MAT1300
MANDATORY ASSIGNMENT I
PROPOSED SOLUTIONS

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Problem I

(a) The inequality

$$x^y < y^x$$

holds if and only if

$$y\ln(x) = \ln(x^y) < \ln(y^x) = x\ln(y)$$

since ln is strictly increasing, and this holds if and only if

$$f(x) = \frac{\ln(x)}{x} < \frac{\ln(y)}{y} = f(y)$$

since x and y are positive. This proves (1). Claim (3) follows by interchanging the roles of x and y.

Claim (2) follows from claims (1) and (3), since $x^y = y^x$ is false if and only if $x^y < y^x$ or $x^y > y^x$, and $f(x) = f(y)$ is false if and only if $f(x) < f(y)$ or $f(x) > f(y)$. Alternatively, one can repeat the proof of (1) with equality replacing inequality.

(b) By the quotient rule,

$$f'(x) = \frac{(1/x)\cdot x - \ln(x)\cdot 1}{x^2} = \frac{1-\ln(x)}{x^2}$$

since $\ln'(x) = 1/x$. Here $f'(x)$ has the same sign as $1-\ln(x)$, since x^2 is positive. For $x < e$ we have $\ln(x) < 1$, so $1-\ln(x) > 0$. For $x = e$ we have $\ln(x) = 1$, so $1-\ln(x) = 0$. For $x > e$ we have $\ln(x) > 1$, so $1-\ln(x) < 0$.

The mean value theorem. *If $f\colon [a,b] \to \mathbb{R}$ is a continuous function with f differentiable on (a,b), then we can find a $c \in (a,b)$ such that*

$$f(b) - f(a) = f'(c)(b-a)\,.$$

(Implicitly, $a < b$.)

To show that f is strictly increasing on $(0,e]$, let $0 < a < b \le e$. We must show that $f(a) < f(b)$. We apply the mean value theorem to the function $f\colon [a,b] \to \mathbb{R}$ given by $f(x) = \ln(x)/x$, which is continuous on $[a,b]$ and differentiable on (a,b). Hence there exists a $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$. Here

$c \in (a,b) \subset (0,e)$, so $f'(c) > 0$. We also know that $b - a > 0$, so the product $f(b) - f(a) = f'(c)(b-a) > 0$ is positive, and hence $f(b) > f(a)$.

To show that f is strictly decreasing on $[e,\infty)$, we let $e \le a < b < \infty$ and apply the mean value theorem. Hence there exists a $c \in (a,b) \subset (e,\infty)$ such that $f(b) - f(a) = f'(c)(b-a)$. Here $f'(c) < 0$ and $b - a > 0$, so $f(b) - f(a) < 0$ and $f(b) < f(a)$. Hence f is strictly decreasing on $[e,\infty)$.

(c) Since $x = \pi$ is less than $y = \sqrt{10}$, and f is strictly decreasing on (e,∞), we know that $f(x) > f(y)$, so that $\pi^{\sqrt{10}} = x^y > y^x = (\sqrt{10})^\pi$. Hence, $\pi^{\sqrt{10}}$ is the greater number.

(d) $f(e) = \ln(e)/e = 1/e$, while

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \ln(x)/x = \lim_{x\to\infty} (1/x)/1 = \lim_{x\to\infty} 1/x = 0$$

by l'Hôpital's rule.

The intermediate value theorem. *If $f\colon [a,b] \to \mathbb{R}$ is continuous, $f(a) \ge 0$ and $f(b) \le 0$, then there exists a $c \in [a,b]$ such that $f(c) = 0$. (Implicitly, $a \le b$.)*

We first show that f maps (e,∞) into $(0,1/e)$, so that $f((e,\infty)) \subseteq (0,1/e)$. Let $e < x < \infty$. Then $1/e = f(e) > f(x)$ since f is strictly decreasing on $[e,\infty)$. Furthermore, $\ln(x) > \ln(e) = 1 > 0$ and $x > e > 0$ are both positive, so $f(x) = \ln(x)/x > 0$ is also positive. This proves that $0 < f(x) < 1/e$ for all $x \in (e,\infty)$.

It remains to prove that f maps (e,∞) onto $(0,1/e)$, so that for each $t \in (0,1/e)$ there is some $c \in (e,\infty)$ with $f(c) = t$. For this we will use the intermediate value theorem, applied to the function $f(x) - t$. Some maneuvering is needed to find a suitable interval $[a,b]$ with $f(a) > t > f(b)$.

First, we let $a = e$, so that $f(a) = f(e) = 1/e > t$. To find b, we use that $\lim_{x\to\infty} f(x) = 0$. This means that for each $\epsilon > 0$ there is some $M(\epsilon)$ such that for all $x \ge M(\epsilon)$ we have $|f(x)| < \epsilon$. We apply this with $\epsilon = t > 0$. Letting $b = M(t)$, we get that $f(b) < t$.

Now $f(x) - t$ defines a continuous function on $[a,b]$, with $f(a) - t > 0$ and $f(b) - t < 0$. By the intermediate value theorem there is a $c \in [a,b]$ such that $f(c) - t = 0$, so that $f(c) = t$. We cannot have $c = a$, since $f(a) > t$, so $c \in (a,b] \subset (e,\infty)$. This proves that f maps (e,∞) onto $(0,1/e)$.

(e) $f(1) = \ln(1)/1 = 0/1 = 0$. By part (a), the equation $x^y = y^x$ has the same solutions (x,y) as the equation $f(x) = f(y)$. We first assume that $x \in (0,1]$. Then $f(x) \le f(1) = 0$, since f is strictly increasing on $(0,1] \subset (0,e]$. For all $y > 1$ we have $f(y) > 0$, so the equation $f(x) = f(y)$ has no solutions with $y > 1$. For $y \in (0,1]$, the only solution to the equation $f(x) = f(y)$ is $y = x$, since f is strictly increasing on that interval, and therefore injective (= one-to-one).

Next assume that $x = e$. The function f is strictly increasing on $(0,e]$, so $f(y) < f(e)$ for all $y \in (0,e)$. The function f is strictly decreasing on $[e,\infty)$, so $f(e) > f(y)$ for all $y \in (e,\infty)$. Hence there is no other y than $y = e$ such that $f(y) = f(e)$.

(f) Now assume that $x \in (1,e)$. We want to show that the equation $f(x) = f(y)$ has exactly two solutions: one with $y = x$ and one with $y \in (e,\infty)$. It is clear that $y = x$ is one solution, with $y \in (1,e) \subset (0,e]$. We know that f is strictly increasing on $(0,e]$, so this will also be the only solution to $f(x) = f(y)$ with $y \in (0,e]$.

It remains to consider $y \in (e, \infty)$. By assumption $x \in (1, e)$, so $0 = f(1) < f(x) < f(e) = 1/e$, since f is strictly increasing on $(1, e) \subset (0, e]$. By part (d), the function f maps (e, ∞) onto the interval $(0, 1/e)$, so for each number $t \in (0, 1/e)$ there is a $c \in (e, \infty)$ with $f(c) = t$. We apply this with $t = f(x)$, which we have seen lies in $(0, 1/e)$. We let $y = c$ be the corresponding number in (e, ∞) with $f(y) = f(c) = t = f(x)$. This y gives one solution to the equation $f(x) = f(y)$, with $y \in (e, \infty)$. This is the only such solution, since f is strictly decreasing on $[e, \infty)$.

Problem II

(a) As rational numbers,

$$x_2 = \frac{1}{2}(3 + \frac{10}{3}) = \frac{1}{2} \cdot \frac{9 + 10}{3} = \frac{19}{6}$$

and

$$\begin{aligned} x_3 &= \frac{1}{2}(\frac{19}{6} + \frac{10}{(19/6)}) = \frac{1}{2}(\frac{19}{6} + \frac{60}{19}) \\ &= \frac{1}{2} \cdot \frac{19 \cdot 19 + 6 \cdot 60}{6 \cdot 19} = \frac{361 + 360}{2 \cdot 114} = \frac{721}{228} . \end{aligned}$$

These have decimal expansions

$$\frac{19}{6} = 3.166\ 666\ \dots$$

and

$$\frac{721}{228} = 3.162\ 280\ \dots .$$

(The rounded answers 3.166 667 and 3.162 281 are also acceptable.)

(b) We compute:

$$\begin{aligned} x_{n+1} &= \frac{1}{2}\big(x_n + \frac{10}{x_n}\big) = \frac{1}{2}\big(\sqrt{10} + y_n + \frac{10}{\sqrt{10} + y_n}\big) \\ &= \frac{1}{2}\big(\sqrt{10} + y_n + \frac{10 - y_n^2}{\sqrt{10} + y_n} + \frac{y_n^2}{\sqrt{10} + y_n}\big) \\ &= \frac{1}{2}\big(\sqrt{10} + y_n + \frac{(\sqrt{10} + y_n)(\sqrt{10} - y_n)}{\sqrt{10} + y_n} + \frac{y_n^2}{\sqrt{10} + y_n}\big) \\ &= \frac{1}{2}\big(\sqrt{10} + y_n + \sqrt{10} - y_n + \frac{y_n^2}{\sqrt{10} + y_n}\big) \\ &= \frac{1}{2}\big(2\sqrt{10} + \frac{y_n^2}{\sqrt{10} + y_n}\big) \\ &= \sqrt{10} + \frac{1}{2} \cdot \frac{y_n^2}{\sqrt{10} + y_n} . \end{aligned}$$

Alternatively, we need to show that

$$x_{n+1} = \frac{1}{2}\big(x_n + \frac{10}{x_n}\big) = \frac{1}{2}\big(\sqrt{10} + y_n + \frac{10}{\sqrt{10} + y_n}\big)$$

is equal to

$$\sqrt{10} + \frac{1}{2} \cdot \frac{y_n^2}{\sqrt{10} + y_n} = \frac{1}{2}\big(2\sqrt{10} + \frac{y_n^2}{\sqrt{10} + y_n}\big)$$

so it suffices to check that

$$\sqrt{10} + y_n + \frac{10}{\sqrt{10} + y_n} = 2\sqrt{10} + \frac{y_n^2}{\sqrt{10} + y_n} .$$

Rearranging, this is equivalent to

$$\frac{10}{\sqrt{10} + y_n} - \frac{y_n^2}{\sqrt{10} + y_n} = \sqrt{10} - y_n$$

which follows from

$$10 - y_n^2 = (\sqrt{10} + y_n)(\sqrt{10} - y_n) .$$

Either way, since $x_n = \sqrt{10} + y_n$ is positive for all $n \ge 1$, and $y_n^2 \ge 0$, it follows that $y_n^2/(\sqrt{10}+y_n) \ge 0$. Hence half of this, which is y_{n+1}, is also non-negative. We can rewrite $y_{n+1} \ge 0$ for all $n \ge 1$ as $y_n \ge 0$ for all $n \ge 2$, which is what we wanted to prove. It is then clear that $x_n = \sqrt{10} + y_n \ge \sqrt{10}$ for $n \ge 2$.

(c) From $x_n \ge \sqrt{10}$ we get $x_n^2 \ge 10$ and $x_n \ge 10/x_n$, for all $n \ge 2$. Hence the mean satisfies

$$x_{n+1} = \frac{1}{2}\big(x_n + \frac{10}{x_n}\big) \le \frac{1}{2}(x_n + x_n) = x_n$$

for all $n \ge 2$, so $(x_n)_{n=2}^{\infty}$ is a decreasing sequence.

The fundamental axiom of analysis. *If $a_n \in \mathbb{R}$ for each $n \ge 1$, $A \in \mathbb{R}$ and $a_1 \le a_2 \le a_3 \le \dots$ and $a_n \in A$ for each n, then there exists an $a \in \mathbb{R}$ such that $a_n \to a$ as $n \to \infty$. (Equivalently: every increasing sequence that is bounded above converges to a limit.)*

The sequence $(x_n)_{n=2}^{\infty}$ is decreasing and bounded below by $\sqrt{10}$. Hence The sequence $(a_n)_{n=1}^{\infty}$ with $a_n = -x_{n+1}$ for $n \ge 1$ is increasing and bounded above by $A = -\sqrt{10}$, hence converges to a limit a as $n \to \infty$. Then the sequence $(x_n)_{n=2}^{\infty}$ converges to the limit $-a$ as $n \to \infty$. It follows that the sequence $(x_n)_{n=1}^{\infty}$ also converges to $-a$, since the question of whether a sequence converges does not depend on the first finitely many terms of the sequence. (See Exercise 1.7.)

(d) Let $r = \lim_{n\to\infty} x_n$. Since each $x_n \ge \sqrt{10}$ for $n \ge 2$, we know that $r \ge \sqrt{10}$. Let $h(x) = \frac{1}{2}(x + (10/x))$. This is a continuous function for $x > 0$, hence it is continuous at r. It follows that the image sequence of h applied to $(x_n)_n$, with n-th term $x_{n+1} = h(x_n)$, converges to $h(r)$. (See Lemma 1.15.) But this is a subsequence of the original sequence $(x_n)_n$, with limit r. Hence these two limits are equal, so that

$$r = h(r) = \frac{1}{2}\big(r + \frac{10}{r}\big)$$

(See Lemma 1.6(i).)

We can rewrite this as $2r = r + (10/r)$, so $r = 10/r$ and $r^2 = 10$. Since $r > 0$ we deduce that $r = \sqrt{10}$.

(e) If $0 \le y_n < 10^{-d}$ then $0 \le y_n^2 < (10^{-d})^2 = 10^{-2d}$, and $2(\sqrt{10}+y_n) \ge 2\sqrt{10} \ge 1$, so $y_{n+1} < 10^{-2d}/1 = 10^{-2d}$.

Problem III

(a) By assumption, $\mathbf{z}_k \to \mathbf{z}$ as $k \to \infty$, so for each $\epsilon > 0$ there is a natural number $k_0(\epsilon)$ such that for all $k \geq k_0(\epsilon)$ we have $\|\mathbf{z}_k - \mathbf{z}\| < \epsilon$. In particular, with $\epsilon = 1$, if $k \geq k_0 = k_0(1)$ we have $\|\mathbf{z}_k - \mathbf{z}\| \leq 1$, so $\|\mathbf{z}_k\| \leq \|\mathbf{z}\| + 1$ for all $k \geq k_0$. Let

$$M = \max_{1 \leq k \leq k_0} \|\mathbf{z}_k\| + 1 \,.$$

Then $\|\mathbf{z}_j\| \leq M$ for all natural numbers j. Letting $j \to \infty$, it follows that $\|\mathbf{z}\| \leq M$. Hence $\|\mathbf{y}\| \leq M$ for all $\mathbf{y} \in E$, so E is a bounded set.

(b) Let $(\mathbf{x}_n)_n$ be a sequence in E, converging to a limit $\mathbf{x}$ in $\mathbb{R}^m$. We must prove that $\mathbf{x} \in E$.

First proof: Suppose first that there is a $\mathbf{y} \in E$ such that $\mathbf{x}_n = \mathbf{y}$ for infinitely many n. Numbering these n in increasing order as $n(1) < n(2) < \dots$, we get that $(\mathbf{x}_n)_n$ has the subsequence $(\mathbf{x}_{n(j)})_j$, constant at $\mathbf{y}$. It follows that the limit $\mathbf{x}$ of $(\mathbf{x}_n)_n$ is equal to the limit $\mathbf{y}$ of $(\mathbf{x}_{n(j)})_j$, so $\mathbf{x} = \mathbf{y} \in E$.

Otherwise, the sequence $(\mathbf{x}_n)_n$ takes each value $\mathbf{y} \in E$ only finitely many times. In particular, it takes the value $\mathbf{z}$ only finitely many times, so there is some natural number $n(1)$ such that $\mathbf{x}_{n(1)}$ is not equal to $\mathbf{z}$. There must then be some natural number $k(1)$ such that

$$\mathbf{x}_{n(1)} = \mathbf{z}_{k(1)} \,.$$

Next, there are only finitely many n such that $\mathbf{x}_n$ is equal to the finitely many values $\mathbf{z}$, $\mathbf{z}_1, \dots, \mathbf{z}_{k(1)}$. Hence we can find an $n(2) > n(1)$ and a $k(2) > k(1)$ such that

$$\mathbf{x}_{n(2)} = \mathbf{z}_{k(2)} \,.$$

Continuing by induction, we can for all natural numbers j find an $n(j+1) > n(j)$ and a $k(j+1) > k(j)$ such that

$$\mathbf{x}_{n(j+1)} = \mathbf{z}_{k(j+1)} \,.$$

For there are only finitely many n such that $\mathbf{x}_n$ is equal to the finitely many values $\mathbf{z}$, $\mathbf{z}_1, \dots, \mathbf{z}_{k(j)}$, so we pick $n(j+1)$ to be one of the infinitely many n for which $\mathbf{x}_n$ is equal to a $\mathbf{z}_k$ with $k > k(j)$. Then we set $k(j+1)$ equal to this k.

We have now proved that the subsequence $(\mathbf{x}_{n(j)})_j$ of $(\mathbf{x}_n)_n$ is equal to the subsequence $(\mathbf{z}_{k(j)})_j$ of $(\mathbf{z}_k)_k$. It follows that the limit $\mathbf{x}$ of $(\mathbf{x}_n)_n$ and $(\mathbf{x}_{n(j)})_j$ is equal to the limit of $(\mathbf{z}_{k(j)})_j$, which is equal to the limit $\mathbf{z}$ of $(\mathbf{z}_k)_k$. Hence $\mathbf{x} = \mathbf{z} \in E$, as we wanted to prove.

Second proof: If there is a $\mathbf{y} \in E$ such that $\mathbf{x}_n = \mathbf{y}$ for infinitely many n, we argue as above to deduce that $\mathbf{x} = \mathbf{y} \in E$.

Otherwise, the sequence $(\mathbf{x}_n)_n$ only takes each value in E a finite number of times, and for the rest of the argument we assume this. We know that $\mathbf{z}_k \to \mathbf{z}$ as $k \to \infty$, so for each $\epsilon > 0$ there is a $k_0(\epsilon)$ such that for each $k \geq k_0(\epsilon)$ we have $\|\mathbf{z}_k - \mathbf{z}\| < \epsilon$. Hence $\|\mathbf{z}_k - \mathbf{z}\| \geq \epsilon$ only for (some of) the finitely many k with $k < k_0(\epsilon)$. For each k we know that $\mathbf{x}_n = \mathbf{z}_k$ only for finitely many n, hence there are only finitely many natural numbers n such that $\|\mathbf{x}_n - \mathbf{z}\| \geq \epsilon$. Let $n_0(\epsilon)$ be greater than all of these natural numbers. Then for all $n \geq n_0(\epsilon)$ we have $\|\mathbf{x}_n - \mathbf{z}\| < \epsilon$. Since $\epsilon > 0$ was arbitrary, this proves that $\mathbf{x}_n \to \mathbf{z}$ as $n \to \infty$, so $\mathbf{x} = \mathbf{z}$ is in E, as required.

(c) Yes, $\mathbf{f}(E)$ is closed and bounded in $\mathbb{R}^p$ by Theorem 4.41.

(d) Yes, $\mathbf{f}$ is uniformly continuous by Theorem 4.63.