MAT1300
MANDATORY ASSIGNMENT II
PROPOSED SOLUTIONS

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Problem I

(a) Let $g(\mathbf{x}) = \|\mathbf{x}\|^2 = \mathbf{x} \cdot \mathbf{x}$. By the scalar product rule (Exercise 6.26(ii))

$$Dg(\mathbf{x})(\mathbf{h}) = \mathbf{h} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{h} = 2(\mathbf{x} \cdot \mathbf{h})\,.$$

By the chain rule (Lemma 6.19 or Exercise 6.26(v))

$$D(1/g)(\mathbf{x})(\mathbf{h}) = -Dg(\mathbf{x})(\mathbf{h})/g(\mathbf{x})^2 = -2(\mathbf{x} \cdot \mathbf{h})/g(\mathbf{x})^2\,.$$

By the product rule applied to $\mathbf{f}(\mathbf{x}) = \mathbf{x}(1/g(\mathbf{x}))$ (Exercise 6.26(iii))

$$\begin{aligned} D\mathbf{f}(\mathbf{x})(\mathbf{h}) &= \mathbf{h}(1/g(\mathbf{x})) + \mathbf{x}(-2(\mathbf{x} \cdot \mathbf{h})/g(\mathbf{x})^2) \\ &= \frac{1}{\|\mathbf{x}\|^2}\mathbf{h} - 2\frac{\mathbf{x} \cdot \mathbf{h}}{\|\mathbf{x}\|^4}\mathbf{x} = \frac{1}{\|\mathbf{x}\|^2}\Big(\mathbf{h} - 2\frac{\mathbf{x} \cdot \mathbf{h}}{\|\mathbf{x}\|^2}\mathbf{x}\Big)\,. \end{aligned}$$

(b) The j-th partial derivative of $\mathbf{f}$ at $\mathbf{x}$ is

$$\begin{aligned} D\mathbf{f}(\mathbf{x})(\mathbf{e}_j) &= \frac{1}{\|\mathbf{x}\|^2}\Big(\mathbf{e}_j - 2\frac{\mathbf{x} \cdot \mathbf{e}_j}{\|\mathbf{x}\|^2}\mathbf{x}\Big) \\ &= \frac{1}{\|\mathbf{x}\|^2}\Big(\mathbf{e}_j - 2\frac{x_j}{\|\mathbf{x}\|^2}\mathbf{x}\Big)\,. \end{aligned}$$

where $\mathbf{e}_j$ is the j-th standard basis vector in $\mathbb{R}^m$. Its i-th coordinate is

$$\begin{aligned} \mathbf{f}_{i,j}(\mathbf{x}) &= \frac{1}{\|\mathbf{x}\|^2}\Big((\mathbf{e}_j)_i - 2\frac{x_j}{\|\mathbf{x}\|^2}x_i\Big) \\ &= \frac{1}{\|\mathbf{x}\|^2}\Big(\delta_{ij} - 2\frac{x_i x_j}{\|\mathbf{x}\|^2}\Big) \end{aligned}$$

where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

(c) We compute:

$$\begin{aligned} \alpha(\mathbf{h}) \cdot \alpha(\mathbf{k}) &= \frac{1}{\|\mathbf{x}\|^2}\Big(\mathbf{h} - 2\frac{\mathbf{x} \cdot \mathbf{h}}{\|\mathbf{x}\|^2}\mathbf{x}\Big) \cdot \frac{1}{\|\mathbf{x}\|^2}\Big(\mathbf{k} - 2\frac{\mathbf{x} \cdot \mathbf{k}}{\|\mathbf{x}\|^2}\mathbf{x}\Big) \\ &= \frac{1}{\|\mathbf{x}\|^4}\Big(\mathbf{h} \cdot \mathbf{k} - 2\frac{\mathbf{x} \cdot \mathbf{k}}{\|\mathbf{x}\|^2}\mathbf{x} \cdot \mathbf{h} - 2\frac{\mathbf{x} \cdot \mathbf{h}}{\|\mathbf{x}\|^2}\mathbf{x} \cdot \mathbf{k} + 4\frac{\mathbf{x} \cdot \mathbf{h}}{\|\mathbf{x}\|^2}\frac{\mathbf{x} \cdot \mathbf{k}}{\|\mathbf{x}\|^2}\mathbf{x} \cdot \mathbf{x}\Big) \\ &= \frac{1}{\|\mathbf{x}\|^4}\Big(\mathbf{h} \cdot \mathbf{k} - 4\frac{(\mathbf{x} \cdot \mathbf{k})(\mathbf{x} \cdot \mathbf{h})}{\|\mathbf{x}\|^2} + 4\frac{(\mathbf{x} \cdot \mathbf{h})(\mathbf{x} \cdot \mathbf{k})}{\|\mathbf{x}\|^2}\Big) \\ &= \frac{1}{\|\mathbf{x}\|^4}\mathbf{h} \cdot \mathbf{k}\,. \end{aligned}$$

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(d) For each $\mathbf{h} \in \mathbb{R}^m$ we have

$$\|\alpha(\mathbf{h})\|^2 = \alpha(\mathbf{h}) \cdot \alpha(\mathbf{h}) = \frac{1}{\|\mathbf{x}\|^4}\mathbf{h} \cdot \mathbf{h} = \frac{1}{\|\mathbf{x}\|^4}\|\mathbf{h}\|^2$$

so

$$\|\alpha(\mathbf{h})\| = \frac{1}{\|\mathbf{x}\|^2}\|\mathbf{h}\| .$$

Hence $\|\alpha(\mathbf{h})\|/\|\mathbf{h}\| = 1/\|\mathbf{x}\|^2$ for all $\mathbf{h} \neq \mathbf{0}$, and $\|D\mathbf{f}(\mathbf{x})\| = \|\alpha\|$, the supremum of these numbers, equals their common value $1/\|\mathbf{x}\|^2$.

(e) We apply the mean value inequality (Theorem 6.27) to $\mathbf{f}\colon E \to \mathbb{R}^m$. Clearly E is open and $\mathbf{f}$ is differentiable. By assumption, $\|\mathbf{x}\| \geq R$ for all $\mathbf{x}$ on the line segment joining $\mathbf{a}$ and $\mathbf{b}$, so $\|D\mathbf{f}(\mathbf{x})\| = 1/\|\mathbf{x}\|^2 \leq 1/R^2$ for these $\mathbf{x}$. Hence

$$\|\mathbf{f}(\mathbf{a}) - \mathbf{f}(\mathbf{b})\| \leq (1/R^2)\|\mathbf{a} - \mathbf{b}\| .$$

Substituting $\mathbf{f}(\mathbf{a}) = \mathbf{a}/\|\mathbf{a}\|^2$ and $\mathbf{f}(\mathbf{b}) = \mathbf{b}/\|\mathbf{b}\|^2$ we get the desired result.

Problem II

(a) Clearly $|f(x)| \leq 1$ for all $x \in [0,1]$, so f is bounded. The function f is not continuous at $x = 0$, since we have $f(y) = 1$ for $y = 1/(2\pi n + \pi/2)$ arbitrarily close to 0, but $f(0) \neq 1$.

(b) We show that for each $\epsilon > 0$ there is a dissection $\mathcal{D}$ of $[0,1]$ such that $S(f,\mathcal{D}) - s(f,\mathcal{D}) < \epsilon$. Choose a $c \in (0,1)$ with $2c < \epsilon/2$. Let $f_1 = f|[0,c]$ denote the restriction of f to $[0,c]$, and let $f_2 = f|[c,1]$ denote the restriction of f to $[c,1]$. Let $\mathcal{D}_1 = \{0 < c\}$ be the "trivial" dissection of $[0,c]$. Then

$$\sup\{f(t) \mid 0 \leq t \leq c\} = 1$$
$$\inf\{f(t) \mid 0 \leq t \leq c\} = -1$$

so

$$S(f_1,\mathcal{D}_1) - s(f_1,\mathcal{D}_1) = c - (-c) = 2c < \epsilon/2 .$$

The function $f_2\colon [c,1] \to \mathbb{R}$ is continuous, hence Riemann integrable by Theorem 8.32. Hence there exists a dissection $\mathcal{D}_2 = \{c = x_0 < x_1 < \cdots < x_n = 1\}$ of $[c,1]$ such that

$$S(f_2,\mathcal{D}_2) - s(f_2,\mathcal{D}_2) < \epsilon/2$$

by Lemma 8.13(i). We let

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 = \{0 < c = x_0 < x_1 < \cdots < x_n = 1\}$$

be the combined dissection of $[0,1]$. Then

$$S(f,\mathcal{D}) = S(f_1,\mathcal{D}_1) + S(f_2,\mathcal{D}_2)$$
$$s(f,\mathcal{D}) = s(f_1,\mathcal{D}_1) + s(f_2,\mathcal{D}_2)$$

so

$$S(f,\mathcal{D}) - s(f,\mathcal{D}) = (S(f_1,\mathcal{D}_1) - s(f_1,\mathcal{D}_1)) + (S(f_2,\mathcal{D}_2) - s(f_2,\mathcal{D}_2)) < \epsilon/2 + \epsilon/2 = \epsilon .$$

Hence f is Riemann integrable, by Lemma 8.13(i).

(c) Yes. The argument given in (b) generalizes to such g. If $|g(x)| \leq K$ for all $x \in [a,b]$, we choose $c \in (a,b)$ such that $2K(c-a) < \epsilon/2$. The rest of the proof then goes through as above.