THE TITS BUILDINGS FOR $\mathbb{Z}/p^n$

John Rognes

April 1991

Abstract. We study the algebraic K–theory of a cyclic ring R of the form $\mathbb{Z}/p^n$. The idea is to consider the so–called rank filtration on a space representing the first delooping of K–theory. Each subquotient of this filtration is equivalent to the homotopy quotient of an action of the general linear group $GL_r R$ on the double suspension of a finite complex $B_r R$, which is the rank r Tits building of R. To understand the homotopy type of these subquotients, we equivalently study the homotopy type of the Tits buildings together with the $GL_r R$–action.

Our results include 1) a description of the non–equivariant homotopy type of each Tits building, as a bouquet of spheres all in the same dimension, 2) a computation of the equivariant homology of a subcomplex of $B_r R$ called the small building, which captures the difference between the K–theory of R and the known K–theory of $\mathbb{F}_p$, the finite field with p elements, and 3) a description of the homotopy type of each fixed subcomplex of the Tits building under the action of a subgroup of the general linear group, in a (sufficiently) large class of such subgoups.

Introduction

`Not yet written`

K–theory and Tits Buildings

In this section we will review definitions of K–theory and the rank filtration, and fix some notation.

The K–theory of a ring. Let R be an associative ring with unit. The algebraic K–theory of R can be defined as follows. Let $\mathcal{P}(R)$ be the additive category of finitely generated projective R–modules, viewed as a category with cofibrations and weak equivalences as per [Quillen, §2] and [Waldhausen, §1.9]. Explicitly, a cofibration is the inclusion of a direct summand in the category, and a weak equivalence is an isomorphism. Then following [Waldhausen], we can apply the $S_\bullet$–construction to $\mathcal{P}(R)$, obtaining a simplicial category $wS_\bullet\mathcal{P}(R)$ whose geometric realization is the (first) delooping of K–theory. This means that the higher algebraic K–groups of R can be defined as:

$$K_i R = \pi_{i+1}|wS_\bullet\mathcal{P}(R)| \qquad \text{for } i \geq 0.$$

This definition agrees with that of [Quillen] using the Q–construction. Equivalently, we may think of the K–groups of R as the homotopy groups of the loopspace $\Omega|wS_\bullet\mathcal{P}(R)|$.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Let $\mathcal{F}(R) \subseteq \mathcal{P}(R)$ denote the full subcategory of finitely generated free R–modules. It is a subcategory with cofibrations and weak equivalences in the sense of [Waldhausen, p. 321], and furthermore it is (weakly) cofinal in $\mathcal{P}(R)$ [Grayson], so the induced map

$$|wS_\bullet\mathcal{F}(R)| \to |wS_\bullet\mathcal{P}(R)|$$

is a covering map. If we define the *free K–theory* of R by

$$K_i^f R = \pi_{i+1}|wS_\bullet\mathcal{F}(R)| \qquad \text{for } i \geq 0,$$

it follows that $K_i^f R \to K_i R$ is an isomorphism for $i \geq 1$, while it is an injection for $i = 0$. We shall be considering the free K–theory of R in this paper, but by the above argument this is the same as the usual K–theory in all positive degrees, so we shall henceforth suppress the difference in notation, and use $K_i R$ to denote the free K–groups. Let $BK(R)$ denote the first delooping of free K–theory; $BK(R) = |wS_\bullet\mathcal{F}(R)|$.

The rank filtration. Now suppose that the rank of a finitely generated free R–module is well defined. Concretely, we will assume that R has the *invariant dimension property* [Mitchell], i.e. R^n and R^m are isomorphic as R–modules only if $n = m$. This automatically holds if R is commutative [Atiyah–Macdonald]. It is also clear that R has this property if it is an algebra over a commutative ring, for which it is finitely generated as a module. The examples we have in mind are matrix algebras over commutative rings. Denote the rank of a finitely generated free R–module M by $\mathrm{rank}(M)$. Then if $M' \rightarrowtail M \twoheadrightarrow M''$ is a cofibration sequence in $\mathcal{F}(R)$, i.e. a split short exact sequence of free R–modules, we have

$$\mathrm{rank}(M) = \mathrm{rank}(M') + \mathrm{rank}(M'').$$

We wish to approximate the K–theory of R by introducing a filtration of the delooped K–theory space, such that each successive stage of the filtration is a better approximation to the full K–theory. We do this, following [Quillen] and [Mitchell], by a *rank filtration* of the space $BK(R)$. Inspecting the $S_\bullet$–construction used to define this space, we see that its simplices correspond to diagrams in the category $\mathcal{F}(R)$. For a fixed rank r we can consider the subcomplex $F_r BK(R)$ consisting of simplices corresponding to diagrams involving only free R–modules of rank less then or equal to r. This gives an increasing filtration of spaces $\{F_r BK(R)\}_{r\geq 0}$ exhausting $BK(R)$, and we call $F_r BK(R)$ the rth stage of the rank filtration on the first delooping of the K–theory of R.

There are also other definitions of a rank filtration on K–theory. By analogy to the plus-construction definition of K–theory [Quillen], one can define

$$K_{i,r}^Q R = \pi_i (BGL_r R)^+ \qquad \text{for } i > 0,$$

as soon as a suitable perfect subgroup can be found, for instance if $r \geq 3$. There is also a Volodin construction [Suslin], where

$$K_{i,r}^V R = \pi_{i+1} V(GL_r R, \{T^\alpha\}) \qquad \text{for } i \geq 0,$$

and Suslin shows that these two agree through a range. See Suslin's paper for a definition of the expression above. Lastly there is a spectrum level rank filtration of the K–theory spectrum, different from all the above, which is developed in [Rognes, thesis].

The subquotients. Our approach to understanding the K–groups of R will be to investigate the homotopy type of the subquotients of the rank filtration, i.e. the spaces $F_rBK(R)/F_{r-1}BK(R)$. There is a description of this in terms of the equivariant homotopy type of the rank r Tits building of R [Tits], as noted in [Quillen] and [Mitchell]. We now define the Tits buildings, state the description (proposition 1), and give a proof.

Definition. Suppose R satisfies the invariant dimension property, and fix a rank r. Consider the set of proper, nontrivial free R–submodules M of R^r: $0 \neq M \subset R^r$. Give this set a partial ordering by setting $M \prec N$ if $M \subseteq N$ and the inclusion map is a cofibration in $\mathcal{F}(R)$, i.e. M is included as a direct summand in N with a free complementary summand. Denote this partially ordered set by $\mathcal{O}(R^r)$, and define the rank r *Tits building* of R, B_rR, to be its nerve [Quillen] :

$$B_rR = N\mathcal{O}(R^r).$$

Hence B_rR is a simplicial set, with q–simplices the sequences of cofibrations

$$0 \neq M_0 \subseteq \cdots \subseteq M_q \subset R^r$$

of free R–modules, also known as *flags*.

Proposition 1 [Quillen] [Mitchell]**.**

$$F_rBK(R)/F_{r-1}BK(R) \simeq EGL_rR_+ \wedge_{GL_rR} \Sigma^2 B_rR.$$

Proof. Review the $S_\bullet$–construction from [Waldhausen, §1.3]. $BK(R) = |wS_\bullet\mathcal{F}(R)|$ where $wS_\bullet\mathcal{F}(R)$ is the simplicial category $[q] \mapsto wS_q\mathcal{F}(R)$. An object in $S_q\mathcal{F}(R)$ is a diagram $Ar[q] \to \mathcal{F}(R)$ satisfying certain extra hypotheses. Here $[q] = \{0 < 1 < \cdots < q\}$ is thought of as a category with $q+1$ objects, and $Ar[q]$ is the arrow category associated with $[q]$. Specifically, a diagram $A : Ar[q] \to \mathcal{F}(R)$ associates to each $i \leq j$ a free R–module $A_{i,j} = A(i \to j)$, and A lies in $S_q\mathcal{F}(R)$ if the following three conditions hold: 1) $A_{i,i} = 0$ for each i, 2) the natural map $A_{i,j} \to A_{i,k}$ is a cofibration for each $i \leq j \leq k$, and 3) the commutative square

$$\begin{tikzcd} A_{i,j} \arrow[r] \arrow[d] & A_{i,k} \arrow[d] \\ A_{j,j} \arrow[r] & A_{j,k} \end{tikzcd}$$

is a pushout square for each $i \leq j \leq k$. Such a diagram A is determined up to isomorphism by its restriction across $[q] \hookrightarrow Ar[q]$ taking i in $[q]$ to $(0 \to i)$ in $Ar[q]$. The restriction of a diagram A is then a sequence of cofibrations

$$0 = A_{0,0} \to A_{0,1} \to \cdots \to A_{0,q}.$$

Let $\mathcal{F}_r(R) \subset \mathcal{F}(R)$ denote the full subcategory where the objects have rank at most r. Then $F_rBK(R)$ realizes the simplicial subcategory $wF_rS_\bullet\mathcal{F}(R)$ where

$F_r S_q \mathcal{F}(R) \subset S_q \mathcal{F}(R)$ has objects the diagrams $Ar[q] \to \mathcal{F}(R)$ which factor through $\mathcal{F}_r(R) \subset \mathcal{F}(R)$. Similarly, $F_r BK(R)/F_{r-1}BK(R)$ realizes the simplicial (quotient) category $[q] \mapsto wF_r S_q \mathcal{F}(R)/wF_{r-1} S_q \mathcal{F}(R)$. For fixed q, the latter category has a base point object $*_q$, together with the diagrams $Ar[q] \to \mathcal{F}(R)$ in $S_q\mathcal{F}(R)$ which factor through $\mathcal{F}_r(R)$, but not through $\mathcal{F}_{r-1}(R)$. These are precisely the diagrams A such that the 'top' module $A_{0,q}$ has rank r.

Inspecting the simplicial structure on $wF_r S_\bullet \mathcal{F}(R)/wF_{r-1} S_\bullet \mathcal{F}(R)$, we see that the restriction of diagrams over $[q] \hookrightarrow Ar[q]$ as above induces an equivalence of simplicial categories. Furthermore, the target simplicial category is equivalent to its full simplicial subcategory where the top module $A_{0,q}$ actually is R^r. Thus $F_r BK(R)/F_{r-1}BK(R)$ is homotopy equivalent to the realization of the simplicial category $Y_\bullet$, which in simplicial degree q has the objects $*_q$ and the cofibration sequences (flags)

$$0 = A_{0,0} \to A_{0,1} \to \cdots \to A_{0,q} = R^r,$$

and morphisms the isomorphisms of such diagrams.

Let $X_\bullet$ be the simplicial set which has as q–simplices the object set of Y_q. For any flag $A \in X_q$, the morphisms in Y_q originating at A are precisely characterized by their effect on the top module $A_{0,q} = R^r$, i.e. an element of $GL_r R$. In this situation we call $Y_\bullet$ the (based) *$GL_r R$–translation category* on $X_\bullet$. Using the usual simplicial model for $EGL_r R$ it is straightforward to check that the nerve of Y_q is isomorphic to $EGL_r R_+ \wedge_{GL_r R} X_q$, and upon realizing in the q–direction we obtain:

$$|Y_\bullet| \cong EGL_r R_+ \wedge_{GL_r R} |X_\bullet|.$$

It remains to recognize $|X_\bullet|$ as $\Sigma^2 B_r R$.

Let $\mathcal{O}_*^*(R^r)$ denote the partially ordered set of (not necessarily proper or nontrivial) free R–submodules M of R^r, with $M \prec N$ if $M \subseteq N$ and the inclusion map is a cofibration. Let $\mathcal{O}_*(R^r)$ denote the partially ordered subset of proper submodules, and let $\mathcal{O}^*(R^r)$ denote the partially ordered subset of nontrivial submodules. The nerve of either of these three partially ordered sets is contractible, due to the presence of initial and/or terminal elements.

Now note that $\mathcal{O}(R^r)$ is the intersection of $\mathcal{O}_*(R^r)$ and $\mathcal{O}^*(R^r)$. Furthermore, $|X_\bullet|$ is obtained from the nerve of $\mathcal{O}_*^*(R^r)$ by identifying any flags

$$0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_q \subseteq R^r$$

not beginning with $M_0 = 0$ or ending with $M_q = R^r$ to $*_q$, which amounts to contracting the nerves of $\mathcal{O}_*(R^r)$ and $\mathcal{O}^*(R^r)$ to a point. We conclude :

$$\begin{aligned} |X_\bullet| &= N\mathcal{O}_*^*(R^r)/(N\mathcal{O}_*(R^r) \cup N\mathcal{O}^*(R^r)) \\ &\simeq \Sigma(N\mathcal{O}_*(R^r) \cup N\mathcal{O}^*(R^r)) \\ &\simeq \Sigma^2(N\mathcal{O}_*(R^r) \cap N\mathcal{O}^*(R^r)) \\ &= \Sigma^2 B_r R. \quad \square \end{aligned}$$

Thus an understanding of a weak form of the $GL_r R$–homotopy type of $B_r R$ suffices to describe the subquotients of the rank filtration, in the sense that a $GL_r R$–map which is a non–equivariant homotopy equivalence is viewed as an equivalence.

Tits Buildings for $\mathbb{Z}/p^n$

We now specialize to studying the K–theory of the 'cyclic rings' $\mathbb{Z}/p^n$, with p a prime. The K–groups of the finite fields $\mathbb{F}_p = \mathbb{Z}/p$ are known, as computed by Quillen [Quillen]. Furthermore, the unique ring homomorphism $\mathbb{Z}/p^n \to \mathbb{F}_p$ induces an isomorphism on K–groups away from the prime p, in the sense of localization [Bousfield and Kan]. Thus we are really only interested in the p–component of the K–groups $K_i\mathbb{Z}/p^n$ for $n \geq 2$. [Aisbett] and [Evens and Friedlander] have computations of these groups for most cases with $i \leq 4$, while $K_4\mathbb{Z}/4$ and $K_4\mathbb{Z}/9$ remain unknown.

A transfer argument. For the remainder of this section let $R = \mathbb{Z}/p^n$. On the level of the subquotients of the rank filtration, we are interested in the homotopy type of $EGL_rR_+ \wedge_{GL_rR} \Sigma^2 B_rR$ localized at p. In this situation, there is a standard transfer argument which allows us to reduce from studying the p–localized equivariant GL_rR–homotopy type of B_rR to the analogous thing for the action on B_rR by a p–Sylow subgroup of GL_rR.

Definition. In the case of $GL_r\mathbb{F}_p$, the upper triangular matrices with 1's on the diagonal constitute a p–Sylow subgroup, denoted $U_r \subset GL_r\mathbb{F}_p$. In general, let $G_r = G_rR \subset GL_rR$ be the pullback of $U_r \subset GL_r\mathbb{F}_p$ over the canonical map $GL_rR \to GL_r\mathbb{F}_p$. Explicitly, G_r consists of the matrices with entries which are 0 modulo p below the diagonal, 1 modulo p on the diagonal, and arbitrary above the diagonal. G_r is a p–Sylow subgroup of GL_rR.

There is a bundle map π:

$$EG_{r+} \wedge_{G_r} \Sigma^2 B_rR \simeq EGL_rR_+ \wedge_{G_r} \Sigma^2 B_rR \xrightarrow{\pi} EGL_rR_+ \wedge_{GL_rR} \Sigma^2 B_rR,$$

and a stable equivariant transfer map τ [Adams] going the other way, such that the composite $\pi \circ \tau$ induces multiplication by the index of G_r in GL_rR on homology or stable homotopy. This index is prime to p, so it follows that π is a split surjection when localized at p. Thus to understand $EGL_rR_+ \wedge_{GL_rR} \Sigma^2 B_rR$, we will look at the G_r–homotopy type of B_rR, and appeal to the methods of [Feshbach] to compute what is lost by mapping along π.

The Tits building as a simplicial G_r–set. We have a description of the Tits building B_rR as a simplicial G_r–set.

Definition. Let $\mathbf{r} = \{1, \ldots, r\}$. For any indexing set $I \subseteq \mathbf{r}$ let $R^I \subseteq R^r$ be the *axial* submodule where only the coordinates in $I \subseteq \mathbf{r}$ may be nonzero. Let $P_I \subseteq G_r$ denote the *parabolic* subgroup consisting of matrices $g = (g_{ij}) \in G_r$ such that $g_{ij} = 0$ if $i \notin I$ and $j \in I$. P_I stabilizes $R^I \subseteq R^r$ for the action of G_r on subspaces of R^r. If $\sigma = (I_0 \subseteq \cdots \subseteq I_q)$ is a chain of subsets of $\mathbf{r}$, then P_σ is the *parabolic* subgroup defined by $P_\sigma = P_{I_0 \ldots I_q} = P_{I_0} \cap \cdots \cap P_{I_q}$. P_σ stabilizes the flag $R^{I_0} \subseteq \cdots \subseteq R^{I_q}$ in R^r.

Proposition 2. *As a simplicial G_r–set, the Tits building B_rR is isomorphic to*

$$[q] \longmapsto \coprod_\sigma G_r/P_\sigma$$

where σ runs through the chains $\varnothing \neq I_0 \subseteq \cdots \subseteq I_q \subset \mathbf{r}$. The ith face map ∂_i is induced by $P_\sigma \subset P_{\partial_i \sigma}$ where $\partial_i \sigma = I_0 \subseteq \cdots \subseteq I_{i-1} \subseteq I_{i+1} \subseteq \cdots \subseteq I_q$, while the jth degeneracy map s_j is induced by $P_\sigma = P_{s_j \sigma}$ where $s_j \sigma = I_0 \subseteq \cdots \subseteq I_j = I_j \subseteq \cdots \subseteq I_q$.

Proof. We will show that an arbitrary q–simplex in $B_r R$

$$0 \neq M_0 \subseteq \cdots \subseteq M_q \subset R^r$$

is in the G_r–orbit of an axial flag

$$0 \neq R^{I_0} \subseteq \cdots \subseteq R^{I_q} \subset R^r$$

for some $\varnothing \neq I_0 \subseteq \cdots \subseteq I_q \subset \mathbf{r}$. Next we will check that there is only one such axial flag in each G_r–orbit, which proves that $\sigma = (I_0 \subseteq \cdots \subseteq I_q)$ is uniquely determined. Lastly we observe that the stabilizer of $R^{I_0} \subseteq \cdots \subseteq R^{I_q}$ is $P_{I_0 \dots I_q} \subseteq G_r$, from which the proposition follows.

For convenience, let us reindex the flag, so as to be considering a $(q-2)$–simplex

$$0 = M_0 \subset M_1 \subseteq \cdots \subseteq M_{q-1} \subset M_q = R^r.$$

Let $\varphi : R \to \mathbb{F}_p$ denote the ring homomorphism, as well as the homomorphisms induced by it. For each $i \in \mathbf{r}$ there is a minimal j such that ${\mathbb{F}_p}^i \subseteq \varphi(M_j)$. Then ${\mathbb{F}_p}^i \cap \varphi(M_{j-1}) + {\mathbb{F}_p}^{i-1} \cap \varphi(M_j)$ is a codimension one subspace of ${\mathbb{F}_p}^i \cap \varphi(M_j) = {\mathbb{F}_p}^i$. Choose a $b_i \in M_j$ such that $\varphi(b_j) \in {\mathbb{F}_p}^i$ is *not* in this subspace. Then b_i is part of an extension of an R–basis for M_{j-1} to one for M_j, and we may assume that b_i has ith coordinate equal to 1 modulo p.

For each j, let $I_j = \{i \in \mathbf{r} \mid b_i \in M_j\}$. Proceeding by induction on j we see that $\{b_i\}_{i \in I_j}$ is an R–basis for M_j. Then set $g = (b_1, \dots, b_r) \in G_r$, and observe that $g \cdot R^{I_j} = M_j$ for each j. This proves the first statement

Next we prove that if $g \in G_r$ and $g \cdot R^I = R^J$ then $I = J$. For each i, $\varphi(g) \in U_r$ takes the ith elementary vector $e_i \in {\mathbb{F}_p}^i$ into $\varphi(g) \cdot e_i$ with nonzero ith coordinate. Thus if $i \in I$, $\varphi(g) \cdot e_i \in \varphi(g) \cdot {\mathbb{F}_p}^I = {\mathbb{F}_p}^J$ and so $i \in J$. Thus $I \subseteq J$, and by the same argument for g^{-1}, $I = J$. This proves the second statement, concluding the proof. $\square$

The non–equivariant homotopy type. Next we will analyze the non–equivariant homotopy type of $B_r R$. First we will prove two properties of cofibrations in $\mathcal{F}(R)$ which are special to $R = \mathbb{Z}/p^n$. These will be precisely what are needed in the proof of proposition 3, below. Neither lemma holds true for arbitrary local rings R.

Lemma 1. *Each inclusion $M \subseteq N$ of finitely generated free R–modules is a cofibration.*

Proof. Choose an R–basis $\{b_1, \dots, b_j\}$ for M, and assume $N = R^r$. Write the coordinates of b_s as $(b_s^1, \dots, b_s^r)$. By rescaling, and permuting the coordinates of R^r, we may assume $b_j^r = 1$. Proceed by induction on j.

If $j = 1$, $R^{r-1} \subset R^r$ is a complementary free summand to $M = Rb_1 \subset R^r$, as required.

If $j > 1$, change the basis $\{b_1, \dots, b_j\}$ for M to $\{c_1, \dots, c_{j-1}, b_j\}$ with

$$c_s = b_s - b_s^r \cdot b_j$$

for $s < j$. Then $\{c_1, \dots, c_{j-1}\}$ spans a free submodule of R^{r-1}. Using induction, it has a complementary free submodule, which works for $M \subseteq N = R^r$ too. $\square$

Definition. If $M \subseteq R^r$ is a free submodule of rank s such that the inclusion map is a cofibration, we call M an *s–plane* in R^r. If $s = 1$, we call M a *line*. A line L is *transverse* to an s–plane $M \subset R^r$ ($s < r$) if $L + M$ is a direct sum in R^r, i.e. $L \cap M = 0$ and $L + M$ is an $(s+1)$–plane.

Lemma 2 (Graph lemma). *Write $R^r = R^{r-1} \oplus R$. For any plane $M \subseteq R^r$, either M contains a line transverse to $R^{r-1} \subset R^r$, or M is the graph of an R–homomorphism $N \to pR \subset R$, with $N \subseteq R^{r-1}$.*

Proof. Let $\{b_1, \dots, b_j\}$ be an R–basis for M, and write $b_s = (b_s^1, \dots, b_s^r)$ as before. Let π and π' denote the projections $R^r \to R$ and $R^r \to R^{r-1}$ respectively. It suffices to prove that if all $b_s^r \in pR$, then $\pi'(b_1), \dots, \pi'(b_j)$ are linearly independent.

So suppose $\sum_{i=1}^j r_i \pi'(b_i) = 0$ for some $r_i \in R$. Then $\pi(\sum_{i=1}^j r_i b_i) \in pR$. If $\pi(\sum_{i=1}^j r_i b_i) \neq 0$, there is a maximal $e > 0$ such that $\pi(\sum_{i=1}^j r_i b_i) \in p^e R$. Then $\sum_{i=1}^j (p^{n-e} r_i) b_i = 0$ and by freeness $p^{n-e} r_i = 0$ for all i. Thus we can choose $t_i \in R$ with $p^e t_i = r_i$, and $\pi(p^e \sum_{i=1}^i t_i b_i) \in p^e R - p^{e+1} R$. Hence $\pi(\sum_{i=1}^i t_i b_i) \notin pR$, contradicting the hypothesis $b_s^r \in pR$ for all s. We conclude that $\sum_{i=1}^j r_i b_i = 0$ and all $r_i = 0$, proving independence. $\square$

We now come to the principal result of this section.

Proposition 3. *$B_r R$ has the homotopy type of a bouquet (one–point wedge) of $(r-2)$–spheres.*

Proof. To begin with, we introduce some notation. If $N \subseteq M$ is an inclusion of arbitrary finitely generated R–modules, let $B(N, M)$ denote the partially ordered set of finitely generated free R–modules U such that $N \subset U \subset M$, where both inclusions are cofibrations. The partial ordering is given by $U \prec V$ if $U \subseteq V$ and the inclusion is a cofibration. Also let $B(N, M)$ denote the nerve of this partially ordered set, so that $B_r R = B(0, R^r)$.

Write $R^r = R^{r-1} \oplus R$ as before, and note that there is a filtration of R by the powers of the maximal ideal pR :

$$R \supset pR \supset \cdots \supset p^l R \supset \cdots \supset p^n R = 0.$$

Let $\pi : R^r \to R$ be the projection on the last coordinate, as before. We filter the partially ordered set $B(0, R^r)$, and hence its nerve, as follows :

$$B_r^l = \{U \in B(0, R^r) \mid \pi(U) \subseteq p^l R\}$$

with the induced ordering. Then

$$B(0, R^r) = B_r^0 \supset B_r^1 \supset \cdots \supset B_r^n \simeq *.$$

B^n_r is contractible as it has a terminal element. We filter $B^l_r \supset B^{l+1}_r$ for $l = 0, 1, \dots, n-1$ by

$$F_s = \{U \in B^l_r \mid U \in B^{l+1}_r \text{ or } \operatorname{rank}(U) \geq s\},$$

suppressing r and l from the notation. Then

$$B^l_r = F_1 \supset F_2 \supset \cdots \supset F_{r-1} \supset F_r = B^{l+1}_r.$$

The partially ordered set F_s is obtained from F_{s+1} by adjoining the s–planes V of R^r in B^l_r but not in B^{l+1}_r. The nondegenerate simplices in the nerve of F_s, not in the nerve of F_{s+1}, are thus flags of the form :

$$0 \neq U_0 \subset U_1 \subset \cdots \subset U_q \subset R^r$$

where some $U_i = V$. Specifically $U_0, \dots, U_{i-1}$ lie in $B(0, V) \cap B^{l+1}_r = B(0, V \cap (R^{r-1} \oplus p^{l+1}R))$, and $U_{i+1}, \dots, U_q$ lie in $B(V, R^r) \cap B^l_r = B(V, R^{r-1} \oplus p^l R)$. The graph lemma implies the following two claims :

Claim 1. $B(0, V \cap (R^{r-1} \oplus p^{l+1}R)) \cong B(0, R^{s-1} \oplus pR) \cong B^1_s$.

Claim 2. $B(V, R^{r-1} \oplus p^l R) \cong B(0, R^{k-s-1} \oplus p^l R) \cong B^l_{r-s}$.

Hence a simplex in the quotient of nerves F_s/F_{s+1} can be viewed as the join of a simplex in B^1_s, a vertex V, and a simplex in B^l_{r-s}, with the faces opposite to V collapsed to $*$:

$$F_s/F_{s+1} \cong \bigvee_V \Sigma(B^1_s * B^l_{r-s}) \cong \bigvee_V \Sigma^2(B^1_s \wedge B^l_{r-s})$$

where V runs over the s–planes of R^r in $B^l_r - B^{l+1}_r$.

Now we can prove by induction that all B^l_r are $(r-3)$–connected. This is trivial to check for $r \leq 2$, and each $B^n_r \simeq *$ is certainly $(r-3)$–connected. By the formula above, with $0 < s < r$, F_s/F_{s+1} is $2 + (s-3) + 1 + (r-s+3) = (r-3)$–connected for each s. Thus by downward induction on l, the result holds for each r.

Thus each B^l_r is an $(r-3)$–connected $(r-2)$–complex, and must have the homotopy type of a bouquet of $(r-2)$–spheres. The case $l = 0$ is the statement of the proposition. □

It remains to prove the two claims from the graph lemma.

Proof of claim 1. Let $V \subset R^r$ be an s–plane in $B^l_r - B^{l+1}_r$. In the case $l = 0$ we can choose a basis $\{b_1, \dots, b_s\}$ for V with $\pi(b_s) = 1$ and $\pi(b_i) = 0$ for $i < s$. Then $V \cap (R^{r-1} \oplus pR) = R\{b_1, \dots, b_{s-1}\} \oplus pR\{b_s\} \cong R^{s-1} \oplus pR$.

If $l > 0$, V is the graph of an R–homomorphism $f : W \to pR \subset R$ with $W \subseteq R^{r-1}$. We can take a basis $\{w_1, \dots, w_s\}$ for W with $f(w_s) = p^l$ and $f(w_i) = 0$ for $i < s$. Then clearly $V \cap (R^{r-1} \oplus p^{l+1}R) \cong R\{w_1, \dots, w_{s-1}\} \oplus pR\{w_s\} \cong R^{s-1} \oplus pR$. □

Proof of claim 2. If $l = 0$, the statement amounts to $B(R^s, R^r) \cong B(0, R^{r-s})$, which is clear.

If $l > 0$, V is the graph of $f : W \to pR \subset R$ as in the preceding proof. Extend the basis $\{w_1, \ldots, w_s\}$ for W to $\{w_1, \ldots, w_r\}$, a basis for R^r, with $w_r = e_r$. We can then change basis for R^r by replacing w_s with $w_s + p^l \cdot e_r$ to reduce to the case $f = 0$. This case is clear. $\square$

To complete the description of the non–equivariant homotopy type of B_rR we give a counting argument for finding the number of wedges in the bouquet.

Notation. Let $<r>_{p,n} = (p^{rn} - p^{r(n-1)})/(p^n - p^{n-1})$ denote the number of lines in R^r. Then

$$\left\langle {r \atop s} \right\rangle_{p,n} = \frac{<r>_{p,n} \cdot \ldots \cdot <r-s+1>_{p,n}}{<s>_{p,n} \cdot \ldots \cdot <1>_{p,n}}$$

is the number of s–planes in R^r.

Proposition 4. *For $0 = s_0 \le s_1 \le \cdots \le s_{q-1} \le s_q = r$ there are*

$$\left\langle {s_1 \atop s_0} \right\rangle_{p,n} \cdot \ldots \cdot \left\langle {s_q \atop s_{q-1}} \right\rangle_{p,n} = \prod_{i=1}^{q} \left\langle {s_i \atop s_{i-1}} \right\rangle_{p,n}$$

flags $0 \ne V_1 \subseteq \cdots \subseteq V_{q-1} \subset R^r$ with $\operatorname{rank}(V_i) = s_i$ representing $(q-2)$–simplices in B_rR. Counting nondegenerate simplices, the Euler characteristic is

$$\chi B_rR = \sum_{0=s_0<\cdots<s_q=r} (-1)^q \prod_{i=1}^{q} \left\langle {s_i \atop s_{i-1}} \right\rangle_{p,n} .$$

Consequently, $B_rR \simeq \bigvee_{\beta_r} S^{r-2}$ with $\beta_r = (-1)^r(\chi B_rR - 1)$. $\square$

It is also possible to give a recursive formula for the number of wedges, denoted β_r above, by following the proof of proposition 3. Lastly, one can produce a formula based on proposition 2, by expressing the order of P_σ for varying σ.

Small Buildings

In this section, we will construct a covering of B_rR by subcomplexes homeomorphic to S^{r-2} called apartments. In the case $R = \mathbb{F}_p$, these are the subcomplexes of $B_r\mathbb{F}_p$ making this complex into a building in the abstract sense [Tits]. With $R = \mathbb{Z}/p^n$, the ring homomorphism $R \to \mathbb{F}_p$ maps the apartments of B_rR onto those of $B_r\mathbb{F}_p$. As the covering of $B_r\mathbb{F}_p$ is well understood, we choose to organize the covering apartments in B_rR by what apartments they map to in $B_r\mathbb{F}_p$. The preimage of one apartment in $B_r\mathbb{F}_p$ is then the union of such a gathering of apartments in B_rR, which forms a subcomplex of B_rR called the *small building* b_rR. We begin an analysis of the equivariant homotopy type of B_rR through that of b_rR.

Apartments.

Definition. Let the *standard apartment* $A = A_rR \subset B_rR$ denote the subcomplex consisting of simplices which are axial flags in R^r :

$$0 \ne R^{I_0} \subseteq \cdots \subseteq R^{I_q} \subset R^r.$$

The *apartments* of B_rR are the translates $g \cdot A$, for $g \in GL_rR$. They form a covering of B_rR. We often identify a simplex in the standard apartment A with the corresponding chain $(I_0 \subseteq \cdots \subseteq I_q)$ of subsets of $\mathbf{r}$.

Lemma. $A \cong S^{r-2}$.

Proof. A is isomorphic to the nerve of the partially ordered set of proper nontrivial subsets of $\mathbf{r}$, i.e. the boundary of an $(r-1)$–simplex. $\square$

To analyze B_rR in terms of a covering by apartments, it is convenient to introduce some notation for various subcomplexes of A.

Notation. If ω is a partial ordering on $\mathbf{r}$, call $I \subseteq \mathbf{r}$ *convex* [Goodwillie] if every element of $\mathbf{r}$ less than an element of I, in the ordering ω, already is in I. Let $A_\omega \subseteq A$ denote the subcomplex consisting of axial flags

$$0 \neq R^{I_0} \subseteq \cdots \subseteq R^{I_q} \subset R^r$$

where each I_i is convex under ω. In the special case that ω partitions $\mathbf{r}$ into a collection of unrelated equivalence classes, we use π to denote the partition, and write $A_\pi \subseteq A$ as above.

If T is a collection of $r \times r$ R–matrices, let $A_T = A_{\omega_T}$ where ω_T is the partial ordering on $\mathbf{r}$ generated by $i \prec j$ whenever $g_{ij} \neq 0$ for some $g \in T$.

Lemma 3. *If π partitions $\mathbf{r}$ into $|\pi|$ equivalence classes, $A_\pi \cong S^{|\pi|-2}$. If ω is not a partition, i.e. there exists $i \prec j$ in ω with $j \not\prec i$, then $A_\omega \simeq *$. If $e \in T$, then $A_T = \bigcap_{g\in T} g \cdot A$.*

Proof. See [Rognes, thesis, proposition 9.1 and lemma 10.6]. $\square$

Let $R = \mathbb{Z}/p^n$. Later in the section we will restrict to $R = \mathbb{Z}/p^2$.

Notation. Let $N_r = N_rR$ be the kernel of $GL_rR \twoheadrightarrow GL_r\mathbb{F}_p$, or equivalently of $G_r \twoheadrightarrow U_r$. Let $Q_\sigma = N_r \cap P_\sigma$, where $\sigma = (I_0 \subseteq \cdots \subseteq I_q)$ can be thought of as a simplex of A.

Lemma. *The G_r–translates $\{g \cdot A\}_{g\in G_r}$ of A cover B_rR. Similarly $\{u \cdot A\}_{u\in U_r}$ cover $B_r\mathbb{F}_p$. The preimage of $A \subset B_r\mathbb{F}_p$ by $B_rR \to B_r\mathbb{F}_p$ is the union of the apartments $\{g \cdot A\}$ with $\in N_r$.*

Proof. Clear from proposition 2. $\square$

Corollary [Quillen]. *There is a U_r–homotopy equivalence :*

$$B_r\mathbb{F}_p \simeq \bigvee_{u\in U_r} S^{r-2}.$$

Proof. $B_r\mathbb{F}_p$ is covered by the apartments $u \cdot A \cong S^{r-2}$ with $u \in U_r$. Consider a (multiple) intersection $\bigcap_{u\in T} u \cdot A = A_T$, where we may assume $e \in T$. If T also contains upper triangular matrices not equal to the identity, ω_T cannot possibly be a partition of $\mathbf{r}$, whence by lemma 3, $A_T \simeq *$. $\square$

The small buildings.

Definition. The *small building* b_rR is the union of the apartments $g \cdot A \subset B_rR$ for $g \in N_r$. If $u \in U_r$, the translate $u \cdot b_rR \subset B_rR$ is well defined, and these cover B_rR.

We now look at the N_r–homotopy type of B_rR, or rather the homotopy type of $EN_{r+} \wedge_{N_r} \Sigma^2 B_rR$, through the covering of B_rR by the N_r–complexes $u \cdot b_rR$.

Lemma. *As a simplicial N_r–set, b_rR equals*

$$[q] \longmapsto \coprod_{\sigma} N_r/Q_\sigma$$

where σ runs through the chains $\varnothing \neq (I_0 \subseteq \cdots \subseteq I_q) \subset \mathbf{r}$.

Proof. Immediate from proposition 2. $\square$

Corollary 1. *There is a spectral sequence, with*

$$E^1_{s,t} = \bigoplus_{\sigma} H_t(Q_\sigma; \mathbb{F}_p)$$

(group homology) where σ runs through the chains $\varnothing = I_0 \subset \cdots \subset I_s = \mathbf{r}$, converging to

$$\widetilde{H}_*(EN_{r+} \wedge_{N_r} \Sigma^2 b_rR; \mathbb{F}_p).$$

The differentials $d^1|_\sigma : H_*(Q_\sigma;\mathbb{F}_p) \hookrightarrow H_*(Q_{\partial_i\sigma};\mathbb{F}_p)$ commute with the inclusion into $H_*(N_r;\mathbb{F}_p)$.

Proof. $\Sigma^2 b_rR$ has nondegenerate q–simplices $\coprod_\sigma N_r/Q_\sigma$, with $\sigma = (\varnothing = I_0 \subset \cdots \subset I_q = \mathbf{r})$ a nondegenerate simplex in $\Sigma^2 A$, for $q > 0$. There is also a single base point in degree 0. The spectral sequence is that associated to the skeletal filtration on $\Sigma^2 b_rR$, with $EN_{r+} \wedge_{N_r} (-)$ applied, and we recognize the E^1–term by

$$\begin{aligned}
E^1_{s,t} &= \widetilde{H}_{s+t}(EN_{r+} \wedge_{N_r} (\Sigma^2 b_rR^{(s)}/\Sigma^2 b_rR^{(s-1)}); \mathbb{F}_p) \\
&\cong \widetilde{H}_{s+t}(EN_{r+} \wedge_{N_r} \Sigma^s(\coprod_\sigma N_r/Q_{\sigma\ +}); \mathbb{F}_p) \\
&\cong \bigoplus_\sigma H_t(EN_r \times_{N_r} N_r/Q_\sigma; \mathbb{F}_p) \\
&\cong \bigoplus_\sigma H_t(Q_\sigma; \mathbb{F}_p).
\end{aligned}$$

The observation about the differentials is immediate from the simplicial structure on b_rR. $\square$

For the remainder of this section, assume $R = \mathbb{Z}/p^2$. Then N_r is an elementary abelian p–group of rank r^2, with one $\mathbb{Z}/p$–factor corresponding to each matrix entry (i,j). In this case we can compute the E^2–term of the spectral sequence above, and prove that it collapses there.

Definition. Let $C_* = H_*(\mathbb{Z}/p;\mathbb{F}_p)$ denote the graded Hopf algebra. As a $\mathbb{F}_p$–vector space it has a generator in each nonnegative degree. By the Künneth formula, $H_* = H_*(N_r;\mathbb{F}_p)$ is additively generated by monomials $\{z\}$ which are (tensor) products of one of the generators in C_* for each matrix entry (i,j) in N_r. We say that a monomial z *involves* the matrix entries (i,j) for which a generator of positive degree occurs.

Let $A_z \subseteq A$ denote the subcomplex A_{ω_z} where ω_z is generated by $i \prec j$ if z involves the (i,j)th entry. If π is a partition of $\mathbf{r}$, let $J_*(\pi) \subset H_*$ denote the H_*–comodule which is additively generated by the monomials z such that $A_z = A_\pi$, i.e. $\omega_z = \pi$. These are the π–*mixing* monomials in H_*.

Proposition 5. *The spectral sequence in corollary 1 collapses at the E^2–term :*

$$E^2_{s,*} = \bigoplus_{\pi} J_*(\pi)$$

where π runs over the partitions of $\mathbf{r}$ *into $|\pi| = s$ equivalence classes. Hence*

$$\widetilde{H}_*(EN_{r+} \wedge_{N_r} \Sigma^2 b_r R; \mathbb{F}_p) \cong \bigoplus_{\pi} \Sigma^{|\pi|} J_*(\pi).$$

Proof. The E^1–term (E^1, d^1) splits over the monomials z. Fix a z, and consider the corresponding summand of E^1. It contains a copy of $\mathbb{F}_p\{z\}$ precisely for each $\sigma \neq *$ in $\Sigma^2 A_z$, and d^1 agrees with the differential in the reduced simplicial chain complex $\widetilde{C}_*(\Sigma^2 A_z; \mathbb{F}_p)$. Hence the contribution to the E^2–term is $\widetilde{H}_*(\Sigma^2 A_z; \mathbb{F}_p)$. By lemma 3, this is 0 unless ω_z is a partition π, in which case the homology is a copy of $\mathbb{F}_p$ in degree $|\pi|$. Hence

$$E^2_{**} \cong \bigoplus_{z} \widetilde{H}_*(\Sigma^2 A_z) \otimes \mathbb{F}_p\{z\}$$

equals the expression in the statement in the proposition, with the sum reindexed by π in place of z.

It remains to prove that the spectral sequence collapses. Consider a partition π of $\mathbf{r}$, and a monomial z in $J_*(\pi) \subset H_*$ representing a generator (π, z) in $E^2_{|\pi|,|z|}$. Let $Q_z \subseteq N_r$ be the intersection of all Q_σ for $\sigma \in A_z$. Then (π, z) is realized in the homology of $BQ_{z+} \wedge \Sigma^2 A_z$. Also there is for each $\sigma \in A_z$ an inclusion :

$$BQ_z \subseteq BQ_\sigma \xrightarrow{\simeq} EN_r \times_{N_r} N_r/Q_\sigma$$

extending to

$$BQ_{z+} \wedge \Sigma^2 A_z \hookrightarrow EN_{r+} \wedge_{N_r} \Sigma^2 b_r R\ .$$

Associated to the skeletal filtration on $\Sigma^2 A_z$ and $\Sigma^2 b_r R$, we have two spectral sequences (the 'left' and the 'right' ones, respectively) with a natural map between them, corresponding to the last inclusion above. The spectral sequence on the left side collapses at the E^2–term, and as the class (π, z) is realized on this E^2 term, it supports no higher differentials on either side. As this holds for all π and z, all these classes survive to E^∞ on the right side, and the proposition follows. □

Example. Let $r = 2$ and $R = \mathbb{Z}/4$. There are two partitions $\pi_1 = \{\{1,2\}\}$ and $\pi_2 = \{\{1\},\{2\}\}$ of $\mathbf{r}$, and $J_*(\pi_1) \cong H_* \cdot x_{12}x_{21}$, $J_*(\pi_2) \cong \mathbb{F}_2[x_{11}, x_{22}]$. Here x_{ij} denotes the generator in degree 1 of $H_*(\mathbb{Z}/2; \mathbb{F}_2) \cong \mathbb{F}_2[x]$, corresponding to the (i,j)th entry in N_2. Then

$$\widetilde{H}_*(EN_{2+} \wedge_{N_2} \Sigma^2 b_2 \mathbb{Z}/4; \mathbb{F}_2) \cong \Sigma^1 H_* \cdot x_{12}x_{21} \oplus \Sigma^2 \mathbb{F}_2[x_{11}, x_{22}].$$

The covering. We would now like to extend this result to $\Sigma^2 B_r R$. Recall the covering of $B_r R$ by $\{u \cdot b_r R\}$ for $u \in U_r$. Let $T \subseteq U_r$ be nonempty. Consider the (multiple) intersection

$$\bigcap_{u \in T} EN_{r+} \wedge_{N_r} \Sigma^2 (u \cdot b_r R) = EN_{r+} \wedge_{N_r} \Sigma^2 \bigcap_{u \in T} u \cdot b_r R \, .$$

If $t \in T$, $\bigcap_{u \in T} u \cdot b_r R = t \cdot \bigcap_{u \in t^{-1}T} u \cdot b_r R$ and $e \in t^{-1}T$, so let us assume $e \in T$. By lemma 3 we have the following extension of corollary 1 :

Lemma. *Let $e \in T \subseteq U_r$. There is a spectral sequence with*

$$E^1_{s,t} = \bigoplus_{\sigma} H_t(Q_\sigma; \mathbb{F}_p)$$

where σ runs through the nondegenerate s–simplices of $\Sigma^2 A_T$, converging to

$$\widetilde{H}_*(EN_{r+} \wedge_{N_r} \Sigma^2 \bigcap_{u \in T} u \cdot b_r R; \mathbb{F}_p). \quad \square$$

We extend the notion of mixing monomials :

Definition. Let $e \in T \subseteq U_r$ and let π be a partition of $\mathbf{r}$. Let $J_*(\pi, T) \subset H_*$ denote the H_*–comodule which is additively generated by the monomials z such that $A_z \cap A_T = A_\pi$. These are the (π, T)–*mixing* monomials in H_*. If $e \notin T$ but T is nonempty, pick $t \in T$ and set $J_*(\pi, T) = J_*(\pi, t^{-1}T)$.

Proposition 6. *Let $e \in T \subseteq U_r$. The spectral sequence in the lemma above collapses at the E^2–term :*

$$E^2_{s,*} = \bigoplus_{\pi} J_*(\pi)$$

where π runs over the partitions of $\mathbf{r}$ into $|\pi| = s$ equivalence classes. Hence

$$\widetilde{H}_*(EN_{r+} \wedge_{N_r} \Sigma^2 \bigcap_{u \in T} u \cdot b_r R; \mathbb{F}_p) \cong \bigoplus_{\pi} \Sigma^{|\pi|} J_*(\pi, T).$$

This isomorphism is also true if $e \notin T$.

Proof. Redo the proof of proposition 5, but replace all references to σ running through the simplices of $\Sigma^2 A$ with σ running through the simplices of $\Sigma^2 A_T$. $\square$

Example. We continue the example of $r = 2$ and $R = \mathbb{Z}/4$. $U_2 = \{e, u\}$, where $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Then $J_*(\pi, \{e\}) = J_*(\pi, \{u\}) = J_*(\pi)$ for either partition π, while $J_*(\pi_1, \{e, u\}) = H_* \cdot x_{21}$ and $J_*(\pi_2, \{e, u\}) = 0$.

Corollary 2. *There is a Mayer–Vietoris spectral sequence (see e.g.* [Rognes, thesis, definition 9.4]*) with E^1–term*

$$E^1_{s,*} = \bigoplus_{T} \bigoplus_{\pi} \Sigma^{|\pi|} J_*(\pi, T)$$

where the sum runs over $T \subseteq U_r$ with $(s+1)$ elements, converging to the N_r–homology of $\Sigma^2 B_r R$, namely $\widetilde{H}_*(EN_{r+} \wedge_{N_r} \Sigma^2 B_r R; \mathbb{F}_p)$. $\square$

Example. Still $r = 2$ and $R = \mathbb{Z}/4$. The only differential in the spectral sequence above is $d^1 : \Sigma^1 J_*(\pi_1, \{e, u\}) \to \Sigma^1 J_*(\pi_1, \{e\}) \oplus \Sigma^1 J_*(\pi_1, \{u\})$, mapping $H_* \cdot x_{21}$ into the diagonal of $H_* \cdot x_{12}x_{21} \oplus H_* \cdot x_{12}x_{21}$ by the map commuting with the inclusion into H_* for each summand.

Dept. of Math, Univ. of Oslo, Norway