MAT3500/4500 Topology
Autumn 2010
Solutions to the Mandatory Assignment

(1) The open subsets are $\emptyset$, $\{\mathbf{n}\}$, $\{\mathbf{s}\}$, $\{\mathbf{n}, \mathbf{s}\}$, $\{\mathbf{e}, \mathbf{n}, \mathbf{s}\}$, $\{\mathbf{n}, \mathbf{w}, \mathbf{s}\}$ and D.

(2) No, D is not Hausdorff. The points $\mathbf{e}$ and $\mathbf{n}$ cannot be separated by disjoint neighborhoods. (Don't say that one set is disjoint: a pair of sets can be disjoint!)

(3) Yes, D is compact. It has only finitely many open subsets, so any collection of open sets is already a finite collection. (Be careful with the wording here! Distinguish between a finite covering by subsets and a covering by finite subsets. Also distinguish between a cover and the set it covers.)

(4) No, there is no homeomorphism $D \cong A \times B$ where A and B are 2-point spaces.

Proof 1: The open subsets of $A \times B$ are unions of products $U \times V$ with U open in A and V open in B. The open one-point sets in $A \times B$ cannot be unions of proper, nontrivial subsets, and must therefore be of the form $U \times V$ with U and V open one-point sets. If $D \cong A \times B$, then the number of open one-point sets in D (namely 2) equals the number of open one-point sets in $A \times B$, which equals the product of the number of open one-points sets in A and the number of open one-points sets in B. Since 2 only factors as $2 \cdot 1$ or $1 \cdot 2$, A must have 2 open points (be discrete) and B must have 1 open point (a Sierpinski topology), or vice versa. In either case $A \times B$ will have 3 open two-point sets, unlike D, which has only 1 open two-point set.

Proof 2: If A has the trivial topology, then the open sets of $A \times B$ are of the form $A \times V$ where V is open in B, so there are at most 4 of these, unlike D which has 7 open subsets. Similarly, B cannot have the trivial topology. If neither A nor B have the trivial topology, then there is an open point $a \in A$ and an open point $b \in B$. Then $\{a\} \times B$ and $A \times \{b\}$ are 2 different open two-point subsets of $A \times B$. Since D only has 1 open two-point subset, it cannot be homeomorphic to $A \times B$.

(Note that the open subsets of $A \times B$ are not in general of the form $U \times V$, but will be unions of such products.)

(5) The closed subsets are $\emptyset$, $\{\mathbf{e}\}$, $\{\mathbf{w}\}$, $\{\mathbf{e}, \mathbf{w}\}$, $\{\mathbf{e}, \mathbf{n}, \mathbf{w}\}$, $\{\mathbf{e}, \mathbf{w}, \mathbf{s}\}$ and D.

(6) The closures are: $\overline{\{\mathbf{e}\}} = \{\mathbf{e}\}$, $\overline{\{\mathbf{n}\}} = \{\mathbf{e}, \mathbf{n}, \mathbf{w}\}$, $\overline{\{\mathbf{w}\}} = \{\mathbf{w}\}$ and $\overline{\{\mathbf{s}\}} = \{\mathbf{e}, \mathbf{w}, \mathbf{s}\}$.

(7) Let $f\colon D \to \mathbb{R}$ be a map. Let $a = f(\mathbf{n})$ and $b = f(\mathbf{s})$. Since $\{a\} \subset \mathbb{R}$ is closed, the preimage $f^{-1}(a) \subset D$ is closed and contains $\{\mathbf{n}\}$. Hence it contains the closure $\{\mathbf{e}, \mathbf{n}, \mathbf{w}\}$, so that $f(\mathbf{e}) = f(\mathbf{n}) = f(\mathbf{w}) = a$.

Since $\{b\} \subset \mathbb{R}$ is closed, the preimage $f^{-1}(b) \subset D$ is closed and contains $\{\mathbf{s}\}$. Hence it contains the closure $\{\mathbf{e}, \mathbf{w}, \mathbf{s}\}$, so that $f(\mathbf{e}) = f(\mathbf{w}) = f(\mathbf{s}) = b$. It follows that $a = f(\mathbf{e}) = b$ and f is constant.

(8) No. For any map $r\colon D \to C \subset \mathbb{R}^2$ the components $r_1 = \pi_1 \circ r$ and $r_2 = \pi_2 \circ r$ are maps $D \to \mathbb{R}$, hence are constant by the previous problem. Hence r must be constant, so $p \circ r\colon D \to D$ is constant, and not equal to the identity map of D.

(9) Yes, p is open. Let $U \subset C$ be open.

If $\mathbf{e} \in p(U)$ then $(x, 0) \in U$ for some $x > 0$. Since U is open, we have $(x, y) \in U$ for some $y > 0$, as well as for some $y < 0$. Hence $\mathbf{n} \in p(U)$ and $\mathbf{s} \in p(U)$, so $\{\mathbf{e}, \mathbf{n}, \mathbf{s}\} \subset U$.

If $\mathbf{w} \in p(U)$ then $(x, 0) \in U$ for some $x < 0$. Since U is open, we have $(x, y) \in U$ for some $y > 0$, as well as for some $y < 0$. Hence $\mathbf{n} \in p(U)$ and $\mathbf{s} \in p(U)$, so $\{\mathbf{n}, \mathbf{w}, \mathbf{s}\} \subset U$.

Since $\{\mathbf{n}\}$ and $\{\mathbf{s}\}$ are open in D, it follows that $p(U)$ contains a neighborhood of each of its points, hence is open.

(10) No, p is not closed. For instance, the singleton set $L = \{(0, 1)\} \subset C$ is closed, but its image $p(L) = \{\mathbf{n}\} \subset D$ is not closed. (Do not say that a set is open when you mean to say that it is not closed!)

(11) Yes, D is connected. By (1) and (5) the only subsets that are both open and closed are $\emptyset$ and D itself.

(12) Let $v\colon [0, \pi] \to C$ be given by $v(t) = (\cos t, \sin t)$. This is a continuous path from $v(0) = (1, 0)$ to $v(\pi) = (-1, 0)$. Hence the composite $p \circ v\colon [0, \pi] \to D$ is a continuous path from $p(1, 0) = \mathbf{e}$ to $p(-1, 0) = \mathbf{w}$. Note that $(p \circ v)(t) = \mathbf{n}$ for all $t \in (0, \pi)$.

(13) The composite $p \circ \alpha$ factors as $k \circ p$, with $k\colon D \to D$ given by $k(\mathbf{e}) = \mathbf{e}$, $k(\mathbf{n}) = \mathbf{s}$, $k(\mathbf{w}) = \mathbf{w}$ and $k(\mathbf{s}) = \mathbf{n}$. Then k is continuous since $p \circ \alpha$ is continuous and p is a quotient map.

The composite $p \circ \beta$ does not factor through p, since not all points in $p^{-1}(\mathbf{n}) = \{(x, y) \mid y > 0\}$ have the same image under $p \circ \beta$. For example, $(p \circ \beta)(0, 1) = \mathbf{e}$, while $(p \circ \beta)(1, 1) = \mathbf{n}$.

(14) A homeomorphism $h\colon D \to D$ must take each open point, i.e., $\mathbf{n}$ or $\mathbf{s}$, to an open point. Similarly, it must take each closed point, i.e., $\mathbf{e}$ or $\mathbf{w}$, to a closed point. The four possible permutations of D satsfying this restriction are

(a) The identity mapping $(\mathbf{e}, \mathbf{n}, \mathbf{w}, \mathbf{s})$ to $(\mathbf{e}, \mathbf{n}, \mathbf{w}, \mathbf{s})$.

(b) The transposition $k = (\mathbf{ns})$ mapping $(\mathbf{e}, \mathbf{n}, \mathbf{w}, \mathbf{s})$ to $(\mathbf{e}, \mathbf{s}, \mathbf{w}, \mathbf{n})$.

(c) The transposition $\ell = (\mathbf{ew})$ mapping $(\mathbf{e}, \mathbf{n}, \mathbf{w}, \mathbf{s})$ to $(\mathbf{w}, \mathbf{n}, \mathbf{e}, \mathbf{s})$.

(d) The permutation $k \circ \ell = (\mathbf{ns})(\mathbf{ew})$ mapping $(\mathbf{e}, \mathbf{n}, \mathbf{w}, \mathbf{s})$ to $(\mathbf{w}, \mathbf{s}, \mathbf{e}, \mathbf{n})$.

We saw in (13) that k is continuous. It is its own inverse, hence is a homeomorphism. The transposition ℓ is also continuous (by a similar argument), and its own inverse, hence a homeomorphism. It follows that the composite $k \circ \ell$ is a homeomorphism. Thus all four of these maps are homeomorphisms.