$ algebra in spectra. Then E_*X is an algebra over the operad $E_*\mathcal{F}$ in E_* modules. If we also assume that $X = F$ is an E-module spectrum, then F_* is also an $E_*\mathcal{F}$ algebra, an the multiplication map $E_*F \to F_*$ is an $E_*\mathcal{F}$ algebra map.

Now let $T \to \mathcal{F}$ be the resolution supplied by Theorem 2.1. Let X be an $\mathcal{F}$ algebra in spectra and suppose that F is an $\mathcal{F}$ algebra in spectra which is also an E module spectrum. Suppose we have an $\mathcal{F}$ algebra map $\phi : X \to F$. This amounts to choosing a basepoint $\phi \in \mathcal{S}^{\mathcal{F}}(X, F)$ for the space of $\mathcal{F}$ algebra maps from X to F. Note that the composition

$$E_*X \xrightarrow{E_*\phi} E_*F \to F_*$$

makes F_* into an E_*X module over the operad $E_*\mathcal{F}$.

The following result is discussed in detail in [20].

Theorem 4.3 *Suppose $E_*\mathcal{F}(n)$ is projective as an E_* module for all $n \geq 0$. Then there is a second quadrant spectral sequence abutting to $\pi_{t-s}(\mathcal{S}^{\mathcal{F}}(X, F), \phi)$ with E^2 term*

$$E_2^{0,0} = \mathrm{Hom}_{E_*\mathcal{F}}(E_*X, F_*)$$

and

$$E_2^{s,t} = D^s_{E_*T}(E_*X, F_{*+t}) \qquad t > 0.$$

This is a Bousfield-Kan spectral sequence, as we will see in the next paragraph. The standard references are [7] and [6]. The latter work, for example, implies the following result:

Corollary 4.4 *Assume the hypotheses of Theorem 4.3. Then there are succesively defined obstructions to realizing a map $f \in \mathrm{Hom}_{E_*\mathcal{F}}(E_*X, F_*)$ in the groups*

$$D^{s+1}_{E_*T}(E_*X, F_{*+s}) \qquad s \geq 1.$$

In particular, if these groups are all zero, then the Hurewicz map

$$\pi_0(\mathcal{S}^{\mathcal{F}}(X, E)) \to \mathrm{Hom}_{E_*\mathcal{F}}(E_*X, F_*) \tag{4.4}$$

is surjective. If, in addition, the groups

$$D^s_{E_*T}(E_*X, F_{*+s}) = 0$$

for $s \geq 1$, the Hurewicz map of Equation 4.4 is a bijection.

The spectral sequence of Theorem 4.3 is constructed as follows. We may regard X as a consant simplicial T algebra and take the Stover resolution $P_T(X) \to X$ guaranteed by Theorem 3.7. We then obtain a cosimplicial space

$$\mathcal{S}^T(P_T(X), F)^\bullet = \{\mathcal{S}^{T_n}(P_T(X)_n, F)\}^n \cong \{\mathcal{S}^{\mathcal{F}}(\mathcal{F} \otimes_{T_n} P_T(X)_n, F)\}^n$$

and the map $\phi : X \to F$ supplies this with the basepoint. The Bousfield-Kan spectral sequence now reads

$$\pi^s \pi_t \mathcal{S}^T(P_T(X), F)^\bullet \Longrightarrow \pi_{t-s} \mathrm{Tot}\mathcal{S}^T(P_T(X), F)^\bullet.$$

One uses standard adjunction arguments, Theorem 1.11, Theorem 2.1, and Theorem 2.2 to show

$$\mathrm{Tot}\mathcal{S}^T(P_T(X), F)^\bullet \cong \mathcal{S}^{|T|}(|P_T(X)|, F) \simeq \mathcal{S}^{\mathcal{F}}(X, F).$$

To identify the E_2 term one must calculate $\pi_t \mathcal{S}^{T_s}(P_T(X)_s, F)$. If we let F^{S^t} denote the spectrum of maps from the space S^t to the spectrum F, then F^{S^t} has a natural structure of an $\mathcal{F}$ algebra and is an E-module spectrum. Thus, because of Theorem 3.7, the choice of the spectra used to build $\mathcal{A}$ resolutions (Definition 3.2) and the conditions on the operads T_s from Theorem 2.1, we have, for $t > 0$:

$$\begin{aligned}
\pi_t \mathcal{S}^{T_s}(P_T(X)_s, F) &= \pi \mathcal{S}^{T_s}(T_s(Z_s), F)\\
&= \pi_t \mathcal{S}(Z_s, F^{S^t})\\
&= \mathrm{Hom}_{E_*}(E_* Z_s, \pi_* F^{S^t})\\
&= \mathrm{Hom}_{E_* T_s}(E_* T_s(E_* Z_s), \Lambda_{F_*}(x_t))\\
&= \mathrm{Der}_{E_* T_s}(E_* T_s(E_* Z_s), F_{*+t}).
\end{aligned}$$

The result then follows from the definition of André-Quillen cohomology; see Equation 4.3

As an amusing reduction of this theory, one can consider the case of the inclusion of the unit operad $\mathbf{1} \to \mathcal{L}$. An algebra over $\mathbf{1}$ is simply a spectrum, and an $E_*\mathbf{1}$ algebra is an E_* module. The spectral sequence of Theorem 4.3 becomes the universal coefficient spectral sequence

$$\mathrm{Ext}^s_{E_*}(E_* X, F_{*+t}) \Longrightarrow \pi_{t-s}\mathcal{S}(X, F) \cong F^{s-t}X \tag{4.5}$$

and the obstructions of Corollary 4.4 to realizing an E_* module map $E_* X \to F_*$ lie in

$$\mathrm{Ext}^{s+1}_{E_*}(E_* X, F_{*+s}) \qquad s \geq 1. \tag{4.6}$$

Compare the remark after Proposition 3.6.

5 Obstructions to algebra structures

We consider the following basic problem: let E_* be a homology theory satisfying the Adams condition and $\mathcal{F}$ an operad over the linear isometries operad. Suppose Λ is an $E_*\mathcal{F}$ algebra in E_*E-comodules. Is there is a spectrum $Z \in \mathcal{S}^{\mathcal{F}}$ so that $E_* Z \cong \Lambda$ as $E_*\mathcal{F}$ algebras? The purpose of this section is to develop an obstruction theory for approaching this question. This is a variant of the obstruction theory posited by Dwyer, Kan, and Stover [12]; it has been worked out in some detail by David Blanc [5]. One should also compare the results of H.-J. Baues [4].

In practice, one is actually concerned with a refinement of the basic problem; namely, one has a spectrum Z_0 and a specified $E_*\mathcal{F}$ algebra structure on $\Lambda = E_* Z_0$ and one wonders if there is a spectrum $Z \in \mathcal{S}^{\mathcal{F}}$ and a weak equivalence of spectra $Z \to Z_0$ inducing an isomorphism of $E_*\mathcal{F}$

algebras $E_*Z \cong E_*Z_0$. In general, this would lead to a seconday obstruction problem which we will relegate to the end: see Proposition 5.6. For now we concentrate on the simpler problem.

In order to realize Λ, we first resolve the operad $\mathcal{F}$ as in Theorem 2.1; thus we have an augmented simplicial operad $T \to \mathcal{F}$ satisfying the conclusions of that result. We assume that $E_*\mathcal{F}(n)$ is a projective E_*-module for all n. We are going to try to construct a simplicial algebra $Y \in s\mathcal{S}^T$ so that

$$\pi_i E_* Y \cong \begin{cases} \Lambda, & i = 0, \\ 0 & i \neq 0. \end{cases}$$

Then $Z = \mathcal{F} \otimes_{|T|} |Y'|$ where Y' is an appropriately cofibrant model for Y. To make that calculation we would use Theorem 1.11, Theorem 2.4, and the spectral sequence of Equation 3.3.

In order to attempt to construct Y, we would like to build a tower of fibrations

$$\cdots Y(n) \to Y(n-1) \to \cdots \to Y(1) \to Y(0) \tag{5.1}$$

is the $\mathcal{A}$-resolution model category $s\mathcal{S}^T$ that would behave much like a Postnikov tower for Y; in particular, we would ask that for all $A \in \mathcal{A}$

$$\pi_i^\natural(Y(n), A) \cong \begin{cases} \pi_i^\natural(Y, A), & i \le n, \\ 0 & i > n. \end{cases}$$

For it is with respect to these "natural" homotopy groups that objects in $s\mathcal{S}^T$ have Postnikov sections, not with respect to the groups $\pi_i(\cdot, A)$. (See Proposition 3.5 for the difference.) Of course, in order to do this, we need to know what the groups $\pi_i^\natural(Y, A)$ would be, and this is where we introduce the first of the assumptions on E_* and the algebra Λ. We are motivated by the following simple observation; the proof is an Adams-Novikov spectral sequence argument.

Lemma 5.1 *If E_*E is flat over E_* and X is a spectrum so that there is an E_* module N_* and and an isomorphism of E_*E comodules $E_*X \cong E_*E \otimes_{E_*} N_*$. Then for all finite cellular spectra A there is an isomorphism*

$$[A, X] \cong \mathrm{Hom}_{E_*E}(E_*A, E_*X) \cong \mathrm{Hom}_{E_*}(E_*A, N_*).$$

Thus, if we require that our $E_*\mathcal{F}$ algebra Λ be an extended E_*E comodule. Then we would know that for our putative simplicial spectrum Y there was an isomorphism:

$$\pi_0^\natural(Y; A) \cong \pi_0(Y; A) \cong \mathrm{Hom}_{E_*E}(E_*A, \Lambda).$$

In addition, if we stipulate that $\pi_i(Y; A) = 0$ for all $i > 0$ we would have, as in Equation 3.2, that

$$\pi_i E_* Y = 0 \qquad i > 0$$

and, because of the spiral exact sequence of Proposition 3.5,

$$\pi_i^\natural(Y; A) \cong \mathrm{Hom}_{E_*E}(E_*\Sigma^i A, \Lambda) \cong \mathrm{Hom}_{E_*E}(E_*A, \Lambda_{*+i})$$

Therefore, the proposal is to try to build a tower of fibrations as in Equation 5.1 so that

$$\pi_i^\natural(Y(n), A) \cong \begin{cases} \mathrm{Hom}_{E_*E}(E_*A, \Lambda_{*+i}), & i \le n, \\ 0 & i > n. \end{cases} \tag{5.2}$$

Furthermore, the natural map

$$\pi^\natural_{i-1}(Y(n);\Sigma A) \to \pi^\natural_i(Y(n);A) \tag{5.3}$$

is to be an isomorphism for $i \leq n$, and the projection

$$\pi^\natural_i(Y(n);A) \to \pi^\natural_i(Y(n-1);A) \tag{5.4}$$

is an isomorphism for $i < n$.

It turns out that $Y(0)$ always exists. Let $\mathbf{H}o(s\mathcal{S}^T)$ denote the homotopy category of $s\mathcal{S}^T$ derived from the $\mathcal{A}$ resolution model category structure. Note that the $E_*\mathcal{F}$ algebra Λ may be regarded as a constant object in simplicial algebras over E_*T in simplicial E_*E comodules.

Proposition 5.2 *The functor from $\mathbf{H}o(s\mathcal{S}^T)$ to the category of sets given by*

$$X \mapsto \mathrm{Hom}_{E_*T}(E_*X,\Lambda)$$

is representable; that is, there is an object $B\Lambda \in s\mathcal{S}^T$ and a natural isomorphism

$$[X, B\Lambda] \cong \mathrm{Hom}_{E_*T}(E_*X,\Lambda).$$

The proof is an exercise in Brown representability. The proof in Spanier [41] §7.7 goes through *mutatis mutandis*. See also Heller [21].

We can now calculate that $B\Lambda$ is a candidate for $Y(0)$:

$$\begin{aligned}
\pi^\natural_i(B\Lambda;A) &\cong [A\wedge \Delta^i/\partial\Delta^i, B\Lambda]_{s\mathcal{S}}\\
&\cong [T(A\wedge \Delta^i/\partial\Delta^i), B\Lambda]_{s\mathcal{S}^T}\\
&\cong \mathrm{Hom}_{E_*T}(E_*(T(A\wedge \Delta^i/\partial\Delta^i)),\Lambda)\\
&\cong \mathrm{Hom}_{E_*T}(E_*(T)(E_*(A\wedge \Delta^i/\partial\Delta^i)),\Lambda)\\
&\cong \mathrm{Hom}_{E_*E}(E_*(A\wedge \Delta^i/\partial\Delta^i),\Lambda)\\
&\cong \mathrm{Hom}_{E_*E}(\pi_0 E_*(A\wedge \Delta^i/\partial\Delta^i),\Lambda)\\
&\cong \begin{cases} \mathrm{Hom}_{E_*E}(E_*A,\Lambda), & i=0,\\ 0 & i>0.\end{cases}
\end{aligned}$$

The isomorphism as the fourth line uses exactly that we have chosen T as in Theorem 2.1.

We now come to the inductive step: assuming the existence of $Y(n-1)$ satisfying the inductive hypothesis suggested by Equation 5.2 – 5.4, we would like to produce $Y(n)$. Here is where the obstruction appears. A calculation with the spiral exact sequence shows

$$\pi_i(Y(n-1);A) \cong \begin{cases} \mathrm{Hom}_{E_*E}(E_*A,\Lambda), & i=0,\\ \mathrm{Hom}_{E_*E}(E_*A,\Lambda_{*+n}), & i=n+1,\\ 0 & i\neq 0, n+1.\end{cases}$$

We would like to kill the top homotopy group using a suitable Eilenberg-Mac Lane object. These are supplied by the following result. If $X \to B\Lambda$ is a morphism in $s\mathcal{S}^T$, then E_*X is an E_*T algebra (in comodules) over $E_*B\Lambda$ and hence over $\Lambda \cong \pi_0 E_*B\Lambda$. Let $\mathbf{H}o(s\mathcal{S}^T/B\Lambda)$ be the homotopy category of simplicial T-algebras over $B\Lambda$ in the $\mathcal{A}$-resolution model category structure.

Proposition 5.3 *Let M be a Λ module over the operad $E_*\mathcal{F}$. The functor on $\mathbf{Ho}(s\mathcal{S}^T/B\Lambda)$ which assigns to $X \to B\Lambda$ the André-Quillen cohomology group*

$$D^n_{E_*T/E_*E}(E_*X, M)$$

is representable; that is, there is an object $E_\Lambda(M,n) \to B\Lambda$ in $s\mathcal{S}^T/B\Lambda$ and a natural isomorphism

$$[X, E_\Lambda(M,n)]_{s\mathcal{S}^T/B\Lambda} \cong D^n_{E_*T/E_*E}(E_*X, M).$$

Again one use Brown representability; in fact, the functor on $\mathbf{Ho}(s\mathcal{S}^T)$ given by

$$X \longmapsto \bigsqcup_{\varphi: E_*X \to \Lambda} D^n_{E_*T/E_*E}(E_*X, M_\varphi) \tag{5.5}$$

is representable by $E_\Lambda(M,n)$ in $\mathbf{Ho}(s\mathcal{S}^T)$. The symbol M_φ indicates that M is an E_*T module via φ.

Remark 5.4 We emphasize that the simplicial E_*T algebra $E_*E_\Lambda(M,n)$ is not a model for the Eilenberg-Mac Lane object $K_\Lambda(M,n)$ of the previous section (see, for example, Equation 4.2). Indeed, using Equation 5.5 and a calculation such as we did above to compute $\pi^\natural_* B\Lambda$ one can show that $\pi^\natural_i(E_\Lambda(M,n), A) = 0$ for $i \neq 0, n$. Furthermore,

$$\pi^\natural_0(E_\Lambda(M,n), A) \cong \mathrm{Hom}_{E_*E}(E_*A, \Lambda)$$

with isomorphism induced by the augmentation $E_\Lambda(M,n) \to B\Lambda$, and

$$\pi^\natural_n(E_\Lambda(M,n), A) = \mathrm{Hom}_{E_*E}(E_*A, M).$$

It follows from this, the spiral exact sequence and a colimit argument as in Equation 3.2 that

$$\pi_i E_* E_\Lambda(M,n) \cong \begin{cases} \Lambda, & i = 0, \\ \Lambda_{*+1}, & i = 2 \\ M, & i = n \\ M_{*+1}, & i = n+2 \\ 0 & \text{otherwise.} \end{cases}$$

But one has

$$\pi_i K_\Lambda(M,n) \cong \begin{cases} \Lambda, & i = 0, \\ M, & i = n \\ 0 & \text{otherwise.} \end{cases}$$

There is a natural map $E_*E_\Lambda(M,n) \to K_\Lambda(M,n)$ over Λ which induces an isomorphism on π_0 and π_n.

We would now like to produce a map over $B\Lambda$

$$\epsilon_n : Y(n-1) \to E_\Lambda(\Lambda_{*+n}, n+1) \tag{5.6}$$

which induces an isomorphism on $\pi_{n+1}(\cdot, A)$ for all $A \in \mathcal{A}$. To do we first need to decide what module structure Λ_{*+n} should have as Λ-module over E_*T.

Notice that if $X \in s\mathcal{S}^T$, then E_*X is an algebra over the simplicial operad E_*T; hence, π_*E_*X is a graded algebra over $\pi_*E_*T \cong E_*\mathcal{F}$. In particular, π_pE_*X is an π_0E_*X module over the operad $E_*\mathcal{F}$ and, hence, over the simplicial operad E_*T via the augmentation $E_*T \to E_*\mathcal{F}$. Note also that $E_*|X|$ is an $E_*|T| \cong E_*\mathcal{F}$ algebra. It is believable that the spectral sequence of Equation 3.3

$$\pi_*E_*X \Longrightarrow E_*|X|$$

is a spectral sequence of E_*F algebras. This is proved by showing, among other things, that the long exact sequence obtained from the spiral exact sequence of Proposition 3.5

$$\begin{aligned}\cdots \to \operatorname{colim}_\alpha \pi^\natural_{n-1}(X, \Sigma^{*+1}DE_\alpha) \to \operatorname{colim}_\alpha \pi^\natural_n(X, \Sigma^*DE_\alpha)\\ \to \pi_nE_*X \to \operatorname{colim}_\alpha \pi^\natural_{n-2}(X, \Sigma^{*+1}DE_\alpha) \to \cdots\end{aligned}$$

is an exact sequence of Λ-modules over $E_*\mathcal{F}$. Applying these remarks to $X = Y(n-1)$ we note that

$$\pi_{n+1}E_*Y(n-1) \cong \Lambda_{*+n}$$

has the canonical structure as an Λ module: the algebra multiplication makes Λ into a module over itself, and this is simply the shifted version of that structure.

We now come to the obstruction to producing the map ϵ_n of Equation 5.6. Consider the simplicial E_*T algebra $E_*Y(n-1)$. This has two non-vanishing homotopy groups:

$$\pi_iE_*Y(n-1) \cong \begin{cases} \Lambda, & i = 0,\\ \Lambda_{*+n}, & i = n\\ 0 & \text{otherwise.}\end{cases}$$

In otherwords, the homotopy groups of the E_*T algebra $E_*Y(n-1)$ are isomorphic to those of $K_\Lambda(\Lambda_{*+n}, n+1)$. The question is whether these are realized by a weak equivalence of simplicial E_*-algebras.

The category of simplicial E_*T algebras has Postnikov sections; in particular $P_nE_*Y(n-1) \simeq \Lambda$ and there is a single k-invariant in André-Quillen cohomology

$$k_n \in D^{n+2}_{E_*T/E_*E}(P_nE_*Y(n-1), \Lambda_{*+n}) \cong D^{n+2}_{E_*T/E_*E}(\Lambda, \Lambda_{*+n}).$$

This k-invariant is the obstruction to construting ϵ_n and, hence, $Y(n)$. We chose to encapsulate this whole discussion in the following result. The reduction to André-Quillen cohomology in E_* modules is a result of Proposition 4.2.

Theorem 5.5 *Let Λ be an $E_*\mathcal{F}$ algebra in E_*E comodules. Suppose E_*E is flat over E_* and that Λ is an extended E_*E comodule on the E_* module N. Then there are succesively defined obstructions*

$$k_n \in D^{n+2}_{E_*T/E_*E}(\Lambda, \Lambda_{*+n}) \cong D^{n+2}_{E_*T}(\Lambda, N_{*+n})$$

to finding an $\mathcal{F}$-algebra spectrum X so that $E_*X \cong \Lambda$.

Proof. Here is a sketch. Suppose $k_n = 0$. Then we can choose a weak equivalence of simplicial E_*T algebras

$$\theta : E_*Y(n-1) \xrightarrow{\simeq} K_\Lambda(\Lambda_{*+n}, n+1).$$

The map f defines a cohomology class in

$$D^{n+1}_{E_*T/E_*E}(Y(n-1), \Lambda_{*+n}) \cong [Y(n-1), E_\Lambda(\Lambda_{*+n}, n)]_{s\mathcal{S}/B\Lambda}.$$

This cohomology class defines the desired map ϵ_n.

We now turn to the more refined question raised at the beginning of the section: suppose one has a spectrum Z_0 and an $E_*\mathcal{F}$ algebra structure on E_*Z_0. Is there a $\mathcal{F}$-algebra spectrum Z and a weak equivalence $Z_0 \to Z$ of spectra inducing an isomorphism of $E_*\mathcal{F}$ algebras $E_*Z_0 \cong E_*Z$?

There are several approaches to this problem. One might, for example, try to build the spectra $Y(n)$ as above equipped with maps

$$i_n : X \to Y(n)$$

where X is cofibrant model for Z_0 in the $\mathcal{A}$ resolution model category structure of $s\mathcal{S}$. These maps would be required to be an isomorphism on $\pi_0(\cdot, A)$ and to fit together as n varies. This leads to a secondary obstruction. However, the following is good enough for us.

Proposition 5.6 *Let Λ be an $E_*\mathcal{F}$ algebra in E_*E comodules. Suppose E_*E is flat over E_* and that Λ is an extended E_*E comodule. Given a spectrum Z_0, an $\mathcal{F}$ algebra spectrum X, and an isomorphism of $E_*\mathcal{F}$ algebras in E_*E comodules*

$$f : E_*Z_0 \xrightarrow{\cong} E_*X \cong \Lambda$$

there are succesively defined obstructions

$$\chi_n \in \mathrm{Ext}^{n+1}_{E_*E}(\Lambda, \Lambda_{*+n})$$

to realizing f as a map of spectra.

This is a consequence of the Adams-Novikov spectral sequence. If $\Lambda = E_*E \otimes_{E_*} N$, then

$$\mathrm{Ext}^{n+1}_{E_*E}(\Lambda, \Lambda_{*+n}) \cong \mathrm{Ext}^{n+1}_{E_*}(\Lambda, N_{*+n}).$$

As in Corollary 4.4, if

$$\mathrm{Ext}^{n+1}_{E_*}(\Lambda, N_{*+n}) = 0 = \mathrm{Ext}^{n}_{E_*}(\Lambda, N_{*+n})$$

for $n \geq 1$, then the map $f : Z_0 \to X$ exists and is unique up to homotopy. This arises in the following setting.

Proposition 5.7 *Suppose $E_0 \subseteq E_*$ is a local ring with maximal ideal m and that*

1. *Λ and N are flat as E_* modules, and*
2. *N is complete in the sense that $N \cong \lim_q N/m^q N$.*

Then

$$\mathrm{Ext}^{s}_{E_*}(\Lambda, N_{*+t}) = 0$$

for all t and all $s \geq 1$.

Proof. Compare Proposition 6.7 below. It is sufficient to consider the case $t = 0$. Filter N by the powers of m. The hypotheses now guarantee a spectral sequence

$$\mathrm{Ext}^p_{E_*/m}(E_*/m \otimes_{E_*}, m^q N/m^{q+1}N) \Longrightarrow \mathrm{Ext}^p_{E_*}(\Lambda, N).$$

But if $p > 1$,

$$\mathrm{Ext}^p_{E_*/m}(E_*/m \otimes_{E_*}, m^q N/m^{q+1}N) = 0$$

since E_*/m is a graded field.

6 Computing with E_∞ operads.

If $\mathcal{L}$ is the linear isometries operad, then Theorem 2.1 supplies an augmented simplicial operad $T \to \mathcal{L}$ so that the augmented simplicial operad $E_*T \to E_*\mathcal{L}$ is an algebraic E_∞ operad in a sense to be defined shortly. Since is it the simplicial operad T and the methods of the previous section that we will use to attempt to impose E_∞ operad structures on spectra, we need to be able to compute the André-Quillen cohomology functor $D^*_{E_*T}$. The purpose of this section is to reduce that computation, at least in some cases, to the calculation of ordinary André-Quillen (co-)homology. The main result is the two spectral sequences supplied by the Propositions 6.4 and 6.5 below. Note that $E_*\mathcal{L}$ is the commutative algebra operad **Comm** in E_* modules; hence ordinary André-Quillen cohomology is cohomology over the operad $E_*\mathcal{L}$.

We will first say what we mean by an E_∞ operad.

Definition 6.1 *For any commutative ring k (possibly graded) an E_∞ operad $\mathcal{E}$ is a simplicial operad in k-modules equipped with a weak equivalence $\mathcal{E} \to$ **Comm** and so that for each $q \geq 0$, $\mathcal{E}(q)$ is a cofibrant (i.e., level-wise projective) simplicial $k[\Sigma_q]$ module.*

There is a canonical such operad—namely a cofibrant model for **Comm** in the category of simplicial k-operads —but we don't need that much structure in this discussion.

If V_* is a cofibrant simplicial k-module, the shuffle chain equivalence of normalized chain complexes

$$\underbrace{N(V) \otimes \cdots \otimes N(V)}_{n} \to \underbrace{N(V \otimes \cdots \otimes V)}_{n}$$

is Σ_n equivariant; thus if S is any simplicial k-module operad, the normalized object $NS = \{NS(k)\}_{k \geq 0}$ is an operad of k-chain complexes. In particular, if $\mathcal{E}$ is an E_∞ operad in the sense of Definition 6.1, then $N\mathcal{E}$ is an E_∞ operad in the category of chain complexes over k (as studied in [24], [15], among other places). More is true, if V is a simplicial k-module, and $\mathcal{C}$ is a simplicial operad, then there is a natural map of chain complexes

$$NC(q) \otimes_{k\Sigma_k} NV^{\otimes q} \to N(\mathcal{C}(q) \otimes_{k\Sigma_k} V^{\otimes q}) \tag{6.1}$$

and if $\mathcal{C}(q)$ is cofibrant as an $k\Sigma_q$ module and V is cofibrant as an k-module, this is a quasi-isomorphism. In shorthand,

$$N\mathcal{C}(NV) \to N\mathcal{C}(V) \tag{6.2}$$

is a quasi-isomorphism of $N\mathcal{C}$ algebras. Furthermore if A is any simplicial $\mathcal{C}$-algebra, NA is an $N\mathcal{C}$ algebra via

$$N\mathcal{C}(NA) \to N\mathcal{C}(A) \to NA.$$

From these considerations, and from [29], it immediately follows that if k is an algebra over a field $\mathbb{F}$ of characteristic $p > 0$, and $\mathcal{E}$ is a simplicial E_∞ operad, then the homotopy of any simplicial $\mathcal{E}$-algebra is an "allowable" algebra over the Dyer-Lashof algebra. Thus, if $A \in s\mathcal{A}^{\mathcal{E}}$, then $\pi_* A$ is a graded commutative algebra equipped with operations

$$Q^i : \pi_n A \to \pi_{n+i} A, \qquad i \geq 0, p = 2$$

or

$$\beta^\epsilon Q^i : \pi_n A \to \pi_{n+2i(p-1)-\epsilon} A \qquad i \geq 0, \epsilon = 0, 1, p > 2$$

subject to the Adem and Cartan formulas of [8], §I.1. Unstable in this context means, at $p = 2$,

$$Q^i(x) = \begin{cases} 0 & i < \deg(x) \\ x^2 & i = \deg(x) \end{cases}$$

and at $p > 2$

$$\beta^\epsilon Q^i(x) = \begin{cases} 0 & 2i - \epsilon < \deg(x) \\ x^p & 2i = \deg(x) \text{ and } \epsilon = 0. \end{cases}$$

This condition arises because we are dealing with a normalized object, not an arbitrary algebra over an E_∞ operad in chain complexes.

Also if k is not the prime field $\mathbb{F}_p$, these operations are not k-linear; if $\phi : k \to k$ is the Frobenius, $a \in k$ and $x \in \pi_* A$, then

$$\beta^\epsilon Q^i(ax) = \phi(a)\beta^\epsilon Q^i(x). \tag{6.3}$$

There is an obvious category $\mathcal{UR}$ of allowable algebras over the Dyer-Lashof algebra. The forgetful functor $\mathcal{UR} \to n\mathcal{M}_k$ to graded k-modules has a left adjoint $\mathcal{S}_{\mathcal{R}}$, and it follows from the quasi-isomorphisms of 6.1 and 6.2 and the calculations of [29] that if $\mathcal{E}$ is an E_∞ operad and $V \in s\mathcal{M}_k$, then the natural map

$$\mathcal{S}_{\mathcal{R}}(\pi_* V) \to \pi_* \mathcal{E}(V) \tag{6.4}$$

is an isomorphism in $\mathcal{UR}$ provided that $\pi_* V$ is a graded projective k-modules. Note that this isomorphism does not depend on $\mathcal{E}$. A spectral sequence argument now shows that for V a cofibrant simplicial k-module, $\pi_* \mathcal{E}(V)$ is independent of the E_∞ operad $\mathcal{E}$ and we have:

Proposition 6.2 *Let $\mathcal{E} \to \mathcal{E}'$ be a morphism of E_∞ operads over an $\mathbb{F}$ algebra k, where $\mathbb{F}$ is a field of positive characteristic. Then the forgetful functor*

$$s\mathcal{A}^{\mathcal{E}'} \to s\mathcal{A}^{\mathcal{E}}$$

and its adjoint $\mathcal{E}' \otimes_{\mathcal{E}} (\cdot)$ descend to a Quillen equivalence of homotopy categories

$$\mathbf{Ho}(s\mathcal{A}^{\mathcal{E}'}) \simeq \mathbf{Ho}(s\mathcal{A}^{\mathcal{E}}).$$

The algebra $S_{\mathcal{R}}(W)$ has a simple description, at least when W is a graded projective k-module. See [8] §I.1. The operations $\beta^\epsilon Q^i$ can be assembled into an algebra $\mathcal{R}$ over $\mathbb{F}_p$ using the Adem relations (see [8], §I.2). This is the Dyer-Lashof algebra. If $W \in n\mathcal{M}_{\mathbb{F}}$ is a graded k module, let

$$\mathcal{R}(W) = \mathcal{R} \otimes_{\mathbb{F}_p} W/U \tag{6.5}$$

where U is the sub-$\mathcal{R}$-module generated by elements of the form

$$\begin{array}{lll} Q^i \otimes x, & i < \deg(x) & (p = 2) \\ \beta^\epsilon Q^i \otimes x, & 2i - \epsilon < \deg(x) & p > 2. \end{array}$$

Then, if W is a graded projective k-module

$$S_{\mathcal{R}}(W) = S(\mathcal{R}(W))/I \tag{6.6}$$

where S is the graded symmetric algebra functor over k and I is the ideal generated by the elements

$$\begin{array}{lll} Q^i(x) - x^2 & \deg(x) = i, & p = 2 \\ Q^i(x) - x^p & \deg(x) = 2i, & p > 2 \end{array}$$

In particular $S_{\mathcal{R}}(W)$ is a free graded commutative k-algebra.

If $\Gamma \in \mathcal{UR}$ then Γ is, among other things, a graded commutative algebra and, as such, we can form its André-Quillen homology $D_*\Gamma$ as a graded commutative algebra: $D_*\Gamma = \pi_*\mathbf{L}\Omega\Gamma$ where $\mathbf{L}\Omega\Gamma$ is the cotangent complex as a graded commutative algebra. As usual (cf. [30],[16]), this object inherits structure from the Dyer-Lashof operations. We next outline this structure.

Let $\mathcal{U}$ be the category of $\mathbb{N}$-graded allowable modules over the Dyer-Lashof algebras $\mathcal{R}$. This are graded k-modules and $\mathcal{R}$ acts with the Frobenius twist as in Equation 6.3, and allowable means that

$$\begin{array}{rlcl} Q^i(x) = 0 & \text{if} & i \leq \deg(x) & (p = 2) \\ \beta^\epsilon Q^i(x) = 0 & \text{if} & 2i - \epsilon < \deg(x) \text{ or } 2i = \deg(x) & (p > 2) \end{array}$$

If $\Gamma \in \mathcal{UR}$ then $\mathcal{U}(\Gamma)$ is the category of objects M which are at once in $\mathcal{U}$, and graded Γ-modules subject to the compatibility condition that the multiplication map

$$\Gamma \otimes_k M \to M$$

is a morphism in $\mathcal{U}$.

Such structures arise naturally as follows: If $M \in \mathcal{U}(\Gamma)$, let $\mathrm{Der}_{\mathcal{R}}(\Gamma, M)$ be the module of commutative k-algebra derivations that commute with the elements of $\mathcal{R}$. The following is proved with a minor variation of the (entirely standard) techniques of [16] §1.

Lemma 6.3 *Let $\Gamma \in \mathcal{UR}$. The graded module $\Omega\Gamma$ of commutative algebra derivations is naturally an object in $\mathcal{U}(\Gamma)$ and there is a natural isomorphism*

$$\mathrm{Der}_{\mathcal{R}}(\Gamma, M) \cong \mathrm{Hom}_{\mathcal{U}(\Gamma)}(\Omega(\Gamma), M).$$

The functor which assigns to an algebra $\Gamma \in \mathcal{UR}$ the module of derivations $\mathrm{Der}_{\mathcal{R}}(\Gamma, M)$ has non-abelian right derived functors. Choose a cofibrant X model for $\Gamma \in \mathcal{UR}$ regarded as a constant object in $s\mathcal{UR}$. Then these derived functors are a kind of André-Quillen cohomology:

$$D^n_{\mathcal{R}}\Gamma = \pi^n \mathrm{Der}_{\mathcal{R}}(X, M). \tag{6.7}$$

This cohomolgy can be dissected. We may assume $X_q = S_{\mathcal{R}}(V_q)$ for some $V_q \in n\mathcal{M}_{\mathbb{F}}$ which is level-wise projective as a k-module; hence, as a simplicial graded commutative algebra, $X \to \Gamma$ is still a cofibrant model for Γ. Thus

$$\mathbf{L}\Omega(V) \simeq \Gamma \otimes_X \Omega X \tag{6.8}$$

acquires, by Lemma 6.3, the structure of a simplicial $s\mathcal{U}(\Gamma)$ module—in fact, $\Gamma \otimes_X \Omega X$ is a cofibrant object in $s\mathcal{U}(\Gamma)$. Thus the ordinary André-Quillen homology

$$D_*\Gamma \cong \pi_* \mathbf{L}\Omega(\Gamma) \tag{6.9}$$

is a graded object in $\mathcal{U}(\Gamma)$, and this structure is independent of the choice of X. (This type of argument has been axiomatized in [30].)

This noted, it is not surprising that the natural isomorphism of Lemma 6.3 yields a composite functor spectral sequence:

Proposition 6.4 *Let $\Gamma \in \mathcal{UR}$. Then there is a spectral sequence*

$$\mathrm{Ext}^{\mathcal{U}(\Gamma)}_p(D_q(\Gamma), M) \Rightarrow D^{p+q}_{\mathcal{R}}(\Gamma, M).$$

Again, see [30]. This is important because of the following result. Let k be an algebra over a field of positive characteristic, and $\mathcal{E}$ a simplicial E_∞ operad over k. If A is a simplicial $\mathcal{E}$ algebra and M is a $\pi_0 A$ module (over the operad $\pi_0\mathcal{E} = \mathbf{Comm}$), then M is an object in $\mathcal{U}(\pi_* A)$.

Proposition 6.5 *Let $\mathcal{E}$ be an E_∞ operad over a field $\mathbb{F}$-algebra k, where $\mathbb{F}$ is a field of characteristic $p > 0$. Let $A \in s\mathcal{A}^{\mathcal{E}}$. Then there is a spectral sequence*

$$D^p_{\mathcal{R}}(\pi_* A, M)^q \Rightarrow D^{p+q}_{\mathcal{E}}(A, M).$$

Proof. Here is an outline of the proof. We may assume A is cofibrant. Let $P^{\mathcal{E}}_\bullet A \in s(s\mathcal{A}^T)$ be a simplicial resolution of A by $\mathcal{E}$ algebras of the form $\mathcal{E}(W)$ where W is a cofibrant simplicial k module with the property that $\pi_* W$ is a projective k-module. Here resolution means that

$$\pi_*\pi_* P^{\mathcal{E}}_\bullet A \cong \pi_* A$$

via the augmentation. It is possible to construct such by a Stover resolution argument. (Compare section 2 and see [18]). Note that this and Equation 6.4 implies that

$$\pi_* P^{\mathcal{E}}_\bullet A \cong \pi_* A$$

is a cofibrant model for $\pi_* A$ as a simplicial object in $\mathcal{UR}$.

Taking the geometric realization, which is possible because $s\mathcal{A}^{\mathcal{E}}$ is a simplicial model category, we obtain a weak equivalence

$$|P^{\mathcal{E}}_{\bullet}A| \longrightarrow A$$

and hence a spectral sequence

$$\pi^p D^q_{\mathcal{E}}(P^{\mathcal{E}}_{\bullet}(A), M) \Longrightarrow D^{p+q}_{\mathcal{E}}(|P^{\mathcal{E}}_{\bullet}A|, M) \cong D^{p+q}_{\mathcal{E}}(A, M).$$

The last isomorphism follows because $|P^{\mathcal{E}}_{\bullet}A|$ is cofibrant. The claim is that

$$D^*_{\mathcal{E}}(\mathcal{E}(W), M) \cong \mathrm{Der}_{\mathcal{R}}(S_{\mathcal{R}}(\pi_* W), M).$$

This is easily verified, completing the proof.

In our applications, we will encounter simplicial algebras of the form E_*X where X is some simplicial algebra over some simplicial operad over the linear isometries operad. In this case, the ground ring will be $k = E_*$ and very rarely will this be an algebra over a field of characteristic p. Therefore, we close this section with two results intended to reduce calculations to the case considered above.

The first is this. Suppose $m \subseteq k$ is an ideal with the property that k/m is an algebra over a field of chacteristic p. Then if $\mathcal{E}$ is an E_∞ operad over k in the sense of Definition 6.1, then $k/m \otimes_k \mathcal{E}$ is an E_∞ operad over k/m. Furthermore, if A is a simplicial $\mathcal{E}$ algebra, then $k/m \otimes_k A$ is a simplicial $k/m \otimes_k \mathcal{E}$ algebra. If M is module over

$$\pi_0(k/m \otimes_k A) \cong k/m \otimes_k \pi_0 A$$

then M is a module over $\pi_0 A$ and we'd like to use these facts to compute $D^*_{\mathcal{E}}(A, M)$. If X is a cofibrant $\mathcal{E}$ algebra, then X is cofibrant as simplicial k module; hence if $X \to A$ is a weak equivalence of $\mathcal{E}$ algebras with X cofibrant, then $k/m \otimes_k X$ is a model for the derived tensor product $k/m \otimes^{\mathbb{L}}_k A$. The precise result is then:

Proposition 6.6 *Let A be a simplicial $\mathcal{E}$ algebra over an E_∞ operad over k and let M be a $k/m \otimes_k \pi_0 A$ module. Then there is a natural isomorphism*

$$D^*_{\mathcal{E}}(A, M) \cong D^*_{k/m \otimes_k \mathcal{E}}(k/m \otimes^{\mathbb{L}}_k A, M).$$

If $\pi_* A$ is flat over k, then

$$\pi_*(k/m \otimes^{\mathbb{L}}_k A) \cong k/m \otimes_k \pi_* A.$$

The last statement of this result follows from the spectral sequence

$$\mathrm{Tor}^p_k(k/m, \pi_q A) \Longrightarrow \pi_{p+q}(k/m \otimes^{\mathbb{L}}_k A).$$

of [33]§II.6.

Now suppose M is simply a module over $\pi_0 A$ and suppose that M is flat as a k module. Then we can filter the module M by powers of the ideal $m \subseteq M$ to get a spectral sequence. The exact result is this:

Proposition 6.7 *There is a spectral sequence*

$$E_1^{p,q} = D^p_{k/m\otimes_k\mathcal{E}}(k/m \otimes^{\mathbb{L}}_k A, m^qM/m^{q+1}M) \Longrightarrow \lim_q D^p_{\mathcal{E}}(A, M/m^qM).$$

If M is complete in the sense that $M \cong \lim_q M/m^qM$ and there is an r so that for all (p,q) we have $E_r^{p,q} = E_\infty^{p,q}$, then

$$\lim_q D^p_{\mathcal{E}}(A, M/m^qM) \cong D^q_{\mathcal{E}}(A, M).$$

7 The E_∞ ring structure on the Lubin-Tate spectra

In the section we apply the technology developed in the previous sections to show that the techniques used by Haynes Miller and the second author (cf. [37]) to show that the Lubin-Tate spectra is an A_∞ ring spectrum actually show that these spectra are E_∞.

We begin with some generalities. Suppose we have a commutative ring R and a formal group law G over R. Then we may form a graded ring $E_* = R[u^{\pm 1}]$ where $R = E_0$ and $u \in E_2$. One gets a new formal group law over E_* defined by

$$\bar{G}(x,y) = u^{-1}G(ux, uy). \tag{7.1}$$

Note that G and $\bar{G}$ determine each other, since u is invertible; however, the formal group law $\bar{G}$ is of the sort that which is classified by a map of graded rings $MU_* \to E_*$. If this map satisfies the hypotheses of the Landweber exact functor theorem [25], then we get a homology theory with coefficients E_* and with

$$E_*X \cong E_* \otimes_{MU_*} MU_*X. \tag{7.2}$$

Furthermore, $E_*(\cdot)$ satisfies the conditions specified by Adams to guarantee that $E_*(\cdot)$ has a universal coefficient spectral sequence; see Definition 3.1. The assertion is implicit in [1] and is spelled out in [37]. Note that if F_* is another Landweber exact homology theory, then

$$MU_*F \cong MU_*MU \otimes_{MU_*} F_*$$

and Equation 7.2 yields

$$E_*F \cong E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} F_*. \tag{7.3}$$

Note that in this case, the Landweber exact functor theorem guarantees that E_*F is a flat E_* module. We will call these *two-periodic* Landweber exact theories.

Now suppose that R is a local ring with maximal ideal m and that R/m is a field k of characteristic p. Then $E_*/m = E_0/m \otimes_{E_0} E_* = k[u^{\pm 1}]$ and $E_*X/m = E_0/m \otimes_{E_0} E_*X$ is a module over $k[u^{\pm 1}]$. Indeed, there is an isomorphism

$$E_*X/m \cong k[u^{\pm 1}] \otimes_k E_0X/m. \tag{7.4}$$

If X is a commutive ring spectrum, then E_*X is an E_* algebra and E_*X/m is an $k[u^{\pm 1}]$ algebra. Hence, we have a ring homomorphism given by the Frobenius

$$\phi = (\cdot)^p : E_*X/m \to E_*X/m.$$

This is not an $k[u^{\pm 1}]$ algebra homomophism; however we can adapt it to one by noting that there is a commutative diagram

$$\begin{tikzcd} E_*/m = k[u^{\pm 1}] \arrow[r] \arrow[d, "\phi"] & E_*X/m \arrow[d, "\phi"] \\ E_*/m = k[u^{\pm 1}] \arrow[r] & E_*X/m. \end{tikzcd}$$

If we write $(k[u^{\pm 1}])_\phi$ for the module structure on $k[u^{\pm 1}]$ induced by ϕ, we get a morphism E_*/m algebras

$$\phi : (k[u^{\pm 1}])_\phi \otimes_{k[u^{\pm 1}]} E_*X/m \to E_*X/m$$

which we call the *relative Frobenius.* In formulas, we have

$$\phi(a \otimes x) = ax^p.$$

The main computation is this.

Proposition 7.1 *Suppose the induced relative Frobenius in degree zero*

$$k_\phi \otimes_k E_0X/m \to E_0X/m$$

is an isomorphism. Then the André-Quillen homology of E_*X/m as a commutative $k[u^{\pm 1}]$ algebra vanishes:

$$D_*((E_*X/m)/(k[u^{\pm 1}])) = 0.$$

Proof. We first reduce to a computation in degree zero. Indeed, Equation 7.4 and flat base change [32] imply,

$$D_*((E_*X/m)/(k[u^{\pm 1}])) \cong D_*((k[u^{\pm 1}] \otimes_k E_0X/m)/(k[u^{\pm 1}])) \cong k[u^{\pm 1}] \otimes_k D_*((E_0X/m)/k).$$

One now shows $D_*((E_0X/m)/k) = 0$.

Since the relative Frobenius is an isomorphism, one gets an isomorphism

$$D_*\phi : D_*((k_\phi \otimes_k E_0X/m)/k) \xrightarrow{\cong} D_*((E_0X/m)/k).$$

However, for any k-algebra A the relative Frobenius induces the zero map on André-Quillen homology; in fact, the claim is that the relative Frobenius induces the zero map on the module of Kaehler differentials, so that the induced map on the cotangent complex which computes $D_*(A/k)$ is identically zero.

To verify the claim, let

$$\Omega(\phi) : \Omega(k_\phi \otimes_k A/k) \to \Omega(A/k)$$

be the induced map. This is a morphism of $k_\phi \otimes_k A$ modules and the source is generated, as such a module, by elements of the form $d(a \otimes x)$ where

$$d : k_\phi \otimes_k A \to \Omega(k_\phi \otimes_k A/k)$$

is the universal k derivation. One calculates

$$\begin{aligned}\Omega(\phi)(d(a\otimes x)) &= d(\phi(a\otimes x))\\ &= d(ax^p)\\ &= pax^{p-1}dx = 0\end{aligned}$$

and this completes the argument.

This result is input to the spectral sequences of the previous section. Let $T \to \mathcal{L}$ be the resolution of the linear isometries operad supplied by Theorem 2.1, so that E_*T is a simplicial E_∞ operad in E_* modules, as in Definition 6.1. Combining the previous result and Propositions 6.4, 6.5, 6.6, and 6.7, we have:

Theorem 7.2 *Let X be a homotopy associative, homotopy commutative ring spectrum and let M be a graded E_*X module. Suppose*

1. *E_*X is flat as an E_* module;*
2. *the relative Frobenius $k_\phi \otimes_k E_0X/m \to E_0X/m$ is an isomorphism; and*
3. *M is m-adically complete; that is, $M \cong \lim_q M/m^qM$.*

Then the André-Quillen cohomology of E_*X as a constant E_*T algebra with coefficients in M is identically zero:

$$D^*_{E_*T}(E_*X, M) = 0.$$

This result has the following immediate consequences.

Corollary 7.3 *Let X be an E_∞ ring spectrum and let F be an E module spectrum which is also an E_∞ ring spectrum. Suppose*

1. *the relative Frobenius $k_\phi \otimes_k E_0X/m \to E_0X/m$ is an isomorphism; and*
2. *F_* is m-adically complete; that is, $F_* \cong \lim_q F_*/m^q$.*

Then

$$\pi_0\mathcal{S}^{\mathcal{L}}(X, E) \cong \mathrm{Hom}_{E_*-alg}(E_*X, F_*)$$

and for any choice of basepoint and any $i > 0$

$$\pi_i\mathcal{S}^{\mathcal{L}}(X, E) = 0.$$

Proof. Since E_*F is automatically a flat E_* module, this follows from Theorem 7.2 and Theorem 4.3.

Corollary 7.4 *Suppose that E_* is m-adically complete and that the relative Frobenius*

$$k_\phi \otimes_k E_0E/m \to E_0E/m$$

is an isomorphism. Then there is a unique E_∞ structure on E realizing the commutative algebra structure on E_*E.

Proof. Applying Theorem 5.5 and Proposition 4.2, we have that the obstructions to the existence of some E_∞ ring spectrum X with $E_*X \cong E_*E$ lie in André-Quillen cohomology groups

$$D^{n+2}_{E_*T}(E_*E, E_{*+n})$$

By Theorem 7.2, these obstruction groups vanish. By Propostions 5.6 and 5.7, there is a weak equivalence $X \to E$ realizing the isomorphism of E_* algebras $E_*E \to E_*X$. This proves the existence of the E_∞ structure. Uniqueness is supplied by Corollary 4.4; in fact, given two E_∞ structures (E,η) and (E,η') on E, Corollary 4.4 supplies a weak equivalence of E_∞ ring spectra

$$f : (E,\eta) \longrightarrow (E,\eta')$$

and f is itself unique up to homotopy through E_∞ maps.

The main examples of spectra satisfying the conclusions are the spectra that arise from Lubin-Tate deformation theory. There is a lengthy exposition of this in [37], and we only give a brief synopsis.[2]

Let k be a perfect field of characteristic $p > 0$, and let G be formal group law over k of finite height. A *deformation* of the pair (k, G) consists of a complete local ring A with maximal ideal $m \subseteq A$, a formal group law H over A and a map of fields $j : k \to A/m$ so that $j^*G = q^*H$, where $q : A \to A/m$ is the quotient maps. Two deformations (H_1, i_1) and (H_2, i_2) over A are said to be $\star$-isomorphic if $i_1 = i_1 : k \to A$ and there is an isomorphism of formal group laws $\psi : H_1 \to H_2$ which reduces to the identity map of

$$q^*H_1 = i_1^*G = i_2^*G = q^*H_2.$$

Write $\mathrm{Def}_G(A)$ for the $\star$-isomorphism class of deformations of G over A.

One of the main results of [27] is that the functor

$$A \longmapsto \mathrm{Def}_G(A)$$

is representable in the category of complete local rings and continuous maps; or, put another way, there is a universal deformation of (k, G). This consists of a complete local ring $R = R(k, G)$ and a formal group H over R with the property that for every complete local ring A, there is a natural isomorphism

$$\mathrm{Hom}_{cont}(R, A) \cong \mathrm{Def}_G(A)$$

sending $f : R \to B$ to f^*H_0. Actually, Lubin and Tate prove more; for example, they show that a deformation has no non-trivial $\star$-self-isomorphisms.

[2]Other authors can add to this discussion; for example, Haynes Miller has an elegant exposition of Lubin-Tate theory that includes simple criteria in terms of the Kodaira-Spencer map for characterizing the universal deformation. In this setting the role of $\star$-isomorphisms is clarified and their essential uniqueness becomes obvious.

This last statement can be rephrased as follows. If A is a ring, let $\mathrm{fgl}(A)$ be the groupoid of formal groups on A: the objects are formal group laws over A and the morphisms are isomorphisms of formal group laws. Then the following diagram is a homotopy pull-back of groupoids.

$$
\begin{tikzcd}
\mathrm{Def}_G(A) \arrow[r] \arrow[d] & \mathrm{fgl}(A) \arrow[d] \\
\mathrm{Hom}_{rings}(k, A/m) \arrow[r] & \mathrm{fgl}(A/m)
\end{tikzcd} \tag{7.5}
$$

The set $\mathrm{Hom}_{rings}(k, A/m)$ is regarded as a discrete groupoid. To say this is a homotopy pull-back of groupoids is equivalent to saying that the resulting square of classifying spaces is a homotopy pull-back.

The ring $R(k, G)$ is easy to describe. If G has height n, then $R(k, G) = W(k)[[u_1, \ldots, u_{n-1}]]$ where $W(k)$ is the ring of Witt vector on k. The maximal ideal is $m = (p, u_1, \ldots, u_{n-1})$ and $R/m \cong k$; thus, $R(k, G)$ depends only on k. However, the formal group law H over $R(k, G)$ does depend on G. If $f : R(k, G) \to A$ classifies a deformation (H, i) of G, then $f(u_i)$ is the ith coefficient of the p-series of H.

The assignment $(k, G) \mapsto R(k, G)$ is functorial in the following sense. Let $\mathcal{FG}$ be the category with objects pairs (k, G) of where G is a finite height formal group over k and morphisms pairs $(i, f) : (k_1, G_1) \to (k_2, G_2)$ with $i : k_2 \to k_1$ a morphism of fields and $f : G_1 \to i^*G_2$ an isomorphism. Note there are no morphisms in $\mathcal{FG}$ between objects of different heights. If (i, f) is a morphism in $\mathcal{FG}$, we define a map $R(k_2, G_2) \to R(k_1, G_1)$ as follows. If $f' \in R(k_1, G_1)[[t]]^\times$ is any lift of the isomorphism $f \in k[[t]]^\times$ and H_1 is the canonical formal group over $R(k_1, G_1)$, then $(f')_* H_1$ is an element of $\mathrm{Def}_{G_2}(R(k_1, G_1))$ and this deformation is independent, up to $\star$-isomorphism, of the choice of f'. We can now represent this by a map $R(k_2, G_2) \to R(k_1, G_1)$ and we now have a contravariant functor from $\mathcal{FG}$ to complete local rings.

We next define the graded ring $E(k, G)_* = R(k, G)[u^{\pm 1}]$ and promote H to a formal group $\bar{H}$ over $E(k, G)_*$, as in Equation 7.1. The next result defines the Lubin-Tate homology theories.

Proposition 7.5 *The ring homomorphism $MU_* \to E(k, G)_*$ determined by the formal group $\bar{H}$ is Landweber exact, so there is a homology theory*

$$X \mapsto E(k, G)_*(X) = E(k, G)_* \otimes_{MU_*} MU_* X.$$

This is proved in [37], but it was known well before that. Let $E(k, G)$ be the representing spectrum of this theory.

Over the field $\mathbb{F}_p$ there is a p-typical formal group law, called the Honda formal group law, whose p series is exactly x^{p^n}. For example, if $n = 1$, the Honda formal group law is the multiplicative formal group law. It is customary to extend this over the field $\mathbb{F}_{p^n}$, because all automorphisms of the resulting formal group G_n are realized over that field in the sense that extending over a larger field yields no new automorphisms. (See [34], §A.2 or [26], §VI. 7). The resulting spectrum $E_n = E(\mathbb{F}_{p^n}, G_n)$ is the canonical Lubin-Tate homology theory usually used to govern the chromatic layers in stable homotopy theory.

We now restrict attention to fields k which are algebraic over the prime field $\mathbb{F}_p$. We do this so that any ring homomorphism $f : (A, m_A) \to (B, m_B)$ between complete local rings with residue fields of this type will be, in fact, continuous. To show this, one need only show that $f(m_A) \subseteq m_B$;

however, if $K \subseteq A$ is the kernel of f, then $A/K \to B/m_B$ is one-to-one. The fact that B/m_B is algebraic over $\mathbb{F}_p$ implies A/K is a field, or $K = m_A$.

To shorten notation, we now let $E = E(k_1, G_1)$ and $F = E(k_2, G_2)$. Let $m = m_E \subseteq R(k_1, G_1) = E_0$ and $m_F \subseteq R(k_2, G_2)$ be the maximal ideals. Among other things, we want to discuss the relative Frobenius

$$(k_1)_\phi \otimes_{k_1} (E_0F)/m = (E_0/m)_\phi \otimes_{E_0/m} (E_0F)/m \to E_0F/m.$$

This will be done by calculating and characterizing the set

$$\mathrm{Hom}_{E_*alg}(E_*F, A).$$

Fix a graded ring A; we assume that A is of the form $R[u^{\pm}1]$ where $R = A_0$ is a complete local ring. We would like to calculate the set of graded ring maps $E_*F \to A$. One uses two facts. First, there is natural isomorphism of sets (see [34], §A.1)

$$\mathrm{Hom}_{graded\ rings}(MU_*MU, A) \cong \{(f, g : MU_* \to A, \psi : f^*G_{MU} \to g^*G_{MU}\}$$

where f, g are maps of graded rings, G_{MU} is the universal formal group law and ψ is a strict isomorphism of formal group laws. Second, Landweber exactness supplies a ring map

$$MU_*MU \to E_*F \cong E_* \otimes_{MU_*} MU_*MU \otimes_{MU_*} F_*.$$

These two facts and the fact that E and F are 2-periodic homology theories, and the fact that A is two-periodic ring now imply an isomorphism

$$\mathrm{Hom}_{graded\ rings}(E_*F, A) \cong \{(\psi_1 : E_0 \to A_0, \psi_2 : F_0 \to A_0, g : \psi_1^*H_1 \to \psi_2^*H_2)\}.$$

Here H_1 and H_2 are the canonical formal group laws over $E_0 = R(k_1, G_1)$ and $F_0 = R(k_2, G_2)$ respectively, and g is an isomorphism of formal groups. Put another way, there is a homotopy pull-back diagram of groupoids

$$\begin{tikzcd}
\mathrm{Hom}_{graded\ rings}(E_*F, A) \arrow[r] \arrow[d] & \mathrm{Hom}_{rings}(E_0, A_0) \arrow[d] \\
\mathrm{Hom}_{rings}(F_0, A_0) \cong \mathrm{Def}_{G_2}(A_0) \arrow[r] & \mathrm{fgl}(A_0).
\end{tikzcd}$$

Now, using the description of $\mathrm{Def}_{G_2}(A)$ given in Diagram 7.5 we obtain a pull-back diagram of groupoids

$$\begin{tikzcd}
\mathrm{Hom}_{graded\ rings}(E_*F, A) \arrow[r] \arrow[d] & \mathrm{Hom}_{rings}(E_0, A_0) \arrow[d] \\
\mathrm{Hom}_{rings}(k_2, A_0/m_A) \arrow[r] & \mathrm{fgl}(A_0/m_A).
\end{tikzcd} \tag{7.6}$$

From this, one easily concludes that

$$\mathrm{Hom}_{rings}(E_*F, A) \cong \coprod_{\psi : E_0 \to A_0} \mathcal{FG}((A_0/m_A, \psi^*G_1), (k_2, G_2)). \tag{7.7}$$

Now suppose A_0 is an $R(k_1, G_1) = E_0$ algebra. This is equivalent to specifying a ring homomorphism $E_0 \to A_0$, or, equivalently a deformation of G_1 over A_0. Let H be the resulting formal group law over A. Thus, Equation 7.7 immediately implies the following result.

Proposition 7.6 *IF A_0 is an E_0 algebra, then the natural map*

$$\mathrm{Hom}_{E_*-alg}(E_*F, A) \to \mathcal{FG}((A_0/m_A, H), (k_2, G_2))$$

is an isomorphism.

In particular, if $A = E_*$, one has

$$\mathrm{Hom}_{E_*-alg}(E_*F, E_*) \cong \mathcal{FG}((k_1, G_1), (k_2, G_2)). \tag{7.8}$$

For example, if $k_1 = k_2 = \mathbb{F}_{p^n}$ and $G_1 = G_2$ is height n Honda formal group law, one recovers the Morava stabilizer group – the group of automorphisms of this formal group law.

Note that the target of this isomorphism is empty unless the formal groups G_1 and G_2 have the same height. From this it follows that $E_*F = 0$ unless the two formal groups have the same height. So we assume that the height of G_1 equals the height of G_2 and this common value is $n < \infty$. Then the Diagram 7.6 yields a homotopy pull-back diagram of groupoids

$$\begin{array}{ccc} \mathrm{Hom}_{rings}(E_*F, A) & \longrightarrow & \mathrm{Hom}_{rings}(k_2, A_0/m_A) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{rings}(k_1, A_0/m_A) & \longrightarrow & \mathrm{fgl}_n(A_0/m_A), \end{array}$$

where $\mathrm{fgl}_n(B)$ is the sub-groupoid of $\mathrm{fgl}(B)$ with objects the formal group laws of height n. Now suppose that A_0 is discrete and of characteristic p. Then this diagram becomes a pull-back diagram of groupoids

$$\begin{array}{ccc} \mathrm{Hom}_{rings}(E_0F/m, A_0) & \longrightarrow & \mathrm{Hom}_{rings}(k_2, A_0) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{rings}(k_1, A_0) & \longrightarrow & \mathrm{fgl}_n(A_0). \end{array} \tag{7.9}$$

Let φ^n_B denote the n-fold Frobenius on a ring B of characteristic p. Then, since k_1 and k_2 are perfect

$$(\phi^n_{A_0})^* = (\phi^n_{k_\varepsilon})^* : \mathrm{Hom}_{rings}(k_\varepsilon, A_0) \to \mathrm{Hom}_{rings}(k_e, A_0)$$

is an isomorphism for $\varepsilon = 1$ and $\varepsilon = 2$. The principal lemma needed at this point is

Lemma 7.7 *The n-fold Frobenius induces an equivalence of categories*

$$(\phi^n_{A_0})^* : fgl_n(A_0) \longrightarrow fgl_n(A_0).$$

See [37] §21. The proof isn't hard. If G is any formal group law over A_0, then there is a natural map of formal groups $G \to (\phi^n)^*G$. If G is of height n, then the p-series $[p] : G \to G$ can be uniquely factored

$$G \to (\phi^n)^*G \xrightarrow{f_G} G$$

with f_G an isomorphism and $f_{(\cdot)}$ supplies the equivalence of categories.

Lemma 7.7 and the homotopy pull-back diagram 7.9 imply that the Frobenius on E_0F/m is an isomorphism, and thus one has

Proposition 7.8 *Let $E_* = E(k_1, G_1)$ and $F_* = E(k_2, G_2)$ where G_1 and G_2 are formal groups of finite height over fields k_1 and k_2 which are algebraic over the prime field $\mathbb{F}_p$. Then the relative Frobenius*

$$\phi : (k_1)_\phi \otimes_{k_1} E_0F/m \to E_0F/m$$

is an isomorphism.

In the case $E_0 = F_0 = R(\mathbb{F}_{p^n}, G_n)$, with G_n the height n Honda formal group one can calculate

$$E_0F = \mathbb{F}_{p^n} \otimes_{\mathbb{F}_p} \mathbb{F}_{p^n}[a_0, a_1, a_2, \ldots]/(a_0^{p^n} - 1, a_1^{p^n} - a_1, a_2^{p^n} - a_2, \ldots).$$

Proposition 7.8 and Corollaries 7.3 and 7.4 immediately imply

Theorem 7.9 *Let $E_* = E(k, G)$ where G is a formal group of finite height over a field k which is algebraic over $\mathbb{F}_p$. Then there is a unique E_∞ ring structure on E realizing the commutative algebra structure on E_*.*

Theorem 7.10 *Let $E_* = E(k_1, G_1)$ and $F_* = E(k_2, G_2)$ where G_1 and G_2 are formal groups of finite height over fields k_1 and k_2 which are algebraic over the prime field k_p. Then the space $\mathcal{S}^{\mathcal{L}}(E, F)$ of E_∞ maps from E to F has weakly contractible components and*

$$\pi_0\mathcal{S}^{\mathcal{L}}(E, F) \cong \mathcal{FG}((k_1, G_1), (k_2, G_2)).$$

In particular, note that $\pi_0\mathcal{S}^{\mathcal{L}}(E_n, E_n)$ is the Morava stabilizer group.

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