K-Theory **32:** 181–191, 2004.

The Additivity Theorem in *K*-theory

KEVIN CHARLES JONES, YOUNGSOO KIM, ANDREA H. MHOON, REKHA SANTHANAM, BARRY J. WALKER and DANIEL R. GRAYSON
University of Illinois at Urbana-Champaign, IL, USA. e-mail: kcjones@students.uiuc.edu, ykim33; mhoon; rekha; bjwalker; dan@math.uiuc.edu

(Received: February 2004)

Abstract. We present a method for converting Theorem B style proofs in algebraic *K*-theory to Theorem A style proofs and apply it to the additivity theorem.

Mathematics Subject Classification: 19D06.

Key words: additivity theorem, theorem A style, theorem B style

1. Introduction

The additivity theorem is a central theorem in *K*-theory. It was originally proved by Quillen [5] for exact categories using the Q-construction. Later, Waldhausen [7] proved it again using the S-construction, generalizing it to apply to categories with cofibrations and weak equivalences.

McCarthy proved an analogue of the additivity theorem in the context of cyclic homology [4, Theorem 3.5.1]. It was not straightforward for McCarthy to transfer the proofs of Quillen and Waldhausen to this setting because they used Quillen's Theorem *B* [5], so he had to devise a new proof of the additivity theorem, presented separately in [4]. For him, the crucial difference in style between the two proofs is that the proofs of Quillen and Waldhausen are *Theorem B style* proofs, whereas the new proof was a *Theorem A style* proof.

A *Theorem A style* proof is one that uses the realization lemma [6, Lemma 5.1], or one of the theorems close to being logically equivalent to it, such as Quillen's [5, Theorem *A*], Waldhausen's [7, Lemma 1.4.A], Gillet-Grayson's [1, Theorem A'], or McCarthy's [4, Proposition 3.4.5]. The hypothesis of all these theorems is that some naive combinatorial approximations to the homotopy fibers of a map are all contractible, and the result is that the map is a homotopy equivalence.

A *Theorem B style* proof is one that uses the fibration lemma [6, Lemma 5.2], or one of the theorems close to being logically equivalent to it, such as

Quillen's [5, Theorem *B*], Waldhausen's [7, Lemma 1.4.B], or Gillet-Grayson's [1, Theorem *B*′]. The hypothesis of all these theorems is that the base change maps between naive combinatorial approximations to the homotopy fibers of a map are all homotopy equivalences, and the result is a fibration sequence incorporating the map.

In this paper, we add a theorem (Theorem $\hat{A}$ of Section 2) to the list of theorems usable in a Theorem A style proof that makes it easy to convert Waldhausen's proof [7, Theorem 1.4.2] of the additivity theorem from a Theorem B style proof to a Theorem A style proof. We also introduce a simplicial version of it (Theorem $\hat{A}'$ of section 3) that can be used to convert the proof of additivity in [3, Theorem 5.1.2], which in turn, is based on Waldhausen's proof.

NOTATION. The nerve of a category $\mathscr{C}$ will be denoted by $N\mathscr{C}$. Given a functor $\mathscr{C} \xrightarrow{g} \mathscr{D}$ of small categories, we will use g (an abuse of notation) for the induced map of simplical sets $N\mathscr{C} \xrightarrow{g} N\mathscr{D}$. For $C \in N_p\mathscr{C}$ and $0 \leqslant i \leqslant p$, the i-th vertex is denoted by C_i, so that C will represent a chain $C_0 \to C_1 \to \cdots \to C_p$ of objects and morphisms of $\mathscr{C}$. Let Δ denote the category of finite nonempty ordered sets, let $[n] \in \Delta$ denote the ordered set $\{0 < 1 < \cdots < n\}$, and let Δ^n be the simplicial set represented by $[n]$. We use $*$ to mean any simplex of Δ^0. For simplicial sets S and T, the external product $S \boxtimes T$ is the bisimplicial set defined by $(S \boxtimes T)_{p,q} = S_p \times T_q$. By Yoneda's lemma, we may identify a simplex $t \in T_n$ with a map $t \colon \Delta^n \to T$; for an arrow $i \colon [m] \to [n]$ the simplex $i^*(t)$ will be identified with the composite map $ti \colon \Delta^m \to T$, so we will usually write ti for $i^*(t)$.

2. Converting Waldhausen's Proof

In this section, we show how to convert Waldhausen's proof of the additivity theorem to a Theorem A style proof.

Given a functor $\mathscr{C} \xrightarrow{g} \mathscr{E}$ of small categories and an object E of $\mathscr{E}$, we write g/E for the category [5, p. 93] with objects $(C, gC \xrightarrow{e} E)$, where C is an object of $\mathscr{C}$ and $gC \xrightarrow{e} E$ is an arrow in $\mathscr{E}$; an arrow is a map $C \to C'$ making the evident triangle commute. Analogously, $E \backslash g$ will be a category with objects $(E \xrightarrow{e} gC, C)$. The projection functor $(C, gC \xrightarrow{e} E) \mapsto C$ will be denoted by $g/E \xrightarrow{\pi} \mathscr{C}$.

THEOREM $\hat{A}$. *Let $\mathscr{D} \xleftarrow{f} \mathscr{C} \xrightarrow{g} \mathscr{E}$ be functors of small categories. If the composite functor $f\pi \colon g/E \to \mathscr{D}$ is a homotopy equivalence for each object $E \in \mathscr{E}$ then the functor $(f, g) \colon \mathscr{C} \to \mathscr{D} \times \mathscr{E}$ is a homotopy equivalence.*

Proof. Our proof is modeled on Quillen's proof of Theorem A [5, p. 95], which amounts to the special case where $\mathscr{D}$ is trivial.

It will suffice to show that the map of bisimplicial sets

$$\begin{array}{rccc} (f,g)\boxtimes 1: & N\mathscr{C}\boxtimes\Delta^0 & \longrightarrow & (N\mathscr{D}\times N\mathscr{E})\boxtimes\Delta^0 \\ & (C,*) & \mapsto & [(fC,gC),*] \end{array}$$

is a homotopy equivalence.

First, we define bisimplicial sets X^g and Y, analogous to those introduced by Quillen, as follows, where $p,q\in\mathbb{N}$, with the evident face and degeneracy maps.

$$X^g_{pq} := \{(C,e,E)|C\in N_p\mathscr{C}, E\in N_q\mathscr{E}, e\in \mathrm{Hom}_{\mathscr{E}}(gC_p,E_0)\}$$

$$Y_{pq} := \{(D,E',e,E)|D\in N_p\mathscr{D}, E'\in N_p\mathscr{E}, E\in N_q\mathscr{E}, e\in \mathrm{Hom}_{\mathscr{E}}(E'_p,E_0)\}$$

Consider the following commutative diagram.

$$\begin{array}{ccccc} N\mathcal{C}\boxtimes\Delta^0 & \xleftarrow{\eta} & X^g & & \\ \big\downarrow{\scriptstyle (f,g)\boxtimes 1} & & \big\downarrow{\scriptstyle\beta} & \searrow^{\alpha} & \\ (N\mathcal{D}\times N\mathcal{E})\boxtimes\Delta^0 & \xleftarrow{\delta} & Y & \xrightarrow{\gamma} & N\mathcal{D}\boxtimes N\mathcal{E} \end{array}$$

The maps are given by these formulas.

$$\begin{aligned} \eta(C,e,E) &= (C,*) \\ \beta(C,e,E) &= (fC,gC,e,E) \\ \alpha(C,e,E) &= (fC,E) \\ \delta(D,E',e,E) &= [(D,E'),*] \\ \gamma(D,E',e,E) &= (D,E) \end{aligned}$$

It will be enough to show that $\alpha,\gamma,\eta,\delta$ are homotopy equivalences. Here is the argument for α and γ. Fix q and consider the following diagram of simplicial sets, where h and k are defined to make the diagram commute.

$$\begin{array}{ccc} X^g_{\cdot q} & \xrightarrow{\cong} & \coprod_{E\in N_q\mathscr{E}} N(g/E_0) \\ \big\downarrow{\scriptstyle \alpha_{\cdot q}} & & \big\downarrow{\scriptstyle h} \\ (N\mathscr{D}\boxtimes N\mathscr{E})_{\cdot q} & \xrightarrow[\cong]{} & N\mathscr{D}\times N_q\mathscr{E} \\ \big\uparrow{\scriptstyle \gamma_{\cdot q}} & & \big\uparrow{\scriptstyle k} \\ Y_{\cdot q} & \xrightarrow{\cong} & \coprod_{E\in N_q\mathscr{E}} (N\mathscr{D}\times N(1_{\mathscr{E}}/E_0)) \end{array} \tag{1}$$

Here the horizontal arrows are the obvious isomorphisms of simplicial sets. We point out that h is a disjoint union of homotopy equivalences induced by composite functors $g/E_0 \xrightarrow{\pi} \mathscr{C} \xrightarrow{f} \mathscr{D}$, each of which is a homotopy equivalence by hypothesis, and k is a disjoint union of nerves of maps of the form

$\mathscr{D} \times (1_{\mathscr{E}}/E_0) \to \mathscr{D}$, each of which is a homotopy equivalence because $1_{\mathscr{E}}/E_0$ is always contractible. Hence $\alpha_{\cdot q}$ and $\gamma_{\cdot q}$ are homotopy equivalences for each q. From the realization lemma [6, Lemma 5.1], it then follows that α and γ are homotopy equivalences.

The arguments for η and δ are similar. Fix p instead of q and consider the following isomorphisms.

$$X^g_{p\cdot} \overset{\cong}{\to} \coprod_{gC \in N_p \mathscr{C}} N(gC_p \backslash 1_{\mathscr{E}}),$$

$$Y_{p\cdot} \overset{\cong}{\to} \coprod_{E' \in N_p \mathscr{E}} [N\mathscr{D} \times N(E'_p \backslash 1_{\mathscr{E}})]$$

The crucial point is that $gC_p \backslash 1_{\mathscr{E}}$ and $E'_p \backslash 1_{\mathscr{E}}$ are always contractible. □

Now we rewrite Theorem $\hat{A}$ so it can be applied to simplicial sets. Let $f \colon X \to Y$ be a map of simplicial sets. For any $y \in Y_n$ the simplicial set $f/(n, y)$ is defined by Waldhausen [7, 1.4] as the following pullback.

$$\begin{array}{ccc} f/(n,y) & \xrightarrow{\pi} & X \\ \downarrow & & \downarrow f \\ \Delta^n & \xrightarrow{y} & Y \end{array} \tag{2}$$

In [7, p. 337] Waldhausen's lemma 1.4 A states that if $f/(n, y)$ is contractible for every (n, y) then f is a homotopy equivalence; lemma 1.4 B states that if for every $i \colon [m] \to [n]$, and every $y \in Y_n$, the induced map $f/[m, i^*(y)] \to f/(n, y)$ is a homotopy equivalence then for every (n, y) the diagram (2) is homotopy cartesian. These results are derived from Quillen's theorems A and B, respectively, using the simplex category of a simplicial set, which is defined as follows.

DEFINITION 2.1 For any simplicial set Y, define the category $\mathrm{Simp}(Y)$ with objects (n, y) where $n \in \mathbb{N}$ and $y \in Y_n$, and with morphisms $(n, y) \to (n', y')$ given by commutative diagrams of the following form.

$$\begin{array}{ccc} \Delta^n & \longrightarrow & \Delta^{n'} \\ & {\scriptstyle y}\searrow \quad \swarrow{\scriptstyle y'} & \\ & Y & \end{array}$$

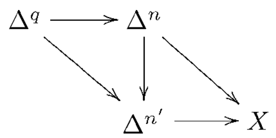
LEMMA 2.2 *If X is a simplicial set, then there is a natural homotopy equivalence $X \sim N\,\mathrm{Simp}(X)$.*

Proof. There is a proof in [7, p. 359]; this proof is extracted from [2, IV, section 5.1]. We identify a p-simplex of $N\mathrm{Simp}(X)$ with a diagram $\Delta^{n_0} \to \cdots \to \Delta^{n_p} \to X$ and we identify a q-simplex of X with a map $\Delta^q \to X$. To interpolate between these two spaces, we introduce the bisimplicial set V

whose (p,q)-simplices are the diagrams of the form $\Delta^q \to \Delta^{n_0} \to \cdots \to \Delta^{n_p} \to X$. There are evident forgetful maps $N_p\,\mathrm{Simp}(X) \xleftarrow{L_{p,q}} V_{p,q} \xrightarrow{M_{p,q}} X_q$ which yield maps $N\,\mathrm{Simp}(X) \boxtimes \Delta^0 \xleftarrow{L} V \xrightarrow{M} \Delta^0 \boxtimes X$ of bisimplicial sets.

Fixing p, the simplicial set $V_{p\cdot}$ is isomorphic to a disjoint union, indexed by the simplices $\Delta^{n_0} \to \cdots \to \Delta^{n_p} \to X$ of $N\,\mathrm{Simp}(X)$, of simplicial sets Δ^{n_0}. The map $L_{p\cdot}$ is similarly a disjoint union of maps of the form $\Delta^{n_0} \to \Delta^0$, and is thus a homotopy equivalence.

Fix q. For any $x \in X_q$, let $\mathcal{G}_x$ be the category whose objects are those pairs of arrows $\Delta^q \to \Delta^n \to X$ whose composite is x; arrows between $\Delta^q \to \Delta^n \to X$ and $\Delta^q \to \Delta^{n'} \to X$ are commutative diagrams of the following form.

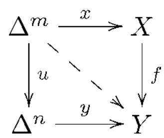


Then $V_{\cdot q}$ is isomorphic to a disjoint union, indexed by simplices $x \in X_q$, of simplicial sets $N\mathcal{G}_x$. Now $N\mathcal{G}_x$ has an initial object, namely $\Delta^q \xrightarrow{1} \Delta^q \xrightarrow{x} X$, and hence is contractible. The map $M_{\cdot q}$ is a disjoint union of the maps of the form $N\mathcal{G}_x \to \Delta^0$, and thus $M_{\cdot q}$ is a homotopy equivalence.

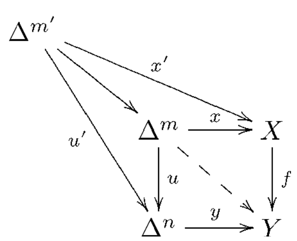
In both cases, we conclude that L and M are homotopy equivalences by the realization lemma [6, Lemma 5.1]. □

THEOREM $\hat{A}$*. *Let $(f,g)\colon X \longrightarrow Y \times T$ be a map of simplicial sets. If the composite $f/(n,y) \xrightarrow{\pi} X \xrightarrow{g} T$ is a homotopy equivalence for all $n \in \mathbb{N}$ and for all $y \in Y_n$ then (f,g) is a homotopy equivalence.*

Proof. We observe that $\mathrm{Simp}[f/(n,y)]$ is naturally isomorphic to $\mathrm{Simp}(f)/(n,y)$ for any $y \in Y_n$. This is easy to see since the objects in both categories are essentially commutative diagrams of the form



and morphisms are essentially commutative diagrams of the following form.



Applying the above observation and Lemma 2.2 we see that

$$\mathrm{Simp}(f)/(n,y) \xrightarrow{\mathrm{Simp}(g)\circ\mathrm{Simp}(\pi)} \mathrm{Simp}(T)$$

is a homotopy equivalence.

Now apply Theorem $\hat{A}$ to the functor

$$\mathrm{Simp}(X) \xrightarrow{[\mathrm{Simp}(f),\mathrm{Simp}(g)]} \mathrm{Simp}(Y) \times \mathrm{Simp}(T)$$

to conclude that $[\mathrm{Simp}(f), \mathrm{Simp}(g)]$ is a homotopy equivalence.

Finally, from lemma 2.2 and the natural isomorphism $\mathrm{Simp}(Y \times T) \cong \mathrm{Simp}(Y) \times \mathrm{Simp}(T)$ we see that (f, g) is a homotopy equivalence. □

With the preliminaries done, we now prove the additivity theorem using Theorem $\hat{A}^*$, thus converting the proof to a Theorem A style proof.

Let $\mathscr{C}$ be a category with cofibrations and weak equivalences [7, 1.2], $\mathscr{E}$ be the category of cofibration sequences of $\mathscr{C}$, and $\star$ be a chosen initial and final object of $\mathscr{C}$. We use the notation $S.\mathscr{C}$ for Waldhausen's S-construction and $S.\mathscr{C}$ for the associated simplicial category with cofibrations and weak equivalences [7, 1.3]. We will use $wS_n\mathscr{C}$ to denote the category of weak equivalences of $S_n\mathscr{C}$ and likewise for $wS_n\mathscr{E}$.

THEOREM 2.3. [Additivity Theorem (7, Theorem 1.4.2)]. *The map* $wS.\mathscr{E} \xrightarrow{s,q} wS.\mathscr{C} \times wS.\mathscr{C}$ *that sends the cofibration sequence* $(A \rightarrowtail C \twoheadrightarrow B)$ *to* (A, B) *is a homotopy equivalence.*

Waldhausen deduces this theorem from the following lemma.

LEMMA 2.4. [7, Lemma 1.4.3] *The map* $S.\mathscr{E} \xrightarrow{s,q} S.\mathscr{C} \times S.\mathscr{C}$ *of simplicial sets is a homotopy equivalence.*

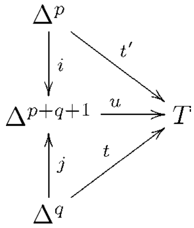
Proof. In order to apply Theorem $\hat{A}^*$, we will need to verify that for all n in $\mathbb{N}$ and for all A' in $S_n\mathscr{C}$ the map $s/(n, A') \xrightarrow{q\pi} S.\mathscr{C}$ is a homotopy equivalence. However, in Waldhausen's proof of the Sublemma to Lemma 1.4.3 in [7, 1.4] he shows that $q\pi$ has a simplicial homotopy inverse. One needs to observe only that the map he calls p is our $q\pi$ and the map he calls f is our s. □

3. Converting A Simplicial Proof

In this section, we show how to convert the proof of the additivity theorem found in [3] to a Theorem A style proof. We begin by reviewing the naive homotopy fibers introduced in [1].

Let T be a simplicial set, and suppose $t' \in T_p$ and $t \in T_q$. The notation $u: t' \Rrightarrow t$ means $u \in T_{p+q+1}$ and $ui = t'$ and $uj = t$, where i is the map induced by the order-preserving map $[p] \to [p+q+1]$ which sends $[p]$ onto the first $p+1$ elements of $[p+q+1]$ and j is the map induced by the map

$[q] \to [p+q+1]$ which sends $[q]$ onto the last $q+1$ elements. We illustrate the meaning of $u\colon t' \Rrightarrow t$ diagrammatically, as follows.



(3)

We use this notation to rephrase the definitions in [1, Section 1, p. 577].

DEFINITION 3.1. For a map $g\colon X \to T$ of simplicial sets and a simplex t of T, the naive homotopy fiber $g|t$ is a simplicial set defined by

$$(g|t)_n = \{(x, u : gx \Rrightarrow t) \mid x \in X_n\}$$

with the evident face and degeneracy maps. We can also define $t|g$ dually by

$$(t|g)_n = \{(u : t \Rrightarrow gx, x) \mid x \in X_n\}.$$

If X and T happen to be nerves of categories, then the simplex $u\colon t \Rrightarrow gx$ gives rise to a collection of arrows $t_i \to gx_j$, which our notation is intended to suggest.

There is a projection map $\pi\colon g|t \to X$ defined by $\pi(x, u : gx \Rrightarrow t) = x$. Letting t vary leads to a bisimplicial set $g|T$ defined as follows.

DEFINITION 3.2 For a map $g\colon X \to T$ of simplicial sets, define the bisimplicial set $g|T$ by

$$(g|T)_{p,q} = \{(x, u : gx \Rrightarrow t) \mid x \in X_p, t \in T_q\}.$$

In case $X = T$ and $g = 1_T$, we will write $T|t$ for $1_T|t$ and $T|T$ for $1_T|T$. The following theorem is a generalization of Theorem $\hat{A}$ that handles simplicial sets.

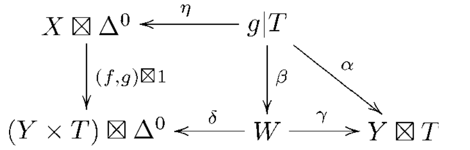
THEOREM $\hat{A}'$ *Let $f\colon X \to Y$ and $g\colon X \to T$ be maps of simplicial sets. If, for any simplex t of T, the composite map $\psi_t\colon (g|t) \xrightarrow{\pi} X \xrightarrow{f} Y$ is a homotopy equivalence, then $(f, g)\colon X \to Y \times T$ is a homotopy equivalence. The same conclusion holds if $g|t$ is replaced by $t|g$ in the hypothesis.*

Proof. We prove just the first part. The proof is completely analogous to the proof of Theorem $\hat{A}$. Define the bisimplicial set W by

$$W_{p,q} = \{(y, u : t' \Rrightarrow t) \mid y \in Y_p, t' \in T_p, \text{ and } t \in T_q\}.$$

The face and degeneracy maps are defined so that $W \cong (Y \boxtimes \Delta^0) \times (T|T)$.

We have a commutative diagram



where the maps are defined as follows.

$$\begin{aligned}
\eta(x, u : gx \Rrightarrow t) &= (x, *)\\
\alpha(x, u : gx \Rrightarrow t) &= (fx, t)\\
\beta(x, u : gx \Rrightarrow t) &= (fx, u : gx \Rrightarrow t)\\
\gamma(y, u : t' \Rrightarrow t) &= (y, t)\\
\delta(y, u : t' \Rrightarrow t) &= [(y, t'), *]
\end{aligned}$$

Once we show that α, γ, δ and η are homotopy equivalences it follows that (f, g) is a homotopy equivalence, by commutativity of the diagram. We shall show that each map is a homotopy equivalence by applying the realization lemma [6, Lemma 5.1].

Fixing q, we have the following commutative diagram of simplicial sets.

$$\begin{array}{ccc}
(g|T)_{\cdot q} & \xrightarrow{\alpha_{\cdot q}} & (Y \boxtimes T)_{\cdot q}\\
\cong\downarrow & & \downarrow\cong\\
\coprod_{t\in T_q} (g|t) & \xrightarrow{\coprod \psi_t} & \coprod_{t\in T_q} Y
\end{array}$$

The vertical maps are the obvious isomorphisms of simplicial sets. The bottom map is a disjoint union of homotopy equivalences, by hypothesis, so $\alpha_{\cdot q}$ is a homotopy equivalence. By the realization lemma, α is a homotopy equivalence.

Similarly γ, δ and η are shown to be homotopy equivalences by the following diagrams.

$$\begin{array}{ccc}
W_{\cdot q} & \xrightarrow{\gamma_{\cdot q}} & (Y \boxtimes T)_{\cdot q}\\
\cong\downarrow & & \downarrow\cong\\
\coprod_{t\in T_q} Y \times (T|t) & \longrightarrow & \coprod_{t\in T_q} Y
\end{array}$$

$$\begin{array}{ccc}
W_{p\cdot} & \xrightarrow{\delta_{p\cdot}} & ((Y \times T) \boxtimes \Delta^0)_{p\cdot}\\
\cong\downarrow & & \downarrow\cong\\
\coprod_{\substack{y\in Y_p\\ t'\in T_p}} (t'|T) & \longrightarrow & \coprod_{\substack{y\in Y_p\\ t'\in T_p}} \Delta^0
\end{array}
\qquad
\begin{array}{ccc}
(g|T)_{p\cdot} & \xrightarrow{\eta_{p\cdot}} & (X \boxtimes \Delta^0)_{p\cdot}\\
\cong\downarrow & & \downarrow\cong\\
\coprod_{x\in X_p} (gx|T) & \longrightarrow & \coprod_{x\in X_p} \Delta^0
\end{array}$$

The bottom maps are homotopy equivalences because $T|t, t'|T$, and $gx|T$ are contractible [1, Lemma 1.4]. This completes the proof. □

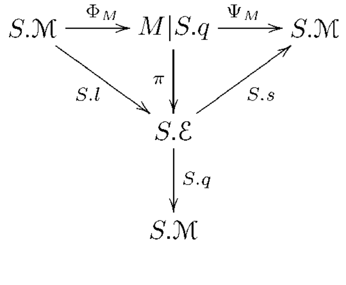
Now we wish to provide another Theorem A style proof of additivity using the naive homotopy fibers introduced in 3.1. We use Theorem $\vec{A}$ to convert the Theorem B style proof in [3], which is in turn based on Waldhausen's proof in [7].

For the rest of this section, $\mathcal{M}$ will be a small exact category with a chosen zero object 0. Let $K(\mathcal{M})$ denote a space whose homotopy groups are the K-groups, for example, $K(\mathcal{M}) = \Omega|\mathcal{S}.\mathcal{M}|$. Let $\mathcal{E}$ be the category whose objects are the short exact sequences $E\colon 0 \to M \to N \to P \to 0$ of objects in $\mathcal{M}$. Define s, t, and $q\colon \mathcal{E} \to \mathcal{M}$ to be the exact functors sending E to M, N, and P, respectively.

THEOREM 3.3 (Additivity Theorem). *The map* $K(\mathcal{E}) \overset{(s,q)}{\to} K(\mathcal{M}) \times K(\mathcal{M})$ *is a homotopy equivalence.*

Proof. It is enough to show that the induced map $S.\mathcal{E} \overset{(s,q)}{\to} S.\mathcal{M}^2$ is a homotopy equivalence. We review some of the details of the proof in [3].

For a fixed $m \geqslant 0$, consider $M \in S_m\mathcal{M}$. For $n \geqslant 0$, an n-simplex of $(M|S.q)$ is a pair (P, E) where P is an $(m+n+1)$-simplex of $S.\mathcal{M}$, E is an n-simplex of $S.\mathcal{E}$, and they are related by $P\colon M \Rrightarrow qE$. An equivalent condition is $Pi = M$ and $Pj = qE$, where i and j are analogous to the maps given in diagram (3). Consider the map $p\colon \Delta^{m+n+1} \to \Delta^m$ sending all vertices coming from Δ^n to the top element of Δ^m while acting as the identity on vertices $\{0, ..., m\}$ coming from Δ^m. Consider also the exact functor $l\colon \mathcal{M} \to \mathcal{E}$ sending N to $0 \to N \overset{1}{\to} N \to 0 \to 0$. Now define a map $\Phi_M\colon S.\mathcal{M} \to (M|S.q)$ by sending $N \in S_n\mathcal{M}$ to (Mp, lN). Note that $qlN = 0$, so to check that $\Phi_M(N)$ is in $(M|S.q)_n$, we need to check that $Mpj = 0$, and this follows from the fact that pj factors through a one element set, together with the remark that the only 0-simplex of $S.\mathcal{M}$ is 0. Next, define $\pi\colon (M|S.q) \to S.\mathcal{E}$ by sending (P, E) to E and $\Psi_M\colon (M|S.q) \to S.\mathcal{M}$ to be $S.s \circ \pi$. The maps we've just defined fit into the following diagram.



The maps Ψ_M and Φ_M are simplicial homotopy inverses. Indeed, one can see by inspection that $\Psi_M \circ \Phi_M$ is the identity, so we will construct a simplicial homotopy from $\Phi_M \circ \Psi_M$ to $1_{M|S.q}$, as a map $H\colon \Delta^1 \times (M|S.q) \to (M|S.q)$ as in [3]. Observe that $(\Phi_M \circ \Psi_M)(P, E) = \Phi_M(sE) = (Mp, lsE) = (Pip, lsE)$. Suppose $[\tau, (P, E)] \in {\Delta^1}_n \times (M|S.q)_n$. We want the first component of

$H[\tau,(P,E)]$ to interpolate between P and Pip. We first must make an order-preserving map $h_\tau\colon \Delta^{m+n+1} \to \Delta^{m+n+1}$ which interpolates between 1 and ip. The map h_τ is defined on vertices in the following way:

$$h_\tau(a) = \begin{cases} a\colon & a \in [m] \\ a\colon & a = m+1+b, \quad b \in [n], \ \tau(b) = 1 \\ m\colon & a = m+1+b, \quad b \in [n], \ \tau(b) = 0 \end{cases}.$$

Note that $h_0 = ip$ and $h_1 = 1$, so we can now set $P^\tau = Ph_\tau$. Also, $h_\tau i = i$, so $P^\tau i = M$, which is necessary for P^τ to be the first component of an element of $M|S.q$.

Next, we need to construct the second component, E^τ. Note that for all $a \in [m+n+1]$, $h_\tau(a) \leqslant a$. So there is a natural transformation $h_\tau \to 1$, which induces a map $P^\tau j = Ph_\tau j \to Pj = qE$. Identifying E with the exact sequence $0 \to sE \to tE \to qE \to 0$, we can now form the pullback of that sequence along the induced map.

$$\begin{array}{lccccccccc} E^\tau\colon & 0 & \longrightarrow & sE^\tau = sE & \longrightarrow & tE^\tau & \longrightarrow & P^\tau j = qE^\tau & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \\ E\colon & 0 & \longrightarrow & sE & \longrightarrow & tE & \longrightarrow & qE & \longrightarrow & 0 \end{array}$$

As suggested by our notation, E^τ is defined to be this pullback. By definition, $P^\tau j = qE^\tau$, and since we already know that $P^\tau i = M$, the pair (P,E) is an element of $M|S.q$. It is shown in [3] that this is indeed a simplicial homotopy, and so we can now apply Theorem $\hat{A}'$ to $\Psi_M = S.s \circ \pi$, which shows that the map $S.\mathscr{E} \xrightarrow{(s,q)} S.\mathscr{M}^2$ is a homotopy equivalence. $\square$

Acknowledgements

Jones and Kim were supported by University of Illinois, Mhoon by NSF grant DMS 99-83160, Santhanam by NSF grant DMS 03-06429, and Grayson by NSF grant DMS 03-11378. We thank Randy McCarthy for useful feedback during the preparation of this paper.

References

1. Gillet, H. and Grayson, D. R.: The loop space of the Q-construction, Illinois *J. Math.* **31** (4) (1987), 574–597. MR **89h:** 18012 1, 6, 7.
2. Goerss, P. G. and Jardine, J. F.: *Simplicial Homotopy Theory, Progress in Mathematics,* Vol. 174, Birkhäuser Verlag, Basel, (1999). MR **2001d:** 55012 4.
3. Grayson, D. R.: *Algebraic K-theory, notes for math 416, available at http://www.math.uiuc.edu/~dan/Courses/2003/Spring/416/, May* (2003) 2, 6, 8, 9.

4. McCarthy, R.: On fundamental theorems of algebraic *K*-theory, *Topology* **32** (2). (1993), 325–328. MR **94e:** 19002 1.
5. Quillen, D.: Higher algebraic *K*-theory. I, *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst.,* Seattle, Wash., 1972), Springer, Berlin (1973), pp. 85–147. Lecture Notes in Math., **341.** MR **49** #2895 1, 2.
6. Waldhausen, F.: Algebraic *K*-theory of generalized free products. I, II, *Ann. of Math.* **108** (2) (1978), 1, 135–204. MR **58** #16845a 1, 3, 4, 7.
7. Wardhausen, F.: Algebraic *K*-theory of spaces, *Algebraic and geometric topology* (New Brunswick, N.J., 1983), Lecture Notes in Math., **1126,** Springer, Berlin, 1985, pp. 318–419. MR **86m:**18011 1, 3, 4, 5, 8.