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On Fundamental Theorems of Algebraic K-Theory

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Abstract. In this work we present proofs of basic theorems in Quillen's algebraic K-theory of exact categories. The proofs given here are simpler and more straight-forward than the originals.

Key words. Exact categories, categories with cofibrations and weak equivalencies, localization theorem.

0. Introduction

The object of this paper is to present proofs of the cofinality theorem, the resolution theorem, and the devissage theorem, and a localization theorem, starting each time from a basic fibration sequence up-to-homotopy constructed by Waldhausen in [4] as a part of his treatment of the K-theory of categories with cofibrations and weak equivalences. In his paper [1], Grayson broaches the idea that these theorems should perhaps be obtained by short arguments branching off a core construction or core theorem. His core construction is the fibration-sequence-up-to-homotopy associated to a dominant functor between two exact categories, a situation somewhat more restrictive and more difficult to handle than the more general situation treated by Waldhausen. Here we are showing that by means of a little work, one can dispense with the dominance condition. (One could, in fact, axiomatize each situation to a maximum level of generality, but this seems pointless, in view of intended applications.) The philosophical consequence of all this is that the additivity theorem (see below) is promoted to the status of the most basic theorem in algebraic K-theory.

1. Recollections

In this section we recall from [4], Chapter 1, various definitions and basic theorems. Associated to any category with cofibrations $\mathscr{C}$ and with a specified subcategory of weak equivalences $w\mathscr{C}$ is its K-theory space $\Omega|wS.\mathscr{C}|$ and we will be using a few properties of the K-theory functor.

DEFINITION 1.1. A category with cofibrations is a pointed category $\mathscr{C}$ (i.e., a category equipped with a distinguished zero object) together with a subcategory co$\mathscr{C}$ satisfying axioms Cof 1, Cof 2, and Cof 3.

Cof 1: The isomorphisms of $\mathscr{C}$ are cofibrations (so that co$\mathscr{C}$ contains all the objects of $\mathscr{C}$.)

Cof 2: For every $A \in \mathscr{C}$, the arrow $0 \rightarrowtail A$ is a cofibration.

Cof 3: Cofibrations admit cobase changes. That is if $A \rightarrowtail B$ is a cofibration and $A \to C$ is any arrow, then the pushout $C \bigcup_A B$ exists in $\mathscr{C}$ and the arrow $C \rightarrowtail C \bigcup_A B$ is also in co$\mathscr{C}$.

In [4], geometrical examples of this situation are most important. We will be concerned with the family of examples obtained from an exact category $\mathscr{M}$ [2], p. 91, by selecting a zero object and by declaring the subcategory of admissible monomorphisms in $\mathscr{M}$ to be the cofibrations.

A functor $f: \mathscr{C} \to \mathscr{C}'$ between two categories with cofibrations is *exact* if it takes 0 to $0'$, cofibrations to cofibrations, and pushout diagrams to pushout diagrams.

DEFINITION 1.2. A category wC of weak equivalences in a category $\mathscr{C}$ with cofibrations shall mean a subcategory $w\mathscr{C}$ of $\mathscr{C}$ satisfying

Weq 1: The isomorphisms of $\mathscr{C}$ are in $w\mathscr{C}$.

Weq 2: (Gluing lemma) If in the commutative diagram

$$\begin{array}{ccccc} B & \leftarrowtail & A & \to & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \leftarrowtail & A' & \to & C' \end{array}$$

the horizontal arrows on the left are cofibrations, and all three vertical arrows are in $w\mathscr{C}$, then the induced map

$$B \bigcup_A C \to B' \bigcup_{A'} C'$$

is a map in $w\mathscr{C}$.

In this paper, we will be interested only in the minimal choice of a subcategory of weak equivalences, namely, the case $w\mathscr{C} = i\mathscr{C} =$ the subcategory of isomorphisms. Conventional usage drops the explicit mention of the cofibrations in the notation and one refers to a category of cofibrations $\mathscr{C}$ with weak equivalences $w\mathscr{C}$, or even to a category with cofibrations and weak equivalences $\mathscr{C}$.

From a category $\mathscr{C}$ with cofibrations and weak equivalences $w\mathscr{C}$, one constructs its K-theory as follows. Consider the partially ordered set of pairs $(i,j)\,(0 \leqslant i \leqslant j \leqslant n)$, where $(i,j) \leqslant (i',j')$ if and only if $i \leqslant i'$ and $j \leqslant j'$. (This poset may be identified with the arrow category $\mathrm{Ar}[n]$ where $[n]$ denotes the ordered set $0 < 1 < \cdots < n$ viewed as a category.)

Consider the functors

$$\begin{aligned} &A: \mathrm{Ar}[n] \to \mathscr{C}, \\ &(i,j) \mapsto A_{j/i} \end{aligned}$$

having the properties that $A_{j/j} = 0$ for all j, and that for every triple $i \leqslant j \leqslant k$,

$A_{j/i} \rightarrowtail A_{k/i}$ is a cofibration and

$$\begin{array}{ccc} A_{j/i} & \rightarrowtail & A_{k/i} \\ \downarrow & & \downarrow \\ A_{j/j} & \rightarrowtail & A_{k/j} \end{array}$$

is a pushout. In other words,

$$A_{k/i}/A_{j/i} \approx A_{k/j}.$$

The category of these functors and their natural transformations is $S_n\mathscr{C}$, and the subcategory of these functors where the components $A_{j/i} \to A'_{j/i}$ of a natural transformation $A \to A'$ lie in $w\mathscr{C}$ is denoted $wS_n\mathscr{C}$.

So far we have a simplicial category

$$S.\mathscr{C}: \Delta^{\mathrm{op}} \to (\mathrm{cat}),$$
$$[n] \mapsto wS_n\mathscr{C}$$

and we make the following definition.

DEFINITION 1.3. The algebraic K-theory of the category with cofibrations $\mathscr{C}$, with respect to the weak equivalences $w\mathscr{C}$, is the pointed space $\Omega|wS.\mathscr{C}|$.

Again we will be concerned with the special case where $\mathscr{C}$ is an exact category considered as a category with cofibrations in the canonical way, and the weak equivalences will be the isomorphisms. For explication of the relation of this construction with the Q-construction, see [4], pp. 375–376.

One of the important properties of this concept of a category with cofibrations and weak equivalences is that it is preserved by certain constructions, the first two of which are in

DEFINITION 1.4. $F_m\mathscr{C}$ is the category in which an object is a sequence of cofibrations

$$A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_m$$

in $\mathscr{C}$, and a morphism is a natural transformation of diagrams. $F_m^+\mathscr{C}$ is the category equivalent to $F_m\mathscr{C}$ in which an object consists of an object of $F_m\mathscr{C}$ plus a choice of a quotients $A_{j/i} = A_j/A_i$ for each $0 \leqslant i < j \leqslant m$.

PROPOSITION 1.5. *$F_m\mathscr{C}$ and $F_m^+\mathscr{C}$ are categories with cofibrations, where a cofibration in either category is a transformation of diagrams $A \rightarrowtail A'$ such that $A_i \rightarrowtail A'_i$ and $A'_i \bigcup_{A_i} A_{i+1} \rightarrowtail A'_{i+1}$ are cofibrations in $\mathscr{C}$. Moreover, the forgetful map $F_m^+\mathscr{C} \to F_m\mathscr{C}$ is an exact equivalence, and the 'subquotient' maps*

$$\begin{array}{lll} q_j: F_m\mathscr{C} \to \mathscr{C} & \text{and} & q_{j/i}: F_m^+\mathscr{C} \to \mathscr{C} \\ A \mapsto A_j, & & A \mapsto A_j/A_i \end{array}$$

are exact.

It follows from this that $S.\mathscr{C}$ is actually a simplicial category with cofibrations and weak equivalences (i.e., for each n, $S_n\mathscr{C}$ is a category with cofibrations and weak equivalences), so the wS construction may be repeated.

A third categorical construction preserving the extra structure is the extension construction $E(\mathscr{A},\mathscr{C},\mathscr{B})$ associated to a category with cofibrations and weak equivalences $\mathscr{C}$ containing subcategories $\mathscr{A}$ and $\mathscr{B}$ such that the inclusions $\mathscr{A}\to\mathscr{C},\mathscr{B}\to\mathscr{C}$ are exact.

$E(\mathscr{A},\mathscr{C},\mathscr{B})$ is the category of diagrams in $\mathscr{C}$

$$\begin{array}{ccc} A & \rightarrowtail & C \\ \downarrow & & \downarrow \\ * & \rightarrowtail & B \end{array}$$

or cofibration sequences

$$A \rightarrowtail C \twoheadrightarrow B$$

with $A\in\mathscr{A}, B\in\mathscr{B}$, and the maps are the maps of diagrams. As a category, $E(\mathscr{A},\mathscr{C},\mathscr{B})$ is the pullback of the diagram

$$F_1^+\mathscr{C}\to\mathscr{C}\times\mathscr{C}\leftarrow\mathscr{A}\times\mathscr{B}$$

and we define the cofibrations and weak equivalences in $E(\mathscr{A},\mathscr{C},\mathscr{B})$ by pulling back

$$\mathrm{co}(F_1^+\mathscr{C})\to\mathrm{co}(\mathscr{C}\times\mathscr{C})=\mathrm{co}\mathscr{C}\times\mathrm{co}\mathscr{C}\leftarrow\mathrm{co}(\mathscr{A}\times\mathscr{B})=\mathrm{co}(\mathscr{A})\times\mathrm{co}(\mathscr{B})$$

and

$$w(F_1^+\mathscr{C})\to w(\mathscr{C}\times\mathscr{C})=w\mathscr{C}\times w\mathscr{C}\leftarrow w(\mathscr{A}\times\mathscr{B})=w\mathscr{A}\times w\mathscr{B}.$$

Then the three projections

$$s,t,q\colon E(\mathscr{A},\mathscr{C},\mathscr{B})\to\mathscr{A},\mathscr{C},\mathscr{B}$$

are all exact functors.

The first important result of all this is the additivity theorem (which we will use explicitly later).

THEOREM 1.6 ([4], pp. 331 and 336). *The subobject and quotient maps s and q induce a homotopy equivalence*

$$wS.E(\mathscr{A},\mathscr{C},\mathscr{B})\to wS.\mathscr{A}\times wS.\mathscr{B}$$

DEFINITION 1.7. ([4], p. 343) Let $f\colon\mathscr{A}\to\mathscr{B}$ be an exact functor of categories with cofibrations and weak equivalences. Then $S_n(f\colon\mathscr{A}\to\mathscr{B})$ is the pullback of

$$S_n\mathscr{A}\xrightarrow{S_nf}S_n\mathscr{B}\xleftarrow{d_0}S_{n+1}\mathscr{B}.$$

$S_n(f\colon\mathscr{A}\to\mathscr{B})$ is a category with cofibrations and weak equivalences in a natural way in which an object may be visualized as a chain of cofibrations

$$B_1\rightarrowtail B_2\rightarrowtail\cdots\rightarrowtail B_{n+1}$$

together with a way of writing each quotient B_j/B_i as $f(A_{j-1/i-1})$, as well as the induced maps between the quotients. Each $S_n(f\colon \mathscr{A} \to \mathscr{B})$ contains $\mathscr{B}$ (as the chain of identities and quotients written as $f(0)$), so we obtain a sequence of simplical categories with cofibrations and weak equivalences

$$\mathscr{B} \to S.(f\colon \mathscr{A} \to \mathscr{B}) \to S.\mathscr{A}.$$

The main theorem combined with corollaries is the following,

THEOREM 1.8. ([4], pp. 343, 345) (i) *The sequence*

$$wS.\mathscr{B} \to wS.S.(f\colon \mathscr{A} \to \mathscr{B}) \to wS.S.\mathscr{A}$$

is a fibration up to homotopy.

(ii) *If $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ are exact functors of categories with cofibrations and weak equivalences then the square*

$$\begin{array}{ccc} wS.\mathscr{B} & \to & wS.S.(\mathscr{A} \to \mathscr{B}) \\ \downarrow & & \downarrow \\ wS.\mathscr{C} & \to & wS.S.(\mathscr{A} \to \mathscr{C}) \end{array}$$

is homotopy Cartesian.

The last result we will have occasion to use is the following:

PROPOSITION 1.9 ([4], p. 335). *If $i\mathscr{C}$ denotes the isomorphism category of $\mathscr{C}$, then*

$$iS.\mathscr{C} \simeq s.\mathscr{C},$$

where $s.\mathscr{C}$ = the simplicial set of objects of the simplicial category $iS.\mathscr{C}$. Moreover, if f_1 and f_2 are isomorphic exact functors from $\mathscr{C}$ to $\mathscr{D}$, then the induced maps $s.f_1$ and $s.f_2\colon s.\mathscr{C} \to s.\mathscr{D}$ are homotopic.

2. The Cofinality Theorem

Here we suppose that $\mathscr{A}$ is an exact subcategory of the exact category $\mathscr{B}$. We will say $\mathscr{A}$ is cofinal in $\mathscr{B}$ if $\mathscr{A}$ is extension closed in $\mathscr{B}$, meaning that if $0 \to A' \to B \to A'' \to 0$ is exact in $\mathscr{B}$ and A' and A'' are in $\mathscr{A}$, then so is B, and if for each $B \in \mathscr{B}$ there is a $B' \in \mathscr{B}$ so that $B \oplus B'$ is isomorphic to an object of $\mathscr{A}$. For example, the category $\mathscr{A}$ of finitely generated free R modules is cofinal in the category $\mathscr{B}$ of finitely generated projective R modules. For simplicity, we will assume that $\mathscr{A}$ is isomorphism closed in $\mathscr{B}$, so that any object of $\mathscr{B}$ isomorphic to an object of $\mathscr{A}$ is itself in $\mathscr{A}$.

THEOREM 2.1. *Suppose $\mathscr{A}$ is cofinal in $\mathscr{B}$ and let $G = K_0(\mathscr{B})/K_0(\mathscr{A})$. Then there is a fibration-sequence-up-to-homotopy*

$$iS.\mathscr{A} \to iS.\mathscr{B} \to BG.$$

Proof. In spirit, we follow Waldhausen's proof of a 'strong cofinality' theorem [4],

p. 346. By Waldhausen's fibration theorem quoted above there is a homotopy Cartesian square

$$\begin{array}{ccc} iS.\mathscr{A} & \to & iS.S.(\mathscr{A} \to \mathscr{A}) \simeq * \\ \downarrow & & \downarrow \\ iS.\mathscr{B} & \to & iS.S.(\mathscr{A} \to \mathscr{B}) \end{array}$$

so the cofinality theorem follows once we identify $iS.S.(\mathscr{A} \to \mathscr{B})$ with the classifying space BG.

The basic trick here is that up to homotopy $iS.S.(\mathscr{A} \to \mathscr{B})$ is the same as $(n \mapsto iS.(S_n\mathscr{A} \to S_n\mathscr{B}))$. The reason for this is that the categories $(S_m(S_n(\mathscr{A} \to \mathscr{B})))$ and $S_n(S_m\mathscr{A} \to S_m\mathscr{B})$ are equivalent. To see this, one observes that an object of the first category may be considered as a diagram

$$\begin{array}{ccc} 0 \rightarrowtail & \cdots \rightarrowtail & 0 \\ \downarrow & & \downarrow \\ B_{1,0} \rightarrowtail & \cdots \rightarrowtail & B_{1,m} \\ \downarrow & & \downarrow \\ B_{n,0} \rightarrowtail & \cdots \rightarrowtail & B_{n,m} \end{array}$$

satisfying certain conditions, together with choices for the quotients. But everything may be symmetrically described, so essentially by 'reversal of priorities' we get our equivalence of categories. In this proof we will consider the simplicial space

$$n \mapsto |iS.(S_n\mathscr{A} \to S_n\mathscr{B}|$$

and will prove it is homotopy equivalent to the simplicial set BG, the homogeneous bar construction on G.

We will derive this result after a three-step analysis. The most work goes into stage one, which is the proof of the following lemma.

LEMMA 2.2. *If $\mathscr{A}$ is cofinal in $\mathscr{B}$ and G denotes $K_0(\mathscr{B})/K_0(\mathscr{A})$ then*

$$\pi_0|iS.(\mathscr{A} \to \mathscr{B})| \cong G$$

and each component of $|iS.(\mathscr{A} \to \mathscr{B})|$ is contractible.

Stage two is a proof that if $\mathscr{A}$ is cofinal in $\mathscr{B}$, then $S_n\mathscr{A}$ is cofinal in $S_n\mathscr{B}$. Stage three uses the additivity theorem to verify that $K_0(S_n\mathscr{B})/K_0(S_n\mathscr{A}) \cong G^n$ in a natural way.

Proof of 2.2. To calculate π_0 we first observe that the function

$$iS_0(\mathscr{A} \to \mathscr{B}) = i\mathscr{B} \to G$$

sending B to $[B] + K_0(\mathscr{A})$ induces a well-defined map from $\pi_0|iS.(\mathscr{A} \to \mathscr{B})|$ to G. This is because the presence of i indicates that isomorphic objects of $\mathscr{B}$ are connected by one simplices, and because a one simplex in the cofibration direction is a diagram

$$B_0 \rightarrowtail B_1 \to B_1/B_0$$

with B_1/B_0 in $\mathscr{A}$, so that

$$[B_1] + K_0(\mathscr{A}) = [B_0] + [B_1/B_0] + K_0(\mathscr{A}) = [B_0] + K_0(\mathscr{A}).$$

Now this map is onto. Since any element $g \in K_0(\mathscr{B})$ can be written $g = [B_1] - [B_2]$ for objects B_1 and B_2 of $\mathscr{B}$, and since there is B_2' in $\mathscr{B}$ such that $B_2 \oplus B_2' \in \mathscr{A}$ we get

$$\begin{aligned} g + K_0(\mathscr{A}) &= [B_1] - [B_2] + K_0(\mathscr{A}) \\ &= [B_1 \oplus B_2'] - [B_2 \oplus B_2'] + K_0(\mathscr{A}) \\ &= [B_1 \oplus B_2'] + K_0(\mathscr{A}). \end{aligned}$$

Thus, each element of G is represented by an object of $\mathscr{B}$.

This map is also one-to-one. For if $[B_1] = [B_2]$ in G, then $[B_1] - [B_2] = [A_1] - [A_2]$ for some pair of objects A_1, A_2 in $\mathscr{A}$. Then in $K_0(\mathscr{B})[B_1] + [A_2] = [B_2] + [A_1]$ and by a standard manipulation there is an object $\bar{B}$ of $\mathscr{B}$ so that

$$B_1 \oplus \bar{B} \oplus A_2 \cong B_2 \oplus \bar{B} \oplus A_1.$$

Using cofinality again there is $\bar{B}'$ so that $\bar{B} \oplus \bar{B}' = \bar{A}$ in $\mathscr{A}$. The following diagram which illustrates three one-cells of $|iS.(\mathscr{A} \to \mathscr{B})|$ shows that B_1 and B_2 are in the same path component.

$$B_1 \to B_1 \oplus \bar{A} \oplus A_2 \cong B_2 \oplus \bar{A} \oplus A_1 \leftarrow B_2.$$

Now we adapt the argument of [4] to prove that each component of $|iS.\mathscr{A} \to \mathscr{B})| \cong |s.(\mathscr{A} \to \mathscr{B})|$ is contractible. Let $|s.(\mathscr{A} \to \mathscr{B})|_B$ denote the component represented by $B \in \mathscr{B}$, and observe that a choice of sums $A \oplus B$ for $A \in \mathscr{A}$ defines a map

$$T(B)\colon s.(\mathscr{A} \to \mathscr{A}) \to s.(\mathscr{A} \to \mathscr{B})_B$$

sending $(A_0 \rightarrowtail \cdots \rightarrowtail A_n$, choices) to $(A_0 \oplus B \rightarrowtail \cdots \rightarrowtail A_n \oplus B$, 'same' choices).

Given a diagram

$$\begin{tikzcd} s.(\mathscr{A} \to \mathscr{A}) \arrow[r, "T(B)"] & s.(\mathscr{A} \to \mathscr{B})_B, \\ L \arrow[u] \arrow[r] & K \arrow[u] \end{tikzcd}$$

where L and K are finite simplicial sets, we will show that after a homotopy of the diagram the map $K \to s.(\mathscr{A} \to \mathscr{B})_B$ factors through $T(B)$. This will imply that $\pi_*(|T(B)|) = 0$, and since $s.(\mathscr{A} \to \mathscr{A})$ is contractible, we deduce $\pi_*(s.(\mathscr{A} \to \mathscr{B})_B) = 0$ for $* \geqslant 0$. Hence, each component of $|s.(\mathscr{A} \to \mathscr{B})|$ is contractible.

Suppose first that $L = \emptyset$ and $K = \Delta^n$. Let the generating simplex of Δ^n have image

$$\sigma = B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n$$

(plus choices in $\mathscr{A}$ for B_i/B_j). In our calculation of $\pi_0(|iS.(\mathscr{A} \to \mathscr{B})|)$, we observed that there is an $A \in \mathscr{A}$ such that $B_0 \oplus A \cong B \oplus A$. So, if B' is such that $B \oplus B' \in \mathscr{A}$, then also $B_0 \oplus A \oplus B' \in \mathscr{A}$.

Thus, moving

$$\sigma = (B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_n, \text{ choices})$$

to

$$(B_0 \oplus A \oplus B') \oplus B \rightarrowtail \cdots \rightarrowtail (B_n \oplus A \oplus B') \oplus B.$$

moves σ into the image of $T(B)$. (Since $\mathscr{A}$ is extension closed in $\mathscr{B}$, $B_i \oplus A \oplus B' \in \mathscr{A}$ for all i, $0 \leqslant i \leqslant n$, by induction.)

Now for a diagram in which K has only finitely many nondegenerate simplices $\{\sigma\}$, choose for each σA_σ as above and let $A = \oplus_\sigma A_\sigma$. Now move everything in K and L by $A \oplus B' \oplus B \in \mathscr{A}$. The map $L \to s.(\mathscr{A} \to \mathscr{A})$ moves inside $s.(\mathscr{A} \to \mathscr{A})$ and after the motion the map $K \to s.(\mathscr{A} \to \mathscr{B})_B$ factors through T_B. Proposition 1.9 above is used here everytime we claim isomorphic exact functors induce homotopic maps on the s level. This concludes the proof of the lemma.

We next assert that if $\mathscr{A}$ is cofinal in $\mathscr{B}$ then $S_n\mathscr{A}$ is cofinal in $S_n\mathscr{B}$. Let

$$B = \begin{array}{ccccccc}
0 \to B_{1/0} & \rightarrowtail & B_{2/0} & \rightarrowtail & \cdots & \rightarrowtail & B_{n/0} \\
 & & \downarrow & & & & \downarrow \\
0 & \rightarrowtail & B_{2/1} & \rightarrowtail & \cdots & \rightarrowtail & B_{n/1} \\
 & & \downarrow & & & & \downarrow \\
 & & 0 & \rightarrowtail & \cdots & \rightarrowtail & B_{n/2} \\
 & & & & & & \downarrow \\
 & & & & & & \vdots \\
 & & & & & & \downarrow \\
 & & & & & 0 \rightarrowtail & B_{n/n-1} \\
 & & & & & & \downarrow \\
 & & & & & & 0
\end{array}$$

be an object of $S_n\mathscr{B}$. By cofinality of $\mathscr{A}$ in $\mathscr{B}$ there are objects $B'_{i/i-1}$ such that $B_{i/i-1} \oplus B'_{i/i-1} \in \mathscr{A}$ for $1 \leqslant i \leqslant n$. Using the standard injections and projections, put

$$B' = \begin{array}{ccccccc}
0 \to B'_{1/0} & \rightarrowtail & B'_{1/0} \oplus B'_{2/1} & \rightarrowtail & \cdots & \rightarrowtail & B'_{1/0} \oplus \cdots \oplus B'_{n/n-1} \\
\downarrow & & \downarrow & & & & \downarrow \\
0 & \rightarrowtail & B'_{2/1} & \rightarrowtail & \cdots & \rightarrowtail & B'_{2/1} \oplus \cdots \oplus B'_{n/n-1} \\
 & & \downarrow & & & & \downarrow \\
 & & 0 & \rightarrowtail & \cdots & & \vdots \\
 & & & & & & \downarrow \\
 & & & & & 0 \rightarrowtail & B'_{n/n-1} \\
 & & & & & & \downarrow \\
 & & & & & & 0
\end{array}$$

B' is fairly clearly an object of $S_n\mathscr{B}$ and $B \oplus B' \in S_n\mathscr{A}$ using the extension closure of

$\mathscr{A}$ in $\mathscr{B}$. Notice also that an exact sequence

$$C' \rightarrowtail C \twoheadrightarrow C''$$

in $S_n\mathscr{B}$ implies exact sequences

$$C'_{i/j} \rightarrowtail C_{i/j} \twoheadrightarrow C''_{i/j}$$

in $\mathscr{B}$, so that if C' and C'' are in $S_n\mathscr{A}$ then so is C. It is also clear that $S_n\mathscr{A}$ is isomorphism-closed in $S_n\mathscr{B}$, so we have retrieved all the hypotheses of the main lemma.

On to step three where we identify $K_0(S_n\mathscr{B})/K_0(S_n\mathscr{A})$ with $(K_0(\mathscr{B})/K_0(\mathscr{A}))^n = G^n$. We observe that there is an exact sequence of endofunctors of $S_n\mathscr{B}$

$$0 \to j' \to \mathrm{Id} \to j'' \to 0$$

where, using the notations above,

$$j'(B) = \begin{array}{ccccccccc} 0 & \to & B_{1/10} & \rightarrowtail & B_{2/0} & \rightarrowtail \cdots \rightarrowtail & B_{n-1/0} & = & B_{n-1/0} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & \rightarrowtail & B_{2/1} & \rightarrowtail \cdots \rightarrowtail & B_{n-1/1} & = & B_{n-1/1} \\ & & & & \downarrow & & \downarrow & & \downarrow \\ & & & & 0 & & & & \vdots \\ & & & & & & 0 & = & 0 \\ & & & & & & & & \downarrow \\ & & & & & & & & 0 \end{array}$$

and

$$j''(B) = \begin{array}{ccccccc} 0 & \to & 0 & \to \cdots & 0 & \to & B_{n/n-1} \\ & & \downarrow & & \downarrow & & \| \\ & & 0 & \to \cdots & 0 & \to & B_{n/n-1} \\ & & & & \vdots & & \vdots \\ & & & & 0 & \to & B_{n/n-1} \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

According to one of the interpretations of the additivity theorem, the exact sequence of functors implies a homotopy equivalence

$$iS.S_n\mathscr{B} \simeq iS.S_{n-1}\mathscr{B} \times iS.\mathscr{B}$$

and, by induction

$$iS.S_n\mathscr{B} \simeq \prod^n iS.\mathscr{B}.$$

Thus

$$\begin{aligned} K_0(S_n\mathscr{B}) &= \pi_1(iS.S_n\mathscr{B}) \\ &\simeq \prod^n K_0(\mathscr{B}) \end{aligned}$$

and the isomorphism is induced by

$$B \to (B_{1/0}, B_{2/1}, \ldots, B_{n/n-1}).$$

Recall that the cofinality theorem follows from the identification of the simplicial space

$$n \mapsto |iS.(S_n\mathscr{A} \to S_n\mathscr{B})|$$

with the bar construction BG with $G = K_0(\mathscr{B})/K_0(\mathscr{A})$. Lemma 2.2 and the calculations above give us a homotopy equivalence

$$(n \mapsto |iS.(S_n\mathscr{A} \to S_n\mathscr{B})| \to (n \mapsto G^n = BG_n)$$

in each degree induced from

$$B \mapsto ([B_{1/0}], [B_{2/1}], \ldots, [B_{n/n-1}]).$$

Now it is easy to see these maps are compatible with the face and degeneracy maps. For instance, a review of the definitions gives us

$$d_i B \mapsto ([B_{1/0}], \ldots, [B_{i+1/i-1}], \ldots, [B_{n/n-1}]$$

if $2 \leqslant i \leqslant n-1$. But from B itself we have an exact sequence

$$0 \to B_{i/i-1} \rightarrowtail B_{i+1/i-1} \to B_{i+1/i} \to 0$$

so that

$$[B_{i+1/i-1}] = [B_{i/i-1}] + [B_{i+1/i}] \quad \text{in } G = K_0(\mathscr{B})/K_0(\mathscr{A}).$$

Hence, we have a global homotopy equivalence by the realization lemma (Lemma 5.1, p. 164 of [3])

$$\begin{aligned} |iS.S.(\mathscr{A} \to \mathscr{B})| &\cong |n \mapsto |iS.(S_n\mathscr{A} \to S_n\mathscr{B})|| \\ &\cong |BG|, \end{aligned}$$

and the proof is complete. □

3. The Resolution Theorem

THEOREM 3.1. *Assume that $\mathscr{A}$ is a full exact subcategory of $\mathscr{B}$, and that $\mathscr{A}$ is closed in $\mathscr{B}$ under exact sequences, extensions, and cokernels. Assume that any $B \in \mathscr{B}$ has a resolution*

$$0 \to B \to A \to A'' \to 0$$

with A and A'' in $\mathscr{A}$. Then the map

$$iS\cdot\mathscr{A} \to iS\cdot\mathscr{B}$$

is a homotopy equivalence.

$\mathscr{A}$ is closed under exact sequences means that a sequence $0 \to A' \to A \to A'' \to 0$ of objects of $\mathscr{A}$ which is exact in $\mathscr{B}$ is exact in $\mathscr{A}$. Here $\mathscr{A}$ is closed under extension means that if $0 \to A' \to B \to A'' \to 0$ is an exact sequence in $\mathscr{B}$ with A' and A'' in $\mathscr{A}$ then B is also in $\mathscr{A}$. That $\mathscr{A}$ is closed under cokernels means that if $0 \to A' \to A \to B \to 0$ is exact in $\mathscr{B}$, then B is in $\mathscr{A}$.

Proof. Consider the fibration sequence up-to-homotopy

$$iS.\mathscr{A} \to iS.\mathscr{B} \to iS.S.(\mathscr{A} \to \mathscr{B}).$$

To prove the theorem it is enough to prove that $iS.S.(\mathscr{A} \to \mathscr{B})$ is contractible. The argument follows the format established in Section 2 of this paper, but it is slightly simpler, more akin to the argument of Proposition 1.5.9 in [4].

Since $|iS.S.(\mathscr{A} \to \mathscr{B})| \cong |s.S.(\mathscr{A} \to \mathscr{B})|$ it suffices to show $s.S.(\mathscr{A} \to \mathscr{B})$ is contractible. As above, we may consider this bisimplicial set as the simplicial set of simplicial sets

$$n \to s.(S_n\mathscr{A} \to S_n\mathscr{B})$$

so it will suffice to show that for each n, $s.(S_n\mathscr{A} \to S_n\mathscr{B})$ is contractible (Lemma 5.1, p. 164 of [3].) This is achieved in two steps by proving the following assertion.

FIRST ASSERTION. *If $\mathscr{A} \subset \mathscr{B}$ satisfies the hypotheses of the theorem then so does $S_n\mathscr{A} \subset S_n\mathscr{B}$.*

SECOND ASSERTION. *If $\mathscr{A} \subset \mathscr{B}$ satisfies the hypotheses of the theorem, then $s.(\mathscr{A} \to \mathscr{B})$ is contractible.*

To verify the first assertion begin by recalling that $S_n\mathscr{A}$ may be thought of as the exact category in which an object is a chain of admissible monomorphisms of $\mathscr{A}$

$$A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n,$$

plus choices for the quotients A_i/A_j, and in which an admissible monomorphism

$$A' = (A'_1 \rightarrowtail A'_2 \rightarrowtail \cdots \rightarrowtail A'_n, \text{ choices}) \rightarrowtail A = (A_1 \rightarrowtail \cdots \rightarrowtail A_n, \text{ choices})$$

is a ladder diagram of admissible monomorphisms

$$\begin{array}{ccc} A'_1 & \rightarrowtail \cdots \rightarrowtail & A'_n \\ \downarrow & & \downarrow \\ A_1 & \rightarrowtail \cdots \rightarrowtail & A_n \end{array}$$

satisfying the extra condition that

$$A_i \bigoplus_{A'_i} A'_{i+1} \rightarrowtail A_{i+1}$$

is also an admissible monomorphism. (By Lemma 1.1.3 of [4], these conditions imply that

$$A_i \bigoplus_{A'_i} A'_j \rightarrowtail A_j \quad \text{and} \quad A_{j/i} \bigoplus_{A'_{j/i}} A'_{k/i} \rightarrowtail A_{k/i}$$

are admissible monomorphisms when $i < j$ and $i < j < k$, respectively. ($A_{j/i}$ indicates a chosen quotient.)) Thus, everything not written out here takes care of itself under the constructions we make. The closure of $S_n\mathscr{A}$ in $S_n\mathscr{B}$ under exact sequences, extensions, and cokernels is a consequence of the definition of exact sequences (i.e., admissible monomorphism) in the two categories together with the appropriate closure property of $\mathscr{A}$ in $\mathscr{B}$.

Now we have to check the resolution condition. Let

$$B = (B_1 \rightarrowtail \cdots \rightarrowtail B_n, \text{ choices})$$

denote an object of $S_n\mathscr{B}$. We have to produce an exact sequence

$$B \rightarrowtail A \to A''$$

in $S_n\mathscr{B}$. The construction is made inductively, as follows. Suppose that we have a partial resolution

$$\begin{array}{ccccccccccc}
B_1 & \rightarrowtail & B_2 & \rightarrowtail & \cdots & \rightarrowtail & B_i & \rightarrowtail & B_{i+1} & \rightarrowtail \cdots \rightarrowtail & B_n \\
\downarrow & & \downarrow & & & & \downarrow & & & & \\
A_1 & \rightarrowtail & A_2 & \rightarrowtail & \cdots & \rightarrowtail & A_i & & & & \\
\downarrow & & \downarrow & & & & \downarrow & & & & \\
A_1'' & \rightarrowtail & A_2'' & \rightarrowtail & \cdots & \rightarrowtail & A_i'' & & & &
\end{array}$$

where the upper squares satisfy

$$A_j \bigoplus_{B_j} B_{j+1} \rightarrowtail A_{j+1}$$

is admissible for $1 \leqslant j \leqslant i$. Note that

$$B_i \rightarrowtail \mathrm{A}_i \to A_i''$$

is a resolution of B_i. By hypotheses we can resolve

$$A_i \bigoplus_{B_i} B_{i+1} \rightarrowtail A_{i+1} \to C_{i+1}$$

with A_{i+1}, C_{i+1} in $\mathscr{A}$, and we claim that

$$B_{i+1} \rightarrowtail A_{i+1} \to A_{i+1}''$$

is a resolution of B_{i+1} with A_{i+1} and A_{i+1}'' in $\mathscr{A}$, where

$$(B_{i+1} \rightarrowtail A_{i+1}) = (B_{i+1} \rightarrowtail A_i \bigoplus_{B_i} B_{i+1} \rightarrowtail A_{i+1})$$

and

$$A_{i+1}'' = \operatorname{coker}(B_{i+1} \rightarrowtail A_{i+1}).$$

Granting this for the moment, we can now tack onto the old diagram one more column and proceed, by induction

$$\begin{tikzcd}
B_1 \arrow[r, tail] \arrow[d, tail] & B_2 \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & B_i \arrow[r, tail] \arrow[d, tail] & B_{i+1} \arrow[r, tail] \arrow[d, tail] & \cdots \arrow[r, tail] & B_n \\
A_1 \arrow[r, tail] \arrow[d] & A_2 \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & A_i \arrow[r, tail] \arrow[d] & A_{i+1} \arrow[d] & & \\
A''_1 \arrow[r, tail] & A''_2 \arrow[r, tail] & \cdots \arrow[r, tail] & A''_i \arrow[r, tail] & A''_{i+1} & &
\end{tikzcd}$$

That $A_i \rightarrowtail A_{i+1}$ is admissible follows from the preservation of admissible monomorphisms by pushouts and the fact that a composite of admissible monomorphisms is admissible. That $A''_i \rightarrowtail A''_{i+1}$ is admissible is also a consequence of the preservation by pushouts property, so all we really are left with is to show that $A''_{i+1} \in \mathscr{A}$.

To see this consider the iterated pushout

$$\begin{tikzcd}
0 & A_i \arrow[l] \arrow[r, equal] & A_i \\
0 \arrow[u, equal] \arrow[d, equal] & B_i \arrow[l] \arrow[r, tail] \arrow[u, tail] \arrow[d, tail] & A_i \arrow[u, tail] \arrow[d, tail] \\
0 & B_{i+1} \arrow[l] \arrow[r, tail] & A_{i+1}
\end{tikzcd}$$

Evaluating rows one obtains the pushout

$$\begin{tikzcd}
0 \\
A''_i \arrow[u] \arrow[d] \\
A''_{i+1}
\end{tikzcd}$$

and thus coker $(A''_i \rightarrow A''_{i+1})$ for the value of the iterated pushout. Evaluating columns first one obtains

$$0 \leftarrow A_i \bigoplus_{B_i} B_{i+1} \rightarrow A_{i+1}$$

from which one obtains C_{i+1} for the value of the iterated pushout. But the iterated pushouts must be the same, so we can restate the computations in the form of an exact sequence.

$$0 \rightarrow A''_i \rightarrow A''_{i+1} \rightarrow C_{i+1} \rightarrow 0.$$

So, from closure of $\mathscr{A}$ in $\mathscr{B}$ under extensions we obtain that $A''_{i+1} \in \mathscr{A}$, as needed

The proof of assertion two is in [1], in the proof of Theorem 4.1, and it is short so we repeat the argument here for completeness.

First one notices that $s.(\mathscr{A} \rightarrow \mathscr{B})$ is homotopy equivalent to the nerve of the category $\mathscr{C}$ in which the objects are those of $\mathscr{B}$ and in which an arrow from B to B' is an admissible monomorphism $B \rightarrowtail B'$ such that $B'/B \in \mathscr{A}$. If $m\mathscr{A}$ denotes subcategory of admissible monomorphisms of $\mathscr{A}$, then there is an inclusion

$$G\colon m\mathscr{A} \to \mathscr{C}.$$

Since $m\mathscr{A}$ has 0 for an initial object, it is contractible, so we can prove $\mathscr{C} \cong s\cdot(\mathscr{A} \to \mathscr{B})$ is contractible by proving G is a homotopy equivalence.

We appeal to Quillen's Theorem A according to which it suffices to show contractibility of each fibre B/G, in which an object is a pair ($A \in \mathscr{A}$, $B \rightarrowtail A$, with $A/B \in \mathscr{A}$).

Choose a resolution

$$0 \to B \to A_0 \to A_0'' \to 0$$

of B, and for each $(A, B \rightarrowtail A)$ in B/G choose a pushout $A_0 \underset{B}{\oplus} A$ and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & B \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A_0 \arrow[r] \arrow[d] & A_0 \underset{B}{\oplus} A \arrow[r] \arrow[d] & A'' \arrow[r] & 0. \\
 & A_0'' \arrow[r, equal] \arrow[d] & A_0'' \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where we have written $A'' = A/B$. We see that $A_0 \oplus_B A \in \mathscr{A}$ by extension closure, and by an argument like the one we made in the proof of assertion one $B \rightarrowtail A_0 \oplus_B A$ is an object of B/G. Moreover, the other arrows amount to natural transformations

$$(B \rightarrowtail A) \to (B \rightarrowtail A_0 \oplus_B A) \leftarrow (B \rightarrowtail \mathrm{A}_0)$$

linking the identity on B/G to the constant functor on B/G whose value is $B \to A_0$. Thus B/G is contractible, $s.(\mathscr{A} \to \mathscr{B})$ is also, and we are done.

4. The Devissage Theorem

In this section $\mathscr{B}$ is an Abelian category and $\mathscr{A} \subset \mathscr{B}$ is a full Abelian subcategory. The first example to keep in mind is the one in which $\mathscr{B}$ is the category of finite Abelian p-torsion groups, and $\mathscr{A}$ is the subcategory of elementary Abelian p-groups.

THEOREM 4.1. *Suppose that $\mathscr{A}$ is closed in $\mathscr{B}$ under direct sum, subobject, and quotient* *object. If every object B of $\mathscr{B}$ has a finite filtration*

$$0 = B_{-1} \rightarrowtail B_o \rightarrowtail \cdots \rightarrowtail B_p = B$$

whose consecutive quotients B_i/B_{i-1} are in $\mathscr{B}$, then $iS.\mathscr{A} \to iS.\mathscr{B}$ is a homotopy equivalence.

$\mathscr{A}$ is closed under direct sum means that $A_1 \oplus A_2 \in \mathscr{A}$ if A_1 and A_2 are in $\mathscr{A}$. That $\mathscr{A}$ is closed under subobject, (quotient) object means that if

$$0 \to B' \rightarrowtail A \to B'' \to 0$$

is exact in $\mathscr{B}$ and $A \in \mathscr{A}$ then $B' \in \mathscr{A}(B'' \in \mathscr{A})$.

Proof. Again we consider the standard fibration sequence to homotopy

$$iS.\mathscr{A} \to iS.\mathscr{B} \to iS.S.(\mathscr{A} \to \mathscr{B})$$

and develop the proof along the lines of the proof in Section 3.

Contractibility of $iS.S.(\mathscr{A} \to \mathscr{B})$ is equivalent to the contractibility of its bisimplicial set of objects $s.S.(\mathscr{A} \to \mathscr{B})$, which may be viewed as the simplicial set of simplicial sets

$$n \to s.(S_n\mathscr{A} \to S_n\mathscr{B}).$$

Thus it suffices to show that for each n, $s.(S_n\mathscr{A} \to S_n\mathscr{B})$ is contractible. Again there are two steps to the proof:

FIRST ASSERTION: *If $\mathscr{A} \subset \mathscr{B}$ satisfies the closure and filtration hypotheses of the theorem, then so does $S_n\mathscr{A} \subset S_n\mathscr{B}$ for any n.*

SECOND ASSERTION: *If $\mathscr{A} \subset \mathscr{B}$ satisfies the closure and filtration hypotheses of the theorem then $s.(\mathscr{A} \to \mathscr{B})$ is contractible.*

We begin the proof of the first assertion by stating that the phrase '$B' \rightarrowtail B \twoheadrightarrow B''$ is exact in $S_n\mathscr{B}$' will mean that $B' \rightarrowtail B$ is an admissible monomorphism of $S_n\mathscr{B}$ and that there is a pushout square

$$\begin{array}{ccc} B' & \rightarrowtail & B \\ \downarrow & & \downarrow \\ 0 & \to & B'' \end{array}$$

in $S_n\mathscr{B}$. Consequently, part of the data of an exact sequence in $S_n\mathscr{B}$ is a family of pushout squares or short exact sequences in $\mathscr{B}$

$$0 \to B'_{i/j} \rightarrowtail B_{i/j} \to B''_{i/j} \to 0.$$

It is clear that if $\mathscr{A}$ is closed in $\mathscr{B}$ under subobject and quotient object then $S_n\mathscr{A}$ is similarly closed in $S_n\mathscr{B}$. It is also clear that $S_n\mathscr{A}$ is closed under sum, since the sum of diagrams in $S_n\mathscr{A}$ is computed 'pointwise'.

Now we have to produce a nice filtration of

$$B = \begin{array}{ccccc} 0 \to B_{1/10} & \rightarrowtail & \cdots & \rightarrowtail & B_{n/0} \\ \downarrow & & & & \downarrow \\ 0 & \rightarrowtail & \cdots & \rightarrowtail & B_{n/1} \\ & & & & \vdots \\ & & & 0 \to & B_{n/n-1} \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

$= B_1 \rightarrowtail \cdots \rightarrowtail B_n$, plus choices,

for simplicity of notation. By hypothesis we can filter B_n,

$$0 = B_{n,-1} \rightarrowtail B_{n,0} \rightarrowtail \cdots \rightarrowtail B_{n,p} = B_n$$

with $B_{n,j}/B_{n,j-1} \in \mathscr{A}$. If we put

$$\begin{aligned} B_{i,j} &= \text{pullback } (B_i \rightarrowtail B_n \leftarrowtail B_{n,j}) \\ &= \text{kernel } (B_i \oplus B_{n,j} \to B_n) \end{aligned}$$

we get a lattice diagram

$$\begin{array}{ccccc} B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_n \\ \uparrow & & & & \uparrow \\ B_{1,p-1} & \rightarrowtail & \cdots & \rightarrowtail & B_{n,p-1} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ B_{1,0} & \rightarrowtail & \cdots & \rightarrowtail & B_{n,0} \end{array}$$

Choices for the cokernels of the horizontal monomorphisms may be made so that we get a diagram

$$B_0 \rightarrowtail B_1 \rightarrowtail \cdots \rightarrowtail B_{p-1} \rightarrowtail B_p = B$$

in $S_n\mathscr{B}$. Now each of these arrows is, in fact, an admissible monomorphism in $S_n\mathscr{B}$, because one also has

$$B_{i,j} \cong \text{kernel } (B_{0,j+1} \oplus B_{i+1,j} \to B_{i+1,j+1}),$$

which implies the admissibility condition

$$B_{i,j+1} \underset{B_{i,j}}{\oplus} B_{i+1,j} \rightarrowtail B_{i+1,j+1}$$

is satisfied. And, since $\mathscr{A}$ is closed under subobjects and quotient objects

$$B_{i,j+1}/B_{i,j} \to B_{n,j+1}/B_{n,j}$$

implies first $B_{1,j+1}/B_{i,j} \in \mathscr{A}$ and then all the unwritten quotients of subquotients by subquotients are in $\mathscr{A}$, too, so that $B_{j+1}/B_j \in S_n\mathscr{A}$, as required.

Now we can go to work on the proof of the second assertion, modifying the ideas in [1] somewhat so as to permit the use of different technical ideas and to avoid another bibliographic reference. The object is to show the contractibility of the simplicial set $s.(\mathscr{A} \to \mathscr{B})$ which in degree q consists of diagrams

$$B = \begin{array}{ccccccc} B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_q \\ \downarrow & & \downarrow & & & & \downarrow \\ 0 & \rightarrowtail & A_{1/10} & \rightarrowtail & \cdots & \rightarrowtail & A_{q/0} \\ & & & & & & \vdots \\ & & & & 0 & \rightarrowtail & A_{q/q-1} \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$

in $S_{q+1}\mathscr{B}$, where $B_j/B_i \approx A_{j/i} \in \mathscr{A}$.

To show $s.(\mathscr{A} \to \mathscr{B})$ is contractible, it suffices to show that the last vertex functor

$$L\colon \operatorname{simp}(s.(\mathscr{A} \to \mathscr{B})) \to m\mathscr{B}$$

from the category of simplices of $s.(\mathscr{A} \to \mathscr{B})$ to the category $m\mathscr{B}$ of monomorphisms of $\mathscr{B}$ is a homotopy equivalence, since $m\mathscr{B}$ has zero for an initial object and is therefore contractible. (For information about the category of simplices construction, we refer to [4] pp. 355 and 359.)

We appeal to Quillen's Theorem A, according to which it suffices to show the categories $L/\bar{B}$ are contractible. In our situation, an object of $\operatorname{simp}(s\cdot(\mathscr{A} \to \mathscr{B}))$ is a pair $(q, B \in s_q(\mathscr{A} \to \mathscr{B}))$ and a map $(q, B) \to (r, B')$ is a map $\alpha\colon [q] \to [r]$ in Δ such that $\alpha^*(B') = B$. The functor L sends B to B_q and sends a map as above to $B_q = B'_{\alpha(q)} \rightarrowtail B'_r$. Thus an object of $L/\bar{B}$ is a pair

$$((q, B); B_q \rightarrowtail \bar{B})$$

and a map

$$((q, B); B_q \rightarrowtail \bar{B}) \to ((r, B'); B_r \rightarrowtail \bar{B})$$

is $\alpha\colon [q] \to [r]$ in Δ such that $\alpha^*(B') = B$ and

$$\begin{tikzcd} B_q = B'_{\alpha(q)} \arrow[rr, tail] \arrow[dr, tail] & & B'_r \arrow[dl, tail] \\ & \bar{B} & \end{tikzcd}$$

commutes.

Contemplating the definitions, one sees that $L/\bar{B}$ is equivalent to $\operatorname{simp}(N_{\bar{B}})$, where $N_{\bar{B}}$ is the simplicial set in which a q simplex is a $q + 1$ simplex of $N(m\mathscr{B})$ of the form

$$B_0 \rightarrowtail \cdots \rightarrowtail B_q \rightarrowtail \bar{B}$$

satisfying that $B_q/B_0 \in \mathscr{A}$, and the face and degeneracy operators act to delete and replicate B_i's. ($N_{\bar{B}}$ is a simplicial set because $\mathscr{A}$ is closed under subobject and quotient object.) Then we have $|L/\bar{B}| \simeq |\text{simp}(N_{\bar{B}}))\simeq |N_{\bar{B}}|$ the last equivalence by a general property of the category of simplices construction ([4], p. 359), so it suffices to show $N_{\bar{B}} \simeq *$.

Following [1] closely, pick a filtration of $\bar{B}$

$$0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n = \bar{B}$$

such that $C_i/C_{i-1} \in \mathscr{A}$ for $0 < i \leqslant n$ and use C_i to define a self map

$$F_i\colon N_{\bar{B}} \to N_{\bar{B}},$$

$$F_i(B_0 \rightarrowtail \cdots \rightarrowtail B_q \rightarrowtail \bar{B}) = (B_0 + C_i \rightarrowtail \cdots \rightarrowtail B_q + C_i \rightarrowtail \bar{B}),$$

where

$$B_j + C_i = B_j \oplus C_i/\ker(B_j \oplus C_i \to \bar{B}).$$

This works because $B_q/B_0 \twoheadrightarrow B_q + C_i/B_0 + C_i$ and $\mathscr{A}$ is closed under quotient object. Clearly F_0 is the identity and F_n is constant, so we need homotopies from F_{i-1} to F_i. These are obtained in a standard way, by noting that a q simplex of $N_{\bar{B}} \times \Delta[1]$ consists of $B_0 \to \cdots \to B_q \to \bar{B}$ in $N_{\bar{B}}$ and $\alpha\colon [q] \to [1]$ in Δ. The homotopy from F_{i-1} to F_i sends the q simplex of $N_{\bar{B}} \times \Delta[1]$ to

$$B_0 + C_{i-1} \rightarrowtail \cdots \rightarrowtail B_t + C_{i-1} \rightarrowtail B_{t+1} + C_i \rightarrowtail \cdots B_q + C_i \rightarrowtail \bar{B},$$

where

$$0 = \alpha(0) = \cdots = \alpha(t),\ 1 = \alpha(t+1) = \cdots = \alpha(q).$$

This works because $\mathscr{A}$ is closed under sums and quotients and because $B_q + C_i/B_0 + C_{i-1}$ is a quotient of $B_q/B_0 \oplus C_i/C_{i-1}$.

5. The Localization Theorem

In this section, R is a ring and $S \subset R$ is a multiplicative set of central nonzero divisors. $\mathscr{P}_R$ denotes the exact category of finitely generated projective left R-modules, and $\mathscr{M}_R$ denotes the exact category of finitely generated R-modules. $\mathscr{M}$ is the full subcategory of $\mathscr{P}_{S^{-1}R}$ consisting of those objects isomorphic to $S^{-1}P$ for some $P \in \mathscr{P}_R$. This is also exact category in a natural way, since all exact sequences in $\mathscr{P}_{S^{-1}R}$ split.

Consider also $\mathscr{P}$, the full subcategory of $\mathscr{M}_R$ consisting of the objects P' of projective dimension $\leqslant 1$, with $S^{-1}P' \in \mathscr{M}$. Here $\mathscr{P}$ is closed under extension and is thus an exact category.

We have the localization functor $F\colon \mathscr{P} \to \mathscr{M}$ sending P' to $S^{-1}P'$ and we let $\mathscr{H} \subset \mathscr{P}$ be the full subcategory whose objects are those H such that $S^{-1}H \cong 0$. Note that $\mathscr{H}$ is clearly closed under extensions, so it inherits the structure of an exact category from $\mathscr{P}$. To use the S construction for K-theory, we point $\mathscr{M}$ by selecting one zero object 0. We assume for convenience that $S^{-1}H = 0$ for each $H \in \mathscr{H}$. The localization theorem is as follows.

THEOREM 5.1. *There is a fibration-sequence up-to-homotopy*

$$|iS.\mathscr{H}| \to |iS.\mathscr{P}| \to |iS.\mathscr{M}|$$

and thus a long exact sequence

$$\to K_{i+1}(S^{-1}R) \to K_i(\mathscr{H}) \to K_i(R) \to K_i(S^{-1}R) \to$$

This theorem follows from the cofinality theorem which identifies $\pi_{i+1}|iS.\mathscr{M}|$ with $K_i(S^{-1}R)$, the resolution theorem which identifies $\pi_{i+1}|iS.\mathscr{P}|$ with $K_i(R)$, and the following theorem.

First, we have from Theorem 1.8 a fibration up to homotopy

$$|s.\mathscr{H}| \to |s.\mathscr{P}| \to |s.S.(\mathscr{H} \to \mathscr{P})|$$

and the main part of localization theorem is as follows.

THEOREM 5.2. *The localization functor $F: \mathscr{P} \to \mathscr{M}$ induces a homotopy equivalence*

$$|s.S.(\mathscr{H} \to \mathscr{P})| \to |N.iS.\mathscr{M}|$$

of realizations of bisimplicial sets.

Proof. Localization obviously induces a map of bisimplicial sets

$$((m, n) \to s_m S_n(\mathscr{H} \to \mathscr{P})) \to ((m, n) \to s_m i_n(\mathscr{M})),$$

where i_n is the category with cofibrations in which an object is a chain of n isomorphisms in $\mathscr{M}$ and a cofibration is a commuting ladder of cofibrations.

By the standard trick of reversal of priorities, the domain and range can be rewritten and the map above replaced by the localization induced map

$$((m, n) \to s_n(S_m\mathscr{H} \to S_m\mathscr{P})) \to \{(m, n) \to N_n(iS_m(\mathscr{M}))\},$$

where N_n is the degree n part of the nerve of a category. Now notice that $s_n(S_m\mathscr{H} \to S_m\mathscr{P})$ is, by neglect of data, homotopy equivalent to the nerve of the category $m(S_m\mathscr{P}, S_m\mathscr{H})$ whose objects are those of $S_m\mathscr{P}$ and in which a monomorphism is a cofibration in $S_m\mathscr{P}$ such that the quotient object is in $S_m\mathscr{H}$. (This defines a category because of the 'pointwise' way of computing quotients in $S_m\mathscr{P}$ and because $\mathscr{H}$ is closed under extensions.)

Thus, by the realization lemma, to prove the theorem it suffices to show that for each $r \geqslant 0$ the localization induced functor

$$F_r: m(S_r\mathscr{P}, S_r\mathscr{H}) \to iS_r\mathscr{M}$$

realizes to a homotopy equivalence.

For this, it suffices to demonstrate that for each $M \in i(S_r\mathscr{M})$ the comma category F_r/M is contractible. But this is a consequence of the facts that each F_r/M is nonempty and filtering, which we prove below.

For $r = 0$, there is nothing to prove, and for $r = 1$ the argument is extractable from Grayson, [1] and it goes as follows. Here F_1/M has objects $P' \xrightarrow{f'} M$, where the arrow is an R module map which localizes to an isomorphism and arrows commuting triangles

$$\begin{tikzcd} P' \arrow[r, tail] \arrow[dr, "f'"'] & P'' \arrow[d, "f''"] \\ & M \end{tikzcd}$$

where the monomorphism has cokernel in $\mathscr{H}$.

By definition of $\mathscr{M}$, there is a projective module P such that $S^{-1}P = M$. Thus F_1/M is nonempty. Now, given two objects

$$P' \xrightarrow{f'} M \xleftarrow{f''} P''$$

of F_1/M we can find $s \in S$ and maps g' and g'' so that

$$\begin{tikzcd} P \arrow[r, "s"] \arrow[d, "g'"] & P \arrow[d] & P \arrow[l, "s"'] \arrow[d, "g''"] \\ P' \arrow[r, "f'"] & M & P'' \arrow[l, "f''"'] \end{tikzcd}$$

commutes. Since P is projective and s is a nonzero divisor, multiplication by s is an admissible monomorphism and $P \to M$ is injective. It follows that g' and g'' are injective, so we have exact sequences

$$0 \to P \xrightarrow{g'} P' \to T' \to 0,$$

$$0 \to P \xrightarrow{g''} P'' \to T'' \to 0.$$

Since P is projective and the projective dimensions of P' and P'' are less than or equal to 1, it follows that T', $T'' \in \mathscr{H} \subset \mathscr{P}$. Thus g', g'' are in $m(\mathscr{P}, \mathscr{H})$, and we have constructed an object

$$P \xrightarrow{s} P \xrightarrow{f} M$$

which maps to the two given objects

$$f' : P' \to M, \qquad f' : P'' \to M.$$

Now suppose that we have two arrows in F_1/M_n

$$h_1, h_2 : (P' \xrightarrow{f'} M) \rightrightarrows (P'' \xrightarrow{f''} M).$$

We find a third object and map g out of it such that $h_1 g = h_2 g$ in F_1/M. Starting with $P \xrightarrow{f} M$ as above, find s and

$$g : (P \xrightarrow{s} P \xrightarrow{f} M) \to (P' \xrightarrow{f'} M) \text{ in } \mathscr{H}.$$

Now $S^{-1}\ker(h_1 g - h_2 g) = 0$, together with the fact that P, being projective, has no S-torsion, implies that $h_1 g = h_2 g$, as desired.

For $r > 1$, we extend the arguments in the following manner. An object of F_r/M amounts to a map of diagrams of R modules

$$P' = \begin{array}{ccc} 0 \to P'_{1/0} & \rightarrowtail \cdots \rightarrowtail & P'_{r/0} \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail \cdots \rightarrowtail & P'_{r/1} \\ & & \vdots \\ & & 0 \to P'_{r/r-1} \\ & & \downarrow \\ & & 0 \end{array} \xrightarrow{f'} M = \begin{array}{ccc} 0 \to M_{1/0} & \rightarrowtail \cdots \rightarrowtail & M_{r/0} \\ \downarrow & & \downarrow \\ 0 & \rightarrowtail \cdots \rightarrowtail & M_{r/1} \\ & & \vdots \\ & & 0 \to M_{r/r-1} \\ & & \downarrow \\ & & 0 \end{array}$$

which localizes to an isomorphism. F_r/M is nonempty, since we may find projectives Q_i such that $S^{-1}Q_i \approx M_{i/i-1}$, $1 \leqslant i \leqslant r$. Then putting $P_{i/j} = \oplus_{j<k\leqslant i} Q_k$, choosing maps in the obvious way, and using the lifting property of projectives we obtain a diagram $P \xrightarrow{f} M$ in F_r/M.

Given two objects, $f'\colon P' \to M$ and $f''\colon P'' \to M$, we construct as above diagrams

$$\begin{array}{ccccc} Q_{i'} & \xrightarrow{s_i} & Q_i & \xleftarrow{s_i} & Q_i \\ \downarrow{\scriptstyle g_i} & & \downarrow & & \downarrow \\ P'_{i/i-1} & \xrightarrow{f_{i/i-1}} & M_{i/i-1} & \xleftarrow{f''_{i/i-1}} & P''_{i/i-1} \end{array}$$

for $1 \leqslant i \leqslant r$. Assembly of these diagrams in the manner above yields

$$(f'\colon P' \to M) \leftarrow (f \cdot \mathrm{s}\colon P \to M) \to (f''\colon P'' \to M)$$

a diagram in $m(S_r\mathscr{P}, S_r\mathscr{H})$ as desired.

Given two maps

$$h_1, h_2\colon (f'\colon P' \to M) \rightrightarrows (f''\colon P'' \to M),$$

we put together

$$g\colon (P \xrightarrow{t} P \xrightarrow{f} M) \to (f'\colon P' \to M)$$

such that $h_1 g = h_2 g$, using the argument above pointwise.

Throughout this paper we have been occupied essentially with the problem of proving that two inequivalent categories have the same K-theory. We close by mentioning a case where two categories are shown to have the same K-theory by showing they are, in fact, equivalent categories. The situation is the derivation of the localization-completion Mayer–Vietoris sequence. With notations as in the beginning of the section, one starts with the diagram

$$\begin{array}{ccc} R & \to & S^{-1}R \\ \downarrow & & \downarrow \\ \hat{R} & \to & S^{-1}\hat{R}_S \end{array}$$

where $\hat{R}_S$ is the S-adic completion of R. According to Karoubi [5] the extension of scalars induces a functor $\mathscr{H} \to \hat{\mathscr{H}}$ where $\hat{\mathscr{H}}$ is the category of S-torsion $\hat{R}_S$ modules of homological dimension $\leqslant 1$. Moreover, this functor is an equivalence of categories, so that one obtains a ladder diagram

$$\begin{array}{ccccccc} \to K_i(\mathscr{H}) & \to & K_i(R) & \to & K_i(S^{-1}R) & \to \\ \downarrow\cong & & \downarrow & & \downarrow & \\ \to K_i(\hat{\mathscr{H}}) & \to & K_i(\hat{R}_S) & \to & K_i(S^{-1}\hat{R}_S) & \to \end{array}$$

and then the Mayer–Vietoris sequence. So it is clear that more usual categorical considerations pop up in K-theory, too.

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