HOMOTOPY THEORY OF Γ-SPACES, SPECTRA, AND BISIMPLICIAL SETS

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In [Segal 1], Graeme Segal introduced the concept of a Γ-space and proved that a certain homotopy category of Γ-spaces is equivalent to the usual homotopy category of connective spectra. Our main purpose is to show that there is a full-fledged homotopy theory of Γ-spaces underlying Segal's homotopy category. We do this by giving Γ-spaces the structure of a closed model category, i.e. defining "fibrations," "cofibrations," and "weak equivalences" for Γ-spaces so that Quillen's theory of homotopical algebra can be applied. Actually, we give two such structures (3.5, 5.2) leading to a "strict" and a "stable" homotopy theory of Γ-spaces. The former has had applications, cf. [Friedlander], but the latter is more closely related to the usual homotopy theory of spectra.

In our work on Γ-spaces, we have adopted the "chain functor" viewpoint of [Anderson]. However, we do not require our Γ-spaces to be "special," cf. §4, because "special" Γ-spaces are not closed under direct limit constructions. We have included in §§4,5 an exposition, and slight generalization, of the Anderson-Segal results on the construction of homology theories from Γ-spaces, and on the equivalence of the homotopy categories of Γ-spaces and connective spectra.

To set the stage for our work on Γ-spaces, we have given in §2 an exposition of spectra from the standpoint of homotopical algebra. We have also included an appendix (§B) on bisimplicial sets, where we outline some well-known basic results needed in this paper and prove a rather strong fibration theorem (B.4) for diagonals of bisimplicial sets. We apply B.4 to prove a generalization of

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Quillen's spectral sequence for a bisimplicial group. In another appendix (§A), we develop some homotopical algebra which we use to construct our "stable" model categories.

The paper is organized as follows:

We work "simplicially" and refer the reader to [May 1] for the basic facts of simplicial theory.

§1. A brief review of homotopical algebra

For convenience we recall some basic notions of homotopical algebra ([Quillen 1,2]) used repeatedly in this paper.

Definition 1.1 ([Quillen 2, p. 233]). A closed model category consists of a category $\mathcal{C}$ together with three classes of maps in $\mathcal{C}$ called fibrations, cofibrations, and weak equivalences, satisfying CM1 - CM5 below. A map f in $\mathcal{C}$ is called a trivial cofibration if f is a cofibration and weak equivalence, and called a trivial fibration if f is a fibration and weak equivalence.

CM1. $\mathcal{C}$ is closed under finite limits and colimits.

CM2. For $W \xrightarrow{f} X \xrightarrow{g} Y$ in $\mathcal{C}$, if any two of f, g, and gf are weak equivalences, then so is the third.

CM3. If f is a retract of g and g is a weak equivalence, fibration, or cofibration, then so is f.

CM4. Given a solid arrow diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ B & \longrightarrow & Y \end{array}$$

where i is a cofibration and p is a fibration, then the filler exists if either i or j is a weak equivalence.

CM5. Any map f can be factored as $f = pi$ and $f = qi$ with i a trivial cofibration, p a fibration, j a cofibration, and q a trivial fibration.

The above axioms are equivalent to the earlier more complicated ones in [Quillen 1] and are motivated in part by Example 1.3 below. They allow one to "do homotopy theory" in $\mathcal{C}$. The homotopy category $\mathrm{Ho}\mathcal{C}$ can be obtain from $\mathcal{C}$ by giving formal inverses to the weak equivalences. More explicitly, the objects of $\mathrm{Ho}\mathcal{C}$ are those of $\mathcal{C}$ and the set of morphisms, $\mathrm{Ho}\mathcal{C}(X,Y) = [X,Y]$, can be obtained as follows: first choose weak equivalences $X' \to X$ and $Y \to Y'$ where X' is cofibrant (i.e. $\emptyset \to X'$ is a cofibration where $\emptyset \in \mathcal{C}$ is initial) and Y' is fibrant (i.e. $Y' \to e$ is a fibration where $e \in \mathcal{C}$ is terminal); then $[X,Y] \approx [X',Y']$ and $[X',Y'] = \mathcal{C}(X',Y')/\sim$ where $\sim$ is the "homotopy relation" ([Quillen 1, I.1]). Thus $\mathrm{Ho}\mathcal{C}$ is equivalent to the category $\mathrm{ho}\mathcal{C}$ whose objects are the fibrant-cofibrant objects of $\mathcal{C}$ and whose maps are homotopy classes of maps in $\mathcal{C}$. The homotopy relation is especially manageable when $\mathcal{C}$ is a closed simplicial model category ([Quillen 1, II.2]), i.e. for objects $V,W \in \mathcal{C}$ there is a natural simplicial set $\mathrm{HOM}(V,W)$ ($= \mathrm{HOM}_{\mathcal{C}}(V,W)$) which has the properties of a function complex with vertices corresponding to the maps $V \to W$ in $\mathcal{C}$. For V cofibrant and W fibrant, one then has $[V,W] \approx \pi_0\mathrm{HOM}(V,W)$.

It will be convenient to have

Definition 1.2. A closed model category $\mathcal{C}$ is proper if whenever a square

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ B & \xrightarrow{g} & D \end{array}$$

is a pushout with i a cofibration and f a weak equivalence, then g is a weak equivalence; and whenever the square is a pullback with j a fibration and g a weak equivalence, then f is a weak equivalence.

Some needed results on proper closed model categories are proved in Appendix A, and we conclude this review with

Example 1.3. Let (s.sets) and (s.sets$_*$) denote the categories of unpointed and pointed simplicial sets respectively. These are proper closed simplicial model categories, where the cofibrations are the injections, the fibrations are the Kan fibrations, the weak equivalences are the maps whose geometric realizations are homotopy equivalences, $\mathrm{HOM}_{(s.sets)}(X,Y)_n$ consists of the maps $X \times \Delta[n] \to Y$ in (s.sets), and $\mathrm{HOM}_{(s.sets_*)}(X,Y)_n$ consists of the maps $X \wedge (\Delta[n] \cup *) \to Y$ in (s.sets$_*$). Note that the Kan complexes are the fibrant objects and all objects are cofibrant. The associated homotopy categories Ho(s.sets) and Ho(s.sets$_*$) are equivalent to the unpointed and pointed homotopy categories of CW complexes respectively. For $X \in$ (s.sets$_*$) we will let $\pi_i X$ denote $\pi_i|X|$ where $|X|$ is the geometric realization of X.

§2. Closed model category structures for spectra

To set the stage for our study of Γ-spaces, we now discuss spectra from the standpoint of homotopical algebra. Although spectra in the sense of [Kan] admit a closed model category structure (cf. [Brown]), these spectra are not very closely related to Γ-spaces and don't seem to form a closed simplicial model category. For our purposes the appropriate spectra are old-fashioned ones equipped with a suitable model category structure. After developing that structure, we show that it gives a stable homotopy theory equivalent to the usual one.

Definition 2.1. A spectrum $\underset{\sim}{X}$ consists of a sequence $\underset{\sim}{X}^n \in (\text{s.sets}_*)$ for $n \geq 0$ and maps $\sigma^n\colon S^1 \wedge \underset{\sim}{X}^n \to \underset{\sim}{X}^{n+1}$ in (s.sets_*), where $S^1 = \Delta[1]/\dot{\Delta}[1] \in (\text{s.sets}_*)$. A map $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ of spectra consists of maps $f^n\colon \underset{\sim}{X}^n \to \underset{\sim}{Y}^n$ in (s.sets_*) for $n \geq 0$ such that $\sigma^n(1 \wedge f^n) = f^{n+1}\sigma^n$; and (spectra) denotes the category of spectra.

The sphere spectrum $\underset{\sim}{S}$ is the obvious spectrum with $\underset{\sim}{S}^0 = S^0 = \Delta[0] \cup *$, $\underset{\sim}{S}^1 = S^1$, $\underset{\sim}{S}^2 = S^1 \wedge S^1$, $\underset{\sim}{S}^3 = S^1 \wedge S^1 \wedge S^1, \ldots$.

For $K \in (\text{s.sets})$ and $\underset{\sim}{X} \in (\text{spectra})$, $\underset{\sim}{X} \wedge K$ is the obvious spectrum with $(\underset{\sim}{X} \wedge K)^n = \underset{\sim}{X}^n \wedge K$ for $n \geq 0$; and for $\underset{\sim}{X}, \underset{\sim}{Y} \in (\text{spectra})$, $\text{HOM}(\underset{\sim}{X}, \underset{\sim}{Y})$ is the obvious simplicial set whose n-simplices are maps $\underset{\sim}{X} \wedge (\Delta[n] \cup *) \to \underset{\sim}{Y}$ in (spectra).

A map $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ in (spectra) is a strict weak equivalence (resp. strict fibration) if $f^n\colon \underset{\sim}{X}^n \to \underset{\sim}{Y}^n$ is a weak equivalence (resp. fibration) in (s.sets_*) for $n \geq 0$; and f is a strict cofibration if the induced maps

$$\underset{\sim}{X}^0 \to \underset{\sim}{Y}^0 \qquad \underset{\sim}{X}^{n+1} \coprod_{S^1 \wedge \underset{\sim}{X}^n} S^1 \wedge \underset{\sim}{Y}^n \longrightarrow \underset{\sim}{Y}^{n+1}$$

are cofibrations in (s.sets_*) for $n \geq 0$. (This implies that each $f^n\colon \underset{\sim}{X}^n \to \underset{\sim}{Y}^n$ is a cofibration.) We let $(\text{spectra})^{\text{strict}}$ denote the category (spectra) equipped with these "strict" classes of maps.

Proposition 2.2. $(\text{spectra})^{\text{strict}}$ is a proper closed simplicial model category.

The proof is straightforward. Of course the associated homotopy category $\text{Ho}(\text{spectra})^{\text{strict}}$ is not equivalent to the usual stable homotopy category because it has too many homotopy types.

To obtain the usual stable theory, we call a map $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ in (spectra) a stable weak equivalence if $f_*\colon \pi_*\underset{\sim}{X} \approx \pi_*\underset{\sim}{Y}$ where $\pi_*\underset{\sim}{X} = \varinjlim_n \pi_{*+n} \underset{\sim}{X}^n$; and call f a stable cofibration if f is a strict cofibration. Call $\underset{\sim}{X} \in (\text{spectra})$ an Ω-spectrum if for each $n \geq 0$ the geometric realization $|S^1| \wedge |\underset{\sim}{X}^n| \approx |S^1 \wedge \underset{\sim}{X}^n| \xrightarrow{|\sigma^n|} |\underset{\sim}{X}^{n+1}|$ induces a weak homotopy equivalence $|\underset{\sim}{X}^n| \to |\underset{\sim}{X}^{n+1}|^{|S^1|}$. Then choose a functor $Q\colon (\text{spectra}) \to (\text{spectra})$ and a natural transformation $\eta\colon 1 \to Q$ such that $\eta\colon \underset{\sim}{X} \to Q\underset{\sim}{X}$ is a stable weak equivalence and $Q\underset{\sim}{X}$ is an Ω-spectrum for each $\underset{\sim}{X} \in (\text{spectra})$. For instance one can let $Q\underset{\sim}{X}$ be the obvious spectrum with

$$(Q\underset{\sim}{X})^n = \lim_{i \to \infty} \text{Sing}\, \Omega^i |\underset{\sim}{X}^{n+i}|$$

where Sing is the singular functor. Now call $f\colon \underset{\sim}{X} \to \underset{\sim}{Y}$ a stable fibration if f is a strict fibration and for $n \geq 0$

$$\begin{array}{ccc} \underset{\sim}{X}^n & \xrightarrow{\eta} & (Q\underset{\sim}{X})^n \\ \downarrow f^n & & \downarrow (Qf)^n \\ \underset{\sim}{Y}^n & \xrightarrow{\eta} & (Q\underset{\sim}{Y})^n \end{array}$$

is a homotopy fibre square in (s.sets_*), cf. A.2. When all the $\underset{\sim}{Y}^n$ are connected this is actually equivalent to saying that f is a strict fibration with fibre on Ω-spectrum. Let $(\text{spectra})^{\text{stable}}$ denote the category (spectra) equipped with stable weak equivalences, stable fibrations, and stable cofibrations.

Theorem 2.3. $(\text{spectra})^{\text{stable}}$ is a proper closed simplicial model category.

Proof. The usual arguments of stable homotopy theory show that if

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ B & \xrightarrow{g} & D \end{array}$$

is a pushout in (spectra) with $f_*: \pi_* A \cong \pi_* C$ and with each $i^n: A^n \to B^n$ a cofibration in (s.sets_*), then $g_*: \pi_* B \cong \pi_* D$; and if the square is a pullback with $g_*: \pi_* B \cong \pi_* D$ and with each $j^n: C^n \to D^n$ a fibration in (s.sets_*) then $f_*: \pi_* A \cong \pi_* C$. Moreover, a map $f: X \to Y$ in (spectra) is a stable weak equivalence iff $Qf: QX \to QY$ is a strict weak equivalence. The result now follows by using Theorem A.7 and the simpliciality criterion SM7(b) of [Quillen 1, II.2].

Note that our definition of "stable fibration" does not actually depend on the choice of Q, because the fibrations in a closed model category are determined by the trivial cofibrations.

2.4. The stable homotopy category. By 2.5 below, $\text{Ho(spectra)}^{\text{stable}}$ is the usual stable homotopy category; and by model category theory, it is equivalent to the "concrete" category $\text{ho(spectra)}^{\text{stable}}$ of fibrant-cofibrant spectra in $\text{(spectra)}^{\text{stable}}$ and homotopy classes of maps. Note that a spectrum $X \in \text{(spectra)}^{\text{stable}}$ is fibrant iff X is an Ω-spectrum with each X^n a Kan complex, and X is cofibrant iff each $\sigma: S^1 \wedge X^n \to X^{n+1}$ is an injection. Also, it is easy to show that Q induces an equivalence

$$\text{Ho(spectra)}^{\text{stable}} \xrightarrow{\approx} \text{Ho}(\Omega\text{-spectra})^{\text{strict}}$$

where $\text{Ho}(\Omega\text{-spectra})^{\text{strict}}$ is the full subcategory of Ω-spectra in $\text{Ho(spectra)}^{\text{strict}}$.

2.5. Equivalence of various stable homotopy theories

We wish to show that our model category $\text{(spectra)}^{\text{stable}}$ gives a

homotopy theory equivalent to that for (Kan's spectra) developed in [Kan] and [Brown]. Recall that Kan's spectra are like pointed simplicial sets, except that they have simplices in both positive and negative degrees, and have operators d_i and s_i for all $i \geq 0$. They arise as "direct limits" of Kan's prespectra, which are sequences $K^0, K^1, K^2, \ldots$ in (s.sets_*) together with maps $SK^n \to K^{n+1}$ for $n \geq 0$. Here, $S(-)$ is the "small" suspension functor given in [Kan,2.2]; so for $K \in (\text{s.sets}_*)$, the non-basepoint non-degenerate simplices of $(SK)_i$ correspond to those of K_{i-1} but have trivial i^{th} faces.

It is difficult to relate our spectra to Kan's in a purely simplicial way, because the suspension functors $S(-)$ and $S^1 \wedge (-)$ are very different. Thus we will need the intermediate category (top. spectra) defined as in 2.1, but using pointed topological spaces and the topological suspension. We will also need the category (Kan's prespectra) defined as in 2.1, but using the "small" suspension functor $S(-)$ as indicated above. Our categories (top. spectra) and (Kan's presepctra) differ from those discussed in [Kan], because we put no injectivity conditions on the structural maps; but there are still adjoint functors

$$\text{(spectra)} \underset{\text{Sing}}{\overset{|\ |}{\rightleftarrows}} \text{(top. spectra)}$$

$$\underset{\text{Sing}}{\overset{|\ |}{\leftrightarrows}} \text{(Kan's prespectra)} \underset{\text{Ps}}{\overset{\text{Sp}}{\rightleftarrows}} \text{(Kan's spectra)}$$

defined as in [Kan, §§3,4], where the upper arrows are the left adjoints. In particular, the realization and singular functors induce adjoint functors between (spectra) and (top. spectra), where the structural maps are handled using the natural homeomorphism $|S^1 \wedge K| \cong |S^1| \wedge |K|$ for $K \in (\text{s.sets}_*)$. We define closed model category structures on (top. spectra) and (Kan's prespectra) by mimicing the construction of $\text{(spectra)}^{\text{stable}}$; in the construction for (top. spectra), we use the standard model category structure on pointed topological spaces, c.f. [Quillen 1, II.3]. The above pairs of adjoint functors all satisfy the hypotheses of [Quillen 1, I.4, Th. 3], and thus induce "equivalences of homotopy theories;" in particular, the four stable homotopy categories are equivalent. We remark that, unlike (spectra) and (top. spectra), the categories (Kan's prespectra) and (Kan's spectra) do not seem to have reasonable closed <u>simplicial</u> model category structures.

§3. <u>The strict homotopy theory of Γ-spaces</u>.

In this section, we introduce Γ-spaces and verify that they admit a "strict" model category structure similar to that of spectra. Not only does this "strict" model category structure admit applications (cf. [Friedlander]), but also it enables us to subsequently construct the "stable" model category structure on the category of Γ-spaces (whose homotopy category is the homotopy category of connected spectra).

We adopt D. Anderson's viewpoint in defining Γ-spaces. Let Γ^0 denote the category of finite pointed sets and pointed maps; Γ^0 is the dual of the category considered by G. Segal [Segal 1]. For $n \geq 0$, let n^+ denote the set $\{0,1,\ldots,n\}$ with basepoint $0 \in n^+$.

<u>Definition 3.1</u>. Let $\underline{C}$ be a pointed category with initial-terminal object $*$. A <u>Γ-object</u> over $\underline{C}$ is a functor $\underset{\sim}{A}: \Gamma^0 \to \underline{C}$ such that $\underset{\sim}{A}(0^+) = *$. A <u>Γ-space</u> is a Γ-object over the category $(s.sets_*)$ of pointed simplicial sets. $\Gamma^0\underline{C}$ is the category of Γ-objects over $\underline{C}$.

The reader should consult [Friedlander], [Segal 1] for interesting examples of Γ-topological spaces, Γ-spaces, and Γ-varieties.

For notational convenience, we shall sometimes view a Γ-object over $\underline{C}$ as a functor from the full subcategory of Γ^0 whose objects are the sets n^+, $n \geq 0$. Such a functor is the restriction of a functor $\Gamma^0 \to \underline{C}$ (determined up to canonical equivalence).

We begin our consideration of $\Gamma^0(s.sets_*)$, the category of Γ-spaces, by introducing some categorical constructions. For $\underset{\sim}{A} \in \Gamma^0(s.sets_*)$ and $K \in (s.sets_*)$, define $\underset{\sim}{A} \wedge K \in \Gamma^0(s.sets_*)$ by

$$(\underset{\sim}{A} \wedge K)(n^+) = \underset{\sim}{A}(n^+) \wedge K \qquad \text{for } n \geq 0$$

and define $\underset{\sim}{A}^K \in \Gamma^0(s.sets_*)$ by

$$\underset{\sim}{A}^K(n^+) = \underset{\sim}{A}(n^+)^K \qquad \text{for } n \geq 0$$

If $\underset{\sim}{A}, \underset{\sim}{B} \in \Gamma^0(s.sets_*)$, we define $HOM(\underset{\sim}{A},\underset{\sim}{B}) \in (s.sets_*)$ by

$$HOM(\underset{\sim}{A},\underset{\sim}{B})_n = \operatorname{Hom}_{\Gamma^0(s.sets_*)}(\underset{\sim}{A} \wedge (\Delta[n] \cup *), \underset{\sim}{B}).$$

<u>Definition 3.2</u>. Let $i_n: \Gamma^0_n \to \Gamma^0$ denote the inclusion of the full subcategory of all finite sets with no more than n non-basepoint elements. Let

$$T_n: \Gamma^0(s.sets_*) \to \Gamma^0_n(s.sets_*)$$

be the <u>n-truncation</u> functor defined by sending $\underset{\sim}{A}: \Gamma^0 \to (s.sets_*)$ to $\underset{\sim}{A} \circ i_n: \Gamma^0_n \to (s.sets_*)$. The left adjoint of T_n

$$sk_n: \Gamma^0_n(s.sets_*) \to \Gamma^0(s.sets_*)$$

is called the <u>n-skeleton</u> functor and is given for $\underset{\sim}{A} \in \Gamma^0_n(s.sets_*)$ by

$$(sk_n \underset{\sim}{A})(m^+) = \operatorname*{colim}_{\substack{k^+ \to m^+ \\ k \leq n}} \underset{\sim}{A}(k^+).$$

The right adjoint of T_n

$$csk_n\colon \Gamma_n^O(s.sets_*) \to \Gamma^O(s.sets_*)$$

is called the n-<u>coskeleton</u> functor and is given for $A \in \Gamma_n^O(s.sets_*)$ by

$$(csk_n A)(m^+) = \lim_{\substack{m^+ \to j^+ \\ j \leq n}} A(j^+).$$

We shall frequently commit a slight abuse of notation and let $sk_n A$ $csk_n A$ denote $sk_n \cdot T_n(A)$ $csk_n \cdot T_n(A)$ for $A \in \Gamma^O(s.sets_*)$.

Our construction of the strict model category for Γ-spaces depends on the following model category structure for G-equivariant homotopy theory for the groups $G = \Sigma_n$ (the groups of pointed automorphisms of n^+). For any group G, we let $G(s.sets_*)$ denote the category of pointed simplicial sets with left G-action (or, equivalently, of simplicial objects over pointed left G-sets). For $X, Y \in G(s.sets_*)$, $HOM(X,Y)$ denotes the simplicial set defined by

$$HOM(X,Y)_n = Hom_{G(s.sets_*)}(X \wedge (\Delta[n] \cup *), Y)$$

where G acts trivially on $\Delta[n] \cup *$.

<u>Proposition 3.3</u>. For any G, the category $G(s.sets_*)$ is a proper closed simplicial model category when provided with the following additional structure: a G-<u>weak equivalence</u> (respectively, a G-<u>fibration</u>) is a map $f\colon X \to Y$ in $G(s.sets_*)$ which is a weak equivalence (resp., fibration) in $(s.sets_*)$; a G-<u>cofibration</u> is a map $f\colon X \to Y$ in $G(s.sets_*)$ which is injective and for which G acts freely on the simplices not in the image of f.

The proof of Proposition 3.3 is straight-forward; indeed, this model category is a case of that defined in [Quillen 1, II.4].

The role of Σ_n-equivariance is revealed by the following proposition, whose straight-forward proof we omit (the notation of the proposition has been chosen to fit the proof of Theorem 3.5).

<u>Proposition 3.4</u>. For $B, X \in \Gamma_n^O(s.sets_*)$, let $u_{n-1}\colon T_{n-1}B \to T_{n-1}X$ be a map in $\Gamma_{n-1}^O(s.sets_*)$. A map $u^n\colon B(n^+) \to X(n^+)$ in $(s.sets_*)$ determines a prolongation of u_{n-1} to $u\colon B \to X$ in $\Gamma_n^O(s.sets_*)$ if and only if u^n is a Σ_n-equivariant map which fills in the following commutative diagram in $\Sigma_n(s.sets_*)$:

(3.4.1)
$$\begin{array}{ccccc} (sk_{n-1}B)(n^+) & \to & B(n^+) & \to & (csk_{n-1}B)(n^+) \\ \downarrow{\scriptstyle sk_{n-1}(u_{n-1})} & & \vdots & & \downarrow{\scriptstyle csk_{n-1}(u_{n-1})} \\ (sk_{n-1}X)(n^+) & \to & X(n^+) & \to & (csk_{n-1}X)(n^+) \end{array}$$

Proposition 3.4 should motivate the following model category structure on $\Gamma^O(s.sets_*)$.

<u>Theorem 3.5</u>. The category of Γ-spaces becomes a proper closed simplicial model category (denoted $\Gamma^O(s.sets_*)^{strict}$), when provided with the following additional structure: a map $f\colon A \to B \in \Gamma^O(s.sets_*)$ is called a <u>strict weak equivalence</u> if $f(n^+)\colon A(n^+) \to B(n^+)$ is a $(\Sigma_n$-) weak equivalence for $n \geq 1$; $f\colon A \to B$ is called a <u>strict cofibration</u> if the induced map

(3.5.1)
$$(sk_{n-1}B)(n^+) \coprod_{(sk_{n-1}A)(n^+)} A(n^+) \longrightarrow B(n^+)$$

is a Σ_n-cofibration for $n \geq 1$; and a map $f\colon A \to B$ is called a <u>strict fibration</u> if the induced map

(3.5.2)
$$A(n^+) \longrightarrow (csk_{n-1}A)(n^+) \prod_{(csk_{n-1}B)(n^+)} B(n^+)$$

is a $(\Sigma_n$-) fibration for $n \geq 1$.

This model category structure is similar to that obtained by

C. Reedy for simplicial objects over a closed model category, and our proof will somewhat resemble his.

<u>Proof</u>. Because finite limits, finite colimits, and weak equivalences in $\Gamma^{0}(\text{s.sets}_*)^{\text{strict}}$ are defined level-wise, <u>CM1</u> and <u>CM2</u> are immediately verified. Similarly, <u>CM3</u> for $\Gamma^{0}(\text{s.sets}_*)^{\text{strict}}$ follows directly from <u>CM3</u> for $\Sigma_n(\text{s.sets}_*)$ for each $n > 0$.

To prove one half of <u>CM4</u> (we omit the similar proof of the other half) for $\Gamma^{0}(\text{s.sets}_*)^{\text{strict}}$, let

$$(3.5.3)\qquad \begin{array}{ccc} \underset{\sim}{A} & \longrightarrow & \underset{\sim}{X} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ \underset{\sim}{B} & \longrightarrow & \underset{\sim}{Y} \end{array}$$

be a diagram in $\Gamma^{0}(\text{s.sets}_*)^{\text{strict}}$ such that i is a strict trivial cofibration and p is a strict fibration. A filler $u\colon \underset{\sim}{B} \to \underset{\sim}{X}$ is constructed inductively by finding fillers $u_n\colon T_n\underset{\sim}{B} \to T_n\underset{\sim}{X}$ for the truncations $T_n(3.5.3)$ of diagram (3.5.3) for $n \geq 1$. These truncated fillers are obtained by applying Propositions 3.3 and 3.4 together with the facts that $(sk_{n-1}\underset{\sim}{B})(n^+) \coprod_{(sk_{n-1}\underset{\sim}{A})(n^+)} \underset{\sim}{A}(n^+) \to \underset{\sim}{B}(n^+)$ is trivial Σ_n-cofibration and $\underset{\sim}{X}(n^+) \to (csk_{n-1}\underset{\sim}{X})(n^+) \times_{(csk_{n-1}\underset{\sim}{X})(n^+)} \underset{\sim}{Y}(n^+)$ is a $(\Sigma_n\text{-})$fibration. The second fact is immediate, and the first follows since $(sk_{n-1}\underset{\sim}{A})(n^+) \to (sk_{n-1}\underset{\sim}{B})(n^+)$ is a trivial cofibration as in the proof of 3.7 below.

To prove one half of <u>CM5</u> (we omit the similar proof of the other half), we must factor a map $f\colon \underset{\sim}{A} \to \underset{\sim}{B}$ in $\Gamma^{0}(\text{s.sets}_*)$ as $f = p\cdot i$ where i is a strict trivial cofibration and p is a strict fibration. Suppose inductively that we have a factorization

$$T_{n-1}\underset{\sim}{A} \to T_{n-1}\underset{\sim}{C} \to T_{n-1}\underset{\sim}{B} \in \Gamma^{0}_{n-1}(\text{s.sets}_*)$$

for some $n \geq 1$. Using the closed model category structure on $\Sigma_n(\text{s.sets}_*)$ given by Proposition 3.3, we obtain a factorization in $\Sigma_n(\text{s.sets}_*)$

$$(sk_{n-1}\underset{\sim}{C})(n^+) \coprod_{(sk_{n-1}\underset{\sim}{A})(n^+)} \underset{\sim}{A}(n^+) \xrightarrow{\alpha} K \xrightarrow{\beta} (csk_{n-1}\underset{\sim}{C})(n^+) \times_{(csk_{n-1}\underset{\sim}{B})(n^+)} \underset{\sim}{B}(n^+)$$

of the canonical map with α a trivial Σ_n-cofibration and β a fibration. The desired factorization $\underset{\sim}{A} \to \underset{\sim}{C} \to \underset{\sim}{B}$ is now obtained by induction using 3.4 and the following lemma (whose proof is immediate); the map $\underset{\sim}{A} \to \underset{\sim}{C}$ is a strict trivial cofibration by a patching argument as in the proof of 3.7 below.

<u>Lemma 3.6</u>. For $\underset{\sim}{C} \in \Gamma^{0}_{n-1}(\text{s.sets}_*)$, let

$$(sk_{n-1}\underset{\sim}{C})(n^+) \to K \to (csk_{n-1}\underset{\sim}{C})(n^+)$$

be a factorization in $\Sigma_n(\text{s.sets}_*)$ of the canonical map. Then $\underset{\sim}{C}$ prolongs to an object $\underset{\sim}{C}' \in \Gamma^{0}_{n}(\text{s.sets}_*)$ with $\underset{\sim}{C}'(n^+) = K$ such that the given factorization equals the canonical one for $\underset{\sim}{C}'$.

This completes the proof of <u>CM5</u>, and thus of the fact that $\Gamma^{0}(\text{s.sets}_*)^{\text{strict}}$ is a closed model category. To prove that $\Gamma^{0}(\text{s.sets}_*)^{\text{strict}}$ is a simplicial closed model category, it suffices to prove for each fibration $p\colon \underset{\sim}{A} \to \underset{\sim}{B}$ in $\Gamma^{0}(\text{s.sets}_*)$ and each cofibration $i\colon K \to L$ in (s.sets) that the induced map in $\Gamma^{0}(\text{s.sets}_*)^{\text{strict}}$

$$\mu\colon \underset{\sim}{A}^{L} \to \underset{\sim}{A}^{K} \times_{\underset{\sim}{B}^{K}} \underset{\sim}{B}^{L}$$

is a fibration which is trivial whenever either p or i is trivial. This follows easily from the closed model category properties of (s.sets_*), because the maps of type (3.5.2) associated with μ are given by the maps $D^{L} \to D^{K} \times_{E^{K}} E^{L}$ in (s.sets_*) induced by

$$D = \underset{\sim}{A}(n^+) \to (\mathrm{csk}_{n-1}\underset{\sim}{A})(n^+) \underset{(\mathrm{csk}_{n-1}\underset{\sim}{B})(n^+)}{\times} \underset{\sim}{B}(n^+) = E.$$

Finally, to prove that $\Gamma^0(\text{s.sets}_*)^{\text{strict}}$ is a proper simplicial closed model category, it suffices to prove the following lemma and then employ the fact that (s.sets_*) is a proper closed model category (one proceeds level-by-level, since strict weak equivalences are determined levelwise).

Lemma 3.7. If $f\colon \underset{\sim}{A} \to \underset{\sim}{B}$ is a cofibration (resp., fibration) in $\Gamma^0(\text{s.sets}_*)^{\text{strict}}$, then

$$(\mathrm{sk}_m\underset{\sim}{A})(n^+) \to (\mathrm{sk}_m\underset{\sim}{B})(n^+) \quad (\text{resp., } (\mathrm{csk}_m\underset{\sim}{A})(n^+) \to (\mathrm{csk}_m\underset{\sim}{B})(n^+))$$

is a cofibration (resp., fibration) in (s.sets_*) for all $m,n \geq 0$.

Proof. We treat the cofibration case and omit the similar proof of the fibration case. Assuming inductively that $(\mathrm{sk}_{m-1}\underset{\sim}{A})(n^+) \to (\mathrm{sk}_{m-1}\underset{\sim}{B})(n^+)$ is a cofibration, we will show that $(\mathrm{sk}_m\underset{\sim}{A})(n^+) \to (\mathrm{sk}_m\underset{\sim}{B})(n^+)$ is a cofibration. There is a push-out square

$$\begin{array}{ccc} \coprod_S \mathrm{sk}_{m-1}\underset{\sim}{A}(S) & \to & \mathrm{sk}_{m-1}\underset{\sim}{A}(n^+) \\ \downarrow & & \downarrow \\ \coprod_S \mathrm{sk}_m\underset{\sim}{A}(S) & \to & \mathrm{sk}_m\underset{\sim}{A}(n^+) \end{array}$$

where S runs through the pointed subsets of n^+ with exactly m non-basepoint elements. Note that for $n < m$ the sums on the left are trivial, and for $n \geq m$ the maps $\mathrm{sk}_{m-1}\underset{\sim}{A}(S) \to \mathrm{sk}_m\underset{\sim}{A}(S)$ are equivalent to the canonical maps $\mathrm{sk}_{m-1}\underset{\sim}{A}(m^+) \to \underset{\sim}{A}(m^+)$. The fact that $\mathrm{sk}_m\underset{\sim}{A}(n^+) \to \mathrm{sk}_m\underset{\sim}{B}(n^+)$ is a cofibration now follows from the following lemma applied to the natural map from the above push-out square to the analogous push-out square for $\underset{\sim}{B}$.

3.8. Reedy's patching lemma ([Reedy]).

Let

$$\begin{array}{ccccc} A_2 & \longleftarrow & A_1 & \longrightarrow & A_3 \\ \downarrow f_2 & & \downarrow f_1 & & \downarrow f_3 \\ B_2 & \longleftarrow & B_1 & \longrightarrow & B_3 \end{array}$$

be a diagram in a closed model category, e.g. (s.sets_*). If f_3 and $A_2 \coprod_{A_1} B_1 \to B_2$ are cofibrations (resp. trivial cofibrations), then $A_2 \coprod_{A_1} A_3 \to B_2 \coprod_{B_1} B_3$ is a cofibration (resp. trivial cofibration).

This follows since the maps

$$A_2 \coprod_{A_1} A_3 \to A_2 \coprod_{A_1} B_3 \approx (A_2 \coprod_{A_1} B_1) \coprod_{B_1} B_3 \to B_2 \coprod_{B_1} B_3$$

are cofibrations (resp. trivial cofibrations). Of course, there is also a dual result.

We observe in passing that Theorem 3.5 is valid more generally for Γ-objects over certain other pointed model categories $\underline{C}$ besides (s.sets_*). To obtain such a generalization, one must be able to impose a suitable model category structure on the category $\Sigma_n\underline{C}$ of left Σ_n-objects over $\underline{C}$ for each $n \geq 1$. In general, this may not be feasible; however, in favorable cases (e.g., when $\underline{C}$ is Quillen's model category of pointed topological spaces [Quillen 1, II.3]), $\Sigma_n\underline{C}$ has a closed model category structure such that a map f in $\Sigma_n\underline{C}$ is a Σ_n-fibration if and only if f is a fibration in $\underline{C}$, and f is a Σ_n-weak equivalence if and only if f is a weak equivalence in $\underline{C}$. (The Σ_n-cofibrations are then determined by closure, and are cofibrations in $\underset{\sim}{C}$). In these favorable cases, one obtains a closed model category $\Gamma^O\underset{\sim}{C}^{strict}$ as in Theorem 3.5.

Finally, we remark that Γ^O(s.sets$_*$) admits a second reasonable "strict" model category structure. This is obtained by [Bousfield-Kan, p. 314] and has weak equivalences (resp. fibrations) given by the termwise weak equivalences (resp. fibrations). However, our version seems to be more useful in applications and allows the symmetric groups to play a more explicit role.

§4. The construction of homology theories from Γ-spaces

In this section we give an exposition, and slight generalization, of some results of [Anderson] and [Segal 1]. In particular, we show that a Γ-space $\underset{\sim}{A}$ induces a generalized homology theory $h_*(\ ;\underset{\sim}{A})$ which can be directly computed when $\underset{\sim}{A}$ is "(very) special" by using $\underset{\sim}{A}$ as a chain functor. The constructions and proofs in this section will be used in §5 to compare Γ-spaces with spectra and to develop the "stable" model category structure for Γ-spaces.

We begin by showing that a Γ-space $\underset{\sim}{A}: \Gamma^O \to$ (s.sets$_*$) prolongs successively to functors $\underset{\sim}{A}$: (sets$_*$) → (s.sets$_*$), $\underset{\sim}{A}$: (s.sets$_*$) → (s.sets$_*$), and $\underset{\sim}{A}$: (spectra) → (spectra). For $W \in$ (sets$_*$) define $\underset{\sim}{A}(W) \in$ (s.sets$_*$) by

$$\underset{\sim}{A}(W) = \operatorname*{colim}_{\substack{V \subset W \\ V \in \Gamma^O}} \underset{\sim}{A}(V).$$

For $K \in$ (s.sets$_*$) define $\underset{\sim}{A}K \in$ (s.sets$_*$) by $(\underset{\sim}{A}K)_n = (\underset{\sim}{A}K_n)_n$ for $n \geq 0$ with the obvious face and degeneracy operators. Thus $\underset{\sim}{A}K$ is the diagonal of the bisimplicial set $(\underset{\sim}{A}K_*)_*$, cf. Appendix B. In order to prolong $\underset{\sim}{A}$ to spectra, note that for $K, L \in$ (s.sets$_*$) there is a natural simplicial map $L \wedge \underset{\sim}{A}K \to \underset{\sim}{A}(L \wedge K)$ sending $x \wedge y \in L_n \wedge (\underset{\sim}{A}K_n)_n$ to the image of y under the map $\underset{\sim}{A}(x \wedge \_)_n: \underset{\sim}{A}(K_n)_n \to \underset{\sim}{A}(L_n \wedge K_n)$. Now for $\underset{\sim}{X} \in$ (spectra)

define $\underset{\sim}{A}\underset{\sim}{X} \in$ (spectra) by $(\underset{\sim}{A}\underset{\sim}{X})^n = \underset{\sim}{A}(\underset{\sim}{X}^n)$ with the obvious structural maps

$$S^1 \wedge \underset{\sim}{A}(\underset{\sim}{X}^n) \to \underset{\sim}{A}(S^1 \wedge \underset{\sim}{X}^n) \to \underset{\sim}{A}(\underset{\sim}{X}^{n+1}).$$

Finally, for $K, L \in$ (s.sets$_*$) and $\underset{\sim}{X} \in$ (spectra), there are pairings

$$(\underset{\sim}{A}K) \wedge L \to \underset{\sim}{A}(K \wedge L) \in (\text{s.sets}_*)$$

$$(\underset{\sim}{A}\underset{\sim}{X}) \wedge L \to \underset{\sim}{A}(\underset{\sim}{X} \wedge L) \in (\text{spectra})$$

whose definitions are now obvious. In particular, $\underset{\sim}{A}$ preserves the simplicial homotopy relation for maps in (s.sets$_*$) and (spectra).

A Γ-space $\underset{\sim}{A}$ determines a spectrum $\underset{\sim}{A}\underset{\sim}{S}$ where $\underset{\sim}{S}$ is the sphere spectrum, and we let $h_*(\ ;\underset{\sim}{A})$ be the associated homology theory, i.e. $h_*(K;\underset{\sim}{A}) = \pi_*(\underset{\sim}{A}\underset{\sim}{S}) \wedge K$ for $K \in$ (s.sets$_*$). An alternative construction of $\tilde{h}_*(K;\underset{\sim}{A})$ is given by

<u>Lemma 4.1.</u> If $\underset{\sim}{A}$ is a Γ-space and $K \in$ (s.sets$_*$), then the map $(\underset{\sim}{A}\underset{\sim}{S}) \wedge K \to \underset{\sim}{A}(\underset{\sim}{S} \wedge K)$ is a stable weak equivalence, cf. 2.3, and thus

$$\tilde{h}_*(K;\underset{\sim}{A}) \cong \operatorname*{colim}_n \pi_{*+n}\underset{\sim}{A}(S^n \wedge K).$$

The proof is in 4.8. To give an even more direct construction of $\tilde{h}_*(K;\underset{\sim}{A})$, we must put conditions on $\underset{\sim}{A}$. A Γ-space $\underset{\sim}{A}$ is <u>special</u> if the obvious map $\underset{\sim}{A}(V \vee W) \to \underset{\sim}{A}V \times \underset{\sim}{A}W$ is a weak equivalence for $V, W \in \Gamma^O$. This is equivalent to requiring that for $n \geq 1$ the map

$$\underset{\sim}{A}(p_1) \times \cdots \times \underset{\sim}{A}(p_n): \underset{\sim}{A}(n^+) \to \underset{\sim}{A}(1^+) \times \cdots \times \underset{\sim}{A}(1^+)$$

is a weak equivalence where $p_i: n^+ \to 1^+$ is defined by $p_i(i) = 1$ and $p_i(j) = 0$ for $j \neq i$. For $\underset{\sim}{A}$ special, $\pi_0\underset{\sim}{A}(1^+)$ is an abelian monoid with multiplication

$$\pi_0 \underset{\sim}{A}(1^+) \times \pi_0 \underset{\sim}{A}(1^+) \xleftarrow[\simeq]{(p_1)_* \times (p_2)_*} \pi_0 \underset{\sim}{A}(2^+) \xrightarrow{\mu_*} \pi_0 \underset{\sim}{A}(1^+)$$

where $\mu: 2^+ \to 1^+$ is defined by $\mu(0) = 0$, $\mu(1) = 1$, and $\mu(2) = 1$. A Γ-space $\underset{\sim}{A}$ is <u>very special</u> if $\underset{\sim}{A}$ is special and $\pi_0 \underset{\sim}{A}(1^+)$ is an abelian group.

The following theorem shows that a very special Γ-space can be used as a chain functor.

<u>Theorem 4.2</u>. (cf. [Anderson, p. 3], [Segal, 1, 1.4]). If $\underset{\sim}{A}$ is a very special Γ-space and $K \in (\text{s.sets}_*)$, then $\underset{\sim}{A}(\underset{\sim}{S} \wedge K)$ is an Ω-spectrum and $\tilde{h}_*(K; \underset{\sim}{A}) \simeq \pi_* \underset{\sim}{A}K$.

This is an easy consequence of 4.1 and

<u>Lemma 4.3</u>. If $\underset{\sim}{A}$ is a very special Γ-space and $L \subset K \in (\text{s.sets}_*)$, then

$$\underset{\sim}{A}L \to \underset{\sim}{A}K \to \underset{\sim}{A}(K/L)$$

is a homotopy fibration, i.e. $\underset{\sim}{A}K$ maps by a weak equivalence to the homotopy theoretic fibre of $\underset{\sim}{A}K \to \underset{\sim}{A}(K/L)$.

<u>Proof</u>. It suffices to show that the bisimplicial square

$$\begin{array}{ccc} (\underset{\sim}{A}L_*)_* & \to & (\underset{\sim}{A}K_*)_* \\ \downarrow & & \downarrow \\ * & \longrightarrow & \underset{\sim}{A}(K_*/L_*) \end{array}$$

satisfies the hypotheses of Theorem B.4. The termwise homotopy fibre square condition follows since $\underset{\sim}{A}$ is special. The remaining conditions follow by B.3.1, because the maps

$$\pi_0^v(\underset{\sim}{A}K_*)_* \to \pi_0^v \underset{\sim}{A}(K_*/L_*)_*$$

$$\pi_t^v((\underset{\sim}{A}K_*)_*)_{free} \to \pi_0^v(\underset{\sim}{A}K_*)_* \quad \text{for } t \geq 1$$

$$\pi_t^v(\underset{\sim}{A}(K_*/L_*)_*)_{free} \to \pi_0^v \underset{\sim}{A}(K_*/L_*)_* \quad \text{for } t \geq 1$$

are fibrations since they are surjective homomorphisms of simplicial groups.

We now wish to generalize Theorem 4.2 to the case of a Γ-space $\underset{\sim}{A}$ which is merely special. For such $\underset{\sim}{A}$, the map $\underset{\sim}{A}(K \vee L) \to \underset{\sim}{A}K \times \underset{\sim}{A}L$ is a weak equivalence for $K, L \in (\text{s.sets}_*)$ by B.2. Thus $\pi_0 \underset{\sim}{A}K$ is an abelian monoid with multiplication given by

$$\pi_0 \underset{\sim}{A}K \times \pi_0 \underset{\sim}{A}K \xleftarrow{\simeq} \pi_0 \underset{\sim}{A}(K \vee K) \xrightarrow{\mu_*} \pi_0 \underset{\sim}{A}K$$

where $\mu: K \vee K \to K$ is the folding map.

<u>Theorem 4.4</u> (cf. [Segal, 1.4]). Let $\underset{\sim}{A}$ be a special Γ-space and $K \in (\text{s.sets}_*)$. Then $\underset{\sim}{A}(\underset{\sim}{S} \wedge K)$ is an Ω-spectrum above its 0^{th} term and thus $\tilde{h}_*(K; \underset{\sim}{A}) \simeq \pi_{*+1} \underset{\sim}{A}(S^1 \wedge K)$. If $\pi_0 \underset{\sim}{A}K$ is an abelian group, then $\underset{\sim}{A}(\underset{\sim}{S} \wedge K)$ is an Ω-spectrum and thus $\tilde{h}_*(K; \underset{\sim}{A}) \simeq \pi_* \underset{\sim}{A}K$.

<u>Proof</u>. Let $\underset{\sim}{B}$ be the Γ-space with $\underset{\sim}{B}(n^+) = \underset{\sim}{A}(n^+ \wedge S^1 \wedge K)$ for $n \geq 0$, and note that $\underset{\sim}{B}$ is very special. Hence $\underset{\sim}{B}\underset{\sim}{S}$ is an Ω-spectrum by 4.2, and the first statement follows since $\underset{\sim}{B}\underset{\sim}{S}$ gives the portion of $\underset{\sim}{A}(\underset{\sim}{S} \wedge K)$ above its 0^{th} term. The second statement follows similarly using the Γ-space $\underset{\sim}{C}$ with $\underset{\sim}{C}(n^+) = \underset{\sim}{A}(n^+ \wedge K)$.

We now turn to the proof of Lemma 4.1 which asserts that the map $(\underset{\sim}{A}\underset{\sim}{S}) \wedge K \to \underset{\sim}{A}(\underset{\sim}{S} \wedge K)$ is a stable weak equivalence. Although our proof is somewhat indirect, it allows us to introduce some notions needed in §5. It is based on the following general criterion.

Lemma 4.5. In a closed simplicial model category $\mathcal{C}$, e.g. $(\text{spectra})^{\text{stable}}$, a map $f: A \to B$ between cofibrant objects is a weak equivalence $\Longleftrightarrow$ $f^*: \mathrm{HOM}(B,X) \to \mathrm{HOM}(A,X)$ is a weak equivalence in (s.sets) for all fibrant $X \in \mathcal{C}$.

Proof. f is a weak equivalence $\Longleftrightarrow$ $f^*: [B,X] \approx [A,X]$ for all fibrant $X \in \mathcal{C}$ $\Longleftrightarrow$ $f^*: [B,X^K] \approx [A,X^K]$ for all $K \in$ (s.sets) and fibrant $X \in \mathcal{C}$ $\Longleftrightarrow$ $f^*: [K,\mathrm{HOM}(B,X)] \approx [K,\mathrm{HOM}(A,X)]$ for all $K \in (\text{s.sets}_*)$ and fibrant $X \in \mathcal{C}$ $\Longleftrightarrow$ $f^*: \mathrm{HOM}(B,X) \to \mathrm{HOM}(A,X)$ is a weak equivalence for all fibrant $X \in \mathcal{C}$.

To effectively apply 4.5 in our case we need an adjointness lemma. For $\underset{\sim}{X},\underset{\sim}{Y} \in$ (spectra) define a Γ-space $\Phi(\underset{\sim}{X},\underset{\sim}{Y})$ by

$$\Phi(\underset{\sim}{X},\underset{\sim}{Y})(V) = \mathrm{HOM}_{(\text{spectra})}(\underset{\sim}{X}^V,\underset{\sim}{Y})$$

for $V \in \Gamma^0$ where $\underset{\sim}{X}^V = \underset{\sim}{X} \times \cdots \times \underset{\sim}{X} \in$ (spectra) is the product of copies of $\underset{\sim}{X}$ indexed by the non-basepoint elements in V.

Lemma 4.6. For $\underset{\sim}{X},\underset{\sim}{Y} \in$ (spectra) and $\underset{\sim}{A} \in \Gamma^0(\text{s.sets}_*)$, there is a natural simplicial isomorphism

$$\mathrm{HOM}_{(\text{spectra})}(\underset{\sim}{A}\underset{\sim}{X},\underset{\sim}{Y}) \approx \mathrm{HOM}_{\Gamma^0(\text{s.sets}_*)}(\underset{\sim}{A},\Phi(\underset{\sim}{X},\underset{\sim}{Y}))$$

Proof. For a functor $T: \Gamma^0 \to (\text{sets}_*)$ with $T(0^+) = *$ and $W \in (\text{sets}_*)$, there is a natural isomorphism

$$\Big(\coprod_{n \geq 0} W^{n^+} \wedge Tn^+\Big)/\sim \xrightarrow{\approx} \overline{T}W$$

where

$$\overline{T}(W) = \operatorname*{colim}_{\substack{V \subset W \\ V \in \Gamma^0}} T(V)$$

and where $\sim$ is the equivalence relation generated by setting $\omega_*(x) \sim \omega^*(x)$ for each $\omega: m^+ \to n^+$ in Γ^0 and each $x \in W^{n^+} \wedge Tm^+$ using

$$W^{m^+} \wedge Tm^+ \xleftarrow{\omega^*} W^{n^+} \wedge Tm^+ \xrightarrow{\omega_*} W^{n^+} \wedge Tn^+.$$

Thus there is a natural isomorphism

$$\underset{\sim}{A}\underset{\sim}{X} \approx \Big(\coprod_{n \geq 0} \underset{\sim}{X}^{n^+} \wedge \underset{\sim}{A}(n^+)\Big)/\sim \ \in (\text{spectra})$$

and the lemma follows easily.

To prove 4.1 using 4.5, we need a final technical lemma which will also be used in §5.

Lemma 4.7. Let $f: \underset{\sim}{B} \to \underset{\sim}{C}$ be a map of Γ-spaces, and let $\underset{\sim}{X}$ be a spectrum. Then:

(i) If f is a strict weak equivalence, then so is $f_*: \underset{\sim}{B}\underset{\sim}{X} \to \underset{\sim}{C}\underset{\sim}{X}$.

(ii) If $f: B(n^+) \to \underset{\sim}{C}(n^+)$ is an injection for each $n \geq 0$, then $f_*: \underset{\sim}{B}\underset{\sim}{X} \to \underset{\sim}{C}\underset{\sim}{X}$ is a strict cofibration.

(iii) If $\underset{\sim}{X}$ is strictly cofibrant and $g: \underset{\sim}{Y} \to \underset{\sim}{Z}$ is a strict fibration of spectra, then $g_*: \Phi(\underset{\sim}{X},\underset{\sim}{Y}) \to \Phi(\underset{\sim}{X},\underset{\sim}{Z})$ is a strict fibration.

Proof. Part (i) follows from B.2, and (ii) is reasonably straightforward. For (iii), it suffices to show that g_* has the right lifting property for each strict trivial cofibration $f: \underset{\sim}{B} \to \underset{\sim}{C}$ of Γ-spaces. This follows from 4.6 using (i) and (ii).

4.8. Proof of 4.1. By 4.7(i) we can assume $\underset{\sim}{A}$ is a strictly cofibrant Γ-space. To show $(\underset{\sim}{A}S) \wedge K \to \underset{\sim}{A}(\underset{\sim}{S} \wedge K)$ is a stable weak equivalence, it suffices by 4.5, 4.6, and 4.7(ii) to show that the map

$$\mathrm{HOM}(\underset{\sim}{A},\Phi(\underset{\sim}{S} \wedge K,\underset{\sim}{X})) \approx \mathrm{HOM}(\underset{\sim}{A}(\underset{\sim}{S} \wedge K),\underset{\sim}{X})$$

$$\longrightarrow \mathrm{HOM}((\underset{\sim}{A}S) \wedge K,\underset{\sim}{X}) \approx \mathrm{HOM}(\underset{\sim}{A},\Phi(\underset{\sim}{S},\underset{\sim}{X}^K))$$

is a weak equivalence for each stably fibrant spectrum $\underset{\sim}{X}$. Now $\Phi(\underset{\sim}{S} \wedge K,\underset{\sim}{X})$ and $\Phi(\underset{\sim}{S},\underset{\sim}{X}^K)$ are strictly fibrant by 4.7(iii), and it suffices by the dual of 4.5 to show that the map $\Phi(\underset{\sim}{S} \wedge K,\underset{\sim}{X}) \to \Phi(\underset{\sim}{S},\underset{\sim}{X}^K)$ is a strict weak equivalence. This follows by 4.5 since the maps

$$(\underset{\sim}{S} \times \cdots \times \underset{\sim}{S}) \wedge K \to (\underset{\sim}{S} \wedge K) \times \cdots \times (\underset{\sim}{S} \wedge K) \in (\text{spectra})^{\text{stable}}$$

are weak equivalence's of cofibrant objects.

We conclude this section by noting that the functor $\underset{\sim}{A}: (\text{s.sets}_*) \to (\text{s.sets}_*)$ has homotopy theoretic significance even when $\underset{\sim}{A}$ is not special.

Proposition 4.9. For $\underset{\sim}{A} \in \Gamma^0(\text{s.sets}_*)$, if $f: K \to L \in (\text{s.sets}_*)$ is a weak equivalence then so is $\underset{\sim}{A}f: \underset{\sim}{A}K \to \underset{\sim}{A}L$. Thus $\underset{\sim}{A}$ induces a functor

$$\mathrm{Ho}\underset{\sim}{A}: \mathrm{Ho}(\text{s.sets}_*) \to \mathrm{Ho}(\text{s.sets}_*).$$

The proof is very similar to that of 4.1.

Corollary 4.10. For $\underset{\sim}{A} \in \Gamma^0(\text{s.sets}_*)$, if $K \in (\text{s.sets}_*)$ is n-connected for some $n \geq 0$ then so is $\underset{\sim}{A}K$.

Proof. This is clear when $K_i = *$ for $i \leq n$, and the general case now follows by 4.9.

§5. The stable homotopy theory of Γ-spaces

Following Graeme Segal, we will show that the strict homotopy category of very special Γ-spaces is equivalent to the stable homotopy category of connective spectra. Then we will develop a "stable" model category structure for Γ-spaces such that the associated homotopy category is equivalent to that of connective spectra.

By 4.6 there are functors

$$(-)\underset{\sim}{S}: \Gamma^0(\text{s.sets}_*) \rightleftarrows (\text{spectra}): \Phi(\underset{\sim}{S},-)$$

with $(-)\underset{\sim}{S}$ left adjoint to $\Phi(\underset{\sim}{S},-)$; indeed, there is a natural isomorphism

$$\mathrm{HOM}(\underset{\sim}{A}\underset{\sim}{S},\underset{\sim}{X}) \approx \mathrm{HOM}(\underset{\sim}{A},\Phi(\underset{\sim}{S},\underset{\sim}{X}))$$

for a Γ-space $\underset{\sim}{A}$ and spectrum $\underset{\sim}{X}$. By 4.7 and the dual of 4.5, $(-)\underset{\sim}{S}$ preserves weak equivalences and cofibrations in $\Gamma^0(\text{s.sets}_*)^{\text{strict}}$, while $\Phi(\underset{\sim}{S},-)$ preserves weak equivalences between fibrant objects and fibrations in $(\text{spectra})^{\text{strict}}$. Thus by [Quillen 1, I.4], there are induced adjoint functors

$$L^{\text{strict}}: \mathrm{Ho}\Gamma^0(\text{s.sets}_*)^{\text{strict}} \rightleftarrows \mathrm{Ho}(\text{spectra})^{\text{strict}}: R^{\text{strict}}$$

where $L^{\text{strict}}(\underset{\sim}{A}) = \underset{\sim}{A}\underset{\sim}{S}$ for $\underset{\sim}{A} \in \mathrm{Ho}\Gamma^0(\text{s.sets}_*)^{\text{strict}}$ and $R^{\text{strict}}(\underset{\sim}{X}) = \Phi(\underset{\sim}{S},\underset{\sim}{X}')$ for $\underset{\sim}{X} \in \mathrm{Ho}(\text{spectra})^{\text{strict}}$ where $\underset{\sim}{X} \to \underset{\sim}{X}'$ is a strict weak equivalence with $\underset{\sim}{X}'$ strictly fibrant. Now let

$$\mathrm{Ho}(\text{v.s. }\Gamma\text{-spaces})^{\text{strict}} \subset \mathrm{Ho}\Gamma^0(\text{s.sets}_*)^{\text{strict}}$$

$$\mathrm{Ho}(\text{c. }\Omega\text{-spectra})^{\text{strict}} \subset \mathrm{Ho}(\text{spectra})^{\text{strict}}$$

denote the full subcategories given by the very special Γ-spaces and the connective Ω-spectra respectively, where a spectrum X is called connective if $\pi_i X = 0$ for $i < 0$.

Theorem 5.1. (cf. [Anderson, pp. 4,5], [Segal 1, 1.4]). The adjoint functors L^{strict} and R^{strict} restrict to adjoint equivalences

$$L^{strict}: Ho(\text{v.s. } \Gamma\text{-spaces})^{strict} \rightleftarrows Ho(\text{c. } \Omega\text{-spectra})^{strict}: R^{strict}.$$

Moreover, $Ho(\text{c. } \Omega\text{-spectra})^{strict}$ is equivalent to the usual homotopy category of connective spectra.

Proof. The first statement is proved by combining the four facts below, and the last follows from 2.4. If A is a very special Γ-space, then AS is a connective Ω-spectrum by 4.2 and 4.10. If X is a strictly fibrant Ω-spectrum, then $\Phi(S,X)$ is a very special Γ-space by 4.5 since the maps $S \vee \cdots \vee S \to S \times \cdots \times S$ are weak equivalences in $(\text{spectra})^{stable}$. If A is a very special Γ-space and $AS \to X$ is a strict weak equivalence with X strictly fibrant, then the natural map $A \to \Phi(S,X)$ is a strict weak equivalence, because both A and $\Phi(S,X)$ are very special and the map

$$(AS)^0 = A(1^+) \to \Phi(S,X)(1^+) = X^0 \epsilon (\text{s.sets}_*)$$

is a weak equivalence. Similarly, if $A \to \Phi(S,X)$ is a strict weak equivalence for some strictly fibrant connective Ω-spectrum X, then the natural map $AS \to X$ is a weak equivalence.

We now wish to use our strict homotopy theory of Γ-spaces to build a corresponding stable theory, just as we previously used our strict homotopy theory of spectra to build a stable theory in 2.3.

Theorem 5.2. The category of Γ-spaces becomes a closed

simplicial model category (denoted $\Gamma^0(\text{s.sets}_*)^{stable}$) when provided with the following additional structure: a map $f: A \to B \epsilon \Gamma^0(\text{s.sets}_*)$ is called a stable weak equivalence if $f_*: \pi_* AS \approx \pi_* BS$; $f: A \to B$ is called a stable cofibration if it is a strict cofibration; and $f: A \to B$ is called a stable fibration if it has the right lifting property for the stable trivial cofibrations.

Following the proof we will say more about stable fibrations in 5.7. Our proof will rely on the formal machinery developed in Appendix A. Let Q: (spectra) $\to$ (spectra) and $\eta: 1 \to Q$ be such that, for each spectrum X, $\eta_X: X \to QX$ is a stable weak equivalence and QX is a stably fibrant spectrum, cf. §2. Now define
$T: \Gamma^0(\text{s.sets}_*) \to \Gamma^0(\text{s.sets}_*)$ by $TA = \Phi(S, QAS)$ and let $\eta: 1 \to T$ be the canonical transformation. Note that for each Γ-space A, $\eta_A: A \to TA$ is a stable weak equivalence and TA is strictly fibrant and very special. Using the terminology of Appendix A, the T-equivalences, T-cofibrations, and T-fibrations in $\Gamma^0(\text{s.sets}_*)^{strict}$ are the same as the stable weak equivalences, stable cofibrations, stable fibrations, respectively. Moreover, for $\eta: 1 \to T$, the conditions (A.4) and (A.5) clearly hold although (A.6) doesn't, cf. 5.7. Thus by A.8(i) all the closed model category axioms hold in $\Gamma^0(\text{s.sets}_*)^{stable}$ except possibly for the "trivial cofibration, fibration" part of CM5. To verify an important case of that part, we use the following substitute for (A.6).

Lemma 5.3. For a pull-back square

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow & & \downarrow j \\ B & \xrightarrow{k} & Y \end{array}$$

in $\Gamma^0(\text{s.sets}_*)$, suppose j is a strict fibration with X and Y very special and with

$$\pi_0 X(1^+) \approx \pi_0 X(S) \xrightarrow{j_*} \pi_0 Y(S) \approx \pi_0 Y(1^+)$$

onto. If k is a stable weak equivalence, then so is h.

<u>Proof</u>. For $K \in (\text{s.sets}_*)$ consider the induced square

$$\begin{array}{ccc} A(K_*)_* & \rightarrow & X(K_*)_* \\ \downarrow & & \downarrow \\ B(K_*)_* & \rightarrow & Y(K_*)_* \end{array}$$

of bisimplicial sets. As in the proof of 4.3, $X(K_*)_*$ and $Y(K_*)_*$ satisfy the π_*-Kan condition and $j_*\colon \pi_0^V X(K_*)_* \to \pi_0^V Y(K_*)_*$ is a fibration. Thus by B.4

$$\begin{array}{ccc} AK & \longrightarrow & XK \\ \downarrow & & \downarrow \\ BK & \longrightarrow & YK \end{array}$$

is a homotopy fibre square in (s.sets_*), and the lemma follows easily.

Now using 5.3 in place of (A.6), the argument in A.10 shows that if $f\colon X \to Y$ is a map in $\Gamma^0(\text{s.sets}_*)$ with $f_*\colon \pi_0 X(S) \to \pi_0 Y(S)$ onto, then f can be factored as $f = pi$ where i is a stable trivial cofibration and p is a stable fibration. The following lemma will complete the proof of <u>CM5</u>, and Theorem 5.2 will then follow using the criterion SM7(b) of [Quillen 1, II.2].

<u>Lemma 5.4</u>. Each map $f\colon A \to B$ in $\Gamma^0(\text{s.sets}_*)$ can be factored as $A \xrightarrow{u} C \xrightarrow{v} B$ where $u_*\colon \pi_0 AS \to \pi_0 CS$ is onto and v is a stable fibration.

To prove this (in 5.6) we will first show that the functor

$$\pi_0(-)(S)\colon \Gamma^0(\text{s.sets}_*) \to (\text{ab. gps.})$$

has a right adjoint. For an abelian group M, let M be the usual very special Γ-space such that $M(V)_n = M^V$ for $V \in \Gamma^0$ and $n \geq 0$, where M^V is the product of copies of M indexed by the non-basepoint elements of V. Clearly MS is an Eilenberg-MacLane spectrum of type $(M,0)$, and we identify $\pi_0 MS$ with M.

<u>Lemma 5.5</u>. For a Γ-space A and an abelian group M, the obvious map

$$\text{Hom}_{\Gamma^0(\text{s.sets}_*)}(A,M) \to \text{Hom}_{(\text{ab. gps.})}(\pi_0 AS, M)$$

is a bijection.

<u>Proof</u>. In $\Gamma^0(\text{s.sets}_*)$, let $B \to A$ be a strict weak equivalence with B strictly cofibrant. In the square

$$\begin{array}{ccc} \text{Hom}_{\Gamma^0(\text{s.sets}_*)}(A,M) & \to & \text{Hom}_{(\text{ab. gps.})}(\pi_0 AS, M) \\ \downarrow & & \downarrow \\ \text{Hom}_{\Gamma^0(\text{s.sets}_*)}(B,M) & \to & \text{Hom}_{(\text{ab. gps.})}(\pi_0 BS, M) \end{array}$$

the right map is bijective since $\pi_0 AS \approx \pi_0 BS$, and the left map is bijective since

$$\text{Hom}_{(\text{s.sets}_*)}(A(n^+), M(n^+)) \approx \text{Hom}_{(\text{sets}_*)}(\pi_0 A(n^+), M^{n^+})$$

$$\approx \text{Hom}_{(\text{sets}_*)}(\pi_0 B(n^+), M^{n^+}) \approx \text{Hom}_{(\text{s.sets}_*)}(B(n^+), M(n^+)).$$

The lemma now follows since the bottom map of the square is a composite of bijections

$$\operatorname{Hom}(B,M) \xrightarrow{\approx} \pi_0 \mathrm{HOM}(B,M) \xrightarrow{\approx 1} \pi_0 \mathrm{HOM}(B,\Phi(S,MS))$$

$$\xrightarrow{\approx} \pi_0 \mathrm{HOM}(BS,MS) \xrightarrow{\approx 2} \operatorname{Hom}(\pi_0 BS, M)$$

where 1 holds by 5.1 and the dual of 4.5, and 2 holds since $\pi_0 \mathrm{HOM}(BS,MS)$ is the set of homotopy classes from the connective spectrum BS to the Eilenberg-MacLane spectrum MS in $(\text{spectra})^{\text{stable}}$.

5.6. Proof of 5.4. It will suffice to inductively construct a descending sequence of Γ-spaces

$$B = C^0 \supset C^1 \supset C^2 \supset \cdots \supset C^\alpha \supset \cdots$$

indexed by the ordinal numbers and such that: $f(A) \subset C^\alpha$ for all α; the inclusion $C^\alpha \xrightarrow{\subset} B$ is a stable fibration for all α; and, for sufficiently large α, $C^\alpha = C^{\alpha+1}$ and $f_*\colon \pi_0 AS \to \pi_0 C^\alpha S$ is onto. Given $C^\alpha \subset B$ with $f(A) \subset C^\alpha$, define $C^{\alpha+1} \subset B$ by the pull-back

$$\begin{array}{ccc} C^{\alpha+1} & \longrightarrow & M^\alpha \\ \downarrow \subset & & \downarrow \subset \\ C^\alpha & \longrightarrow & (\pi_0 C^\alpha S) \end{array}$$

where M^α is the image of $f_*\colon \pi_0 AS \to \pi_0 C^\alpha S$ and where the bottom map corresponds via 5.5 to the identity on $\pi_0 C^\alpha S$. Note that $C^{\alpha+1} \xrightarrow{\subset} C^\alpha$ is a stable fibration because $M^\alpha \xrightarrow{\subset} (\pi_0 C^\alpha S)$ is one by an argument using 5.5, and note that $f(A) \subset C^{\alpha+1}$. Given a limit ordinal λ and given $C^\alpha \subset B$ with $f(A) \subset C^\alpha$ for all $\alpha < \lambda$, define

$C^\lambda \subset B$ by $C^\lambda = \bigcap_{\alpha<\lambda} C^\alpha$, and note that $f(A) \subset C^\alpha$. This completes the inductive construction of $\{C^\alpha\}$, and the desired properties are easily verified.

This concludes the proof of Theorem 5.2 and we next discuss

5.7. Stable fibrations of Γ-spaces. By A.9, a sufficient condition for a Γ-space map $f\colon A \to B$ to be a stable fibration is that f be a strict fibration and that

$$\begin{array}{ccc} A & \xrightarrow{\eta} & TA \\ \downarrow f & & \downarrow Tf \\ B & \xrightarrow{\eta} & TB \end{array}$$

be a homotopy fibre square in $\Gamma^0(\text{s.sets}_*)^{\text{strict}}$. When $f_*\colon \pi_0 AS \to \pi_0 BS$ is onto, this condition is also necessary by the argument of A.10; but it is not always necessary. To give an example, we first note that an abelian monoid M determines a Γ-space M with $M(V)_n = M^V$ for $V \in \Gamma^0$ and $n \geq 0$. Letting $\overline{M}$ denote the universal abelian group generated by M, we note that the Γ-space map $M \to \overline{M}$ is a stable weak equivalence, because $\pi_* MS^n \xrightarrow{\approx} \pi_* \overline{M}S^n$ for $n \geq 1$ by [Spanier, Corollary 5.7]. Now let M be the abelian monoid given by

$$M = \{n \in Z \mid n \geq 0\} \cup \{0'\}$$

with the usual addition for the non-negative integers and with $0' + 0' = 0$, $0' + 0 = 0'$, $0' + n = n$ for $n \geq 1$. Note that $\overline{M} = Z$, and let $D = \{0, 0'\} \subset M$. Using the pull-back square

$$\begin{array}{ccc} \underset{\sim}{D} & \longrightarrow & \underset{\sim}{O} \\ \big\downarrow \subset & & \big\downarrow \\ \underset{\sim}{M} & \longrightarrow & \underset{\sim}{Z} \end{array}$$

in $\Gamma_0(\text{s.sets}_*)$, one sees that $\underset{\sim}{D} \xrightarrow{\subset} \underset{\sim}{M}$ is a stable fibration although it doesn't satisfy the sufficient condition mentioned above. Since $\underset{\sim}{M} \to \underset{\sim}{Z}$ is a stable weak equivalence and $\underset{\sim}{D} \to \underset{\sim}{O}$ is not, this square also shows that (A.6) fails in our Γ-space context.

For the adjoint functors

$$(-)\underset{\sim}{S}: \Gamma^0(\text{s.sets}_*) \rightleftarrows (\text{spectra}): \Phi(\underset{\sim}{S},-)$$

it is now easy to verify that $(-)\underset{\sim}{S}$ preserves weak equivalences and cofibrations in $\Gamma^0(\text{s.sets}_*)^{\text{stable}}$, while $\Phi(\underset{\sim}{S},-)$ preserves weak equivalences between fibrant objects and fibrations in $(\text{spectra})^{\text{stable}}$. Thus by [Quillen 1, I.4] there are induced adjoint functors

$$L^{\text{stable}}: \text{Ho}\Gamma^0(\text{s.sets}_*)^{\text{stable}} \rightleftarrows \text{Ho}(\text{spectra})^{\text{stable}}: R^{\text{stable}}$$

and we let

$$\text{Ho}(\text{c.spectra})^{\text{stable}} \subset \text{Ho}(\text{spectra})^{\text{stable}}$$

denote the full subcategory given by the connective spectra. It is now easy to prove

<u>Theorem 5.8</u>. The adjoint functors L^{stable} and R^{stable} restrict to adjoint equivalences

$$L^{\text{stable}}: \text{Ho}\Gamma^0(\text{s.sets}_*)^{\text{stable}} \rightleftarrows \text{Ho}(\text{c.spectra})^{\text{stable}}: R^{\text{stable}}.$$

Thus the stable homotopy category of Γ-spaces is equivalent to the usual connective homotopy category of spectra. Moreover, it is easy to show that T induces an equivalence

$$\text{Ho}\Gamma^0(\text{s.sets}_*)^{\text{stable}} \xrightarrow{\simeq} \text{Ho}(\text{v.s. }\Gamma\text{-spaces})^{\text{strict}}$$

just as Q induced an equivalence

$$\text{Ho}(\text{spectra})^{\text{stable}} \xrightarrow{\simeq} \text{Ho}(\Omega\text{-spectra})^{\text{strict}}$$

in 2.4.

Appendix A. Proper closed model categories

In this appendix we outline some formal results on proper closed model categories (cf. 1.2) which we use in §§2,5 to pass from our "strict" to our "stable" model category structures on spectra and Γ-spaces. Some familiar examples of proper closed model categories are the (pointed) simplicial sets, (pointed) topological spaces, and simplicial groups, all equipped with the standard model structures ([Quillen 1, II.3]); however, as noted in [Quillen 2, p. 241], some closed model categories are not proper.

Our first result may be viewed as a generalization of the factorization axiom CM5 (see 1.1).

Proposition A.1. Let $\mathcal{C}$ be a proper closed model category and let $f: X \to Y$ in $\mathcal{C}$. For each factorization $[f] = vu$ in $\mathrm{Ho}\mathcal{C}$ there is a factorization $f = ji$ in $\mathcal{C}$ such that i is a cofibration, j is a fibration, and the factorization $[f] = [j][i]$ is equivalent to $[f] = vu$ in $\mathrm{Ho}\mathcal{C}$ (i.e. there exists an isomorphism w in $\mathrm{Ho}\mathcal{C}$ such that $wu = [i]$ and $[j]w = v$.)

Proof. First suppose X is cofibrant and Y is fibrant. Then choose a fibrant-cofibrant object $W \in \mathcal{C}$ and maps $X \xrightarrow{\alpha} W \xrightarrow{\beta} Y$ in $\mathcal{C}$ such that $[f] = [\beta][\alpha]$ and such that this factorization is equivalent to $[f] = vu$ in $\mathrm{Ho}\mathcal{C}$. Using CM5 and the homotopy extension theorem ([Quillen, HA, Ch. I, p. 1.7]), one then constructs the desired factorization $f = ji$. In the general case, choose weak equivalences $s: X' \to X$ and $t: Y \to Y'$ with X' cofibrant and Y' fibrant. Then apply the special case to give a factorization $tfs = \theta\omega$ where ω is a cofibration, θ is a fibration, and the factorization $[f] = ([t]^{-1}[\theta])([\omega][s]^{-1})$ is equivalent to $[f] = vu$ in $\mathrm{Ho}\mathcal{C}$. Now, using the properness of $\mathcal{C}$ and CM5, it is not hard to construct the desired factorization of f.

A.2. Homotopy fibre squares. In a proper closed model category $\mathcal{C}$, a commutative square

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow w \\ B & \xrightarrow{v} & D \end{array}$$

is a homotopy fibre square if for some factorization $C \xrightarrow{i} W \xrightarrow{p} D$ of w with i a weak equivalence and p a fibration, the map $A \to B \times_D W$ is a weak equivalence. This easily implies that for any factorization $B \xrightarrow{j} V \xrightarrow{q} D$ of v with j a weak equivalence and q a fibration, the map $A \to V \times_D C$ is a weak equivalence. Thus in our definition we could have replaced "some" by "any" or used v in place of w. It is not hard to verify the following expected results. In a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & E \\ \downarrow & \mathrm{I} & \downarrow & \mathrm{II} & \downarrow \\ B & \longrightarrow & D & \longrightarrow & F, \end{array}$$

if I and II are homotopy fibre squares, so is the combined square III; and if II and III are homotopy fibre squares, so is I. If a map between homotopy fibre squares has weak equivalences at the three corners away from the upper left, then it has a weak equivalence at the upper left. A retract of a homotopy fibre square is a homotopy fibre square.

Although it does not depend on properness, we also need.

A.3. The model category $\mathcal{C}^{\mathrm{Pairs}}$. Let $\mathcal{C}$ be a closed model category, and let $\mathcal{C}^{\mathrm{Pairs}}$ be the category whose objects are the maps in $\mathcal{C}$

and whose maps are commutative squares in $\mathcal{C}$. A map

$$\begin{array}{ccc} A_0 & \xrightarrow{f_0} & B_0 \\ \downarrow i & & \downarrow j \\ A_1 & \xrightarrow{f_1} & B_1 \end{array}$$

from i to j in $\mathcal{C}^{\text{Pairs}}$ will be called a <u>weak equivalence</u> (resp. <u>fibration</u>) if f_0 and f_1 are weak equivalences (resp. fibrations), and a <u>cofibration</u> if $f_0: A_0 \to B_0$ and $(f_1, j): A_1 \coprod_{A_0} B_0 \to B_1$ are cofibrations. (This implies that $f_1: A_1 \to B_1$ is also a cofibration.) One easily shows that $\mathcal{C}^{\text{Pairs}}$ is a closed model category which is proper if $\mathcal{C}$ is proper.

We now develop the machinery which allows us to pass from our "strict" to our "stable" model category structures on spectra and Γ-spaces. Let $\mathcal{C}$ be a proper closed model category, let $Q: \mathcal{C} \to \mathcal{C}$ be a functor, and let $\eta: 1 \to Q$ be a natural transformation. A map $f: X \to Y$ in $\mathcal{C}$ will be called a <u>Q-equivalence</u> if $Qf: QX \to QY$ is a weak equivalence, a Q-<u>cofibration</u> if f is a cofibration, and a Q-<u>fibration</u> if the filler exists in each commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow i & \nearrow & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

where i is a Q-cofibration and Q-equivalence. We wish to show that $\mathcal{C}^Q$ is a proper closed model category, where $\mathcal{C}^Q$ denotes $\mathcal{C}$ equipped with its Q-equivalences, Q-cofibrations, and Q-fibrations. For this we need:

(A.4) If $f: X \to Y$ is a weak equivalence in $\mathcal{C}$, then so is

$Qf: QX \to QY$.

(A.5) For each $X \in \mathcal{C}$ the maps $\eta_{QX}, Q\eta_X: QX \to QQX$ are weak equivalences in $\mathcal{C}$.

(A.6) For a pull-back square

$$\begin{array}{ccc} A & \xrightarrow{h} & X \\ \downarrow & & \downarrow j \\ B & \xrightarrow{k} & Y \end{array}$$

in $\mathcal{C}$, if j is a Q-fibration and k is a Q-equivalence, then h is a Q-equivalence; and the dual condition holds for a push-out square.

<u>Theorem A.7</u>. Suppose (A.4), (A.5), and (A.6). Then $\mathcal{C}^Q$ is a proper closed model category. Moreover, a map $f: X \to Y$ in $\mathcal{C}$ is a Q-fibration $\Longleftrightarrow$ f is a fibration and

$$\begin{array}{ccc} X & \xrightarrow{\eta} & QX \\ \downarrow & & \downarrow Qf \\ Y & \xrightarrow{\eta} & QY \end{array}$$

is a homotopy fibre square in $\mathcal{C}$.

The proof is completed in A.10 after the following lemmas. In our Γ-space context, (A.6) does not quite hold and we use these lemmas directly.

<u>Lemma A.8</u>. Suppose (A.4). Then:

(i) $\mathcal{C}^Q$ satisfies <u>CM1</u>-<u>CM4</u> and the "cofibration, trivial fibration" part of <u>CM5</u>.

(ii) A map $f\colon X \to Y$ in $\mathcal{C}$ is a trivial fibration in $\mathcal{C}^Q \iff f$ is a trivial fibration in $\mathcal{C}$.

(iii) If $f\colon X \to Y$ is a fibration in $\mathcal{C}$ and both $\eta\colon X \to QX$ and $\eta\colon Y \to QY$ are weak equivalences, then f is a Q-fibration.

Proof. Statement (i) follows using (ii). In (ii), "$\Longleftarrow$" is clear and "$\Longrightarrow$" follows by first factoring f as $f = ji$ with i a cofibration and j a trivial fibration, and then noting that f is a retract of j by a lifting argument using the fact that i is a Q-equivalence. For (iii), it suffices to show that the filler exists in each commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

with i a trivial cofibration in $\mathcal{C}^Q$. Viewing this as a map from i to f in $\mathcal{C}^{\text{Pairs}}$, we apply A.1 and A.3 to factor it as

$$\begin{array}{ccccc} A & \longrightarrow & V & \longrightarrow & X \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f} \\ B & \longrightarrow & W & \longrightarrow & Y \end{array}$$

where h is isomorphic to Qi in $\text{Ho}(\mathcal{C}^{\text{Pairs}})$. Then h is a weak equivalence, so we apply CM5 to h and use CM4 to obtain the desired filler.

Now, A.8(iii) easily implies

Lemma A.9. Suppose (A.4) and (A.5). If $f\colon X \to Y$ is a fibration in $\mathcal{C}$ and

$$\begin{array}{ccc} X & \xrightarrow{\eta} & QX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Qf} \\ Y & \xrightarrow{\eta} & QY \end{array}$$

is a homotopy fibre square, then f is a Q-fibration.

A.10. Proof of A.7. We wish to factor a map $f\colon X \to Y$ in $\mathcal{C}$ as $f = ji$ where j is a Q-fibration and i is a Q-cofibration and Q-equivalence. First factor Qf as $Qf = vu$ where u is a weak equivalence and v is a fibration. Then let $f = v'u'$ be the factorization of f induced by $\eta\colon X \to QX$ and $\eta\colon Y \to QY$; and factor u' as $u' = ki$ where i is a cofibration and k is a trivial fibration. Then the factorization $f = (v'k)i$ has the desired properties, since $v'k$ satisfies the hypotheses of A.8(iii) and i is a Q-equivalence by (A.4)-(A.6). The "$\Longleftarrow$" part of A.7 is A.9, and the "$\Longrightarrow$" part follows by using the above procedure to factor f as $f = (v'k)i$, and then noting that f is a retract of $v'k$.

Appendix B. Bisimplicial sets

For convenience we have gathered here various definitions and results on bisimplicial sets which are used elsewhere in this paper. Much of this material is well-known, and the main innovation is the fibre square theorem (B.4) for diagonals of bisimplicial sets. As a consequence of that theorem we deduce a generalization of Quillen's spectral sequence ([Quillen 3]).

Let Δ be the category whose objects are the finite ordered sets $[m] = \{0,1,\ldots,m\}$ for $m \geq 0$, and whose morphisms are the non-decreasing maps. A bisimplicial set is a functor $\Delta^0 \times \Delta^0 \to (\text{sets})$, and these form a category (bis. sets). One can think of a bisimplicial set X as a collection of sets $X_{m,n}$ for $m,n \geq 0$ together with horizontal and vertical face and degeneracy operators $d_i^h: X_{m,n} \to X_{m-1,n}$, $s_i^h: X_{m,n} \to X_{m+1,n}$, $d_j^v: X_{m,n} \to X_{m,n-1}$, $s_j^v: X_{m,n} \to X_{m,n+1}$ for $0 \leq i \leq m$ and $0 \leq j \leq n$, where the horizontal and vertical operators commute, and the usual simplicial identities hold horizontally and vertically.

In practice, many constructions in algebraic topology can be achieved by first forming an appropriate bisimplicial set and then applying the diagonal functor

$$\text{diag}: (\text{bis. sets}) \to (\text{s.sets})$$

where diag X is given by the sets $X_{m,m}$ for $m \geq 0$ with operators $d_i = d_i^h d_i^v$ and $s_i = s_i^h s_i^v$. For example, if K and L are simplicial sets, there is an obvious bisimplicial set $K \underset{\sim}{\times} L$ with $(K \underset{\sim}{\times} L)_{m,n} = K_m \times L_n$, and $\text{diag}(K \underset{\sim}{\times} L) = K \times L$. Many other examples are given, at least implicitly, in [Artin-Mazur], [Bousfield-Kan, XII], [Dress], [May 2], [Segal 2], and elsewhere. Most of these examples lead to interesting homotopy or (co)homology spectral sequences.

The main results for bisimplicial sets involve the relation between the vertical simplicial terms and the diagonal, i.e. between the $X_{m,*}$ and diag X. (Of course, there are immediate corollaries with "vertical" replaced by "horizontal.") To understand these results one should first note that the construction of diag X is deceptively simple, and diag X may actually be viewed as the "total complex" or "realization" of X. Specifically, let Tot X be the simplicial set obtained from the disjoint union $\coprod_{m \geq 0} \Delta[m] \times X_{m,*}$ by identifying the simplex $(a, \theta^* x) \in \Delta[m] \times X_{m,*}$ with $(\theta_* a, x) \in \Delta[n] \times X_{n,*}$ for each $\theta: [m] \to [n]$ in Δ. Now the classical Eilenberg-Zilber-Cartier theorem ([Dold-Puppe, p. 213]) for bisimplicial abelian groups has the following well-known analogue for bisimplicial sets.

<u>Proposition B.1</u>. For a bisimplicial set X, there is a natural simplicial isomorphism Ψ: Tot X $\approx$ diag X.

<u>Proof</u>. The desired map Ψ: Tot X $\to$ diag X is induced by the maps $\Delta[m] \times X_{m,*} \to$ diag X sending $(\theta^* \iota^m, x) \in \Delta[m]_n \times X_{m,n}$ to $\theta^* x \in X_{n,n}$ for $\theta: [n] \to [m]$ in Δ. One checks explicitly that Ψ is iso whenever $X = \Delta[m] \underset{\sim}{\times} \Delta[n]$, i.e. X is freely generated by an (m,n)-simplex. The proposition then follows by a direct limit argument.

In view of B.1, the following fundamental theorem is not surprising.

<u>Theorem B.2</u>. Let f: X $\to$ Y be a map of bisimplicial sets such that $f_{m,*}: X_{m,*} \to Y_{m,*}$ is a weak equivalence for each $m \geq 0$. Then diag(f): diag X $\to$ diag Y is a weak equivalence.

This was proved in [Bousfield-Kan, p. 335], but a more direct proof using a patching argument is in [Tornehave] and [Reedy].

The diagonal functor not only preserves termwise weak equivalences of bisimplicial sets, but also clearly preserves termwise cofibre squares. To state a similar, but more complicated, result for termwise fibre squares, we will need

<u>B.3. The π_*-Kan condition</u>. This is a condition on a bisimplicial set X which holds automatically when each $X_{m,*}$ is connected, and in many other cases. Roughly speaking, it requires that the vertical homotopy groups of X satisfy Kan's extension condition horizontally. More precisely, for $m,t \geq 1$ and $a \in X_{m,0}$ consider the homomorphisms

$$(d_i^h)_*: \pi_t(X_{m,*}, a) \to \pi_t(X_{m-1,*}, d_i^h a) \quad 0 \leq i \leq m$$

where the homotopy groups of a simplicial set are defined to be those of its geometric realization. We say X satisfies the π_t-<u>Kan</u>

<u>condition at</u> $a \in X_{m,0}$ if for every collection of elements

$$\{x_i \in \pi_t(X_{m-1,*}, d_i^h a)\}_{i=0,1,\ldots,k-1,k+1,\ldots,m}$$

which satisfy the compatibility condition $(d_i^h)_* x_j = (d_{j-1}^h)_* x_i$ for $i < j$, $i \neq k$, $j \neq k$, there exists an element $x \in \pi_t(X_{m,*}, a)$ such that $(d_i^h)_* x = x_i$ for all $i \neq k$. We say X satisfies the <u>π_*-Kan condition</u> if for each $m, t \geq 1$ it satisfies the π_t-Kan condition at each $a \in X_{m,0}$.

To see that X satisfies the π_*-Kan condition when each $X_{m,*}$ connected, one makes the following two observations. First, if $a, b \in X_{m,0}$ are in the same component of $X_{m,*}$, then the π_t-Kan condition for X at a is clearly equivalent to that at b. Second, if $a \in X_{m,0}$ can be expressed as $a = s_0^h \cdots s_0^h e$ for some $e \in X_{0,0}$, then X satisfies the π_t-Kan condition at a for all $t \geq 1$, because any simplicial group satisfies the ordinary Kan condition. Note also that if $X, Y \in$ (bis. sets) are related by a termwise weak equivalence $X \to Y$, then X satisfies the π_*-Kan condition if and only if Y does.

It is easy to show that a bisimplicial set X satisfies the π_*-Kan condition if it has a bisimplicial group structure. To give a more general criterion we use the following notation. For a simplicial set K and $t \geq 1$, let $\pi_t(K)_{free}$ denote the set of unpointed homotopy classes of maps from a t-sphere to $|K|$, and let $\theta\colon \pi_t(K)_{free} \to \pi_0 K$ be the obvious surjection. We call K <u>simple</u> if each component of $|K|$ is a simple space. It is now an easy exercise to prove

(B.3.1). Let X be a bisimplicial set with $X_{m,*}$ simple for $m \geq 0$. Then X satisfies the π_*-Kan condition if and only if the simplicial map $\theta\colon \pi_t^v(X)_{free} \to \pi_0^v X$ is a fibration for each $t \geq 1$.

To state our fibre square theorem, we recall that a commutative square

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow w \\ B & \longrightarrow & D \end{array}$$

of simplicial sets is a <u>homotopy fibre square</u> (see A.2) if for some factorization $C \xrightarrow{i} W \xrightarrow{p} D$ of w with i a weak equivalence and p a (Kan) fibration, the map $A \to B \times_D W$ is a weak equivalence. Also, for a bisimplicial set X, we let $\pi_i^v X$ be the simplicial set with $(\pi_i^v X)_m = \pi_i X_{m,*}$.

<u>Theorem B.4.</u> Let

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

be a commutative square of bisimplicial sets such that the terms $V_{m,*}$, $W_{m,*}$, $X_{m,*}$, and $Y_{m,*}$ form a homotopy fibre square for each $m \geq 0$. If X and Y satisfy the π_*-Kan condition and if $\pi_0^v X \to \pi_0^v Y$ is a fibration, then

$$\begin{array}{ccc} \operatorname{diag} V & \to & \operatorname{diag} X \\ \downarrow & & \downarrow \\ \operatorname{diag} W & \to & \operatorname{diag} Y \end{array}$$

is a homotopy fibre square.

Note that the hypotheses on X and Y hold automatically when the terms $X_{m,*}$ and $Y_{m,*}$ are all connected. Some other interesting, but more specialized, versions of this theorem have been proved in [May 2, §12] and [Segal 2]; and some extensions and applications have been obtained by T. Gunnarson in his thesis work. Before starting to prove B.4, we apply it to generalize Quillen's spectral sequence for bisimplicial groups [Quillen 3].

<u>Theorem B.5</u>. Let X be a bisimplicial set satisfying the π_*-Kan condition, and let $*\in X_{0,0}$ be a base vertex (whose degeneracies are taken as the basepoints of the sets $X_{m,n}$.) Then there is a first quadrant spectral sequence $\{E^r_{s,t}\}_{r\geq 2}$ converging to $\pi_{s+t}(\mathrm{diag}\ X)$ with $E^2_{s,t} = \pi^h_s\pi^v_t X$. The term $E^r_{s,t}$ is a set for $t + s = 0$, a group for $t + s = 1$, and an abelian group for $t + s \geq 2$. Convergence has the obvious meaning, e.g. there is an isomorphism of sets $E^\infty_{0,0} \cong \pi_0 \mathrm{diag}\ X$ and a short exact sequence $1 \to E^\infty_{0,1} \to \pi_1 \mathrm{diag}\ X \to E^\infty_{1,0} \to 1$ of groups.

<u>Proof</u>. By B.2 we can assume each $X_{m,*}$ is a Kan complex, and by B.4 there is a homotopy fibre square

$$\begin{array}{ccc} \mathrm{diag}(F_tX) & \to & \mathrm{diag}(P_tX) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{diag}(P_{t-1}X) \end{array}$$

for $t \geq 0$ where P_tX is the bisimplicial set given by the t^{th}-Postnikov sections of the terms $X_{m,*}$ (taking $P_{-1}X = *$), and where F_tX is the fibre of $P_tX \to P_{t-1}X$. Let $\{E^r_{s,t}\}$ be the associated spectral sequence with $E^2_{s,t} = \pi_{s+t}\mathrm{diag}(F_tX)$. The convergence result follows since $\mathrm{diag}X \to \mathrm{diag}P_tX$ is iso in dimensions $\leq t$ and onto elsewhere, and it remains to show $\pi_{s+t}\mathrm{diag}(F_tX) \cong \pi^h_s\pi^v_tX$. Since $(F_tX)_{m,n} = *$ for $n < t$, there is a natural bisimplicial map $F_tX \to K(\pi^v_tX,t)$ where $K(\pi^v_tX,t)$ is given by the minimal Eilenberg-MacLane complexes $K(\pi_tX_{m,*},t)$. By B.2, we now have

$$\pi_{s+t}\mathrm{diag}F_tX \cong \pi_{s+t}\mathrm{diag}K(\pi^v_tX,t)$$

and the required isomorphism

$$\pi_{s+t}\mathrm{diag}K(\pi^v_tX,t) \cong \pi^h_s\pi^v_tX$$

follows for $t \geq 2$ from [Dold-Puppe, p. 213], and for $t = 0$ trivially. The remaining case $t = 1$ will follow by showing $\pi_*BG \cong \pi_{*-1}G$ for a simplicial group G, where

$$BG = \mathrm{diag}K(G_*,1)_*.$$

The natural principal fibrations

$$K(G_n,0) \to L(G_n,1) \to K(G_n,1)$$

with $|L(G_n,1)| \simeq *$ induce a principal fibration

$$G = \mathrm{diag}K(G_*,0)_* \to \mathrm{diag}L(G_*,1)_* \to \mathrm{diag}K(G_*,1) = BG$$

and $|\mathrm{diag}L(G_*,1)_*| \simeq *$ by an argument using B.2. Thus $\pi_*BG \approx \pi_{*-1}G$.

To prove B.4 we need a model category structure on (bis. sets). For $X,Y\in$(bis. sets), let $HOM(X,Y)$ be the simplicial set whose n-simplices are the bisimplicial maps $X \otimes \Delta[n] \to Y$ where $(X \otimes \Delta[n])_{m,*} = X_{m,*} \times \Delta[n]$.

<u>Theorem B.6</u>. The category (bis. sets) is a proper closed simplicial model category when provided with the following additional structure: a map $f: X \to Y$ in (bis. sets) is called a <u>weak equivalence</u> if $f_{m,*}: X_{m,*} \to Y_{m,*}$ is a weak equivalence in (s.sets) for each $m \geq 0$; f is called a <u>cofibration</u> if it is injective; and f is called a

<u>fibration</u> if $f_{0,*}: X_{0,*} \to Y_{0,*}$ is a fibration and for each $m \geq 1$ the simplicial square

$$\begin{array}{ccc} X_{m,*} & \xrightarrow{d} & M_m X \\ \downarrow{\scriptstyle f_{m,*}} & & \downarrow{\scriptstyle M_m f} \\ Y_{m,*} & \xrightarrow{d} & M_m Y \end{array}$$

induces a fibration $X_{m,*} \to Y_{m,*} \times_{M_m Y} M_m X$ where $(M_m X)_n$ is the set of $(m+1)$-tuples $(x_0,\ldots,x_m)$ in $X_{m-1,n}$ such that $d_i^h x_j = d_{j-1}^h x_i$ for $i < j$, and where $d: X_{m,*} \to M_m X$ is given by $d(x) = (d_0^h x,\ldots,d_m^h x)$.

This theorem follows from [Reedy]; the proof is similar to that of 3.5. We remark that if $f: X \to Y$ is a fibration in (bis. sets), then each $f_{m,*}: X_{m,*} \to Y_{m,*}$ is a fibration in (s.sets), but not conversely.

<u>Proof of B.4</u>. By <u>CM5</u> (cf. §1) and B.2, we can suppose that the given square

$$\begin{array}{ccc} V & \longrightarrow & X \\ \downarrow & & \downarrow \\ W & \longrightarrow & Y \end{array}$$

is a pull-back with $X \to Y$ a fibration and with X,Y fibrant. Since the diagonal functor preserves pull-backs, B.4 follows from

<u>Proposition B.7</u>. Let $X,Y \in$(bis. sets) be fibrant objects satisfying the π_*-Kan condition, and let $f: X \to Y$ be a fibration. If $f_*: \pi_0^v X \to \pi_0^v Y$ is a fibration in (s.sets), then so is diag f: diagX $\to$ diagY.

To prove B.7, we begin by noting that the diagonal functor has a

left adjoint

$$L: (\text{s.sets}) \to (\text{bis.sets})$$

given by

$$L(K)_{m,n} = \operatorname*{colim}_{([m],[n])\to([i],[i])\in\Delta\times\Delta} K_i.$$

To construct L(K) more explicitly, we use the bisimplicial map $c: L(K) \to K \underset{\sim}{\times} K$ adjoint to the diagonal $K \to K \times K$. Although c is not always injective, we have

<u>Lemma B.8</u>. If K is the simplicial set associated with an ordered simplicial complex (cf. [May 1, 1.4]), then $c: L(K) \to K \underset{\sim}{\times} K$ is an injection onto the bisimplicial subset generated by all $(x,x) \in K \underset{\sim}{\times} K$.

<u>Proof</u>. Suppose $(\theta_1^* x_1, \varphi_1^* x_1) = (\theta_2^* x_2, \varphi_2^* x_2)$ in $K \underset{\sim}{\times} K$ where x_1, x_2 are non-degenerate simplices of K and $\theta_1, \varphi_1, \theta_2, \varphi_2$ are maps in Δ. The injectivity of c follows because there exist factorizations $\theta_1 = \gamma_1\sigma$, $\varphi_1 = \gamma_1\tau$, $\theta_2 = \gamma_2\sigma$, $\varphi_2 = \gamma_2\tau$ in Δ such that $\gamma_1^* x_1 = \gamma_2^* x_2$. (Take $\gamma_1^* x_1$ to be the "largest common face" of x_1 and x_2.) The result on the image of c is obvious.

We next use B.8 to show

<u>Lemma B.9</u>. Let $f: X \to Y$ be a bisimplicial fibration such that $f_{*,n}: X_{*,n} \to Y_{*,n}$ is a fibration for each $n \geq 0$. Then diag f is a fibration.

<u>Proof</u>. It suffices to show that diag f has the right lifting property (RLP) for the maps $\Delta^k[n] \xrightarrow{\subset} \Delta[n]$ with $n \geq 1$ and $0 \leq k \leq n$, where $\Delta^k[n]$ is the simplicial subset of $\Delta[n]$ generated by the faces

d_i for $i \neq k$. By adjointness, it now suffices to show that f has the RLP for the bisimplicial maps $L\Delta^k[n] \to L\Delta[n]$ with $n \geq 1$ and $0 \leq k \leq n$. Using B.8 we factor these maps as

$$L\Delta^k[n] \xrightarrow{\subset} \Delta^k[n] \underset{\sim}{\times} \Delta[n] \xrightarrow{\subset} \Delta[n] \underset{\sim}{\times} \Delta[n] \approx L\Delta[n]$$

and we observe that the left map is a trivial cofibration in (bis.sets). The result now follows since f has the RLP for each of the factor maps.

Continuing with the proof of B.7, we must reformulate B.9 using "matching" objects. For $m \geq 1$, $0 \leq s_1 < \cdots < s_r \leq m$, and a bisimplicial set X, let $M_m^{(s_1,\ldots,s_r)}X$ denote the "matching" simplicial set whose n-simplices are the r-tuples $(x_{s_1},\ldots,x_{s_r})$ in $X_{m-1,n}$ such that $d_i^h x_j = d_{j-1}^h x_i$ for each $i < j$ in $\{s_1,\ldots,s_r\}$. Also let $d\colon X_{m,*} \to M_m^{(s_1,\ldots,s_r)}X$ be the simplicial map with $d(x) = (d_{s_1}^h x,\ldots,d_{s_r}^h x)$. It will be convenient to write $M_m^k X$ for $M_m^{(0,\ldots,\hat{k},\ldots,m)}X$.

Lemma B.10. Let $f\colon X \to Y$ be a bisimplicial fibration such that the square

$$\begin{array}{ccc} X_{m,*} & \xrightarrow{d} & M_m^k X \\ \downarrow{\scriptstyle f_{m,*}} & & \downarrow{\scriptstyle M_m^k f} \\ Y_{m,*} & \xrightarrow{d} & M_m^k Y \end{array}$$

induces a surjection

$$\pi_0 X_{m,*} \to \pi_0(Y_{m,*} \times_{M_m^k Y} M_m^k X)$$

for $m \geq 1$ and $0 \leq k \leq m$. Then diag f is a fibration.

Proof. Since f is a fibration and $\Delta^k[m] \underset{\sim}{\times} \Delta[0] \xrightarrow{\subset} \Delta[m] \underset{\sim}{\times} \Delta[0]$ is a cofibration, the map $X_{m,*} \to Y_{m,*} \times_{M_m^k Y} M_m^k X$, is a fibration by SM7 in [Quillen 1, II.2], and it is onto by our π_0-hypothesis. Hence, $X_{*,n} \to Y_{*,n}$ is a fibration for $n \geq 0$, and the result follows from B.9.

To verify the hypotheses of B.10 in our situation, we need

Lemma B.11. Let X be a fibrant bisimplicial set satisfying the π_*-Kan condition, and let $a = (a_{s_1},\ldots,a_{s_r})$ be a vertex of $M_m^{(s_1,\ldots,s_r)}X$ where $1 \leq r \leq m$, $0 \leq s_1 < \cdots < s_r \leq m$. Then for $t \geq 0$ the obvious map

$$\pi_t(M_m^{(s_1,\ldots,s_r)}X,a) \to \pi_t(X_{m-1,*},a_{s_1}) \times \cdots \times \pi_t(X_{m-1,*},a_{s_r})$$

is an injection whose image consists of the elements $(u_{s_1},\ldots,u_{s_r})$ such that $(d_i^h)_* u_j = (d_{j-1}^h)_* u_i$ for each $i < j$ in $\{s_1,\ldots,s_r\}$. Moreover, $d\colon X_{m,*} \to M_m^{(s_1,\ldots,s_r)}X$ is a fibration.

Proof. Using SM7 as in B.10, one shows that d is a fibration. Then the lemma follows by induction on r using the fibre squares

$$\begin{array}{ccc} M_m^{(s_1,\ldots,s_r)}X & \longrightarrow & X_{m-1,*} \\ \downarrow & & \downarrow \\ M_m^{(s_1,\ldots,s_{r-1})}X & \longrightarrow & M_{m-1}^{(s_1,\ldots,s_{r-1})}X \end{array}$$

for $r \geq 2$.

Finally we can give

Proof of B.7. Consider the square of simplicial sets

$$\begin{array}{ccc} X_{m,*} & \xrightarrow{d} & M_m^k X \\ \Big\downarrow f_{m,*} & & \Big\downarrow M_m^k f \\ Y_{m,*} & \xrightarrow{d} & M_m^k Y \end{array}$$

for $m \geq 1$ and $0 \leq k \leq m$. For each vertex $a \in Y_{m,0}$ we show that

$$d_*: \pi_1(Y_{m,*}, a) \to \pi_1(M_m^k Y, da)$$

is onto by using B.11 to compute $\pi_1(M_m^k Y, da)$ and using π_1-Kan condition for Y at a. Thus there is an isomorphism

$$\pi_0(Y_{m,*} \times_{M_m^k Y} M_m^k X) \approx \pi_0 Y_{m,*} \times_{\pi_0 M_m^k Y} \pi_0 M_m^k X$$

and we conclude that

$$\pi_0 X_{m,*} \to \pi_0(Y_{m,*} \times_{M_m^k Y} M_m^k X)$$

is onto by using B.11 in the case $t = 0$ and the hypothesis that $\pi_0^V X \to \pi_0^V Y$ is a fibration. Now B.7 follows from B.10.

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Algebraic and Geometric Connecting Homomorphisms in the Adams Spectral Sequence

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Let E be a commutative ring spectrum such that E_*E is flat over π_*E and such that, for any spectra X and Y, $[X, Y \wedge E] \cong \mathrm{Hom}_{E_*E}(E_*X, E_*Y \otimes_{\pi_*E} E_*E)$ (see, e.g., [1, §13 and §16]).

If $A \to B \to C$ is a cofiber sequence such that (1) is short exact

$$(1) \qquad 0 \to E_*A \to E_*B \to E_*C \to 0$$

then there is an algebraically defined connecting homomorphism

$$\partial : \mathrm{Ext}^{s,t}_{E_*E}(M, E_*C) \to \mathrm{Ext}^{s+1,t}_{E_*E}(M, E_*A)$$

for any E_*E comodule M. When $M = E_*X$, these Ext groups are E_2 terms of Adams spectral sequences and we may ask:

(a) Does ∂ commute with differentials in the Adams spectral sequence?

(b) Does ∂ converge to the homomorphism $\delta_* : [X, C] \to [X, \Sigma A]$ induced by the geometric connecting map $\delta : C \to \Sigma A$?

It is possible to answer (b) without answering (a) (see [2, Theorem 1.7]). We show here that δ induces ∂ in the most natural possible way, answering (a) and (b) affirmatively.

The canonical Adams resolution of a spectrum Y with respect to E is defined by requiring that $Y_{i+1} \to Y_i \to Y_i \wedge E$ be a cofibration for each $i \geq 0$.

<u>Lemma</u>: The connecting map $\delta : C \to \Sigma A$ induces a map D of Adams resolutions with a shift of filtration: