# THE SEIBERG–WITTEN MODULI SPACE

**The Seiberg–Witten equations.**

Let $X$ be a smooth oriented Riemannian 4-manifold. Let $P \to X$ be the principal $SO(4)$-bundle associated to the tangent bundle $TX \to X$, and let $\tilde{P} \to X$ be a choice of $Spin^c$-structure on $X$, *i.e.* a principal $Spin^c(4)$-bundle with a $Spin^c(4) \to SO(4)$-equivariant bundle map to $P \to X$. Let $\mathbb{C}^2 \to S^{\pm}(\tilde{P}) \to X$ be the plus and minus complex spin bundles associated to the $Spin^c$-structure, and let $\mathbb{C}^1 \to \mathcal{L} \to X$ be their associated determinant bundle.

Let

$$\mathcal{C} = \mathcal{A}(\mathcal{L})_{L^2_2} \times \Gamma(S^+(\tilde{P}))_{L^2_2}$$

be the *configuration space*, where $\mathcal{A}(\mathcal{L})$ denotes the affine space of $U(1)$-connections $A$ on $\mathcal{L}$, and $\Gamma(S^+(\tilde{P}))$ denotes the space of (smooth) sections $\psi$ in the plus complex spin bundle $S^+(\tilde{P}) \to X$, *i.e.* a positive spinor field. The subscript $L^2_2$ denotes completion with respect to the $L^2_2$-Sobolev norm. The tangent space of $\mathcal{C}$ at $(A, \psi)$ is

$$T_{(A,\psi)}\mathcal{C} = \Omega^1(X, i\mathbb{R})_{L^2_2} \times \Gamma(S^+(\tilde{P}))_{L^2_2}.$$

Let

$$\mathcal{D} = \Omega^2_+(X; i\mathbb{R})_{L^2_1} \times \Gamma(S^-(\tilde{P}))_{L^2_1},$$

where $\Omega^2_+(X; i\mathbb{R})$ denotes the self-dual 2-forms on $X$ with coefficients in $i\mathbb{R} = u(1)$, *i.e.* those in the $+1$-eigenspace for the involution induced by the Hodge $*$-operation, or equivalently for the multiplication by the complex unit $\omega_{\mathbb{C}}$. As above $\Gamma(S^-(\tilde{P}))$ denotes the space of sections in the minus complex spinor bundle $S^-(\tilde{P})$, *i.e.* a negative spinor field. The subscript $L^2_1$ denotes completion with respect to the $L^2_1$-Sobolev norm. The tangent space of $\mathcal{D}$ at $(\omega, \mu)$ is

$$T_{(\omega,\mu)}\mathcal{D} = \Omega^2(X, i\mathbb{R}))_{L^2_1} \times \Gamma(S^-(\tilde{P}))_{L^2_1}.$$

Let $F \colon \mathcal{C} \to \mathcal{D}$ be the *Seiberg–Witten functional* given by

$$F(A, \psi) = (F^+_A - q(\psi), \partial\!\!\!/_A \psi).$$

Here $F^+_A$ is the self-dual (positive) part of the curvature 2-form of the connection $A$, and $q(\psi) = \psi \otimes \psi^* - \frac{1}{2}|\psi|^2 \cdot \mathrm{id}$ is the trace 0 part of the rank 1 endomorphism $\psi \otimes \psi^* \colon \mu \mapsto \langle \mu, \psi \rangle \psi$, viewed as a self-dual 2-form through the identification

$$\Omega^2_+(X; \mathbb{C}) \cong \mathrm{End}_{\mathbb{C}}(S^+(\tilde{P}))_0,$$

and $\partial\!\!\!/_A$ is the Dirac operator for the Levi–Civita connection on $P \to X$ and the connection $A$ on $\mathcal{L}$. Then $F$ is smooth. The derivative of $F$ at $(A, \psi)$ has matrix

$$DF_{(A,\psi)} = \begin{bmatrix} P_+ d & Dq_w \\ \cdot \frac{1}{2}\psi & \partial\!\!\!/_A \end{bmatrix}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX where $P_+ \colon \Omega^2 \to \Omega^2_+$ denotes projection to the plus part.

Let the *gauge group*

$$\mathcal{G} = \Omega^0(X, S^1)_{L^2_3}$$

be the Sobolev $L^2_3$-completion of the space of (smooth) maps $\sigma \colon X \to S^1$. We identify $\sigma$ with the gauge equivalence $\tilde{P} \to \tilde{P}$ over $P$ given by right scalar multiplication through $S^1 \subset Spin^c(4)$. Then $\mathcal{G}$ acts smoothly on $\mathcal{C}$ and $\mathcal{D}$ by the formulas

$$\begin{aligned} (A, \psi) \cdot \sigma &= (\det(\sigma)^* A, S^+(\sigma^{-1})(\psi)) = (A + 2\sigma^{-1} d\sigma, \sigma^{-1} \cdot \psi) \\ (\omega, \mu) \cdot \sigma &= (\omega, S^-(\sigma^{-1})(\mu)) = (\omega, \sigma^{-1} \cdot \mu), \end{aligned}$$

and the map $F$ is $\mathcal{G}$-equivariant.

**Local structure of $\mathcal{B}$.**

The action of $\mathcal{G}$ on $\mathcal{C}$ has slices. Let $(A, \psi) \in \mathcal{C}$ be a configuration. Then the action map $m \colon \mathcal{G} \to \mathcal{C}$ has derivative

$$Dm = Dm_{\mathrm{id}} \colon T_{\mathrm{id}}\mathcal{G} \to T_{(A,\psi)}\mathcal{C}$$

at the identity given by $Dm(f) = (2df, -f \cdot \psi)$. Let $K \subset T_{(A,\psi)}\mathcal{C}$ be the orthogonal complement of the image of $Dm$, with respect to the $L^2$-inner product on $T_{(A,\psi)}\mathcal{C}$. Equivalently $K$ is the kernel of the adjoint $(Dm)^*$ to $Dm$. Then for $U \subset K$ a (small) $Stab_{(A,\psi)}$-invariant neighbourhood of $0 \in K$ the subspace

$$S = (A, \psi) + U \subset \mathcal{C}$$

is a slice for the action of $\mathcal{G}$ on $\mathcal{C}$.

Let $\mathcal{B} = \mathcal{C}/\mathcal{G}$ be the *orbit space* for the gauge action. Let $\mathcal{N} = F^{-1}(0) \subset \mathbb{C}$ be the *solution space* of configurations satisfying the Seiberg–Witten equations $F(A, \psi) = 0$. This subspace is $\mathcal{G}$-invariant. Let $\mathcal{M} = \mathcal{N}/\mathcal{G} \subset \mathcal{B}$ be the *Seiberg–Witten moduli space* of gauge equivalence classes of solutions to the Seiberg–Witten equations. When it makes sense, the *Seiberg–Witten invariant* of $X$ (with the given $Spin^c$-structure) will be defined as the bordism class of $\mathcal{M} \to \mathcal{B}$.

$$\begin{array}{ccccc} \mathcal{G} & & \mathcal{G} & & \\ \downarrow & & \downarrow & & \\ \mathcal{N} & \longrightarrow & \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} & & \\ \mathcal{M} & \longrightarrow & \mathcal{B} & & \end{array}$$

A configuration $(A, \psi)$ is *reducible* if $\psi \equiv 0$ and *irreducible* otherwise. Let $\mathcal{C}^* \subset \mathcal{C}$ be the open subset of irreducible configurations, This subspace is $\mathcal{G}$-invariant. Similarly let $\mathcal{N}^* \subset \mathcal{N}$, $\mathcal{B}^* \subset \mathcal{B}$ and $\mathcal{M}^* \subset \mathcal{M}$ be the open subsets of (gauge equivalence classes of) irreducible configurations in the given spaces. Write $[A, \psi]$ for the gauge equivalence class of $(A, \psi)$.

The action of $\mathcal{G}$ is free on the irreducible configurations $\mathcal{C}^*$, and has stabilizer $S^1 \subset \mathcal{G}$ on the reducible configurations. Hence $\mathcal{B}^*$ can be given the structure of a

(Hausdorff) Banach manifold with local parametrizations $S \subset \mathcal{C}^* \to \mathcal{B}^*$ near $[A, \psi]$, when $S$ is a slice for the action at $(A, \psi)$. Its tangent space there is then

$$T_{[A,\psi]}\mathcal{B}^* = \operatorname{coker}(Dm) = T_{(A,\psi)}\mathcal{C}/Dm(T_{\mathrm{id}}\mathcal{G}) \cong K \subset T_{(A,\psi)}\mathcal{C}.$$

Since $\mathcal{G}$ acts freely with slices on $\mathcal{C}^*$ there is a fiber bundle $\mathcal{G} \to \mathcal{C}^* \to \mathcal{B}^*$. This bundle is universal because $\mathcal{C}^*$ is contractible, so $\mathcal{B}^*$ has the homotopy type of

$$B\,\mathrm{Map}(X, S^1) \simeq \mathbb{C}P^\infty \times BH^1(X; \mathbb{Z}),$$

suitably completed.

**Local structure of the moduli space.**

Let $(A, \psi) \in \mathcal{N}^*$ be an irreducible solution. We have an elliptic complex

$$0 \to T_{\mathrm{id}}\mathcal{G} \xrightarrow{Dm_{\mathrm{id}}} T_{(A,\psi)}\mathcal{C} \xrightarrow{DF_{(A,\psi)}} T_0\mathcal{D} \to 0$$

denoted $\mathcal{E} = \mathcal{E}(A, \psi)$. Here $Dm$ is injective because the action of $\mathcal{G}$ on $(A, \psi)$ is free, so $H^0(\mathcal{E}) = 0$.

We say that $\mathcal{M}$ is *smooth at* $[A, \psi]$ if $F\colon \mathcal{C} \to \mathcal{D}$ is a submersion at $(A, \psi)$, *i.e.* if $DF_{(A,\psi)}$ is surjective. Then $H^2(\mathcal{E}) = 0$.

Now suppose that $(A, \psi)$ is a smooth irreducible solution. Then by the implicit function theorem $\mathcal{N} \subset \mathcal{C}$ is a Banach submanifold locally near $(A, \psi)$, with tangent space

$$T_{(A,\psi)}\mathcal{N} = \ker(DF) \subset T_{(A,\psi)}\mathcal{C}.$$

Then the orbit of $\mathcal{G}$ acting on $(A, \psi)$ is contained in $\mathcal{N}$ and has tangent space $\mathrm{im}(Dm)$. Then $K \cap T_{(A,\psi)}\mathcal{N}$ is its orthogonal complement in $T_{(A,\psi)}\mathcal{N}$, and gives a slice for the $\mathcal{G}$-action on $\mathcal{N}$ at $(A, \psi)$. For the action map of $\mathcal{G}$ on the slice has derivative

$$(K \cap T_{(A,\psi)}\mathcal{N}) \times T_{\mathrm{id}}\mathcal{G} \to T_{(A,\psi)}\mathcal{N} \subset T_{(A,\psi)}\mathcal{C}$$

at $((A, \psi), \mathrm{id})$, which is the inclusion on the first factor and $Dm$ on the second. We claim that this map is an isomorphism. It is injective because $K \cap \mathrm{im}(Dm) = 0$. Given $n \in T_{(A,\psi)}\mathcal{N}$ write $n = k + g$ with $k \in K$ and $g \in \mathrm{im}(Dm)$. Then $g \in T_{(A,\psi)}\mathcal{N}$ so $k = n - g \in K \cap T_{(A,\psi)}\mathcal{N}$, so the map is also surjective.

Hence $\mathcal{N} \to \mathcal{M}$ is locally a fiber bundle with structure group $\mathcal{G}$, near $[A, \psi]$, and the smooth irreducible points of $\mathcal{M}$ form a (Hausdorff) Banach manifold locally modelled on $K \cap T_{(A,\psi)}\mathcal{N}$. It has tangent space

$$T_{[A,\psi]}\mathcal{M} = \ker(DF)/\,\mathrm{im}(Dm) \cong K \cap T_{(A,\psi)}\mathcal{N}$$

equal to the *Zariski tangent space* $H^1(\mathcal{E})$ at $[A, \psi]$.

An application of the Atiyah–Singer index theorem shows that $H^1(\mathcal{E})$ has finite dimension

$$d = \frac{1}{4}(c_1^2(\mathcal{L}) - 2\chi(X) - 3\,\mathrm{sign}(X))$$

and so in fact $\mathcal{M}$ is locally a $d$-dimensional (ordinary) Hausdorff manifold near its smooth irreducible points.