Homology operations in the topological cyclic homology of a point

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We consider the commutative $\mathbb{S}$-algebra given by the topological cyclic homology of a point. The induced Dyer–Lashof operations in mod p homology are shown to be non-trivial for $p = 2$, and an explicit formula is given. As a part of the calculation, we are led to compare the fixed point spectrum $\mathbb{S}^G$ of the sphere spectrum and the algebraic K-theory spectrum of finite G-sets, as structured ring spectra.

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Introduction

Let $A(\star) = K(\mathbb{S})$ denote Waldhausen's algebraic K-theory of a point [23]. It is a commutative $\mathbb{S}$-algebra, in the sense of Elmendorf, Kriz, Mandell and May [7], and the algebraic K-theory $A(X)$ of any space X, or more generally the algebraic K-theory $K(R)$ of any $\mathbb{S}$-algebra R, is a module spectrum over it. Hence it makes sense to carefully study the commutative $\mathbb{S}$-algebra structure of $A(\star)$, or equivalently its structure as an E_∞ ring spectrum. To the eyes of mod p homology, the primary incarnation of this structure is the Pontryagin algebra structure on $H_*(A(\star))$, together with the multiplicative Dyer–Lashof operations $Q^i\colon\ H_*(A(\star)) \to H_{*+i}(A(\star))$, as defined by Bruner, May, McClure and Steinberger [2]. Here and elsewhere we write $H_*(E)$ for the mod p homology $H_*(E; \mathbb{F}_p)$ of a spectrum E.

The additive structure of $H_*(A(\star))$ is known for $p = 2$ and for p an odd regular prime, by the second author's papers [18] and [19], but at present the Pontryagin product and Dyer–Lashof operations are not known for this E_∞ ring spectrum. There is, however, a very good approximation to Waldhausen's algebraic K-theory, given by the cyclotomic trace map to the topological cyclic homology of Bökstedt, Hsiang and Madsen [1]. This is a natural map $trc\colon\ K(R) \to TC(R; p)$, which we write as $trc\colon\ A(\star) \to TC(\star; p)$

in the special case when $R = \mathbb{S}$, where $TC(\star;p) = TC(\mathbb{S};p)$ is the topological cyclic homology of a point. By a theorem of Dundas [5], there is a homotopy cartesian square

$$\begin{tikzcd}
A(\star) \arrow[r] \arrow[d, "trc"] & K(\mathbb{Z}) \arrow[d, "trc"] \\
TC(\star;p) \arrow[r] & TC(\mathbb{Z};p)
\end{tikzcd}$$

(after p-adic completion) of commutative $\mathbb{S}$-algebras [9, Sect. 6], and this square is the basis for our additive understanding of $H_*(A(\star))$.

We are therefore led to study the commutative $\mathbb{S}$-algebra structure of $TC(\star;p)$, including the Pontryagin algebra structure and the Dyer–Lashof operations on its mod p homology. Like in the case of algebraic K-theory, the topological cyclic homology $TC(X;p)$ of any space X, and more generally the topological cyclic homology $TC(R;p)$ of any $\mathbb{S}$-algebra R, is a module spectrum over $TC(\star;p)$, and this provides a second motivation for the study of $TC(\star;p)$. In the present paper, we determine the Dyer–Lashof operations in $H_*(TC(\star;p))$ in the case when $p = 2$, as explained in Theorem 0.2 and Corollary 0.3 below.

A third motivation stems from ideas of Jack Morava [17], to the effect that there may be a spectral enrichment of the algebro-geometric category of mixed Tate motives, given by A-theoretic [24] or TC-theoretic [6] correspondences, followed by stabilization. The trace map $A(\star) \to TC(\star;p) \to THH(\star) = \mathbb{S}$ defines a fiber functor to the category of $\mathbb{S}$-modules, with Tannakian automorphism group realized through its Hopf algebra of functions, which will be of the form $\mathbb{S} \wedge_{A(\star)} \mathbb{S}$ or $\mathbb{S} \wedge_{TC(\star;p)} \mathbb{S}$. Rationally, this is well compatible with Deligne's results on the Tannakian group of mixed Tate motives over the integers [4]. A calculational analysis of the commutative $\mathbb{S}$-algebras $\mathbb{S} \wedge_{A(\star)} \mathbb{S}$ or $\mathbb{S} \wedge_{TC(\star;p)} \mathbb{S}$ clearly depends heavily on a proper understanding of the commutative $\mathbb{S}$-algebra structures of $A(\star)$ and $TC(\star;p)$.

Let $\mathbb{T}$ be the circle group and let $C_{p^n} \subset \mathbb{T}$ be the (cyclic) subgroup of order p^n. The spectrum $TC(\star;p)$ is defined as the homotopy inverse limit of a diagram

$$\cdots \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S}^{C_{p^{n+1}}} \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S}^{C_{p^n}} \overset{R}{\underset{F}{\rightrightarrows}} \cdots \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S}^{C_p} \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S} \tag{0–1}$$

of E_∞ ring spectra, where $\mathbb{S}^{C_{p^n}}$ denotes the C_{p^n}-fixed points of the $\mathbb{T}$-equivariant sphere spectrum, the maps labeled R are restriction maps, and the maps labeled F are Frobenius maps. See Bökstedt, Hsiang and Madsen [1] or Hesselholt and Madsen [10] for the construction of these maps. Similarly, let $TC^{(1)}(\star;p)$ denote the homotopy limit of the subdiagram

$$\mathbb{S}^{C_p} \overset{R}{\underset{F}{\rightrightarrows}} \mathbb{S} \ , \tag{0–2}$$

that is, the homotopy equalizer of R and F. The canonical maps

$$TC(\star;p) \xrightarrow{f_1} TC^{(1)}(\star;p) \xrightarrow{g_1} \mathbb{S}^{C_p} \tag{0–3}$$

are then maps of E_∞ ring spectra.

The unit $\eta\colon \mathbb{S} \to TC(\star;p)$ and the restriction $R\colon \mathbb{S}^{C_p} \to \mathbb{S}$ let us split off a copy of $\mathbb{S}$ from each term in (0–3). Let $\mathbb{C}P^\infty_{-1}$ be the Thom spectrum of the negative tautological complex line bundle $-\gamma^1_{\mathbb{C}}$ over $\mathbb{C}P^\infty$. Its suspension $\Sigma\mathbb{C}P^\infty_{-1}$ is equivalent to the homotopy fiber of the dimension-shifting $\mathbb{T}$-transfer map $t_{\mathbb{T}}\colon \Sigma^\infty\Sigma(\mathbb{C}P^\infty_+) \to \mathbb{S}$, see Knapp [13, 2.9] or Lemma 1.1 below. We define the spectrum L^∞_{-1} to be the homotopy fiber of the C_p-transfer $t_p\colon \Sigma^\infty(BC_p)_+ \to \mathbb{S}$. For $p = 2$, there is an equivalence $L^\infty_{-1} \simeq \mathbb{R}P^\infty_{-1}$, where $\mathbb{R}P^\infty_{-1}$ is the Thom spectrum of the negative tautological real line bundle $-\gamma^1_{\mathbb{R}}$ over $\mathbb{R}P^\infty$. The mod p homology groups of these spectra are well known:

$$\begin{aligned} H_*(\Sigma\mathbb{C}P^\infty_{-1}) &\cong \mathbb{F}_p\{\Sigma\beta_k \mid k \geq -1\} \\ H_*(L^\infty_{-1}) &\cong \mathbb{F}_p\{\alpha_k \mid k \geq -1\} \\ H_*(\Sigma^\infty(BC_p)_+) &\cong \mathbb{F}_p\{\alpha_k \mid k \geq 0\} \end{aligned}$$

Here $\Sigma\beta_k$ has degree $2k+1$ and α_k has degree k.

Lemma 0.1 *After p-completion, diagram (0–3) is homotopy equivalent to a diagram*

$$\mathbb{S} \vee \Sigma\mathbb{C}P^\infty_{-1} \xrightarrow{1\vee f} \mathbb{S} \vee L^\infty_{-1} \xrightarrow{1\vee g} \mathbb{S} \vee \Sigma^\infty(BC_p)_+ \,.$$

In particular, the Pontryagin product on $H_*(TC(\star;p))$ is trivial.

Applying homology gives a sequence

$$H_*(\mathbb{S}) \oplus H_*(\Sigma\mathbb{C}P^\infty_{-1}) \xrightarrow{1\oplus f_*} H_*(\mathbb{S}) \oplus H_*(L^\infty_{-1}) \xrightarrow{1\oplus g_*} H_*(\mathbb{S}) \oplus H_*(\Sigma^\infty(BC_p)_+) \,.$$

Here f_* sends $\Sigma\beta_k$ to α_{2k+1} for $k \geq -1$, and g_* is the identity on α_k for $k \geq 0$, while α_{-1} maps to zero.

We now state our main result, which concerns the Dyer–Lashof operations in the mod p spectrum homology $H_*(TC(\star;p))$ for $p = 2$. The calculations will be done in the auxiliary E_∞ ring spectra $\mathbb{S}^{C_2}$ and $TC^{(1)}(\star;2)$.

Theorem 0.2 *The Dyer–Lashof operations Q^i in $H_*(TC^{(1)}(\star;2))$ and $H_*(\mathbb{S}^{C_2})$ are given by the formula*

$$Q^i(\alpha_j) = \binom{2^N + i - 1}{2^N + j}\alpha_{i+j} \,,$$

where $j \geq -1$ and i is any integer, and N is sufficiently large.

Corollary 0.3 *The Dyer–Lashof operations Q^{2i} in $H_*(TC(\star;2))$ are given by the formula*

$$Q^{2i}(\Sigma\beta_j) = \binom{2^N+i-1}{2^N+j}\Sigma\beta_{i+j}\,,$$

where $j \geq -1$ and i is any integer, and N is sufficiently large. The operations Q^{2i+1} are all zero for degree reasons.

Note that the binomial coefficients used in the theorem and corollary can be evaluated to

$$\binom{2^N+i-1}{2^N+j} \equiv \begin{cases} \binom{i-1}{j} & \text{for } i > j \geq 0 \\ 1 & \text{for } (i,j) = (0,-1) \\ 0 & \text{otherwise} \end{cases}$$

modulo 2, for all sufficiently large N. In particular $Q^0(\alpha_{-1}) = \alpha_{-1}$ and $Q^0(\Sigma\beta_{-1}) = \Sigma\beta_{-1}$.

We prove Lemma 0.1 in Section 1 and Theorem 0.2 in Section 3, after a homological comparison of E_∞ ring structures in Section 2. Corollary 0.3 follows immediately from the lemma and the theorem.

1 Topological cyclic homology of a point

In this preliminary section we review the calculation of $TC(\star;p)$ from Bökstedt, Hsiang and Madsen [1, 5.17], in order to describe the map f_1 to $TC^{(1)}(\star;p)$.

For each $n \geq 1$ the Segal–tom Dieck splitting tells us that the norm–restriction homotopy cofiber sequence $\Sigma^\infty(BC_{p^n})_+ \xrightarrow{N} \mathbb{S}^{C_{p^n}} \xrightarrow{R} \mathbb{S}^{C_{p^{n-1}}}$ is canonically split. The homotopy limit $TR(\star;p) = \operatorname{holim}_{n,R} \mathbb{S}^{C_{p^n}}$ of the R-maps in (0–1) thus factors as $TR(\star;p) \simeq \prod_{n\geq 0}\Sigma^\infty(BC_{p^n})_+$. Let $pr_n\colon TR(\star;p) \to \Sigma^\infty(BC_{p^n})_+$ denote the n-th projection, and let $\widetilde{TR}(\star;p) \simeq \prod_{n\geq 1}\Sigma^\infty(BC_{p^n})_+$ be the homotopy fiber of pr_0. There is a vertical map of horizontal homotopy fiber sequences

(1–1)
$$\begin{tikzcd} TC(\star;p) \arrow[r, "\pi"] \arrow[d, "f_1"'] & TR(\star;p) \arrow[r, "F-R"] \arrow[d, "p_1"] & TR(\star;p) \arrow[d, "pr_0"] \\ TC^{(1)}(\star;p) \arrow[r, "g_1"] & \mathbb{S}^{C_p} \arrow[r, "F-R"] & \mathbb{S} \end{tikzcd}$$

and the augmentation $TC(\star;p) \to \mathbb{S}$ factors as $R \circ p_1 \circ \pi = pr_0 \circ \pi$. Replacing the left hand square by the homotopy fibers of the augmentations to $\mathbb{S}$, we get a second vertical map of horizontal homotopy fiber sequences

(1–2)
$$\begin{tikzcd} \widetilde{TC}(\star;p) \arrow[r] \arrow[d, "f_2"'] & \widetilde{TR}(\star;p) \arrow[r, "T-I"] \arrow[d, "pr_1"] & TR(\star;p) \arrow[d, "pr_0"] \\ \widetilde{TC}^{(1)}(\star;p) \arrow[r, "g_2"] & \Sigma^\infty(BC_p)_+ \arrow[r, "t_p"] & \mathbb{S}\,. \end{tikzcd}$$

In the upper row we have used that $F-R$ restricted along the inclusion $I\colon \widetilde{TR}(\star;p) \to TR(\star;p)$ is homotopic to $T-I$, where T is the product of the C_p-transfer maps $\Sigma^\infty(BC_{p^n})_+ \to \Sigma^\infty(BC_{p^{n-1}})_+$ for all $n \geq 1$. See [1, (5.18)]. In the lower row we have used that $F-R$ restricted along $N\colon \Sigma^\infty(BC_p)_+ \to \mathbb{S}^{C_p}$ is homotopic to the C_p-transfer map t_p.

There is a third vertical map of horizontal homotopy fiber sequences

(1–3)
$$\begin{tikzcd} \operatorname{holim}_n \Sigma^\infty(BC_{p^n})_+ \arrow[r] \arrow[d, "f_3"'] & TR(\star;p) \arrow[r, "T-1"] \arrow[d, "p_1"] & TR(\star;p) \arrow[d, "pr_0"] \\ \operatorname{hofib}(F-2R) \arrow[r, "g_3"] & \mathbb{S}^{C_p} \arrow[r, "F-2R"] & \mathbb{S}\,. \end{tikzcd}$$

Replacing its left hand square by the homotopy fibers of the augmentations to $\mathbb{S}$, we also recover diagram (1–2).

Lemma 1.1 *There are equivalences*

$$\widetilde{TC}(\star;p) \simeq \operatorname{hofib}(t_{\mathbb{T}}\colon\ \Sigma^\infty\Sigma(\mathbb{C}P^\infty_+) \to \mathbb{S}) \simeq \Sigma\mathbb{C}P^\infty_{-1}$$

(after p-completion) and

$$\widetilde{TC}^{(1)}(\star;p) \simeq \operatorname{hofib}(t_p\colon\ \Sigma^\infty(BC_p)_+ \to \mathbb{S}) = L^\infty_{-1}\,.$$

When $p=2$, $L^\infty_{-1} \simeq \mathbb{R}P^\infty_{-1}$.

Proof The dimension-shifting $\mathbb{T}$-transfer maps for the bundles $BC_{p^n} \to B\mathbb{T}$ induce an equivalence $\Sigma^\infty\Sigma(\mathbb{C}P^\infty_+) \simeq \operatorname{holim}_n \Sigma^\infty(BC_{p^n})_+$ after p-completion [1, 5.15]. The augmentation $\operatorname{holim}_n \Sigma^\infty(BC_{p^n})_+ \to \mathbb{S}$ then gets identified with $t_{\mathbb{T}}$, which implies the first claim.

There is a $\mathbb{T}$-equivariant homotopy cofiber sequence $S^0 \xrightarrow{z} S^{\mathbb{C}} \xrightarrow{t} \mathbb{T}_+ \wedge S^1$, where z is the zero-inclusion. The right hand map t is the Pontryagin–Thom collapse associated to the standard embedding $\mathbb{T} \subset \mathbb{C}$, as in Lewis, May and Steinberger [14, II.5.1]. The dimension-shifting $\mathbb{T}$-transfer $t_{\mathbb{T}}\colon \Sigma^\infty\Sigma(\mathbb{C}P^\infty_+) \to \mathbb{S}$ for $E\mathbb{T} \to B\mathbb{T}$ is constructed as the balanced smash product

$$1 \wedge_{\mathbb{T}} \Sigma^{1-\mathbb{C}}(t)\colon\ E\mathbb{T}_+ \wedge_{\mathbb{T}} \Sigma^{1-\mathbb{C}}(S^{\mathbb{C}}) \to E\mathbb{T}_+ \wedge_{\mathbb{T}} \Sigma^{1-\mathbb{C}}(\mathbb{T}_+ \wedge S^1)\,,$$

see [14, II.7.5]. Hence its homotopy fiber is $E\mathbb{T}_+ \wedge_{\mathbb{T}} \Sigma^{1-\mathbb{C}}(S^0) \cong \Sigma\mathbb{C}P^\infty_{-1}$.

The proof that $\mathbb{R}P^\infty_{-1}$ is the homotopy fiber of t_p for $p = 2$ is essentially the same. □

Proof of Lemma 0.1 Under the identifications of Lemma 1.1, the maps $f\colon \Sigma\mathbb{C}P^\infty_{-1} \to L^\infty_{-1}$ and $g\colon L^\infty_{-1} \to \Sigma^\infty(BC_p)_+$ correspond to the maps f_2 and g_2 in diagram (1–2), respectively.

The C_p-transfer map t_p induces multiplication by p on π_0, and the zero map in mod p homology, so $\pi_{-1}(f)$ is surjective, f_* maps $\Sigma\beta_{-1}$ to α_{-1}, and g_* maps α_k to α_k for all $k \geq 0$. It remains to see that g_*f_* maps $\Sigma\beta_k$ to α_{2k+1} for $k \geq 0$. This is clear from diagram (1–3), since g_3f_3 agrees in positive degrees with the $\mathbb{T}$-transfer map $\Sigma^\infty\Sigma(\mathbb{C}P^\infty_+) \to \Sigma^\infty(BC_p)_+$, which has this behavior on homology. □

2 Algebraic K-theory of finite G-sets

In this section we will compare the algebraic K-theory spectrum of finite G-sets with the G-fixed points of the sphere spectrum, as structured ring spectra. Before we state the result we recall some of the definitions involved.

The K-theory construction we use is that of Elmendorf and Mandell [8]. When the input category is a bipermutative category $\mathcal{C}$, their machine produces a symmetric spectrum $K(\mathcal{C})$, in the sense of Hovey, Shipley and Smith [11], with an action of the simplicial Barratt–Eccles operad. We will use the same notation for the geometrically realized symmetric spectrum in topological spaces, which has an action

$$\kappa_j\colon\ E\Sigma_j \ltimes_{\Sigma_j} K(\mathcal{C})^{\wedge j} \to K(\mathcal{C})$$

of the operad $E\Sigma$ consisting of the contractible Σ_j-free spaces $E\Sigma_j$. As usual, $E\Sigma_j$ can be defined as the nerve $N\tilde{\Sigma}_j$ of the translation category $\tilde{\Sigma}_j$, for $j \geq 0$. The K-theory construction itself is somewhat involved, but all we need to know is that the zeroth space $K(\mathcal{C})_0$ is the nerve $N\mathcal{C}$ of $\mathcal{C}$, so the zeroth space of $E\Sigma_j \ltimes_{\Sigma_j} K(\mathcal{C})^{\wedge j}$ is the nerve of $\tilde{\Sigma}_j \ltimes_{\Sigma_j} \mathcal{C}^j$, and the action of $E\Sigma$ on $K(\mathcal{C})_0$ is given by the maps $\lambda_j\colon\ E\Sigma_j \ltimes_{\Sigma_j} N\mathcal{C}^{\wedge j} \to N\mathcal{C}$ that are induced by the functors taking an object $(\sigma; a_1, \dots, a_j)$ in $\tilde{\Sigma}_j \ltimes_{\Sigma_j} \mathcal{C}^j$ to the object $a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(j)}$ in $\mathcal{C}$ (see [8, Sect. 8]). Here $\otimes$ denotes the product in the bipermutative structure on $\mathcal{C}$. Hence there is a commutative diagram

$$\begin{tikzcd}
E\Sigma_j \ltimes_{\Sigma_j} K(\mathcal{C})^{\wedge j} \arrow[rr, "\kappa_j"] & & K(\mathcal{C}) \\
E\Sigma_j \ltimes_{\Sigma_j} (\Sigma^\infty N\mathcal{C})^{\wedge j} \arrow[u] \arrow[r, "\cong"] & \Sigma^\infty(E\Sigma_j \ltimes_{\Sigma_j} N\mathcal{C}^{\wedge j}) \arrow[r, "\Sigma^\infty \lambda_j"] & \Sigma^\infty N\mathcal{C} \arrow[u]
\end{tikzcd}$$

for each $j \geq 0$.

Let G be a finite group, and let $\mathcal{E}^G$ denote the category of finite G-sets and G-equivariant bijections. This is a symmetric bimonoidal category under disjoint union and cartesian product, taking (X, Y) to $X \coprod Y$ and $X \times Y$, respectively. We give $X \times Y$ the diagonal G-action. There is a functorially defined bipermutative category $\Phi\mathcal{E}^G$, and a natural equivalence $\mathcal{E}^G \to \Phi\mathcal{E}^G$ [16, VI.3.5]. It follows that there is a homotopy commutative diagram

(2–1)
$$\begin{tikzcd}
E\Sigma_j \ltimes_{\Sigma_j} K(\Phi\mathcal{E}^G)^{\wedge j} \arrow[rr, "\kappa_j"] & & K(\Phi\mathcal{E}^G) \\
E\Sigma_j \ltimes_{\Sigma_j} (\Sigma^\infty N\mathcal{E}^G)^{\wedge j} \arrow[u, "1 \ltimes \epsilon^{\wedge j}"] \arrow[r, "\cong"] & \Sigma^\infty(E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j}) \arrow[r, "\Sigma^\infty \lambda_j"] & \Sigma^\infty N\mathcal{E}^G \arrow[u, "\epsilon"']
\end{tikzcd}$$

for each $j \geq 0$, where

$$\lambda_j \colon\ E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j} \to N\mathcal{E}^G$$

is induced by the functor $\tilde{\Sigma}_j \ltimes_{\Sigma_j} (\mathcal{E}^G)^j \to \mathcal{E}^G$ that takes $(\sigma; X_1, \dots, X_j)$ to the cartesian product

$$X_{\sigma^{-1}(1)} \times (X_{\sigma^{-1}(2)} \times \cdots \times (X_{\sigma^{-1}(j-1)} \times X_{\sigma^{-1}(j)}) \dots) .$$

Let $\mathcal{U}$ be a complete G-universe, and let $\mathcal{L}$ denote the linear isometries operad with spaces $\mathcal{L}(j)$ consisting of linear isometries $\mathcal{U}^j \to \mathcal{U}$, where $\mathcal{U}^j$ denotes the direct sum of j copies of $\mathcal{U}$. There is an action of G on each $\mathcal{L}(j)$ given by conjugation, and this gives $\mathcal{L}$ the structure of an E_∞ G-operad in the sense of Lewis, May and Steinberger [14, VII.1.1]. The E_∞ ring structure on the G-equivariant sphere spectrum $\mathbb{S}_G = \Sigma^\infty_G S^0$ is given by an action

$$\zeta_j \colon\ \mathcal{L}(j) \ltimes_{\Sigma_j} \mathbb{S}_G^{\wedge j} \to \mathbb{S}_G$$

of this operad (where, for once, $\ltimes$ denotes the twisted half-smash product in Lewis–May spectra). It is compatible with a corresponding action

$$\omega_j \colon\ \mathcal{L}(j) \ltimes_{\Sigma_j} Q_G(S^0)^{\wedge j} \to Q_G(S^0)$$

on the underlying infinite loop space $Q_G(S^0) = \Omega^\infty \mathbb{S}_G = \operatorname{colim}_{V\subset\mathcal{U}} \Omega^V S^V$, in the sense that the following diagram commutes.

$$\tag{2–2} \begin{tikzcd} \mathcal{L}(j) \ltimes_{\Sigma_j} (\Sigma^\infty Q_G(S^0))^{\wedge j} \arrow[r, "\cong"] \arrow[d, "1\ltimes\epsilon^{\wedge j}"'] & \Sigma^\infty(\mathcal{L}(j) \ltimes_{\Sigma_j} Q_G(S^0)^{\wedge j}) \arrow[r, "\Sigma^\infty\omega_j"] & \Sigma^\infty Q_G(S^0) \arrow[d, "\epsilon"] \\ \mathcal{L}(j) \ltimes_{\Sigma_j} \mathbb{S}_G^{\wedge j} \arrow[rr, "\zeta_j"] & & \mathbb{S}_G \end{tikzcd}$$

Here ω_j sends an element in $\mathcal{L}(j) \ltimes_{\Sigma_j} Q_G(S^0)^{\wedge j}$ represented by $(f; g_1, \dots, g_j)$, where $f\colon \mathcal{U}^j \to \mathcal{U}$ and $g_i\colon S^{V_i} \to S^{V_i}$, to the element represented by the composite of the following maps.

$$S^{f(V_1\oplus\cdots\oplus V_j)} \xleftarrow[\cong]{f_*} S^{V_1\oplus\cdots\oplus V_j} \xrightarrow{g_1\wedge\cdots\wedge g_j} S^{V_1\oplus\cdots\oplus V_j} \xrightarrow[\cong]{f_*} S^{f(V_1\oplus\cdots\oplus V_j)}$$

By taking G-fixed points we get the non-equivariant E_∞ ring spectrum $\mathbb{S}^G = (\mathbb{S}_G)^G$ with an action

$$\xi_j\colon\ \mathcal{L}^G(j) \ltimes_{\Sigma_j} (\mathbb{S}^G)^{\wedge j} \to \mathbb{S}^G$$

of the non-equivariant E_∞ operad $\mathcal{L}^G$ of G-equivariant isometries. The corresponding infinite loop space $\Omega^\infty(\mathbb{S}^G)$ is the space $Q_G(S^0)^G = \operatorname{colim}_{V\subset\mathcal{U}} (\Omega^V S^V)^G$, with the inherited $\mathcal{L}^G$-action

$$\eta_j\colon\ \mathcal{L}^G(j) \ltimes_{\Sigma_j} (Q_G(S^0)^G)^{\wedge j} \to Q_G(S^0)^G\,.$$

Next we recall the definition of the Dyer–Lashof operations Q^i. Let $C_*(-)$ denote the cellular chains functor, from either CW complexes or CW spectra to chain complexes. Let E be a spectrum with an action of an E_∞ operad $\mathcal{O}$, and let W_* be the standard free C_p-resolution of $\mathbb{F}_p$ with basis elements e_i in degree i. There is a chain map $W_* \to C_*(\mathcal{O}(p))$ lifting the identity on $\mathbb{F}_p$, unique up to homotopy, and we also denote the image of e_i under this map by e_i. Let $x \in H_q(E)$ be represented by a cycle $z \in C_q(E)$. Now consider the image of the cycle $e_i \otimes z^{\otimes p}$ under the map

$$C_*(\mathcal{O}(p)) \otimes_{\Sigma_p} C_*(E)^{\otimes p} \xrightarrow{\cong} C_*(\mathcal{O}(p) \ltimes_{\Sigma_p} E^{\wedge p}) \xrightarrow{\xi_{p*}} C_*(E)\,,$$

and denote its image in homology by $Q_i(x)$. Here ξ_p is the E_∞ structure map. Then for $p = 2$ define $Q^i(x) = 0$ when $i < q$, and

$$Q^i(x) = Q_{i-q}(x)$$

when $i \geq q$. For $p > 2$ define $Q^i(x) = 0$ when $2i < q$, and

$$Q^i(x) = (-1)^i \nu(q) \cdot Q_{(2i-q)(p-1)}(x)$$

when $2i \geq q$, where $\nu(q) = (-1)^{q(q-1)(p-1)/4}((\frac{1}{2}(p-1))!)^q$. See Bruner, May, McClure and Steinberger [2, Ch. III] for more details.

The spectra $\mathbb{S}^G$ and $K(\Phi\mathcal{E}^G)$ should be equivalent as E_∞ ring spectra, but we will only need the following weaker result.

Lemma 2.1 *There is an equivalence $\mathbb{S}^G \simeq K(\Phi\mathcal{E}^G)$ of spectra such that the induced isomorphism $H_*(\mathbb{S}^G) \cong H_*(K(\Phi\mathcal{E}^G))$ commutes with the Dyer–Lashof operations.*

Proof Our first goal is to construct a commutative diagram

(2–3)
$$\begin{tikzcd}
 & E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j} \arrow[r, "\lambda_j"] & N\mathcal{E}^G & \\
E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G_{tr})^{\wedge j} \arrow[ur, "1\ltimes\phi^{\wedge j}"] \arrow[r] \arrow[ddr, "1\ltimes\psi^{\wedge j}"'] & (E\Sigma_j \times \mathcal{L}^G(j)) \ltimes_{\Sigma_j} D_{\mathcal{U}}^{\wedge j} \arrow[u, "\simeq"] \arrow[d, "\simeq"'] \arrow[r, "\mu_j"] & D_{\mathcal{U}} \arrow[u, "\simeq"'] \arrow[d, "\simeq"'] & N\mathcal{E}^G_{tr} \arrow[l] \arrow[ul, "\phi"'] \arrow[ddl, "\psi"] \\
 & \mathcal{L}^G(j) \ltimes_{\Sigma_j} C_{\mathcal{U}}^{\wedge j} \arrow[r, "\nu_j"] \arrow[d] & C_{\mathcal{U}} \arrow[d] & \\
 & \mathcal{L}^G(j) \ltimes_{\Sigma_j} (Q_G(S^0)^G)^{\wedge j} \arrow[r, "\eta_j"] & Q_G(S^0)^G . &
\end{tikzcd}$$

We start by describing the space $C_{\mathcal{U}}$. Let V be an indexing space in $\mathcal{U}$. For each finite G-set X, consider the space $E_V(X)$ of X-tuples of distance-reducing embeddings of V in V, closed under the action of G. More precisely, this is the space of G-equivariant maps $\coprod_X V \to V$ such that the restriction to each summand $g\colon V \to V$ is an embedding that satisfies $|g(v) - g(w)| \leq |v - w|$ for all $v, w \in V$. Let $K_V(X)$ be the space of paths $[0,1] \to E_V(X)$ such that the embeddings at the endpoint 0 are identities, and the embeddings at 1 have disjoint images. Now let

$$K_{\mathcal{U}}(X) = \operatorname*{colim}_{V \subset \mathcal{U}} K_V(X)\,.$$

These are G-equivariant versions of the spaces in the Steiner operad [22]. The group $\mathrm{Aut}^G(X)$ acts freely on $K_{\mathcal{U}}(X)$ by permuting the embeddings, and the space $C_{\mathcal{U}}$ is to be the disjoint union

$$C_{\mathcal{U}} = \coprod_{[X]} K_{\mathcal{U}}(X)/\,\mathrm{Aut}^G(X)$$

where X ranges over all isomorphism classes of finite G-sets.

The action of the operad $\mathcal{L}^G$ on $C_{\mathcal{U}}$ is defined as follows. Let $f\colon \mathcal{U}^j \to \mathcal{U}$ be a G-linear isometry, and let $[g_i]$, $1 \leq i \leq j$, be elements in $C_{\mathcal{U}}$, represented by paths of X_i-tuples of embeddings $g_i \in K_{\mathcal{U}}(X_i)$. Denote the component paths of embeddings that constitute g_i by g_{i,x_i}, where $x_i \in X_i$. The resulting element $\nu_j(f;[g_1],\dots,[g_j])$ in $C_{\mathcal{U}}$ is represented by an element in $K_{\mathcal{U}}(X_1 \times \cdots \times X_j)$, which on the summand indexed by $(x_1,\dots,x_j)$ is given by $f \circ (g_{1,x_1} \times \cdots \times g_{j,x_j}) \circ f^{-1}$.

There is a map $C_{\mathcal{U}} \to Q_G(S^0)^G$, given by evaluating a Steiner path in $E_V(X)$ at 1 to get a G-equivariant embedding $e\colon \coprod_X V \to V$, and then applying a folded Pontryagin–Thom construction to obtain a G-equivariant map $q\colon S^V \to S^V$, which is a point in $Q_G(S^0)^G$. Given the distance-reducing embedding e, let $S^V \to \bigvee_X S^V$ be the G-equivariant map that is given by e^{-1} on the image of e in $V \subset S^V$ and maps the remainder of S^V to the base point of $\bigvee_X S^V \supset \coprod_X V$. Let $\bigvee_X S^V \to S^V$ be the fold map that is the identity on each summand. The folded Pontryagin–Thom construction q is the composite of these two G-maps. If we permute the embeddings indexed by X we get the same element in $Q_G(S^0)^G$, so our map is well-defined. A comparison of definitions shows that this construction is compatible with the $\mathcal{L}^G$-actions on $C_{\mathcal{U}}$ and $Q_G(S^0)^G$, so the lower square in (2–3) commutes.

Let
$$D_{\mathcal{U}} = \coprod_{[X]} (E\operatorname{Aut}^G(X) \times K_{\mathcal{U}}(X))/\operatorname{Aut}^G(X)\,,$$

where $\operatorname{Aut}^G(X)$ acts diagonally on the product. The nerve $N\mathcal{E}^G$ splits as a sum of components

$$N\mathcal{E}^G \simeq \coprod_{[X]} B\operatorname{Aut}^G(X)\,, \tag{2–4}$$

where the disjoint union is over the isomorphism classes of G-sets X. Projection on the first factor in $D_{\mathcal{U}}$ followed by this homotopy equivalence gives the map $D_{\mathcal{U}} \to N\mathcal{E}^G$ in (2–3), while the map $D_{\mathcal{U}} \to C_{\mathcal{U}}$ is the projection on the second factor. There is an induced action of the product operad $E\Sigma \times \mathcal{L}^G$ on $D_{\mathcal{U}}$, defined as follows. Let $(e,f) \in E\Sigma_j \times \mathcal{L}^G(j)$ and let $(e_i,f_i) \in E\operatorname{Aut}^G(X) \times C_{\mathcal{U}}(X)$ represent elements in $D_{\mathcal{U}}$, for $1 \le i \le j$. The image under μ_j is the element represented by $(\lambda_j(e;e_1,\dots,e_j), \nu_j(f;f_1,\dots,f_j))$. This makes the upper and middle squares in (2–3) commute.

Let
$$N\mathcal{E}^G_{tr} = \Big(\coprod_{(H)} (E\operatorname{Aut}^G(G/H) \times K_{\mathcal{U}}(G/H))/\operatorname{Aut}^G(G/H)\Big)_+\,,$$

where the coproduct is taken over the conjugacy classes of subgroups H of G. The map $N\mathcal{E}^G_{tr} \to D_{\mathcal{U}}$ in (2–3) is the inclusion of the components indexed by the isomorphism classes of transitive G-sets.

The maps ϕ and ψ are defined by commutativity of the right hand triangles in the diagram. We claim that the adjoints

$$\Sigma^\infty N\mathcal{E}^G_{tr} \to K(\mathcal{E}^G) \simeq K(\Phi\mathcal{E}^G) \tag{2–5}$$

$$\Sigma^\infty N\mathcal{E}^G_{tr} \to \mathbb{S}^G \tag{2–6}$$

of the maps ϕ and ψ, respectively, are both equivalences. Here $K(\mathcal{E}^G)$ is the additive K-theory spectrum of $\mathcal{E}^G$, with zeroth space $K(\mathcal{E}^G)_0 = N\mathcal{E}^G$, which only depends on the additive symmetric monoidal structure of $\mathcal{E}^G$. There is an equivalence

$$\Sigma^\infty N\mathcal{E}^G_{tr} \simeq \bigvee_{(H)} \Sigma^\infty BW_GH_+ ,$$

where $W_GH = N_GH/H \cong \mathrm{Aut}^G(G/H)$ is the Weyl group of H and the wedge sum is over the conjugacy classes of subgroups of G. By Waldhausen's additivity theorem [23, 1.3.2] applied to a suitable filtration of $\mathcal{E}^G$ according to stabilizer types, there is a splitting

$$K(\mathcal{E}^G) \simeq \bigvee_{(H)} K(\mathcal{E}(W_GH)) ,$$

where $\mathcal{E}(W_GH)$ is the category of finite free W_GH-sets and equivariant bijections. The map (2–5) is equivalent under these identifications to the wedge sum of the maps

$$\Sigma^\infty BW_GH_+ \to K(\mathcal{E}(W_GH)) \tag{2–7}$$

that are left adjoint to the inclusions $BW_GH_+ \to N\mathcal{E}(W_GH) = K(\mathcal{E}(W_GH))_0$.

The Barratt–Priddy–Quillen–Segal theorem [20, 3.6] says that each of the maps (2–7) is an equivalence, hence (2–5) is an equivalence. The map (2–6) is an equivalence by the Segal–tom Dieck splitting [14, V.11.2]. The composition of these two equivalences is the equivalence $\mathbb{S}^G \simeq K(\Phi\mathcal{E}^G)$ referred to in the statement of the lemma.

We apply the suspension spectrum functor Σ^∞ to the diagram (2–3), combine it with diagram (2–1) and the G-fixed part of (2–2), take homology, and end up with the following commutative diagram.

(2–8)

$$\begin{tikzcd}
 & H_*(\Sigma_j; H_*(K(\Phi\mathcal{E}^G))^{\otimes j}) \arrow[r, "\kappa_{j*}"] & H_*(K(\Phi\mathcal{E}^G)) & \\
H_*(\Sigma_j; H_*(N\mathcal{E}^G_{tr})^{\otimes j}) \arrow[ur, "\cong"] \arrow[r] \arrow[dr, "\cong"'] & H_*(\Sigma_j; \tilde{H}_*(D_{\mathcal{U}})^{\otimes j}) \arrow[u] \arrow[d] \arrow[r, "\mu_{j*}"] & \tilde{H}_*(D_{\mathcal{U}}) \arrow[u, "\epsilon_1"'] \arrow[d, "\epsilon_2"] & H_*(N\mathcal{E}^G_{tr}) \arrow[l] \arrow[ul, "\phi_*"', "\cong"] \arrow[dl, "\psi_*", "\cong"'] \\
 & H_*(\Sigma_j; H_*(\mathbb{S}^G)^{\otimes j}) \arrow[r, "\xi_{j*}"] & H_*(\mathbb{S}^G) &
\end{tikzcd}$$

We need the fact that ϵ_1 and ϵ_2 have the same kernel. In fact, all summands in $\tilde{H}_*(D_{\mathcal{U}})$ indexed by G-sets with more than one orbit map to zero under both ϵ_1 and ϵ_2. This follows from the fact that Pontryagin products and additive Dyer–Lashof operations vanish after stabilization. More precisely, a decomposition of a G-set $X = \coprod_{i=1}^k n_i(G/H_i)$, where the H_i lie in distinct conjugacy classes, induces a factorization

$$B\,\mathrm{Aut}^G(X) \cong \prod_{i=1}^k B(\Sigma_{n_i} \wr W_G H_i)\,.$$

The homology group $H_*(B(\Sigma_{n_i} \wr W_G H_i)) \subset H_*(N\mathcal{E}^G)$ is generated by $H_*(BW_G H_i)$ under iterated Pontryagin products and Dyer–Lashof operations, see Cohen, Lada and May [3, I.4.1], which all map to zero under ϵ_1 and ϵ_2 unless $k = 1$ and $n_1 = 1$.

Let $x \in H_*(K(\Phi\mathcal{E}^G))$, and let $y \in H_*(\mathbb{S}^G)$ be the element corresponding to x under $\psi_* \circ \phi_*^{-1}$, via an element $z \in H_*(N\mathcal{E}^G_{tr})$. We need to show that the image $Q_i(x)$ of $e_i \otimes x^{\otimes p}$ under the top map corresponds, via the isomorphism, to the image $Q_i(y)$ of $e_i \otimes y^{\otimes p}$ under the bottom map. The element $e_i \otimes z^{\otimes p} \in H_*(\Sigma_p; H_*(N\mathcal{E}^G_{tr})^{\otimes p})$ maps to an element $Q_i(z) \in \tilde{H}_*(D_{\mathcal{U}})$, which further maps to $Q_i(x)$ and $Q_i(y)$ under ϵ_1 and ϵ_2, respectively. Let $w \in H_*(N\mathcal{E}^G_{tr})$ map to $Q_i(x)$ under ϕ_*. Since the maps ϵ_1 and ϵ_2 have the same kernel, the elements $Q_i(z)$ and w have the same image in $H_*(\mathbb{S}^G)$, which implies the result. ◻

Remark 2.2 The additive equivalence $\mathbb{S}^G \simeq K(\mathcal{E}^G) \simeq K(\Phi\mathcal{E}^G)$ of spectra can be realized as the G-fixed part of a G-equivalence $\mathbb{S}_G \simeq K_G(\mathcal{E})$ of G-spectra, for example using Shimakawa's construction [21] of G-equivariant K-theory spectra. Presumably this is a G-equivalence of E_∞ ring G-spectra.

3 Proof of the main theorem

Recall the E_∞ structure maps $\lambda_j\colon E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j} \to N\mathcal{E}^G$. We have inclusions $B\,\mathrm{Aut}^G(G) \to N\mathcal{E}^G$ and $\delta\colon B\,\mathrm{Aut}^G(G^j) \to N\mathcal{E}^G$, corresponding to the summands indexed by $X = G$ and $X = G^j = G \times \cdots \times G$, respectively, in the decomposition (2–4) of $N\mathcal{E}^G$. Restricting λ_j to these summands, we have a commutative diagram

(3–1)
$$\begin{tikzcd}
E\Sigma_j \ltimes_{\Sigma_j} (N\mathcal{E}^G)^{\wedge j} \arrow[rr, "\lambda_j"] & & N\mathcal{E}^G \\
E\Sigma_j \ltimes_{\Sigma_j} B\,\mathrm{Aut}^G(G)^j \arrow[u] \arrow[r, "\cong"] & B(\Sigma_j \ltimes \mathrm{Aut}^G(G)^j) \arrow[r, "B\phi"] & B\,\mathrm{Aut}^G(G^j)\,, \arrow[u, "\delta"']
\end{tikzcd}$$

where the homomorphism ϕ sends an element $(\sigma; f_1, \dots, f_j)$ in $\Sigma_j \ltimes \mathrm{Aut}^G(G)^j$ to the G-automorphism $f_{\sigma^{-1}(1)} \times \cdots \times f_{\sigma^{-1}(j)}$ of G^j.

We write $\Sigma_j \wr \mathrm{Aut}^G(G) \cong \Sigma_j \wr G$ for the wreath product $\Sigma_j \ltimes \mathrm{Aut}^G(G)^j$. The free G-set G^j splits into $k = |G|^{j-1}$ orbits, and we fix a G-equivariant bijection $G^j \cong \coprod_k G$. This induces an isomorphism $\mathrm{Aut}^G(G^j) \cong \mathrm{Aut}^G(\coprod_k G)$, and we also have $\mathrm{Aut}^G(\coprod_k G) \cong \Sigma_k \wr G$. Thus we get a commutative diagram

$$\begin{tikzcd} B(\Sigma_j \ltimes \mathrm{Aut}^G(G)^j) \arrow[r, "B\phi"] & B\,\mathrm{Aut}^G(G^j) \\ B(\Sigma_j \wr G) \arrow[r, "B\phi"] \arrow[u, "\cong"] & B(\Sigma_k \wr G) \arrow[u, "\cong"] \end{tikzcd} \tag{3–2}$$

where we also write ϕ for the induced homomorphism $\Sigma_j \wr G \to \Sigma_k \wr G$.

Now we specialize to the case $p = 2$. First we study the Dyer–Lashof operation $Q^2\colon\ H_1(\mathbb{S}^{C_2}) \to H_3(\mathbb{S}^{C_2})$.

Lemma 3.1 *The operation Q^2 in $H_*(\mathbb{S}^{C_2})$ satisfies $Q^2(\alpha_1) = \alpha_3$.*

Proof Let $C = C_2$. By Lemma 2.1, we may instead compute Q^2 in $H_*(K(\Phi\mathcal{E}^C))$. We let $j = 2$, combine diagrams (2–1), (3–1) and (3–2), apply homology, and end up with the upper half of the diagram

$$\begin{tikzcd} H_*(E\Sigma_2 \ltimes_{\Sigma_2} K(\Phi\mathcal{E}^C)^{\wedge 2}) \arrow[r, "\kappa_{2*}"] & H_*(K(\Phi\mathcal{E}^C)) \\ H_*(B(\Sigma_2 \wr C)) \arrow[r, "B\phi_*"] \arrow[u] & H_*(B(\Sigma_2 \wr C)) \arrow[u, "\epsilon_* \circ \delta_*"'] \\ H_*(B(\Sigma_2 \times C)) \arrow[r, "B\psi_*"] \arrow[u, "Bd_*"] & H_*(B(C \times C))\,. \arrow[u, "B\iota_*"'] \end{tikzcd} \tag{3–3}$$

The vertical homomorphisms in the lower square are induced by the homomorphism $d = 1 \times \Delta$ that sends (σ, x) to $(\sigma; x, x)$, and the inclusion ι of the subgroup $C \times C = C^2$ in $\Sigma_2 \wr C = \Sigma_2 \ltimes C^2$. The homomorphism ψ is the restriction of ϕ to $\Sigma_2 \times C$. It is easily checked that ψ takes values in the subgroup $C \times C$ (since $p = 2$) and is given by $\psi(\sigma, x) = (x, \sigma x)$, using the description of ϕ given after diagram (3–1). We have $B\psi_*(e_i \otimes 1) = 1 \otimes \alpha_i$ and

$$B\psi_*(1 \otimes \alpha_j) = \Delta_*(\alpha_j) = \sum_{s+t=j} \alpha_s \otimes \alpha_t\,,$$

which combine to give

$$B\psi_*(e_i \otimes \alpha_j) = \sum_{s+t=j} \alpha_s \otimes (\alpha_i * \alpha_t)\,,$$

where $*$ denotes the Pontryagin product in $H_*(BC)$ induced by the topological group multiplication $BC \times BC \to BC$. We recall that $\alpha_i * \alpha_t = \binom{i+t}{i}\alpha_{i+t}$.

By May's paper [15, 9.1] the map Bd_* is given by

$$Bd_*(e_i \otimes \alpha_j) = \sum_k e_{i+2k-j} \otimes Sq^k_*(\alpha_j) \otimes Sq^k_*(\alpha_j)\,.$$

Recall that $Sq^k_*(\alpha_j) = \binom{j-k}{k}\alpha_{j-k}$, where Sq^k_* denotes the dual of the Steenrod operation Sq^k. In particular $Bd_*(e_1 \otimes \alpha_2) = e_1 \otimes \alpha_1 \otimes \alpha_1$, which further maps to $Q_1(\alpha_1)$ in the upper right hand corner of (3–3). But now $Q_1(\alpha_1)$ is also the image of $e_1 \otimes \alpha_2$ under $\epsilon_* \circ \delta_* \circ B\iota_* \circ B\psi_*$. Using the description of $B\psi_*$ above, we see that $Q_1(\alpha_1)$ equals

$$\sum_{s+t=2} \epsilon_*\delta_*(1 \otimes \alpha_s \otimes (\alpha_1 * \alpha_t))\,. \tag{3–4}$$

The map ϵ_* vanishes on decomposables with respect to the product in $H_*(N\mathcal{E}^C)$ induced by the additive symmetric monoidal structure on $\mathcal{E}^C$. The element $\delta_*(1 \otimes \alpha_s \otimes (\alpha_1 * \alpha_t))$ is the image of

$$\alpha_s \otimes (\alpha_1 * \alpha_t) \in H_*(B\,\mathrm{Aut}^C(C) \times B\,\mathrm{Aut}^C(C))$$

in $H_*(B\,\mathrm{Aut}^C(C \coprod C)) \subset H_*(N\mathcal{E}^C)$ under the map induced by disjoint union, thus the only non-zero term in (3–4) is the one with $s = 0$ and $t = 2$, and $Q^2(\alpha_1) = Q_1(\alpha_1) = \epsilon_*(1 \otimes \alpha_0 \otimes (\alpha_1 * \alpha_2)) = \alpha_3$. □

Proof of Theorem 0.2 We now turn to the operations in $H_*(TC^{(1)}(\star;2)) = \mathbb{F}_2 \oplus H_*(\mathbb{R}P^\infty_{-1})$. The general formula for the Q^i will follow from Lemma 3.1 and the Nishida relations, which say in particular (see [2, III.1.1]) that

$$Sq^{i+j+1}_* Q^i(\alpha_j) = \sum_k \binom{2^N - j - 1}{2^N - i - 2j - 2 + 2k} Q^{k-j-1} Sq^k_*(\alpha_j)\,, \tag{3–5}$$

where N is sufficiently large. When $k \geq j+2$ the element $Sq^k_*(\alpha_j)$ is zero for degree reasons, and when $k \leq j$ the fact that Q^{k-j-1} vanishes on classes in degree higher than $k-j-1$ implies that $Q^{k-j-1}Sq^k_*(\alpha_j) = 0$. Hence the sum in (3–5) simplifies to the single term

$$Sq^{i+j+1}_* Q^i(\alpha_j) = \binom{2^N - j - 1}{2^N - i} Q^0 Sq^{j+1}_*(\alpha_j)$$

for $k = j+1$, where N is large.

Bob Bruner has observed that

$$\binom{2^N - j - 1}{2^N - i} \equiv \binom{2^N + i - 1}{2^N + j}$$

mod 2, for large N. Here is a quick proof. Let x_k denote the k'th bit in the binary expansion of a natural number x. Then $\binom{2^N-j-1}{2^N-i} \equiv 1$ if and only if $(2^N - i)_k = 1$ implies $(2^N - j - 1)_k = 1$ for all k, and $\binom{2^N+i-1}{2^N+j} \equiv 1$ if and only if $(2^N + j)_k = 1$ implies $(2^N + i - 1)_k = 1$ for all k. But for N large compared to i, j and k the bit $(2^N - i)_k$ is complementary to $(2^N + i - 1)_k$, and $(2^N - j - 1)_k$ is complementary to $(2^N + j)_k$, so $\binom{2^N-j-1}{2^N-i} \equiv 1$ if and only if $\binom{2^N+i-1}{2^N+j} \equiv 1$.

The operations Sq^k_* in $H_*(\mathbb{R}P^\infty_{-1})$ are given by the formula

$$Sq^k_*(\alpha_j) = \binom{j-k}{k} \alpha_{j-k} \,.$$

This follows by the corresponding formula for $\mathbb{R}P^\infty$ and James periodicity. More precisely, a theorem of James [12] says that given $m \le n$, there is a positive integer M, depending only on $n - m$, such that $\mathbb{R}P^{n+\ell}_{m+\ell} \simeq \Sigma^\ell \mathbb{R}P^n_m$ when ℓ is a positive multiple of 2^M. The space $\mathbb{R}P^n_m$ is the stunted projective space $\mathbb{R}P^n/\mathbb{R}P^{m-1}$. If we now define the spectrum $\mathbb{R}P^n_{-1}$ to be $\Sigma^{-\ell}\Sigma^\infty \mathbb{R}P^{n+\ell}_{\ell-1}$ for such ℓ (depending on n), we have that $\mathbb{R}P^\infty_{-1} = \operatorname{colim}_n \mathbb{R}P^n_{-1}$. The Steenrod operations in $H_*(\mathbb{R}P^\infty_{-1})$ can now be calculated from the operations in $H_*(\mathbb{R}P^{n+\ell}_{\ell-1})$, and the stated formula follows by noting that the relevant binomial coefficients are 2^M-periodic in the numerator.

In particular $Sq^{j+1}_*(\alpha_j) = \alpha_{-1}$ for all $j \ge -1$, and we have

$$Sq^{i+j+1}_* Q^i(\alpha_j) = \binom{2^N + i - 1}{2^N + j} Q^0(\alpha_{-1}) \,.$$

If $Q^0(\alpha_{-1})$ were zero, it would follow that $Q^i(\alpha_j) = 0$ for all i and j, since Sq^{i+j+1}_* is an isomorphism to dimension -1. But this contradicts Lemma 3.1. Hence $Q^0(\alpha_{-1}) = \alpha_{-1}$, and the formula stated in the theorem follows. □

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