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On cyclic fixed points of spectra

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Abstract For a finite p-group G and a bounded below G-spectrum X of finite type mod p, the G-equivariant Segal conjecture for X asserts that the canonical map $X^G \to X^{hG}$, from G-fixed points to G-homotopy fixed points, is a p-adic equivalence. Let C_{p^n} be the cyclic group of order p^n. We show that if the C_p-equivariant Segal conjecture holds for a C_{p^n}-spectrum X, as well as for each of its geometric fixed point spectra $\Phi^{C_{p^e}}(X)$ for $0 < e < n$, then the C_{p^n}-equivariant Segal conjecture holds for X. Similar results also hold for weaker forms of the Segal conjecture, asking only that the canonical map induces an equivalence in sufficiently high degrees, on homotopy groups with suitable finite coefficients.

Keywords Segal conjecture · cyclic p-group · fixed points · Tate construction · smash power · topological Hochschild homology

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1 Introduction

Let p be any prime number. Graeme Segal's Burnside ring conjecture [1] for a finite p-group G asserts that if $X = S_G$ is the genuinely G-equivariant sphere spectrum, then the canonical map $X^G \to X^{hG} = F(EG_+, X)^G$ is a p-adic equivalence. For cyclic groups $G = C_p$ of prime order the conjecture

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was proved by Lin [15] and Gunawardena [11], [2]. Thereafter Ravenel [19], [20] gave an inductive proof of Segal's conjecture for finite cyclic p-groups $G = C_{p^n}$ of order p^n, starting from Lin and Gunawardena's theorems. Ravenel's result was superseded by Carlsson's proof [7] of the Segal conjecture for all finite p-groups, but as we shall show here, Ravenel's methods are also of interest in a more general context, where X is a quite general G-spectrum.

As was elucidated by Miller and Wilkerson [18], Ravenel's methods give two proofs of the Segal conjecture for cyclic groups—one computational using the modified Adams spectral sequence, and one non-computational, using explicit geometric constructions. The object of this paper is to generalize Ravenel's geometric proof of the Segal conjecture to show that $X^G \to X^{hG}$ is "close to" a p-adic equivalence for $G = C_{p^n}$, assuming that $X^C \to X^{hC}$ and similar maps are "close to" such an equivalence for $C = C_p$. Our main technical results are Theorems 2.4 and 2.5. Their statements involve (W, k)-coconnected maps and geometric fixed points, which are discussed in Definitions 2.1 and 2.3, respectively. See Example 2.2 for more on how a (W, k)-coconnected map is close to a p-adic equivalence.

In the special cases $X = B^{\wedge p^n}$ and $X = THH(B)$, where $B^{\wedge p^n}$ is a specific C_{p^n}-equivariant model for the p^n-th smash power of a symmetric spectrum B, and $THH(B)$ is the topological Hochschild homology of a symmetric ring spectrum B, the geometric fixed points are well understood, as explained in Theorems 2.7 and 2.8, respectively. In the special cases $W = S^{-1}/p^\infty$ and $W = F(V, S)$, where V is a finite p-torsion spectrum, the (W, k)-coconnected maps are well understood in terms of p-completion and homotopy with V-coefficients, as explained in Examples 2.9 and 2.10, respectively. In the doubly special case when $X = THH(B)$ and $W = S^{-1}/p^\infty$, our results recover the main theorem of Tsalidis [24].

2 Statement of results

We first formalize the notion of being close to a p-adic equivalence. Throughout the paper we assume that a pair (W, k) has been chosen as in the following definition. The hypothesis on W ensures that the function spectrum $F(W, Y)$ is contractible whenever the p-adic completion $Y^\wedge_p$ is contractible.

Definition 2.1 Let S^{-1}/p^∞ be a Moore spectrum with homology $\mathbb{Z}/p^\infty$ concentrated in degree -1, so that $F(S^{-1}/p^\infty, Y) = Y^\wedge_p$ for each spectrum Y. Let W be an object in the localizing ideal [13, Def. 1.4.3(d)] of spectra generated by S^{-1}/p^∞, i.e., the smallest thick subcategory of spectra that contains S^{-1}/p^∞ and is closed under arbitrary wedge sums, as well as under smash products with arbitrary spectra.

Let k be an integer, or the symbol $-\infty$. We say that a spectrum Y is *(W, k)-coconnected* if $\pi_* F(W, Y) = 0$ for all $* \geq k$. We say that a map of spectra $f \colon Y_1 \to Y_2$ is *(W, k)-coconnected* if $\mathrm{hofib}(f)$ is (W, k)-coconnected, or equivalently, if $\pi_* F(W, Y_1) \to \pi_* F(W, Y_2)$ is injective for $* = k$ and an isomorphism for all $* > k$.

Example 2.2 The most obvious choice for W is the Moore spectrum S^{-1}/p^∞ itself, in which case $F(W, Y) = Y_p^\wedge$, so a map $f\colon Y_1 \to Y_2$ is (W, k)-coconnected if and only if the p-completed map $f_p^\wedge\colon (Y_1)_p^\wedge \to (Y_2)_p^\wedge$ induces an injection on π_* for $* = k$ and an isomorphism for $* > k$. When $k = -\infty$, this is the same as being a p-adic equivalence.

Alternatively, we may take $W = F(V, S)$, where V is a finite CW spectrum whose integral homology is p-torsion, in which case $F(W, Y) \simeq V \wedge Y$ by Spanier–Whitehead duality. In this case $f\colon Y_1 \to Y_2$ is (W, k)-coconnected if and only if the map $1 \wedge f\colon V \wedge Y_1 \to V \wedge Y_2$ induces an injection $V_*(Y_1) = \pi_*(V \wedge Y_1) \to \pi_*(V \wedge Y_2) = V_*(Y_2)$ for $* = k$ and an isomorphism for $* > k$. The Smith–Toda complexes $V(m)$ for $m \geq 0$, see [22] and [23], are examples of such finite p-torsion spectra.

Next, we recall some comparison maps between fixed points, homotopy fixed points, geometric fixed points and Tate constructions.

Definition 2.3 Let $C = C_p \subset C_{p^n} = G$ and $\bar{G} = G/C \cong C_{p^{n-1}}$. Let $\lambda = \mathbb{C}(1)$ be the basic faithful G-representation of complex rank one, and S^λ its one-point compactification. Let $\infty\lambda$ be the direct sum of a countable number of copies of λ. Its unit sphere $S(\infty\lambda) = EG$ is a free contractible G-CW space, and its one-point compactification $S^{\infty\lambda} = \widetilde{EG}$ sits in a G-homotopy cofiber sequence $EG_+ \to S^0 \to \widetilde{EG}$, where the first map collapses EG to the non-basepoint.

Let X be a G-spectrum, in the sense of [14], and consider the vertical map

$$\begin{tikzcd}
EG_+ \wedge X \arrow[r] \arrow[d, "\simeq_G"] & X \arrow[r] \arrow[d] & \widetilde{EG} \wedge X \arrow[d] \\
EG_+ \wedge F(EG_+, X) \arrow[r] & F(EG_+, X) \arrow[r] & \widetilde{EG} \wedge F(EG_+, X)
\end{tikzcd}$$

of horizontal G-homotopy cofiber sequences. Passing to G-fixed point spectra we obtain a vertical map

$$\begin{tikzcd}
X_{hG} \arrow[r, "N"] \arrow[d, equal] & X^G \arrow[r, "R"] \arrow[d, "\Gamma_n"] & \Phi^C(X)^{\bar{G}} \arrow[d, "\hat{\Gamma}_n"] \\
X_{hG} \arrow[r, "N^h"] & X^{hG} \arrow[r, "R^h"] & X^{tG}
\end{tikzcd}$$

of horizontal homotopy cofiber sequences, often called the norm–restriction sequences [10, Diag. (C), (D)]. Here

$X_{hG} = EG_+ \wedge_G X$	(homotopy orbits)
$X^{hG} = F(EG_+, X)^G$	(homotopy fixed points)
$X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G$	(Tate construction)

and there is a preferred $\bar{G}$-equivariant equivalence

$$[\widetilde{EG} \wedge X]^C \xrightarrow{\simeq} \Phi^C(X) \qquad \text{(geometric fixed points)}$$

inducing the upper right hand equivalence $[\widetilde{EG} \wedge X]^G \simeq \Phi^C(X)^{\bar{G}}$. For more details, see e.g. [12, Prop. 2.1].

The right hand square above is homotopy cartesian, so Γ_n is (W,k)-coconnected if and only if $\hat{\Gamma}_n$ is (W,k)-coconnected. This observation can be combined with the conclusions of all of the theorems below.

We briefly write $H_*(X) = H_*(X; \mathbb{F}_p)$ for the mod p homology of any spectrum, and say that $H_*(X)$ is of finite type if each group $H_m(X)$ is finite.

Theorem 2.4 *Let X be a G-spectrum with $\pi_*(X)$ bounded below and $H_*(X)$ of finite type. Suppose that $\Gamma_1\colon X^C \to X^{hC}$ and $\Gamma_{n-1}\colon \Phi^C(X)^{\bar{G}} \to \Phi^C(X)^{h\bar{G}}$ are (W,k)-coconnected maps. Then $\Gamma_n\colon X^G \to X^{hG}$ is (W,k)-coconnected.*

Informally, this theorem asserts that if $X^C \to X^{hC}$ and $Y^{\bar{G}} \to Y^{h\bar{G}}$ are close to p-adic equivalences, for $Y = \Phi^C(X)$, then $X^G \to X^{hG}$ is close to a p-adic equivalence.

Theorem 2.5 *Let X be a C_{p^n}-spectrum. Suppose, for each of the geometric fixed point spectra*

$$Y = X\,,\ \Phi^{C_p}(X)\,,\ \dots\,,\ \Phi^{C_{p^{n-1}}}(X)\,,$$

that $\pi_*(Y)$ is bounded below, $H_*(Y)$ is of finite type and $\Gamma_1\colon Y^{C_p} \to Y^{hC_p}$ is (W,k)-coconnected. Then $\Gamma_n\colon X^{C_{p^n}} \to X^{hC_{p^n}}$ is (W,k)-coconnected.

The proofs of Theorems 2.4 and 2.5 are given near the end of Section 3. The bounded below and finite type mod p hypotheses enter in the proof of Proposition 3.8, where we make use of the convergence of an inverse limit of Adams spectral sequences.

The following construction was introduced by the first author, in the context of functors with smash product (FSPs). See [5, §3] for a published account. We are principally interested in the case $r = p^n$.

Definition 2.6 Let B be any symmetric spectrum. The r-th smash power $B^{\wedge r}$ can be defined as a C_r-spectrum by the construction

$$B^{\wedge r} = sd_r THH(B)_0 = THH(B)_{r-1}$$

from [12, §2.4]. Its V-th space is defined by a homotopy colimit

$$(B^{\wedge r})_V = \operatorname*{hocolim}_{(i_1,\dots,i_r)\in I^r} \mathrm{Map}(S^{i_1} \wedge \dots \wedge S^{i_r}, B_{i_1} \wedge \dots \wedge B_{i_r} \wedge S^V)\,,$$

and C_r cyclically permutes the smash factors, in addition to its natural action on S^V.

To ensure that $B^{\wedge r}$ has the same naively equivariant homotopy type as the ordinary r-fold smash product $B \wedge \cdots \wedge B$, it suffices to assume that B is flat and convergent, see e.g. [17, Lem. 5.5]. Hereafter, when referring to $B^{\wedge r}$ we always assume that B has first been replaced by an equivalent flat and convergent symmetric spectrum. In [17, Thm. 5.13], the third and fourth authors prove that $\Gamma_1 \colon (B^{\wedge p})^{C_p} \to (B^{\wedge p})^{hC_p}$ is a p-adic equivalence whenever $\pi_*(B)$ is bounded below and $H_*(B)$ is of finite type. This provides the inductive beginning for the following application of Theorem 2.5.

Theorem 2.7 *Let B be a symmetric spectrum with $\pi_*(B)$ bounded below and $H_*(B)$ of finite type. Then*

$$\Gamma_n \colon (B^{\wedge p^n})^{C_{p^n}} \to (B^{\wedge p^n})^{hC_{p^n}}$$

is a p-adic equivalence, for each $n \geq 1$.

When B is a symmetric ring spectrum, its topological Hochschild homology $THH(B)$ is a $\mathbb{T}$-spectrum [12, §2.4], where $\mathbb{T}$ is the circle group. It is not true in general that $\Gamma_1 \colon THH(B)^{C_p} \to THH(B)^{hC_p}$ is a p-adic equivalence, see e.g. [12, Prop. 5.3] and [21, Thm. 4.7], but when it is "approximately" true, then the following theorem is useful.

Theorem 2.8 *Let B be a connective symmetric ring spectrum with $H_*(B)$ of finite type, and suppose that*

$$\Gamma_1 \colon THH(B)^{C_p} \to THH(B)^{hC_p}$$

is (W, k)-coconnected. Then

$$\Gamma_n \colon THH(B)^{C_{p^n}} \to THH(B)^{hC_{p^n}}$$

is (W, k)-coconnected, for each $n \geq 2$.

The proofs of Theorems 2.7 and 2.8 are given at the end of Section 3.

In the case $B = S$ there is a G-equivariant equivalence $THH(S) \simeq S_G$, and Γ_1 is a p-adic equivalence by the classical Segal conjecture. Also in the cases $B = MU$ (the complex cobordism spectrum) and $B = BP$ (the Brown–Peterson spectrum) it turns out that Γ_1 for $THH(B)$ is a p-adic equivalence, as the third and fourth authors show in [16, Thm. 1.1]. This provides examples with $k = -\infty$ for the following special case.

Example 2.9 Taking $W = S^{-1}/p^\infty$, the assumption in Theorem 2.8 is that the p-completed map $\Gamma_1 \colon (THH(B)^{C_p})^\wedge_p \to (THH(B)^{hC_p})^\wedge_p$ is k-coconnected, i.e., that it induces an injection on π_k and an isomorphism on π_* for $* > k$, and the conclusion is that the p-completed map

$$\Gamma_n \colon (THH(B)^{C_{p^n}})^\wedge_p \to (THH(B)^{hC_{p^n}})^\wedge_p$$

is also k-coconnected, for all $n \geq 2$. This recovers a theorem of Tsalidis [24, Thm. 2.4].

Example 2.10 Taking $W = F(V, S)$ and $V = V(1) = S/(p, v_1)$, the Smith–Toda complex of chromatic type 2 (for p odd), the assumption in Theorem 2.8 is that

$$V(1)_*(\Gamma_1)\colon\ V(1)_*THH(B)^{C_p} \to V(1)_*THH(B)^{hC_p}$$

is k-coconnected, and the conclusion is that

$$V(1)_*(\Gamma_n)\colon\ V(1)_*THH(B)^{C_{p^n}} \to V(1)_*THH(B)^{hC_{p^n}}$$

is also k-coconnected, for all $n \geq 2$. This recovers the generalization of Tsalidis' theorem used by Ausoni and the fourth author [3, Thm. 5.7] in the special case when $B = \ell$, the Adams summand of connective p-local complex K-theory, and $k = 2p - 2$. The generalized result is used again in [4, Cor. 5.9], for $B = \ell/p = k(1)$, the first connective Morava K-theory.

3 Constructions and proofs

Definition 3.1 Let $\bar\lambda$ be the basic faithful $\bar G$-representation of complex rank one. Like in Definition 2.3, we let $E\bar G = S(\infty\bar\lambda)$ and $\widetilde{E\bar G} = S^{\infty\bar\lambda}$. The usual map from the homotopy colimit to the categorical colimit is a $\bar G$-equivalence $\operatorname{hocolim}_j S^{j\bar\lambda} \xrightarrow{\simeq} S^{\infty\bar\lambda} = \widetilde{E\bar G}$. The pullback of $\bar\lambda$ along the canonical projection $G \to \bar G$ is the p-th tensor power $\lambda^p = \mathbb{C}(p)$ of λ, and we get a G-equivalence

$$\operatorname*{hocolim}_j S^{j\lambda^p} \xrightarrow{\simeq} S^{\infty\lambda^p} = \widetilde{E\bar G}\,,$$

where the right hand side is implicitly viewed as a G-space by pullback along $G \to \bar G$. The G-map $S^{j\lambda^p} \to S^{(j+1)\lambda^p}$ in the colimit system is given by smashing $S^{j\lambda^p}$ with the one-point compactification $z\colon\ S^0 \to S^{\lambda^p}$ of the inclusion $\{0\} \subset \lambda^p$.

Lemma 3.2 *Let X be a G-spectrum. There is a natural homotopy cofiber sequence*

$$\operatorname{holim}_j\,(\Sigma^{-j\lambda^p}X)^G \longrightarrow (X^C)^{\bar G} \xrightarrow{\quad\Gamma_{n-1}\quad} (X^C)^{h\bar G}\,,$$

where the right hand map is Γ_{n-1} for the $\bar G$-spectrum X^C.

Proof By mapping the $\bar G$-homotopy cofiber sequence $E\bar G_+ \to S^0 \to \widetilde{E\bar G}$ into X^C, we get the homotopy (co-)fiber sequence

$$F(\widetilde{E\bar G}, X^C)^{\bar G} \to (X^C)^{\bar G} \xrightarrow{\Gamma_{n-1}} F(E\bar G_+, X^C)^{\bar G}\,.$$

At the left hand side we have a natural chain of equivalences

$$\begin{aligned} F(\widetilde{E\bar G}, X^C)^{\bar G} &\simeq F(\operatorname*{hocolim}_j S^{j\bar\lambda}, X^C)^{\bar G} \simeq \operatorname*{holim}_j F(S^{j\bar\lambda}, X^C)^{\bar G} \\ &\simeq \operatorname*{holim}_j(\Sigma^{-j\bar\lambda}(X^C))^{\bar G} \simeq \operatorname*{holim}_j((\Sigma^{-j\lambda^p}X)^C)^{\bar G} \simeq \operatorname*{holim}_j\,(\Sigma^{-j\lambda^p}X)^G\,. \end{aligned}$$

This gives the asserted homotopy cofiber sequence.

Proposition 3.3 *Let X be a G-spectrum. There is a vertical map of homotopy cofiber sequences*

$$\begin{array}{ccccc}
\operatorname{holim}_j \Phi^C(\Sigma^{-j\lambda^p} X)^{\bar G} & \longrightarrow & \Phi^C(X)^{\bar G} & \xrightarrow{\Gamma_{n-1}} & \Phi^C(X)^{h\bar G} \\
\downarrow & & \downarrow{\scriptstyle \hat\Gamma_n} & & \downarrow{\scriptstyle (\hat\Gamma_1)^{h\bar G}} \\
\operatorname{holim}_j (\Sigma^{-j\lambda^p} X)^{tG} & \longrightarrow & X^{tG} & \xrightarrow{\Gamma_{n-1}} & (X^{tC})^{h\bar G}\,.
\end{array}$$

The right hand horizontal maps are Γ_{n-1} for the $\bar G$-spectra $\Phi^C(X) \simeq [\widetilde{EG} \wedge X]^C$ and $X^{tC} = [\widetilde{EG} \wedge F(EG_+, X)]^C$, respectively.

Proof We replace X in the lemma above by the G-spectra $\widetilde{EG} \wedge X$ and $\widetilde{EG} \wedge F(EG_+, X)$. This gives the two claimed homotopy cofiber sequences, in view of the $\bar G$-equivalences

$$[\Sigma^{-j\lambda^p}(\widetilde{EG} \wedge X)]^C \simeq [\widetilde{EG} \wedge \Sigma^{-j\lambda^p} X]^C \simeq \Phi^C(\Sigma^{-j\lambda^p} X)$$

and

$$[\Sigma^{-j\lambda^p}(\widetilde{EG} \wedge F(EG_+, X))]^C \simeq [\widetilde{EG} \wedge F(EG_+, \Sigma^{-j\lambda^p} X)]^C = (\Sigma^{-j\lambda^p} X)^{tC}\,,$$

respectively. These follow from the G-dualizability of $S^{j\lambda^p}$.

Lemma 3.4 *If $\hat\Gamma_1 \colon \Phi^C(X) \to X^{tC}$ is (W, k)-coconnected, then $(\hat\Gamma_1)^{h\bar G}$ is (W, k)-coconnected.*

Proof This is a special case of a more general result. The homotopy fixed point spectral sequence

$$E^2_{s,t} = H^{-s}(G; \pi_t(Y)) \Longrightarrow \pi_{s+t}(Y^{hG})$$

shows that Y^{hG} is k-coconnected whenever Y is a k-coconnected G-spectrum. Commutation of function spectra, homotopy fibers and homotopy fixed points shows that $f^{hG} \colon Y_1^{hG} \to Y_2^{hG}$ is (W, k)-coconnected whenever $f \colon Y_1 \to Y_2$ is a (W, k)-coconnected G-map. The lemma follows by applying this to the case of the $\bar G$-map $\hat\Gamma_1$.

Definition 3.5 The *Greenlees filtration* [9, p. 437] of $\widetilde{EG} = S^{\infty\lambda}$ is an integer-indexed G-cellular filtration of spectra, whose $2i$-th term is $S^{i\lambda}$ for each i. The $(2i+1)$-th term is obtained from $S^{i\lambda}$ by attaching a single G-free $(2i+1)$-cell, and $S^{(i+1)\lambda}$ is in turn obtained from it by attaching a single G-free $(2i+2)$-cell. The composite G-map $S^{i\lambda} \to S^{(i+1)\lambda}$ is given by smashing $S^{i\lambda}$ with the one-point compactification $\tau \colon S^0 \to S^\lambda$ of the inclusion $\{0\} \subset \lambda$. The Greenlees filtration induces an increasing filtration of $X^{tG} = [\widetilde{EG} \wedge F(EG_+, X)]^G$, and a tower of homotopy cofibers with $(2i+1)$-th term

$$X^{tG}\langle i\rangle = [\widetilde{EG}/S^{i\lambda} \wedge F(EG_+, X)]^G\,, \tag{3.1}$$

which we call the Tate tower. The associated spectral sequence is the homological G-equivariant Tate spectral sequence

$$\hat{E}^2_{s,t} = \widehat{H}^{-s}(G; H_t(X))$$

converging to the continuous homology groups

$$H^c_*(X^{tG}) = \lim_i H_*(X^{tG}\langle i \rangle)$$

of X^{tG}, when X is a bounded below spectrum with $H_*(X)$ of finite type. See [17, Def. 2.3, Prop. 4.15]. Note that i tends to $-\infty$ in this limit. We shall also refer to the continuous cohomology groups

$$H_c^*(X^{tG}) = \operatorname*{colim}_i H^*(X^{tG}\langle i \rangle)\,,$$

and note that $H^c_*(X^{tG}) \cong H^*_c(X^{tG})^*$ (the Hom dual) when $H_*(X)$ is bounded below and of finite type, because then each $H_*(X^{tG}\langle i \rangle)$ is also of finite type.

Definition 3.6 Let the G-map $\xi\colon\ S^\lambda \to S^{\lambda^p}$ of representation spheres be the suspension of the standard degree p covering map $\pi\colon\ S(\lambda) \to S(\lambda^p)$ of unit circles, as in the following vertical map of horizontal G-homotopy cofiber sequences:

$$\begin{tikzcd}
S(\lambda)_+ \arrow[r] \arrow[d, "\pi_+"] & S^0 \arrow[r, "\tau"] \arrow[d, equal] & S^\lambda \arrow[d, "\xi"] \\
S(\lambda^p)_+ \arrow[r] & S^0 \arrow[r, "z"] & S^{\lambda^p}\,.
\end{tikzcd}$$

We note that ξ is not induced by a linear map. It has degree p on the top cell, so $\xi_*\colon\ H_*(S^\lambda) \to H_*(S^{\lambda^p})$ is the zero homomorphism, since we work with reduced homology and mod p coefficients.

Proposition 3.7 *Let X be a G-spectrum with $H_*(X)$ bounded below. Then*

$$\lim_j H^c_*((\Sigma^{-j\lambda^p} X)^{tG}) = \lim_{i,j} H_*((\Sigma^{-j\lambda^p} X)^{tG}\langle i \rangle) = 0$$

and

$$\operatorname*{colim}_j H_c^*((\Sigma^{-j\lambda^p} X)^{tG}) = \operatorname*{colim}_{i,j} H^*((\Sigma^{-j\lambda^p} X)^{tG}\langle i \rangle) = 0\,.$$

Proof In the notation of (3.1) we have a natural equivalence

$$\nu\colon\ (\Sigma^{-j\lambda^p} X)^{tG}\langle i \rangle \xrightarrow{\simeq} (\Sigma^{j(\lambda-\lambda^p)} X)^{tG}\langle i-j \rangle$$

for each i and j. It is obtained from the G-equivalence

$$\frac{S^{\infty\lambda} \wedge S^{j\lambda}}{S^{i\lambda}} \wedge F(EG_+, \Sigma^{-j\lambda^p} X) \xrightarrow{\simeq} \frac{S^{\infty\lambda}}{S^{(i-j)\lambda}} \wedge F(EG_+, S^{j\lambda} \wedge \Sigma^{-j\lambda^p} X)$$

(see [14, III.1]) by passage to G-fixed points. Under these equivalences, the z-tower map

$$z\colon\ (\Sigma^{-(j+1)\lambda^p}X)^{tG}\langle i\rangle \to (\Sigma^{-j\lambda^p}X)^{tG}\langle i\rangle$$

induced by smashing with $z\colon S^0 \to S^{\lambda^p}$ is compatible with the composite of the Tate tower map

$$\tau\colon\ (\Sigma^{(j+1)(\lambda-\lambda^p)}X)^{tG}\langle i{-}j{-}1\rangle \to (\Sigma^{(j+1)(\lambda-\lambda^p)}X)^{tG}\langle i{-}j\rangle$$

induced by smashing with $\tau\colon S^0 \to S^\lambda$, and the ξ-tower map

$$\xi\colon\ (\Sigma^{(j+1)(\lambda-\lambda^p)}X)^{tG}\langle i{-}j\rangle \to (\Sigma^{j(\lambda-\lambda^p)}X)^{tG}\langle i{-}j\rangle$$

induced by smashing with $\xi\colon S^\lambda \to S^{\lambda^p}$, in the sense that the following diagram commutes up to homotopy:

$$\begin{tikzcd}
{[\widetilde{EG}/S^{i\lambda} \wedge F(EG_+, \Sigma^{-(j+1)\lambda^p}X)]^G} \arrow[dr, "\nu", "\simeq"'] \arrow[ddd, "z"'] & \\
& {[\widetilde{EG}/S^{(i-j-1)\lambda} \wedge F(EG_+, \Sigma^{(j+1)(\lambda-\lambda^p)}X)]^G} \arrow[d, "\tau"] \\
& {[\widetilde{EG}/S^{(i-j)\lambda} \wedge F(EG_+, \Sigma^{(j+1)(\lambda-\lambda^p)}X)]^G} \arrow[dd, "\xi"] \\
{[\widetilde{EG}/S^{i\lambda} \wedge F(EG_+, \Sigma^{-j\lambda^p}X)]^G} \arrow[dr, "\simeq", "\nu"'] & \\
& {[\widetilde{EG}/S^{(i-j)\lambda} \wedge F(EG_+, \Sigma^{j(\lambda-\lambda^p)}X)]^G\,.}
\end{tikzcd}$$

To see this, note that the diagram

$$\begin{tikzcd}
S^\lambda \wedge F(EG_+, S^0 \wedge X) \arrow[r, "\nu", "\simeq"'] \arrow[dd, "z"'] & S^0 \wedge F(EG_+, S^\lambda \wedge X) \arrow[d, "\tau"] \\
& S^\lambda \wedge F(EG_+, S^\lambda \wedge X) \arrow[d, "\xi"] \\
S^\lambda \wedge F(EG_+, S^{\lambda^p} \wedge X) \arrow[r, equal] & S^\lambda \wedge F(EG_+, S^{\lambda^p} \wedge X)
\end{tikzcd}$$

commutes up to G-homotopy (because the twist map $S^\lambda \wedge S^\lambda \cong S^\lambda \wedge S^\lambda$ is G-homotopic to the identity), replace X with $\Sigma^{j\lambda-(j+1)\lambda^p}X$, smash with $\widetilde{EG}/S^{(i-j-1)\lambda}$ and pass to G-fixed points.

Passing to homology, we get that z_* is strictly compatible with the composite $\xi_*\tau_*$ under the isomorphisms ν_*. Next pass to continuous homology by forming limits over i along the homomorphisms τ_*. Then the homomorphism

$$z_*\colon\ H^c_*((\Sigma^{-(j+1)\lambda^p}X)^{tG}) \to H^c_*((\Sigma^{-j\lambda^p}X)^{tG})$$

is identified with the homomorphism

$$\xi_*\colon\ H^c_*((\Sigma^{(j+1)(\lambda-\lambda^p)}X)^{tG}) \to H^c_*((\Sigma^{j(\lambda-\lambda^p)}X)^{tG})\,, \tag{3.2}$$

so it suffices to show that the limit over j of the latter homomorphisms is zero. Let

$$\hat{E}^2_{s,t}(j) = \widehat{H}^{-s}(G; H_t(\Sigma^{j(\lambda-\lambda^p)}X)) \Longrightarrow H^c_{s+t}((\Sigma^{j(\lambda-\lambda^p)}X)^{tG})$$

be the homological Tate spectral sequence for the j-th term in the ξ-tower.

By naturality of the Tate spectral sequence, the homomorphism ξ_* above is compatible with the spectral sequence map $\hat{E}^2_{**}(j+1) \to \hat{E}^2_{**}(j)$ that is induced on Tate cohomology by the G-module homomorphism

$$\xi_*\colon\ H_*(\Sigma^{(j+1)(\lambda-\lambda^p)}X) \to H_*(\Sigma^{j(\lambda-\lambda^p)}X)\,.$$

This homomorphism is zero, since $\xi_*\colon\ H_*(S^\lambda) \to H_*(S^{\lambda^p})$ is zero. Hence the map of spectral sequences is also zero. It follows that the homomorphism ξ_* in (3.2) strictly reduces the Tate filtration ($= s$) of each nonzero continuous homology class. Equivalently, ξ_* strictly increases the vertical degree ($= t$) of the spectral sequence representative of each nonzero class.

By assumption, there is an integer ℓ such that $H_t(X) = 0$ for all $t < \ell$. Then $\hat{E}^2_{s,t}(j) = \hat{E}^\infty_{s,t}(j) = 0$ for $t < \ell$ and any j. If $x = (x_j)_j$ is an arbitrary element of $\lim_j H^c_*((\Sigma^{j(\lambda-\lambda^p)}X)^{tG})$, then $x_j = \xi^m_*(x_{j+m})$ for each $m \ge 0$. If x_j is represented in vertical degree t, then x_{j+m} must be represented in vertical degree $\le (t-m)$. Choosing m so large that $t-m < \ell$, it follows that $x_{j+m} = 0$, which implies $x_j = 0$. Repeating the argument for each j we see that $x = 0$, so $\lim_j H^c_*((\Sigma^{j(\lambda-\lambda^p)}X)^{tG})$ must be the trivial group.

Let $M = \operatorname{colim}_j H^*_c((\Sigma^{-j\lambda^p}X)^{tG})$. Then the Hom dual M^* is the limit group we just showed is zero, and M injects into its double Hom dual M^{**}, so $M = 0$ as well.

Proposition 3.8 *Let X be a G-spectrum such that $\pi_*(X)$ is bounded below and $H_*(X)$ is of finite type. Then the p-adic completion $Y^\wedge_p$ of*

$$Y = \operatorname*{holim}_j\, (\Sigma^{-j\lambda^p}X)^{tG}$$

is contractible. Hence the map $\Gamma_{n-1}\colon\ X^{tG} \to (X^{tC})^{h\bar{G}}$ is a p-adic equivalence.

Proof The spectrum Y is the homotopy limit over i and j of the spectra

$$(\Sigma^{-j\lambda^p}X)^{tG}\langle i\rangle = [\widetilde{EG}/S^{i\lambda} \wedge F(EG_+, \Sigma^{-j\lambda^p}X)]^G\,,$$

which can be rewritten as

$$(\widetilde{EG}/S^{i\lambda} \wedge \Sigma^{-j\lambda^p}X)_{hG}$$

by the Adams equivalence [14, II.8.4], since $\widetilde{EG}/S^{i\lambda}$ is a free G-CW spectrum. Each of these is bounded below with mod p homology of finite type. Hence there is an inverse limit Adams spectral sequence

$$E_2^{**} = \mathrm{Ext}_{\mathscr{A}}^{**}(M, \mathbb{F}_p) \Longrightarrow \pi_*(Y^\wedge_p)$$

converging to the p-adic homotopy of that homotopy limit (see [8, Prop. 7.1] and [17, Prop. 2.2]), where

$$M = \operatorname*{colim}_j H_c^*((\Sigma^{-j\lambda^p} X)^{tG}) .$$

The latter $\mathscr{A}$-module was shown to be zero in Proposition 3.7, hence the E_2-term is zero and $Y^\wedge_p$ is contractible. The second conclusion follows from Proposition 3.3.

Proof of Theorem 2.4 Consider the diagram in Proposition 3.3. By assumption, the maps

$$\Gamma_{n-1}\colon\ \Phi^C(X)^{\bar{G}} \to \Phi^C(X)^{h\bar{G}} \qquad \text{and} \qquad \Gamma_1\colon\ X^C \to X^{hC}$$

are (W,k)-coconnected. Hence $\hat{\Gamma}_1\colon\ \Phi^C(X) \to X^{tC}$ is (W,k)-coconnected, so by Lemma 3.4 also

$$(\hat{\Gamma}_1)^{h\bar{G}}\colon\ \Phi^C(X)^{h\bar{G}} \to (X^{tC})^{h\bar{G}}$$

is (W,k)-coconnected. By Proposition 3.8, the map $\Gamma_{n-1}\colon\ X^{tG} \to (X^{tC})^{h\bar{G}}$ is a p-adic equivalence, hence $(W,-\infty)$-coconnected, by our standing assumption that W is in the localizing ideal of spectra generated by S^{-1}/p^∞. It follows easily that $\hat{\Gamma}_n\colon\ \Phi^C(X)^{\bar{G}} \to X^{tG}$ is (W,k)-coconnected, which is equivalent to $\Gamma_n\colon\ X^G \to X^{hG}$ being (W,k)-coconnected.

Proof of Theorem 2.5 This follows by induction on n, using Theorem 2.4 and the observation that

$$\Phi^{C_p}(\Phi^{C_{p^e}}(X)) \cong \Phi^{C_{p^{e+1}}}(X)$$

for all $0 \le e < n$.

Proof of Theorem 2.7 This follows from Theorem 2.5 in the case $X = B^{\wedge p^n}$, $W = S^{-1}/p^\infty$ and $k = -\infty$, once we show that for each $0 \le e < n$ there is a $C_{p^{n-e}}$-equivalence

$$Y = \Phi^{C_{p^e}}(B^{\wedge p^n}) \simeq B^{\wedge p^{n-e}} ,$$

the right hand side is bounded below with mod p homology of finite type, and $\Gamma_1\colon\ Y^{C_p} \to Y^{hC_p}$ is a p-adic equivalence. The first claim follows from the proof in simplicial degree 0 of [12, Prop. 2.5]. Writing $Y \simeq Z^{\wedge p}$, where $Z = B^{\wedge p^{n-e-1}}$ is bounded below with $H_*(Z)$ of finite type, the other claims also follow, since $\Gamma_1\colon\ (Z^{\wedge p})^{C_p} \to (Z^{\wedge p})^{hC_p}$ is a p-adic equivalence by [17, Thm. 5.13], generalizing [6, §II.5].

Proof of Theorem 2.8 There is a $C_{p^{n-1}}$-equivalence

$$r\colon \Phi^{C_p} THH(B) \xrightarrow{\simeq} THH(B)$$

(the cyclotomic structure map of $THH(B)$, see [12, §2.5]), whose e-fold iterate is a $C_{p^{n-e}}$-equivalence $\Phi^{C_{p^e}}(THH(B)) \simeq THH(B)$. It is clear from the simplicial definition that $THH(B)$ is connective and has mod p homology of finite type, hence the theorem follows from Theorem 2.5.

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