TWO-VECTOR BUNDLES DEFINE A FORM OF ELLIPTIC COHOMOLOGY

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Abstract. We prove that for well-behaved small rig categories $\mathcal{R}$ (also known as bimonoidal categories) the algebraic K-theory space, $K(H\mathcal{R})$, of the K-theory ring spectrum of $\mathcal{R}$ is equivalent to $\mathcal{K}(\mathcal{R}) \simeq \mathbb{Z} \times |BGL(\mathcal{R})|^+$, where $GL(\mathcal{R})$ is the monoidal category of weakly invertible matrices over $\mathcal{R}$.

To achieve this, we solve the long-standing problem of group completing within the context of rig categories. More precisely, we construct an additive group completion $\bar{\mathcal{R}}$ of $\mathcal{R}$ that retains the multiplicative structure, *i.e.*, that remains a rig category.

In particular, this proves the conjecture from [BDR] that $K(ku)$ is the K-theory of the 2-category of complex 2-vector spaces. Hence, the work of Christian Ausoni and the fourth author on $K(ku)$ [AR, A] shows that the theory of virtual 2-vector bundles as in [BDR, Theorem 4.10] qualifies as a form of elliptic cohomology.

1. Introduction and main result

In telescopic complexity 0, 1 and ∞ there are cohomology theories that possess a geometric definition: de Rham cohomology of manifolds is given in terms of differential forms, cohomology classes in real and complex K-theory are classes of virtual vector bundles, and complex cobordism has a geometric definition *per se*. In order to understand phenomena of intermediate telescopic complexity, it is desirable to have geometric interpretations for such cohomology theories as well.

In [BDR] it was conjectured that virtual 2-vector bundles provide a geometric interpretation of a cohomology theory of telescopic complexity 2 which qualifies as a form of elliptic cohomology. More precisely, it was conjectured that the algebraic K-theory of a commutative rig category $\mathcal{R}$ is equivalent to the algebraic K-theory of the ring spectrum associated with $\mathcal{R}$. The case of virtual 2-vector bundles arises when $\mathcal{R}$ is the category of finite dimensional complex vector spaces, with $\oplus$ and $\otimes_{\mathbf{C}}$ as sum and multiplication. This, together with the analysis of the K-theory of complex topological K-theory due to Ausoni and the fourth author, and the (now proven) Quillen–Lichtenbaum conjecture for the integers, gives the desired relation to elliptic cohomology.

In this paper we prove the conjecture from [BDR].

Let $\mathcal{R}$ be a rig category (also known as a bimonoidal category), *i.e.*, a category with two operations $\oplus$ and $\otimes$ satisfying the axioms of a rig (ring without negative elements) up to coherent natural isomorphisms, see Definitions 2.1 and 2.5 below for details. In analogy with Quillen's definition of the algebraic K-theory space $K(A) = \Omega B(\coprod_n BGL_n(A))$ of a ring A, the algebraic K-theory of $\mathcal{R}$ was defined in [BDR] as $\mathcal{K}(\mathcal{R}) = \Omega B(\coprod_n |BGL_n(\mathcal{R})|)$ where B and GL_n are versions of the bar construction and the general linear group appropriate for rig categories.

On the other hand, forgetting the multiplicative structure, $\mathcal{R}$ has an underlying symmetric monoidal category, and so it makes sense to speak about its K-theory spectrum $H\mathcal{R}$ with respect to $\oplus$. The K-theory spectrum construction $H\mathcal{R}$ is a direct extension of the usual Eilenberg–Mac Lane construction, and can, since $\mathcal{R}$ is a rig category, be endowed with the structure of a strict ring spectrum, for instance through the model given by Elmendorf and Mandell in [EM]. Hence, we may speak about its algebraic K-theory space $K(H\mathcal{R})$. We prove that, under certain mild restrictions on $\mathcal{R}$, there is an equivalence

$$\mathcal{K}(\mathcal{R}) \simeq K(H\mathcal{R})\,.$$

In the special situation where $\mathcal{R}$ is a ring (*i.e.*, $\mathcal{R}$ is discrete as a category and has negative elements), this is the standard assertion that the K-theory of a ring is equivalent to the K-theory of its associated

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Eilenberg-Mac Lane spectrum. The key difficulty in establishing the equivalence above lies in proving that the lack of negative elements makes no difference for algebraic K-theory, even for rig categories.

More precisely, we prove the following result:

Theorem 1.1. *Let $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, \otimes, 1_{\mathcal{R}})$ be a small topological rig category satisfying the following conditions:*

1. *All morphisms in $\mathcal{R}$ are isomorphisms.*
2. *For every object $X \in \mathcal{R}$ the translation functor $X \oplus (-)$ is faithful.*

Then the algebraic K-theory space of $\mathcal{R}$ as a rig category,

$$\mathcal{K}(\mathcal{R}) = \Omega B\Big(\coprod_{n\geqslant 0} |BGL_n(\mathcal{R})|\Big) \simeq \mathbb{Z} \times |BGL(\mathcal{R})|^+ ,$$

is weakly equivalent to the algebraic K-theory space of the strict ring spectrum associated to $\mathcal{R}$,

$$K(H\mathcal{R}) = \Omega B\Big(\coprod_{n\geqslant 0} BGL_n(H\mathcal{R})\Big) \simeq \mathbb{Z} \times BGL(H\mathcal{R})^+ .$$

Addendum 1.2. *We will prove that $|BGL(\mathcal{R})|$ and $BGL(H\mathcal{R})$ are weakly equivalent before applying the plus construction.*

The conditions (1) and (2) on $\mathcal{R}$ are not restrictive for the applications we have in mind, and are associated with the fact that we at certain points have chosen to work with variants of the Grayson–Quillen model for K-theory. Probably, the restrictions can be removed if one uses another technological platform, such as Jardine's model [Ja] or a variant of Segal's construction we describe in the appendix. A thorough discussion of such generalizations would lengthen the proofs, and so we refrain from exploring these questions further until good applications demand this level of generality.

As stated, we permit $\mathcal{R}$ to be a topological rig category. To ensure that condition (2) is appropriate in this enriched context, we assume that we apply the singular functor to get a simplicial rig category, before entering it into Grayson's machine.

Among those rig categories that satisfy the requirements of Theorem 1.1 are the following 'standard' ones, usually considered in the context of K-theory constructions.

- If $\mathcal{R}$ is the discrete category (having only identity morphisms) with objects the elements of a ring with unit, R, then $H\mathcal{R}$ is the Eilenberg-Mac Lane spectrum HR.
- The sphere spectrum S is the algebraic K-theory spectrum of the small rig category of finite sets $\mathcal{E}$. The objects of $\mathcal{E}$ are the finite sets $\mathbf{n} = \{1, \ldots, n\}$ with $n \geqslant 0$. Here the convention is that 0 is the empty set. Morphisms from $\mathbf{n}$ to $\mathbf{m}$ are only non-trivial for $n = m$ and in this case they consist of the symmetric group on n letters. The algebraic K-theory of S is equivalent to Waldhausen's A-theory of a point $A(*)$ [W], and so gives information about diffeomorphisms of high dimensional disks. Thus we obtain that
$$A(*) \simeq K(S) \simeq \mathcal{K}(\mathcal{E}) \simeq \mathbb{Z} \times |BGL(\mathcal{E})|^+ .$$
- For a ring A we consider the following small rig category of finitely generated free A-modules, $\mathcal{F}(A)$. Objects of $\mathcal{F}(A)$ are the finitely generated free A-modules A^n for $n \geqslant 0$. The set of morphisms from A^n to A^m is empty unless $n = m$, and the morphisms from A^n to itself are the A-module automorphisms of A^n, *i.e.*, $GL_n(A)$. Our result allows us to identify the two-fold iterated algebraic K-theory of A with $\mathbb{Z} \times |BGL(\mathcal{F}(A))|^+$.
- The case that started the project is the category of 2-vector spaces of Kapranov and Voevodsky [KV], viewed as modules over the rig category $\mathcal{V}$ of complex (Hermitian) vector spaces. Here $\mathcal{V}$ has objects $\mathbb{C}^n$ for $n \geqslant 0$, and the automorphism space of $\mathbb{C}^n$ is the unitary group $U(n)$. This identifies $K(H\mathcal{V}) = K(ku)$ with $\mathbb{Z} \times |BGL(\mathcal{V})|^+$, which was called the K-theory of the 2-category of complex 2-vector spaces in [BDR]. Ausoni's calculations [A] show that $K(ku_p)$ has telescopic complexity 2 for every prime $\geqslant 5$, and thus qualifies as a form of elliptic cohomology.
- Replacing the complex numbers by the reals yields an identification of $K(ko)$ with the K-theory of the 2-category of real 2-vector spaces.
- Considering other subgroups of $GL_n(\mathbb{C})$ or $GL_n(\mathbb{R})$ as morphisms on a category with objects $\mathbf{n} = \{1, \ldots, n\}$ with $n \geqslant 0$ gives a large variety of K-theory spectra that are in the range of our result. For a sample of such species have a look at [M2, pp. 161–167].

Here is an outline of the proof of Theorem 1.1. We want to replace $\mathcal{R}$ with a group complete model (a ring category). The standard approaches, for instance Grayson–Quillen's $(-\mathcal{R})\mathcal{R}$ [G1], yield models that are symmetric monoidal categories with respect to an additive structure, but which have no meaningful multiplicative structure [Th2]. We will instead use an iteration of the Grayson–Quillen model to obtain a cubical model, $(-\mathcal{R})^\bullet\mathcal{R}$, which is a *graded* bimonoidal category, in a sense we make precise. A variant of Thomason's homotopy colimit turns this graded bimonoidal category into a group completion of $\mathcal{R}$, $\bar{\mathcal{R}}$, that still carries a bimonoidal structure. The full construction of $\bar{\mathcal{R}}$ takes up sections 2 to 6.

This model comes with a natural transformation $\mathcal{R} \to \bar{\mathcal{R}}$ of rig categories, which allows us to compare the bar construction $BGL_n(\mathcal{R})$ with $BGL_n(\bar{\mathcal{R}})$ and to prove that they become equivalent in the limit as n goes to infinity. We do this by proving that the homotopy fibre $B(*, GL(\mathcal{R}), GL(\bar{\mathcal{R}}))$ of the map from $BGL(\mathcal{R})$ to $BGL(\bar{\mathcal{R}})$ is contractible. Here we only need the $GL(\mathcal{R})$-module structure of $GL(\bar{\mathcal{R}})$, which allows us to replace $\bar{\mathcal{R}}$ by a group completion that is easier to handle. We compare weakly invertible matrices over $\bar{\mathcal{R}}$ with those over $H\bar{\mathcal{R}}$, and obtain that $|GL_n(\bar{\mathcal{R}})| \simeq GL_n(H\bar{\mathcal{R}})$. With these ingredients at hand we can quite easily show that $|BGL(\mathcal{R})|$ is equivalent to $BGL(H\mathcal{R})$, which yields the comparison of $\mathcal{K}(\mathcal{R})$ with $K(H\mathcal{R})$.

The structure of the paper is as follows. We discuss graded versions of bipermutative and strictly bimonoidal categories and their morphisms in section 2. In section 3 we identify an iterated version of the Grayson–Quillen model as a bipermutative (resp. strictly bimonoidal) category that is graded over the category I of finite sets and injective functions.

Thomason's homotopy colimit of symmetric monoidal categories is defined in a non-unital (or zeroless) setting. We extend this to the unital setting by constructing a derived version of it in section 4, and in section 5 we show that the homotopy colimit of a graded bipermutative (resp. strictly bimonoidal) category is almost a bipermutative (resp. strictly bimonoidal) category — it only lacks a zero. Section 6 describes how the results obtained so far combine to lead to a multiplicative group completion of (symmetric) bimonoidal categories.

In section 7 we discuss the monoidal category $GL_n(\mathcal{R})$ of (weakly) invertible matrices over a strictly bimonoidal category $\mathcal{R}$, together with similar categories of matrices over various models for the group completion of $\mathcal{R}$, as well as the module structure over $GL_n(\mathcal{R})$ of these categories. Section 8 recalls the definition of the bar construction of monoidal categories as in [BDR] and introduces a version with coefficients in a module. We construct a contraction of this one-sided bar construction in the case relevant to our proof (section 9) and show in section 10 that weakly invertible matrices do not distinguish between a ring category, $\bar{\mathcal{R}}$, and its associated K-theory, $H\bar{\mathcal{R}}$. Finally, in section 11 we fit the pieces together and prove the main theorem. In the appendix we sketch an alternative construction of a multiplicative group completion device based on Segal's approach to K-theory.

In contrast to $K(H\mathcal{R})$, which is built in a two-stage process, the K-theory of the (strictly) bimonoidal category $\mathcal{R}$ is built using both monoidal structures at once, so in this sense $\mathcal{K}(\mathcal{R})$ is a model that is easier to understand and handle than $K(H\mathcal{R})$.

A piece of notation: if $\mathcal{C}$ is any small category, then the expression $X \in \mathcal{C}$ is short for "X is an object of $\mathcal{C}$" and likewise for morphisms and diagrams.

2. Bipermutative and rig categories

For the definition of a permutative category see for instance [EM, 3.1] or [M1, §4]; compare also [ML, XI.1]. Since our permutative categories are typically going to be the underlying additive symmetric monoidal categories of categories with some further structure, we call the neutral element "zero" or simply 0.

We consider lax and strict symmetric monoidal functors F between two permutative categories $(\mathcal{M}, \oplus, 0_\mathcal{M}, \tau_\mathcal{M})$ and $(\mathcal{N}, \oplus, 0_\mathcal{N}, \tau_\mathcal{N})$. A *lax symmetric monoidal functor* is a functor F in the sense of [ML, XI.2], *i.e.*, there are morphisms

$$f_{a,a'} \colon F(a) \oplus F(a') \to F(a \oplus a')$$

for all objects $a, a' \in \mathcal{M}$ that are natural in a and a', there is a morphism

$$n \colon 0_\mathcal{N} \to F(0_\mathcal{M}),$$

and these structure maps fulfill coherence conditions which are spelled out in [ML, XI.2]; in particular

$$\begin{tikzcd} F(a) \oplus F(a') \arrow[r, "f_{a,a'}"] \arrow[d, "\tau_\mathcal{N}(F(a),F(a'))"'] & F(a \oplus a') \arrow[d, "F(\tau_\mathcal{M}(a,a'))"] \\ F(a') \oplus F(a) \arrow[r, "f_{a',a}"] & F(a' \oplus a) \end{tikzcd}$$

commutes for all $a, a' \in \mathcal{M}$. Let Perm be the category of small permutative categories and lax symmetric monoidal functors.

A *strict symmetric monoidal functor* has furthermore to satisfy that the morphisms $f_{a,a'}$ and n are identities, so that

$$F(a \oplus a') = F(a) \oplus F(a') \quad \text{and} \quad F(0_{\mathcal{M}}) = 0_{\mathcal{N}}$$

[ML, XI.2]. We denote the category of small permutative categories and strict symmetric monoidal functors by Perm(Strict).

Since any symmetric monoidal category is naturally equivalent to a permutative category, we lose no generality by only considering permutative categories. Note that we consider the unital situation, unless explicitly stated otherwise.

Roughly speaking, a rig category $\mathcal{R}$ consists of a symmetric monoidal category $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, \tau_{\mathcal{R}})$ together with a functor $\mathcal{R} \times \mathcal{R} \to \mathcal{R}$ called "multiplication" and denoted by $\otimes$ or $\cdot$. Note that the multiplication is not a map of monoidal categories. The multiplication has a unit $1_{\mathcal{R}} \in \mathcal{R}$, multiplying by $0_{\mathcal{R}}$ is the zero map, multiplying with $1_{\mathcal{R}}$ is the identity map, and the multiplication is (left and right) distributive over $\oplus$ up to appropriate coherencies. If we pose the additional requirement that our rig categories are commutative (up to coherent isomorphisms), then this coincides with what is often called a symmetric bimonoidal category. Laplaza spelled out the coherence conditions in [L, pp. 31–35].

According to [M2, VI, Proposition 3.5] any commutative rig category is equivalent in the appropriate sense to a "bipermutative category", and a similar rigidification result holds for bimonoidal categories. Our main theorem is equivalent to the statement with "commutative rig category" (resp. "rig category") replaced by "bipermutative category" (resp. "strictly bimonoidal category") everywhere, and this is what we prove. The reader will find the axioms for a bipermutative category in Definition 2.1 below as the special case of a "0-graded bipermutative category" where 0 is the one-point category.

Otherwise one may for instance consult [EM, 3.6]. Note that we demand strict left distributivity. One word of warning: in [M2, VI, Definition 3.3], May's right distributivity law is precisely what we (and [EM]) call the left distributivity law.

We will focus on the bipermutative case in the course of this paper and indicate what has to be adjusted in the strictly bimonoidal case.

If $\mathcal{R}$ is a small rig category such that $\pi_0(\mathcal{R})$ is a ring (has additive inverses), then we call $\mathcal{R}$ a *ring category*. Elmendorf and Mandell's ring categories are not ring categories in our sense, but non-symmetric rig categories. In the course of this paper we have to resolve rig categories simplicially. If $\mathcal{R}$ is a small simplicial rig category such that $\pi_0(|\mathcal{R}|)$ is a ring, then we call $\mathcal{R}$ a *simplicial ring category* (even though it is not a simplicial object in the category of ring categories).

If $\mathcal{R}$ is a strictly bimonoidal category, a left $\mathcal{R}$-module is a permutative category $\mathcal{M}$ together with a multiplication $\mathcal{R} \times \mathcal{M} \to \mathcal{M}$ that is strictly associative and coherently distributive as spelled out in [EM, 9.1.1].

2.1. ***J*-graded bipermutative categories and strictly bimonoidal categories.** We want to additively group complete a rig category $\mathcal{R}$, in such a way that the outcome still possesses a multiplicative structure, *i.e.*, so as to produce a ring category. There are constructions for additive group completions of $\mathcal{R}$, *e.g.* the Grayson–Quillen construction $G(\mathcal{R}) = (-\mathcal{R})\mathcal{R}$, but they are known to have bad multiplicative behavior [Th2]. If we perform this group completion more than once, then there is no further change up to homotopy equivalence. So we might alternatively consider the homotopy colimit of the iterated group completions $G^{(n)}(\mathcal{R}) = (-\mathcal{R})^n\mathcal{R}$, and still just have group completed additively. However, such a naïve homotopy colimit construction will not carry a decent multiplicative structure.

Precisely what is needed to ensure that the homotopy colimit retains multiplicative structure is that the sequential diagram $n \mapsto G^{(n)}(\mathcal{R})$ extends to a diagram indexed over the category I of finite sets and injective functions, *i.e.*, that the iterated G-construction produces an I-graded bipermutative category or I-graded strictly bimonoidal category, in the sense soon to be defined.

In order to avoid setting up a huge machine for graded rig categories, we always assume that the input to our machinery has been transformed to an equivalent bipermutative or strictly bimonoidal category before we start.

Definition 2.1. Let $(J, +, 0, c_J)$ be a small permutative category. A *J-graded bipermutative category* is a functor X from J to the category Perm(Strict) of small permutative categories and strict symmetric monoidal functors, together with the following data and subject to the following conditions, where we denote the permutative structure of $X(i)$ by $(X(i), \oplus, 0_i, \gamma_\oplus)$.

(1) A functor

$$\otimes \colon X(i) \times X(j) \to X(i+j),$$

i.e., for all $(A, B) \in X(i) \times X(j)$ there is an object $A \otimes B$ in $X(i+j)$ and for any pair of morphisms, $f \colon A \to A'$ and $g \colon B \to B'$, there is a morphism $f \otimes g \colon A \otimes B \to A' \otimes B'$ satisfying the usual requirements.

We require that for any pair of morphisms in J, $\varphi \colon i \to k$ and $\psi \colon j \to \ell$, the following diagram commutes:

$$\begin{tikzcd} X(i) \times X(j) \arrow[r, "\otimes"] \arrow[d, "X(\varphi)\times X(\psi)"'] & X(i+j) \arrow[d, "X(\varphi+\psi)"] \\ X(k) \times X(\ell) \arrow[r, "\otimes"] & X(k+\ell)\,. \end{tikzcd}$$

(2) An object 1 of $X(0)$ such that the composition of the inclusion $\{1\} \times X(j) \to X(0) \times X(j)$ followed by $\otimes \colon X(0) \times X(j) \to X(0+j) = X(j)$ equals the projection isomorphism $\{1\} \times X(j) \cong X(j)$, and likewise for the map from $X(j) \times \{1\}$.

(3) Isomorphisms

$$\gamma_\otimes = \gamma_\otimes^{A,B} \colon A \otimes B \longrightarrow X(c_J^{j,i})(B \otimes A)$$

in $X(i+j)$, for all $A \in X(i)$ and $B \in X(j)$, such that

$$\begin{tikzcd} A \otimes B \arrow[r, "\gamma_\otimes^{A,B}"] \arrow[d, "f\otimes g"'] & X(c_J^{j,i})(B \otimes A) \arrow[d, "X(c_J^{j,i})(g\otimes f)"] \\ A' \otimes B' \arrow[r, "\gamma_\otimes^{A',B'}"] & X(c_J^{j,i})(B' \otimes A') \end{tikzcd}$$

commutes and $X(c_J^{j,i})(\gamma_\otimes^{B,A}) \circ \gamma_\otimes^{A,B}$ is equal to the identity on $A \otimes B$:

$$\begin{tikzcd} A \otimes B \arrow[rr, "\mathrm{id}_{A\otimes B}"] \arrow[dr, "\gamma_\otimes^{A,B}"'] & & X(c_J^{j,i})X(c_J^{i,j})(A \otimes B) \\ & X(c_J^{j,i})(B \otimes A)\,. \arrow[ur, "X(c_J^{j,i})(\gamma_\otimes^{B,A})"'] & \end{tikzcd}$$

For φ and ψ as above we require that

$$X(\varphi+\psi)(\gamma_\otimes^{A,B}) = \gamma_\otimes^{X(\varphi)(A),X(\psi)(B)}.$$

In other words, the multiplicative twist $\gamma_\otimes$ is natural in i and j.

In addition, $\gamma_\otimes^{A,1}$ and $\gamma_\otimes^{1,A}$ agree with the identity morphism on A for all objects A.

(4) The composition $\otimes$ is associative and the diagram

$$\begin{tikzcd} A \otimes B \otimes C \arrow[r, "\gamma_\otimes^{A\otimes B,C}"] \arrow[d, "\mathrm{id}\otimes\gamma^{B,C}"'] & X(c_J^{k,i+j})(C \otimes A \otimes B) \arrow[d, "X(c_J^{k,i+j})(\gamma^{C,A}\otimes\mathrm{id})"] \\ X(\mathrm{id}+c_J^{k,j})(A \otimes C \otimes B) \arrow[r, equal] & X(c_J^{k,i+j})X(c_J^{i,k}+\mathrm{id})(A \otimes C \otimes B) \end{tikzcd}$$

commutes for all objects (compare [ML, p. 254, (7a)]).

(5) For each $i \in J$ the zero object 0_i annihilates everything multiplicatively, *i.e.*, $\{0_i\} \times X(j) \to X(i) \times X(j) \to X(i+j)$ is the constant map to 0_{i+j}. Here the first map is the inclusion and the second is $\otimes$.

(6) Left distributivity holds strictly, *i.e.*,

$$
\begin{tikzcd}
(X(i) \times X(i)) \times X(j) \arrow[r, "\oplus \times \mathrm{id}"] \arrow[d, "\Delta"] & X(i) \times X(j) \arrow[dd, "\otimes"] \\
(X(i) \times X(j)) \times (X(i) \times X(j)) \arrow[d, "\otimes \times \otimes"] & \\
X(i+j) \times X(i+j) \arrow[r, "\oplus"] & X(i+j)
\end{tikzcd}
$$

commutes, where $\oplus$ is the monoidal structure and Δ is the diagonal on $X(j)$ combined with the identity on $X(i) \times X(i)$ followed by a twist. We denote these instances of identities by d_ℓ.

(7) The right distributivity transformation, d_r, is given in terms of d_ℓ and $\gamma_\otimes$ as

$$d_r = \gamma_\otimes \circ d_\ell \circ (\gamma_\otimes \oplus \gamma_\otimes)\,.$$

(Here, we suppress the twist $X(c_J)$ in the notation.) Thus, for all i and j and $A \in X(i)$, $B, C \in X(j)$ the following diagram defines d_r:

$$
\begin{tikzcd}
A \otimes B \oplus A \otimes C \arrow[r, "\gamma_\otimes^{A,B} \oplus \gamma_\otimes^{A,C}"] \arrow[d, "d_r"'] & X(c_J^{j,i})(B \otimes A) \oplus X(c_J^{j,i})(C \otimes A) \arrow[d, equal] \\
A \otimes (B \oplus C) \arrow[d, equal] & X(c_J^{j,i})(B \otimes A \oplus C \otimes A) \arrow[d, "X(c_J^{j,i})(d_\ell)=X(c_J^{j,i})(\mathrm{id})=\mathrm{id}"] \\
X(c_J^{j,i})X(c_J^{i,j})(A \otimes (B \oplus C)) & X(c_J^{j,i})((B \oplus C) \otimes A)\,. \arrow[l, "X(c_J^{j,i})(\gamma_\otimes^{B\oplus C,A})"']
\end{tikzcd}
$$

(8) The diagram

$$
\begin{tikzcd}
(A \otimes B) \oplus (A \otimes B') \arrow[r, "d_r"] \arrow[d, "\gamma_\oplus"'] & A \otimes (B \oplus B') \arrow[d, "\mathrm{id} \otimes \gamma_\oplus"] \\
(A \otimes B') \oplus (A \otimes B) \arrow[r, "d_r"] & A \otimes (B' \oplus B)
\end{tikzcd}
$$

commutes for all objects. The analogous diagram for d_ℓ also commutes.

Due to the definition of d_r in terms of $\gamma_\otimes$ and the identity d_ℓ, it suffices to demand that $\gamma_\oplus \circ (\gamma_\otimes \oplus \gamma_\otimes) = (\gamma_\otimes \oplus \gamma_\otimes) \circ \gamma_\oplus$ and $(\gamma_\oplus \otimes \mathrm{id}) \circ \gamma_\otimes = \gamma_\otimes \circ (\mathrm{id} \otimes \gamma_\oplus)$.

(9) The distributivity transformations are associative, *i.e.*, the diagram

$$
\begin{tikzcd}
(A \otimes B \otimes C) \oplus (A \otimes B \otimes C') \arrow[d, "d_r"'] \arrow[dr, "d_r"] & \\
A \otimes ((B \otimes C) \oplus (B \otimes C')) \arrow[r, "\mathrm{id} \otimes d_r"] & A \otimes B \otimes (C \oplus C')
\end{tikzcd}
$$

commutes for all objects.

(10) The following pentagon diagram commutes

$$\begin{tikzcd}[column sep=small]
 & (A \otimes (B \oplus B')) \oplus (A' \otimes (B \oplus B')) \arrow[dr, "d_\ell"] & \\
(A \otimes B) \oplus (A \otimes B') \oplus (A' \otimes B) \oplus (A' \otimes B') \arrow[ur, "d_r \oplus d_r"] \arrow[d, "\mathrm{id} \oplus \gamma_\oplus \oplus \mathrm{id}"'] & & (A \oplus A') \otimes (B \oplus B') \\
(A \otimes B) \oplus (A' \otimes B) \oplus (A \otimes B') \oplus (A' \otimes B') \arrow[dr, "d_\ell \oplus d_\ell"'] & & \\
 & ((A \oplus A') \otimes B) \oplus ((A \oplus A') \otimes B') \arrow[uur, "d_r"'] &
\end{tikzcd}$$

for all objects $A, A' \in X(i)$ and $B, B' \in X(j)$.

Remark 2.2. Notice that in Definition 2.1, the condition (1) only says that we have a natural transformation

$$\otimes \colon X \times X \Rightarrow X \circ +$$

of functors $J \times J \to Cat$, and condition (3) demands a modification

$$\begin{tikzcd}
X \times X \arrow[r, Rightarrow, "c_{Cat}"] \arrow[d, Rightarrow, "\otimes"] \arrow[dr, phantom, "\overset{\gamma_\otimes}{\Rightarrow}"] & (X \times X) \circ tw_J \arrow[d, Rightarrow, "\otimes"] \\
X \circ + & X \circ + \circ tw_J \arrow[l, Rightarrow, "X(c_J)"]
\end{tikzcd}$$

where c_{Cat} is the symmetric structure on Cat (with respect to product) and tw_J is the interchange of factors on $J \times J$.

Remark 2.3. Note that for an $A \in X(j)$ and σ an automorphism of j in J, σ will not give rise to an automorphism of A in $X(j)$ in general.

In the following we will denote a J-graded bipermutative category $X \colon J \to \mathrm{Perm(Strict)}$ by $X^\bullet$ if the category J is clear from the context. For the one-point category $J = 0$, a J-graded bipermutative category is the same as a bipermutative category. Thus every J-graded bipermutative category $X^\bullet$ comes with a bipermutative category $X(0)$, and $X^\bullet$ can be viewed as a functor $J \to X(0)$-modules.

Example 2.4. We consider the small bipermutative category of finite sets, whose objects are the finite sets of the form $\mathbf{n} = \{1, \dots, n\}$ for $n \geqslant 0$ and $\mathbf{0} = \varnothing$, and whose morphisms are functions.

Disjoint union of sets gives rise to a permutative structure

$$\mathbf{n} \oplus \mathbf{m} := \mathbf{n} \sqcup \mathbf{m}$$

and we identify $\mathbf{n} \sqcup \mathbf{m}$ with $\mathbf{n} + \mathbf{m}$. For functions $f \colon \mathbf{n} \to \mathbf{n}'$ and $g \colon \mathbf{m} \to \mathbf{m}'$ we define $f \oplus g$ as the map on the disjoint union $f \sqcup g$ which we will denote by $f + g$. The additive twist $c_\oplus$ is given by the shuffle maps

$$\chi(n, m) \colon \mathbf{n} + \mathbf{m} \longrightarrow \mathbf{m} + \mathbf{n}$$

with

$$\chi(n, m)(i) = \begin{cases} m + i & \text{for } \ i \leqslant n \\ i - n & \text{for } \ i > n. \end{cases}$$

Multiplication of sets is defined via

$$\mathbf{n} \otimes \mathbf{m} := \mathbf{nm}\,.$$

If we identify the element $(i-1) \cdot m + j$ in $\mathbf{nm}$ with the pair (i, j) with $i \in \mathbf{n}$ and $j \in \mathbf{m}$, then the function $f \otimes g$ is given as

$$(i, j) \mapsto (f(i), g(j)),$$

and the multiplicative twist

$$c_\otimes \colon \mathbf{n} \otimes \mathbf{m} \longrightarrow \mathbf{m} \otimes \mathbf{n}$$

sends (i,j) to (j,i). The empty set is a strict unit for the addition and the set $\mathbf{1}$ is a strict unit for the multiplication. Left distributivity is the identity and the right distributivity law is given by the resulting permutation

$$d_r = c_\otimes \circ d_\ell \circ (c_\otimes \oplus c_\otimes)\,.$$

For later reference we denote d_r by ξ.

Considering only the subcategory of bijections, instead of arbitrary functions, results in the bipermutative category of finite sets $\mathcal{E}$ that we discussed in the introduction. Later, we will consider the bipermutative category of finite sets and surjective functions.

Definition 2.5. A *J-graded strictly bimonoidal category* is a functor $X\colon J \to \mathrm{Perm(Strict)}$ to the category of permutative categories and strict symmetric monoidal functors, satisfying the conditions of Definition 2.1, except that we do not require the existence of the natural isomorphism $\gamma_\otimes$, and the right distributivity transformation d_r is not given in terms of d_ℓ. Axiom (7) of Definition 2.1 has to be replaced by the following condition.

(7') The diagram

$$\begin{tikzcd}
A \otimes B \otimes C \oplus A \otimes B' \otimes C \arrow[r, "d_\ell"] \arrow[d, "d_r"'] & (A \otimes B \oplus A \otimes B') \otimes C \arrow[d, "d_r \otimes \mathrm{id}"] \\
A \otimes (B \otimes C \oplus B' \otimes C) \arrow[r, "\mathrm{id} \otimes d_\ell"] & A \otimes (B \oplus B') \otimes C
\end{tikzcd}$$

commutes for all objects.

In the J-graded bipermutative case condition (7') follows from the other axioms.

Definition 2.6. A *lax morphism of bipermutative categories*, $g\colon X \to Y$, is a lax symmetric monoidal functor from $(X, \oplus, 0_X, c_\oplus)$ to $(Y, \oplus, 0_Y, c_\oplus)$ together with a structure of a lax symmetric monoidal functor from $(X, \otimes, 1_X, c_\otimes)$ to $(Y, \otimes, 1_Y, c_\otimes)$, that respects the distributivity laws.

We therefore have a binatural transformation $\eta_\oplus$ from $(- \oplus -) \circ (g,g)$ to $g \circ (- \oplus -)$, *i.e.*,

$$\eta_\oplus = \eta_\oplus(A,B)\colon g(A) \oplus g(B) \to g(A \oplus B) \text{ for } A, B \in X,$$

and a corresponding binatural transformation from $(- \otimes -) \circ (g,g)$ to $g \circ (- \otimes -)$

$$\eta_\otimes = \eta_\otimes(A,B)\colon g(A) \otimes g(B) \to g(A \otimes B) \text{ for } A, B \in X,$$

and we require that these interact with $c_\oplus$ and $c_\otimes$ and that the following diagram (and the analogous one for d_r) commutes

$$\begin{tikzcd}
g(A) \otimes g(B) \oplus g(A') \otimes g(B) \arrow[r, equal, "d_\ell = \mathrm{id}"] \arrow[d, "\eta_\otimes \oplus \eta_\otimes"'] & (g(A) \oplus g(A')) \otimes g(B) \arrow[r, "\eta_\oplus \otimes \mathrm{id}"] & g(A \oplus A') \otimes g(B) \arrow[d, "\eta_\otimes"] \\
g(A \otimes B) \oplus g(A' \otimes B) \arrow[r, "\eta_\oplus"'] & g(A \otimes B \oplus A' \otimes B) \arrow[r, equal, "g(d_\ell) = \mathrm{id}"'] & g((A \oplus A') \otimes B)
\end{tikzcd}$$

for all objects $A, A', B \in X$, *i.e.*, we have

$$\eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\eta_\oplus \otimes \mathrm{id})$$

and

$$g(\gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)) \circ \eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\mathrm{id} \otimes \eta_\oplus) \circ \gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)\,.$$

For a *lax morphism of strictly bimonoidal categories* we demand that g is lax monoidal with respect to $\otimes$, lax symmetric monoidal with respect to $\oplus$ and that

$$g(d_r) \circ \eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\mathrm{id} \otimes \eta_\oplus) \circ d_r \quad \text{and} \quad g(d_\ell) \circ \eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\eta_\oplus \otimes \mathrm{id}) \circ d_\ell.$$

Definition 2.7. A *lax morphism of J-graded bipermutative categories*, $g\colon X^\bullet \to Y^\bullet$, consists of a natural transformation g from $X^\bullet$ to $Y^\bullet$ that is compatible with the bifunctors $\oplus, \otimes$ and the units. In detail, we require that there are transformations $\eta_\oplus$ from $(- \oplus -) \circ (g \times g)$ to $g \circ (- \oplus -)$

$$\begin{tikzcd}
X(i) \times X(j) \arrow[r, "\oplus"] \arrow[d, "g(i) \times g(j)"'] & X(i+j) \arrow[d, "g(i+j)"] \\
Y(i) \times Y(j) \arrow[r, "\oplus"'] \arrow[ur, Rightarrow, "\eta_\oplus^{i,j}"] & Y(i+j),
\end{tikzcd}$$

and $\eta_\otimes$ from $(- \otimes -) \circ (g, g)$ to $g \circ (- \otimes -)$. These commute with $\gamma_\oplus$ and $\gamma_\otimes$ and they are binatural with respect to i and j and morphisms in $X(i), X(j)$.

The functor g respects the distributivity constraints in that it fulfills

$$\eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\eta_\oplus \otimes \mathrm{id})$$

and

$$g(d_r) \circ \eta_\oplus \circ (\eta_\otimes \oplus \eta_\otimes) = \eta_\otimes \circ (\mathrm{id} \otimes \eta_\oplus) \circ d_r \,.$$

For a *lax morphism of J-graded strictly bimonoidal categories* there is no requirement on g concerning the multiplicative twist $\gamma_\otimes$.

3. The Grayson–Quillen model as a graded bipermutative category

Let I be the skeleton of the category of finite sets and injective maps, *i.e.*, objects are finite sets $\mathbf{n} = \{1, \dots, n\}$ for $n \geqslant 0$ with the convention that $\mathbf{0} = \varnothing$ and morphisms are injective functions of finite sets. We define the sum of two objects $\mathbf{n}$ and $\mathbf{m}$ as $\mathbf{n} + \mathbf{m}$ and use the twist maps $\chi(n, m)$ defined in Example 2.4. Then $(I, +, \mathbf{0}, \chi)$ is a permutative category.

We remodel the Grayson–Quillen model [G1] for the group completion of a permutative category to suit our multiplicative needs. Our cubical model will avoid the problems with the multiplicative structure. In the Grayson–Quillen model an element (A, B) should be thought of as $A - B$, because on the level of π_0 an element $(A, 0)$ is inverse to $(0, A)$. Then the naïve guess for how to multiply elements is dictated by the rule that $(A - B)(C - D) = (AC + BD) - (AD + BC)$. This, however, does not lead to a decent multiplicative structure, because this product is not functorial. We will choose models where a product of two elements in the Grayson–Quillen model will be a 2-dimensional cube, and thus elements in a product are spread out in order to avoid the "phoniness" of the multiplication [Th2]. Since our permutative structures are to be thought of as additive, we use expressions like $(-\mathcal{M})\mathcal{M}$ where Grayson–Quillen would have written $\mathcal{M}^{-1}\mathcal{M}$.

Definition 3.1. Let $(\mathcal{M}, \oplus, 0_\mathcal{M}, \tau_\mathcal{M})$ be a small permutative category. For $n \geqslant 0$, let $(-\mathcal{M})^n\mathcal{M}$ be the following permutative category. The objects are n-cubes of objects of $\mathcal{M}$, *i.e.*, functions from the power set of the set $\mathbf{n} = \{1, \dots, n\}$ to the set of objects of $\mathcal{M}$. We use the pointwise addition of cubes, *i.e.*, we define the sum of two n-cubes C and C' by $(C \oplus C')_S = C_S \oplus C'_S$ for all $S \subset \mathbf{n}$.

If $\varphi \colon \mathbf{m} \to \mathbf{n} \in I$ is a non-bijective injection and C an m-cube, we call the n-cube $\varphi_*(C)$ with $\varphi_*(C)_S = C_{\varphi^{-1}(S)}$ an *elementary degenerate n-cube*, and if $f \colon C \to C'$ is an isomorphism of m-cubes (*i.e.*, for each $S \subset \mathbf{m}$ an isomorphism $f_S \colon C_S \to C'_S$ in $\mathcal{M}$), the map $\varphi_*(f) \colon \varphi_*(C) \to \varphi_*(C')$ given by $\varphi_*(f)_S = f_{\varphi^{-1}(S)}$ is an *elementary degenerate isomorphism*. A *degenerate n-cube* is an n-cube of the form $in_{1*}(D_1) \oplus in_{2*}(D_2) \oplus \dots \oplus in_{n*}(D_n)$ where $in_k \colon \mathbf{n-1} \to \mathbf{n}$ is the injection missing $k \in \mathbf{n}$. A *degenerate isomorphism* is similarly an isomorphism of the form $in_{1*}(f_1) \oplus \dots \oplus in_{n*}(f_n)$.

A morphism in $(-\mathcal{M})^n\mathcal{M}$ from C to C' is an equivalence class of pairs (D, α) where D is a degenerate n-cube and α is a collection of maps $\alpha_S \colon C_S \oplus D_S \to C'_S \in \mathcal{M}$ for all $S \subset \mathbf{n}$. A pair (D, α) is equivalent to a pair (D', α') if there is a degenerate isomorphism $f \colon D \to D'$ such that the diagrams

$$\begin{tikzcd} C_S \oplus D_S \arrow[dd, "\mathrm{id} \oplus f_S"'] \arrow[dr, "\alpha_S"] & \\ & C'_S \\ C_S \oplus D'_S \arrow[ur, "\alpha'_S"'] & \end{tikzcd}$$

commute for all $S \subset \mathbf{n}$. We write $[D, \alpha]$ for this equivalence class. The composite

$$C \xrightarrow{[D,\alpha]} C' \xrightarrow{[E,\beta]} C''$$

is represented by

$$C \oplus \bigoplus\nolimits_k in_{k*}(D_k \oplus E_k) \cong C \oplus D \oplus E \xrightarrow{\alpha \oplus \mathrm{id}} C' \oplus E \xrightarrow{\beta} C'' \,.$$

We define 0_n as the cube that has $(0_n)_S = 0_\mathcal{M} \in \mathcal{M}$ for all $S \subset \mathbf{n}$. As $\mathcal{M}$ was permutative, the addition of cubes defines permutative structures on the $(-\mathcal{M})^n\mathcal{M}$ for all $n \geqslant 0$.

Definition 3.2. Let the *Grayson–Quillen functor* G: Perm $\to$ Perm be defined by $G(\mathcal{M}) = (-\mathcal{M})\mathcal{M}$.

Lemma 3.3. *Let $\mathcal{M}$ be a small permutative category. Then there is a natural isomorphism between $(-\mathcal{M})^n\mathcal{M}$ and the n-th iterate $G^{(n)}(\mathcal{M})$ of the Grayson–Quillen functor, being the identity on objects.* □

Consider the transformation $in_*\colon id \to G$ given by

$$\mathcal{M} \xrightarrow{a\mapsto[a,0]} G(\mathcal{M}),$$

and the twist $\tau_*\colon GG \to GG$ given by transposition of matrices: $\tau_*(C)_S = C_{\tau^{-1}S}$ (where $\tau\colon \mathbf{2}\to\mathbf{2}$ is the twist) and $\tau_*[in_{1*}(x_1)\oplus in_{2*}(x_2), f] = [in_{1*}(x_2)\oplus in_{2*}(x_1), \tau_* f]$.

Since $(G\,in_*)_{\mathcal{M}} = (\tau_* in_*)_{G\mathcal{M}}\colon G\mathcal{M}\to GG\mathcal{M}$, these natural transformations give rise to a functor $G^{(-)}$ from the category I of finite sets and injective functions to the endomorphism category of Perm by sending $\mathbf{n}$ to $\mathcal{M}\mapsto G^{(n)}(\mathcal{M})$.

More concretely, if $\varphi\colon \mathbf{m}\to\mathbf{n}\in I$, we define $\varphi_*\colon (-\mathcal{M})^m\mathcal{M}\to(-\mathcal{M})^n\mathcal{M}$ by sending an m-cube C to the n-cube $\varphi_*(C)$ given by $\varphi_*(C)_S := C_{\varphi^{-1}(S)}$. We define the effect of φ on a morphism $[D,\alpha]$ from C to C' in $(-\mathcal{M})^m\mathcal{M}$ to be $[\varphi_*(D),\varphi_*\alpha]$ with $(\varphi_*\alpha)_S = \alpha_{\varphi^{-1}(S)}$. There is a slight technicality: in order for $\varphi_*(D)$ to be written as a sum of elementary degenerate m-cubes in the prescribed order, one must permute the summands and compose with the proper isomorphisms. This is taken care of by the other viewpoint.

Definition 3.4. Let $(-\mathcal{M})^\bullet\mathcal{M}$ be the resulting functor $I\to \mathrm{Perm(Strict)}$, taking $\mathbf{n}$ to $(-\mathcal{M})^n\mathcal{M}$ and φ to φ_*.

We are concerned with the homotopy properties of this construction.

Definition 3.5. Let $\mathrm{Perm}^{\mathrm{nz}}$ be the category of small permutative categories without zero. This is a category with a symmetric addition satisfying all the axioms for a permutative category except that all mention of the zero object is skipped. Likewise, a morphism in $\mathrm{Perm}^{\mathrm{nz}}$ is defined exactly as a lax symmetric monoidal functor in Perm except that all mention of the zero object is skipped.

Definition 3.6. A *weak equivalence* of simplicial small categories is a functor $\mathcal{C}\to\mathcal{D}$ such that the diagonal of the associated map of nerves $N\mathcal{C}\to N\mathcal{D}$ is a weak equivalence of simplicial sets.

A lax symmetric monoidal functor $\mathcal{C}\to\mathcal{D}$ is an *unstable equivalence* if it induces a weak equivalence of simplicial sets $N\mathcal{C}\to N\mathcal{D}$.

A lax symmetric monoidal functor $\mathcal{C}\to\mathcal{D}$ is a *stable equivalence* if it induces a stable equivalence $\mathrm{Spt}\,\mathcal{C}\to\mathrm{Spt}\,\mathcal{D}$. Here Spt is any one of the (equivalent) group completion machines, for instance the one in [Th3] and [Th4, 1.6]; in particular it is a functor from symmetric monoidal categories and lax symmetric monoidal functors to connective spectra and spectrum maps.

A lax symmetric monoidal natural transformation $X\Rightarrow X'$ of functors $J\to\mathrm{Perm}^{\mathrm{nz}}$ is an *unstable (resp. stable) equivalence* if $X(j)\to X'(j)$ is an unstable (resp. stable) equivalence for each $j\in J$.

Lemma 3.7. *Let $(\mathcal{M},\oplus,0_{\mathcal{M}},\tau_{\mathcal{M}})$ be a small permutative category.*

1. *If $\mathcal{M}$ is a groupoid (all morphisms in $\mathcal{M}$ are isomorphisms) and translation is faithful (*i.e.*, for each pair $a,b\in\mathcal{M}$ the map $\mathcal{M}(a,a)\mapsto\mathcal{M}(a\oplus b,a\oplus b)$ is injective), then $\mathcal{M}\to(-\mathcal{M})\mathcal{M}$ is a stable equivalence.*
2. *For every $\varphi\colon\mathbf{m}\to\mathbf{n}$ in I with $m>0$ the induced map*
$$\varphi_*\colon(-\mathcal{M})^m\mathcal{M}\to(-\mathcal{M})^n\mathcal{M}$$
is an unstable equivalence.

Remark 3.8. In [G2, p. 166] Grayson proves a statement that is similar in spirit to our second claim.

Proof. The first claim is [G1, p. 228].

For the second claim, it suffices to show that $(in_*)_{G\mathcal{M}}\colon G\mathcal{M}\to GG\mathcal{M}$ is an equivalence. Consider the map $T\colon GG\mathcal{M}\to G\mathcal{M}$ sending $A=\left[\begin{smallmatrix} a_\varnothing & a_1\\ a_2 & a_{12}\end{smallmatrix}\right]$ to $[a_\varnothing\oplus a_{12}, a_1\oplus a_2]$, and a morphism represented by

$$\begin{bmatrix} f_\varnothing & f_1\\ f_2 & f_{12}\end{bmatrix} : \left[\begin{smallmatrix} a_\varnothing & a_1\\ a_2 & a_{12}\end{smallmatrix}\right]\oplus\left[\begin{smallmatrix} x_\varnothing & x_1\\ x_\varnothing & x_1\end{smallmatrix}\right]\oplus\left[\begin{smallmatrix} y_\varnothing & y_\varnothing\\ y_2 & y_2\end{smallmatrix}\right]\to\begin{bmatrix} b_\varnothing & b_1\\ b_2 & b_{12}\end{bmatrix}\in\mathcal{M}^4$$

to the morphism represented by the pair

$$a_\varnothing\oplus a_{12}\oplus x_\varnothing\oplus y_\varnothing\oplus x_1\oplus y_2\cong(a_\varnothing\oplus x_\varnothing\oplus y_\varnothing)\oplus(a_{12}\oplus x_1\oplus y_2)\xrightarrow{f_\varnothing\oplus f_{12}} b_\varnothing\oplus b_{12},$$
$$a_1\oplus a_2\oplus x_\varnothing\oplus y_\varnothing\oplus x_1\oplus y_2\cong(a_1\oplus x_1\oplus y_\varnothing)\oplus(a_2\oplus x_\varnothing\oplus y_2)\xrightarrow{f_1\oplus f_2} b_1\oplus b_2\,.$$

Whereas Tin_* is the identity on $G\mathcal{M}$, the composite in_*T sends $\left[\begin{smallmatrix} a_\varnothing & a_1 \\ a_2 & a_{12} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} a_\varnothing \oplus a_{12} & a_1 \oplus a_2 \\ 0 & 0 \end{smallmatrix}\right]$ which is connected to the identity by a chain of natural transformations

$$\begin{tikzcd}
\left[\begin{smallmatrix} a_\varnothing \oplus a_{12} & a_1 \oplus a_2 \\ 0 & 0 \end{smallmatrix}\right] \arrow[r, "{\left[\left[\begin{smallmatrix} 0 & a_{12} \\ 0 & a_{12} \end{smallmatrix}\right], \mathrm{id}\right]}"] & \left[\begin{smallmatrix} a_\varnothing \oplus a_{12} & a_1 \oplus a_2 \oplus a_{12} \\ 0 & a_{12} \end{smallmatrix}\right] & \\
& \left[\begin{smallmatrix} a_\varnothing & a_1 \oplus a_2 \\ 0 & a_{12} \end{smallmatrix}\right] \arrow[u, "{\left[\left[\begin{smallmatrix} a_{12} & a_{12} \\ 0 & 0 \end{smallmatrix}\right], \mathrm{id}\right]}"] \arrow[r, "{\left[\left[\begin{smallmatrix} a_2 & 0 \\ a_2 & 0 \end{smallmatrix}\right], \mathrm{id}\right]}"] & \left[\begin{smallmatrix} a_\varnothing \oplus a_2 & a_1 \oplus a_2 \\ a_2 & a_{12} \end{smallmatrix}\right] \\
& & \left[\begin{smallmatrix} a_\varnothing & a_1 \\ a_2 & a_{12} \end{smallmatrix}\right]. \arrow[u, "{\left[\left[\begin{smallmatrix} a_2 & a_2 \\ 0 & 0 \end{smallmatrix}\right], \mathrm{id}\right]}"]
\end{tikzcd}$$

□

Proposition 3.9. *Let $(\mathcal{R}, \oplus, 0_\mathcal{R}, c_\oplus, \otimes, 1_\mathcal{R}, c_\otimes)$ be a bipermutative category. Then the functor $(-\mathcal{R})^\bullet\mathcal{R} = \{\mathbf{n} \mapsto (-\mathcal{R})^n\mathcal{R}\}_{n\geqslant 0}$ is an I-graded bipermutative category.*

Proof. In the following let f, g and h be the following morphisms: $f = [X, \alpha]\colon A \to A'$, *i.e.*, $\alpha\colon A \oplus X \to A'$, $g = [Y, \beta]\colon B \to B'$ and $h = [Z, \delta]\colon C \to C'$.

First we make the structure of $(-\mathcal{R})^\bullet\mathcal{R}$ as a functor into Perm(Strict) explicit. As for $(-\mathcal{R})\mathcal{R}$, the definition of the morphism $f \oplus g$ involves an additive twist:

$$f \oplus g := [A \oplus B \oplus X \oplus Y \xrightarrow{\mathrm{id}\oplus\gamma_\oplus\oplus\mathrm{id}} A \oplus X \oplus B \oplus Y \xrightarrow{\alpha\oplus\beta} A' \oplus B'] .$$

We consider the isomorphism $\gamma_\oplus\colon X \oplus Y \to Y \oplus X$. The commutativity of the diagram

$$\begin{tikzcd}
A \oplus B \oplus X \oplus Y \arrow[r, "\mathrm{id}_{A\oplus B}\oplus\gamma_\oplus"] \arrow[d, "\mathrm{id}\oplus\gamma_\oplus\oplus\mathrm{id}"'] & A \oplus B \oplus Y \oplus X \arrow[r, "\gamma_\oplus\oplus\mathrm{id}_{Y\oplus X}"] & B \oplus A \oplus Y \oplus X \arrow[d, "\mathrm{id}\oplus\gamma_\oplus\oplus\mathrm{id}"] \\
A \oplus X \oplus B \oplus Y \arrow[rr, "\gamma_\oplus^{A\oplus X, B\oplus Y}"] \arrow[d, "\alpha\oplus\beta"'] & & B \oplus Y \oplus A \oplus X \arrow[d, "\beta\oplus\alpha"] \\
A' \oplus B' \arrow[rr, "\gamma_\oplus"] & & B' \oplus A'
\end{tikzcd}$$

shows that $\gamma_\oplus$ is natural. It is straightforward to show that the addition fulfills all requirements of a permutative structure.

The multiplicative structure and its interplay with the additive structure are more subtle, and we will give more details on that part of the proof.

First, we define $\otimes$ on objects. We use the natural inclusion $i\colon \mathbf{n} \to \mathbf{n} + \mathbf{m}$ which sends $x \in \mathbf{n}$ to x and the inclusion $j\colon \mathbf{m} \to \mathbf{n} + \mathbf{m}$ which maps $y \in \mathbf{m}$ to $n + y$. For a subset $U \subset \mathbf{n} + \mathbf{m}$ we define

$$(A \otimes B)_U := A_{i^{-1}U} \otimes B_{j^{-1}U}$$

for all $A \in (-\mathcal{R})^n\mathcal{R}$ and $B \in (-\mathcal{R})^m\mathcal{R}$.

We have to define $\otimes$ also on morphisms, so that it becomes a functor $\otimes\colon (-\mathcal{R})^n\mathcal{R} \times (-\mathcal{R})^m\mathcal{R} \to (-\mathcal{R})^{n+m}\mathcal{R}$. In addition we must show that it fulfills the naturality conditions of property (1) in Definition 2.1.

We have to define the product $f \otimes g$ for morphisms in $(-\mathcal{R})^\bullet\mathcal{R}$. The cube

$$A \otimes Y \oplus X \otimes B \oplus X \otimes Y$$

is degenerate and we define $f \otimes g$ to be the equivalence class of

$$A \otimes B \oplus A \otimes Y \oplus X \otimes B \oplus X \otimes Y \xrightarrow{d_r \oplus d_r} A \otimes (B \oplus Y) \oplus X \otimes (B \oplus Y) \xrightarrow{d_\ell} (A \oplus X) \otimes (B \oplus Y) \xrightarrow{\alpha\otimes\beta} A' \otimes B' .$$

Note that this is equivalent to defining $f \otimes g$ as

$$A \otimes B \oplus X \otimes B \oplus A \otimes Y \oplus X \otimes Y \xrightarrow{d_\ell \oplus d_\ell} (A \oplus X) \otimes B \oplus (A \oplus X) \otimes Y \xrightarrow{d_r} (A \oplus X) \otimes (B \oplus Y) \xrightarrow{\alpha\otimes\beta} A' \otimes B'$$

because $d_r \circ (d_\ell \oplus d_\ell) \circ (\mathrm{id} \oplus c_\oplus \oplus \mathrm{id}) = d_\ell \circ (d_r \oplus d_r)$ holds in $\mathcal{R}$ and therefore the corresponding equation

$$d_r \circ (d_\ell \oplus d_\ell) \circ (\mathrm{id} \oplus \gamma_\oplus \oplus \mathrm{id}) = d_\ell \circ (d_r \oplus d_r)$$

holds in $(-\mathcal{R})^\bullet\mathcal{R}$.

The naturality with respect to morphisms in I is straightforward: given injections $\varphi\colon \mathbf{n} \to \mathbf{n}'$ and $\psi\colon \mathbf{m} \to \mathbf{m}'$, they give an injection on $\mathbf{n}+\mathbf{m}$, $\varphi+\psi$. Then for every $U' \subset \mathbf{n}'+\mathbf{m}'$

$$\begin{aligned}(\varphi+\psi)_*(A\otimes B)_{U'} = (A\otimes B)_{(\varphi+\psi)^{-1}(U')} &= A_{i^{-1}(\varphi+\psi)^{-1}(U')} \otimes B_{j^{-1}(\varphi+\psi)^{-1}(U')}\\ &= A_{\varphi^{-1}(i')^{-1}(U')} \otimes B_{\psi^{-1}(j')^{-1}(U')}.\end{aligned}$$

Here $i'\colon \mathbf{n}' \hookrightarrow \mathbf{n}'+\mathbf{m}'$ and $j'\colon \mathbf{m}' \hookrightarrow \mathbf{n}'+\mathbf{m}'$ are inclusions analogous to i and j. This is precisely the value of $((\varphi_*A)\otimes(\psi_*B))_{U'}$.

We will check that

$$f\otimes g = (\mathrm{id}\otimes g)\circ(f\otimes \mathrm{id})$$

and

$$(f'\otimes\mathrm{id})\circ(f\otimes\mathrm{id}) = (f'\circ f)\otimes\mathrm{id}$$

for $f' = [\alpha'\colon A'\oplus X' \to A'']$. Together with the properties $f\otimes g = (f\otimes\mathrm{id})\circ(\mathrm{id}\otimes g)$ and $(\mathrm{id}\otimes g')\circ(\mathrm{id}\otimes g) = \mathrm{id}\otimes(g'\circ g)$ for $g' = [\beta'\colon B'\oplus Y' \to B'']$ which hold as well (but whose proof we leave to the reader), this shows that $\otimes$ defines a bifunctor.

In the diagrams we will omit the tensor signs and just concatenate symbols and we write $+$ for $\oplus$ to ease notation.

Note that the representative we chose for the morphism $f\otimes\mathrm{id}$ has source

$$AB + A0 + XB + X0 = AB + XB\,.$$

Consider the diagram

$$\begin{tikzcd}
AB+AY+XB+XY \arrow[r, "\mathrm{id}+\gamma_\oplus+\mathrm{id}"] \arrow[d, "d_r+d_r"'] & AB+XB+AY+XY \arrow[r, "\mathrm{id}+\mathrm{id}+d_\ell"] \arrow[d, "d_\ell+d_\ell"] & AB+XB+(A+X)Y \arrow[dl, "d_\ell+\mathrm{id}"] \arrow[d, "\mathrm{id}+\mathrm{id}+\alpha\otimes\mathrm{id}"] \\
A(B+Y)+X(B+Y) \arrow[d, "d_\ell"'] & (A+X)B+(A+X)Y \arrow[dl, "d_r"] \arrow[d, "\alpha\otimes\mathrm{id}+\alpha\otimes\mathrm{id}"] \arrow[dr, "\mathrm{id}+\alpha\otimes\mathrm{id}"] & AB+XB+A'Y \arrow[d, "d_\ell+\mathrm{id}"] \\
(A+X)(B+Y) \arrow[d, "\alpha\otimes\beta"'] \arrow[dr, "\alpha\otimes\mathrm{id}"] & A'B+A'Y \arrow[d, "d_r"] & (A+X)B+A'Y \arrow[l, "\alpha\otimes\mathrm{id}+\mathrm{id}"] \\
A'B' & A'(B+Y)\,. \arrow[l, "\mathrm{id}\otimes\beta"] &
\end{tikzcd}$$

The left vertical composition corresponds to $f\otimes g$ whereas the right vertical composition (starting in $AB+XB+A'Y$) followed by the map at the bottom gives a representative for $(\mathrm{id}\otimes g)\circ(f\otimes\mathrm{id})$. The map at the top is a map of the form $\mathrm{id}\oplus\varepsilon$ with ε being an isomorphism and therefore the two maps $f\otimes g$ and $(\mathrm{id}\otimes g)\circ(f\otimes\mathrm{id})$ are equivalent.

The second property is easier to see because the diagram describing $(f'\circ f)\otimes\mathrm{id}$ and $(f'\otimes\mathrm{id})\circ(f\otimes\mathrm{id})$

$$\begin{tikzcd}
AB+XB+X'B \arrow[r, "\mathrm{id}+d_\ell"] \arrow[d, "d_\ell+\mathrm{id}"'] & AB+(X+X')B \arrow[d, "d_\ell"] \\
(A+X)B+X'B \arrow[r, "d_\ell"] \arrow[d, "f\otimes\mathrm{id}+\mathrm{id}"'] & (A+X+X')B \arrow[d, "(f+\mathrm{id})\otimes\mathrm{id}"] \\
A'B+X'B \arrow[r, "d_\ell"] \arrow[d, "d_\ell"'] & (A'+X')B \arrow[d, "f'\otimes\mathrm{id}"] \\
(A'+X')B \arrow[r, "f'\otimes\mathrm{id}"] & A''B
\end{tikzcd}$$

visibly commutes.

As we assume that $\otimes$ is strictly associative in $\mathcal{R}$, the multiplication of cubes is also strictly associative on objects:

If A, B, C are cubes of sizes n, m and l, then

$$((A\otimes B)\otimes C)_U = (A_{i^{-1}U}\otimes B_{j^{-1}U})\otimes C_{k^{-1}U} = A_{i^{-1}U}\otimes(B_{j^{-1}U}\otimes C_{k^{-1}U}) = (A\otimes(B\otimes C))_U\,.$$

Here, U is a subset of $\mathbf{n}+\mathbf{m}+\mathbf{l}$ and i,j,k denote the appropriate inclusions.

We have to show that $\otimes$ is strictly associative on morphisms in $(-\mathcal{R})^\bullet\mathcal{R}$. The representative we chose for the morphism $(f\otimes g)\otimes h$ has source

$$ABC+ABZ+(AY+XB+XY)C+(AY+XB+XY)Z$$

whereas the one for $f\otimes(g\otimes h)$ has source

$$ABC+A(BZ+YC+YZ)+XBC+X(BZ+YC+YZ).$$

However, both morphisms pass via the object $(A+X)(B+Y)(C+Z)$, *i.e.*, we can factor $(f\otimes g)\otimes h$ as

$$ABC+ABZ+(AY+XB+XY)C+(AY+XB+XY)Z \xrightarrow{w_1} (A+X)(B+Y)(C+Z) \xrightarrow{\alpha\otimes\beta\otimes\delta} A'B'C'$$

and $f\otimes(g\otimes h)$ as

$$ABC+A(BZ+YC+YZ)+XBC+X(BZ+YC+YZ) \xrightarrow{w_2} (A+X)(B+Y)(C+Z) \xrightarrow{\alpha\otimes\beta\otimes\delta} A'B'C'$$

where w_1, w_2 are isomorphisms.

In order to see that $w_2^{-1}\circ w_1$ is of the form $\mathrm{id}\oplus\varepsilon$ it is useful to compare the sources of the two maps via

$$\begin{tikzcd}
ABC+ABZ+(AY+XB+XY)C+(AY+XB+XY)Z \arrow[d, "\mathrm{id}+\mathrm{id}+d_\ell^{-1}+d_\ell^{-1}"] \\
ABC+ABZ+AYC+XBC+XYC+AYZ+XBZ+XYZ \arrow[d, "\mathrm{id}+\mathrm{id}+\sigma"] \\
ABC+ABZ+AYC+AYZ+XBC+XBZ+XYC+XYZ \arrow[d, "\mathrm{id}+d_r+\mathrm{id}+d_r"] \\
ABC+A(BZ+YC+YZ)+XBC+X(BZ+YC+YZ)
\end{tikzcd}$$

where σ is the appropriate permutation acting by permuting the summands. A diagram chase shows that this map is $w_2^{-1}\circ w_1$ and therefore we obtain that

$$(f\otimes g)\otimes h = f\otimes(g\otimes h)\,.$$

Recall the shuffle permutation $\chi(n,m)\colon \mathbf{n}+\mathbf{m}\to\mathbf{m}+\mathbf{n}$ from Example 2.4. For the multiplicative twist transformation $\gamma_\otimes$ as in Definition 2.1 (3) we have to specify a natural isomorphism from $A\otimes B$ to $\chi(m,n)_*(B\otimes A)$ for every $A\in(-\mathcal{R})^n\mathcal{R}$ and $B\in(-\mathcal{R})^m\mathcal{R}$. For $U\subset\mathbf{n}+\mathbf{m}$ and $U_1=i^{-1}(U)$, $U_2=j^{-1}(U)$ the isomorphism $\gamma_\otimes$ is the multiplicative twist

$$c_\otimes\colon A_{U_1}\otimes B_{U_2}\to B_{U_2}\otimes A_{U_1}\,.$$

It is then obvious that $\gamma_\otimes\circ\gamma_\otimes=\mathrm{id}$. If A or B is the multiplicative unit, then each of the twist maps $c_\otimes$ reduces to the identity map and thus $\gamma_\otimes$ is the identity. It is straightforward to check the remaining properties of 2.1 (4) and we leave this to the reader.

In the following we will omit the maps $\chi(n,m)$ from the notation.

Consider again morphisms in $(-\mathcal{R})^\bullet\mathcal{R}$, f and g as above. In the diagram

$$\begin{tikzcd}
AB+XB+AY+XY \arrow[r, "\mathrm{id}+(\gamma_\otimes)^{\oplus 3}"] \arrow[drr, "\gamma_\otimes^{\oplus 4}"] \arrow[dd, bend left, "d_r\circ(d_\ell+d_\ell)"] \arrow[ddrr, "d_\ell\circ(\gamma_\otimes+\gamma_\otimes)\circ(d_\ell+d_\ell)"] & AB+BX+YA+YX \arrow[dr, "\gamma_\otimes+\mathrm{id}+\mathrm{id}+\mathrm{id}"] & \\
AB+AY+XB+XY \arrow[u, "\mathrm{id}+\gamma_\oplus+\mathrm{id}"] \arrow[d, "d_\ell\circ(d_r+d_r)"] & & BA+BX+YA+YX \arrow[d, "d_\ell\circ(d_r+d_r)"] \\
(A+X)(B+Y) \arrow[d, "\alpha\otimes\beta"] & & (B+Y)(A+X) \arrow[ll, "\gamma_\otimes"] \arrow[d, "\beta\otimes\alpha"] \\
A'B' \arrow[rr, "\gamma_\otimes"] & & B'A'
\end{tikzcd}$$

the composition $\gamma_\otimes \circ (f \otimes g)$ is represented by the counterclockwise composition starting at $AB + AY + XB + XY$ and one representative for $(g \otimes f) \circ \gamma_\otimes$ runs clockwise starting at $AB + BX + YA + YX$. As d_r is defined in terms of $\gamma_\otimes$ and d_ℓ we get that

$$d_\ell \circ (d_r \oplus d_r) \circ \gamma_\otimes^{\oplus 4} = d_\ell \circ (\gamma_\otimes \oplus \gamma_\otimes) \circ (d_\ell \oplus d_\ell)$$

and therefore the diagram commutes and the multiplicative twist is natural with respect to morphisms in $(-\mathcal{R})^\bullet\mathcal{R}$.

We have to prove that $\gamma_\otimes$ is natural with respect to maps in I. Given injections $\varphi\colon \mathbf{n} \to \mathbf{n}'$, $\psi\colon \mathbf{m} \to \mathbf{m}'$ we have to show that $(\varphi + \psi)_*$ applied to the isomorphism

$$\gamma_\otimes\colon A \otimes B \to \chi(m, n)_*(B \otimes A)$$

of $(n + m)$-cubes equals the isomorphism $\gamma_\otimes\colon \varphi_* A \otimes \psi_* B \to \chi(m', n')_*(\psi_* B \otimes \varphi_* A)$ of $(n' + m')$-cubes. Noting that $(\varphi + \psi) \circ \chi(m, n) = \chi(m', n') \circ (\psi + \varphi)$, we find that at each subset $U \subset \mathbf{n}' + \mathbf{m}'$ both isomorphisms are equal to $c_\otimes$ applied to $A_{\varphi^{-1}(U_1)} \otimes B_{\psi^{-1}(U_2)}$, where $U_1 = i^{-1}(U)$ and $U_2 = j^{-1}(U)$.

Obviously, tensoring with a zero cube annihilates everything.

We claim that the left distributivity isomorphism d_ℓ can be chosen to be the identity on $(-\mathcal{R})^\bullet\mathcal{R}$ and we define d_r as $\gamma_\otimes \circ d_\ell \circ (\gamma_\otimes \oplus \gamma_\otimes)$ according to Definition 2.1 (7).

We already proved that the twist transformation $\gamma_\otimes$ is natural with respect to morphisms in I, hence d_r is natural.

Thus, for left distributivity we have to identify $(A \oplus B) \otimes C$ with $A \otimes C \oplus B \otimes C$ for all $A, B \in (-\mathcal{R})^n\mathcal{R}$ and $C \in (-\mathcal{R})^m\mathcal{R}$. On every subset $U \subset \mathbf{n} + \mathbf{m}$ the latter takes value

$$(A \otimes C)_U \oplus (B \otimes C)_U = (i_* A)_U \otimes (j_* C)_U \oplus (i_* B)_U \otimes (j_* C)_U\,.$$

As $\mathcal{R}$ is supposed to be a bipermutative category, this term is equal to

$$((i_* A)_U \oplus (i_* B)_U) \otimes (j_* C)_U$$

which is precisely $((A \oplus B) \otimes C)_U$. We also have to show that

$$(f \oplus g) \otimes h = f \otimes h \oplus g \otimes h\,.$$

This follows from the commutativity of the following solid diagram (using cycle notation for permutations of summands)

$$\begin{tikzcd}
AC + AZ + BC + BZ + XC + XZ + YC + YZ \arrow[r, bend left, "(d_\ell \oplus \mathrm{id}) \circ (\mathrm{id} \oplus d_\ell^{\oplus 3}) \circ (2,3)(6,7)"] \arrow[d, "(3,5)(4,6)"'] & (A+B)C + (A+B)Z + (X+Y)C + (X+Y)Z \arrow[d, "d_r \oplus d_r"] \\
AC + AZ + XC + XZ + BC + BZ + YC + YZ \arrow[d, "d_r^{\oplus 4}"'] & (A+B)(C+Z) + (X+Y)(C+Z) \arrow[d, "(\mathrm{id} \otimes \delta)^{\oplus 2}"] \\
A(C+Z) + X(C+Z) + B(C+Z) + Y(C+Z) \arrow[ur, dotted, "(d_\ell \oplus d_\ell) \circ (2,3)"'] \arrow[d, "(\mathrm{id} \otimes \delta)^{\oplus 4}"'] & (A+B)C' + (X+Y)C' \arrow[d, "d_\ell"] \\
AC' + XC' + BC' + YC' \arrow[ur, dotted, "(d_\ell \oplus d_\ell) \circ (2,3)"'] \arrow[d, "d_\ell \oplus d_\ell"'] & (A+B+X+Y)C' \arrow[d, "(2,3) \otimes \mathrm{id}"] \\
(A+X)C' + (B+Y)C' \arrow[r, dotted, "d_\ell"] \arrow[d, "(\alpha \otimes \mathrm{id}) \oplus \beta \otimes \mathrm{id}"'] & (A+X+B+Y)C' \arrow[d, "(\alpha \oplus \beta) \otimes \mathrm{id}"] \\
A'C' + B'C' \arrow[r, "d_\ell"] & (A'+B')C'\,.
\end{tikzcd}$$

The subdiagram at the bottom commutes because d_ℓ is natural in $\mathcal{R}$ and for the same reason the subdiagram involving δ commutes. The diagram on the third level corresponds to (8) of Definition 2.1 for d_ℓ, and commutes by direct inspection. For the top subdiagram naturality of $\gamma_\oplus$ yields that $(2,3) \circ d_r^{\oplus 4} = d_r^{\oplus 4} \circ (3,5)(4,6)$ and therefore $(d_\ell \oplus d_\ell) \circ (2,3) \circ d_r^{\oplus 4} \circ (3,5)(4,6) = (d_\ell \oplus d_\ell) \circ d_r^{\oplus 4}$ and the commutativity of this subdiagram follows from the pentagon axiom for d_r and d_ℓ.

What is left to check in Definition 2.1 are properties (8), (9) and (10). These follow from a direct inspection we leave to the reader. ☐

Proposition 3.10. *Let $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, c_{\oplus}, \otimes, 1_{\mathcal{R}})$ be a strictly bimonoidal category. Then the functor*

$$(-\mathcal{R})^{\bullet}\mathcal{R} = \{\mathbf{n} \mapsto (-\mathcal{R})^{n}\mathcal{R}\}$$

is an I-graded strictly bimonoidal category.

Proof. We have to define a right distributivity map d_r, prove that it is natural and show that property (7') of Definition 2.5 holds.

We define d_r to be the pointwise application of the right distributivity map d_r of $\mathcal{R}$. As property (7') holds in $\mathcal{R}$, it also holds in $(-\mathcal{R})^{\bullet}\mathcal{R}$.

Let f, g, h be morphisms in $(-\mathcal{R})^{\bullet}\mathcal{R}$ as in the proof of Proposition 3.9. Again, we use cycle notation for the permutations induced by combinations of instances of $\gamma_{\oplus}$. For the naturality of d_r we have to show that the following solid diagram

$$\begin{tikzcd}
AB+AC+AY+XB+XY+AZ+XC+XZ \arrow[r, bend left, "(d_r\oplus\mathrm{id})\circ(\mathrm{id}\oplus d_r^3)\circ(4,5,7,6)"] \arrow[d, "(2,5,4,3)"'] & A(B+C)+A(Y+Z)+X(B+C)+X(Y+Z) \arrow[d, "d_r\oplus d_r"] \\
AB+AY+XB+XY+YC+YZ+XC+XZ \arrow[d, "d_r^{\oplus 4}"'] & A(B+C+Y+Z)+X(B+C+Y+Z) \arrow[d, "\mathrm{id}\otimes(2,3)\oplus\mathrm{id}\otimes(2,3)"] \\
A(B+Y)+X(B+Y)+A(C+Z)+X(C+Z) \arrow[r, dotted, "(d_r+d_r)\circ(2,3)"] \arrow[d, "(\mathrm{id}\otimes\beta)^{\oplus 2}\oplus(\mathrm{id}\otimes\delta)^{\oplus 2}"'] & A(B+Y+C+Z)+X(B+Y+C+Z) \arrow[d, "(\mathrm{id}\otimes(\beta+\delta))^{\oplus 2}"] \\
AB'+XB'+AC'+XC' \arrow[r, dotted, "(d_r+d_r)\circ(2,3)"] \arrow[d, "d_\ell\oplus d_\ell"'] & A(B'+C')+X(B'+C') \arrow[d, "d_\ell"] \\
(A+X)B'+(A+X)C' \arrow[r, dotted, "d_r"] \arrow[d, "(\alpha\otimes\mathrm{id})^{\oplus 2}"'] & (A+X)(B'+C') \arrow[d, "\alpha\otimes\mathrm{id}"] \\
A'B'+A'C' \arrow[r, "d_r"] & A'(B'+C')
\end{tikzcd}$$

commutes.

The bottom square commutes because d_r is natural in $\mathcal{R}$ and so does the square on the third layer. The square on the fourth layer commutes because of the pentagon relation. For the top subdiagram note that the equations

$$\begin{aligned}
(2,3) \circ d_r^{\oplus 4} &= d_r^{\oplus 4} \circ (3,5)(4,6) \\
(\mathrm{id} \otimes (2,3))^{\oplus 2} \circ (d_r + d_r) \circ d_r^{\oplus 4} &= (d_r + d_r) \circ d_r^{\oplus 4} \circ (2,3)(6,7)
\end{aligned}$$

hold: the first one follows from the naturality of $\gamma_{\oplus}$ and the second one is a consequence of property (8) of Definition 2.1. Naturality then follows from the identities

$$(2,3)(6,7)(4,5,7,6) = (2,3)(5,6,4) = (3,5)(4,6)(2,5,4,3)$$

in the symmetric group on 8 letters. ☐

4. Hocolim-lemmata

We briefly recall Thomason's homotopy colimit construction in the case of a functor from a small category J to the category $\mathrm{Perm}^{\mathrm{nz}}$.

The forgetful functor $U\colon \mathrm{Perm} \to \mathrm{Perm}^{\mathrm{nz}}$ has a left adjoint $F\colon \mathrm{Perm}^{\mathrm{nz}} \to \mathrm{Perm}$ given by $F(S) = S_{+}$, the category obtained by adding a disjoint zero (called "+" to distinguish it from old zeros that might live in S).

4.1. **The non-unital case.** Let $X\colon J \to \mathrm{Perm}^{\mathrm{nz}}$ be a functor. An object in $\mathrm{hocolim}_J X$ is an expression like $n[(a_1, X_1), \dots, (a_n, X_n)]$ where $n \geqslant 1$ is a natural number, the a_i are objects of J and the X_i are objects of $X(a_i)$. A morphism from $n[(a_1, X_1), \dots, (a_n, X_n)]$ to $m[(b_1, Y_1), \dots, (b_m, Y_m)]$ consists of three parts: a surjection ψ from the set $\{1, \dots, n\}$ to $\{1, \dots, m\}$, morphisms $\ell_i\colon a_i \to b_{\psi(i)}$ for $1 \leqslant i \leqslant n$ and morphisms ϱ_j in $X(b_j)$ from $\bigoplus_{\psi(i)=j} X(\ell_i)(X_i)$ to Y_j. By abuse of notation, we will write $(\psi, \ell_i, \varrho_j)$ to signify this morphism.

The category $\mathrm{hocolim}_J X$ is permutative (without a zero) if one defines the addition to be given by concatenation (compare [Th3, p. 1632]).

As a matter of fact, if $\mathrm{Perm}^{\mathrm{nz}}(\mathrm{Strict})$ is the subcategory of $\mathrm{Perm}^{\mathrm{nz}}$ with all objects, but with strict monoidal functors as morphisms, the universal property in [Th3, pp. 1632–1633] says that $\mathrm{hocolim}_J$ is left adjoint to the composite functor $\mathrm{Perm}^{\mathrm{nz}}(\mathrm{Strict}) \to \mathrm{Perm}^{\mathrm{nz}} \to (\mathrm{Perm}^{\mathrm{nz}})^J$, where the first functor is the forgetful one and the second functor assigns the constant J-diagram (= functor from J). (Actually, this is even true on the level of 2-categories.)

Let Cat denote the category of all small categories. Recall the free functor $P\colon \mathrm{Cat} \to \mathrm{Perm}^{\mathrm{nz}}(\mathrm{Strict})$ with $P\mathcal{C} = \coprod_{n>0} \widetilde{\Sigma_n} \times_{\Sigma_n} \mathcal{C}^{\times n}$ where $\widetilde{\Sigma_n}$ is the translation category of the symmetric group Σ_n.

Lemma 4.1. *The free functor $P\colon \mathit{Cat} \to \mathit{Perm}^{\mathrm{nz}}(\mathrm{Strict})$ sends unstable equivalences to unstable equivalences.*

Proof. This follows from the natural isomorphism of nerves $NP\mathcal{C} \cong \coprod_{n>0} N\widetilde{\Sigma_n} \times_{\Sigma_n} (N\mathcal{C})^{\times n}$ and the fact that $E\Sigma_n = N\widetilde{\Sigma_n}$ is a free Σ_n-space. □

Lemma 4.2. *Let $F\colon X \to Y$ be an unstable (resp. stable) equivalence in $(\mathit{Perm}^{\mathrm{nz}})^J$. Then*

$$\mathrm{hocolim}_J F\colon\ \mathrm{hocolim}_J X \to \mathrm{hocolim}_J Y$$

is an unstable (resp. stable) equivalence. If $X\colon J \to \mathit{Perm}^{\mathrm{nz}}$ is a constant functor and J is contractible, then

$$X(j) \to \mathrm{hocolim}_J X$$

is an unstable equivalence.

Let I be the category of finite sets and injections and $\mathbf{m} \in I$. If $X\colon I \to \mathit{Perm}^{\mathrm{nz}}$ is a functor such that any $\varphi\colon \mathbf{m} \to \mathbf{n} \in I$ is sent to an unstable (resp. stable) equivalence $X(\varphi)\colon X(\mathbf{m}) \to X(\mathbf{n})$, then the canonical map $X(\mathbf{m}) \to \mathrm{hocolim}_I X$ is an unstable (resp. stable) equivalence.

Proof. The stable version follows from the main theorem 4.1 in [Th3] since homotopy colimits of spectra preserve stable equivalences.

The unstable version follows from the proof of the main theorem 4.1 in [Th3]: if $F\colon X \to Y$ is an unstable equivalence in $(\mathrm{Perm}^{\mathrm{nz}})^J$, then $PF\colon PX \to PY$ is also an unstable equivalence by Lemma 4.1. Furthermore, there is a natural isomorphism $P\,\mathrm{hocolim}_J \cong \mathrm{hocolim}_J P$ where the leftmost hocolim is in Cat. The homotopy colimit in Cat preserves unstable equivalences, and hence $\mathrm{hocolim}_J PX \to \mathrm{hocolim}_J PY$ is an unstable equivalence. If X is a diagram in $\mathrm{Perm}^{\mathrm{nz}}(\mathrm{Strict})$, then X has a simplicial resolution coming from the free-forgetful pair between Cat and $\mathrm{Perm}^{\mathrm{nz}}(\mathrm{Strict})$. Thomason's argument [Th3, pp. 1641–1644] shows that the homotopy colimit respects this resolution and hence we get the statement for diagrams in $\mathrm{Perm}^{\mathrm{nz}}(\mathrm{Strict})$. We can then extend this to general functors to $\mathrm{Perm}^{\mathrm{nz}}$ as in [Th3, p. 1645].

The last statement is a weak version of Bökstedt's Lemma [Bö, 9.1] which holds for homotopy colimits in Cat since it holds for homotopy colimits in simplicial sets, and by the argument above using the resolution by free permutative categories, it also holds in $\mathrm{Perm}^{\mathrm{nz}}$. □

4.2. **The case with zero.** We shall need a version of the homotopy colimit for permutative categories with zero. Thomason comments that such a homotopy colimit with zero is not a homotopy functor, unless the category is "well based". Hence we must derive our functor to get a homotopy invariant version. One option would be to use the free-forgetful pair to resolve everything in sight by free permutative categories with zero, but since we shall be concerned with more delicate structure in our categories, we choose a less drastic approach.

Recall the adjoint functor $F\colon \mathrm{Perm}^{\mathrm{nz}} \to \mathrm{Perm}$ of the forgetful functor U given by $F(S) = S_+$. Since U and F are adjoints, we get a simplicial resolution ("monadic resolution") Z as usual: if $S \in \mathrm{Perm}$ and $[q] \in \Delta^{\mathrm{op}}$ then $Z_q S = (FU)^{q+1}(S)$ with simplicial operations derived from the unit and counit of the adjunction. The counit $FU(S) \to S$ induces a map $Z(S) \to S$ of simplicial symmetric monoidal categories with zero.

Lemma 4.3. *If $S \in Perm$, then $Z(S) \to S$ is an unstable equivalence.*

Proof. The map of simplicial symmetric monoidal categories $UZ(S) \to US$ has an extra degeneracy induced by $id \to UF$. Hence the map of nerves $NZ(S) \to N(S)$ has an extra degeneracy, since the nerve only depends on the underlying category, and so it is a weak equivalence. □

We will not define the categorical homotopy colimit on Perm, but in special cases (including all we will need) it is given in terms of the ordinary homotopy colimit. The following is a formal consequence of the universal property of the homotopy colimit.

Lemma 4.4. *If J is any small category and X and Y are functors from J to the category $Perm^{\mathrm{nz}}$, then $(\mathrm{hocolim}_J X)_+$ is functorial in transformations $f : X_+ \to Y_+$ of permutative categories with zero.*

Proof. Let $\mathcal{M}$ be an object of Perm(Strict), which we can view as a constant functor $U\mathcal{M}$ from J to the category $\mathrm{Perm}^{\mathrm{nz}}$. The universal property of the (permutative) homotopy colimit [Th3, pp. 1626–1627] is that natural transformations of functors $J \to \mathrm{Perm}^{\mathrm{nz}}$ from X to $U\mathcal{M}$ correspond to strict maps from $\mathrm{hocolim}_J X$ to $U\mathcal{M}$.

Hence we get isomorphisms

$$\begin{aligned}(1)\qquad (\mathrm{Perm})^J(X_+,\mathcal{M}) \cong (\mathrm{Perm}^{\mathrm{nz}})^J(X,U\mathcal{M}) \quad &\cong \mathrm{Perm}^{\mathrm{nz}}(\mathrm{Strict})(\mathrm{hocolim}_J X, U\mathcal{M}) \\ &\cong \mathrm{Perm}(\mathrm{Strict})((\mathrm{hocolim}_J X)_+,\mathcal{M}).\end{aligned}$$

Even though we have not defined the homotopy colimit with zero, this shows that whatever its definition is, its value on X_+ is $(\mathrm{hocolim}_J X)_+$. More to the point, we get a map

$$(\mathrm{Perm})^J(X_+,Y_+) \to (\mathrm{Perm})^J(X_+,(\mathrm{hocolim}_J Y)_+) \cong \mathrm{Perm}(\mathrm{Strict})((\mathrm{hocolim}_J X)_+,(\mathrm{hocolim}_J Y)_+)$$

where the first map comes via the isomorphism (1) from the identity on $(\mathrm{hocolim}_J Y)_+$ and the last one is an instance of the isomorphism (1). This map gives the desired result. □

Let IsolPerm(Strict) denote the category of permutative categories with an isolated zero (*i.e.*, in the image of F) and strict symmetric monoidal functors. Lemma 4.4 implies that the homotopy colimit defines a functor:

Lemma 4.5. *The assignment*

$$FX \mapsto \mathrm{hocolim}^u_J FX := F\,\mathrm{hocolim}_J X$$

defines a functor $\mathrm{hocolim}^u_J$ *from the full subcategory of $Perm^J$ generated by the functors $X\colon J \to Perm$ that factor through* IsolPerm(Strict).

The proof of Lemma 4.4 shows that this homotopy colimit has a universal property similar to the unbased homotopy colimit, and since the unbased homotopy colimit preserves (un)stable equivalences, so does $\mathrm{hocolim}^u$.

This allows us to define a derived version of the homotopy colimit with zero.

Definition 4.6. The *derived homotopy colimit*

$$D\,\mathrm{hocolim}_J\colon \mathrm{Perm}^J \to \mathrm{IsolPerm}(\mathrm{Strict})^{\Delta^{\mathrm{op}}}$$

is defined by

$$D\,\mathrm{hocolim}_J X = \mathrm{hocolim}^u_J ZX = \{[q] \mapsto \mathrm{hocolim}^u_J (FU)^{q+1}X\}\,.$$

The construction deserves its name.

Lemma 4.7. *Let $X \to Y$ be a stable (resp. unstable) equivalence in $Perm^J$. Then $Z_qX \to Z_qY$ is a stable (resp. unstable) equivalence for each q, and hence the induced map*

$$D\,\mathrm{hocolim}_J X \to D\,\mathrm{hocolim}_J Y$$

is a stable (resp. unstable) equivalence, too.

Let $\mathbf{m}$ be an object of the category I of finite sets and injections. If $X\colon I \to Perm$ is a functor such that any $\varphi\colon \mathbf{m} \to \mathbf{n} \in I$ is sent to an unstable (resp. stable) equivalence $X(\varphi)\colon X(\mathbf{m}) \to X(\mathbf{n})$, then the canonical chain

$$X(\mathbf{m}) \xleftarrow{\ \sim\ } ZX(\mathbf{m}) \longrightarrow D\,\mathrm{hocolim}_I X$$

is a stable (resp. unstable) equivalence.

We are now ready for a key proposition:

Proposition 5.1. *Let J be a permutative category, and let $\mathcal{C}^\bullet$ be a J-graded bipermutative category. Then $D\operatorname{hocolim}_J \mathcal{C}^\bullet$ is a simplicial bipermutative category, and*

$$\mathcal{C}^0 \xleftarrow{\sim} Z\mathcal{C}^0 \longrightarrow D\operatorname{hocolim}_J \mathcal{C}^\bullet$$

are maps of simplicial bipermutative categories. The same statement holds when replacing "bipermutative" by "strictly bimonoidal".

Furthermore, for each $i \in J$, the canonical maps

$$\mathcal{C}^i \xleftarrow{\sim} Z\mathcal{C}^i \longrightarrow D\operatorname{hocolim}_J \mathcal{C}^\bullet$$

are maps of $Z\mathcal{C}^0$-modules.

Proof. If $\mathcal{C}^\bullet$ is a J-graded bipermutative category, then so is $FU\mathcal{C}^\bullet$, and $Z\mathcal{C}^\bullet$ becomes a simplicial J-graded bipermutative category. By Lemma 5.2 which we will prove below, we get that $\operatorname{hocolim}_J U(FU)^q\mathcal{C}^\bullet$ becomes a zeroless bipermutative category for each q. Hence $\operatorname{hocolim}^u_J Z_q\mathcal{C}^\bullet = F\operatorname{hocolim}_J U(FU)^q\mathcal{C}^\bullet$ is a bipermutative category, and all the simplicial structure maps are maps of bipermutative categories. Therefore $D\operatorname{hocolim}_J \mathcal{C}^\bullet$ becomes a simplicial bipermutative category. Likewise, for each q Lemma 5.2 below guarantees that

$$Z_q\mathcal{C}^0 \to \operatorname{hocolim}^u_J Z_q\mathcal{C}^\bullet$$

is a map of bipermutative categories and that

$$Z_q\mathcal{C}^i \to \operatorname{hocolim}^u_J Z_q\mathcal{C}^\bullet$$

is a map of $Z_q\mathcal{C}^0$-modules, so we are done by functoriality. $\square$

Lemma 5.2. *Let J be a permutative category. If $\mathcal{C}^\bullet$ is a J-graded bipermutative category, then Thomason's homotopy colimit of permutative categories $\operatorname{hocolim}_J \mathcal{C}^\bullet$ is a zeroless bipermutative category. The natural map $\mathcal{C}^0 \to \operatorname{hocolim}_J \mathcal{C}^\bullet$ is a lax map of zeroless bipermutative categories. Furthermore, for each $i \in J$, the canonical map*

$$\mathcal{C}^i \longrightarrow \operatorname{hocolim}_J \mathcal{C}^\bullet$$

is a map of $\mathcal{C}^0$-modules.

If $\mathcal{C}^\bullet$ is a J-graded strictly bimonoidal category, then $\operatorname{hocolim}_J \mathcal{C}^\bullet$ is a zeroless strictly bimonoidal category with a lax map of zeroless strictly bimonoidal categories $\mathcal{C}^0 \to \operatorname{hocolim}_J \mathcal{C}^\bullet$, and $\mathcal{C}^0$-module maps $\mathcal{C}^i \to \operatorname{hocolim}_J \mathcal{C}^\bullet$.

Proof. Thomason showed that the homotopy colimit is a permutative category without zero. There is an obvious twist map

$$\begin{aligned}\tau_\oplus\colon n[(x_1,X_1),\dots,(x_n,X_n)] \oplus m[(y_1,Y_1),\dots,(y_m,Y_m)]&\\ \to m[(y_1,Y_1),\dots,(y_m,Y_m)] \oplus n[(x_1,X_1),\dots,(x_n,X_n)]&\end{aligned}$$

that is given by $(\chi(n,m),\mathrm{id},\mathrm{id})$.

For convenience we introduce the following symbolic notation: let $[X]$ be shorthand notation for $n[(x_1,X_1),\dots,(x_n,X_n)]$ and similarly $[Y]$ for $m[(y_1,Y_1),\dots,(y_m,Y_m)]$. Then we denote $[X]\oplus[Y]$ by $\left[\begin{smallmatrix}X\\Y\end{smallmatrix}\right]$, which should be read as "first X then Y". The twist map $\tau_\oplus$ is then symbolically given by

$$\tau_\oplus\colon \begin{bmatrix}X\\Y\end{bmatrix} \cong \begin{bmatrix}Y\\X\end{bmatrix}.$$

In order to distinguish the multiplicative structure of $\mathcal{C}^\bullet$ from the one on the homotopy colimit, we denote the bifunctor $\otimes$ on $\mathcal{C}^\bullet$ by $\cdot$ or just by juxtaposition of objects. The multiplicative bifunctor $\otimes$ on the homotopy colimit is then given by matrix multiplication. We define

$$\begin{aligned}&n[(x_1,X_1),\dots,(x_n,X_n)] \otimes m[(y_1,Y_1),\dots,(y_m,Y_m)]\\ &\qquad := nm[(x_1+y_1,X_1Y_1),\dots,(x_1+y_m,X_1Y_m),\dots,(x_n+y_1,X_nY_1),\dots,(x_n+y_m,X_nY_m)].\end{aligned}$$

Again, we use shorthand notation for that and write

$$[XY] := [X]\otimes[Y].$$

The element $1 := 1[(0,1)]$ is a unit for $\otimes$. With this structure $(\operatorname{hocolim}_J \mathcal{C}^\bullet, \otimes, 1)$ is a strict monoidal category.

We define the twist map $\tau_\otimes$ for $\otimes$ as follows, as a composite of two morphisms. Let $\gamma_\otimes$ denote the twist map for the multiplication in $\mathcal{C}^\bullet$. First, we apply $\gamma_\otimes$ on every entry of the form X_iY_j. The triple $(\mathrm{id}_{\{1,\ldots,nm\}}, c_J^{x_i,y_j}, \gamma_\otimes)$ defines a morphism

$$\begin{aligned} nm&[(x_1+y_1, X_1Y_1), \ldots, (x_1+y_m, X_1Y_m), \ldots, (x_n+y_1, X_nY_1), \ldots, (x_n+y_m, X_nY_m)] \\ &\to nm[(y_1+x_1, Y_1X_1), \ldots, (y_m+x_1, Y_mX_1), \ldots, (y_1+x_n, Y_1X_n), \ldots, (y_m+x_n, Y_mX_n)] \end{aligned}$$

where c_J is the twist in the permutative category J. (To be precise, $\gamma_\otimes$ maps X_iY_j to $(c_J^{y_j,x_i})_*(Y_jX_i)$, whereas the third coordinate of the morphism should map $(c_J^{x_i,y_j})_*(X_iY_j)$ to Y_jX_i, so $\gamma_\otimes$ is really an abbreviation for $(c_J^{x_i,y_j})_*(\gamma_\otimes)$.)

Second, we postcompose these maps with the morphism given by $(\sigma_{n,m}, \mathrm{id}_{y_j+x_i}, \mathrm{id})$, where $\sigma_{n,m} \in \Sigma_{nm}$ is the permutation that induces matrix transposition.

We write the composition $\tau_\otimes = (\sigma_{n,m}, \mathrm{id}_{y_j+x_i}, \mathrm{id}) \circ (\mathrm{id}_{\{1,\ldots,nm\}}, c_J^{x_i,y_j}, \gamma_\otimes)$ symbolically as

$$\tau_\otimes \colon [XY] \cong [YX]\,.$$

As matrix transposition squares to the identity, $c_J^{y_j,x_i} \circ c_J^{x_i,y_j} = \mathrm{id}$ and $\gamma_\otimes^2 = \mathrm{id}$, we obtain that $\tau_\otimes^2 = \mathrm{id}$. If X is the multiplicative unit, then we have that $\sigma_{1,m}$ is the identity in Σ_m and c_J^{0,y_j} is the identity as well. Similarly one shows that $\tau_\otimes$ gives the identity morphism if Y is the multiplicative unit. We leave it to the reader to check the remaining properties of 2.1 (4).

Writing out $([X]\otimes[Y]) \oplus ([X']\otimes[Y])$ and $([X]\oplus[X'])\otimes Y$ we get the same object, and we define left distributivity d_ℓ to be the identity map between these two expressions. The right distributivity d_r involves a reordering of elements. We have to have

$$d_r \colon ([X]\otimes[Y]) \oplus ([X]\otimes[Y']) = \begin{bmatrix} XY \\ XY' \end{bmatrix} \longrightarrow [X]\otimes([Y]\oplus[Y']) = [X]\begin{bmatrix} Y \\ Y' \end{bmatrix} = [XY, XY']\,.$$

Here $[XY, XY']$ is shorthand notation for

$$n(m+m')[(x_1+y_1, X_1Y_1), \ldots, (x_1+y'_{m'}, X_1Y'_{m'}), \ldots, (x_n+y_1, X_nY_1), \ldots, (x_n+y'_{m'}, X_nY'_{m'})]\,.$$

The elements in the source occur in the ordering

$$(nm+nm')[(x_1+y_1, X_1Y_1), \ldots, (x_n+y_m, X_nY_m), (x_1+y'_1, X_1Y'_1), \ldots, (x_n+y'_{m'}, X_nY'_{m'})]\,,$$

thus the source and the target do *not* agree, but they differ by a suitable permutation $\xi \in \Sigma_{nm+nm'}$. Thus we define d_r as $(\xi, \mathrm{id}, \mathrm{id})$. Note that ξ is the right distributivity isomorphism in the bipermutative category of finite sets and surjective maps as defined in Example 2.4.

We have to check that the so defined distributivity transformation d_r coincides with $\tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes)$. The twist terms $\gamma_\otimes$ and c_J occur twice in the composition, so they reduce to the identity. What is left is a permutation that is caused by $\tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes)$ and this is precisely ξ.

Since the isomorphisms d_ℓ, d_r and $\tau_\oplus$ are all of the form $(\sigma, \mathrm{id}, \mathrm{id})$ for some permutation σ, properties (8) to (10) of Definition 2.1 follow from the ones in the bipermutative category of finite sets and surjections.

This finishes the proof that the zeroless bipermutative category structure works fine on objects.

We have to establish that $\oplus$ and $\otimes$ are bifunctors on $\mathrm{hocolim}_J\,\mathcal{C}^\bullet$, and that the associativity and distributivity laws and the additive and multiplicative twists are natural.

For $\oplus$ this is straightforward and can be found in [Th3]: suppose given two morphisms

$$(\psi, \ell_i, \varrho_j) \colon n[(x_1, X_1), \ldots, (x_n, X_n)] \to n'[(x'_1, X'_1), \ldots, (x'_{n'}, X'_{n'})]$$

and

$$(\varphi, k_i, \pi_j) \colon m[(y_1, Y_1), \ldots, (y_m, Y_m)] \to m'[(y'_1, Y'_1), \ldots, (y'_{m'}, Y'_{m'})]$$

in the homotopy colimit, with $\psi \colon \mathbf{n} \to \mathbf{n}'$, $\ell_i : x_i \to x'_{\psi(i)}$ and $\varrho_j : \bigoplus_{\psi(i)=j} \mathcal{C}(\ell_i)(X_i) \to X'_j$, and $\varphi \colon \mathbf{m} \to \mathbf{m}'$ with corresponding k_i and π_j. Then there is a surjection $\psi + \varphi$ from $\mathbf{n}+\mathbf{m}$ to $\mathbf{n}'+\mathbf{m}'$, and we can recycle the morphisms ℓ_i and k_i to give corresponding morphisms in J. In the third coordinate we can use the morphisms ϱ_j and π_j to get new ones, because the preimages of $\mathbf{n}'$ and $\mathbf{m}'$ under ψ and φ are disjoint. Taken together, this results in a morphism from the sum $(n+m)[(x_1, X_1), \ldots, (y_m, Y_m)]$ to the sum $(n'+m')[(x'_1, X'_1), \ldots, (y'_{m'}, Y'_{m'})]$. It is straightforward to see that $\oplus$ defines a bifunctor, that the associativity law for $\oplus$ is natural, and that the additive twist $\tau_\oplus$ is natural.

For the remainder of this proof let us denote the elements in the set $\mathbf{nm} = \{1, \ldots, nm\}$ as pairs (i,j) with $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. The tensor product of the maps $(\psi, \ell_i, \varrho_j)$ and (φ, k_i, π_j) has three coordinates. On the first, we take the product of the surjections, *i.e.*,

$$\mathbf{nm} \ni (i,j) \mapsto (\psi(i), \varphi(j)) \in \mathbf{n}'\mathbf{m}',$$

and on the second we take the sum $\ell_i + k_j \colon x_i + y_j \to x'_{\psi(i)} + y'_{\varphi(j)}$ of the maps ℓ_i and k_j in J.

The third coordinate of the morphism $(\psi, \ell_i, \varrho_j) \otimes (\varphi, k_i, \pi_j)$ has to be a map

$$\bigoplus_{(\psi(i),\varphi(j))=(r,s)} \mathcal{C}(\ell_i + k_j)(X_i \cdot Y_j) = \bigoplus_{(\psi(i),\varphi(j))=(r,s)} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \longrightarrow X'_r \cdot Y'_s\,.$$

Here, the sum is taken with respect to the lexicographical ordering of the indices (i,j). Consider the following diagram.

$$\begin{tikzcd}
& \bigoplus_{(\psi(i),\varphi(j))=(r,s)} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \arrow[dl, "\mathrm{id}"'] \arrow[dr, "\sigma"] & \\
\bigoplus_{\psi(i)=r} \bigoplus_{\varphi(j)=s} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \arrow[d, "\bigoplus_{\psi(i)=r} d_r"'] & & \bigoplus_{\varphi(j)=s} \bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot \mathcal{C}(k_j)(Y_j) \arrow[d, "\bigoplus_{\varphi(j)=s} d_\ell"] \\
\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot \Big(\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j)\Big) \arrow[d, "\bigoplus_{\psi(i)=r} \mathrm{id}_{\mathcal{C}(\ell_i)(X_i)} \cdot \pi_s"'] & & \bigoplus_{\varphi(j)=s} \Big(\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i)\Big) \cdot \mathcal{C}(k_j)(Y_j) \arrow[d, "\bigoplus_{\varphi(j)=s} \varrho_r \cdot \mathrm{id}_{\mathcal{C}(k_j)(Y_j)}"] \\
\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i) \cdot Y'_s \arrow[d, "d_\ell"'] & & \bigoplus_{\varphi(j)=s} X'_r \cdot \mathcal{C}(k_j)(Y_j) \arrow[d, "d_r"] \\
\Big(\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i)\Big) \cdot Y'_s \arrow[dr, "\varrho_r \otimes \mathrm{id}"'] & & X'_r \cdot \Big(\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j)\Big) \arrow[dl, "\mathrm{id} \otimes \pi_s"] \\
& X'_r \cdot Y'_s &
\end{tikzcd}$$

The σ is an appropriate permutation of the summands.

The distributivity laws in $\mathcal{C}^\bullet$ are natural with respect to morphisms in $\mathcal{C}^\bullet$ and therefore we have that

$$d_\ell \circ \left(\bigoplus_{\psi(i)=r} \mathrm{id}_{\mathcal{C}(\ell_i)(X_i)} \cdot \pi_s\right) = \left(\left(\mathrm{id}_{\bigoplus_{\psi(i)=r} \mathcal{C}(\ell_i)(X_i)}\right) \cdot \pi_s\right) \circ d_\ell\,,$$
$$d_r \circ \left(\bigoplus_{\varphi(j)=s} \varrho_r \cdot \mathrm{id}_{\mathcal{C}(k_j)(Y_j)}\right) = \left(\varrho_r \cdot \left(\mathrm{id}_{\bigoplus_{\varphi(j)=s} \mathcal{C}(k_j)(Y_j)}\right)\right) \circ d_r\,.$$

We use the generalized pentagon equation

$$d_\ell \circ \bigoplus_{\psi(i)=r} d_r = d_r \circ \bigoplus_{\varphi(j)=s} d_\ell \circ \sigma$$

to see that the diagram commutes. We define the tensor product of the two maps to be the composition given by either of the branches.

Note that for $(\psi, \ell_i, \varrho_j) \otimes \mathrm{id}$ the definition reduces to $(\varrho_j \cdot \mathrm{id}) \circ d_\ell$, and similarly the third coordinate of $\mathrm{id} \otimes (\varphi, k_i, \pi_j)$ is $(\mathrm{id} \cdot \pi_j) \circ d_r$. In particular, the tensor product of identity morphisms is an identity morphism.

Compositions of morphisms in the homotopy colimit involve an additive twist [Th3, p. 1631]. For

$$(\psi', \ell'_i, \varrho'_j) \colon n'[(x'_1, X'_1), \ldots, (x'_{n'}, X'_{n'})] \to n''[(x''_1, X''_1), \ldots, (x''_{n''}, X''_{n''})]$$

the morphism $\bigoplus_{\psi'\psi(i)=r} \mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i) \longrightarrow X''_r$ is given as a composition. First, one has to permute the summands

$$\sigma\colon \bigoplus_{\psi'\psi(i)=r} \mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i) \to \bigoplus_{\psi'(k)=r} \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_k\ell_i)(X_i)\,.$$

Then, as we assumed that $\mathcal{C}$ is a functor to Perm(Strict), we know that

$$\bigoplus_{\psi'(k)=r}\ \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_k\ell_i)(X_i) = \bigoplus_{\psi'(k)=r}\ \bigoplus_{\psi(i)=k} \mathcal{C}(\ell'_k)\mathcal{C}(\ell_i)(X_i) = \bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)\left(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i)(X_i)\right).$$

Finally, we apply the morphism

$$\bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)(\varrho_k)\colon \bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)\left(\bigoplus_{\psi(i)=k} \mathcal{C}(\ell_i)(X_i)\right) \longrightarrow \bigoplus_{\psi'(k)=r} \mathcal{C}(\ell'_k)(X'_k)$$

and prolong this map with ϱ'_r to end up in X''_r.

In order to prove that the tensor product actually defines a bifunctor, we will show that

$$(\psi,\ell_i,\varrho_j)\otimes(\varphi,k_i,\pi_j) = ((\psi,\ell_i,\varrho_j)\otimes \mathrm{id})\circ(\mathrm{id}\otimes(\varphi,k_i,\pi_j)) = (\mathrm{id}\otimes(\varphi,k_i,\pi_j))\circ((\psi,\ell_i,\varrho_j)\otimes\mathrm{id})$$

and

$$((\psi',\ell'_i,\varrho'_j)\otimes\mathrm{id})\circ((\psi,\ell_i,\varrho_j)\otimes\mathrm{id}) = ((\psi',\ell'_i,\varrho'_j)\circ(\psi,\ell_i,\varrho_j))\otimes\mathrm{id}$$

and leave it to the reader to check the remaining identity.

The first equation is straightforward to see, because $((\psi,\ell_i,\varrho_j)\otimes\mathrm{id})\circ(\mathrm{id}\otimes(\varphi,k_i,\pi_j))$ corresponds to the left branch of the diagram above and the other composition is given by the right branch.

For the second equation we have to check that $(((\varrho'\circ\varrho)\cdot\mathrm{id})\circ d_\ell)_s = ((\varrho'\cdot\mathrm{id})\circ d_\ell\circ(\varrho\cdot\mathrm{id})\circ d_\ell)_s$. Both morphisms have source

$$\bigoplus_{\psi'\psi(i)=s} \mathcal{C}(\ell'_{\psi(i)}\ell_i+\mathrm{id})(X_i\cdot Y_j) = \bigoplus_{\psi'\psi(i)=s} \mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i)\cdot Y_j$$

and the left hand side corresponds to the left branch of the following diagram and the right hand side to the right branch.

$$\begin{tikzcd}
\bigoplus_{\psi'\psi(i)=s}\mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i)\cdot Y_j \arrow[rr, equal] \arrow[d, "d_\ell"'] & & \bigoplus_{\psi'\psi(i)=s}\mathcal{C}(\ell'_{\psi(i)}\ell_i+\mathrm{id})(X_i\cdot Y_j) \arrow[d, "\sigma"] \\
\left(\bigoplus_{\psi'\psi(i)=s}\mathcal{C}(\ell'_{\psi(i)}\ell_i)(X_i)\right)\cdot Y_j \arrow[d, "\sigma\cdot\mathrm{id}"'] & & \bigoplus_{\psi'(k)=s}\bigoplus_{\psi(i)=k}\mathcal{C}(\ell'_{\psi(i)}+\mathrm{id})\mathcal{C}(\ell_i+\mathrm{id})(X_i\cdot Y_j) \arrow[d, equal] \\
\left(\bigoplus_{\psi'(k)=s}\bigoplus_{\psi(i)=k}\mathcal{C}(\ell'_{\psi(i)})\mathcal{C}(\ell_i)(X_i)\right)\cdot Y_j \arrow[d, equal] & & \bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k+\mathrm{id})\left(\bigoplus_{\psi(i)=k}\mathcal{C}(\ell_i+\mathrm{id})(X_i\cdot Y_j)\right) \arrow[d, "\bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k+\mathrm{id})((\varrho_k\cdot\mathrm{id})\circ d_\ell)"] \\
\left(\bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k)\left(\bigoplus_{\psi(i)=k}\mathcal{C}(\ell_i)(X_i)\right)\right)\cdot Y_j \arrow[d, "\left(\bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k)(\varrho_k)\right)\cdot\mathrm{id}"'] & & \bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k)(X'_k)\cdot Y_j \arrow[d, "d_\ell"] \\
\left(\bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k)(X'_k)\right)\cdot Y_j \arrow[rr, equal] \arrow[dr, "\varrho'_s\cdot\mathrm{id}"'] & & \left(\bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k)(X'_k)\right)\cdot Y_j \arrow[dl, "\varrho'_s\cdot\mathrm{id}"] \\
& X''_s\cdot Y_j &
\end{tikzcd}$$

Naturality of d_ℓ in $\mathcal{C}^\bullet$ ensures that d_ℓ can change place with $\bigoplus_{\psi'(k)=s}\mathcal{C}(\ell'_k+\mathrm{id})(\varrho_k\cdot\mathrm{id})$ on the right branch. That $d_\ell\circ\sigma = (\sigma\cdot\mathrm{id})\circ d_\ell$ holds because $\mathcal{C}^\bullet$ satisfies property (8) from Definition 2.1 and hence the diagram commutes.

In order to show that the associativity identification is natural, we have to prove that

$$((\psi^1,\ell^1_i,\varrho^1_j)\otimes(\psi^2,\ell^2_i,\varrho^2_j))\otimes(\psi^3,\ell^3_i,\varrho^3_j) = (\psi^1,\ell^1_i,\varrho^1_j)\otimes((\psi^2,\ell^2_i,\varrho^2_j)\otimes(\psi^3,\ell^3_i,\varrho^3_j))$$

for morphisms in the homotopy colimit. The claim is obvious on the coordinates of the surjections and the morphisms in J.

For proving the identity in the third coordinate of morphisms, note that the naturality of $\otimes$ implies that we can write

$$\begin{aligned}
&((\psi^1,\ell_i^1,\varrho_j^1)\otimes(\psi^2,\ell_i^2,\varrho_j^2))\otimes(\psi^3,\ell_i^3,\varrho_j^3)\\
=&(((\psi^1,\ell_i^1,\varrho_j^1)\otimes \mathrm{id})\otimes \mathrm{id})\circ((\mathrm{id}\otimes(\psi^2,\ell_i^2,\varrho_j^2))\otimes \mathrm{id})\circ((\mathrm{id}\otimes \mathrm{id})\otimes(\psi^3,\ell_i^3,\varrho_j^3))\,.
\end{aligned}$$

Therefore, it suffices to prove the claim for each of the factors. We will show it for the middle one and leave the other ones to the curious reader. Recall that $\mathrm{id}\otimes(\psi^2,\ell_i^2,\varrho_j^2)$ has as third coordinate the composition $(\mathrm{id}\cdot\varrho_j^2)\circ d_r$ and therefore $(\mathrm{id}\otimes(\psi^2,\ell_i^2,\varrho_j^2))\otimes\mathrm{id}$ has third coordinate

$$(((\mathrm{id}\cdot\varrho_j^2)\circ d_r)\cdot\mathrm{id})\circ d_\ell=(\mathrm{id}\cdot\varrho_j^2\cdot\mathrm{id})\circ(\mathrm{id}\cdot d_r)\circ d_\ell\,.$$

But $(\mathrm{id}\cdot d_r)\circ d_\ell=(d_\ell\cdot\mathrm{id})\circ d_r$ (equation (7') of Definition 2.5) holds in $\mathcal{C}^\bullet$, and therefore the third coordinate equals

$$(\mathrm{id}\cdot\varrho_j^2\cdot\mathrm{id})\circ(\mathrm{id}\cdot d_r)\circ d_\ell=(\mathrm{id}\cdot\varrho_j^2\cdot\mathrm{id})\circ(d_\ell\cdot\mathrm{id})\circ d_r$$

which is the third coordinate of $\mathrm{id}\otimes((\psi^2,\ell_i^2,\varrho_j^2)\otimes\mathrm{id})$.

Naturality of the multiplicative twist map can be seen as follows. We have to show that

$$\tau_\otimes\circ((\psi,\ell_i,\varrho_j)\otimes(\varphi,k_i,\pi_j))=((\psi,\ell_i,\varrho_j)\otimes(\varphi,k_i,\pi_j))\circ\tau_\otimes\,.$$

On the first coordinate of the morphisms this reduces to the equality

$$\sigma_{n',m'}\circ(\psi,\varphi)(i,j)=(\varphi(j),\psi(i))=(\varphi,\psi)\circ\sigma_{n,m}(i,j),$$

and on the second coordinate we have the equation

$$c_J\circ(\ell_i+k_j)=(k_j+\ell_i)\circ c_J$$

because c_J is natural. Thus, it remains to prove that the above equation holds in the third coordinate, which amounts to showing that the following diagram commutes.

$$\begin{tikzcd}
\bigoplus_{\psi(i)=r}\bigoplus_{\varphi(j)=s}\mathcal{C}(\ell_i)(X_i)\cdot\mathcal{C}(k_j)(Y_j) \arrow[r, "(\bigoplus\bigoplus\gamma_\otimes)\circ\sigma"] \arrow[dd, "\bigoplus_{\psi(i)=r} d_r"'] & \bigoplus_{\varphi(j)=s}\bigoplus_{\psi(i)=r}\mathcal{C}(k_j)(Y_j)\cdot\mathcal{C}(\ell_i)(X_i) \arrow[d, "\sigma^{-1}"] \\
 & \bigoplus_{\psi(i)=r}\bigoplus_{\varphi(j)=s}\mathcal{C}(k_j)(Y_j)\cdot\mathcal{C}(\ell_i)(X_i) \arrow[d, "\bigoplus_{\psi(i)=r} d_\ell"] \\
\bigoplus_{\psi(i)=r}\mathcal{C}(\ell_i)(X_i)\cdot\left(\bigoplus_{\varphi(j)=s}\mathcal{C}(k_j)(Y_j)\right) \arrow[r, dotted, "\bigoplus_{\psi(i)=r}\gamma_\otimes"] \arrow[d, "\bigoplus_{\psi(i)=r}\mathrm{id}\cdot\pi_s"'] & \bigoplus_{\psi(i)=r}\left(\bigoplus_{\varphi(j)=s}\mathcal{C}(k_j)(Y_j)\right)\cdot\mathcal{C}(\ell_i)(X_i) \arrow[d, "\bigoplus_{\psi(i)=r}\pi_s\cdot\mathrm{id}"] \\
\bigoplus_{\psi(i)=r}\mathcal{C}(\ell_i)(X_i)\cdot Y_s' \arrow[d, "d_\ell"'] & \bigoplus_{\psi(i)=r}Y_s'\cdot\mathcal{C}(\ell_i)(X_i) \arrow[d, "d_r"] \\
\left(\bigoplus_{\psi(i)=r}\mathcal{C}(\ell_i)(X_i)\right)\cdot Y_s' \arrow[d, "\varrho_r\cdot\mathrm{id}"'] & Y_s'\cdot\left(\bigoplus_{\psi(i)=r}\mathcal{C}(\ell_i)(X_i)\right) \arrow[d, "\mathrm{id}\cdot\varrho_r"] \\
X_r'\cdot Y_s' \arrow[r, "\gamma_\otimes"] & Y_s'\cdot X_r'
\end{tikzcd}$$

The top diagram commutes because d_r is defined in terms of d_ℓ and $\gamma_\otimes$. For the bottom diagram we apply the same argument together with the naturality of $\gamma_\otimes$.

We have to check that left distributivity is the identity on morphisms. Consider three morphisms as above. When we focus on the surjections $\psi^1\colon n\to n'$, $\psi^2\colon m\to m'$, and $\psi^3\colon \ell\to\ell'$, we see that a condition like $(\psi^1+\psi^2)\psi^3(i,j)=(r,s)$ only affects either the preimage of $n'\ell'$ or the preimage of $m'\ell'$ in $(n+m)\ell$, but never both. Therefore, the third coordinate of the morphism

$$((\psi^1,\ell_i^1,\varrho_j^1)\oplus(\psi^2,\ell_i^2,\varrho_j^2))\otimes(\psi^3,\ell_i^3,\varrho_j^3)$$

is either a third coordinate of $(\psi^1,\ell_i^1,\varrho_j^1)\otimes(\psi^3,\ell_i^3,\varrho_j^3)$ or of $(\psi^2,\ell_i^2,\varrho_j^2)\otimes(\psi^3,\ell_i^3,\varrho_j^3)$ and thus left distributivity is the identity on morphisms.

In the J-graded bipermutative case the naturality of the right distributivity isomorphism follows from the one of d_ℓ and the multiplicative twist. In the bipermutative and the strictly bimonoidal case right distributivity is given by $(\xi, \mathrm{id}, \mathrm{id})$. Therefore naturality of d_r in the bipermutative setting proves naturality in the strictly bimonoidal setting as well.

This finishes the proof that the homotopy colimit $\mathrm{hocolim}_J\, \mathcal{C}^\bullet$ is a bipermutative category without zero.

We now prove the remaining statements of the lemma. There is a natural functor G from $\mathcal{C}^0$ to $\mathrm{hocolim}_J\, \mathcal{C}^\bullet$ which sends $X \in \mathcal{C}^0$ to $1[(0, X)]$. Note that the functor G is strict (symmetric) monoidal with respect to $\otimes$, because $G(1) = 1[(0, 1)]$ and

$$G(X) \otimes G(Y) = 1[(0, X)] \otimes 1[(0, Y)] = 1[(0 + 0, X \otimes Y)] = 1[(0, X \otimes Y)] = G(X \otimes Y)\,.$$

However, G is only lax symmetric monoidal with respect to $\oplus$: there is a binatural morphism $\eta_\oplus$ from $G(X) \oplus G(Y) = 1[(0, X)] \oplus 1[(0, Y)] = 2[(0, X), (0, Y)]$ to $G(X \oplus Y) = 1[(0, X \oplus Y)]$ given by the canonical surjection ψ from $\mathbf{2}$ to $\mathbf{1}$ and identity morphisms in the other two components, but of course this map is not an isomorphism. We have to show that the functor G respects the distributivity constraints $d_\ell = \mathrm{id}$ and d_r. In our situation we have that $\eta_\otimes = \mathrm{id}$, thus we have to check that

$$\eta_\oplus = \eta_\oplus \otimes \mathrm{id}$$

and

$$(\mathrm{id} \otimes \eta_\oplus) \circ \tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes) = G(\tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes)) \circ \eta_\oplus\,.$$

The first equation is just stating the fact that

$$\begin{tikzcd}
2[(0, XY), (0, X'Y)] \arrow[r, equal] \arrow[d, "\eta_\oplus = (\psi, \mathrm{id}, \mathrm{id})"'] & 2[(0, X), (0, X')] \otimes 1[(0, Y)] \arrow[d, "\eta_\oplus \otimes \mathrm{id} = (\psi, \mathrm{id}, \mathrm{id}) \otimes \mathrm{id}"] \\
1[0, XY \oplus X'Y] \arrow[r, equal] & 1[0, (X \oplus X')Y]
\end{tikzcd}$$

commutes.

For the right distributivity law we should observe that the multiplicative twist $\tau_\otimes$ on the homotopy colimit reduces to the morphism $(\mathrm{id}, c_J, \gamma_\otimes)$ in the case of elements of length 1 in the homotopy colimit, and that $c_J^{0,0} = \mathrm{id}$. Furthermore, $\mathrm{id} \otimes (\psi, \mathrm{id}, \mathrm{id}) = (\psi, \mathrm{id}, \mathrm{id})$ holds. Therefore

$$\begin{aligned}
(\mathrm{id} \otimes \eta_\oplus) \circ d_r =& (\mathrm{id} \otimes (\psi, \mathrm{id}, \mathrm{id})) \circ \tau_\otimes \circ (\tau_\otimes \oplus \tau_\otimes) \\
=& (\psi, \mathrm{id}, \mathrm{id}) \circ (\mathrm{id}, \mathrm{id}, \gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)) \\
=& (\mathrm{id}, \mathrm{id}, \gamma_\otimes \circ (\gamma_\otimes \oplus \gamma_\otimes)) \circ (\psi, \mathrm{id}, \mathrm{id}) = G(d_r) \circ \eta_\oplus\,.
\end{aligned}$$

The claim about the module structure is obvious.

As the right distributivity on the homotopy colimit is of the form $(\xi, \mathrm{id}, \mathrm{id})$, the above proof carries over to the strictly bimonoidal case. $\square$

Lemma 5.3. *If $g\colon \mathcal{C}^\bullet \to \mathcal{D}^\bullet$ is a lax morphism of J-graded bipermutative categories (resp. J-graded strictly bimonoidal categories) then it induces a lax morphism of zeroless bipermutative categories (resp. zeroless strictly bimonoidal categories) $g_*\colon \mathrm{hocolim}_J\, \mathcal{C}^\bullet \to \mathrm{hocolim}_J\, \mathcal{D}^\bullet$.*

Proof. Of course, we define $g_*\colon \mathrm{hocolim}_J\, \mathcal{C}^\bullet \to \mathrm{hocolim}_J\, \mathcal{D}^\bullet$ as

$$g_*(n[(x_1, A_1), \ldots, (x_n, A_n)]) := n[(x_1, g(A_1)), \ldots, (x_n, g(A_n))]\,.$$

Note that with this definition g_* is *strict* symmetric monoidal with respect to $\oplus$ even if g was only lax symmetric monoidal.

For a morphism $(\psi, \ell_i, \varrho_j)$ from $n[(x_1, A_1), \ldots, (x_n, A_n)]$ to $m[(y_1, B_1), \ldots, (y_m, B_m)]$ we define an induced morphism

$$g_*(n[(x_1, A_1), \ldots, (x_n, A_n)]) \to g_*(m[(y_1, B_1), \ldots, (y_m, B_m)])$$

as follows: we keep the surjection ψ and the maps ℓ_i. For

$$\varrho_j\colon \bigoplus_{\psi(i)=j} A_i \to B_j$$

we take the composition

$$\varrho_j^g\colon \bigoplus_{\psi(i)=j} g(A_i) \xrightarrow{\eta_\oplus} g(\bigoplus_{\psi(i)=j} A_i) \xrightarrow{g(\varrho_j)} g(B_j)$$

and obtain a morphism $(\psi, \ell_i, \varrho_j^g)$ on the homotopy colimit. The naturality of $\eta_\oplus$ ensures that composition of morphisms is well-defined.

Let $n[(x_1, A_1), \ldots, (x_n, A_n)]$ and $m[(y_1, B_1), \ldots, (y_m, B_m)]$ be two objects in $\mathrm{hocolim}_J\, \mathcal{C}^\bullet$. Applying $g_* \circ (- \otimes -)$ yields

$$nm[(x_1 + y_1, g(A_1 \otimes B_1)), \ldots, (x_n + y_m, g(A_n \otimes B_m))]$$

whereas the composition $(- \otimes -) \circ (g_*, g_*)$ gives

$$nm[(x_1 + y_1, g(A_1) \otimes g(B_1)), \ldots, (x_n + y_m, g(A_n) \otimes g(B_m))] \,.$$

Thus, we can use $(\mathrm{id}, \mathrm{id}, \eta_\otimes)$ to obtain a natural transformation $\eta_\otimes^h$ from $(-\otimes-)\circ(g_*, g_*)$ to $g_* \circ (-\otimes-)$. This transformation inherits all properties from $\eta_\otimes$, in particular, $\eta_\otimes^h$ is lax symmetric monoidal if $\eta_\otimes$ was so.

It remains to check the properties concerning the distributivity laws. As d_ℓ is the identity on the J-graded bipermutative category and on the homotopy colimit, and $\eta_\oplus$ is the identity on the homotopy colimit, the equalities reduce to

$$(\eta_\otimes^h \oplus \eta_\otimes^h) = \eta_\otimes^h \tag{2}$$

and

$$g_*(d_r) \circ (\eta_\otimes^h \oplus \eta_\otimes^h) = \eta_\otimes^h \circ d_r \,. \tag{3}$$

The first equation is straightforward to check.

The right distributivity law in the homotopy colimit is given by $d_r = (\xi, \mathrm{id}, \mathrm{id})$ and $\eta_\otimes^h \oplus \eta_\otimes^h$ is equal to

$$\eta_\otimes^h \oplus \eta_\otimes^h = (\mathrm{id}_{\mathbf{nm}}, \mathrm{id}_{x_i + y_j}, \eta_\otimes) \oplus (\mathrm{id}_{\mathbf{nm}'}, \mathrm{id}_{x_i + z_k}, \eta_\otimes) \,.$$

As addition in the homotopy colimit is given by concatenation, this shows that we can simplify the above expression to $(\mathrm{id}_{\mathbf{nm}+\mathbf{nm}'}, \mathrm{id}_{x_i+u_r}, \eta_\otimes)$ where u_r is either of the form y_j or z_k. As d_r differs from the identity only in the first coordinate, and $\eta_\otimes^h \oplus \eta_\otimes^h$ only in the third coordinate, these maps commute. □

6. A multiplicative group completion device

Collecting the results we are ready to define our multiplicative group completion.

Theorem 6.1. *If $\mathcal{R}$ is a commutative rig category (or a rig category) satisfying the conditions of Theorem 1.1, then*

$$\bar{\mathcal{R}} = D\, \mathrm{hocolim}_I (-\mathcal{R})^\bullet \mathcal{R}$$

is a simplicial commutative ring category (resp. a simplicial ring category), where $(-\mathcal{R})^\bullet\mathcal{R}$ is the I-graded bipermutative category (resp. I-graded strictly bimonoidal category) of Proposition 3.9 applied to the bipermutative category (resp. strictly bimonoidal category) associated with $\mathcal{R}$.

The rig maps

$$\mathcal{R} \xleftarrow{\sim} Z\mathcal{R} \longrightarrow \bar{\mathcal{R}}$$

are stable equivalences. Furthermore, the maps

$$(-\mathcal{R})\mathcal{R} \xleftarrow{\sim} Z(-\mathcal{R})\mathcal{R} \xrightarrow{\sim} \bar{\mathcal{R}}$$

form a chain of unstable equivalences of $Z\mathcal{R}$-modules.

Proof. We may assume that $\mathcal{R}$ is a bipermutative category or a strictly bimonoidal category. By Lemma 3.7 we know that $\mathcal{R} \to (-\mathcal{R})\mathcal{R}$ is a stable equivalence, and that for each $\varphi\colon \mathbf{m} \to \mathbf{n}$ with $m > 0$ the induced map $\varphi_*\colon (-\mathcal{R})^m\mathcal{R} \to (-\mathcal{R})^n\mathcal{R}$ is an unstable equivalence, and so we are done by Lemma 4.2 and Proposition 5.1. □

7. Connection to $GL_n(-\mathcal{R})\mathcal{R}$

Let $\mathcal{R}$ be a strictly bimonoidal category.

Definition 7.1. The *category of $n \times n$-matrices over $\mathcal{R}$*, $M_n(\mathcal{R})$, is defined as follows. The objects of $M_n(\mathcal{R})$ are matrices $X = (X_{i,j})_{i,j=1}^n$ of objects of $\mathcal{R}$ and morphisms from $X = (X_{i,j})_{i,j=1}^n$ to $Y = (Y_{i,j})_{i,j=1}^n$ are matrices $F = (F_{i,j})_{i,j=1}^n$ where each $F_{i,j}$ is a morphism in $\mathcal{R}$ from $X_{i,j}$ to $Y_{i,j}$.

Lemma 7.2. *For a strictly bimonoidal category $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, c_{\oplus}, \otimes, 1_{\mathcal{R}})$ the category $M_n(\mathcal{R})$ is a monoidal category with respect to the matrix multiplication bifunctor*

$$M_n(\mathcal{R}) \times M_n(\mathcal{R}) \xrightarrow{\cdot} M_n(\mathcal{R})$$

$$(X_{i,j})_{i,j=1}^n \cdot (Y_{i,j})_{i,j=1}^n = (Z_{i,j})_{i,j=1}^n \quad \textit{with } Z_{i,j} = \bigoplus_{k=1}^n X_{i,k} \otimes Y_{k,j}\,.$$

The unit of this structure is given by the unit matrix object E_n which has $1_{\mathcal{R}} \in \mathcal{R}$ as diagonal entries and $0_{\mathcal{R}} \in \mathcal{R}$ in the other places.

The property of $\mathcal{R}$ being bimonoidal gives $\pi_0(\mathcal{R})$ the structure of a rig, and its (additive) group completion $Gr(\pi_0(\mathcal{R})) = (-\pi_0\mathcal{R})\pi_0\mathcal{R}$ is a ring.

Definition 7.3. We define the *weakly invertible $n \times n$-matrices over $\pi_0(\mathcal{R})$*, $GL_n(\pi_0\mathcal{R})$, to be the $n \times n$-matrices over $\pi_0(\mathcal{R})$ that are invertible as matrices over $Gr(\pi_0\mathcal{R})$.

Note that we can define $GL_n(\pi_0\mathcal{R})$ by the pullback square

$$\begin{tikzcd}
GL_n(\pi_0\mathcal{R}) \arrow[r] \arrow[d, tail] & GL_n(Gr(\pi_0\mathcal{R})) \arrow[d, tail] \\
M_n(\pi_0\mathcal{R}) \arrow[r] & M_n(Gr(\pi_0\mathcal{R}))
\end{tikzcd}$$

Definition 7.4. The *category of weakly invertible $n \times n$-matrices over $\mathcal{R}$*, $GL_n(\mathcal{R})$, is the full subcategory of $M_n(\mathcal{R})$ with objects all matrices $X = (X_{i,j})_{i,j=1}^n \in M_n(\mathcal{R})$ whose matrix of π_0-classes $[X] = ([X_{i,j}])_{i,j=1}^n$ is contained in $GL_n(\pi_0\mathcal{R})$.

Matrix multiplication is of course compatible with the property of being weakly invertible. Thus, the category $GL_n(\mathcal{R})$ inherits a monoidal structure from $M_n(\mathcal{R})$.

However, even if our base category is not bimonoidal it still makes sense to talk about matrices and even weakly invertible matrices, as long as π_0 of that category is a rig. In particular, we can consider $M_n(-\mathcal{R})\mathcal{R}$ and $GL_n(-\mathcal{R})\mathcal{R}$. Recall that $(-\mathcal{R})\mathcal{R}$ has a bifunctor $\oplus$ which turns it into a permutative category and recall the $\mathcal{R}$-module structure on $(-\mathcal{R})\mathcal{R}$ defined by $A(B, C) := (AB, AC)$ for $A \in \mathcal{R}$ and $(B, C) \in (-\mathcal{R})\mathcal{R}$.

Lemma 7.5. *Assume that $\mathcal{R}$ satisfies the conditions of Theorem 1.1. Then the categories $GL_n(-\mathcal{R})\mathcal{R}$, $GL_nZ(-\mathcal{R})\mathcal{R}$ and $GL_n(\bar{\mathcal{R}})$ are weakly equivalent as modules over $GL_n(Z\mathcal{R})$.*

Proof. We define the $GL_n(Z\mathcal{R})$-action on $M_n(-\mathcal{R})\mathcal{R}$, $M_nZ(-\mathcal{R})\mathcal{R}$ and $M_n(\bar{\mathcal{R}})$ via

$$(X_{i,j})_{i,j=1}^n (Y_{i,j})_{i,j=1}^n = (W_{i,j})_{i,j=1}^n$$

with $W_{i,j} = \bigoplus_{k=1}^n X_{i,k}Y_{k,j}$ where $X_{i,k}Y_{k,j}$ is given by the $Z\mathcal{R}$-module structure of $(-\mathcal{R})\mathcal{R}$, $Z(-\mathcal{R})\mathcal{R}$ or $\bar{\mathcal{R}}$, respectively. Multiplicativity of the determinant then ensures that this passes to a well-defined module structure on the weakly invertible matrices. The weak equivalences from Theorem 6.1 thus combine to give weak equivalences of $GL_n(Z\mathcal{R})$-modules

$$GL_n(-\mathcal{R})\mathcal{R} \xleftarrow{\sim} GL_nZ(-\mathcal{R})\mathcal{R} \xrightarrow{\sim} GL_n(\bar{\mathcal{R}})\,.$$

$\square$

There is a canonical stabilization functor $GL_n(\mathcal{R}) \to GL_{n+1}(\mathcal{R})$ which is induced by taking the block sum with $E_1 \in GL_1(\mathcal{R})$. Let $GL(\mathcal{R})$ be the colimit of the categories $GL_n(\mathcal{R})$.

8. One-sided bar construction

Definition 8.1. Let $(\mathcal{T}, \cdot, 1)$ be a monoidal category and $\mathcal{M}$ a left $\mathcal{T}$-module. The *one-sided bar construction* $B(*, \mathcal{T}, \mathcal{M})$ is the simplicial category whose q-simplices $B_q(*, \mathcal{T}, \mathcal{M})$ are the following category: consider the ordered set $[q]_+ = [q] \sqcup \{\infty\}$, *i.e.*, in addition to the numbers $0 < 1 < \cdots < q$ there is a maximal element ∞. An object a in $B_q(*, \mathcal{T}, \mathcal{M})$ consists of the following data.

(1) For each $0 \leqslant i < j \leqslant q$ there is an object $a_{ij} \in \mathcal{T}$, and for each $0 \leqslant i \leqslant q$ an object $a_{i\infty} \in \mathcal{M}$.
(2) For each $0 \leqslant i < j < k \leqslant \infty$ there is an isomorphism

$$a_{ijk} \colon a_{ij} \cdot a_{jk} \to a_{ik}$$

(in $\mathcal{T}$ if $k < \infty$ and in $\mathcal{M}$ if $k = \infty$) such that if $0 \leqslant i < j < k < l \leqslant \infty$, the following diagram commutes

$$\begin{tikzcd} (a_{ij} \cdot a_{jk}) \cdot a_{kl} \arrow[rr, "\text{struct. iso.}"] \arrow[d, "a_{ijk} \cdot \mathrm{id}"'] & & a_{ij} \cdot (a_{jk} \cdot a_{kl}) \arrow[d, "\mathrm{id} \cdot a_{jkl}"] \\ a_{ik} \cdot a_{kl} \arrow[r, "a_{ikl}"] & a_{il} & a_{ij} \cdot a_{jl} \,. \arrow[l, "a_{ijl}"'] \end{tikzcd}$$

A morphism $f \colon a \to b$ consists of morphisms $f_{ij} \colon a_{ij} \to b_{ij}$ (in $\mathcal{T}$ if $j < \infty$ and in $\mathcal{M}$ if $j = \infty$) such that if $0 \leqslant i < j < k \leqslant \infty$

$$f_{ik} a_{ijk} = b_{ijk}(f_{ij} \cdot f_{jk}) \colon a_{ij} \cdot a_{jk} \to b_{ik} \,.$$

The simplicial structure is gotten as follows: if $\phi \colon [q] \to [p] \in \Delta$ the functor $\phi^* \colon B_p(*, \mathcal{T}, \mathcal{M}) \to B_q(*, \mathcal{T}, \mathcal{M})$ is obtained by precomposing with $\phi_+ = \phi \sqcup \{\infty\}$. So for instance $d_1(a)$ is gotten by deleting all entries with indices containing 1 from the data giving a . In order to allow for degeneracy maps s_i, we use the convention that all objects of the form a_{ii} are the unit of the monoidal structure, and all isomorphisms of the form a_{iik} and a_{ikk} are identities.

A good way to think about this comes from the discrete case ($\mathcal{T}$ is a monoid and $\mathcal{M}$ is a $\mathcal{T}$-set). Then an object $a \in B_q(*, \mathcal{T}, \mathcal{M})$ is uniquely given by the "diagonal" $(a_{01}, a_{12}, \dots, a_{q-1\,q}, a_{q\infty})$, and $B(*, \mathcal{T}, \mathcal{M})$ is isomorphic to the nerve of the category with objects $\mathcal{T}$ and morphisms $a_{1\infty} \to a_{01} \cdot a_{1\infty} = a_{0\infty}$ corresponding to $(a_{01}, a_{1\infty})$.

Example 8.2. (1) If $\mathcal{M}$ is the one-point category $*$, then $B(*, \mathcal{T}, *)$ is isomorphic to the bar construction $B\mathcal{T}$ of [BDR].
(2) If $F \colon \mathcal{T} \to \mathcal{T}'$ is a lax monoidal functor, then $\mathcal{T}'$ may be considered as a $\mathcal{T}$-module, and we write without further ado $B(*, \mathcal{T}, \mathcal{T}')$ for the corresponding bar construction (with F suppressed). In case F is an isomorphism, $B(*, \mathcal{T}, \mathcal{T}')$ is contractible.

We think of elements of $B_q(*, \mathcal{T}, \mathcal{M})$ in terms of triangular arrays of objects, suppressing the isomorphisms, so that a typical element in $B_2(*, \mathcal{T}, \mathcal{M})$ is written

$$\begin{matrix} a_{01} & a_{02} & a_{0\infty} \\ & a_{12} & a_{1\infty} \\ & & a_{2\infty} \end{matrix}$$

with d_1 given by

$$\begin{matrix} a_{02} & a_{0\infty} \\ & a_{2\infty} \,. \end{matrix}$$

The one-sided bar construction is functorial in "natural modules". A natural module is a pair $(\mathcal{T}, \mathcal{M})$ where $\mathcal{T}$ is a monoidal category and $\mathcal{M}$ is a $\mathcal{T}$-module. A morphism $(\mathcal{T}, \mathcal{M}) \to (\mathcal{T}', \mathcal{M}')$ consists of a pair (F, G) where $F \colon \mathcal{T} \to \mathcal{T}'$ is a lax monoidal functor and $G \colon \mathcal{M} \to F^*\mathcal{M}'$ is a map of $\mathcal{T}$-modules, where $F^*\mathcal{M}'$ is $\mathcal{M}'$ endowed with the $\mathcal{T}$-module structure given by restricting along F.

Lemma 8.3. *For each q there is an equivalence of categories between $B_q(*, \mathcal{T}, \mathcal{M})$ and the product category $\mathcal{T}^{\times q} \times \mathcal{M}$.*

Proof. The equivalence is given by the forgetful functor

$$F \colon B_q(*, \mathcal{T}, \mathcal{M}) \to \mathcal{T}^{\times q} \times \mathcal{M}$$

sending a to the "diagonal" $F(a) = (a_{01}, \dots, a_{q-1\,q}, a_{q\infty})$. The inverse is gotten by sending $(a_1, \dots, a_q, a_\infty)$ to the a with $a_{ij} = a_{i+1} \cdot (\cdots (a_{j-1} \cdot a_j) \cdots)$ and a_{ijk} given by the structural isomorphisms. $\square$

Corollary 8.4. *Let $(F, G)\colon (\mathcal{T}, \mathcal{M}) \to (\mathcal{T}', \mathcal{M}')$ be a map of natural modules such that F and G are equivalences of categories. Then the induced map*

$$B(*, F, G)\colon B(*, \mathcal{T}, \mathcal{M}) \to B(*, \mathcal{T}', \mathcal{M}')$$

is a degreewise equivalence of simplicial categories.

Usually $\mathcal{T}^{\times q} \times \mathcal{M}$ is not functorial in $[q]$, but if $(\mathcal{T}, \mathcal{M})$ is strict, the monoidal structure gives a simplicial category

$$B^{\text{strict}}(*, \mathcal{T}, \mathcal{M}) = \{[q] \mapsto \mathcal{T}^{\times q} \times \mathcal{M}\}\,.$$

In this situation Lemma 8.3 reads:

Corollary 8.5. *Let $\mathcal{T}$ be a strict monoidal category and $\mathcal{M}$ a strict $\mathcal{T}$-module. Then there is a degreewise equivalence between the simplicial categories $B(*, \mathcal{T}, \mathcal{M})$ and $B^{strict}(*, \mathcal{T}, \mathcal{M})$.*

Proposition 8.6. *Let $F\colon \mathcal{T} \to \mathcal{G}$ be a strong monoidal functor such that the monoidal structure on $\mathcal{G}$ induces a group structure on $\pi_0\mathcal{G}$. Then*

$$\begin{array}{ccc} B(*, \mathcal{T}, \mathcal{G}) & \longrightarrow & B\mathcal{T} \\ \downarrow & & \downarrow \\ B(*, \mathcal{G}, \mathcal{G}) & \longrightarrow & B\mathcal{G} \end{array}$$

is homotopy cartesian, meaning that it induces a homotopy cartesian diagram upon applying the nerve functor in every degree. The (nerve of the) lower left hand corner is contractible.

Proof. By [JoS] there is a diagram of monoidal categories

$$\begin{array}{ccc} \mathbf{st}\mathcal{T} & \xrightarrow{\sim} & \mathcal{T} \\ {\scriptstyle \mathbf{st}F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathbf{st}\mathcal{G} & \xrightarrow{\sim} & \mathcal{G} \end{array}$$

such that the horizontal maps are monoidal equivalences, and $\mathbf{st}F$ is a strict monoidal functor between strict monoidal categories. Together with Corollaries 8.4 and 8.5 this tells us that we may just as well consider the strict situation, and use the strict bar construction. However, note that the nerve of the strict monoidal category $\mathbf{st}\mathcal{T}$ is a simplicial monoid, and that reversal of priorities gives a natural isomorphism

$$B(*, N\mathbf{st}\mathcal{T}, N\mathbf{st}\mathcal{G}) \cong NB^{\text{strict}}(*, \mathbf{st}\mathcal{T}, \mathbf{st}\mathcal{G}),$$

so that our statement reduces to the statement that

$$B(*, N\mathbf{st}\mathcal{T}, N\mathbf{st}\mathcal{G}) \to B(N\mathbf{st}\mathcal{T}) \to B(N\mathbf{st}\mathcal{G})$$

is a fiber sequence up to homotopy, which is a classical result given that $N\mathbf{st}\mathcal{G}$ is group-like. □

9. Contracting the one-sided bar construction

9.1. **A model for K-theory of $\mathcal{R}$ as an $\mathcal{R}$-module.** In order to construct concrete homotopies, we offer a slight variant of the Grayson–Quillen model where morphisms are not entire equivalence classes. The price is as usual that the resulting object is a two-category. Since there was some confusion about this point while the paper was still at a preprint stage, we emphasize that this is not the construction of Thomason [Th1, 4.3.2] and Jardine [Ja].

Let $(\mathcal{M}, \oplus, 0_{\mathcal{M}}, \tau_{\mathcal{M}})$ be a permutative category written additively. Let $T\mathcal{M}$ be the following 2-category. The objects of $T\mathcal{M}$ are pairs (A^+, A^-) of objects in $\mathcal{M}$, thought of as plus and minus objects in $\mathcal{M}$. Given two objects $A, B \in T\mathcal{M}$, the category of morphisms $T\mathcal{M}(A, B)$ has objects the pairs (X, α) where X is an object in $\mathcal{M}$ and α is a pair of isomorphisms $\alpha^{\pm}\colon A^{\pm} \oplus X \to B^{\pm}$ in $\mathcal{M}$. A morphism from (X, α) to (Y, β) is a map $\phi\colon X \to Y$ such that $\beta(1 \oplus \phi) = \alpha$. Composition $T\mathcal{M}(B, C) \times T\mathcal{M}(A, B) \to T\mathcal{M}(A, C)$ is gotten by sending $((Y, \beta), (X, \alpha))$ to the pair consisting of $X \oplus Y$ and the composite maps

$$A^{\pm} \oplus (X \oplus Y) = (A^{\pm} \oplus X) \oplus Y \xrightarrow{\alpha^{\pm} \oplus \mathrm{id}} B^{\pm} \oplus Y \xrightarrow{\beta^{\pm}} C^{\pm}\,.$$

Composition on morphisms is simply given by addition. Composition is strictly associative because $\mathcal{M}$ is permutative; if $\mathcal{M}$ is merely symmetric monoidal, standard modifications are necessary. Symmetry allows for a symmetric monoidal structure on $T\mathcal{M}$: if we define $(A^+, A^-) \oplus (B^+, B^-) := (A^+ \oplus B^+, A^- \oplus B^-)$, we need the symmetry in order to turn that prescription into a bifunctor.

Consider the Grayson–Quillen version of K-theory $(-\mathcal{M})\mathcal{M}$ as a 2-category with discrete morphism categories. Note that if A and B are objects in $T\mathcal{M}$, then $(-\mathcal{M})\mathcal{M}(A,B) = \pi_0 T\mathcal{M}(A,B)$. (In the case of a topological category, we interpret π_0 as the coequalizer of the source and target maps from the morphism space to the object space.) The assignment that is the identity on objects and otherwise is induced by the projection $T\mathcal{M}(A,B) \to \pi_0 T\mathcal{M}(A,B)$ gives a 2-functor $T\mathcal{M} \to (-\mathcal{M})\mathcal{M}$. It is not hard to see that the morphism categories in $T\mathcal{M}$ are homotopy discrete (in the sense that $T\mathcal{M}(A,B) \to \pi_0 T\mathcal{M}(A,B)$ is a weak equivalence), and so the following result is true:

Lemma 9.1. *Let $\mathcal{M}$ be a permutative category. The 2-functor $T\mathcal{M} \to (-\mathcal{M})\mathcal{M}$ is a weak equivalence. Hence, if all morphisms in $\mathcal{M}$ are isomorphisms and if translation is faithful, then the standard inclusion $\mathcal{M} \to T\mathcal{M}$ is a group completion.*

Note that if $\mathcal{R}$ is a rig category, $T\mathcal{R}$ will not be a rig category (essentially because of the non-strict symmetry in quadratic terms, as in [Th2, p. 572]), but it will still be an $\mathcal{R}$-module:

Lemma 9.2. *Let $(\mathcal{R}, \oplus, 0_{\mathcal{R}}, c_{\oplus}, \otimes, 1_{\mathcal{R}})$ be a strictly bimonoidal category. The map*

$$\mathcal{R} \times T\mathcal{R} \to T\mathcal{R}$$

given on objects by $(A, (B^+, B^-)) \mapsto (A \otimes B^+, A \otimes B^-)$, and on morphisms by sending $\phi\colon A \to B \in \mathcal{R}$ and $(X, \alpha) \in T\mathcal{R}(C, D)$ to the pair consisting of $A \otimes X$ and the map

$$A \otimes C^{\pm} \oplus A \otimes X \xrightarrow{d_r} A \otimes (C^{\pm} \oplus X) \xrightarrow{\phi \otimes \alpha^{\pm}} B \otimes D^{\pm}$$

induces an $\mathcal{R}$-module structure on $T\mathcal{R}$.

We consider $T\mathcal{R}$ as a simplicial category by taking the nerve of each category of morphisms; thus in simplicial degree ℓ, the objects of $T_\ell\mathcal{R}$ are the objects of $T\mathcal{R}$. The morphisms in $T_\ell\mathcal{R}$ from (A^+, A^-) to (B^+, B^-) consist of objects $X^0, \ldots, X^\ell$, a one-morphism $\alpha^{\pm}\colon A^{\pm} \oplus X^0 \to B^{\pm}$, and isomorphisms $\phi^l : X^l \to X^{l-1}$ for $l = 1, \ldots, \ell$. The simplicial structure is given by composing and forgetting ϕ^l's and inserting identity maps.

9.2. **Subdivisions.** We will use the following variant of edgewise subdivision to make room for an explicit simplicial contraction, whose construction begins in subsection 9.4. Consider the shear functor $z\colon \Delta \times \Delta \to \Delta \times \Delta$ given by sending (S,T) to $(T \sqcup S, T)$ where $T \sqcup S$ is the disjoint union with the ordering obtained from T and S with the extra declaration that every object in S is greater than every object in T. If B is a bisimplicial object, we let $z^*B = B \circ z$. The standard inclusion $S \to T \sqcup S$ gives a natural transformation η in $\Delta \times \Delta$ from the identity to z, and hence a natural transformation in bisimplicial sets $\eta^* : z^* \to \mathrm{id}$. Let Ens denote the category of sets and functions.

Lemma 9.3. *For any bisimplicial set X the map $\eta^* : z^*X \to X$ becomes a weak equivalence upon realization.*

Proof. The diagonal of z^*X is equal to the evaluation of X on the opposite of the composite

$$\Delta \xrightarrow{S \mapsto (S,S)} \Delta \times \Delta \xrightarrow{(S,T) \mapsto (S \sqcup S, T)} \Delta \times \Delta\,,$$

so since a map of bisimplicial sets is an equivalence if it is one in every (vertical) degree, it is enough to know that for each fixed $T \in \Delta$ the natural map $\{S \mapsto X(S \sqcup S, T)\} \to \{S \mapsto X(S,T)\}$ is a weak equivalence. But this is a standard weak equivalence from the (second) edgewise subdivision, which is known to be homotopic to a homeomorphism after realization. See [BHM, Lemma 1.1] and the proof of [BHM, Proposition 2.5]. □

Vertices in $z^*(\Delta[p] \times \Delta[q])$ (where products of simplicial sets are viewed as bisimplicial sets, and vertices are $(0,0)$-simplices) are for instance indexed by tuples $((a,b),c)$ where $0 \leqslant a \leqslant b \leqslant p$ and $0 \leqslant c \leqslant q$.

Here are pictures of $z^*(\Delta[2] \times \Delta[0])$ and $z^*(\Delta[2] \times \Delta[1])$:

$$\begin{tikzcd}
((2,2),0) \arrow[r] \arrow[dr] & ((1,2),0) \arrow[r] \arrow[d] & ((1,1),0) \arrow[d] \\
 & ((0,2),0) \arrow[r] \arrow[dr] & ((0,1),0) \arrow[d] \\
 & & ((0,0),0)
\end{tikzcd}$$

$$
\begin{tikzcd}
((2,2),1) \arrow[r] \arrow[dd] \arrow[ddr] \arrow[drr] \arrow[dddrr, bend left] & ((1,2),1) \arrow[rr] \arrow[dd] \arrow[dr] \arrow[dddr] & & ((1,1),1) \arrow[dd] \arrow[dr] \arrow[dddr] & & \\
& & ((0,2),1) \arrow[rr] \arrow[dd] \arrow[drrr] & & ((0,1),1) \arrow[dr] \arrow[dd] & \\
((2,2),0) \arrow[r] \arrow[drr] & ((1,2),0) \arrow[rr] \arrow[dr] & & ((1,1),0) \arrow[dr] & & ((0,0),1) \arrow[dd] \\
& & ((0,2),0) \arrow[rr] \arrow[drrr] & & ((0,1),0) \arrow[dr] & \\
& & & & & ((0,0),0)\,.
\end{tikzcd}
$$

Note that for any bisimplicial set X

$$X_{(p,q)} \cong \mathrm{Hom}_{\text{bisimp. sets}}(\Delta[p] \times \Delta[q], X) = \int_{([s],[t])} \mathrm{Ens}(\Delta([s],[p]) \times \Delta([t],[q]), X_{(s,t)})$$

as a categorical end. Therefore, the right adjoint of z^*, z_*, is given by

$$(z_*X)_{(p,q)} = \mathrm{Hom}_{\text{bisimp. sets}}(\Delta[p] \times \Delta[q], z_*X) \cong \mathrm{Hom}_{\text{bisimp. sets}}(z^*(\Delta[p] \times \Delta[q]), X)$$

and thus

$$\begin{aligned} z_*X &= \{[p],[q] \mapsto \{\text{bisimp. maps } z^*(\Delta[p] \times \Delta[q]) \to X\}\} \\ &= \{[p],[q] \mapsto \int_{([s],[t])} \mathrm{Ens}(\Delta([t] \sqcup [s],[p]) \times \Delta([t],[q]), X_{(s,t)})\}\,. \end{aligned}$$

Let $\eta_* \colon X \to z_*X$ be the natural transformation associated with η. Notice that η_* maps $X_{(0,q)}$ isomorphically to $(z_*X)_{(0,q)}$ for all $q \geqslant 0$, so $(z_*X)_{(0,q)} \cong X_{(0,q)}$.

Lemma 9.4. *In the homotopy category with respect to maps that are weak equivalences on the diagonal, $\eta_* \colon X \to z_*X$ is a split monomorphism.*

Proof. By formal considerations the diagram

$$
\begin{tikzcd}
z^*X \arrow[r, "z^*\eta_*"] \arrow[dr, "\eta^*"'] & z^*z_*X \arrow[d, "\text{adjunction}"] \\
& X
\end{tikzcd}
$$

commutes, and η^* is a weak equivalence on the diagonal. Hence $z^*\eta_*$ (and so η_*) is a split monomorphism in the homotopy category. ☐

9.3. **The bar construction on matrices.** Let $\mathcal{R}$ be a strictly bimonoidal category such that all morphisms are isomorphisms.

Consider the one-sided bar construction $B(*, GL_n(\mathcal{R}), GL_n(T\mathcal{R}))$. In the following, 0 and 1 are short for zero resp. unit matrices over $\mathcal{R}$ of varying size. Viewing $T\mathcal{R}$ as a simplicial category we get that $B(*, GL_n(\mathcal{R}), GL_n(T\mathcal{R}))$ is a bisimplicial category. We are going to show that

$$B(*, GL(\mathcal{R}), GL(T\mathcal{R})) \cong \mathrm{colim}_n\, B(*, GL_n(\mathcal{R}), GL_n(T\mathcal{R}))$$

is contractible, and it is enough to show that $B(*, GL(\mathcal{R}), GL(T_\ell\mathcal{R}))$ is contractible for every ℓ.

To ease readability, we will abandon the cumbersome $\oplus$ and $\otimes$ in favor of the more readable $+$ and $\cdot$ — reminding us of the matrix nature of our efforts.

Fix ℓ once and for all, and let $B^n = B(*, GL_n(\mathcal{R}), GL_n(T_\ell\mathcal{R}))$. An object in B^n_q is a collection m_{ij} of $n \times n$ matrices in $\mathcal{R}$ for $0 \leqslant i < j \leqslant q$ and for each $0 \leqslant i \leqslant q$ a matrix $m_{i\infty}$ in $T_\ell\mathcal{R}$, together with suitably compatible structural isomorphisms $m_{ijk} \colon m_{ij} \cdot m_{jk} \to m_{ik}$. The matrices are drawn from the "weakly invertible components". The matrices $m_{i\infty}$ and the structural isomorphisms relating these need special attention. Each entry is in $T_\ell\mathcal{R}$, so $m_{i\infty}$ can be viewed as a pair $m^{\pm}_{i\infty}$ of matrices, and the structural isomorphism $m_{ij\infty} \colon m_{ij} \cdot m_{j\infty} \to m_{i\infty}$ is a tuple $(m^{\pm}_{ij\infty}, \phi^1_{ij\infty}, \ldots, \phi^\ell_{ij\infty})$, where the $\phi^l_{ij\infty} \colon x^l_{ij\infty} \to$

$x^{l-1}_{ij\infty} \in M_n(\mathcal{R})$ for $l = 1, \dots, \ell$ are matrices of isomorphisms, and the $m^{\pm}_{ij\infty} \colon m_{ij} \cdot m^{\pm}_{j\infty} + x^0_{ij\infty} \to m^{\pm}_{i\infty}$ are isomorphisms.

The commutativity of

$$\begin{tikzcd}
(m_{ij} \cdot m_{jk}) \cdot m_{k\infty} \arrow[rr, "\cong"] \arrow[d, "m_{ijk}\cdot \mathrm{id}"'] & & m_{ij} \cdot (m_{jk} \cdot m_{k\infty}) \arrow[d, "\mathrm{id}\cdot m_{jk\infty}"] \\
m_{ik} \cdot m_{k\infty} \arrow[r, "m_{ik\infty}"'] & m_{i\infty} & m_{ij} \cdot m_{j\infty} \arrow[l, "m_{ij\infty}"]
\end{tikzcd}$$

says that two morphisms from $(m_{ij} \cdot m_{jk}) \cdot m_{k\infty}$ agree: one is an isomorphism with source $(m_{ij} \cdot m_{jk}) \cdot m_{k\infty} + x^l_{ik\infty}$, the other one is an isomorphism with source $(m_{ij} \cdot m_{jk}) \cdot m_{k\infty} + m_{ij} \cdot x^l_{jk\infty} + x^l_{ij\infty}$. Therefore we obtain the following equality.

Lemma 9.5. *In the situation above one has the identity*

$$x^l_{ik\infty} = m_{ij} \cdot x^l_{jk\infty} + x^l_{ij\infty}$$

for $l = 0, 1, \dots, \ell$, and the diagram

$$\begin{tikzcd}
m_{ij} \cdot (m_{jk} \cdot m^{\pm}_{k\infty}) + x^0_{ik\infty} \arrow[r, equal] & m_{ij} \cdot (m_{jk} \cdot m^{\pm}_{k\infty}) + m_{ij} \cdot x^0_{jk\infty} + x^0_{ij\infty} \arrow[d, "\mathrm{id}\cdot m^{\pm}_{jk\infty} + \mathrm{id}"] \\
(m_{ij} \cdot m_{jk}) \cdot m^{\pm}_{k\infty} + x^0_{ik\infty} \arrow[u, "\cong"] \arrow[d, "m_{ijk}\cdot\mathrm{id}+\mathrm{id}"'] & m_{ij} \cdot m^{\pm}_{j\infty} + x^0_{ij\infty} \arrow[d, "m^{\pm}_{ij\infty}"] \\
m_{ik} \cdot m^{\pm}_{k\infty} + x^0_{ik\infty} \arrow[r, "m^{\pm}_{ik\infty}"] & m^{\pm}_{i\infty}
\end{tikzcd}$$

commutes.

(Here the map $\mathrm{id} \cdot m^{\pm}_{jk\infty}$ already incorporates the right distributivity isomorphism d_r, as specified in Lemma 9.2.)

A morphism $\alpha \colon m \to \tilde{m}$ in B^n_q consists of a matrix $\alpha_{ij} \colon m_{ij} \to \tilde{m}_{ij}$ of maps in $\mathcal{R}$ for $0 \leqslant i < j \leqslant q$, and morphisms $(\alpha^{\pm}_{i\infty}, \psi^1_{i\infty}, \dots, \psi^{\ell}_{i\infty}) \colon m^{\pm}_{i\infty} \to \tilde{m}^{\pm}_{i\infty}$ in $T_\ell\mathcal{R}$ for $0 \leqslant i \leqslant q$, all compatible with the structure maps of m and $\tilde{m}$. Here $\alpha^{\pm}_{i\infty}$ is a map $m^{\pm}_{i\infty} + \xi^0_{i\infty} \to \tilde{m}^{\pm}_{i\infty}$, and the $\psi^l_{i\infty}$ for $l = 1, \dots, \ell$ are maps $\xi^l_{i\infty} \to \xi^{l-1}_{i\infty}$ of matrices in $\mathcal{R}$.

The condition $\alpha_{i\infty} m_{ij\infty} = \tilde{m}_{ij\infty}(\alpha_{ij} \cdot \alpha_{j\infty})$ allows us to draw the following conclusion.

Lemma 9.6. *In the situation above one has the identity*

$$x^l_{ij\infty} + \xi^l_{i\infty} = m_{ij} \cdot \xi^l_{j\infty} + \tilde{x}^l_{ij\infty}$$

for each $l = 0, \dots, \ell$.

9.4. **Start of the proof that $B(*, GL(\mathcal{R}), GL(T_\ell\mathcal{R}))$ is contractible.** We will show that $\mathrm{colim}_n B^n = B(*, GL(\mathcal{R}), GL(T_\ell\mathcal{R}))$ is contractible by showing that each matrix stabilization functor $\mathrm{in} \colon B^n \to B^{2n}$ is trivial in the homotopy category. Here $\mathrm{in}(m) = \left[\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix}\right]$.

We regard the simplicial categories B^n and B^{2n} as bisimplicial sets, by way of their respective nerves NB^n and NB^{2n}. To be precise, the (p,q)-simplices of NB^{2n} are $N_pB^{2n}_q$. By Lemma 9.4 it then suffices to show that the composite map $\mathrm{inc} = \eta_* \circ \mathrm{in} \colon NB^n \to z_*NB^{2n}$ is trivial in the homotopy category. As remarked above, $z_*(NB^{2n})_{(0,q)} \cong (NB^{2n})_{(0,q)} = N_0B^{2n}_q$, so the subdivision operator z_* does not make any difference before we start to consider positive-dimensional simplices ($p > 0$) in the nerve direction.

Seeing that the image lies in a single path component is easy: if $m \in N_0B^n_0 = \mathrm{ob}\, GL_n(T_\ell\mathcal{R})$ then there is a path

$$\begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} m & m^- \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} m & m^- \\ (1,1) & 1 \end{bmatrix} \leftarrow \begin{bmatrix} 1 & 0 \\ (0,1) & 1 \end{bmatrix}.$$

The first arrow represents the one-simplex in the bar direction given by the matrix multiplication

$$\begin{bmatrix} 1 & m^- \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m & m^- \\ 0 & 1 \end{bmatrix} \in GL_{2n}(T_\ell\mathcal{R}).$$

The second arrow represents the one-simplex in the nerve direction induced by the map $0 = (0,0) \to (1,1) \in T_0\mathcal{R} \subset T_\ell\mathcal{R}$. The third map represents the one-simplex in the bar direction given by multiplication by

$$\begin{bmatrix} m^+ & m^- \\ 1 & 1 \end{bmatrix} \in GL_{2n}(\mathcal{R}) .$$

The rest of this section extends this path to a full homotopy, from inc via maps jnc and knc to a constant map lnc.

9.5. **The homotopic maps inc and jnc.** Recall that $\ell \geqslant 0$ is fixed, $B^n = B(*, GL_n(\mathcal{R}), GL_n(T_\ell\mathcal{R}))$ is the simplicial category given by the one-sided bar construction, and $NB^n \colon [p],[q] \mapsto N_pB^n_q$ is the bisimplicial set given by its degreewise nerve. We let $\mathrm{inc}\colon NB^n \to z_*NB^{2n}$ be the composite of the matrix stabilization map $\mathrm{in}\colon NB^n \to NB^{2n}$ and the natural map $\eta_*\colon NB^{2n} \to z_*NB^{2n}$.

There is another map $\mathrm{jnc}\colon NB^n \to z_*NB^{2n}$ which is homotopic to inc. On $N_0B^n_q$ it is easy to describe: if $m \in N_0B^n_q$, we declare that $X(m)$ is given by

$$X(m)_{ij} = \begin{cases} \begin{bmatrix} 1 & m^-_{i\infty} \\ 0 & 1 \end{bmatrix} & \text{if } i < j = \infty \\[2ex] \begin{bmatrix} 1 & x^0_{ij\infty} \\ 0 & 1 \end{bmatrix} & \text{if } i < j < \infty \end{cases}$$

and let

$$\mathrm{jnc}(m) = X(m) \cdot \mathrm{inc}(m) \in N_0B^{2n}_q = z_*(NB^{2n})_{(0,q)} .$$

Here

$$\mathrm{jnc}(m)_{ij} = \begin{cases} \begin{bmatrix} m_{i\infty} & m^-_{i\infty} \\ 0 & 1 \end{bmatrix} & \text{if } i < j = \infty \\[2ex] \begin{bmatrix} m_{ij} & x^0_{ij\infty} \\ 0 & 1 \end{bmatrix} & \text{if } i < j < \infty \end{cases}$$

with $\mathrm{jnc}(m)_{ijk} \colon \mathrm{jnc}(m)_{ij} \cdot \mathrm{jnc}(m)_{jk} \to \mathrm{jnc}(m)_{ik}$ being the isomorphisms induced by m_{ijk} as follows: for $k < \infty$ we use the identity $x^0_{ik\infty} = m_{ij} \cdot x^0_{jk\infty} + x^0_{ij\infty}$ from Lemma 9.5 and obtain

$$\begin{bmatrix} m_{ijk} & \mathrm{id} \\ \mathrm{id} & \mathrm{id} \end{bmatrix} \colon \begin{bmatrix} m_{ij} & x^0_{ij\infty} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m_{jk} & x^0_{jk\infty} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{ij} \cdot m_{jk} & m_{ij} \cdot x^0_{jk\infty} + x^0_{ij\infty} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} m_{ik} & x^0_{ik\infty} \\ 0 & 1 \end{bmatrix}$$

and for $k = \infty$ we use the string of isomorphisms

$$\begin{bmatrix} x^\ell_{ij\infty} & 0 \\ 0 & 0 \end{bmatrix} \to \dots \to \begin{bmatrix} x^0_{ij\infty} & 0 \\ 0 & 0 \end{bmatrix}$$

together with the isomorphism

$$\begin{aligned} \begin{bmatrix} m_{ij\infty} & m^-_{ij\infty} \\ \mathrm{id} & \mathrm{id} \end{bmatrix} \colon & \begin{bmatrix} m_{ij} & x^0_{ij\infty} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} m_{j\infty} & m^-_{j\infty} \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} x^0_{ij\infty} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} m_{ij} \cdot m_{j\infty} + x^0_{ij\infty} & m_{ij} \cdot m^-_{j\infty} + x^0_{ij\infty} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} m_{i\infty} & m^-_{i\infty} \\ 0 & 1 \end{bmatrix} . \end{aligned}$$

We notice that the T_ℓ-direction does not add any complications but notational. This continues to be true in general, so we simplify notation by considering only the case $\ell = 0$.

The relevant complications arise when one starts moving in the nerve direction. As the construction of the map jnc is quite involved, we will give some examples first. The impatient reader can skip this part and restart reading in subsection 9.6 where the formula in the general case is given.

As an illustration, let $\ell = 0$, $p = 2$ and $q = 0$ so that

$$m =\!=\!=\!= (m^0 \xleftarrow{(\xi^1,\alpha^1)} m^1 \xleftarrow{(\xi^2,\alpha^2)} m^2) \in N_2B^n_0 = N_2GL_n(T_0\mathcal{R}) .$$

Then $\mathrm{jnc}(m)$ is captured by the picture

$$
\begin{tikzcd}
\begin{bmatrix} m^2 & (m^2)^- \\ 0 & 1 \end{bmatrix} \arrow[r, "\left[\begin{smallmatrix} 1 & \xi^2 \\ 0 & 1 \end{smallmatrix}\right]"] \arrow[dr, "\left[\begin{smallmatrix} 1 & \xi^2+\xi^1 \\ 0 & 1 \end{smallmatrix}\right]"'] & \begin{bmatrix} m^2 & (m^2)^-+\xi^2 \\ 0 & 1 \end{bmatrix} \arrow[r] \arrow[d, "\left[\begin{smallmatrix} 1 & \xi^1 \\ 0 & 1 \end{smallmatrix}\right]"] & \begin{bmatrix} m^1 & (m^1)^- \\ 0 & 1 \end{bmatrix} \arrow[d, "\left[\begin{smallmatrix} 1 & \xi^1 \\ 0 & 1 \end{smallmatrix}\right]"] \\
& \begin{bmatrix} m^2 & (m^2)^-+\xi^2+\xi^1 \\ 0 & 1 \end{bmatrix} \arrow[r] \arrow[dr] & \begin{bmatrix} m^1 & (m^1)^-+\xi^1 \\ 0 & 1 \end{bmatrix} \arrow[d] \\
& & \begin{bmatrix} m^0 & (m^0)^- \\ 0 & 1 \end{bmatrix}
\end{tikzcd}
$$

where the bar direction is written in the "$g \xrightarrow{m} mg$" form, and the unlabeled arrows correspond to the nerve direction, with entries consisting of the appropriate α's.

An even more complicated example, essentially displaying all the complications of the general case: $\ell = 0$, $p = q = 1$, and $\alpha\colon m^1 \to m^0 \in N_1B_1^n$ with $(m^u)^{\pm}_{01\infty}\colon (m^u)_{01} \cdot (m^u)^{\pm}_{1\infty} + (x^u)^0_{01\infty} \to (m^u)^{\pm}_{0\infty}$ (for $u = 0, 1$ running in the nerve direction), $\alpha_{01}\colon m^1_{01} \to m^0_{01}$ and $\alpha^{\pm}_{i\infty}\colon (m^1)^{\pm}_{i\infty} + \xi_{i\infty} \to (m^0)^{\pm}_{i\infty}$.

Then $\mathrm{jnc}(m)$ is the map from

$$
\begin{tikzcd}
z^*(\Delta[1] \times \Delta[1]) = & ((0,0),0) & ((0,1),0) \arrow[l] & ((1,1),0) \arrow[l] \\
& ((0,0),1) \arrow[u] & ((0,1),1) \arrow[l] \arrow[u] & ((1,1),1) \arrow[l] \arrow[u] \arrow[ul]
\end{tikzcd}
$$

sending the (nondegenerate) $((0,1),0) \leftarrow ((1,1),0) \leftarrow ((1,1),1)$ simplex to

$$
\begin{array}{ccc}
\left[\begin{smallmatrix} 1 & \xi_{0\infty} \\ 0 & 1 \end{smallmatrix}\right] & \begin{bmatrix} m^1_{01} & x^1_{01\infty}+\xi_{0\infty} \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} m^1_{0\infty} & (m^1)^-_{0\infty}+\xi_{0\infty} \\ 0 & 1 \end{bmatrix} \\
& \begin{bmatrix} m^1_{01} & x^1_{01\infty} \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} m^1_{0\infty} & (m^1)^-_{0\infty} \\ 0 & 1 \end{bmatrix} \\
& & \begin{bmatrix} m^1_{1\infty} & (m^1)^-_{1\infty} \\ 0 & 1 \end{bmatrix}
\end{array}
\in N_0B_2^{2n}\,,
$$

the $((0,1),0) \leftarrow ((0,1),1) \leftarrow ((1,1),1)$ simplex to

$$
\begin{array}{ccc}
\begin{bmatrix} m^1_{01} & x^0_{01\infty} \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} m^1_{01} & x^1_{01\infty}+\xi_{0\infty} \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} m^1_{0\infty} & (m^1)^-_{0\infty}+\xi_{0\infty} \\ 0 & 1 \end{bmatrix} \\
& \left[\begin{smallmatrix} 1 & \xi_{1\infty} \\ 0 & 1 \end{smallmatrix}\right] & \begin{bmatrix} m^1_{1\infty} & (m^1)^-_{1\infty}+\xi_{1\infty} \\ 0 & 1 \end{bmatrix} \\
& & \begin{bmatrix} m^1_{1\infty} & (m^1)^-_{1\infty} \\ 0 & 1 \end{bmatrix}
\end{array}
\in N_0B_2^{2n}\,,
$$

(here the identity from Lemma 9.6 is used) and the $(1,1)$-simplex

$$
\begin{tikzcd}
((0,0),0) & ((0,1),0) \arrow[l] \\
((0,0),1) \arrow[u] & ((0,1),1) \arrow[l] \arrow[u]
\end{tikzcd}
$$

to

$$
\begin{array}{cc}
\begin{bmatrix} m^0_{01} & x^0_{01\infty} \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} m^0_{0\infty} & (m^0)^-_{0\infty} \\ 0 & 1 \end{bmatrix} \\
& \begin{bmatrix} m^0_{1\infty} & (m^0)^-_{1\infty} \\ 0 & 1 \end{bmatrix}
\end{array}
\xleftarrow{\;\left[\begin{smallmatrix} \alpha_{01} & \mathrm{id} \\ \mathrm{id} & \mathrm{id} \end{smallmatrix}\right] \quad \left[\begin{smallmatrix} \alpha_{0\infty} & \alpha^-_{0\infty} \\ \mathrm{id} & \mathrm{id} \end{smallmatrix}\right] \left[\begin{smallmatrix} \alpha_{1\infty} & \alpha^-_{1\infty} \\ \mathrm{id} & \mathrm{id} \end{smallmatrix}\right]\;}
\begin{array}{cc}
\begin{bmatrix} m^1_{01} & x^0_{01\infty} \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} m^1_{0\infty} & (m^1)^-_{0\infty}+\xi_{0\infty} \\ 0 & 1 \end{bmatrix} \\
& \begin{bmatrix} m^1_{1\infty} & (m^1)^-_{1\infty}+\xi_{1\infty} \\ 0 & 1 \end{bmatrix}
\end{array}
\in N_1B_1^{2n}\,.
$$

Here we have employed the formula $x^1_{01\infty} + \xi_{0\infty} = m^1_{01} \cdot \xi_{1\infty} + x^0_{01\infty}$ of Lemma 9.6.

9.6. **General version of jnc.** Consider

$$
m = (m^0 \xleftarrow{\alpha^1} m^1 \longleftarrow \dots \xleftarrow{\alpha^p} m^p) \in N_pB_q^n\,. \tag{4}
$$

Then $(\alpha^u)_{i\infty}$ is given by the tuple

$$
((\alpha^u)^{\pm}_{i\infty}\colon (m^u)^{\pm}_{i\infty} + (\xi^u)^0_{i\infty} \to (m^{u-1})^{\pm}_{i\infty}\,, \{(\psi^u)^l_{i\infty}\colon (\xi^u)^l_{i\infty} \to (\xi^u)^{l-1}_{i\infty}\})
$$

for $u = 1, \dots, p$, $l = 1, \dots, \ell$, but we simplify notation by setting $\xi^u_{i\infty} = (\xi^u)^0_{i\infty}$, $x^u_{ij\infty} = (x^u)^0_{ij\infty}$, and ignoring the ψ's. Then jnc sends an m as in (4) to the simplex $\mathrm{jnc}(m) \in z_*(NB^n)_{pq}$ with value at the $((a,b),c)$-vertex in $z^*(\Delta[p] \times \Delta[q])$ given by

$$\begin{bmatrix} (m^b)_{c\infty} & (m^b)^-_{c\infty} + \xi^b_{c\infty} + \cdots + \xi^{a+1}_{c\infty} \\ 0 & 1 \end{bmatrix} \in GL_{2n}(T_\ell \mathcal{R})$$

with the convention that the ξ's only occur if $a + 1 \leqslant b$. Higher simplices are given by the structural isomorphisms in m. Note that the elements in the off-diagonal blocks are actually all in $\mathcal{R}$.

More precisely, a triple (ϕ, b, ψ) where $\phi\colon [r] \to [p]$ and $\psi\colon [r] \to [q]$ are in Δ and $\phi(r) \leqslant b \leqslant p$, determines a $(0,r)$-simplex in $z^*(\Delta[p] \times \Delta[q])$, because $z^*(\Delta[p] \times \Delta[q])_{(0,r)} = \Delta([r+1],[p]) \times \Delta([r],[q])$ and ϕ together with b determine an element in the first factor. We see that $\mathrm{jnc}(m)(\phi,b,\psi) \in N_0 B^{2n}_r$ is the element whose $(0 \leqslant i < j \leqslant r)$- and $(0 \leqslant i \leqslant r < j = \infty)$-entries are

$$\begin{bmatrix} (m^b)_{\psi(i)\psi(j)} & x^{\phi(j)}_{\psi(i)\psi(j)\infty} + \xi^{\phi(j)}_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty} \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} (m^b)_{\psi(i)\infty} & (m^b)^-_{\psi(i)\infty} + \xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty} \\ 0 & 1 \end{bmatrix},$$

respectively. (As before, the ξ's only occur when $\phi(i) + 1 \leqslant \phi(j)$ and $\phi(i) + 1 \leqslant b$, respectively.)

Moving in the (nerve $=$) b-direction is easy because it amounts to connecting two values $\mathrm{jnc}(m)(\phi,b,\psi)$ and $\mathrm{jnc}(m)(\phi,b',\psi)$ by morphisms. Since this is determined by the one-skeleton, it is enough to describe the case $b' = b - 1 < b$. On the $(0 \leqslant i < j \leqslant r)$-entries it is induced by $(\alpha^b)_{\psi(i)\psi(j)}\colon (m^b)_{\psi(i)\psi(j)} \to (m^{b-1})_{\psi(i)\psi(j)}$ (in the upper left hand corner, and otherwise the identity), and on the $(0 \leqslant i \leqslant r < j = \infty)$-entries it is given by $(\xi^b_{\psi(i)\infty}, (\alpha^b)_{\psi(i)\infty})\colon (m^b)_{\psi(i)\infty} \to (m^{b-1})_{\psi(i)\infty}$ and $(\alpha^b)^-_{\psi(i)\infty}\colon (m^b)^-_{\psi(i)\infty} + \xi^b_{\psi(i)\infty} \to (m^{b-1})^-_{\psi(i)\infty}$ (in the upper row, and otherwise the identity).

Checking that this is well defined and simplicial amounts to the same kind of checking as we have already encountered, using the same identities. One should notice that at no time during the verifications is the symmetry of addition used. It is used, however, for the isomorphism that renders matrix multiplication associative up to isomorphism.

The simplicial homotopy from inc to jnc is gotten by multiplications (in the bar direction) by matrices of the form

$$\begin{bmatrix} 1 & x^{\phi(j)}_{\psi(i)\psi(j)\infty} + \xi^{\phi(j)}_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & (m^b)^-_{\psi(i)\infty} + \xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty} \\ 0 & 1 \end{bmatrix}.$$

9.7. **The homotopic maps knc and lnc.** Consider the following variant knc of the map jnc: using the same notation as for jnc, when evaluated on (ϕ, b, ψ) where $\phi\colon [r] \to [p]$ and $\psi\colon [r] \to [q]$ are in Δ and $\phi(r) \leqslant b \leqslant p$, $\mathrm{knc}(m)(\phi,b,\psi) \in N_0 B^{2n}_r$ is the element whose $(0 \leqslant i < j \leqslant r)$- and $(0 \leqslant i \leqslant r < j = \infty)$-entries are

$$\begin{bmatrix} (m^b)_{\psi(i)\psi(j)} & x^{\phi(j)}_{\psi(i)\psi(j)\infty} + \xi^{\phi(j)}_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty} \\ 0 & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} (m^b)_{\psi(i)\infty} + (\xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty}, \xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty}) & (m^b)^-_{\psi(i)\infty} + \xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty} \\ (1,1) & 1 \end{bmatrix},$$

respectively. The (A,B)-notation is the "plus-minus" notation for objects in $T_\ell\mathcal{R}$. The entries for $j = \infty$ can be written more concisely as $\begin{bmatrix} (m^b)_{\psi(i)\infty} + (\Xi, \Xi) & (m^b)^-_{\psi(i)\infty} + \Xi \\ (1,1) & 1 \end{bmatrix}$ where $\Xi = \xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty}$. The entries for finite j are the same as for jnc.

There is a natural map (in the nerve direction) from jnc to knc (of the form $(X, \mathrm{id})\colon (A,B) \to (A+X, B+X) \in T_\ell\mathcal{R}$ — induced by the identity), giving a homotopy.

Finally, let lnc: $NB^n \to z_* NB^{2n}$ be induced by the constant map sending any matrix to $\left[\begin{smallmatrix} 1 & 0 \\ (0,1) & 1 \end{smallmatrix}\right]$. Matrix multiplication yields

$$\begin{bmatrix} (m^b)^+_{\psi(i)\infty} + \xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty} & (m^b)^-_{\psi(i)\infty} + \xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty} \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ (0,1) & 1 \end{bmatrix}$$
$$= \begin{bmatrix} (m^b)_{\psi(i)\infty} + (\xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty}, \xi^b_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty}) & (m^b)^-_{\psi(i)\infty} + \xi^c_{\psi(i)\infty} + \cdots + \xi^{\phi(i)+1}_{\psi(i)\infty} \\ (1,1) & 1 \end{bmatrix}.$$

With the same abbreviation as above, this reads

$$\begin{bmatrix} (m^b)^+_{\psi(i)\infty} + \Xi & (m^b)^-_{\psi(i)\infty} + \Xi \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ (0,1) & 1 \end{bmatrix} = \begin{bmatrix} (m^b)_{\psi(i)\infty} + (\Xi, \Xi) & (m^b)^-_{\psi(i)\infty} + \Xi \\ (1,1) & 1 \end{bmatrix}.$$

We obtain a homotopy from lnc to knc.

Hence inc is connected by a chain of homotopies to a constant map. Since $\eta_* \colon \mathrm{id} \to z_*$ is a monomorphism in the homotopy category, this means that the stabilization map in: $B^n \to B^{2n}$ is homotopically trivial, and so $B(*, GL(\mathcal{R}), GL(T_\ell \mathcal{R})) = \operatorname{colim}_n B^n$ is contractible for each $\ell \geqslant 0$. Hence $B(*, GL(\mathcal{R}), GL(T\mathcal{R}))$ is also contractible.

10. The K-theory of a ring category

Given a strictly bimonoidal category $\mathcal{R}$, let $H\mathcal{R}$ be a suitably defined multiplicative version of $\mathrm{Spt}\,\mathcal{R}$ with $H\mathcal{R}$ being an Ω-spectrum from the first level on. For example, we can use the construction in [EM].

Remember that the group-like monoid $GL_n(H\mathcal{R})$ is defined by the pullback

$$\begin{tikzcd} GL_n(H\mathcal{R}) \arrow[r] \arrow[d] & \operatorname{hocolim}_{\mathbf{m}\in I} \Omega^m M_n(H\mathcal{R}(S^m)) \arrow[d] \\ GL_n(\pi_0 H\mathcal{R}) \arrow[r] & M_n(\pi_0 H\mathcal{R}). \end{tikzcd}$$

Lemma 10.1. *If $\bar{\mathcal{R}}$ is a ring category,* i.e., *a rig category with $\pi_0(\bar{\mathcal{R}})$ a ring, then*

$$|GL_n(\bar{\mathcal{R}})| \xrightarrow{\sim} GL_n(H\bar{\mathcal{R}})$$

is a homotopy equivalence.

Proof. By assumption $\pi_0(\bar{\mathcal{R}})$ is isomorphic to $Gr(\pi_0 \bar{\mathcal{R}}) = \pi_0 H\bar{\mathcal{R}}$, so it is enough to show that $|M_n \bar{\mathcal{R}}|$ and $\operatorname{hocolim}_{\mathbf{m}\in I} \Omega^m M_n(H\bar{\mathcal{R}}(S^m))$ are equivalent. Both are n^2-fold products, so it suffices to show that $|\bar{\mathcal{R}}|$ and $\operatorname{hocolim}_{\mathbf{m}\in I} \Omega^m H\bar{\mathcal{R}}(S^m)$ are equivalent. All the structure maps $\Omega^m H\bar{\mathcal{R}}(S^m) \to \Omega^{m'} H\bar{\mathcal{R}}(S^{m'})$ are equivalences for $\mathbf{m} \to \mathbf{m'}$ an injection of nonempty finite sets, and since $\bar{\mathcal{R}}$ is already group-like $|\bar{\mathcal{R}}| \simeq H\bar{\mathcal{R}}(S^0)$ maps by an equivalence to $\Omega H\bar{\mathcal{R}}(S^1)$. ☐

11. The Proof

Let $\mathcal{R}$ be a bipermutative or strictly bimonoidal category that satisfies the conditions of Theorem 1.1. Then we know that there is a group completion of $\mathcal{R}$, $\bar{\mathcal{R}} = D \operatorname{hocolim}_I (-\mathcal{R})^\bullet \mathcal{R}$, which is a ring category.

We know from Lemma 9.2 that $T\mathcal{R}$ is an $\mathcal{R}$-module, and from Lemma 4.3, Theorem 6.1 and Lemma 9.1 that the maps $Z\mathcal{R} \to \mathcal{R}$ and $T\mathcal{R} \to (-\mathcal{R})\mathcal{R} \leftarrow Z(-\mathcal{R})\mathcal{R} \to \bar{\mathcal{R}}$ are unstable equivalences of $Z\mathcal{R}$-modules. It follows that $GL(Z\mathcal{R})$ is equivalent to $GL(\mathcal{R})$, and that $GL(T\mathcal{R})$ is equivalent to $GL(\bar{\mathcal{R}})$ as a $GL(Z\mathcal{R})$-module. In section 9 we showed that the one-sided bar construction $B(*, GL(\mathcal{R}), GL(T\mathcal{R}))$ is contractible, hence $B(*, GL(Z\mathcal{R}), GL(\bar{\mathcal{R}}))$ is contractible, too. But this is the homotopy fibre of $BGL(Z\mathcal{R}) \to BGL(\bar{\mathcal{R}})$, so we have weak equivalences

$$BGL(\mathcal{R}) \xleftarrow{\sim} BGL(Z\mathcal{R}) \xrightarrow{\sim} BGL(\bar{\mathcal{R}}).$$

In section 10 we obtained that

$$|BGL(\bar{\mathcal{R}})| \xrightarrow{\sim} BGL(H\bar{\mathcal{R}}).$$

As $H\mathcal{R}$ is equivalent to $H\bar{\mathcal{R}}$ this yields that in the diagram

$$\begin{tikzcd} {|BGL(\mathcal{R})|} \arrow[d] & {|BGL(Z\mathcal{R})|} \arrow[l, "\sim"'] \arrow[r, "\sim"] & {|BGL(\bar{\mathcal{R}})|} \arrow[d, "\sim"] \\ BGL(H\mathcal{R}) & BGL(HZ\mathcal{R}) \arrow[l, "\sim"'] \arrow[r, "\sim"] & BGL(H\bar{\mathcal{R}}) \end{tikzcd}$$

five out of six arrows are weak equivalences, hence so is $|BGL(\mathcal{R})| \longrightarrow BGL(H\mathcal{R})$. Thus finally we obtain

$$K(H\mathcal{R}) \simeq \mathbb{Z} \times BGL(H\mathcal{R})^+ \simeq \mathbb{Z} \times |BGL(\mathcal{R})^+| \simeq \mathcal{K}(\mathcal{R})\,.$$

12. Appendix: an alternative construction

We sketch an alternative construction of a group completion of a strictly bimonoidal category which works in broader contexts than the construction we gave in sections 2.1 to 6.

First, let us recall a slightly modified version of the Elmendorf–Mandell model of K-theory. There is a precursor of this model in Shimakawa's papers [Sh, pp. 378–379]. Let $(\mathcal{R}, \oplus, 0_\mathcal{R}, c_\oplus, \otimes, 1_\mathcal{R})$ be a small strictly bimonoidal category. For now we focus on the additive structure $(\mathcal{R}, \oplus, 0_\mathcal{R}, c_\oplus)$ of $\mathcal{R}$. The following is taken from [EM, §4].

For finite based sets $X^1_+, \ldots, X^n_+$ with $+$ denoting the basepoint, $\bar{H}\mathcal{R}(X^1_+, \ldots, X^n_+)$ is the category with objects $(C_{\langle S \rangle}, \rho(\langle S \rangle; i, T, U))$ where

- $\langle S \rangle = (S_1, \ldots, S_n)$ is an n-tuple of basepoint-free subsets $S_i \subset X^i$.
- The $C_{\langle S \rangle}$ are objects of $\mathcal{R}$.
- Let $\langle S; i, T \rangle$ denote $(S_1, \ldots, S_{i-1}, T, S_{i+1}, \ldots, S_n)$ for some subset $T \subset S_i$. Then the $\rho(\langle S \rangle; i, T, U)$ are isomorphisms from $C_{\langle S;i,T \rangle} \oplus C_{\langle S;i,U \rangle}$ to $C_{\langle S \rangle}$ for $i = 1, \ldots, n$ and $T, U \subset S_i$ with $T \cap U = \varnothing$ and $T \cup U = S_i$.

The $(C_{\langle S \rangle}, \rho(\langle S \rangle; i, T, U))$ satisfy the following properties.

(1) If one $S_i = \varnothing$ for $i \in \{1, \ldots, n\}$, then $C_{\langle S \rangle} = 0_\mathcal{R}$.
(2) If one of the S_i or T or U is empty, then $\rho(\langle S \rangle; i, T, U) = \mathrm{id}$.
(3) If $c_\oplus$ denotes the twist of the permutative structure $(\mathcal{R}, \oplus, 0_\mathcal{R})$, then

$$\rho(\langle S \rangle; i, T, U) = \rho(\langle S \rangle; i, U, T) \circ c_\oplus\,.$$

(4) The $\rho(\langle S \rangle; i, T, U)$ are associative, *i.e.*, for all $\langle S \rangle, i$ and pairwise disjoint $T, U, V \subset S_i$ with $T \cup U \cup V = S_i$ the diagram

$$\begin{tikzcd}
C_{\langle S;i,T \rangle} \oplus C_{\langle S;i,U \rangle} \oplus C_{\langle S;i,V \rangle} \arrow[r, "\rho(\langle S;i,T\cup U \rangle;i,T,U)\oplus \mathrm{id}"] \arrow[d, "\mathrm{id}\oplus\rho(\langle S;i,U\cup V \rangle;i,U,V)"'] & C_{\langle S;i,T\cup U \rangle} \oplus C_{\langle S;i,V \rangle} \arrow[d, "\rho(\langle S \rangle;i,T\cup U,V)"] \\
C_{\langle S;i,T \rangle} \oplus C_{\langle S;i,U\cup V \rangle} \arrow[r, "\rho(\langle S \rangle;i,T,U\cup V)"] & C_{\langle S \rangle}
\end{tikzcd}$$

commutes.

(5) The $\rho(\langle S \rangle; i, T, U)$ satisfy the pentagon rule, *i.e.*, for $i \neq j$ and $T, U \subset S_i$, $V, W \subset S_j$ with $T \cap U = \varnothing = V \cap W$ the diagram

$$\begin{tikzcd}
{} & C_{\langle S;j,V \rangle} \oplus C_{\langle S;j,W \rangle} \arrow[ddr, "\rho(\langle S \rangle;j,V,W)"] & \\
C_{\langle S;i,T;j,V \rangle} \oplus C_{\langle S;i,U;j,V \rangle} \oplus C_{\langle S;i,T;j,W \rangle} \oplus C_{\langle S;i,U;j,W \rangle} \arrow[ur, "\rho(\langle S;j,V \rangle;i,T,U)\oplus\rho(\langle S;j,W \rangle;i,T,U)"] \arrow[dd, "\mathrm{id}\oplus c_\oplus\oplus\mathrm{id}"'] & & \\
& & C_{\langle S \rangle} \\
C_{\langle S;i,T;j,V \rangle} \oplus C_{\langle S;i,T;j,W \rangle} \oplus C_{\langle S;i,U;j,V \rangle} \oplus C_{\langle S;i,U;j,W \rangle} \arrow[dr, "\rho(\langle S;i,T \rangle;j,V,W)\oplus\rho(\langle S;i,U \rangle;j,V,W)"'] & & \\
& C_{\langle S;i,T \rangle} \oplus C_{\langle S;i,U \rangle} \arrow[uur, "\rho(\langle S \rangle;i,T,U)"'] &
\end{tikzcd}$$

commutes.

Morphisms in the category consist of morphisms $f_{\langle S \rangle} \colon C_{\langle S \rangle} \longrightarrow D_{\langle S \rangle}$ in $\mathcal{R}$ that are the identity if any of the S_i is empty. These morphisms have to commute with the structure maps $\rho(\langle S \rangle; i, T, U)$.

Thus $\bar{H}\mathcal{R}$ is a functor from the n-fold product of the category Γ of finite pointed sets to the category of permutative categories. If $f \colon X_+ \to Y_+$ is a map of finite pointed sets and $(C_{\langle S \rangle}, \rho(\langle S \rangle; i, T, U))$ is an object in $\bar{H}\mathcal{R}$, then $f_*(C_{\langle S \rangle}, \rho(\langle S \rangle; i, T, U))$ is the object that is given by the cube with values

$f_*C_{\langle S\rangle} := C_{\langle f^{-1}(S)\rangle}$ for all subsets S of Y. As f respects the basepoint $+$, this is well-defined because $f^{-1}(T)$ does not contain the basepoint for $T \subset S$. The structure maps $f_*\rho$ are given by

$$\begin{tikzcd} f_*C_{\langle S;i,T\rangle} \oplus f_*C_{\langle S;i,U\rangle} \arrow[r, equal] & C_{\langle f^{-1}S;i,f^{-1}T\rangle} \oplus C_{\langle f^{-1}S;i,f^{-1}U\rangle} \arrow[d, "\rho(\langle f^{-1}S\rangle;i,f^{-1}T,f^{-1}U)"] \\ & C_{\langle f^{-1}S;i,f^{-1}T\cup f^{-1}U\rangle} = f_*C_{\langle S\rangle}\,. \end{tikzcd}$$

The permutative structure on $\bar{H}\mathcal{R}$ is given by sending $(C_{\langle S\rangle}, \rho(\langle S\rangle;i,T,U))$ and $(D_{\langle S\rangle}, \rho'(\langle S\rangle;i,T,U))$ to the object

$$(C_{\langle S\rangle} \oplus D_{\langle S\rangle}, \rho''(\langle S\rangle;i,T,U))$$

where the structure map ρ'' is given by

$$\begin{tikzcd} (C\oplus D)_{\langle S;i,T\rangle} \oplus (C\oplus D)_{\langle S;i,U\rangle} \arrow[r, equal] & C_{\langle S;i,T\rangle} \oplus D_{\langle S;i,T\rangle} \oplus C_{\langle S;i,U\rangle} \oplus D_{\langle S;i,U\rangle} \arrow[d, "\mathrm{id}\oplus c_\oplus \oplus \mathrm{id}"] \\ & C_{\langle S;i,T\rangle} \oplus C_{\langle S;i,U\rangle} \oplus D_{\langle S;i,T\rangle} \oplus D_{\langle S;i,U\rangle} \arrow[d, "\rho(\langle S\rangle;i,T,U)\oplus\rho'(\langle S\rangle;i,T,U)"] \\ & C_{\langle S\rangle} \oplus D_{\langle S\rangle}\,. \end{tikzcd}$$

The above definition is quite close to the one in [EM]; however, we require the gluing maps ρ to be isomorphisms.

The case $n = 1$ is well studied: let m_+ denote the finite pointed set $\{0, 1, \ldots, m\}$ with 0 as basepoint.

Lemma 12.1. [ShSh, Lemma 2.2] *The canonical map*

$$\bar{H}\mathcal{R}(m_+) \longrightarrow \bar{H}\mathcal{R}(1_+) \times \ldots \times \bar{H}\mathcal{R}(1_+)$$

is an equivalence of categories.

Let $X_1, \ldots, X_n$ be finite pointed simplicial sets. We define $\bar{H}\mathcal{R}(X_1, \ldots, X_n)$ to be the n-simplicial permutative category with

$$\bar{H}\mathcal{R}(X_1, \ldots, X_n)_{(\ell_1,\ldots,\ell_n)} := \bar{H}\mathcal{R}((X_1)_{\ell_1}, \ldots, (X_n)_{\ell_n})$$

for $\ell_i \in \Delta$.

Definition 12.2. For a strictly bimonoidal category $\mathcal{R}$ we define $K^n\mathcal{R}$ to be the category that is the limit of the diagram

$$\bar{H}\mathcal{R}(Y_{i_1}, \ldots, Y_{i_n})$$

with $i_j \in \{0, 1, 2\}$ and $Y_0 = \mathbb{S}^1$, $Y_1 = Y_2 = P\mathbb{S}^1$ and $d_0 \colon P\mathbb{S}^1 \to \mathbb{S}^1$.

Here, $\mathbb{S}^1$ is the small simplicial model of the 1-sphere, P denotes the simplicial model of the path space functor which takes a simplicial set X to the simplicial set PX with $PX_n = X_{n+1}$. The map $d_0 \colon X_{n+1} \to X_n$ induces a map $PX \to X$.

More generally, we define for pointed simplicial sets $X_1, \ldots, X_n$

$$\tilde{H}\mathcal{R}(X_1, \ldots, X_n) := \lim \bar{H}\mathcal{R}(Y_{i_1} \wedge X_1, \ldots, Y_{i_n} \wedge X_n)$$

where the Y_{i_j} are as above.

Note that $K^1\mathcal{R}$ corresponds to the classical case ([ShSh, M3, Se]). It is the pullback of the diagram

$$\begin{tikzcd} & \bar{H}\mathcal{R}(P\mathbb{S}^1) \arrow[d, "d_0"] \\ \bar{H}\mathcal{R}(P\mathbb{S}^1) \arrow[r, "d_0"'] & \bar{H}\mathcal{R}(\mathbb{S}^1). \end{tikzcd}$$

Lemma 12.3. *The set of path components $\pi_0(K^1\mathcal{R})$ is an abelian group.*

Proof. The pullback $K^1\mathcal{R}$ is a simplicial permutative category. Therefore $\pi_0(K^1\mathcal{R})$ is an abelian monoid. Switching the two copies of $\bar{H}\mathcal{R}(P\mathbb{S}^1)$ in the defining diagram for $K^1\mathcal{R}$ results in a homotopy inverse which gives $\pi_0(K^1\mathcal{R})$ a group structure. $\square$

There is a natural pairing

$$K^n\mathcal{R} \times K^m\mathcal{R} \longrightarrow K^{n+m}\mathcal{R}$$

which is induced by

$$\begin{array}{rcl} \bar{H}\mathcal{R}(X^1_+, \dots, X^n_+) \times \bar{H}\mathcal{R}(X^{n+1}_+, \dots, X^{n+m}_+) & \longrightarrow & \bar{H}\mathcal{R}(X^1_+, \dots, X^{n+m}_+) \\ (C_{\langle S\rangle}, D_{\langle T\rangle}) & \mapsto & (C \otimes D)_{\langle U\rangle} \end{array}$$

with

$$(C \otimes D)_{(U_1,\dots,U_{n+m})} := C_{(U_1,\dots,U_n)} \otimes D_{(U_{n+1},\dots,U_{n+m})}.$$

The functors K^n are natural with respect to strictly bipermutative functors between bipermutative categories.

Let I be the category of finite sets and injective functions. Any morphism in I can be expressed as a composition of an order preserving injection with a permutation. For a permutation $\sigma \in \Sigma_n$ we obtain from [EM, §4], that the induced map

$$\sigma\colon \bar{H}\mathcal{R}(X^1_+, \dots, X^n_+) \longrightarrow \bar{H}\mathcal{R}(X^{\sigma^{-1}(1)}_+, \dots, X^{\sigma^{-1}(n)}_+)$$

is an equivalence of categories. Thus it induces an equivalence of n-simplicial categories on $K^n\mathcal{R}$.

Let $i\colon n \to n{+}1$ be the standard inclusion which misses the element $n{+}1$. Then Elmendorf and Mandell show in their discussion of Extension Functors [EM, §4], that there is an isomorphism of categories

$$i\colon \bar{H}\mathcal{R}(X^1_+, \dots, X^n_+) \longrightarrow \bar{H}\mathcal{R}(X^1_+, \dots, X^n_+, 1_+)$$

for every n-tuple of pointed sets $(X^1_+, \dots, X^n_+)$ (compare [Sh, p. 380]). This induces a map $K^n\mathcal{R} \to K^{n+1}\mathcal{R}$ as follows. First of all the maps $\bar{H}\mathcal{R}(X^1_+, \dots, X^n_+) \longrightarrow \bar{H}\mathcal{R}(X^1_+, \dots, X^n_+, 1_+)$ induce a map from $K^n\mathcal{R}$ to the limit of the system $\bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, 1_+)$. The natural maps from 1_+ to $(P\mathbb{S}^1)_0 = 1_+$ and $\mathbb{S}^1_0 = +$ then yield the desired map to $K^{n+1}\mathcal{R}$.

One can check that these structure maps fit together to give the following result.

Theorem 12.4. *The assignment $n \mapsto K^n\mathcal{R}$ turns $K^\bullet\mathcal{R}$ into an I-graded bimonoidal category.*

Fixing finite pointed sets $X^1_+, \dots, X^n_+$, $\bar{H}\mathcal{R}(X^1_+, \dots, X^n_+, -)$ is a functor from the category of finite pointed sets to n-fold simplicial categories. Similar to Lemma 12.1 we get that this is a special Γ-space in the sense of Segal (up to some realizations resp. diagonals).

Lemma 12.5. *The canonical inclusion $n \to n+1$ induces a weak equivalence $K^n\mathcal{R} \to K^{n+1}\mathcal{R}$ for $n \geqslant 1$.*

Proof. Note that

$$\tilde{H}\mathcal{R}(1_+) = \lim \bar{H}\mathcal{R}(Y_{i_1}) \sim \Omega\bar{H}\mathcal{R}(\mathbb{S}^1)$$

because the natural map is a homology isomorphism of H-spaces (compare [Se, §4]). We know that $K^n\mathcal{R} \cong \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, 1_+)$. As n is at least one, the defining diagram for this limit admits a flip-map and therefore

$$\lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, 1_+) \sim \Omega \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, \mathbb{S}^1)\,.$$

An argument similar to the one at the beginning of the proof shows that

$$\Omega \lim \bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, \mathbb{S}^1) = \lim \Omega\bar{H}\mathcal{R}(Y_{i_1}, \dots, Y_{i_n}, \mathbb{S}^1) \sim \tilde{H}\mathcal{R}(1_+, \dots, 1_+) = K^{n+1}\mathcal{R}\,.$$

□

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