ALGEBRAIC K-THEORY OF THE FRACTION FIELD OF TOPOLOGICAL K-THEORY

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Abstract. We compute the algebraic K-theory modulo p and v_1 of the S-algebra $\ell/p = k(1)$, using topological cyclic homology. We use this to compute the homotopy cofiber of a transfer map $K(L/p) \to K(L_p)$, which we interpret as the algebraic K-theory of the "fraction field" of the p-complete Adams summand of topological K-theory. The results suggest that there is an arithmetic duality theorem for this fraction field, much like Tate–Poitou duality for p-adic fields.

1. Introduction

In this paper we continue the investigation of the algebraic K-theory of topological K-theory initiated in [AuR02]. There we computed the mod p and v_1 homotopy groups of $K(\ell_p)$ at primes $p \geq 5$, denoted by $V(1)_*K(\ell_p)$ or $K/(p, v_1)_*(\ell_p)$, where ℓ_p is the Adams summand of the p-complete connective complex K-theory spectrum ku_p, with coefficients $\pi_*(\ell_p) = \ell_{p*} = \mathbb{Z}_p[v_1]$, $|v_1| = 2p-2$. Denoting by $P(v_2)$ the polynomial algebra over $\mathbb{F}_p$ generated by v_2, $|v_2| = 2p^2-2$, we showed that $V(1)_*K(\ell_p)$ is essentially a free $P(v_2)$-module on $4p+4$ generators. This is reminiscent, up to a chromatic shift of one in the sense of stable homotopy theory, of the structure of the mod p algebraic K-theory $V(0)_*K(\mathbb{Z}_p) = K/(p)_*(\mathbb{Z}_p)$ of the p-adic integers, which is a free $P(v_1)$-module on $p+3$ generators.

The structure of the algebraic K-theory of $\mathbb{Z}_p$ can be conceptually explained in terms of localization, Galois descent and motivic truncation. First, there is a localization sequence

$$K(\mathbb{Z}/p) \to K(\mathbb{Z}_p) \to K(\mathbb{Q}_p)$$

relating the algebraic K-theory of $\mathbb{Z}_p$ to that of its fraction field $\mathbb{Q}_p = \mathit{ff}(\mathbb{Z}_p)$ and its residue field $\mathbb{Z}/p$. Second, the v_1-periodic mod p algebraic K-theory of $\mathbb{Q}_p$ is the target of a Galois descent spectral sequence [Th85]

$$E^2_{s,t} = H^{-s}_{Gal}(\mathbb{Q}_p; \mathbb{F}_p(t/2)) \Longrightarrow V(0)_{s+t}K(\mathbb{Q}_p)[v_1^{-1}]\,.$$

The coefficient module $\mathbb{F}_p(t/2)$ can be interpreted as $V(0)_tK(\bar{\mathbb{Q}}_p)[v_1^{-1}]$ by Suslin's Theorem [Su84], and this spectral sequence can be thought of as the continuous homotopy fixed-point spectral sequence in mod p homotopy associated to the action of the absolute Galois group of $\mathbb{Q}_p$ on $K(\bar{\mathbb{Q}}_p)$. Third, the unlocalized mod p algebraic K-theory of $\mathbb{Q}_p$ is the target of a motivic spectral sequence [BL:ss], [FS02]

$$E^2_{s,t} = H^{-s}_{mot}(\mathbb{Q}_p; \mathbb{F}_p(t/2)) \Longrightarrow V(0)_{s+t}K(\mathbb{Q}_p)\,,$$

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where

$$H^r_{mot}(\mathbb{Q}_p;\mathbb{F}_p(i)) \cong \begin{cases} H^r_{Gal}(\mathbb{Q}_p;\mathbb{F}_p(i)) & \text{for } r \le i \ , \\ 0 & \text{for } r > i \ . \end{cases}$$

In other words, the mod p motivic cohomology of $\mathbb{Q}_p$ is a specific truncation of its Galois cohomology. In terms of Bloch's higher Chow groups [Bl86], the vanishing of $H^r_{mot}(\mathbb{Q}_p;\mathbb{F}_p(i))$ for $r > i$ follows from the fact that there are no varieties of negative dimension relative to $\mathrm{Spec}(\mathbb{Q}_p)$. The identification with Galois cohomology for $r \le i$ is part of the Beilinson conjectures [BMS87].

Our aim here is to conceptually understand $V(1)_*K(\ell_p)$ in the same way as we understand $V(0)_*K(\mathbb{Z}_p)$, using the Galois theory for commutative S-algebras developed by the second author [Rog08]. The first task is therefore to identify the fraction field $ff(\ell_p)$ of ℓ_p. There is a localization sequence

$$K(\mathbb{Z}_p) \to K(\ell_p) \to K(L_p)\,,$$

the existence of which was conjectured by the second author and established by Blumberg and Mandell [BM08], relating the algebraic K-theory of ℓ_p to that of the periodic p-complete Adams summand L_p. The fact that p is not invertible in $\pi_*(L_p) = L_{p*} = \mathbb{Z}_p[v_1^{\pm 1}]$, and the presence of the nontrivial residue S-algebra $L/p = K(1)$ of L_p, give some indications as to why L_p should not qualify as the fraction field we are after.

A naive guess would be to take the Bousfield localization $L_p[p^{-1}]$ of L_p away from p, inverting p in $\pi_*(L_p)$. However, $L_p[p^{-1}]$ is an $H\mathbb{Q}_p$-algebra, and we cannot possibly recover the v_2-periodic mod p and v_1 homotopy of $K(\ell_p)$ or $K(L_p)$ from that of $K(L_p[p^{-1}])$, which is v_2-torsion. See Remark 3.12. Therefore $L_p[p^{-1}]$ does not qualify as the desired fraction field of ℓ_p. Instead, we have reasons to expect that if $ff(\ell_p)$ is the "correct" fraction field, by which we mean that it will suit the purpose of understanding algebraic K-theory by Galois descent, then its algebraic K-theory $K(ff(\ell_p))$ ought to fit in the following 3×3-diagram of localization cofiber sequences, see Definition 3.9.

(1.1)
$$\begin{tikzcd}
K(\mathbb{Z}/p) \arrow[r, "i_*"] \arrow[d, "\pi_*"] & K(\mathbb{Z}_p) \arrow[r, "j^*"] \arrow[d, "\pi_*"] & K(\mathbb{Q}_p) \arrow[d, "\pi_*"] \\
K(\ell/p) \arrow[r, "i_*"] \arrow[d, "\rho^*"] & K(\ell_p) \arrow[r, "j^*"] \arrow[d, "\rho^*"] & K(p^{-1}\ell_p) \arrow[d, "\rho^*"] \\
K(L/p) \arrow[r, "i_*"] & K(L_p) \arrow[r, "j^*"] & K(ff(\ell_p))\,.
\end{tikzcd}$$

We emphasize that for now $p^{-1}\ell_p$ and $ff(\ell_p)$ are just formal symbols, not the same as the Bousfield localizations $\ell_p[p^{-1}]$ and $L_p[p^{-1}]$. The $V(1)$-homotopy groups of $K(\mathbb{Z}/p)$, $K(\mathbb{Z}_p)$ and $K(\ell_p)$ are known by [Qu72], [HM97] and [AuR02]. It therefore remains to compute $V(1)_*K(\ell/p)$ and the transfer maps i_* and π_* in order to be in a position to evaluate the $V(1)$-homotopy of the iterated cofiber $K(ff(\ell_p))$. This is our main result, corresponding to Theorem 8.10 in the body of the paper.

Theorem 1.2. *There is an isomorphism of $E(\lambda_1, \lambda_2) \otimes P(v_2)$-modules*

$$\begin{aligned} V(1)_* K(\ell/p) \cong P(v_2) \otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{1, \partial\lambda_2, \lambda_2, \partial v_2\} \\ \oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ \oplus P(v_2) \otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\}\,. \end{aligned}$$

Here $|\lambda_1| = |\bar\epsilon_1| = 2p-1$, $|\lambda_2| = 2p^2-1$, $|v_2| = 2p^2-2$, $|\operatorname{dlog} v_1| = 1$, $|\partial| = -1$ and $|t| = -2$. This is a free $P(v_2)$-module of rank $(2p^2 - 2p + 8)$ and of zero Euler characteristic, where $p \geq 5$ is assumed.

As explained in Section 2, the spectrum ℓ/p admits infinitely many structures of an associative ℓ_p-algebra, none of which are commutative. However, all of these ℓ_p-algebras have equivalent underlying S-algebra structures [An:un], and indeed it turns out that the additive structure of $V(1)_*K(\ell/p)$ is independent of the choice of ℓ_p-algebra structure. The classes λ_1, λ_2 and v_2 are present in $V(1)_*K(\ell_p)$, and in the statement above we refer to the $V(1)_*K(\ell_p)$-module structure of $V(1)_*K(\ell/p)$. We prove this theorem by means of the cyclotomic trace map to topological cyclic homology $TC(\ell/p)$. On the way we evaluate $V(1)_*THH(\ell/p)$, where THH denotes topological Hochschild homology, as well as $V(1)_*TC(\ell/p)$, see Proposition 5.4 and Theorem 8.8.

Defining $K(f\!f(\ell_p))$ as the iterated cofiber in diagram (1.1), we can then evaluate its $V(1)$-homotopy. The formulation given in Theorem 9.4 readily implies the following statement.

Theorem 1.3. *Let $p \geq 5$ be a prime. There is an isomorphism of $P(v_2^{\pm 1})$-modules*

$$V(1)_* K(f\!f(\ell_p))[v_2^{-1}] \cong P(v_2^{\pm 1}) \otimes \Lambda_*$$

where

$$\begin{aligned} \Lambda_* \cong E(\partial v_2, \operatorname{dlog} p, \operatorname{dlog} v_1) \\ \oplus E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d \lambda_1 \mid 0 < d < p\} \\ \oplus E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \operatorname{dlog} p \mid 0 < d < p^2 - p, p \nmid d\} \\ \oplus E(\operatorname{dlog} p) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \end{aligned}$$

is displayed in Figure 10.3. Here $|\operatorname{dlog} p| = 1$, and the degrees of the other classes are as in Theorem 1.2. The localization homomorphism

$$V(1)_* K(f\!f(\ell_p)) \to V(1)_* K(f\!f(\ell_p))[v_2^{-1}]$$

is an isomorphism in degrees $* \geq 2p$.

This proves that the iterated cofiber $K(f\!f(\ell_p))$ cannot be a $K(\mathbb{Q}_p)$-module, since $V(1)_*K(\mathbb{Q}_p)$ is a torsion $P(v_2)$-module.

In the final part of this paper, we conjecturally interpret the computation of Theorem 1.3 in terms of Galois descent. Indeed, the second author conjectured that if Ω_1 is a separable closure of $f\!f(\ell_p)$, then there is a homotopy equivalence

$$L_{K(2)} K(\Omega_1) \simeq E_2\,.$$

Here E_2 is Morava's E-theory [GH04] with coefficients $\pi_* E_2 = \mathbb{W}(\mathbb{F}_{p^2})[[u_1]][u^{\pm 1}]$, and $L_{K(2)}$ denotes Bousfield localization with respect to the Morava K-theory $K(2)$ with coefficients $\pi_* K(2) = \mathbb{F}_p[v_2^{\pm 1}]$. The v_2-periodic $V(1)$-homotopy groups of $K(\Omega_1)$ will then be given by

$$V(1)_* K(\Omega_1)[v_2^{-1}] \cong \mathbb{F}_{p^2}[u^{\pm 1}].$$

We therefore conjecture to have a corresponding Galois descent spectral sequence

$$E^2_{s,t} = H^{-s}_{Gal}(\mathit{ff}(\ell_p); \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t} K(\mathit{ff}(\ell_p))[v_2^{-1}]\,,$$

see Conjecture 10.5. Starting with $V(1)_* K(\mathit{ff}(\ell_p))[v_2^{-1}]$ and working backwards, we give a conjectural description of the Galois cohomology of $\mathit{ff}(\ell_p)$ with coefficients in $V(1)_* K(\Omega_1)[v_2^{-1}]$, see Figure 10.3. This fits very well with the example of the Galois cohomology of $\mathbb{Q}_p$ with coefficients in $V(0)_* K(\bar{\mathbb{Q}}_p)[v_1^{-1}]$, with the difference that the absolute Galois group of $\mathit{ff}(\ell_p)$ has cohomological dimension 3 instead of 2. Also, there appears to be a perfect arithmetic duality pairing in the conjectural Galois cohomology of $\mathit{ff}(\ell_p)$, analogous to Tate–Poitou duality for local number fields, which is not present in the analogous cohomology rings for ℓ_p or L_p. This indicates that $\mathit{ff}(\ell_p)$ ought to be a form of S-algebraic two-local field, mixing three different residue characteristics. As argued in section 3 of this paper, we expect very similar results to hold for ℓ_p replaced by ku_p.

The paper is organized as follows. In section 2 we fix our notations, show that ℓ/p admits the structure of an associative ℓ-algebra, and give a similar discussion for ku/p and the periodic versions L/p and KU/p. We give derived algebro-geometric and log-geometric interpretations of these constructions. In section 3, we discuss localization sequences in algebraic K-theory. Section 4 contains the computation of the mod p homology of $THH(\ell/p)$, and in section 5 we evaluate its $V(1)$-homotopy. Sections 6, 7 and 8 deal with the computation of $TC(\ell/p)$ and $K(\ell/p)$ in $V(1)$-homotopy. In section 9 the various computations are assembled in a diagram of localization sequences, to evaluate $V(1)_* K(\mathit{ff}(\ell_p))$. In section 10, the structure of $V(1)_* K(\mathit{ff}(\ell_p))$ is interpreted in terms of Galois descent, yielding a conjectural description of the Galois cohomology of $\mathit{ff}(\ell_p)$.

2. Base change squares of S-algebras

We fix some notations. Let p be a prime, even or odd for now. Write $X_{(p)}$ and X_p for the p-localization and the p-completion, respectively, of any spectrum or abelian group X. Let ku and KU be the connective and the periodic complex K-theory spectra, with homotopy rings $ku_* = \mathbb{Z}[u]$ and $KU_* = \mathbb{Z}[u^{\pm 1}]$, where $|u| = 2$. Let $\ell = BP\langle 1\rangle$ and $L = E(1)$ be the p-local Adams summands, with $\ell_* = \mathbb{Z}_{(p)}[v_1]$ and $L_* = \mathbb{Z}_{(p)}[v_1^{\pm 1}]$, where $|v_1| = 2p-2$. The inclusion $\ell \to ku_{(p)}$ maps v_1 to u^{p-1}. Alternate notations in the p-complete cases are $KU_p = E_1$ and $L_p = \widehat{E(1)}$. These ring spectra are all commutative S-algebras, in the sense that each admits a unique E_∞ ring spectrum structure. See [BR05] for proofs of uniqueness in the periodic cases.

Let ku/p and KU/p be the connective and periodic mod p complex K-theory spectra, with coefficients $(ku/p)_* = \mathbb{Z}/p[u]$ and $(KU/p)_* = \mathbb{Z}/p[u^{\pm 1}]$. These are

2-periodic versions of the Morava K-theory spectra $\ell/p = k(1)$ and $L/p = K(1)$, with $(\ell/p)_* = \mathbb{Z}/p[v_1]$ and $(L/p)_* = \mathbb{Z}/p[v_1^{\pm 1}]$. Each of these can be constructed as the cofiber of the multiplication by p map, as a module over the corresponding commutative S-algebra. For example, there is a cofiber sequence of ku-modules $ku \xrightarrow{p} ku \xrightarrow{i} ku/p$.

Let HR be the Eilenberg–Mac Lane spectrum of a ring R. When R is associative, HR admits a unique associative S-algebra structure, and when R is commutative, HR admits a unique commutative S-algebra structure. The zeroth Postnikov section defines unique maps of commutative S-algebras $\pi\colon ku \to H\mathbb{Z}$ and $\pi\colon \ell \to H\mathbb{Z}_{(p)}$, which can be followed by unique commutative S-algebra maps to $H\mathbb{Z}/p$.

The ku-module spectrum ku/p does not admit the structure of a commutative ku-algebra. It cannot even be an E_2 or H_2 ring spectrum, since the homomorphism induced in mod p homology by the resulting map $\pi\colon ku/p \to H\mathbb{Z}/p$ of H_2 ring spectra would not commute with the homology operation $Q^1(\bar\tau_0) = \bar\tau_1$ in the target $H_*(H\mathbb{Z}/p; \mathbb{F}_p)$ [BMMS86, III.2.3]. Similar remarks apply for KU/p, ℓ/p and L/p. Associative algebra structures, or A_∞ ring spectrum structures, are easier to come by. The following result is a direct application of the methods of [La01, §§9–11]. We adapt the notation of [BJ02, §3] to provide some details in our case.

Proposition 2.1. *The ku-module spectrum ku/p admits the structure of an associative ku-algebra, but the structure is not unique. Similar statements hold for KU/p as a KU-algebra, ℓ/p as an ℓ-algebra and L/p as an L-algebra.*

Proof. We construct ku/p as the (homotopy) limit of its Postnikov tower of associative ku-algebras $P^{2m-2} = ku/(p, u^m)$, with coefficient rings $ku/(p,u^m)_* = ku_*/(p, u^m)$ for $m \geq 1$. To start the induction, $P^0 = H\mathbb{Z}/p$ is a ku-algebra via $i \circ \pi\colon ku \to H\mathbb{Z} \to H\mathbb{Z}/p$. Assume inductively for $m \geq 1$ that $P = P^{2m-2}$ has been constructed. We will define P^{2m} by a (homotopy) pullback diagram

$$\begin{tikzcd} P^{2m} \arrow[r] \arrow[d] & P \arrow[d, "in_1"] \\ P \arrow[r, "d"] & P \vee \Sigma^{2m+1} H\mathbb{Z}/p \end{tikzcd}$$

in the category of associative ku-algebras. Here

$$d \in \mathrm{ADer}_{ku}^{2m+1}(P, H\mathbb{Z}/p) \cong THH_{ku}^{2m+2}(P, H\mathbb{Z}/p)$$

is an associative ku-algebra derivation of P with values in $\Sigma^{2m+1}H\mathbb{Z}/p$, and the group of such can be identified with the indicated topological Hochschild cohomology group of P over ku. We recall that these are the homotopy groups (cohomologically graded) of the function spectrum $F_{P \wedge_{ku} P^{op}}(P, H\mathbb{Z}/p)$. The composite map $pr_2 \circ d\colon P \to \Sigma^{2m+1}H\mathbb{Z}/p$ of ku-modules, where pr_2 projects onto the second wedge summand, is restricted to equal the ku-module Postnikov k-invariant of ku/p in

$$H_{ku}^{2m+1}(P; \mathbb{Z}/p) = \pi_0 F_{ku}(P, \Sigma^{2m+1} H\mathbb{Z}/p)\,.$$

We compute that $\pi_*(P \wedge_{ku} P^{op}) = ku_*/(p, u^m) \otimes E(\tau_0, \tau_{1,m})$, where $|\tau_0| = 1$, $|\tau_{1,m}| = 2m+1$ and $E(-)$ denotes the exterior algebra on the given generators.

(For $p = 2$, the use of the opposite product is essential here [An08, §3].) The function spectrum description of topological Hochschild cohomology leads to the spectral sequence

$$\begin{aligned} E_2^{**} &= \mathrm{Ext}^{**}_{\pi_*(P \wedge_{ku} P^{op})}(\pi_*(P), \mathbb{Z}/p) \\ &\cong \mathbb{Z}/p[y_0, y_{1,m}] \\ &\Longrightarrow THH^*_{ku}(P, H\mathbb{Z}/p)\,, \end{aligned}$$

where y_0 and $y_{1,m}$ have cohomological bidegrees $(1,1)$ and $(1, 2m+1)$, respectively. The spectral sequence collapses at $E_2 = E_\infty$, since it is concentrated in even total degrees. In particular,

$$\mathrm{ADer}^{2m+1}_{ku}(P, H\mathbb{Z}/p) \cong \mathbb{F}_p\{y_{1,m}, y_0^{m+1}\}\,.$$

Additively, $H^{2m+1}_{ku}(P; \mathbb{Z}/p) \cong \mathbb{F}_p\{Q_{1,m}\}$ is generated by a class dual to $\tau_{1,m}$, which is the image of $y_{1,m}$ under left composition with pr_2. It equals the ku-module k-invariant of ku/p. Thus there are precisely p choices $d = y_{1,m} + \alpha y_0^{m+1}$, with $\alpha \in \mathbb{F}_p$, for how to extend any given associative ku-algebra structure on $P = P^{2m-2}$ to one on $P^{2m} = ku/(p, u^{m+1})$. In the limit, we find that there are an uncountable number of associative ku-algebra structures on $ku/p = \mathrm{holim}_m\, P^{2m}$, each indexed by the sequence of choices $\alpha \in \mathbb{F}_p$.

The periodic spectrum KU/p can be obtained from ku/p by Bousfield KU-localization in the category of ku-modules [EKMM97, VIII.4], which makes it an associative KU-algebra. The classification of periodic S-algebra structures is the same as in the connective case, since the original ku-algebra structure on ku/p can be recovered from that on KU/p by a functorial passage to the connective cover. To construct ℓ/p as an associative ℓ-algebra, or L/p as an associative L-algebra, replace u by v_1 in these arguments. □

By varying the ground S-algebra, we obtain the same conclusions about ku/p as a $ku_{(p)}$-algebra or ku_p-algebra, and about ℓ/p as an ℓ_p-algebra. Similarly, for each $\nu \geq 1$ we can realize the ku-module spectrum ku/p^ν, with $(ku/p^\nu)_* = \mathbb{Z}/p^\nu[u]$, as an associative ku-algebra, and with a little care the ku-algebra structures may be so chosen that the p-adic tower

$$\dots \to ku/p^{\nu+1} \to ku/p^\nu \to \dots \to ku/p \tag{2.2}$$

is one of associative ku-algebras. Uncountably many choices of structures remain, though. Again, the same remarks apply over ℓ, where the p-adic tower $\{\ell/p^\nu\}_\nu$ can be chosen to be one of associative ℓ-algebras.

For each choice of ku-algebra structure on ku/p, the zeroth Postnikov section $\pi\colon ku/p \to H\mathbb{Z}/p$ is a ku-algebra map, with the unique ku-algebra structure on the target. Hence there is a commutative square of associative ku-algebras

$$\begin{tikzcd} ku \arrow[r, "i"] \arrow[d, "\pi"] & ku/p \arrow[d, "\pi"] \\ H\mathbb{Z} \arrow[r, "i"] & H\mathbb{Z}/p \end{tikzcd} \tag{2.3}$$

and similarly in the p-local and p-complete cases. In view of the weak equivalence $H\mathbb{Z} \wedge_{ku} ku/p \simeq H\mathbb{Z}/p$, this square expresses the associative $H\mathbb{Z}$-algebra $H\mathbb{Z}/p$ as the base change of the associative ku-algebra ku/p along $\pi\colon ku \to H\mathbb{Z}$. Likewise, there is a commutative square of associative ℓ-algebras

$$\begin{tikzcd} \ell \arrow[r, "i"] \arrow[d, "\pi"] & \ell/p \arrow[d, "\pi"] \\ H\mathbb{Z}_{(p)} \arrow[r, "i"] & H\mathbb{Z}/p \end{tikzcd} \tag{2.4}$$

that expresses $H\mathbb{Z}/p$ as the base change of ℓ/p along $\ell \to H\mathbb{Z}_{(p)}$, and similarly in the p-complete case. By omission of structure, these squares are also diagrams of S-algebras and S-algebra maps.

Remark 2.5. We might think algebro-geometrically about these diagrams. In other words, we would like to view the S-algebras in diagram (2.3) as the rings of functions on a diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathbb{Z}/p) \arrow[r, "i"] \arrow[d, "\pi"] & \mathrm{Spec}(\mathbb{Z}) \arrow[d, "\pi"] \\ \mathrm{Spnc}(ku/p) \arrow[r, "i"] & \mathrm{Spec}(ku) \end{tikzcd}$$

of affine derived schemes, in the sense of Jacob Lurie, or affine brave new schemes in the sense of Toën and Vezzosi. Here $\mathrm{Spec}(ku)$ has the same underlying prime ideal space as $\mathrm{Spec}(\pi_0(ku)) = \mathrm{Spec}(\mathbb{Z})$, but it is ringed in commutative S-algebras, rather than in ordinary commutative rings. Over the open subset $\mathrm{Spec}(\mathbb{Z}[f^{-1}])$, where f is a finite product of primes, the ring of functions is the commutative S-algebra $ku[f^{-1}]$ with coefficient ring $ku_*[f^{-1}]$. However, since ku/p is not a commutative S-algebra, we do not know how to make good sense of its structure space. At best, $i\colon \mathrm{Spnc}(ku/p) \to \mathrm{Spec}(ku)$ can be though of as a map from a non-commutative derived scheme to the derived scheme $\mathrm{Spec}(ku)$. The base change assertion above then says that $\mathrm{Spec}(\mathbb{Z}/p)$ is the fiber of $\mathrm{Spnc}(ku/p)$ over the closed subspace $\mathrm{Spec}(\mathbb{Z})$ in $\mathrm{Spec}(ku)$, which in turn happens to be an honest (commutative) scheme.

Remark 2.6. At this point we recall how Spec of the fraction field $F = ff(R)$ of an integral domain R is obtained geometrically by cutting away all the proper closed subspaces of $\mathrm{Spec}(R)$. The remaining open subset represents the generic point of $\mathrm{Spec}(R)$. For any localization T of R, with $R \subset T \subset F$, we also have $F = ff(T)$, so F is simultaneously the fraction field of R and of T. In the case of $\mathrm{Spec}(ku)$, with $R = ku$, the analogous (sub-)spaces appear to be $\mathrm{Spec}(\mathbb{Z})$ (included by π) and the $\mathrm{Spnc}(ku/p)$ (mapped by i) for all primes p, so Spec of the fraction field $ff(ku)$ should consist of the complement in $\mathrm{Spec}(ku)$ of all of these spaces. In the p-complete case, this means cutting out the two spaces $\mathrm{Spec}(\mathbb{Z}_p)$ and $\mathrm{Spnc}(ku/p)$ from $\mathrm{Spec}(ku_p)$, meeting precisely along their intersection $\mathrm{Spec}(\mathbb{Z}/p)$. The remainder should be $\mathrm{Spec}(ff(ku_p))$. We expect that the complement of $\mathrm{Spec}(\mathbb{Z}_p)$ in $\mathrm{Spec}(ku_p)$ will play the role of $\mathrm{Spec}(KU_p)$, and that the complement of $\mathrm{Spec}(\mathbb{Z}/p)$ in $\mathrm{Spnc}(ku/p)$ will play the role of $\mathrm{Spnc}(KU/p)$, so the S-algebraic fraction field of ku_p is also realized as the S-algebraic fraction field of KU_p, denoted $ff(ku_p) = ff(KU_p) = p^{-1}KU_p$. Similarly, Spec of the fraction field

$$ff(\ell_p) = ff(L_p) = p^{-1}L_p$$

of ℓ_p or L_p would be the complement of $\mathrm{Spec}(\mathbb{Z}_p)$ and $\mathrm{Spnc}(\ell/p)$ in $\mathrm{Spec}(\ell_p)$, meeting at $\mathrm{Spec}(\mathbb{Z}/p)$.

Based on the algebraic K-theory computations to follow, we have strong reasons to believe that the complement $\mathrm{Spec}(p^{-1}ku)$ of $\mathrm{Spnc}(ku/p)$ in $\mathrm{Spec}(ku)$ is not equal to the derived algebraic scheme associated to the commutative ku-algebra $ku[p^{-1}]$ that is obtained by (algebraic) Bousfield localization away from p. See Remark 3.12 for the calculational justification of our expectation. Similarly, in the p-complete case the complement $\mathrm{Spec}(p^{-1}ku_p)$ of $\mathrm{Spnc}(ku/p)$ in $\mathrm{Spec}(ku_p)$ is not equal to $\mathrm{Spec}(ku_p[p^{-1}])$, and in the periodic case the complement $\mathrm{Spec}(p^{-1}KU_p) = \mathrm{Spec}(\mathit{ff}(ku_p))$ is different from $\mathrm{Spec}(KU_p[p^{-1}])$. In fact, we do not expect that $\mathrm{Spec}(p^{-1}ku_p)$ and $\mathrm{Spec}(\mathit{ff}(ku_p))$ are affine derived schemes in the sense that they are equal to $\mathrm{Spec}(B)$ for a commutative S-algebra B. Still, they may be derived stacks, in a sense approximated by the following remark.

Remark 2.7. Since $\mathrm{Spnc}(ku/p)$ is not actually a closed subscheme of $\mathrm{Spec}(ku_p)$, but only maps to the complement of the open subscheme $\mathrm{Spec}(ku_p[p^{-1}])$, we expect that a derived scheme $\mathrm{Spec}(p^{-1}ku_p)$ that is complementary to $\mathrm{Spnc}(ku/p)$, is a derived algebro-geometric object which is intermediate between $\mathrm{Spec}(ku_p[p^{-1}])$ and $\mathrm{Spec}(ku_p)$. In forthcoming work, by the second author together with Clark Barwick and Steffen Sagave, we hope to give good sense to this intermediate object as a logarithmic S-algebra (ku_p, M), where M is a pointed E_∞ space. See also Remark 3.3. This makes (ku_p, M) a logarithmic S-algebra and $\mathrm{Spec}(ku_p, M)$ an affine derived logarithmic scheme. Following Martin Olsson's work in the classical algebraic case [Ol03], this derived logarithmic scheme can be faithfully represented by the derived stack representing the ∞-category of derived logarithmic schemes over it. Hence we expect that this moduli stack of $\mathrm{Spec}(ku_p, M)$ will provide the desired derived stack interpretation of $\mathrm{Spec}(p^{-1}ku_p)$. Related topological logarithmic structures on ku_p will then provide useful interpretations for $\mathrm{Spec}(KU_p)$ and $\mathrm{Spec}(\mathit{ff}(ku_p))$. See [Rog09] for a set of foundations of topological logarithmic geometry, and [Rog:ltc] for the construction of topological cyclic homology of a logarithmic S-algebra.

3. Localization squares in algebraic K-theory

We are interested in the algebraic K-theory spectra of the S-algebras considered in the previous section. For an S-algebra B, we adapt [EKMM97, VI] and let $\mathcal{C}_B$ be the category of finite cell B-modules and B-module maps. This is a category with cofibrations and weak equivalences in the sense of [Wa85, §1], with cofibrations the maps that are isomorphic to the inclusion of a subcomplex, and weak equivalences the homotopy equivalences. The algebraic K-theory of B, denoted $K(B)$, is defined to be the algebraic K-theory of this category with cofibrations and weak equivalences. When $B = HR$ is an Eilenberg–Mac Lane spectrum, it is known that $K(HR) \simeq K(R)$ recovers Quillen's algebraic K-theory of the ring R.

If $\varphi\colon A \to B$ is a map of S-algebras, the exact functor $B \wedge_A (-)\colon \mathcal{C}_A \to \mathcal{C}_B$ induces a map $\varphi^*\colon K(A) \to K(B)$, making $K(-)$ a covariant functor in the S-algebra. We shall follow the variance conventions of algebraic geometry, thinking of $K(-)$ as a contravariant functor in a (not yet generally defined) geometric object $\mathrm{Spec}(-)$ associated to the S-algebra, so that φ^* is derived from the inverse image functor on sheaves. This is done to avoid confusion with the transfer maps $\varphi_*\colon K(B) \to K(A)$, derived from the direct image functor on sheaves.

Suppose that $\varphi\colon A \to B$ makes B a finite cell A-module. Then each finite cell

B-module M can be viewed as an A-module along φ, and each B-cell attachment needed to build M can be achieved by attaching finitely many A-cells, one for each A-cell in B. The resulting exact functor $\mathcal{C}_B \to \mathcal{C}_A$ induces the transfer map $\varphi_* \colon K(B) \to K(A)$.

When A is a commutative S-algebra, the smash product $(-) \wedge_A (-)$ induces a pairing on $\mathcal{C}_A$ that makes $K(A)$ a commutative S-algebra. When A is central in B, so that B is an A-algebra, the smash product makes $K(B)$ a $K(A)$-module, and the maps $\varphi^* \colon K(A) \to K(B)$ and $\varphi_* \colon K(B) \to K(A)$ (when defined) are $K(A)$-module maps. In other words, the transfer is a module map over its target. When B is commutative, then $K(B)$ is likewise a commutative S-algebra, and φ^* is a commutative S-algebra map. The $K(A)$-linearity of φ_* can then be expressed by the projection/reciprocity formula $\varphi_*(\varphi^*(x) \wedge_B y) = x \wedge_A \varphi_*(y)$.

We apply this to the maps $i \colon ku \to ku/p$ and $\pi \colon ku \to H\mathbb{Z}$. Here $ku/p = ku \cup_p e^1$ and $H\mathbb{Z} = ku \cup_u e^3$ in the category of ku-modules, meaning that ku/p is obtained from ku by attaching a 1-cell along p, and that $H\mathbb{Z}$ is obtained from ku by attaching a 3-cell along u. The transfer maps i_* and π_* and their variants then define the upper left hand square in the following commutative diagram, where all rows and columns are cofiber sequences.

$$
\begin{tikzcd}
K(\mathbb{Z}/p) \arrow[r, "i_*"] \arrow[d, "\pi_*"] & K(\mathbb{Z}) \arrow[r, "j^*"] \arrow[d, "\pi_*"] & K(\mathbb{Z}[p^{-1}]) \arrow[d, "\pi_*"] \\
K(ku/p) \arrow[r, "i_*"] \arrow[d, "\rho^*"] & K(ku) \arrow[r, "j^*"] \arrow[d, "\rho^*"] & K(p^{-1}ku) \arrow[d, "\rho^*"] \\
K(KU/p) \arrow[r, "i_*"] & K(KU) \arrow[r, "j^*"] & K(p^{-1}KU)
\end{tikzcd}
\tag{3.1}
$$

The upper row is a cofiber sequence by Quillen's localization theorem for Dedekind domains [Qu73]. Andrew Blumberg and Mike Mandell [BM08] have proved that the middle column is a cofiber sequence, as conjectured by the second author. The result is analogous to that of Quillen, in that $ku/u \simeq H\mathbb{Z}$ and $ku[u^{-1}] \simeq KU$. The same argument shows that the left hand column is a cofiber sequence.

We complete the 3×3 diagram of cofiber sequences by making the following ad hoc definition.

Definition 3.2. Let $K(p^{-1}ku)$ and $K(p^{-1}KU)$ denote the (homotopy) cofibers of the transfer maps $i_* \colon K(ku/p) \to K(ku)$ and $i_* \colon K(KU/p) \to K(KU)$, respectively.

Remark 3.3. The theory of logarithmic structures on S-algebras, see [Rog09], is designed to model intermediate objects between a commutative S-algebra B, like ku, and its localizations, like $ku[p^{-1}]$. A logarithmic structure on B is given by a commutative S-algebra map $S[M] \to B$ from the spherical monoid ring on a "commutative monoid" M, where the latter term should be interpreted in a sufficiently flexible category to also encompass E_∞ spaces. In joint work with Steffen Sagave, the second author has defined a category of logarithmic modules over a logarithmic S-algebra (B, M), and one may define the logarithmic K-theory $K(B, M)$ as the algebraic K-theory of the category of suitably finite objects in that module category. It still remains to be seen if one can realize $K(p^{-1}ku)$ as $K(ku, M)$ for a suitable logarithmic S-algebra (ku, M).

Remark 3.4. There is a similar 3×3 diagram to (3.1), in which the left hand column is replaced by the sum of corresponding columns for all primes p, and i_* is replaced by the sum of the transfer maps. By Quillen's localization sequence the upper right hand term can be written as $K(\mathbb{Q})$. We expect that the lower right hand term then plays the role of the algebraic K-theory of the S-algebraic fraction field of ku, denoted $K(\mathit{ff}(ku))$. The hypothetical fraction field has a valuation with uniformizer u, and the middle right hand term $K(\mathcal{O})$ in the 3×3 diagram plays the role of the algebraic K-theory of an S-algebraic valuation ring $\mathcal{O}$ of $\mathit{ff}(ku)$. However, we do not expect that this fraction field $\mathit{ff}(ku)$ is the rationalization $L_0KU = KU\mathbb{Q}$ of the S-algebra KU. This expectation is again supported by the algebraic K-theory computations that follow.

We shall use the topological cyclic homology $TC(-)$ and the cyclotomic trace map $trc\colon K(B)\to TC(B)$ of [BHM93] to compute the algebraic K-theory spectra and maps in the diagram (3.1) above, after making two modifications. Firstly, we will replace ku by its p-completion ku_p, and secondly we will pass to the Adams summand ℓ_p of ku_p. We will now motivate these two changes, and discuss their effect on the algebraic K-theory diagram.

First, the cyclotomic trace map is very close to a p-complete equivalence when applied to connective p-complete S-algebras B with $\pi_0(B)=\mathbb{Z}/p$ or $\mathbb{Z}_p$, or a finite extension of these. More precisely, $K(B)_p$ is the connective cover of $TC(B)_p$ in these cases [HM97]. Therefore we shall pass to the p-complete situation, replacing ku and $H\mathbb{Z}$ by ku_p and $H\mathbb{Z}_p$, respectively. In view of [Du97], the iterated (homotopy) fiber of the square

$$\begin{array}{ccc} K(ku)_p & \xrightarrow{\kappa^*} & K(ku_p)_p \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ K(\mathbb{Z})_p & \xrightarrow{\kappa^*} & K(\mathbb{Z}_p)_p\,, \end{array}$$

where $\kappa\colon ku\to ku_p$ and $\kappa\colon H\mathbb{Z}\to H\mathbb{Z}_p$ denote the completion maps, is homotopy equivalent to the iterated fiber of the corresponding square of topological cyclic homology spectra, and the latter is trivial, since $TC(B)_p\simeq TC(B_p)_p$. Therefore the fiber of $K(ku)_p\to K(ku_p)_p$ is homotopy equivalent to the fiber of $K(\mathbb{Z})_p\to K(\mathbb{Z}_p)_p$, so the change created by passing to the p-complete case is as well understood for ku and ku_p as in the number ring case.

In fact, our calculations shall concentrate on the v_2-periodic behavior of the algebraic K-theory of the S-algebras related to topological K-theory, rather than the subsequent issues of divisibility by $v_0=p$ and $v_1=u^{p-1}$. We achieve this by working with homotopy with mod p and v_1 coefficients, i.e., with the $V(1)$-homotopy functor $X\mapsto V(1)_*X=\pi_*(V(1)\wedge X)$, where $V(1)=S/(p,v_1)$ is the Smith–Toda complex [Sm70] defined for p odd as the mapping cone of the Adams self-map $v_1\colon \Sigma^{2p-2}V(0)\to V(0)$ of the mod p Moore spectrum $V(0)=S/p=S\cup_p e^1$. Hence there is a cofiber sequence

$$\Sigma^{2p-2}V(0)\xrightarrow{v_1} V(0)\xrightarrow{i_1} V(1)\xrightarrow{j_1}\Sigma^{2p-1}V(0)\,.$$

The Smith–Toda complex $V(1)$ is a homotopy commutative and associative ring spectrum for $p\geq 5$ [Ok84], and its BP-homology satisfies $BP_*V(1)\cong BP_*/(p,v_1)$. In this sense $V(1)$ generalizes the mod p Moore spectrum representing mod p homotopy, and represents mod p and v_1 homotopy. There is an element $v_2\in\pi_{2p^2-2}V(1)$, with image v_2 in $BP_*V(1)$. So $V(1)$ is a finite complex of type 2 in the sense of [HoSm98], with telescope $V(1)[v_2^{-1}]$ the spectrum that represents v_2-periodic homotopy, $V(1)_*(X)[v_2^{-1}]$. The composite map $\beta_{1,1} = i_1 j_1 \colon V(1) \to \Sigma^{2p-1}V(1)$ defines the primary v_1-Bockstein homomorphism acting naturally on $V(1)_*X$.

It is known by [BM94] and [BM95] that $V(1)_*K(\mathbb{Z}_p)$ is a finite $\mathbb{Z}/p$-module. Furthermore, the announced proof of the Bloch–Kato conjecture by Vladimir Voevodsky and Markus Rost also implies the Lichtenbaum–Quillen conjecture on the algebraic K-theory of rings of integers in number fields, which in turn implies that $V(1)_*K(\mathbb{Z})$ is a finite $\mathbb{Z}/p$-module. So the common fiber of the maps $K(\mathbb{Z})_p \to K(\mathbb{Z}_p)_p$ and $K(ku)_p \to K(ku_p)_p$ has (totally) finite $V(1)$-homotopy. Therefore, replacing $K(\mathbb{Z})$ by $K(\mathbb{Z}_p)$ and $K(ku)$ by $K(ku_p)$ in diagram (3.1), and still defining the lower row and the right hand column so as to obtain a 3×3 square of cofibrations, only changes the $V(1)$-homotopy in the diagram by a finite $\mathbb{Z}/p$-module. In this case the upper right hand corner is $K(\mathbb{Q}_p)$, since $\mathbb{Z}_p[p^{-1}] = \mathbb{Q}_p$, and the lower right hand corner $p^{-1}KU_p$ plays the role of the S-algebraic fraction field of ku_p, denoted $\mathit{ff}(ku_p)$. It follows that the v_2-periodic homotopy, given by $V(1)_*K(ku)[v_2^{-1}]$, etc., does not change at all:

$$V(1)_*K(p^{-1}KU)[v_2^{-1}] \cong V(1)_*K(\mathit{ff}(ku_p))[v_2^{-1}]\,. \tag{3.5}$$

Along the same lines, we might have started with the version of diagram (3.1) discussed in Remark 3.4, inverting all primes in $\mathbb{Z}$ rather than just p. This adds a copy of the cofiber sequence $K(\mathbb{Z}/r) \xrightarrow{\pi_*} K(ku/r) \to K(KU/r)$ to the left hand column, for each prime $r \neq p$. Here $\pi^* \colon K(ku/r)_p \to K(\mathbb{Z}/r)_p$ is easily seen to be an equivalence, using the $BGL(-)^+$-model for algebraic K-theory, and from the $F\psi^r$-model for $K(\mathbb{Z}/r)$ from [Qu72] it is clear that $V(1)_*K(\mathbb{Z}/r)$ is finite and concentrated in degrees $0 \le * < 2p-2$. Therefore, replacing the right hand column by the cofiber sequence $K(\mathbb{Q}) \xrightarrow{\pi_*} K(\mathcal{O}) \to K(\mathit{ff}(ku))$, where $\mathcal{O} = \mathcal{O}_{\mathit{ff}(ku)}$ is the valuation ring of the fraction field of ku, only changes the $V(1)$-homotopy in a finite range of degrees, and the v_2-periodic homotopy is unchanged:

$$V(1)_*K(p^{-1}KU)[v_2^{-1}] \cong V(1)_*K(\mathit{ff}(ku))[v_2^{-1}]\,. \tag{3.6}$$

Second, the Hurewicz image of the Bott element u is nilpotent in the mod p homology of the spectrum ku, since $u^{p-1} = v_1$ has Adams filtration 1, and this significantly complicates the algebra involved in computing the topological Hochschild homology $THH(-)$ and the topological cyclic homology $TC(-)$ of ku, compared to the computations for its Adams summand ℓ. Compare [MS93], [AuR02] and [Au05]. Therefore, for the detailed calculations in this paper we shall restrict attention to the Adams summands ℓ_p and ℓ/p of ku_p and ku/p, respectively. The map $L_p \to KU_p$ is a Δ-Galois extension of commutative S-algebras, in the sense of [Rog08, §4.1], where $\Delta = (\mathbb{Z}/p)^\times \cong \mathbb{Z}/(p-1)$ acts on KU_p through the p-adic Adams operations fixing L_p. Generalizing the Lichtenbaum–Quillen conjecture, it is therefore to be expected that the map

$$\Phi \colon K(L_p) \to K(KU_p)^{h\Delta}$$

is (close to) an equivalence after p-adic completion, and similarly for the fraction fields. Indeed, the first expectation is known to be correct: Consider the map of horizontal cofiber sequences

$$
\begin{tikzcd}
K(\mathbb{Z}_p)_p \arrow[r, "\pi_*"] \arrow[d, "p-1"] & K(\ell_p)_p \arrow[r] \arrow[d, "\varphi"] & K(L_p)_p \arrow[d, "\Phi"] \\
K(\mathbb{Z}_p)_p^{h\Delta} \arrow[r, "\pi_*^{h\Delta}"] & K(ku_p)_p^{h\Delta} \arrow[r] & K(KU_p)_p^{h\Delta} ,
\end{tikzcd}
\tag{3.7}
$$

where the lower row is obtained from the Blumberg–Mandell localization sequence by taking Δ-homotopy fixed points (Δ acts trivially on $\mathbb{Z}_p = \pi_0(ku_p)$), and the upper row is its analogue for the Adams summand. The left hand vertical map multiplies by the ramification index $(p-1)$, since the uniformizer v_1 in L_p maps to that power of the uniformizer u in KU_p. More directly, the upper map π_* takes the unit class of $H\mathbb{Z}_p$ to that of the complex $v_1 \colon \Sigma^{2p-2}\ell_p \to \ell_p$, which extends along $\ell_p \to ku_p$ to the class of $v_1 = u^{p-1} \colon \Sigma^{2p-2}ku_p \to ku_p$. This is equivalent to $(p-1)$ times the class of $u \colon \Sigma^2 ku_p \to ku_p$, which is the image of the unit class of $H\mathbb{Z}_p$ under the lower map $\pi_*^{h\Delta}$. Either way, the left hand vertical map is a p-adic equivalence. Thus the middle vertical map φ is an equivalence if and only if the right hand vertical map Φ is one. These equivalent conclusions are indeed precisely verified by the calculations of the first author in [Au05], showing that $V(1)_*K(\ell_p) \cong V(1)_*K(ku_p)^\Delta$, from which the corresponding assertion for p-adic integral homotopy groups follows by a Bockstein argument.

Conversely, the later calculations of [Au:tcku] show that there is an element $b \in V(1)_{2p+2}K(ku)$ with $b^{p-1} = -v_2$, related to $a_{(0,1)} \in K(2)_{2p+2}K(\mathbb{Z}/p, 2)$ [RaWi80, §9] via the composite map

$$K(\mathbb{Z}/p, 2) \to K(\mathbb{Z}, 3) \to BGL_1(ku) \to K(ku)$$

and the unit map $V(1) \to V(1) \wedge E(2) = K(2)$. Furthermore, there is an isomorphism

$$P(b^{\pm 1}) \otimes_{P(v_2^{\pm 1})} V(1)_*K(\ell_p)[v_2^{-1}] \cong V(1)_*K(ku_p)[v_2^{-1}]$$

where $P(x^{\pm 1})$ denotes the Laurent polynomial algebra over $\mathbb{F}_p$ generated by x. The left hand side can also be written as the algebraic extension

$$V(1)_*K(\ell_p)[v_2^{-1}, b]/(b^{p-1} + v_2) .$$

Here Δ acts faithfully on b, fixing $b^{p-1} = -v_2$, so this both shows how the v_2-periodic algebraic K-theory of ku_p is generated by that of ℓ_p and the element b, and conversely how the v_2-periodic algebraic K-theory of ℓ_p can be recovered from that of ku_p as the fixed points of the Galois action. In view of the cofiber sequences (3.7) above, the corresponding formulas also hold with L_p and KU_p in place of ℓ_p and ku_p.

We are therefore confident that ku-based calculations of $K(ku/p)$, $K(ku_p)$ and $K(\mathit{ff}(ku_p))$, like the ℓ-based calculations of $K(\ell/p)$, $K(\ell_p)$ and $K(\mathit{ff}(\ell_p))$ to be presented here, will verify the, for now conjectural, formula

$$P(b^{\pm 1}) \otimes_{P(v_2^{\pm 1})} V(1)_*K(\mathit{ff}(\ell_p))[v_2^{-1}] \cong V(1)_*K(\mathit{ff}(ku_p))[v_2^{-1}] \tag{3.8}$$

and show that $V(1)_*K(\mathit{ff}(\ell_p))[v_2^{-1}]$ can be recovered as the Δ-invariants of the right hand side. Thus we expect these more complicated calculations to give qualitatively the same answers. One reason to perform such more complicated calculations, other than to confirm our expectations, would be to precisely identify the image of $V(1)_*K(\mathit{ff}(ku_p))$ in $V(1)_*K(\mathit{ff}(ku_p))[v_2^{-1}]$, and to check that the map inverting v_2 is very close to being injective, but we will not address this matter in the present paper.

With this we are done with our justifications for concentrating on the following framework. We collect the relevant definitions in one place, for the reader's convenience.

Definition 3.9. Let ℓ_p be the p-complete connective Adams summand, which is a well-defined commutative S-algebra, with $\pi_*\ell_p = \mathbb{Z}_p[v_1]$. For each choice of associative ℓ_p-algebra structure on ℓ/p, the p-complete form

$$\begin{tikzcd} \ell_p \arrow[r, "i"] \arrow[d, "\pi"] & \ell/p \arrow[d, "\pi"] \\ H\mathbb{Z}_p \arrow[r, "i"] & H\mathbb{Z}/p \end{tikzcd}$$

of diagram (2.4) is a base change square of associative ℓ_p-algebras. Each map $i\colon \ell_p \to \ell/p$, $i\colon H\mathbb{Z}_p \to H\mathbb{Z}/p$, $\pi\colon \ell_p \to H\mathbb{Z}_p$ and $\pi\colon \ell/p \to H\mathbb{Z}/p$ makes the target a finite 2-cell complex over the source. The respective transfer maps in algebraic K-theory define the upper left hand square in the diagram (3.10) below.

The upper row, and the left hand and middle columns, are cofiber sequences by the algebraic K-theory localization sequences of Quillen and Blumberg–Mandell, respectively. We define $K(p^{-1}\ell_p)$ and $K(p^{-1}L_p)$ to be the cofibers of the transfer maps i_* in the middle and lower rows, respectively. This makes $K(p^{-1}L_p)$ the iterated cofiber of the upper left hand square. Furthermore, we write $\mathit{ff}(\ell_p) = \mathit{ff}(L_p)$ for the symbol $p^{-1}L_p$, and consider it as an S-algebraic fraction field of ℓ_p. Likewise, we consider the symbol $p^{-1}\ell_p$ as an S-algebraic valuation ring for a valuation on $\mathit{ff}(\ell_p)$ with uniformizer v_1. Then the following diagram is a 3×3 square of cofiber sequences.

(3.10)
$$\begin{tikzcd} K(\mathbb{Z}/p) \arrow[r, "i_*"] \arrow[d, "\pi_*"] & K(\mathbb{Z}_p) \arrow[r, "j^*"] \arrow[d, "\pi_*"] & K(\mathbb{Q}_p) \arrow[d, "\pi_*"] \\ K(\ell/p) \arrow[r, "i_*"] \arrow[d, "\rho^*"] & K(\ell_p) \arrow[r, "j^*"] \arrow[d, "\rho^*"] & K(p^{-1}\ell_p) \arrow[d, "\rho^*"] \\ K(L/p) \arrow[r, "i_*"] & K(L_p) \arrow[r, "j^*"] & K(\mathit{ff}(\ell_p)) \end{tikzcd}$$

Remark 3.11. We gave the existence of a non-trivial S-algebra homomorphism $L_p \to L/p$ as one argument for why L_p is not an S-algebraic field. However, this is in an essential way a non-commutative phenomenon. It follows from [BMMS86, III.4.1] that any commutative S-algebra with $p = 0$ in π_0 is an $H\mathbb{Z}/p$-module, hence any commutative L_p-algebra with $p = 0$ in π_0 is trivial, since $H\mathbb{Z}/p \wedge L_p \simeq *$. Algebraic K-theory is a functor of associative S-algebras, not just of commutative S-algebras, so this should not be perceived as an insurmountable obstacle, but it still illustrates some of the subtleties involved in thinking algebro-geometrically about these S-algebras.

Remark 3.12. It is time to explain why we do not think of $K(p^{-1}ku)$ as the algebraic K-theory of the S-algebra $ku[p^{-1}]$, etc. For primes $p \geq 5$ we computed $V(1)_*K(\ell_p)$ in [AuR02], and we shall compute $V(1)_*K(\ell/p)$ in Theorem 8.10 below. The results are different, also after inverting v_2, so $V(1)_*K(p^{-1}\ell_p)[v_2^{-1}]$ is certainly nonzero. This is a direct summand of $V(1)_*K(p^{-1}ku_p)[v_2^{-1}]$, which is isomorphic to $V(1)_*K(p^{-1}ku)[v_2^{-1}]$, $V(1)_*K(\mathit{ff}(ku_p))[v_2^{-1}]$ and $V(1)_*K(\mathit{ff}(ku))[v_2^{-1}]$, so all of these are also nonzero. On the other hand, ku_p is an S_p-algebra, so the localization $ku_p[p^{-1}]$ is an $S_p[p^{-1}] \simeq H\mathbb{Q}_p$-algebra. Therefore $V(1)_*K(ku_p[p^{-1}])$ is a module over $V(1)_*K(\mathbb{Q}_p)$, which is finite, so $V(1)_*K(ku_p[p^{-1}])[v_2^{-1}]$ is a module over $V(1)_*K(\mathbb{Q}_p)[v_2^{-1}]$, which is zero. Hence $V(1)_*K(ku_p[p^{-1}])[v_2^{-1}] = 0$. Therefore $p^{-1}ku_p \not\simeq ku_p[p^{-1}]$ and $\mathit{ff}(ku_p) \not\simeq KU_p[p^{-1}]$.

4. Topological Hochschild homology

In this and the following sections, we shall compute the $V(1)$-homotopy of the topological Hochschild homology $THH(-)$ and topological cyclic homology $TC(-)$ of the S-algebras in the upper left hand square in diagram (3.10), for primes $p \geq 5$. Passing to connective covers, this also computes the $V(1)$-homotopy of the algebraic K-theory spectra appearing in that square, which by the projection formula for the transfer maps allows us to compute the $V(1)$-homotopy of the remaining entries in the diagram, including $K(p^{-1}L_p) = K(\mathit{ff}(\ell_p))$. With these coefficients, or more generally, after p-adic completion, the functors THH and TC are insensitive to p-completion in the argument, so we shall simplify the notation slightly by working with the associative S-algebras ℓ and $H\mathbb{Z}_{(p)}$ in place of ℓ_p and $H\mathbb{Z}_p$. For ordinary rings R we almost always shorten notations like $THH(HR)$ to $THH(R)$.

The computations follow the strategy of [Bo:zzp], [BM94], [BM95] and [HM97] for $H\mathbb{Z}/p$ and $H\mathbb{Z}$, and of [MS93] and [AuR02] for ℓ. See also [AnR05, §§4–7] for further discussion of the THH-part of such computations. In this section we shall compute the mod p homology of the topological Hochschild homology of ℓ/p as a module over the corresponding homology for ℓ, for any odd prime p.

We write $E(x) = \mathbb{F}_p[x]/(x^2)$ for the exterior algebra, $P(x) = \mathbb{F}_p[x]$ for the polynomial algebra and $P(x^{\pm 1}) = \mathbb{F}_p[x, x^{-1}]$ for the Laurent polynomial algebra on one generator x, and similarly for a list of generators. We will also write $\Gamma(x) = \mathbb{F}_p\{\gamma_i(x) \mid i \geq 0\}$ for the divided power algebra, with $\gamma_i(x) \cdot \gamma_j(x) = (i,j)\gamma_{i+j}(x)$, where $(i,j) = (i+j)!/i!j!$ is the binomial coefficient. We use the obvious abbreviations $\gamma_0(x) = 1$ and $\gamma_1(x) = x$. Finally, we write $P_h(x) = \mathbb{F}_p[x]/(x^h)$ for the truncated polynomial algebra of height h, and recall the isomorphism $\Gamma(x) \cong P_p(\gamma_{p^e}(x) \mid e \geq 0)$ in characteristic p.

We write $H_*(-)$ for homology with mod p coefficients. It takes values in A_*-comodules, where A_* is the dual Steenrod algebra [Mi58]. Explicitly (for p odd),

$$A_* = P(\bar{\xi}_k \mid k \geq 1) \otimes E(\bar{\tau}_k \mid k \geq 0)$$

with coproduct

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{p^i}$$

and

$$\psi(\bar\tau_k) = 1 \otimes \bar\tau_k + \sum_{i+j=k} \bar\tau_i \otimes \bar\xi_j^{p^i} .$$

Here $\bar\xi_0 = 1$, $\bar\xi_k = \chi(\xi_k)$ has degree $2(p^k - 1)$ and $\bar\tau_k = \chi(\tau_k)$ has degree $2p^k - 1$, where χ is the canonical conjugation [MM65]. Then the zeroth Postnikov sections induce identifications

$$\begin{aligned} H_*(H\mathbb{Z}_{(p)}) &= P(\bar\xi_k \mid k \ge 1) \otimes E(\bar\tau_k \mid k \ge 1) \\ H_*(\ell) &= P(\bar\xi_k \mid k \ge 1) \otimes E(\bar\tau_k \mid k \ge 2) \\ H_*(\ell/p) &= P(\bar\xi_k \mid k \ge 1) \otimes E(\bar\tau_0, \bar\tau_k \mid k \ge 2) \end{aligned}$$

as A_*-comodule subalgebras of $H_*(H\mathbb{Z}/p) = A_*$. We often make use of the following A_*-comodule coactions

$$\begin{aligned} \nu(\bar\tau_0) &= 1 \otimes \bar\tau_0 + \bar\tau_0 \otimes 1 \\ \nu(\bar\xi_1) &= 1 \otimes \bar\xi_1 + \bar\xi_1 \otimes 1 \\ \nu(\bar\tau_1) &= 1 \otimes \bar\tau_1 + \bar\tau_0 \otimes \bar\xi_1 + \bar\tau_1 \otimes 1 \\ \nu(\bar\xi_2) &= 1 \otimes \bar\xi_2 + \bar\xi_1 \otimes \bar\xi_1^p + \bar\xi_2 \otimes 1 \\ \nu(\bar\tau_2) &= 1 \otimes \bar\tau_2 + \bar\tau_0 \otimes \bar\xi_2 + \bar\tau_1 \otimes \bar\xi_1^p + \bar\tau_2 \otimes 1 . \end{aligned}$$

The Bökstedt spectral sequences

$$E^2_{**}(B) = HH_*(H_*(B)) \Longrightarrow H_*(THH(B))$$

for the commutative S-algebras $B = H\mathbb{Z}/p$, $H\mathbb{Z}_{(p)}$ and ℓ begin

$$\begin{aligned} E^2_{**}(\mathbb{Z}/p) &= A_* \otimes E(\sigma\bar\xi_k \mid k \ge 1) \otimes \Gamma(\sigma\bar\tau_k \mid k \ge 0) \\ E^2_{**}(\mathbb{Z}_{(p)}) &= H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar\xi_k \mid k \ge 1) \otimes \Gamma(\sigma\bar\tau_k \mid k \ge 1) \\ E^2_{**}(\ell) &= H_*(\ell) \otimes E(\sigma\bar\xi_k \mid k \ge 1) \otimes \Gamma(\sigma\bar\tau_k \mid k \ge 2) . \end{aligned}$$

They are (graded) commutative A_*-comodule algebra spectral sequences, and there are differentials

$$d^{p-1}(\gamma_j \sigma\bar\tau_k) = \sigma\bar\xi_{k+1} \cdot \gamma_{j-p}\sigma\bar\tau_k$$

for $j \ge p$ and $k \ge 0$, see [Bo:zzp], [Hu96] or [Au05, 4.3], leaving

$$\begin{aligned} E^\infty_{**}(\mathbb{Z}/p) &= A_* \otimes P_p(\sigma\bar\tau_k \mid k \ge 0) \\ E^\infty_{**}(\mathbb{Z}_{(p)}) &= H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar\xi_1) \otimes P_p(\sigma\bar\tau_k \mid k \ge 1) \\ E^\infty_{**}(\ell) &= H_*(\ell) \otimes E(\sigma\bar\xi_1, \sigma\bar\xi_2) \otimes P_p(\sigma\bar\tau_k \mid k \ge 2) . \end{aligned}$$

The inclusion of 0-simplices $\eta\colon B \to THH(B)$ is split for commutative B by the augmentation $\epsilon\colon THH(B) \to B$. Thus there are unique representatives in Bökstedt filtration 1, with zero augmentation, for each of the classes σx. They correspond to $1 \otimes x - x \otimes 1$ in the Hochschild complex, or just $1 \otimes x$ in the normalized Hochschild complex. There are multiplicative extensions $(\sigma\bar\tau_k)^p = \sigma\bar\tau_{k+1}$ for $k \ge 0$, so

$$\begin{aligned} H_*(THH(\mathbb{Z}/p)) &= A_* \otimes P(\sigma\bar\tau_0) \\ H_*(THH(\mathbb{Z}_{(p)})) &= H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar\xi_1) \otimes P(\sigma\bar\tau_1) \\ H_*(THH(\ell)) &= H_*(\ell) \otimes E(\sigma\bar\xi_1, \sigma\bar\xi_2) \otimes P(\sigma\bar\tau_2) \end{aligned} \tag{4.1}$$

as A_*-comodule algebras. The A_*-comodule coactions are given by

$$
\begin{aligned}
\nu(\sigma\bar\tau_0) &= 1 \otimes \sigma\bar\tau_0 \\
\nu(\sigma\bar\xi_1) &= 1 \otimes \sigma\bar\xi_1 \\
\nu(\sigma\bar\tau_1) &= 1 \otimes \sigma\bar\tau_1 + \bar\tau_0 \otimes \sigma\bar\xi_1 \\
\nu(\sigma\bar\xi_2) &= 1 \otimes \sigma\bar\xi_2 \\
\nu(\sigma\bar\tau_2) &= 1 \otimes \sigma\bar\tau_2 + \bar\tau_0 \otimes \sigma\bar\xi_2 \,.
\end{aligned}
\tag{4.2}
$$

The map $\pi^*\colon THH(\ell) \to THH(\mathbb{Z}_{(p)})$ takes $\sigma\bar\xi_2$ to 0 and $\sigma\bar\tau_2$ to $(\sigma\bar\tau_1)^p$. The map $i^*\colon THH(\mathbb{Z}_{(p)}) \to THH(\mathbb{Z}/p)$ takes $\sigma\bar\xi_1$ to 0 and $\sigma\bar\tau_1$ to $(\sigma\bar\tau_0)^p$.

The Bökstedt spectral sequence for the associative S-algebra $B = \ell/p$ begins

$$E^2_{**}(\ell/p) = H_*(\ell/p) \otimes E(\sigma\bar\xi_k \mid k \ge 1) \otimes \Gamma(\sigma\bar\tau_0, \sigma\bar\tau_k \mid k \ge 2)\,.$$

It is an A_*-comodule module spectral sequence over the Bökstedt spectral sequence for ℓ, since the ℓ-algebra multiplication $\ell \wedge \ell/p \to \ell/p$ is a map of associative S-algebras. However, it is not itself an algebra spectral sequence, since the product on ℓ/p is not commutative enough to induce a natural product structure on $THH(\ell/p)$. Nonetheless, we will use the algebra structure present at the E^2-term to help in naming classes.

The map $\pi\colon \ell/p \to H\mathbb{Z}/p$ induces an injection of Bökstedt spectral sequence E^2-terms, so there are differentials generated algebraically by

$$d^{p-1}(\gamma_j\sigma\bar\tau_k) = \sigma\bar\xi_{k+1} \cdot \gamma_{j-p}\sigma\bar\tau_k$$

for $j \ge p$, $k = 0$ or $k \ge 2$, leaving

$$E^\infty_{**}(\ell/p) = H_*(\ell/p) \otimes E(\sigma\bar\xi_2) \otimes P_p(\sigma\bar\tau_0, \sigma\bar\tau_k \mid k \ge 2) \tag{4.3}$$

as an A_*-comodule module over $E^\infty_{**}(\ell)$. We need to resolve the A_*-comodule and $H_*(THH(\ell))$-module extensions in order to obtain $H_*(THH(\ell/p))$. This is achieved in Lemma 4.6 below.

The map $\pi^*\colon E^\infty_{**}(\ell/p) \to E^\infty_{**}(\mathbb{Z}/p)$ is an isomorphism in total degrees $\le (2p-2)$ and injective in total degrees $\le (2p^2-2)$. The first class in the kernel is $\sigma\bar\xi_2$. Hence there are unique classes

$$1\ ,\ \bar\tau_0\ ,\ \sigma\bar\tau_0\ ,\ \bar\tau_0\sigma\bar\tau_0\ ,\ \dots\ ,\ (\sigma\bar\tau_0)^{p-1}$$

in degrees $0 \le * \le 2p-2$ of $H_*(THH(\ell/p))$, mapping to classes with the same names in $H_*(THH(\mathbb{Z}/p))$. More concisely, these are the monomials $\bar\tau_0^\delta(\sigma\bar\tau_0)^i$ for $0 \le \delta \le 1$ and $0 \le i \le p-1$, except that the degree $(2p-1)$ case $(\delta, i) = (1, p-1)$ is omitted. The A_*-comodule coaction on these classes is given by the same formulas in $H_*(THH(\ell/p))$ as in $H_*(THH(\mathbb{Z}/p))$, cf. (4.2).

There is also a class $\bar\xi_1$ in degree $(2p-2)$ of $H_*(THH(\ell/p))$ mapping to a class with the same name, and same A_*-coaction, in $H_*(THH(\mathbb{Z}/p))$.

In degree $(2p-1)$ there is a map π^* of extensions from

$$0 \to \mathbb{F}_p\{\bar\xi_1\bar\tau_0\} \to H_{2p-1}(THH(\ell/p)) \to \mathbb{F}_p\{\bar\tau_0(\sigma\bar\tau_0)^{p-1}\} \to 0$$

to

$$0 \to \mathbb{F}_p\{\bar\tau_1, \bar\xi_1\bar\tau_0\} \to H_{2p-1}(THH(\mathbb{Z}/p)) \to \mathbb{F}_p\{\bar\tau_0(\sigma\bar\tau_0)^{p-1}\} \to 0\,.$$

The latter extension is canonically split by the augmentation $\epsilon\colon THH(\mathbb{Z}/p) \to H\mathbb{Z}/p$, which uses the commutativity of the S-algebra $H\mathbb{Z}/p$.

In degree $2p$, the map π^* goes from

$$H_{2p}(THH(\ell/p)) = \mathbb{F}_p\{\bar\xi_1\sigma\bar\tau_0\}$$

to

$$0 \to \mathbb{F}_p\{\bar\tau_0\bar\tau_1\} \to H_{2p}(THH(\mathbb{Z}/p)) \to \mathbb{F}_p\{\sigma\bar\tau_1, \bar\xi_1\sigma\bar\tau_0\} \to 0\,.$$

Again the latter extension is canonically split.

Lemma 4.4. *There is a unique class y in $H_{2p-1}(THH(\ell/p))$ that is represented by $\bar\tau_0(\sigma\bar\tau_0)^{p-1}$ in $E^\infty_{p-1,p}(\ell/p)$ and that maps to $\bar\tau_0(\sigma\bar\tau_0)^{p-1} - \bar\tau_1$ in $H_*(THH(\mathbb{Z}/p))$.*

Proof. This follows by naturality of the suspension operator σ and the multiplicative relation $(\sigma\bar\tau_0)^p = \sigma\bar\tau_1$ in $H_*(THH(\mathbb{Z}/p))$. A class y in $H_{2p-1}(THH(\ell/p))$ represented by $\bar\tau_0(\sigma\bar\tau_0)^{p-1}$ is determined modulo $\bar\xi_1\bar\tau_0$. Its image in $H_{2p-1}(THH(\mathbb{Z}/p))$ thus has the form $\alpha\bar\tau_1 + \bar\tau_0(\sigma\bar\tau_0)^{p-1}$ modulo $\bar\xi_1\bar\tau_0$, for some $\alpha \in \mathbb{F}_p$. The suspension σy lies in $H_{2p}(THH(\ell/p)) = \mathbb{F}_p\{\bar\xi_1\sigma\bar\tau_0\}$, so its image in $H_{2p}(THH(\mathbb{Z}/p))$ is 0 modulo $\bar\tau_0\bar\tau_1$ and $\bar\xi_1\sigma\bar\tau_0$. It is also the suspension of $\alpha\bar\tau_1 + \bar\tau_0(\sigma\bar\tau_0)^{p-1}$ modulo $\bar\xi_1\bar\tau_0$, which equals $\sigma(\alpha\bar\tau_1) + (\sigma\bar\tau_0)^p = (\alpha+1)\sigma\bar\tau_1$. In particular, the coefficient $(\alpha+1)$ of $\sigma\bar\tau_1$ is 0, so $\alpha = -1$. $\square$

Remark 4.5. For $p = 2$ this can alternatively be read off from the explicit form [Wu91] of the commutator for the product μ in ℓ/p. The coequalizer C of the two maps

$$\ell/p \wedge \ell/p \underset{\mu\tau}{\overset{\mu}{\rightrightarrows}} \ell/p$$

maps to (the 1-skeleton of) $THH(\ell/p)$. The commutator $\mu - \mu\tau$ factors as

$$\ell/p \wedge \ell/p \xrightarrow{\beta\wedge\beta} \Sigma\ell/p \wedge \Sigma\ell/p \xrightarrow{\mu} \Sigma^2\ell/p \xrightarrow{v_1} \ell/p$$

where β is the mod p Bockstein associated to the cofiber sequence $\ell \xrightarrow{p} \ell \xrightarrow{i} \ell/p$ and the cofiber of v_1 is $H\mathbb{Z}/p$. We get a map of cofiber sequences

$$\begin{tikzcd}
\ell/p \wedge \ell/p \arrow[r, "\mu-\mu\tau"] \arrow[d, "\mu(\beta\wedge\beta)"'] & \ell/p \arrow[r] \arrow[d, equal] & C \arrow[d] \\
\Sigma^2\ell/p \arrow[r, "v_1"] & \ell/p \arrow[r] & H\mathbb{Z}/p\,,
\end{tikzcd}$$

so there is a class in $H_3(C)$ that maps to $\bar\xi_1 \otimes \bar\xi_1$ in $H_2(\ell/p \wedge \ell/p)$ and to $\bar\xi_1\sigma\bar\xi_1$ in $H_3(THH(\ell/p))$, which also maps to $\bar\xi_2$ in the cofiber of v_1, i.e., whose A_*-coaction contains the term $\bar\xi_2 \otimes 1$. (The classes $\bar\tau_0$ and $\bar\tau_1$ go by the names $\bar\xi_1$ and $\bar\xi_2$ at $p = 2$.)

For odd primes there is a similar interpretation of how the non-commutativity of the product on ℓ/p provides an obstruction to splitting off the 0-simplices from the $(p-1)$-skeleton of $THH(\ell/p)$, where the cyclic permutation of the p factors in the $(p-1)$-simplex $\bar\tau_0(\sigma\bar\tau_0)^{p-1}$, represented by the Hochschild cycle $\bar\tau_0 \otimes \cdots \otimes \bar\tau_0$, plays a similar role to the twist map τ above.

Let

$$H_*(THH(\ell))/(\sigma\bar\xi_1) \cong H_*(\ell) \otimes E(\sigma\bar\xi_2) \otimes P(\sigma\bar\tau_2)$$

denote the quotient algebra of $H_*(THH(\ell))$ by the ideal generated by $\sigma\bar\xi_1$.

Lemma 4.6. *There is an isomorphism of $H_*(THH(\ell))$-modules*

$$H_*(THH(\ell/p)) \cong H_*(THH(\ell))/(\sigma\bar\xi_1) \otimes \mathbb{F}_p\{1, \bar\tau_0, \sigma\bar\tau_0, \bar\tau_0\sigma\bar\tau_0, \dots, (\sigma\bar\tau_0)^{p-1}, y\}\,.$$

Hence $H_*(THH(\ell/p))$ is a free module of rank $2p$ over $H_*(THH(\ell))/(\sigma\bar\xi_1)$, generated by classes

$$1\ ,\ \bar\tau_0\ ,\ \sigma\bar\tau_0\ ,\ \bar\tau_0\sigma\bar\tau_0\ ,\ \dots\ ,\ (\sigma\bar\tau_0)^{p-1}\ ,\ y$$

in degrees 0 through $2p-1$. These generators are represented in $E^\infty_{**}(\ell/p)$ by the classes

$$1\ ,\ \bar\tau_0\ ,\ \sigma\bar\tau_0\ ,\ \bar\tau_0\sigma\bar\tau_0\ ,\ \dots\ ,\ (\sigma\bar\tau_0)^{p-1}\ ,\ \bar\tau_0(\sigma\bar\tau_0)^{p-1}\,,$$

and map under π^* to classes with the same names in $H_*(THH(\mathbb{Z}/p))$, except for y, which maps to

$$\bar\tau_0(\sigma\bar\tau_0)^{p-1} - \bar\tau_1\,.$$

The A_*-comodule coactions are given by

$$\nu((\sigma\bar\tau_0)^i) = 1 \otimes (\sigma\bar\tau_0)^i$$

for $0 \le i \le p-1$,

$$\nu(\bar\tau_0(\sigma\bar\tau_0)^i) = 1 \otimes \bar\tau_0(\sigma\bar\tau_0)^i + \bar\tau_0 \otimes (\sigma\bar\tau_0)^i$$

for $0 \le i \le p-2$, and

$$\nu(y) = 1 \otimes y + \bar\tau_0 \otimes (\sigma\bar\tau_0)^{p-1} - \bar\tau_0 \otimes \bar\xi_1 - \bar\tau_1 \otimes 1\,.$$

Proof. $H_*(\ell/p)$ is freely generated as a module over $H_*(\ell)$ by 1 and $\bar\tau_0$, and the classes $\sigma\bar\xi_2$ and $\sigma\bar\tau_2$ in $H_*(THH(\ell))$ induce multiplication by the same symbols in $E^\infty_{**}(\ell/p)$, as given in (4.3). This generates all of $E^\infty_{**}(\ell/p)$ from the $2p$ classes $\bar\tau_0^\delta(\sigma\bar\tau_0)^i$ for $0 \le \delta \le 1$ and $0 \le i \le p-1$.

We claim that multiplication by $\sigma\bar\xi_1$ acts trivially on $H_*(THH(\ell/p))$. It suffices to verify this on the module generators $\bar\tau_0^\delta(\sigma\bar\tau_0)^i$, for which the product with $\sigma\bar\xi_1$ remains in the range of degrees where the map to $H_*(THH(\mathbb{Z}/p))$ is injective. The action of $\sigma\bar\xi_1$ is trivial on $H_*(THH(\mathbb{Z}/p))$, since $d^{p-1}(\gamma_p\sigma\bar\tau_0) = \sigma\bar\xi_1$ and $\epsilon(\sigma\bar\xi_1) = 0$, from which the claim follows.

The A_*-comodule coaction on each module generator, including y, is determined by that on its image under π^*. In the latter case, the thing to check is that

$$\begin{aligned}(1\otimes\pi^*)(\nu(y)) &= \nu(\pi^*(y)) = \nu(\bar\tau_0(\sigma\bar\tau_0)^{p-1} - \bar\tau_1)\\ &= 1 \otimes \bar\tau_0(\sigma\bar\tau_0)^{p-1} + \bar\tau_0 \otimes (\sigma\bar\tau_0)^{p-1} - 1 \otimes \bar\tau_1 - \bar\tau_0 \otimes \bar\xi_1 - \bar\tau_1 \otimes 1\end{aligned}$$

equals

$$(1\otimes\pi^*)(1 \otimes y + \bar\tau_0 \otimes (\sigma\bar\tau_0)^{p-1} - \bar\tau_0 \otimes \bar\xi_1 - \bar\tau_1 \otimes 1)\,.$$

□

We note that these results do not visibly depend on the particular choice of ℓ-algebra structure on ℓ/p.

5. Passage to $V(1)$-homotopy

For $p \geq 5$ the Smith–Toda complex $V(1)$ is a homotopy commutative ring spectrum. In this section we compute $V(1)_* THH(\ell/p)$ as a module over $V(1)_* THH(\ell)$, for any prime $p \geq 5$. The unique ring spectrum map from $V(1)$ to $H\mathbb{Z}/p$ induces the identification

$$H_*(V(1)) = E(\tau_0, \tau_1)$$

(no conjugations) as A_*-comodule subalgebras of A_*. Here

$$\begin{aligned} \nu(\tau_0) &= 1 \otimes \tau_0 + \tau_0 \otimes 1 \\ \nu(\tau_1) &= 1 \otimes \tau_1 + \xi_1 \otimes \tau_0 + \tau_1 \otimes 1 \,. \end{aligned}$$

For each ℓ-algebra B, $V(1) \wedge THH(B)$ is a module spectrum over $V(1) \wedge THH(\ell)$ and thus over $V(1) \wedge \ell \simeq H\mathbb{Z}/p$, so $H_*(V(1) \wedge THH(B))$ is a sum of copies of A_* as an A_*-comodule. In particular, $V(1)_* THH(B) = \pi_*(V(1) \wedge THH(B))$ is identified with the subgroup of A_*-comodule primitives in

$$H_*(V(1) \wedge THH(B)) \cong H_*(V(1)) \otimes H_*(THH(B))$$

with the diagonal A_*-comodule coaction. We write $v \wedge x$ for the image of $v \otimes x$ under this identification, with $v \in H_*(V(1))$ and $x \in H_*(THH(B))$. Let

$$\begin{aligned} \epsilon_0 &= 1 \wedge \bar\tau_0 + \tau_0 \wedge 1 \\ \epsilon_1 &= 1 \wedge \bar\tau_1 + \tau_0 \wedge \bar\xi_1 + \tau_1 \wedge 1 \\ \lambda_1 &= 1 \wedge \sigma\bar\xi_1 \\ \lambda_2 &= 1 \wedge \sigma\bar\xi_2 \\ \mu_0 &= 1 \wedge \sigma\bar\tau_0 \\ \mu_1 &= 1 \wedge \sigma\bar\tau_1 + \tau_0 \wedge \sigma\bar\xi_1 \\ \mu_2 &= 1 \wedge \sigma\bar\tau_2 + \tau_0 \wedge \sigma\bar\xi_2 \,. \end{aligned} \tag{5.1}$$

These are all A_*-comodule primitive, where defined. By a dimension count,

$$\begin{aligned} V(1)_* THH(\mathbb{Z}/p) &= E(\epsilon_0, \epsilon_1) \otimes P(\mu_0) \\ V(1)_* THH(\mathbb{Z}_{(p)}) &= E(\epsilon_1) \otimes E(\lambda_1) \otimes P(\mu_1) \\ V(1)_* THH(\ell) &= E(\lambda_1, \lambda_2) \otimes P(\mu_2) \end{aligned} \tag{5.2}$$

as commutative $\mathbb{F}_p$-algebras. The map $\pi \colon \ell \to H\mathbb{Z}_{(p)}$ takes λ_2 to 0 and μ_2 to μ_1^p. The map $i \colon H\mathbb{Z}_{(p)} \to H\mathbb{Z}/p$ takes λ_1 to 0 and μ_1 to μ_0^p. Note that $\mu_2 \in V(1)_{2p^2} THH(\ell)$ was simply denoted μ in [AuR02].

In degrees $\leq (2p-2)$ of $H_*(V(1) \wedge THH(\ell/p))$ the classes

$$\mu_0^i := 1 \wedge (\sigma\bar\tau_0)^i \tag{5.3.a}$$

for $0 \leq i \leq p-1$ and

$$\epsilon_0 \mu_0^i := 1 \wedge \bar\tau_0 (\sigma\bar\tau_0)^i + \tau_0 \wedge (\sigma\bar\tau_0)^i \tag{5.3.b}$$

for $0 \le i \le p-2$ are A_*-comodule primitive. These map to the classes $\epsilon_0^\delta \mu_0^i$ in $V(1)_* THH(\mathbb{Z}/p)$ for $0 \le \delta \le 1$ and $0 \le i \le p-1$, except that the degree bound excludes the top case of $\epsilon_0 \mu_0^{p-1}$.

In degree $(2p-1)$ of $H_*(V(1) \wedge THH(\ell/p))$ we have generators $1 \wedge \bar\xi_1 \bar\tau_0$, $\tau_0 \wedge (\sigma\bar\tau_0)^{p-1}$, $\tau_0 \wedge \bar\xi_1$, $\tau_1 \wedge 1$ and $1 \wedge y$. These have coactions

$$\begin{aligned}
\nu(1 \wedge \bar\xi_1 \bar\tau_0) &= 1 \otimes 1 \wedge \bar\xi_1 \bar\tau_0 + \bar\tau_0 \otimes 1 \wedge \bar\xi_1 + \bar\xi_1 \otimes 1 \wedge \bar\tau_0 + \bar\xi_1 \bar\tau_0 \otimes 1 \wedge 1 \\
\nu(\tau_0 \wedge (\sigma\bar\tau_0)^{p-1}) &= 1 \otimes \tau_0 \wedge (\sigma\bar\tau_0)^{p-1} + \tau_0 \otimes 1 \wedge (\sigma\bar\tau_0)^{p-1} \\
\nu(\tau_0 \wedge \bar\xi_1) &= 1 \otimes \tau_0 \wedge \bar\xi_1 + \tau_0 \otimes 1 \wedge \bar\xi_1 + \bar\xi_1 \otimes \tau_0 \wedge 1 + \bar\xi_1 \tau_0 \otimes 1 \wedge 1 \\
\nu(\tau_1 \wedge 1) &= 1 \otimes \tau_1 \wedge 1 + \xi_1 \otimes \tau_0 \wedge 1 + \tau_1 \otimes 1 \wedge 1
\end{aligned}$$

and

$$\nu(1 \wedge y) = 1 \otimes 1 \wedge y + \bar\tau_0 \otimes 1 \wedge (\sigma\bar\tau_0)^{p-1} - \bar\tau_0 \otimes 1 \wedge \bar\xi_1 - \bar\tau_1 \otimes 1 \wedge 1 \,.$$

Hence the sum

$$\bar\epsilon_1 := 1 \wedge y + \tau_0 \wedge (\sigma\bar\tau_0)^{p-1} - \tau_0 \wedge \bar\xi_1 - \tau_1 \wedge 1 \tag{5.3.c}$$

is A_*-comodule primitive. Its image under π^* in $H_*(V(1) \wedge THH(\mathbb{Z}/p))$ is

$$\epsilon_0 \mu_0^{p-1} - \epsilon_1 = 1 \wedge \bar\tau_0 (\sigma\bar\tau_0)^{p-1} + \tau_0 \wedge (\sigma\bar\tau_0)^{p-1} - 1 \wedge \bar\tau_1 - \tau_0 \wedge \bar\xi_1 - \tau_1 \wedge 1 \,.$$

Let

$$V(1)_* THH(\ell)/(\lambda_1) \cong E(\lambda_2) \otimes P(\mu_2)$$

be the quotient algebra of $V(1)_* THH(\ell)$ by the ideal generated by λ_1.

Proposition 5.4. *There is an isomorphism of $V(1)_* THH(\ell)$-modules*

$$V(1)_* THH(\ell/p) = V(1)_* THH(\ell)/(\lambda_1) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar\epsilon_1\} \,,$$

where the classes μ_0^i, $\epsilon_0 \mu_0^i$ and $\bar\epsilon_1$ are defined in (5.3.abc) above. Multiplication by λ_1 is 0, so this is a free module on the $2p$ generators

$$1 \ , \ \epsilon_0 \ , \ \mu_0 \ , \ \epsilon_0\mu_0 \ , \ \dots \ , \ \mu_0^{p-1} \ , \ \bar\epsilon_1$$

over $V(1)_* THH(\ell)/(\lambda_1)$. The map π^* to $V(1)_* THH(\mathbb{Z}/p)$ takes $\epsilon_0^\delta \mu_0^i$ in degree $0 \le \delta + 2i \le 2p-2$ to $\epsilon_0^\delta \mu_0^i$, and takes $\bar\epsilon_1$ in degree $(2p-1)$ to $\epsilon_0 \mu_0^{p-1} - \epsilon_1$.

Proof. Additively, this follows by another dimension count. The multiplication by λ_1 is 0 for degree and filtration reasons: λ_1 has Bökstedt filtration 1 and cannot map to $\bar\epsilon_1$ in Bökstedt filtration $(p-1)$. Similarly in higher degrees. □

We end this section with a suggestive conjectural calculation of the topological Hochschild homology of the fraction field $ff(\ell) = p^{-1}L$, which may play the role of the de Rham complex over $\mathrm{Spec}(ff(\ell))$ in derived algebraic geometry. The calculation is not needed for the rest of the paper. We work with the p-local ℓ, but could equally well have worked with the p-complete ℓ_p.

Thus consider a 3×3 square of cofiber sequences

$$
\begin{tikzcd}
THH(\mathbb{Z}/p) \arrow[r, "i_*"] \arrow[d, "\pi_*"] & THH(\mathbb{Z}_{(p)}) \arrow[r, "j^*"] \arrow[d, "\pi_*"] & THH(\mathbb{Z}_{(p)}|\mathbb{Q}) \arrow[d, "\pi_*"] \\
THH(\ell/p) \arrow[r, "i_*"] \arrow[d, "\rho^*"] & THH(\ell) \arrow[r, "j^*"] \arrow[d, "\rho^*"] & THH(\ell|p^{-1}\ell) \arrow[d, "\rho^*"] \\
THH(\ell/p|L/p) \arrow[r, "i_*"] & THH(\ell|L) \arrow[r, "j^*"] & THH(ff(\ell))
\end{tikzcd}
\tag{5.5}
$$

generated by the upper left hand square. The transfer map $i_* \colon THH(\mathbb{Z}/p) \to THH(\mathbb{Z}_{(p)})$ was properly defined in [HM03], and $THH(\mathbb{Z}_{(p)}|\mathbb{Q})$ is its cofiber. To construct the remaining transfer maps one may use the definition in [BM:loc] of THH in terms of spectral categories. By analogy with the algebraic case, we write $THH(\ell|p^{-1}\ell)$, $THH(\ell/p|L/p)$ and $THH(\ell|L)$ for three of the remaining cofibers in the diagram. We might have denoted the term in the lower right hand corner by something like $THH(\ell|p^{-1}\ell, L|p^{-1}L)$, but for simplicity we abbreviate this to $THH(ff(\ell))$.

In the top row we get [HM03, 2.4.1] a long exact sequence in $V(1)$-homotopy

$$\dots \xrightarrow{\partial} E(\epsilon_0, \epsilon_1) \otimes P(\mu_0) \xrightarrow{i_*} E(\epsilon_1, \lambda_1) \otimes P(\mu_1) \xrightarrow{j^*} E(\operatorname{dlog} p, \epsilon_1) \otimes P(\kappa_0) \xrightarrow{\partial} \dots$$

where $i_*(\epsilon_0 \mu_0^{p-1}) = \lambda_1$, $j^*(\mu_1) = \kappa_0^p$, $\partial(\operatorname{dlog} p) = 1$ and $\partial(\kappa_0) = \epsilon_0$, tensored with the identity on $E(\epsilon_1)$.

In the middle row we expect a long exact sequence

$$\dots \xrightarrow{\partial} \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1\} \otimes E(\lambda_2) \otimes P(\mu_2) \xrightarrow{i_*}$$
$$\xrightarrow{i_*} E(\lambda_1, \lambda_2) \otimes P(\mu_2) \xrightarrow{j^*} E(\operatorname{dlog} p, \lambda_2) \otimes P_p(\kappa_0) \otimes P(\mu_2) \xrightarrow{\partial} \dots$$

where $i_*(\bar{\epsilon}_1) = \lambda_1$, $\partial(\operatorname{dlog} p) = 1$ and $\partial(\kappa_0) = \epsilon_0$, tensored with the identity on $E(\lambda_2)$.

In the middle column we expect [Au05, §10] a long exact sequence

$$\dots \xrightarrow{\partial} E(\epsilon_1, \lambda_1) \otimes P(\mu_1) \xrightarrow{\pi_*} E(\lambda_1, \lambda_2) \otimes P(\mu_2) \xrightarrow{\rho^*} E(\operatorname{dlog} v_1, \lambda_1) \otimes P(\kappa_1) \xrightarrow{\partial} \dots$$

where $\pi_*(\epsilon_1 \mu_1^{p-1}) = \lambda_2$, $\rho^*(\mu_2) = \kappa_1^p$, $\partial(\operatorname{dlog} v_1) = 1$ and $\partial(\kappa_1) = \epsilon_1$, tensored with the identity on $E(\lambda_1)$.

In the right hand column we expect a long exact sequence

$$\dots \xrightarrow{\partial} E(\operatorname{dlog} p, \epsilon_1) \otimes P(\kappa_0) \xrightarrow{\pi_*}$$
$$\xrightarrow{\pi_*} E(\operatorname{dlog} p, \lambda_2) \otimes P_p(\kappa_0) \otimes P(\mu_2) \xrightarrow{\rho^*} E(\operatorname{dlog} p, \operatorname{dlog} v_1) \otimes P(\kappa_0) \xrightarrow{\partial} \dots$$

where $\pi_*(\epsilon_1 \kappa_0^{p^2-p}) = \lambda_2$, $\rho^*(\mu_2) = \kappa_0^{p^2}$, $\partial(\operatorname{dlog} v_1) = 1$ and $\partial(\kappa_0^p) = \epsilon_1$, tensored with the identity on $E(\operatorname{dlog} p)$. Note that this ρ^* is not multiplicative, since it takes the truncated polynomial algebra $P_p(\kappa_0)$ into $P(\kappa_0)$.

This leads to the following formula. When compared with [HM03, 2.4.1], it strongly suggests that $E(\operatorname{dlog} p, \operatorname{dlog} v_1)$ is the $V(1)$-homotopy of a de Rham complex for ℓ with logarithmic poles along (p) and (v_1).

Conjecture 5.6. *There is an isomorphism*

$$V(1)_*THH(\mathit{ff}(\ell)) \cong E(\mathrm{dlog}\, p, \mathrm{dlog}\, v_1) \otimes P(\kappa_0)$$

with $|\,\mathrm{dlog}\, p\,| = |\,\mathrm{dlog}\, v_1| = 1$ *and* $|\kappa_0| = 2$. *Here* $THH(\mathit{ff}(\ell))$ *is defined as the iterated cofiber of the upper left hand square in diagram (5.5),* $\mathrm{dlog}\, p$ *is in the image from* $\pi_1 THH(\ell|p^{-1}\ell)$*, with* $\partial(\mathrm{dlog}\, p) = 1$ *in* $\pi_0 THH(\ell/p)$*,* $\mathrm{dlog}\, v_1$ *is in the image from* $\pi_1 THH(\ell|L)$*, with* $\partial(\mathrm{dlog}\, v_1) = 1$ *in* $\pi_0 THH(\mathbb{Z}_{(p)})$*, and* κ_0 *satisfies* $\kappa_0^{p^2} = \mu_2$*, with* μ_2 *in the image from* $V(1)_{2p^2} THH(\ell)$*.*

6. The C_p-Tate construction

Let $C = C_{p^n}$ denote the cyclic group of order p^n, considered as a closed subgroup of the circle group S^1. For each spectrum X with C-action, $X_{hC} = EC_+ \wedge_C X$ and $X^{hC} = F(EC_+, X)^C$ denote its homotopy orbit and homotopy fixed point spectra, as usual. We now write $X^{tC} = [\widetilde{EC} \wedge F(EC_+, X)]^C$ for the C-Tate construction on X, denoted $t_C(X)^C$ in [GM95] and $\hat{\mathbb{H}}(C, X)$ in [AuR02]. There are C-homotopy fixed point and C-Tate spectral sequences in $V(1)$-homotopy for X, with

$$E^2_{s,t}(C, X) = H^{-s}_{gp}(C; V(1)_t(X)) \Longrightarrow V(1)_{s+t}(X^{hC})$$

and

$$\hat{E}^2_{s,t}(C, X) = \hat{H}^{-s}_{gp}(C; V(1)_t(X)) \Longrightarrow V(1)_{s+t}(X^{tC})\,.$$

We write $H^*_{gp}(C_{p^n}; \mathbb{F}_p) = E(u_n) \otimes P(t)$ and $\hat{H}^*_{gp}(C_{p^n}; \mathbb{F}_p) = E(u_n) \otimes P(t^{\pm 1})$ with u_n in degree 1 and t in degree 2. So u_n, t and $x \in V(1)_t(X)$ have bidegree $(-1, 0)$, $(-2, 0)$ and $(0, t)$ in either spectral sequence, respectively. See [HM03, §4.3] for proofs of the multiplicative properties of these spectral sequences.

We are principally interested in the case when $X = THH(B)$, with the S^1-action given by the cyclic structure. It is a cyclotomic spectrum, in the sense of [HM97], leading to the commutative diagram

$$\begin{tikzcd}
THH(B)_{hC_{p^n}} \arrow[r, "N"] \arrow[d, equal] & THH(B)^{C_{p^n}} \arrow[r, "R"] \arrow[d, "\Gamma_n"] & THH(B)^{C_{p^{n-1}}} \arrow[d, "\hat{\Gamma}_n"] \\
THH(B)_{hC_{p^n}} \arrow[r, "N^h"] & THH(B)^{hC_{p^n}} \arrow[r, "R^h"] & THH(B)^{tC_{p^n}}
\end{tikzcd}$$

of horizontal cofiber sequences. We abbreviate $\hat{E}^2_{**}(C, THH(B))$ to $\hat{E}^2_{**}(C, B)$, etc. When B is a commutative S-algebra, this is a commutative algebra spectral sequence, and when B is an associative A-algebra, with A commutative, then $\hat{E}^*(C, B)$ is a module spectral sequence over $\hat{E}^*(C, A)$. The map R^h corresponds to the inclusion $E^2_{**}(C, B) \to \hat{E}^2_{**}(C, B)$ from the second quadrant to the upper half-plane, for connective B.

In this section we compute $V(1)_*THH(\ell/p)^{tC_p}$ by means of the C_p-Tate spectral sequence in $V(1)$-homotopy for $THH(\ell/p)$. In Propositions 6.8 and 6.9 we show that the comparison map $\hat{\Gamma}_1 \colon V(1)_*THH(\ell/p) \to V(1)_*THH(\ell/p)^{tC_p}$ is $(2p-2)$-coconnected and can be identified with the algebraic localization homomorphism that inverts μ_2.

First we recall the structure of the C_p-Tate spectral sequence for $THH(\mathbb{Z}/p)$, with $V(0)$- and $V(1)$-coefficients. We have $V(0)_*THH(\mathbb{Z}/p) = E(\epsilon_0) \otimes P(\mu_0)$, and with an obvious notation the E^2-terms are

$$\begin{aligned} \hat{E}^2_{**}(C_p, \mathbb{Z}/p; V(0)) &= E(u_1) \otimes P(t^{\pm 1}) \otimes E(\epsilon_0) \otimes P(\mu_0) \\ \hat{E}^2_{**}(C_p, \mathbb{Z}/p) &= E(u_1) \otimes P(t^{\pm 1}) \otimes E(\epsilon_0, \epsilon_1) \otimes P(\mu_0)\,. \end{aligned}$$

In each C-Tate spectral sequence we have a first differential

$$d^2(x) = t \cdot \sigma x\,,$$

see e.g. [Rog98, 3.3]. We easily deduce $\sigma\epsilon_0 = \mu_0$ and $\sigma\epsilon_1 = \mu_0^p$ from (5.1), so

$$\begin{aligned} \hat{E}^3_{**}(C_p, \mathbb{Z}/p; V(0)) &= E(u_1) \otimes P(t^{\pm 1}) \\ \hat{E}^3_{**}(C_p, \mathbb{Z}/p) &= E(u_1) \otimes P(t^{\pm 1}) \otimes E(\epsilon_0\mu_0^{p-1} - \epsilon_1)\,. \end{aligned}$$

Thus the $V(0)$-homotopy spectral sequence collapses at $\hat{E}^3 = \hat{E}^\infty$. By naturality with respect to the map $i_1 \colon V(0) \to V(1)$, all the classes on the horizontal axis of $\hat{E}^3(C_p, \mathbb{Z}/p)$ are infinite cycles, so also the latter spectral sequence collapses at $\hat{E}^3_{**}(C_p, \mathbb{Z}/p) = \hat{E}^\infty_{**}(C_p, \mathbb{Z}/p)$.

We know from [HM97, Prop. 5.3] that the comparison map

$$\hat{\Gamma}_1 \colon V(0)_*THH(\mathbb{Z}/p) \to V(0)_*THH(\mathbb{Z}/p)^{tC_p}$$

takes $\epsilon_0^\delta\mu_0^i$ to $(u_1t^{-1})^\delta t^{-i}$, for all $0 \le \delta \le 1$, $i \ge 0$. In particular, the integral map $\hat{\Gamma}_1 \colon \pi_*THH(\mathbb{Z}/p) \to \pi_*THH(\mathbb{Z}/p)^{tC_p}$ is (-2)-coconnected, meaning that it induces an injection in degree (-2) and an isomorphism in all higher degrees. From this we can deduce the following behavior of the comparison map $\hat{\Gamma}_1$ in $V(1)$-homotopy.

Lemma 6.1. *The map*

$$\hat{\Gamma}_1 \colon V(1)_*THH(\mathbb{Z}/p) \to V(1)_*THH(\mathbb{Z}/p)^{tC_p}$$

takes the classes $\epsilon_0^\delta\mu_0^i$ from $V(0)_*THH(\mathbb{Z}/p)$, for $0 \le \delta \le 1$ and $i \ge 0$, to classes represented in $\hat{E}^\infty_{**}(C_p, \mathbb{Z}/p)$ by $(u_1t^{-1})^\delta t^{-i}$ (on the horizontal axis).

Furthermore, it takes the class $\epsilon_0\mu_0^{p-1} - \epsilon_1$ in degree $(2p-1)$ to a class represented by $\epsilon_0\mu_0^{p-1} - \epsilon_1$ (on the vertical axis).

Proof. The classes $\epsilon_0^\delta\mu_0^i$ are in the image from $V(0)$-homotopy, and we recalled above that they are detected by $(u_1t^{-1})^\delta t^{-i}$ in the $V(0)$-homotopy C_p-Tate spectral sequence for $THH(\mathbb{Z}/p)$. By naturality along $i_1 \colon V(0) \to V(1)$, they are detected by the same (nonzero) classes in the $V(1)$-homotopy spectral sequence $\hat{E}^\infty_{**}(C_p, \mathbb{Z}/p)$.

To find the representative for $\hat{\Gamma}_1(\epsilon_0\mu_0^{p-1} - \epsilon_1)$ in degree $(2p-1)$, we appeal to the cyclotomic trace map from algebraic K-theory, or more precisely, to the commutative diagram

(6.2)
$$\begin{tikzcd} & K(B) \arrow[dl, "tr"'] \arrow[d, "tr_1"] \arrow[dr, "tr"] & \\ THH(B) & THH(B)^{C_p} \arrow[l, "F"'] \arrow[r, "R"] \arrow[d, "\Gamma_1"] & THH(B) \arrow[d, "\hat{\Gamma}_1"] \\ & THH(B)^{hC_p} \arrow[ul] \arrow[r, "R^h"] & THH(B)^{tC_p}\,. \end{tikzcd}$$

The Bökstedt trace map $tr\colon K(B) \to THH(B)$ admits a preferred lift tr_n through each fixed-point spectrum $THH(B)^{C_{p^n}}$, which equalizes the iterated restriction and Frobenius maps R^n and F^n to $THH(B)$ [BHM93]. In particular, the circle action and the σ-operator act trivially on classes in the image of tr.

In the case $B = H\mathbb{Z}/p$, we know that $K(\mathbb{Z}/p)_p \simeq H\mathbb{Z}_p$ after p-adic completion, so $V(1)_*K(\mathbb{Z}/p) = E(\bar\epsilon_1)$, where the v_1-Bockstein of $\bar\epsilon_1$ is -1. The Bökstedt trace image $tr(\bar\epsilon_1) \in V(1)_*THH(\mathbb{Z}/p)$ lies in $\mathbb{F}_p\{\epsilon_1, \epsilon_0\mu_0^{p-1}\}$, has v_1-Bockstein $tr(-1) = -1$ and suspends by σ to 0. Hence

$$tr(\bar\epsilon_1) = \epsilon_0\mu_0^{p-1} - \epsilon_1\,.$$

As we recalled above, the map $\hat\Gamma_1\colon \pi_*THH(\mathbb{Z}/p) \to \pi_*THH(\mathbb{Z}/p)^{tC_p}$ is (-2)-coconnected, so the corresponding map in $V(1)$-homotopy is at least $(2p-2)$-coconnected. Thus it takes $\epsilon_0\mu_0^{p-1} - \epsilon_1$ to a nonzero class in $V(1)_*THH(\mathbb{Z}/p)^{tC_p}$, represented somewhere in total degree $(2p-1)$ of $\hat E^\infty_{**}(C_p, \mathbb{Z}/p)$, in the lower right hand corner of the diagram.

Going down the middle of the diagram, we reach a class $(\Gamma_1 \circ tr_1)(\bar\epsilon_1)$, represented in total degree $(2p-1)$ of the left half-plane C_p-homotopy fixed point spectral sequence $E^\infty_{**}(C_p, \mathbb{Z}/p)$. Its image under the edge homomorphism to $V(1)_*THH(\mathbb{Z}/p)$ equals $(F \circ tr_1)(\bar\epsilon_1) = tr(\bar\epsilon_1)$, hence $(\Gamma_1 \circ tr_1)(\bar\epsilon_1)$ is represented by $\epsilon_0\mu_0^{p-1} - \epsilon_1$ in $E^\infty_{0,2p-1}(C_p, \mathbb{Z}/p)$. Its image under R^h in the C_p-Tate spectral sequence is the generator of $\hat E^\infty_{0,2p-1}(C_p, \mathbb{Z}/p) = \mathbb{F}_p\{\epsilon_0\mu_0^{p-1} - \epsilon_1\}$, hence that generator is the E^∞-representative of $\hat\Gamma_1(\epsilon_0\mu_0^{p-1} - \epsilon_1)$. $\square$

We can lift the algebraic K-theory class $\bar\epsilon_1$ to ℓ/p.

Definition 6.3. The map $\pi\colon \ell/p \to H\mathbb{Z}/p$ is $(2p-2)$-connected, so it induces a $(2p-1)$-connected map $V(1)_*K(\ell/p) \to V(1)_*K(\mathbb{Z}/p) = E(\bar\epsilon_1)$, by [BM94, 10.9]. We can therefore choose a class

$$\bar\epsilon_1^K \in V(1)_{2p-1}K(\ell/p)$$

that maps to the generator $\bar\epsilon_1$ in $V(1)_*K(\mathbb{Z}/p) \cong \mathbb{Z}/p$.

Lemma 6.4. *The Bökstedt trace $tr\colon V(1)_*K(\ell/p) \to V(1)_*THH(\ell/p)$ takes $\bar\epsilon_1^K$ to $\bar\epsilon_1$.*

Proof. In the commutative square

$$\begin{array}{ccc} V(1)_*K(\ell/p) & \xrightarrow{tr} & V(1)_*THH(\ell/p) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ V(1)_*K(\mathbb{Z}/p) & \xrightarrow{tr} & V(1)_*THH(\mathbb{Z}/p) \end{array}$$

the trace image $tr(\bar\epsilon_1^K)$ in $V(1)_*THH(\ell/p)$ must map under π^* to $tr(\bar\epsilon_1) = \epsilon_0\mu_0^{p-1} - \epsilon_1$ in $V(1)_*THH(\mathbb{Z}/p)$, which by Proposition 5.4 characterizes it as being equal to the class $\bar\epsilon_1$. Hence $tr(\bar\epsilon_1^K) = \bar\epsilon_1$. $\square$

Next we turn to the C_p-Tate spectral sequence $\hat E^*(C_p, \ell/p)$ in $V(1)$-homotopy for $THH(\ell/p)$. Its E^2-term is

$$\hat E^2_{**}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm1}) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar\epsilon_1\} \otimes E(\lambda_2) \otimes P(\mu_2)\,.$$

We have $d^2(x) = t \cdot \sigma x$, where

$$\sigma(\epsilon_0^\delta \mu_0^{i-1}) = \begin{cases} \mu_0^i & \text{for } \delta = 1,\ 0 < i < p, \\ 0 & \text{otherwise} \end{cases}$$

is readily deduced from (5.1), and $\sigma(\bar\epsilon_1) = 0$ since $\bar\epsilon_1$ is in the image of tr. Thus

$$\hat{E}^3_{**}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm 1}) \otimes E(\bar\epsilon_1) \otimes E(\lambda_2) \otimes P(t\mu_2)\,. \tag{6.5}$$

We prefer to use $t\mu_2$ rather than μ_2 as a generator, since it represents multiplication by v_2 in all module spectral sequences over $E^*(S^1, \ell)$, by [AuR02, 4.8].

To proceed, we shall use that $\hat{E}^*(C_p, \ell/p)$ is a module over the spectral sequence for $THH(\ell)$. We therefore recall the structure of the latter spectral sequence, from [AuR02, 5.5]. It begins

$$\hat{E}^2_{**}(C_p, \ell) = E(u_1) \otimes P(t^{\pm 1}) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2)\,.$$

The classes λ_1, λ_2 and $t\mu_2$ are infinite cycles, and the differentials

$$d^{2p}(t^{1-p}) = t\lambda_1$$
$$d^{2p^2}(t^{p-p^2}) = t^p\lambda_2$$
$$d^{2p^2+1}(u_1 t^{-p^2}) = t\mu_2$$

up to units in $\mathbb{F}_p$, which we will always suppress, leave the terms

$$\hat{E}^{2p+1}_{**}(C_p, \ell) = E(u_1, \lambda_1, \lambda_2) \otimes P(t^{\pm p}, t\mu_2)$$
$$\hat{E}^{2p^2+1}_{**}(C_p, \ell) = E(u_1, \lambda_1, \lambda_2) \otimes P(t^{\pm p^2}, t\mu_2)$$
$$\hat{E}^{2p^2+2}_{**}(C_p, \ell) = E(\lambda_1, \lambda_2) \otimes P(t^{\pm p^2})$$

with $\hat{E}^{2p^2+2} = \hat{E}^\infty$, converging to $V(1)_* THH(\ell)^{tC_p}$. The comparison map $\hat\Gamma_1$ maps λ_1, λ_2 and μ_2 to λ_1, λ_2 and t^{-p^2}, respectively, inducing the algebraic localization map and identification

$$\hat\Gamma_1 \colon V(1)_* THH(\ell) \to V(1)_* THH(\ell)[\mu_2^{-1}] \cong V(1)_* THH(\ell)^{tC_p}\,.$$

Lemma 6.6. *In $\hat{E}^*(C_p, \ell/p)$, the class $u_1 t^{-p}$ supports the nonzero differential*

$$d^{2p^2}(u_1 t^{-p}) = u_1 t^{p^2-p}\lambda_2$$

and does not survive to the E^∞-term.

Proof. In $\hat{E}^*(C_p, \ell)$, there is such a nonzero differential, up to a unit in $\mathbb{F}_p$, which we have already declared that we will suppress. By naturality along $i\colon \ell \to \ell/p$, it follows that there is also such a differential in $\hat{E}^*(C_p, \ell/p)$. It remains to argue that the target is nonzero. Considering the E^3-term in (6.5), the only possible source of a previous differential hitting $u_1 t^{p^2-p}\lambda_2$ is $\bar\epsilon_1$. But $\bar\epsilon_1$ is in an even column and $u_1 t^{p^2-p}\lambda_2$ is in an odd column. By naturality with respect to the Frobenius (group restriction) map from the S^1-Tate spectral sequence to the C_p-Tate spectral sequence, which takes $\hat{E}^2_{**}(S^1, B)$ isomorphically to the even columns of $\hat{E}^2_{**}(C_p, B)$, any such differential from an even to an odd column must be zero. □

To determine the map $\hat\Gamma_1$ we use naturality with respect to the map $\pi\colon \ell/p \to H\mathbb{Z}/p$.

Lemma 6.7. *The classes $1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}$ and $\bar\epsilon_1$ in $V(1)_*THH(\ell/p)$ map under $\hat\Gamma_1$ to classes in $V(1)_*THH(\ell/p)^{tC_p}$ that are represented in $\hat E^\infty_{**}(C_p, \ell/p)$ by the permanent cycles $(u_1t^{-1})^\delta t^{-i}$ (on the horizontal axis) in degrees $\le (2p-2)$, and by the permanent cycle $\bar\epsilon_1$ (on the vertical axis) in degree $(2p-1)$.*

Proof. In the commutative square

$$\begin{array}{ccc} V(1)_*THH(\ell/p) & \xrightarrow{\hat\Gamma_1} & V(1)_*THH(\ell/p)^{tC_p} \\ \Big\downarrow{\scriptstyle \pi^*} & & \Big\downarrow{\scriptstyle \pi^*} \\ V(1)_*THH(\mathbb{Z}/p) & \xrightarrow{\hat\Gamma_1} & V(1)_*THH(\mathbb{Z}/p)^{tC_p} \end{array}$$

the classes $\epsilon_0^\delta\mu_0^i$ in the upper left hand corner map to classes in the lower right hand corner that are represented by $(u_1t^{-1})^\delta t^{-i}$ in degrees $\le (2p-2)$, and $\bar\epsilon_1$ maps to $\epsilon_0\mu_0^{p-1} - \epsilon_1$ in degree $(2p-1)$. This follows by combining Proposition 5.4 and Lemma 6.1.

The first $(2p-1)$ of these are represented in maximal filtration (on the horizontal axis), so their images in the upper right hand corner must be represented by permanent cycles $(u_1t^{-1})^\delta t^{-i}$ in the Tate spectral sequence $\hat E^\infty_{**}(C_p, \ell/p)$.

The image of the last class, $\bar\epsilon_1$, in the upper right hand corner could either be represented by $\bar\epsilon_1$ in bidegree $(0, 2p-1)$ or by u_1t^{-p} in bidegree $(2p-1, 0)$. However, the last class supports a differential $d^{2p^2}(u_1t^{-p}) = u_1t^{p^2-p}\lambda_2$, by Lemma 6.6 above. This only leaves the other possibility, that $\hat\Gamma_1(\bar\epsilon_1)$ is represented by $\bar\epsilon_1$ in $\hat E^\infty_{**}(C_p, \ell/p)$. $\square$

We proceed to determine the differential structure in $\hat E^*(C_p, \ell/p)$, making use of the permanent cycles identified above.

Proposition 6.8. *The C_p-Tate spectral sequence in $V(1)$-homotopy for $THH(\ell/p)$ has*

$$\hat E^3_{**}(C_p, \ell/p) = E(u_1, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm1}, t\mu_2)\,.$$

It has differentials generated by

$$d^{2p^2-2p+2}(t^{p-p^2} \cdot t^{-i}\bar\epsilon_1) = t\mu_2 \cdot t^{-i}$$

for $0 < i < p$, $d^{2p^2}(t^{p-p^2}) = t^p\lambda_2$ and $d^{2p^2+1}(u_1t^{-p^2}) = t\mu_2$. The following terms are

$$\begin{aligned} \hat E^{2p^2-2p+3}_{**}(C_p, \ell/p) &= E(u_1, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p}) \\ &\quad \oplus E(u_1, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p}, t\mu_2) \\ \hat E^{2p^2+1}_{**}(C_p, \ell/p) &= E(u_1, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\quad \oplus E(u_1, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^2}, t\mu_2) \\ \hat E^{2p^2+2}_{**}(C_p, \ell/p) &= E(u_1, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\quad \oplus E(\bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^2})\,. \end{aligned}$$

The last term can be rewritten as

$$\hat E^\infty(C_p, \ell/p) = \big(E(u_1) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \oplus E(\bar\epsilon_1)\big) \otimes E(\lambda_2) \otimes P(t^{\pm p^2})\,.$$

Proof. We have already identified the E^2- and E^3-terms above. The E^3-term (6.5) is generated over $\hat{E}^3(C_p, \ell)$ by (an $\mathbb{F}_p$-basis for) $E(\bar{\epsilon}_1)$, so the next possible differential is induced by $d^{2p}(t^{1-p}) = t\lambda_1$. But multiplication by λ_1 is trivial in $V(1)_*THH(\ell/p)$, by Lemma 5.4, so $\hat{E}^3(C_p, \ell/p) = \hat{E}^{2p+1}(C_p, \ell/p)$. This term is generated over $\hat{E}^{2p+1}(C_p, \ell)$ by $P_p(t^{-1}) \otimes E(\bar{\epsilon}_1)$. Here $1, t^{-1}, \dots, t^{1-p}$ and $\bar{\epsilon}_1$ are permanent cycles, by Lemma 6.7. Any d^r-differential before d^{2p^2} must therefore originate on a class $t^{-i}\bar{\epsilon}_1$ for $0 < i < p$, and be of even length r, since these classes lie in even columns. For bidegree reasons, the first possibility is $r = 2p^2 - 2p + 2$, so $\hat{E}^3(C_p, \ell/p) = \hat{E}^{2p^2-2p+2}(C_p, \ell/p)$.

Multiplication by v_2 acts trivially on $V(1)_*THH(\ell)$ and $V(1)_*THH(\ell)^{tC_p}$ for degree reasons, and therefore also on $V(1)_*THH(\ell/p)$ and $V(1)_*THH(\ell/p)^{tC_p}$ by the module structure. The class v_2 maps to $t\mu_2$ in the S^1-Tate spectral sequence for ℓ, as recalled above, so multiplication by v_2 is represented by multiplication by $t\mu_2$ in the C_p-Tate spectral sequence for ℓ/p. Applied to the permanent cycles $(u_1 t^{-1})^\delta t^{-i}$ in degrees $\le (2p-2)$, this implies that the products

$$t\mu_2 \cdot (u_1 t^{-1})^\delta t^{-i}$$

must be infinite cycles representing zero, i.e., they must be hit by differentials. In the cases $\delta = 1$, $0 \le i \le p-2$, these classes in odd columns cannot be hit by differentials of odd length, such as d^{2p^2+1}, so the only possibility is

$$d^{2p^2-2p+2}(t^{p-p^2} \cdot (u_1 t^{-1}) t^{-i} \bar{\epsilon}_1) = t\mu_2 \cdot (u_1 t^{-1}) t^{-i}$$

for $0 \le i \le p-2$. By the module structure (consider multiplication by u_1) it follows that

$$d^{2p^2-2p+2}(t^{p-p^2} \cdot t^{-i}\bar{\epsilon}_1) = t\mu_2 \cdot t^{-i}$$

for $0 < i < p$. Hence we can compute from (6.5) that

$$\begin{aligned} \hat{E}^{2p^2-2p+3}_{**}(C_p, \ell/p) = {} & E(u_1) \otimes P(t^{\pm p}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\ & \oplus E(u_1) \otimes P(t^{\pm p}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2) \otimes P(t\mu_2)\,. \end{aligned}$$

This is generated over $\hat{E}^{2p+1}(C_p, \ell)$ by the permanent cycles $1, t^{-1}, \dots, t^{1-p}$ and $\bar{\epsilon}_1$, so the next differential is induced by $d^{2p^2}(t^{p-p^2}) = t^p\lambda_2$. This leaves

$$\begin{aligned} \hat{E}^{2p^2+1}_{**}(C_p, \ell/p) = {} & E(u_1) \otimes P(t^{\pm p^2}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\ & \oplus E(u_1) \otimes P(t^{\pm p^2}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2) \otimes P(t\mu_2)\,. \end{aligned}$$

Finally, $d^{2p^2+1}(u_1 t^{-p^2}) = t\mu_2$ applies, and leaves

$$\begin{aligned} \hat{E}^{2p^2+2}_{**}(C_p, \ell/p) = {} & E(u_1) \otimes P(t^{\pm p^2}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\ & \oplus P(t^{\pm p^2}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2)\,. \end{aligned}$$

For bidegree reasons, $\hat{E}^{2p^2+2} = \hat{E}^\infty$. $\square$

Proposition 6.9. *The comparison map $\hat{\Gamma}_1$ takes the classes $\epsilon_0^\delta \mu_0^i$, $\bar{\epsilon}_1$, λ_2 and μ_2 in $V(1)_* THH(\ell/p)$ to classes in $V(1)_* THH(\ell/p)^{tC_p}$ represented by $(u_1 t^{-1})^\delta t^{-i}$, $\bar{\epsilon}_1$, λ_2 and t^{-p^2} in $\hat{E}^\infty_{**}(C_p, \ell/p)$, respectively. Thus*

$$V(1)_* THH(\ell/p)^{tC_p} \cong \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1\} \otimes E(\lambda_2) \otimes P(\mu_2^{\pm 1})$$

and $\hat{\Gamma}_1$ factors as the algebraic localization map and identification

$$\hat{\Gamma}_1 \colon V(1)_* THH(\ell/p) \to V(1)_* THH(\ell/p)[\mu_2^{-1}] \cong V(1)_* THH(\ell/p)^{tC_p} .$$

In particular, this map is $(2p-2)$-coconnected.

Proof. The action of the map $\hat{\Gamma}_1$ on the classes $1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}$ and $\bar{\epsilon}_1$ was given in Lemma 6.7, and the action on the classes λ_2 and μ_2 was already recalled from [AuR02]. The structure of $V(1)_* THH(\ell/p)^{tC_p}$ is then immediate from the E^∞-term in Proposition 6.8. The top class not in the image of $\hat{\Gamma}_1$ is $\bar{\epsilon}_1 \lambda_2 \mu_2^{-1}$, in degree $(2p-2)$. □

Recall that

$$TF(B) = \operatorname*{holim}_{n,F} THH(B)^{C_{p^n}}$$
$$TR(B) = \operatorname*{holim}_{n,R} THH(B)^{C_{p^n}}$$

are defined as the homotopy limits over the Frobenius and the restriction maps $F, R \colon THH(B)^{C_{p^n}} \to THH(B)^{C_{p^{n-1}}}$, respectively.

Corollary 6.10. *The comparison maps*

$$\Gamma_n \colon THH(\ell/p)^{C_{p^n}} \to THH(\ell/p)^{hC_{p^n}}$$
$$\hat{\Gamma}_n \colon THH(\ell/p)^{C_{p^{n-1}}} \to THH(\ell/p)^{tC_{p^n}}$$

for $n \geq 1$, and

$$\Gamma \colon TF(\ell/p) \to THH(\ell/p)^{hS^1}$$
$$\hat{\Gamma} \colon TF(\ell/p) \to THH(\ell/p)^{tS^1}$$

all induce $(2p-2)$-coconnected maps on $V(1)$-homotopy.

Proof. This follows from a theorem of Tsalidis [Ts98] and Proposition 6.9 above, just like in [AuR02, 5.7]. See also [BBLR:cf]. □

7. Higher fixed points

Let $n \geq 1$. Write $v_p(i)$ for the p-adic valuation of i. Define a numerical function $\rho(-)$ by

$$\rho(2k-1) = (p^{2k+1}+1)/(p+1) = p^{2k} - p^{2k-1} + \cdots - p + 1$$
$$\rho(2k) = (p^{2k+2} - p^2)/(p^2-1) = p^{2k} + p^{2k-2} + \cdots + p^2$$

for $k \geq 0$, so $\rho(-1) = 1$ and $\rho(0) = 0$. For even arguments, $\rho(2k) = r(2k)$ as defined in [AuR02, 2.5].

In all of the following spectral sequences we know that λ_2, $t\mu_2$ and $\bar{\epsilon}_1$ are infinite cycles. For λ_2 and $\bar{\epsilon}_1$ this follows from the C_{p^n}-fixed point analogue of diagram (6.2), by [AuR02, 2.8] and Lemma 6.4. For $t\mu_2$ it follows from [AuR02, 4.8], by naturality.

Theorem 7.1. *The C_{p^n}-Tate spectral sequence $\hat{E}^*(C_{p^n}, \ell/p)$ in $V(1)$-homotopy for $THH(\ell/p)$ begins*

$$\hat{E}^2_{**}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1\} \otimes P(t^{\pm 1}, \mu_2)$$

and converges to $V(1)_*THH(\ell/p)^{tC_{p^n}}$. It is a module spectral sequence over the algebra spectral sequence $\hat{E}^*(C_{p^n}, \ell)$ converging to $V(1)_*THH(\ell)^{tC_{p^n}}$.

There is an initial d^2-differential generated by

$$d^2(\epsilon_0\mu_0^{i-1}) = t\mu_0^i$$

for $0 < i < p$. Next, there are $2n$ families of even length differentials generated by

$$d^{2\rho(2k-1)}(t^{p^{2k-1}-p^{2k}+i} \cdot \bar{\epsilon}_1) = (t\mu_2)^{\rho(2k-3)} \cdot t^i$$

for $v_p(i) = 2k-2$, for each $k = 1, \dots, n$, and

$$d^{2\rho(2k)}(t^{p^{2k-1}-p^{2k}}) = \lambda_2 \cdot t^{p^{2k-1}} \cdot (t\mu_2)^{\rho(2k-2)}$$

for each $k = 1, \dots, n$. Finally, there is a differential of odd length generated by

$$d^{2\rho(2n)+1}(u_n \cdot t^{-p^{2n}}) = (t\mu_2)^{\rho(2n-2)+1} .$$

We shall prove Theorem 7.1 by induction on n. The base case $n = 1$ is covered by Proposition 6.8. We can therefore assume that Theorem 7.1 holds for some fixed $n \geq 1$. First we make the following deduction.

Corollary 7.2. *The initial differential in the C_{p^n}-Tate spectral sequence in $V(1)$-homotopy for $THH(\ell/p)$ leaves*

$$\hat{E}^3_{**}(C_{p^n}, \ell/p) = E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 1}, t\mu_2) .$$

The next $2n$ families of differentials leave the intermediate terms

$$\begin{aligned} \hat{E}^{2\rho(1)+1}_{**}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p}) \\ \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p}, t\mu_2) \end{aligned}$$

(for $m = 1$),

$$\begin{aligned} \hat{E}^{2\rho(2m-1)+1}_{**}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ \oplus \bigoplus_{k=2}^{m} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ \oplus \bigoplus_{k=2}^{m-1} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j\lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2) \end{aligned}$$

for $m = 2, \dots, n$*, and*

$$\begin{aligned}\hat{E}^{2\rho(2m)+1}_{**}(C_{p^n}, \ell/p) = {} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ & \oplus \bigoplus_{k=2}^{m} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ & \oplus \bigoplus_{k=2}^{m} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ & \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m}}, t\mu_2)\end{aligned}$$

for $m = 1, \dots, n$*. The final differential leaves the* $E^{2\rho(2n)+2} = E^\infty$*-term, equal to*

$$\begin{aligned}\hat{E}^{\infty}_{**}(C_{p^n}, \ell/p) = {} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ & \oplus \bigoplus_{k=2}^{n} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ & \oplus \bigoplus_{k=2}^{n} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ & \oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}) \otimes P_{\rho(2n-2)+1}(t\mu_2)\,.\end{aligned}$$

Proof. The statements about the E^3-, $E^{2\rho(1)+1}$- and $E^{2\rho(2)+1}$-terms are clear from Proposition 6.8. For each $m = 2, \dots, n$ we proceed by a secondary induction. The differential

$$d^{2\rho(2m-1)}(t^{p^{2m-1}-p^{2m}+i} \cdot \bar{\epsilon}_1) = (t\mu_2)^{\rho(2m-3)} \cdot t^i$$

for $v_p(i) = 2m-2$ is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-2}}, t\mu_2)$$

of the $E^{2\rho(2m-2)+1} = E^{2\rho(2m-1)}$-term, with homology

$$\begin{aligned} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2m-2\} \otimes P_{\rho(2m-3)}(t\mu_2) \\ & \quad \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2)\,.\end{aligned}$$

This gives the stated $E^{2\rho(2m-1)+1}$-term. Similarly, the differential

$$d^{2\rho(2m)}(t^{p^{2m-1}-p^{2m}}) = \lambda_2 \cdot t^{p^{2m-1}} \cdot (t\mu_2)^{\rho(2m-2)}$$

is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2)$$

of the $E^{2\rho(2m-1)+1} = E^{2\rho(2m)}$-term, with homology

$$\begin{aligned} & E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid v_p(j) = 2m-1\} \otimes P_{\rho(2m-2)}(t\mu_2) \\ & \quad \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m}}, t\mu_2)\,.\end{aligned}$$

This gives the stated $E^{2\rho(2m)+1}$-term. The final differential

$$d^{2\rho(2n)+1}(u_n \cdot t^{-p^{2n}}) = (t\mu_2)^{\rho(2n-2)+1}$$

is non-trivial only on the summand

$$E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)$$

of the $E^{2\rho(2n)+1}$-term, with homology

$$E(\bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^{2n}}) \otimes P_{\rho(2n-2)+1}(t\mu_2)\,.$$

This gives the stated $E^{2\rho(2n)+2}$-term. At this stage there is no room for any further differentials, since the spectral sequence is concentrated in a narrower horizontal band than the vertical height of the following differentials. $\square$

Next we compare the C_{p^n}-Tate spectral sequence with the C_{p^n}-homotopy spectral sequence obtained by restricting the E^2-term to the second quadrant ($s \le 0$, $t \ge 0$). It is algebraically easier to handle the latter after inverting μ_2, which can be interpreted as comparing $THH(\ell/p)$ with its C_p-Tate construction.

In general, there is a commutative diagram

$$\begin{tikzcd}
THH(B)^{C_{p^n}} \arrow[r, "R"] \arrow[d, "\Gamma_n"] & THH(B)^{C_{p^{n-1}}} \arrow[r, "\Gamma_{n-1}"] \arrow[d, "\hat\Gamma_n"] & THH(B)^{hC_{p^{n-1}}} \arrow[d, "\hat\Gamma_1^{hC_{p^{n-1}}}"] \\
THH(B)^{hC_{p^n}} \arrow[r, "R^h"] & THH(B)^{tC_{p^n}} \arrow[r, "G_{n-1}"] & (THH(B)^{tC_p})^{hC_{p^{n-1}}}
\end{tikzcd} \tag{7.3}$$

where G_{n-1} is the comparison map from the $C_{p^{n-1}}$-fixed points to the $C_{p^{n-1}}$-homotopy fixed points of $THH(B)^{tC_p}$, in view of the identification

$$(THH(B)^{tC_p})^{C_{p^{n-1}}} = THH(B)^{tC_{p^n}}\,.$$

We are of course considering the case $B = \ell/p$. In $V(1)$-homotopy all four maps with labels containing Γ are $(2p-2)$-coconnected, by Corollary 6.10, so G_{n-1} is at least $(2p-1)$-coconnected. (We shall see in Lemma 7.11 that $V(1)_*G_{n-1}$ is an isomorphism in all degrees.) By Proposition 6.9 the map $\hat\Gamma_1$ precisely inverts μ_2, so the E_2-term of the C_{p^n}-homotopy fixed point spectral sequence in $V(1)$-homotopy for $THH(\ell/p)^{tC_p}$ is obtained by inverting μ_2 in $E^2_{**}(C_{p^n}, \ell/p)$. We denote it by $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$, even though in later terms only a power of μ_2 is present.

Theorem 7.4. *The C_{p^n}-homotopy fixed point spectral sequence $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$ in $V(1)$-homotopy for $THH(\ell/p)^{tC_p}$ begins*

$$\mu_2^{-1}E^2_{**}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar\epsilon_1\} \otimes P(t, \mu_2^{\pm1})$$

and converges to $V(1)_*(THH(\ell/p)^{tC_p})^{hC_{p^n}}$, which receives a $(2p-2)$-coconnected map $(\hat\Gamma_1)^{hC_{p^n}}$ from $V(1)_*THH(\ell/p)^{hC_{p^n}}$. There is an initial d^2-differential generated by

$$d^2(\epsilon_0\mu_0^{i-1}) = t\mu_0^i$$

for $0 < i < p$. Next, there are $2n$ families of even length differentials generated by

$$d^{2\rho(2k-1)}(\mu_2^{p^{2k}-p^{2k-1}+j} \cdot \bar{\epsilon}_1) = (t\mu_2)^{\rho(2k-1)} \cdot \mu_2^j$$

for $v_p(j) = 2k-2$, for each $k = 1, \dots, n$, and

$$d^{2\rho(2k)}(\mu_2^{p^{2k}-p^{2k-1}}) = \lambda_2 \cdot \mu_2^{-p^{2k-1}} \cdot (t\mu_2)^{\rho(2k)}$$

for each $k = 1, \dots, n$. Finally, there is a differential of odd length generated by

$$d^{2\rho(2n)+1}(u_n \cdot \mu_2^{p^{2n}}) = (t\mu_2)^{\rho(2n)+1} .$$

Proof. The differential pattern follows from Theorem 7.1 by naturality with respect to the maps of spectral sequences

$$\mu_2^{-1}E^*(C_{p^n}, \ell/p) \xleftarrow{\hat{\Gamma}_1^{hC_{p^n}}} E^*(C_{p^n}, \ell/p) \xrightarrow{R^h} \hat{E}^*(C_{p^n}, \ell/p)$$

induced by $\hat{\Gamma}_1^{hC_{p^n}}$ and R^h. The first inverts μ_2 and the second inverts t, at the level of E^2-terms. We are also using that $t\mu_2$, the image of v_2, multiplies as an infinite cycle in all of these spectral sequences. □

Corollary 7.5. *The initial differential in the C_{p^n}-homotopy fixed point spectral sequence in $V(1)$-homotopy for $THH(\ell/p)^{tC_p}$ leaves*

$$\begin{aligned}\mu_2^{-1}E^3_{**}(C_{p^n}, \ell/p) = {} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ & \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm 1}, t\mu_2) .\end{aligned}$$

The next $2n$ families of differentials leave the intermediate terms

$$\begin{aligned}\mu_2^{-1}E^{2\rho(2m-1)+1}_{**}(C_{p^n}, \ell/p) = {} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ & \oplus \bigoplus_{k=1}^{m} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\ & \oplus \bigoplus_{k=1}^{m-1} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^j \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\ & \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-1}}, t\mu_2)\end{aligned}$$

and

$$\begin{aligned}\mu_2^{-1}E^{2\rho(2m)+1}_{**}(C_{p^n}, \ell/p) = {} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ & \oplus \bigoplus_{k=1}^{m} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\ & \oplus \bigoplus_{k=1}^{m} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^j \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\ & \oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m}}, t\mu_2)\end{aligned}$$

for $m = 1, \dots, n$. The final differential leaves the $E^{2\rho(2n)+2} = E^\infty$-term, equal to

$$\begin{aligned}
\mu_2^{-1} E^\infty_{**}(C_{p^n}, \ell/p) = {} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\
& \oplus \bigoplus_{k=1}^{n} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\
& \oplus \bigoplus_{k=1}^{n} E(u_n, \bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\
& \oplus E(\bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}) \otimes P_{\rho(2n)+1}(t\mu_2) \,.
\end{aligned}$$

Proof. The computation of the E^3-term from the E^2-term is straightforward. The rest of the proof goes by a secondary induction on $m = 1, \dots, n$, very much like the proof of Corollary 7.2. The differential

$$d^{2\rho(2m-1)}(\mu_2^{p^{2m}-p^{2m-1}+j} \cdot \bar\epsilon_1) = (t\mu_2)^{\rho(2m-1)} \cdot \mu_2^j$$

for $v_p(j) = 2m-2$ is non-trivial only on the summand

$$E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-2}}, t\mu_2)$$

of the $E^3 = E^{2\rho(1)}$-term (for $m = 1$), resp. the $E^{2\rho(2m-2)+1} = E^{2\rho(2m-1)}$-term (for $m = 2, \dots, n$). Its homology is

$$\begin{aligned}
& E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2m-2\} \otimes P_{\rho(2m-1)}(t\mu_2) \\
& \quad \oplus E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-1}}, t\mu_2) \,,
\end{aligned}$$

which gives the stated $E^{2\rho(2m-1)+1}$-term. The differential

$$d^{2\rho(2m)}(\mu_2^{p^{2m}-p^{2m-1}}) = \lambda_2 \cdot \mu_2^{-p^{2m-1}} \cdot (t\mu_2)^{\rho(2m)}$$

is non-trivial only on the summand

$$E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-1}}, t\mu_2)$$

of the $E^{2\rho(2m-1)+1} = E^{2\rho(2m)}$-term, leaving

$$\begin{aligned}
& E(u_n, \bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid v_p(j) = 2m-1\} \otimes P_{\rho(2m)}(t\mu_2) \\
& \quad \oplus E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m}}, t\mu_2) \,.
\end{aligned}$$

This gives the stated $E^{2\rho(2m)+1}$-term. The final differential

$$d^{2\rho(2n)+1}(u_n \cdot \mu_2^{p^{2n}}) = (t\mu_2)^{\rho(2n)+1}$$

is non-trivial only on the summand

$$E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}, t\mu_2)$$

of the $E^{2\rho(2n)+1}$-term, with homology

$$E(\bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}) \otimes P_{\rho(2n)+1}(t\mu_2)\,.$$

This gives the stated $E^{2\rho(2n)+2}$-term. There is no room for any further differentials, since the spectral sequence is concentrated in a narrower vertical band than the horizontal width of the following differentials, so $E^{2\rho(2n)+2} = E^\infty$. □

Proof of Theorem 7.1. To make the inductive step to $C_{p^{n+1}}$, we use that the first d^r-differential of odd length in $\hat E^*(C_{p^n}, \ell/p)$ occurs for $r = r_0 = 2\rho(2n) + 1$. It follows from [AuR02, 5.2] that the terms $\hat E^r(C_{p^n}, \ell/p)$ and $\hat E^r(C_{p^{n+1}}, \ell/p)$ are isomorphic for $r \le 2\rho(2n) + 1$, via the Frobenius map (taking t^i to t^i) in even columns and the Verschiebung map (taking $u_n t^i$ to $u_{n+1} t^i$) in odd columns. Furthermore, the differential $d^{2\rho(2n)+1}$ is zero in the latter spectral sequence. This proves the part of Theorem 7.1 for $n + 1$ that concerns the differentials leading up to the term

$$\begin{aligned}\hat E^{2\rho(2n)+2}(C_{p^{n+1}}, \ell/p) &= E(u_{n+1}, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2})\\ &\oplus \bigoplus_{k=2}^{n} E(u_{n+1}, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2)\\ &\oplus \bigoplus_{k=2}^{n} E(u_{n+1}, \bar\epsilon_1) \otimes \mathbb{F}_p\{t^j\lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2)\\ &\oplus E(u_{n+1}, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)\,.\end{aligned} \tag{7.6}$$

Next we use the following commutative diagram, where we abbreviate $THH(B)$ to $T(B)$:

$$\begin{tikzcd}(T(B)^{tC_p})^{hC_{p^n}} \arrow[d, "F"] & T(B)^{hC_{p^n}} \arrow[l, "\hat\Gamma_1^{hC_{p^n}}"'] \arrow[d, "F"] & T(B)^{C_{p^n}} \arrow[l, "\Gamma_n"'] \arrow[r, "\hat\Gamma_{n+1}"] \arrow[d, "F"] & T(B)^{tC_{p^{n+1}}} \arrow[d, "F"] \\ T(B)^{tC_p} & T(B) \arrow[l, "\hat\Gamma_1"'] \arrow[r, equal] & T(B) \arrow[r, "\hat\Gamma_1"] & T(B)^{tC_p}\end{tikzcd} \tag{7.7}$$

The horizontal maps all induce $(2p-2)$-coconnected maps in $V(1)$-homotopy for $B = \ell/p$. Here F is the Frobenius map, forgetting part of the equivariance. Thus the map $\hat\Gamma_{n+1}$ to the right induces an isomorphism of $E(\lambda_2) \otimes P(v_2)$-modules in all degrees $> (2p-2)$ from $V(1)_*THH(\ell/p)^{C_{p^n}}$, implicitly identified to the left with the abutment of $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$, to $V(1)_*THH(\ell/p)^{tC_{p^{n+1}}}$, which is the abutment of $\hat E^*(C_{p^{n+1}}, \ell/p)$. The diagram above ensures that the isomorphism induced by $\hat\Gamma_{n+1}$ is compatible with the one induced by $\hat\Gamma_1$. By Proposition 6.9 it takes $\bar\epsilon_1$, λ_2 and μ_2 to $\bar\epsilon_1$, λ_2 and t^{-p^2}, respectively, and similarly for monomials in these classes.

We focus on the summand

$$E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2n-2\} \otimes P_{\rho(2n-1)}(t\mu_2)$$

in $\mu_2^{-1}E^\infty_{**}(C_{p^n}, \ell/p)$, abutting to $V(1)_*THH(\ell/p)^{C_{p^n}}$ in degrees $> (2p-2)$. In the $P(v_2)$-module structure on the abutment, each class μ_2^j with $v_p(j) = 2n-2$, $j > 0$, generates a copy of $P_{\rho(2n-1)}(v_2)$, since there are no permanent cycles in the same total degree as $y = (t\mu_2)^{\rho(2n-1)} \cdot \mu_2^j$ that have lower (= more negative) homotopy fixed point filtration. See Lemma 7.8 below for the elementary verification. The $P(v_2)$-module isomorphism induced by $\hat{\Gamma}_{n+1}$ must take this to a copy of $P_{\rho(2n-1)}(v_2)$ in $V(1)_* THH(\ell/p)^{tC_{p^{n+1}}}$, generated by $t^{-p^2 j}$.

Writing $i = -p^2 j$, we deduce that for $v_p(i) = 2n$, $i < 0$, the infinite cycle $z = (t\mu_2)^{\rho(2n-1)} \cdot t^i$ must represent zero in the abutment, and must therefore be hit by a differential $z = d^r(x)$ in the $C_{p^{n+1}}$-Tate spectral sequence. Here $r \geq 2\rho(2n)+2$.

Since z generates a free copy of $P(t\mu_2)$ in the $E^{2\rho(2n)+2}$-term displayed in (7.6), and d^r is $P(t\mu_2)$-linear, the class x cannot be annihilated by any power of $t\mu_2$. This means that x must be contained in the summand

$$E(u_{n+1}, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)$$

of $\hat{E}^{2\rho(2n)+2}_{**}(C_{p^{n+1}}, \ell/p)$. By an elementary check of bidegrees, see Lemma 7.9 below, the only possibility is that x has vertical degree $(2p-1)$, so that we have differentials

$$d^{2\rho(2n+1)}(t^{p^{2n+1}-p^{2n+2}+i} \cdot \bar{\epsilon}_1) = (t\mu_2)^{\rho(2n-1)} \cdot t^i$$

for all $i < 0$ with $v_p(i) = 2n$. The cases $i > 0$ follow by the module structure over the $C_{p^{n+1}}$-Tate spectral sequence for ℓ. The remaining two differentials,

$$d^{2\rho(2n+2)}(t^{p^{2n+1}-p^{2n+2}}) = \lambda_2 \cdot t^{p^{2n+1}} \cdot (t\mu_2)^{\rho(2n)}$$

and

$$d^{2\rho(2n+2)+1}(u_{n+1} \cdot t^{-p^{2n+2}}) = (t\mu_2)^{\rho(2n)+1}$$

are also present in the $C_{p^{n+1}}$-Tate spectral sequence for ℓ, see [AuR02, 6.1], hence follow in the present case by the module structure. With this we have established the complete differential pattern asserted by Theorem 7.1. □

Lemma 7.8. *For $v_p(j) = 2n-2$, $n \geq 1$, there are no classes in $\mu_2^{-1} E^\infty_{**}(C_{p^n}, \ell/p)$ in the same total degree as $y = (t\mu_2)^{\rho(2n-1)} \cdot \mu_2^j$ that have lower homotopy fixed point filtration.*

Proof. The total degree of y is $2(p^{2n+2} - p^{2n+1} + p - 1) + 2p^2 j \equiv (2p-2) \mod 2p^{2n}$, which is even.

Looking at the formula for $\mu_2^{-1} E^\infty_{**}(C_{p^n}, \ell/p)$ in Corollary 7.5, the classes of lower filtration than y all lie in the terms

$$E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^i \mid v_p(i) = 2n-1\} \otimes P_{\rho(2n)}(t\mu_2)$$

and

$$E(\bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}) \otimes P_{\rho(2n)+1}(t\mu_2) .$$

Those in even total degree and of lower filtration than y are

$$u_n \lambda_2 \cdot \mu_2^i (t\mu_2)^e, \quad \bar{\epsilon}_1 \lambda_2 \cdot \mu_2^i (t\mu_2)^e$$

with $v_p(i) = 2n-1$, $\rho(2n-1) < e < \rho(2n)$, and

$$\mu_2^i (t\mu_2)^e, \quad \bar{\epsilon}_1 \lambda_2 \cdot \mu_2^i (t\mu_2)^e$$

with $v_p(i) \ge 2n$, $\rho(2n-1) < e \le \rho(2n)$.

The total degree of $u_n\lambda_2 \cdot \mu_2^i(t\mu_2)^e$ for $v_p(i) = 2n-1$ is $(-1)+(2p^2-1)+2p^2 i + (2p^2-2)e \equiv (2p^2-2)(e+1) \mod 2p^{2n}$. For this to agree with the total degree of y, we must have $(2p-2) \equiv (2p^2-2)(e+1) \mod 2p^{2n}$, so $(e+1) \equiv 1/(1+p) \mod p^{2n}$ and $e \equiv \rho(2n-1) - 1 \mod p^{2n}$. There is no such e with $\rho(2n-1) < e < \rho(2n)$.

The total degree of $\bar\epsilon_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$ for $v_p(i) = 2n-1$ is $(2p-1)+(2p^2-1)+2p^2 i + (2p^2-2)e \equiv 2p + (2p^2-2)(e+1) \mod 2p^{2n}$. To agree with that of y, we must have $(2p-2) \equiv 2p + (2p^2-2)(e+1) \mod 2p^{2n}$, so $(e+1) \equiv 1/(1-p^2) \mod p^{2n}$ and $e \equiv \rho(2n) \mod p^{2n}$. There is no such e with $\rho(2n-1) < e < \rho(2n)$.

The total degree of $\mu_2^i(t\mu_2)^e$ for $v_p(i) \ge 2n$ is $2p^2 i + (2p^2-2)e \equiv (2p^2-2)e \mod 2p^{2n}$. To agree with that of y, we must have $(2p-2) \equiv (2p^2-2)e \mod 2p^{2n}$, so $e \equiv 1/(1+p) \equiv \rho(2n-1) \mod p^{2n}$. There is no such e with $\rho(2n-1) < e \le \rho(2n)$.

The total degree of $\bar\epsilon_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$ for $v_p(i) \ge 2n$ is $(2p-1)+(2p^2-1)+2p^2 i + (2p^2-2)e$. To agree modulo $2p^{2n}$ with that of y, we must have $e \equiv \rho(2n) \mod p^{2n}$. The only such e with $\rho(2n-1) < e \le \rho(2n)$ is $e = \rho(2n)$. But in that case, the total degree of $\bar\epsilon_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$ is $2p + 2p^2 i + (2p^2-2)(\rho(2n)+1) = 2(p^{2n+2}+p-1) + 2p^2 i$. To be equal to that of y, we must have $2p^2 i + 2p^{2n+1} = 2p^2 j$, which is impossible for $v_p(i) \ge 2n$ and $v_p(j) = 2n-2$. $\square$

Lemma 7.9. *For $v_p(i) = 2n$, $n \ge 1$ and $z = (t\mu_2)^{\rho(2n-1)} \cdot t^i$, the only class in*

$$E(u_{n+1}, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)$$

that can support a differential $d^r(x) = z$ for $r \ge 2\rho(2n)+2$ is (a unit times)

$$x = t^{p^{2n+1}-p^{2n+2}+i} \cdot \bar\epsilon_1 \,.$$

Proof. The class z has total degree $(2p^2-2)\rho(2n-1) - 2i = 2p^{2n+2} - 2p^{2n+1} + 2p - 2 - 2i \equiv (2p-2) \mod 2p^{2n}$, which is even, and vertical degree $2p^2\rho(2n-1)$. Hence x has odd total degree, and vertical degree at most $2p^2\rho(2n-1) - 2\rho(2n) - 1 = 2p^{2n+2} - 2p^{2n+1} - \dots - 2p^3 - 1$. This leaves the possibilities

$$u_{n+1} \cdot t^j(t\mu_2)^e, \quad \bar\epsilon_1 \cdot t^j(t\mu_2)^e, \quad \lambda_2 \cdot t^j(t\mu_2)^e$$

with $v_p(j) \ge 2n$ and $0 \le e < p^{2n} - p^{2n-1} - \dots - p = \rho(2n-1) - \rho(2n-2) - 1$, and

$$u_{n+1}\bar\epsilon_1\lambda_2 \cdot t^j(t\mu_2)^e$$

with $v_p(j) \ge 2n$ and $0 \le e < p^{2n} - p^{2n-1} - \dots - p - 1 = \rho(2n-1) - \rho(2n-2) - 2$.

The total degree of x must be one more than the total degree of z, hence is congruent to $(2p-1)$ modulo $2p^{2n}$.

The total degree of $u_{n+1} \cdot t^j(t\mu_2)^e$ is $-1 - 2j + (2p^2-2)e \equiv -1 + (2p^2-2)e \mod 2p^{2n}$. To have $(2p-1) \equiv -1 + (2p^2-2)e \mod 2p^{2n}$ we must have $e \equiv -p/(1-p^2) \equiv p^{2n} - p^{2n-1} - \dots - p \mod p^{2n}$, which does not happen for e in the allowable range.

The total degree of $\lambda_2 \cdot t^j(t\mu_2)^e$ is $(2p^2-1) - 2j + (2p^2-2)e \equiv (2p^2-1) + (2p^2-2)e \mod 2p^{2n}$. To have $(2p-1) \equiv (2p^2-1) + (2p^2-2)e \mod 2p^{2n}$ we must have $e \equiv -p/(1+p) \equiv \rho(2n-1) - 1 \mod p^{2n}$, which does not happen.

The total degree of $u_{n+1}\bar\epsilon_1\lambda_2 \cdot t^j(t\mu_2)^e$ is $-1 + (2p-1) + (2p^2-1) - 2j + (2p^2-2)e \equiv (2p-1) + (2p^2-2)(e+1) \mod 2p^{2n}$. To have $(2p-1) \equiv (2p-1) + (2p^2-2)(e+1)$

mod $2p^{2n}$ we must have $(e+1) \equiv 0 \mod p^{2n}$, so $e \equiv p^{2n}-1 \mod p^{2n}$, which does not happen.

The total degree of $\bar\epsilon_1 \cdot t^j (t\mu_2)^e$ is $(2p-1)-2j+(2p^2-2)e \equiv (2p-1)+(2p^2-2)e \mod 2p^{2n}$. To have $(2p-1) \equiv (2p-1)+(2p^2-2)e \mod 2p^{2n}$, we must have $e \equiv 0 \mod p^{2n}$, so $e=0$ is the only possibility in the allowable range. In that case, a check of total degrees shows that we must have $j = p^{2n+1}-p^{2n+2}+i$. □

Corollary 7.10. *$V(1)_* THH(\ell/p)^{C_{p^n}}$ is finite in each degree.*

Proof. This is clear by inspection of the E^∞-term in Corollary 7.2. □

Lemma 7.11. *The map G_n induces an isomorphism*

$$V(1)_* THH(\ell/p)^{tC_{p^{n+1}}} \xrightarrow{\cong} V(1)_*(THH(\ell/p)^{tC_p})^{hC_{p^n}}$$

in all degrees. In the limit over the Frobenius maps F, there is a map G inducing an isomorphism

$$V(1)_* THH(\ell/p)^{tS^1} \xrightarrow{\cong} V(1)_*(THH(\ell/p)^{tC_p})^{hS^1} .$$

Proof. As remarked after diagram (7.3), G_n induces an isomorphism in $V(1)$-homotopy above degree $(2p-2)$. The permanent cycle $t^{-p^{2n+2}}$ in $\hat E^\infty_{**}(C_{p^{n+1}}, \ell)$ acts invertibly on $\hat E^\infty_{**}(C_{p^{n+1}}, \ell/p)$, and its image $G_n(t^{-p^{2n+2}}) = \mu_2^{p^{2n}}$ in $\mu_2^{-1} E^\infty_{**}(C_{p^n}, \ell)$ acts invertibly on $\mu_2^{-1} E^\infty_{**}(C_{p^n}, \ell/p)$. Therefore the module action derived from the ℓ-algebra structure on ℓ/p ensures that G_n induces isomorphisms in $V(1)$-homotopy in all degrees. □

Theorem 7.12. *(a) The associated graded of $V(1)_* THH(\ell/p)^{tS^1}$ for the S^1-Tate spectral sequence is*

$$\begin{aligned}
\hat E^\infty_{**}(S^1, \ell/p) = {} & E(\lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\
& \oplus \bigoplus_{k \ge 2} E(\lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\
& \oplus \bigoplus_{k \ge 2} E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\
& \oplus E(\bar\epsilon_1, \lambda_2) \otimes P(t\mu_2) .
\end{aligned}$$

(b) The associated graded of $V(1)_* THH(\ell/p)^{hS^1}$ for the S^1-homotopy fixed point spectral sequence maps by a $(2p-2)$-coconnected map to

$$\begin{aligned}
\mu_2^{-1} E^\infty_{**}(S^1, \ell/p) = {} & E(\lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\
& \oplus \bigoplus_{k \ge 1} E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\
& \oplus \bigoplus_{k \ge 1} E(\bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\
& \oplus E(\bar\epsilon_1, \lambda_2) \otimes P(t\mu_2) .
\end{aligned}$$

(c) The isomorphism from (a) to (b) induced by G takes t^{-i} to μ_0^i for $0 < i < p$ and t^i to μ_2^j for $i + p^2 j = 0$. Furthermore, it takes multiples by $\bar\epsilon_1$, λ_2 or $t\mu_2$ in the source to the same multiples in the target.

Proof. Claims (a) and (b) follow by passage to the limit over n from Corollaries 7.2 and 7.5. Claim (c) follows by passage to the same limit from the formulas for the isomorphism induced by $\hat\Gamma_{n+1}$, which were given below diagram (7.7). □

8. Topological cyclic homology

By definition, there is a fiber sequence

$$TC(B) \xrightarrow{\pi} TF(B) \xrightarrow{R-1} TF(B)$$

inducing a long exact sequence

$$\tag{8.1} \dots \xrightarrow{\partial} V(1)_*TC(B) \xrightarrow{\pi} V(1)_*TF(B) \xrightarrow{R-1} V(1)_*TF(B) \xrightarrow{\partial} \dots$$

in $V(1)$-homotopy. By Corollary 6.10, there are $(2p-2)$-coconnected maps Γ and $\hat{\Gamma}$ from $V(1)_*TF(\ell/p)$ to $V(1)_*THH(\ell/p)^{hS^1}$ and $V(1)_*THH(\ell/p)^{tS^1}$, respectively. We model $V(1)_*TF(\ell/p)$ in degrees $> (2p-2)$ by the map $\hat{\Gamma}$ to the S^1-Tate construction. Then, by diagram (7.3), R is modeled in the same range of degrees by the chain of maps below.

$$\begin{tikzcd}
V(1)_*THH(B)^{tS^1} \arrow[dr, "G"] & V(1)_*THH(B)^{hS^1} \arrow[r, "R^h"] \arrow[d, "(\hat{\Gamma}_1)^{hS^1}"] & V(1)_*THH(B)^{tS^1} \\
& V(1)_*(THH(B)^{tC_p})^{hS^1} &
\end{tikzcd}$$

Here R^h induces a map of spectral sequences

$$E^*(R^h)\colon E^*(S^1, B) \to \hat{E}^*(S^1, B)\,,$$

which at the E^2-term equals the inclusion that algebraically inverts t. When $B = \ell/p$, the left hand map G is an isomorphism by Lemma 7.11, and the middle (wrong-way) map is $(2p-2)$-coconnected.

Proposition 8.2. *In degrees $> (2p-2)$, the homomorphism*

$$E^\infty(R^h)\colon E^\infty(S^1, \ell/p) \to \hat{E}^\infty(S^1, \ell/p)$$

maps

(a) $E(\bar{\epsilon}_1, \lambda_2) \otimes P(t\mu_2)$ identically to the same expression;

(b) $E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^{-j}\} \otimes P_{\rho(2k-1)}(t\mu_2)$ surjectively onto

$$E(\lambda_2) \otimes \mathbb{F}_p\{t^j\} \otimes P_{\rho(2k-3)}(t\mu_2)$$

for each $k \geq 2$, $j = dp^{2k-2}$, $0 < d < p^2 - p$ and $p \nmid d$;

(c) $E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^{-j}\} \otimes P_{\rho(2k)}(t\mu_2)$ surjectively onto

$$E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j\lambda_2\} \otimes P_{\rho(2k-2)}(t\mu_2)$$

for each $k \geq 2$, $j = dp^{2k-1}$ and $0 < d < p$;

(d) the remaining terms to zero.

Proof. Consider the summands of $E^\infty(S^1, \ell/p)$ and $\hat{E}^\infty(S^1, \ell/p)$, as given in Theorem 7.12. Clearly, the first term $E(\lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2)$ goes to zero (these classes are hit by d^2-differentials), and the last term $E(\bar\epsilon_1, \lambda_2) \otimes P(t\mu_2)$ maps identically to the same term. This proves (a) and part of (d).

For each $k \geq 1$ and $j = dp^{2k-2}$ with $p \nmid d$, the term $E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^{-j}\} \otimes P_{\rho(2k-1)}(t\mu_2)$ maps to the term $E(\lambda_2) \otimes \mathbb{F}_p\{t^j\} \otimes P_{\rho(2k-3)}(t\mu_2)$, except that the target is zero for $k = 1$. In symbols, the element $\lambda_2^\delta \mu_2^{-j}(t\mu_2)^i$ maps to the element $\lambda_2^\delta t^j (t\mu_2)^{i-j}$. If $d < 0$, then the t-exponent in the target is bounded above by $dp^{2k-2} + \rho(2k-3) < 0$, so the target lives in the right half-plane and is essentially not hit by the source, which lives in the left half-plane. If $d > p^2 - p$, then the total degree in the source is bounded above by $(2p^2 - 1) - 2dp^{2k} + \rho(2k-1)(2p^2 - 2) < 2p - 2$, so the source lives in total degree $< (2p - 2)$ and will be disregarded. If $0 < d < p^2 - p$, then $\rho(2k-1) - dp^{2k-2} > \rho(2k-3)$ and $-dp^{2k-2} < 0$, so the source surjects onto the target. This proves (b) and part of (d).

Lastly, for each $k \geq 1$ and $j = dp^{2k-1}$ with $p \nmid d$, the term $E(\bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^{-j}\} \otimes P_{\rho(2k)}(t\mu_2)$ maps to the term $E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^j \lambda_2\} \otimes P_{\rho(2k-2)}(t\mu_2)$. The target is zero for $k = 1$. If $d < 0$, then $dp^{2k-1} + \rho(2k-2) < 0$ so the target lives in the right half-plane. If $d > p$, then $(2p-1) + (2p^2 - 1) - 2dp^{2k+1} + \rho(2k)(2p^2 - 2) < 2p - 2$, so the source lives in total degree $< (2p-2)$. If $0 < d < p$, then $\rho(2k) - dp^{2k-1} > \rho(2k-2)$ and $-dp^{2k-1} < 0$, so the source surjects onto the target. This proves (c) and the remaining part of (d). $\square$

Definition 8.3. Let

$$\begin{aligned} A &= E(\bar\epsilon_1, \lambda_2) \otimes P(t\mu_2) \\ B_k &= E(\lambda_2) \otimes \mathbb{F}_p\{t^{dp^{2k-2}} \mid 0 < d < p^2 - p, p \nmid d\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ C_k &= E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp^{2k-1}} \lambda_2 \mid 0 < d < p\} \otimes P_{\rho(2k-2)}(t\mu_2) \end{aligned}$$

for $k \geq 2$ and let D be the span of the remaining monomials in $\hat{E}^\infty(S^1, \ell/p)$. Let $B = \bigoplus_{k\geq 2} B_k$ and $C = \bigoplus_{k \geq 2} C_k$. Then $\hat{E}^\infty(S^1, \ell/p) = A \oplus B \oplus C \oplus D$.

Proposition 8.4. *In degrees $> (2p-2)$, there are closed subgroups $\widetilde{A} = E(\bar\epsilon_1, \lambda_2) \otimes P(v_2)$, $\widetilde{B}_k$, $\widetilde{C}_k$ and $\widetilde{D}$ in $V(1)_*TF(\ell/p)$, represented by A, B_k, C_k and D in $\hat{E}^\infty(S^1, \ell/p)$, respectively, such that the homomorphism induced by the restriction map R*

(a) is the identity on $\widetilde{A}$;
(b) maps $\widetilde{B}_{k+1}$ surjectively onto $\widetilde{B}_k$ for all $k \geq 2$;
(c) maps $\widetilde{C}_{k+1}$ surjectively onto $\widetilde{C}_k$ for all $k \geq 2$;
(d) is zero on $\widetilde{B}_2$, $\widetilde{C}_2$ and $\widetilde{D}$.

In these degrees, $V(1)_*TF(\ell/p) \cong \widetilde{A} \oplus \widetilde{B} \oplus \widetilde{C} \oplus \widetilde{D}$, where $\widetilde{B} = \prod_{k\geq 2} \widetilde{B}_k$ and $\widetilde{C} = \prod_{k \geq 2} \widetilde{C}_k$.

Proof. In terms of the model $THH(\ell/p)^{tS^1}$ for $TF(\ell/p)$, the restriction map R is given in these degrees as the composite of the isomorphism G, computed in Theorem 7.12(c), and the map $\hat{E}^\infty(R^h)$, computed in Proposition 8.2. This gives the desired formulas at the level of E^∞-terms. The rest of the argument is the same as that for Theorem 7.7 of [AuR02], using Corollary 7.10 to control the topologies, and will be omitted. $\square$

Remark 8.5. Here we have followed the basic computational strategy of [BM94], [BM95] and [AuR02]. It would be desirable to have a more concrete construction of the lifts $\widetilde{B}_k$, $\widetilde{C}_k$ and $\widetilde{D}$, in terms of de Rham–Witt operators R, F, V and $d=\sigma$, like in the algebraic case of [HM97] and [HM03].

Proposition 8.6. *In degrees $> (2p-2)$ there are isomorphisms*

$$\begin{aligned}\ker(R-1) &\cong \widetilde{A} \oplus \lim_k \widetilde{B}_k \oplus \lim_k \widetilde{C}_k \\ &\cong E(\bar{\epsilon}_1, \lambda_2) \otimes P(v_2) \\ &\qquad \oplus E(\lambda_2) \otimes \mathbb{F}_p\{t^d \mid 0 < d < p^2 - p, p \nmid d\} \otimes P(v_2) \\ &\qquad \oplus E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \otimes P(v_2)\end{aligned}$$

and $\operatorname{cok}(R-1) \cong \widetilde{A} = E(\bar{\epsilon}_1, \lambda_2) \otimes P(v_2)$. *Hence there is an isomorphism*

$$\begin{aligned}V(1)_* TC(\ell/p) &\cong E(\partial, \bar{\epsilon}_1, \lambda_2) \otimes P(v_2) \\ &\qquad \oplus E(\lambda_2) \otimes \mathbb{F}_p\{t^d \mid 0 < d < p^2 - p, p \nmid d\} \otimes P(v_2) \\ &\qquad \oplus E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \otimes P(v_2)\end{aligned}$$

in these degrees, where ∂ has degree -1 and represents the image of 1 *under the connecting map ∂ in* (8.1).

Proof. By Proposition 8.4, the homomorphism $R-1$ is zero on $\widetilde{A}$ and an isomorphism on $\widetilde{D}$. Furthermore, there is an exact sequence

$$0 \to \lim_k \widetilde{B}_k \to \prod_{k\ge 2} \widetilde{B}_k \xrightarrow{R-1} \prod_{k \ge 2} \widetilde{B}_k \to \lim_k{}^1 \widetilde{B}_k \to 0$$

and similarly for the C's. The derived limit on the right vanishes since each $\widetilde{B}_{k+1}$ surjects onto $\widetilde{B}_k$.

Multiplication by $t\mu_2$ in each B_k is realized by multiplication by v_2 in $\widetilde{B}_k$. Each $\widetilde{B}_k$ is a sum of $2(p-1)^2$ cyclic $P(v_2)$-modules, and since $\rho(2k-3)$ grows to infinity with k their limit is a free $P(v_2)$-module of the same rank, with the indicated generators t^d and $t^d\lambda_2$ for $0 < d < p^2 - p$, $p \nmid d$. The argument for the C's is practically the same.

The long exact sequence (8.1) yields the short exact sequence

$$0 \to \Sigma^{-1} \operatorname{cok}(R-1) \xrightarrow{\partial} V(1)_* TC(\ell/p) \xrightarrow{\pi} \ker(R-1) \to 0\,,$$

from which the formula for the middle term follows. □

Remark 8.7. A more obvious set of $E(\lambda_2) \otimes P(v_2)$-module generators for $\lim_k \widetilde{B}_k$ would be the classes t^{dp^2} in $B_2 \cong \widetilde{B}_2$, for $0 < d < p^2 - p$, $p \nmid d$. Under the canonical map $TF(\ell/p) \to THH(\ell/p)^{C_p}$, modeled here by $THH(\ell/p)^{tS^1} \to (THH(\ell/p)^{tC_p})^{hC_p}$, these map to the classes μ_2^{-d}. Since we are only concerned with degrees $> (2p-2)$ we may equally well use their v_2-power multiplies $(t\mu_2)^d \cdot \mu_2^{-d} = t^d$ as generators, with the advantage that these are in the image of the localization map $THH(\ell/p)^{hC_p} \to (THH(\ell/p)^{tC_p})^{hC_p}$. Hence the class denoted t^d in $\lim_k \widetilde{B}_k$ is chosen so as to map under $TF(\ell/p) \to THH(\ell/p)^{hC_p}$ to t^d in $E^\infty_{**}(C_p; \ell/p)$. Similarly, the class denoted $t^{dp}\lambda_2$ in $\lim_k \widetilde{C}_k$ is chosen so as to map to $t^{dp}\lambda_2$ in $E^\infty_{**}(C_p; \ell/p)$.

The map $\pi\colon \ell/p \to \mathbb{Z}/p$ is $(2p-2)$-connected, hence induces $(2p-1)$-connected maps $\pi^*\colon K(\ell/p) \to K(\mathbb{Z}/p)$ and $\pi^*\colon V(1)_*TC(\ell/p) \to V(1)_*TC(\mathbb{Z}/p)$, by [BM94, 10.9] and [Du97]. Here $TC(\mathbb{Z}/p) \simeq H\mathbb{Z}_p \vee \Sigma^{-1} H\mathbb{Z}_p$ and $V(1)_*TC(\mathbb{Z}/p) \cong E(\partial, \bar{\epsilon}_1)$, so we can recover $V(1)_*TC(\ell/p)$ in degrees $\le (2p-2)$ from this map.

Theorem 8.8. *There is an isomorphism of $E(\lambda_1, \lambda_2) \otimes P(v_2)$-modules*

$$\begin{aligned} V(1)_* TC(\ell/p) \cong P(v_2) \otimes E(\partial, \bar\epsilon_1, \lambda_2) & \\ \oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} & \\ \oplus P(v_2) \otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} & \end{aligned}$$

where $v_2 \cdot \operatorname{dlog} v_1 = \lambda_2$. The degrees are $|\partial| = -1$, $|\bar\epsilon_1| = |\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$ and $|v_2| = 2p^2 - 2$. The formal multipliers have degrees $|t| = -2$ and $|\operatorname{dlog} v_1| = 1$.

The notation $\operatorname{dlog} v_1$ for the multiplier $v_2^{-1}\lambda_2$ is suggested by the relation $v_1 \cdot \operatorname{dlog} p = \lambda_1$ in $V(0)_* TC(\mathbb{Z}_{(p)}|\mathbb{Q})$.

Proof. Only the additive generators t^d for $0 < d < p^2 - p$, $p \nmid d$ from Proposition 8.6 do not appear in $V(1)_* TC(\ell/p)$, but their multiples by λ_2 and positive powers of v_2 do. This leads to the given formula, where $\operatorname{dlog} v_1 \cdot t^d v_2$ must be read as $t^d \lambda_2$. $\square$

By [HM97] the cyclotomic trace map of [BHM93] induces cofiber sequences

$$K(B_p)_p \xrightarrow{trc} TC(B)_p \xrightarrow{g} \Sigma^{-1} H\mathbb{Z}_p \tag{8.9}$$

for each connective S-algebra B with $\pi_0(B_p) = \mathbb{Z}_p$ or $\mathbb{Z}/p$, and thus long exact sequences

$$\cdots \to V(1)_* K(B_p) \xrightarrow{trc} V(1)_* TC(B) \xrightarrow{g} \Sigma^{-1} E(\bar\epsilon_1) \to \dots .$$

This uses the identifications $W(\mathbb{Z}_p)_F \cong W(\mathbb{Z}/p)_F \cong \mathbb{Z}_p$ of Frobenius coinvariants of Witt rings, and applies in particular for $B = H\mathbb{Z}_{(p)}$, $H\mathbb{Z}/p$, ℓ and ℓ/p.

Theorem 8.10. *There is an isomorphism of $E(\lambda_1, \lambda_2) \otimes P(v_2)$-modules*

$$\begin{aligned} V(1)_* K(\ell/p) \cong P(v_2) \otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{1, \partial\lambda_2, \lambda_2, \partial v_2\} & \\ \oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} & \\ \oplus P(v_2) \otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} . & \end{aligned}$$

This is a free $P(v_2)$-module of rank $(2p^2 - 2p + 8)$ and of zero Euler characteristic, where $p \geq 5$ is assumed.

Proof. In the case $B = \mathbb{Z}/p$, $K(\mathbb{Z}/p)_p \simeq H\mathbb{Z}_p$ and the map g is split surjective up to homotopy. So the induced homomorphism to $V(1)_* \Sigma^{-1} H\mathbb{Z}_p = \Sigma^{-1} E(\bar\epsilon_1)$ is surjective. Since $\pi \colon \ell/p \to \mathbb{Z}/p$ induces a $(2p-1)$-connected map in topological cyclic homology, and $\Sigma^{-1} E(\bar\epsilon_1)$ is concentrated in degrees $\leq (2p-2)$, it follows by naturality that also in the case $B = \ell/p$ the map g induces a surjection in $V(1)$-homotopy. The kernel of the surjection $P(v_2) \otimes E(\partial, \bar\epsilon_1, \lambda_2) \to \Sigma^{-1} E(\bar\epsilon_1)$ gives the first row in the asserted formula. $\square$

9. The fraction field

We wish to determine the effect on algebraic K-theory of excising the closed subspaces $\operatorname{Spec}(\mathbb{Z}_p)$ and $\operatorname{Spnc}(\ell/p)$ from $\operatorname{Spec}(\ell_p)$, meeting at the closed point

$\mathrm{Spec}(\mathbb{Z}/p)$. It will be a little cleaner to express the computation in topological cyclic homology. Therefore we consider another 3×3 diagram of cofiber sequences

(9.1)
$$\begin{tikzcd}
TC(\mathbb{Z}/p) \arrow[r, "i_*"] \arrow[d, "\pi_*"] & TC(\mathbb{Z}_{(p)}) \arrow[r, "j^*"] \arrow[d, "\pi_*"] & TC(\mathbb{Z}_{(p)}|\mathbb{Q}) \arrow[d, "\pi_*"] \\
TC(\ell/p) \arrow[r, "i_*"] \arrow[d, "\rho^*"] & TC(\ell) \arrow[r, "j^*"] \arrow[d, "\rho^*"] & TC(\ell|p^{-1}\ell) \arrow[d, "\rho^*"] \\
TC(\ell/p|L/p) \arrow[r, "i_*"] & TC(\ell|L) \arrow[r, "j^*"] & TC(\mathit{ff}(\ell)) ,
\end{tikzcd}$$

generated as usual by the upper left hand square. Like in the THH-case displayed in diagram (5.5), the upper horizontal transfer map i_* was defined in [HM03], and the remaining transfer maps can be defined using [BM:loc].

In the top row, the transfer map i_* from

$$V(1)_*TC(\mathbb{Z}/p) = E(\partial, \bar{\epsilon}_1)$$

to

$$V(1)_*TC(\mathbb{Z}_{(p)}) = E(\partial, \lambda_1) \oplus \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\}$$

is a module map over the target. We claim that $i_*(1) = 0$ and $i_*(\bar{\epsilon}_1) = \lambda_1$, up to a unit in $\mathbb{F}_p$, so

$$V(1)_*TC(\mathbb{Z}_{(p)}|\mathbb{Q}) = E(\partial, \mathrm{dlog}\, p) \oplus \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} ,$$

where the connecting map from (9.1) takes the degree 1 class $\mathrm{dlog}\, p$ to 1. To prove the claim, recall from [HM03] that $v_1 \cdot \mathrm{dlog}\, p = \lambda_1$ in the $V(0)$-homotopy of $TC(\mathbb{Z}_{(p)}|\mathbb{Q})$. Hence λ_1 in $V(1)_*TC(\mathbb{Z}_{(p)})$ maps to zero under j^*, and must be in the image of i_*. The only class that can hit it is $\bar{\epsilon}_1$, up to a unit.

In the middle row, the transfer map i_* from

$$\begin{aligned} V(1)_*TC(\ell/p) = {} & P(v_2) \otimes E(\partial, \bar{\epsilon}_1, \lambda_2) \\ & \oplus P(v_2) \otimes E(\mathrm{dlog}\, v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ & \oplus P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \end{aligned}$$

(computed in Theorem 8.8), to

$$\begin{aligned} V(1)_*TC(\ell) = {} & P(v_2) \otimes E(\partial, \lambda_1, \lambda_2) \\ & \oplus P(v_2) \otimes E(\lambda_2) \otimes \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} \\ & \oplus P(v_2) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \end{aligned}$$

(computed in [AuR02, 8.4]), is also a module map over the target. By naturality in the diagram

$$\begin{tikzcd}
TC(\ell/p) \arrow[r, "i_*"] \arrow[d, "\pi^*"] & TC(\ell) \arrow[r, "j^*"] \arrow[d, "\pi^*"] & TC(\ell|p^{-1}\ell) \arrow[d, "\pi^*"] \\
TC(\mathbb{Z}/p) \arrow[r, "i_*"] & TC(\mathbb{Z}_{(p)}) \arrow[r, "j^*"] & TC(\mathbb{Z}_{(p)}|\mathbb{Q})
\end{tikzcd}$$

we find that $i_*(1) = 0$ and $i_*(\bar\epsilon_1) = \lambda_1$ in $V(1)_*TC(\ell)$, since the vertical maps π^* are all $(2p-1)$-connected.

To be precise, $i_*(\bar\epsilon_1)$ will be a unit times λ_1 plus a multiple of $t^{p^2-p}\lambda_2$, but λ_1 was only defined modulo such indeterminacy in [AuR02, 1.3], and this is a good occasion to make a more definite choice. In p-adic homotopy, $\pi^* \colon \pi_{2p-1}TC(\ell)_p \to \pi_{2p-1}TC(\mathbb{Z}_{(p)})_p \cong \mathbb{Z}_p\{\lambda_1\}$ is surjective with kernel isomorphic to $\pi_{2p-2}(\ell)_p \cong \mathbb{Z}_p$, so

$$\pi_{2p-1}TC(\ell)_p \to V(1)_{2p-1}TC(\ell) = \mathbb{F}_p\{\lambda_1, t^{(p-1)p}\lambda_2\}$$

is surjective, and this ensures that we may, indeed, choose $\lambda_1^K \in K_{2p-1}(\ell_p)_p$ so that its cyclotomic trace image $\lambda_1 \in \pi_{2p-1}TC(\ell)_p$ reduces mod p and v_1 to a unit times $i_*(\bar\epsilon_1)$.

Furthermore, $i_*(t^d v_2) = 0$ for $0 < d < p^2 - p$, $p \nmid d$, since the target is zero in these degrees, namely the even degrees strictly between $2p-2$ and $2p^2-2$ that are not congruent to $-2 \mod 2p$. Similarly, $i_*(t^d v_2 \operatorname{dlog} v_1) = 0$ for the same d. Therefore the cofiber of i_* is

$$\begin{aligned} V(1)_*TC(\ell|p^{-1}\ell) = {} & P(v_2) \otimes E(\partial, \operatorname{dlog} p, \lambda_2) \\ & \oplus P(v_2) \otimes E(\lambda_2) \otimes \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} \\ & \oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \operatorname{dlog} p \mid 0 < d < p^2 - p, p \nmid d\} \\ & \oplus P(v_2) \otimes E(\operatorname{dlog} p) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \end{aligned}$$

where the connecting map from (9.1) takes $\operatorname{dlog} p$ to 1.

In the middle column the transfer map $\pi_* \colon V(1)_*TC(\mathbb{Z}_{(p)}) \to V(1)_*TC(\ell)$ is zero. For the natural map π^* is $(2p-1)$-connected and thus surjective, and the composite $\pi_* \circ \pi^*$ is multiplication by the cyclotomic trace class of $H\mathbb{Z}_{(p)}$ in $\pi_0 TC(\ell)$, realized as the mapping cone of $v_1 \colon \Sigma^{2p-2}\ell \to \ell$, which represents zero already in $\pi_0 K(\ell)$.

We introduce a degree 1 class $\operatorname{dlog} v_1$ in the cofiber $V(1)_*TC(\ell|L)$, and suppose that this notation is compatible with that of Theorem 8.8, in the sense that $v_2 \cdot \operatorname{dlog} v_1 = \lambda_2$ as elements of $V(1)_*TC(\ell|L)$, not just formally as in $V(1)_*TC(\ell/p)$. This is again to be expected by analogy with the case of $V(0)_*TC(\mathbb{Z}_{(p)}|\mathbb{Q})$. Then the terms $E(\partial, \lambda_1) \otimes \mathbb{F}_p\{\lambda_2\}$ and $\mathbb{F}_p\{t^d\lambda_1\lambda_2 \mid 0 < d < p\}$ in $V(1)_*TC(\ell)$ become once divisible by v_2 in the cofiber, which we can write as

$$\begin{aligned} V(1)_*TC(\ell|L) = {} & P(v_2) \otimes E(\partial, \lambda_1, \operatorname{dlog} v_1) \\ & \oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} \\ & \oplus P(v_2) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{t^{dp} v_2 \operatorname{dlog} v_1 \mid 0 < d < p\} \,. \end{aligned}$$

In the left hand column, a very similar argument gives $\pi_* = 0$ and

$$\begin{aligned} V(1)_*TC(\ell/p|L/p) = {} & P(v_2) \otimes E(\partial, \bar\epsilon_1, \operatorname{dlog} v_1) \\ & \oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ & \oplus P(v_2) \otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp} v_2 \operatorname{dlog} v_1 \mid 0 < d < p\} \,. \end{aligned}$$

We are most interested in the right hand column, where again π^* is surjective and $\pi_* \circ \pi^* = 0$, so $\pi_* = 0$.

Proposition 9.2. *There is an exact sequence*

$$0 \to V(1)_*TC(\ell|p^{-1}\ell) \xrightarrow{\rho^*} V(1)_*TC(ff(\ell)) \to V(1)_{*-1}TC(\mathbb{Z}_{(p)}|\mathbb{Q}) \to 0\,.$$

If $v_2 \cdot \operatorname{dlog} v_1 = \lambda_2$, where the (right hand) connecting map takes $\operatorname{dlog} v_1$ to 1, then there is an isomorphism

$$\begin{aligned} V(1)_*TC(ff(\ell)) = P(v_2) &\otimes E(\partial, \operatorname{dlog} p, \operatorname{dlog} v_1) \\ &\oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} \\ &\oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \operatorname{dlog} p \mid 0 < d < p^2 - p, p \nmid d\} \\ &\oplus P(v_2) \otimes E(\operatorname{dlog} p) \otimes \mathbb{F}_p\{t^{dp} v_2 \operatorname{dlog} v_1 \mid 0 < d < p\}\,. \end{aligned}$$

This is a free module over $P(v_2)$ of rank $(2p^2+6)$ and of zero Euler characteristic.

Proof. The assumption in the second clause is that the terms $E(\partial, \operatorname{dlog} p) \otimes \mathbb{F}_p\{\lambda_2\}$ and $\mathbb{F}_p\{t^d\lambda_1\lambda_2 \mid 0 < d < p\}$ in $V(1)_*TC(\ell|p^{-1}\ell)$ become once divisible by v_2 in $V(1)_*TC(ff(\ell))$, and the formula then follows from those for $V(1)_*TC(\mathbb{Z}_{(p)}|\mathbb{Q})$ and $V(1)_*TC(\ell|p^{-1}\ell)$. $\square$

Remark 9.3. We expect to verify the relation $v_2 \cdot \operatorname{dlog} v_1 = \lambda_2$ in $V(1)_*TC(\ell|L)$ and $V(1)_*TC(ff(\ell))$ by means of a logarithmic geometric model for $TC(\ell|L)$, to be discussed in [Rog:ltc].

Theorem 9.4. *There is an exact sequence*

$$\begin{aligned} 0 \to \Sigma^{-2}E(\operatorname{dlog} p, \operatorname{dlog} v_1) \otimes \mathbb{F}_p\{\bar\epsilon_1\} \to V(1)_*K(ff(\ell_p)) &\xrightarrow{trc} \\ \xrightarrow{trc} V(1)_*TC(ff(\ell)) \to \Sigma^{-1}E(\operatorname{dlog} p, \operatorname{dlog} v_1) &\to 0\,. \end{aligned}$$

Thus, if $v_2 \cdot \operatorname{dlog} v_1 = \lambda_2$, then there is an isomorphism of $P(v_2)$-modules

$$V(1)_*K(ff(\ell_p)) \equiv P(v_2) \otimes \Lambda_*$$

modulo the kernel $\Sigma^{-2}E(\operatorname{dlog} p, \operatorname{dlog} v_1) \otimes \mathbb{F}_p\{\bar\epsilon_1\}$ of trc, where

$$\begin{aligned} \Lambda_* = E(&\partial v_2, \operatorname{dlog} p, \operatorname{dlog} v_1) \\ &\oplus E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d\lambda_1 \mid 0 < d < p\} \\ &\oplus E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \operatorname{dlog} p \mid 0 < d < p^2 - p, p \nmid d\} \\ &\oplus E(\operatorname{dlog} p) \otimes \mathbb{F}_p\{t^{dp} v_2 \operatorname{dlog} v_1 \mid 0 < d < p\}\,. \end{aligned}$$

See Figure 10.3 below for a chart of Λ_*.

Proof. To pass to algebraic K-theory, we consider the map from diagram (3.10) to diagram (9.1) induced by the cyclotomic trace map in (8.9). The cofiber is the 3×3 diagram of cofiber sequences with a copy of $\Sigma^{-1}H\mathbb{Z}_p$ in each corner of the upper left hand square, and $V(1)_*\Sigma^{-1}H\mathbb{Z}_p = \Sigma^{-1}E(\bar\epsilon_1)$. The transfer maps i_* and π_* in this diagram are all trivial, since the natural maps i^* induce the isomorphisms

$W(\mathbb{Z}_p)_F \cong W(\mathbb{Z}/p)_F$ and the maps π^* induce identity maps. So there is a long exact sequence

$$\cdots \to V(1)_* K(ff(\ell_p)) \xrightarrow{trc} V(1)_* TC(ff(\ell)) \xrightarrow{g} \Sigma^{-1} E(\bar{\epsilon}_1, \operatorname{dlog} p, \operatorname{dlog} v_1) \to \dots .$$

We claim that the image of g equals $\Sigma^{-1} E(\operatorname{dlog} p, \operatorname{dlog} v_1)$. It is clear that the term $E(\operatorname{dlog} p, \operatorname{dlog} v_1) \otimes \mathbb{F}_p\{\partial\}$ in $V(1)_* TC(ff(\ell))$ has this image. We must argue that $\Sigma^{-1} E(\operatorname{dlog} p, \operatorname{dlog} v_1) \otimes \mathbb{F}_p\{\bar{\epsilon}_1\}$ is not in the image of g.

In degree $(2p-2)$ the class $\Sigma^{-1}\bar{\epsilon}_1$ could only be hit by $t\lambda_1 \cdot \operatorname{dlog} v_1$, but $t\lambda_1$ is the image of α_1 in $\pi_{2p-3}(S)$ and g takes $\operatorname{dlog} v_1$ to zero, so g also takes $t\lambda_1 \cdot \operatorname{dlog} v_1$ to zero, since it is a map of S-modules. In degree $(2p-1)$ the class $t^{p^2-p} v_2 \operatorname{dlog} v_1$ comes from $t^{p^2-p}\lambda_2$ in $V(1)_* TC(\ell)$, so by naturality it cannot map to $\Sigma^{-1}\bar{\epsilon}_1 \operatorname{dlog} p$ or $\Sigma^{-1}\bar{\epsilon}_1 \operatorname{dlog} v_1$. In degree $2p$ the class $t^{p^2-p} v_2 \operatorname{dlog} p \operatorname{dlog} v_1$ comes from $t^{p^2-p}\lambda_2 \operatorname{dlog} p$ in $V(1)_* TC(\ell | p^{-1}\ell)$, so it cannot hit $\Sigma^{-1}\bar{\epsilon}_1 \operatorname{dlog} p \operatorname{dlog} v_1$. This exhausts all possibilities, and concludes the proof. □

The low-degree v_2-divisibility assumptions needed for Proposition 9.2 and Theorem 9.4 become irrelevant upon inverting v_2, since the $V(1)$-homotopy of both $TC(\mathbb{Z}_{(p)}|\mathbb{Q})$ and $\Sigma^{-1} H\mathbb{Z}_p$ is v_2-torsion. Hence we have the following unconditional result, for primes $p \geq 5$.

Theorem 9.5. *In v_2-periodic homotopy there are isomorphisms of $P(v_2^{\pm 1})$-modules*

$$\begin{aligned} V(1)_* TC(\ell | p^{-1}\ell)[v_2^{-1}] &\cong V(1)_* TC(ff(\ell))[v_2^{-1}] \cong V(1)_* K(p^{-1}\ell_p)[v_2^{-1}] \\ &\cong V(1)_* K(ff(\ell_p))[v_2^{-1}] \cong P(v_2^{\pm 1}) \otimes \Lambda_* , \end{aligned}$$

where Λ_* is as in Theorem 9.4.

10. Galois cohomology

For motivation, let $F \to E$ be a G-Galois extension of local or global number fields, i.e., finite extensions of $\mathbb{Q}$ or $\mathbb{Q}_p$ for some prime p. There is an induced G-action on $K(E)$ and a natural map $K(F) \to K(E)^{hG}$. In their simplest form, the Lichtenbaum–Quillen conjectures predict that this map induces an isomorphism in mod p homotopy, in sufficiently high degrees. In other words, the abutment of the homotopy fixed point spectral sequence

$$E^2_{s,t} = H^{-s}_{gp}(G; V(0)_t K(E)) \Longrightarrow V(0)_{s+t} K(E)^{hG}$$

is conjectured to agree with $V(0)_* K(F)$ in sufficiently high degrees. Thomason [Th85, 0.1, 4.1] proved these conjectures for algebraic K-theory with the Bott element inverted (in much greater generality than the present one), or equivalently, for the v_1-periodic algebraic K-theory functor $V(0)_* K(-)[v_1^{-1}]$. Hence there is a spectral sequence

$$E^2_{s,t} = H^{-s}_{gp}(G; V(0)_t K(E)[v_1^{-1}]) \Longrightarrow V(0)_{s+t} K(F)[v_1^{-1}]$$

for each Galois extension E of F, with $G = \operatorname{Gal}(E/F)$. Passing to the colimit over all such Galois extensions contained in a separable closure $\bar{F}$ of F still gives a spectral sequence

$$E^2_{s,t} = H^{-s}_{cont}(G_F; V(0)_t K(\bar{F})[v_1^{-1}]) \Longrightarrow V(0)_{s+t} K(F)[v_1^{-1}] .$$

Here $H^r_{cont}(G_F; M)$ equals the continuous group cohomology of the absolute Galois group $G_F = \mathrm{Gal}(\bar{F}/F)$, for any discrete G_F-module M, which is also denoted $H^r_{Gal}(F; M)$ and known as Galois cohomology. By Suslin's theorem [Su84], $K(\bar{F})_p \simeq ku_p$, so $V(0)_t K(\bar{F})[v_1^{-1}] \cong V(0)_t KU = \mathbb{F}_p(t/2)$ equals the G_F-module $\mu_p^{\otimes(t/2)}$ for t even, and 0 for t odd. Here $\mathbb{F}_p(1) = V(0)_2 K(\bar{F}) \cong \mu_p(\bar{F}) = \mu_p$ is the group of p-th roots of unity in $\bar{F}$, considered as the p-torsion in $K_1(\bar{F}) = \bar{F}^\times$. Thus the spectral sequence above can be rewritten as the Galois descent spectral sequence

$$E^2_{s,t} = H^{-s}_{Gal}(F; \mathbb{F}_p(t/2)) \Longrightarrow V(0)_{s+t} K(F)[v_1^{-1}]\,. \tag{10.1}$$

We will conjecture below that there are similar results for G-Galois extensions $f\!f(A) \to f\!f(B)$ of the fraction fields of a class of $K(1)$-local commutative S-algebras, containing L_p and KU_p, when considering the v_2-periodic algebraic K-theory functor $V(1)_* K(-)[v_2^{-1}]$. Recall from (3.5) and (3.6) that this functor does not distinguish between the fraction fields of L and L_p, or between the fraction fields of KU, $KU_{(p)}$ or KU_p, since it vanishes on finite, global and local fields in the ordinary sense. Note that we here shift our focus from connective spectra like ℓ and ℓ_p, which are convenient for topological cyclic homology calculations, to $K(1)$-local spectra like L_p, which are convenient for Galois theory. As discussed in Remark 2.6, $f\!f(\ell_p) = f\!f(L_p)$ and $f\!f(ku_p) = f\!f(KU_p)$, so the shift is only one of perspective.

More precisely, we might consider a $K(1)$-local pro-Galois extension $L_{K(1)}S \to C$ of commutative S-algebras in the sense of [Rog08, 4.1], with profinite Galois group H, and a pair $K \subset N$ of closed subgroups of H, with K normal in N. Letting $A = C^{hN}$, $B = C^{hK}$ and $G = N/K$ we obtain a diagram

$$J_p = L_{K(1)}S \to A \xrightarrow{G} B \to C\,.$$

Here J_p is the p-complete image-of-J spectrum, which plays an analogous initial role in the $K(1)$-local category as $H\mathbb{Q} = L_0 S$ does in the rational category. There is a $K(1)$-local pro-Γ-Galois extension $J_p \to L_p$, where $\Gamma = 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^\times$ acts on L_p through p-adic Adams operations, and $\mathrm{Spec}(J_p) = [\mathrm{Spec}(L_p)/\Gamma]$ is essentially the derived orbit stack for the corresponding Γ-action on the derived scheme $\mathrm{Spec}(L_p)$. Similar remarks apply in a non-commutative sense to $J/p \to L/p$, so $\mathrm{Spec}(f\!f(J_p))$ may be interpreted as the derived orbit stack for Γ acting on $\mathrm{Spec}(f\!f(L_p))$. In both cases, some care in interpretation is needed, since Γ is a profinite group.

For such a $K(1)$-local G-Galois extension $A \to B$ as above, we expect that the map $K(A) \to K(B)^{hG}$ induces an isomorphism in $V(1)$-homotopy, in sufficiently high degrees, so that there is a weak equivalence

$$V(1)_* K(A)[v_2^{-1}] \xrightarrow{\simeq} V(1)_* K(B)^{hG}[v_2^{-1}]$$

and a homotopy fixed point spectral sequence

$$E^2_{s,t} = H^{-s}_{gp}(G; V(1)_t K(B)[v_2^{-1}]) \Longrightarrow V(1)_{s+t} K(A)[v_2^{-1}]\,.$$

In the special case of the Δ-Galois extension $L_p \to KU_p$, this conjecture is somewhat trivially compatible with the predicted formula (3.8), since $H^r_{gp}(\Delta; P(b^{\pm 1}))$ equals $P(v_2^{\pm 1})$ for $r = 0$ and is 0 otherwise.

Considering A and B as valuation rings in their fraction fields, when defined, these extensions correspond to unramified Galois extensions of the fraction fields. It is difficult to use the obstruction theories of [GH04] or [Rob03] to construct ramified extensions in commutative S-algebras, since the ramification creates highly nontrivial obstruction groups of André–Quillen type. In fact, there are only a few unramified Galois extensions of L_p and KU_p. By [BR08, 1.1, 7.9], the extended Lubin–Tate spectrum KU_p^{nr}, with $\pi_* KU_p^{nr} = \mathbb{W}(\bar{\mathbb{F}}_p)[u^{\pm 1}]$, is the maximal pro-Galois extension of KU_p, both as commutative S-algebras and as $K(1)$-local commutative S-algebras.

More generally, we suppose that there is a notion of $K(1)$-local Galois extensions in an extended framework that contains the fraction fields of commutative S-algebras discussed in Section 3, but which also allows for ramified Galois extensions. Then let Ω_1 be a maximal connected $K(1)$-local pro-Galois extension of $\mathit{ff}(L_p)$, or equivalently of $\mathit{ff}(J_p)$. In other words, Ω_1 is to be a separable closure of the fraction field of L_p (or J_p). Then for each intermediate such $K(1)$-local G-Galois extension

$$\mathit{ff}(J_p) = \mathit{ff}(L_{K(1)}S) \to \mathit{ff}(A) \xrightarrow{G} \mathit{ff}(B) \to \Omega_1$$

(where $\mathit{ff}(B)$ or $K(\mathit{ff}(B))$ might only exist in the extended framework), we expect to have a weak equivalence

$$V(1)_* K(\mathit{ff}(A))[v_2^{-1}] \xrightarrow{\simeq} V(1)_* K(\mathit{ff}(B))^{hG}[v_2^{-1}]$$

and a homotopy fixed point spectral sequence

$$E^2_{s,t} = H^{-s}_{gp}(G; V(1)_t K(\mathit{ff}(B))[v_2^{-1}]) \Longrightarrow V(1)_{s+t} K(\mathit{ff}(A))[v_2^{-1}]\,,$$

as above. Fixing $\mathit{ff}(A)$ and passing to the colimit over all such Galois extensions contained in Ω_1, we are then led to a spectral sequence

$$E^2_{s,t} = H^{-s}_{cont}(G_{\mathit{ff}(A)}; V(1)_t K(\Omega_1)[v_2^{-1}]) \Longrightarrow V(1)_{s+t} K(\mathit{ff}(A))[v_2^{-1}]\,,$$

where $G_{\mathit{ff}(A)} = \mathrm{Gal}(\Omega_1/\mathit{ff}(A))$ is the absolute Galois group, defined as the limit of the Galois groups G above.

In the case $A = L_p$ for $p \geq 5$, we achieved a concrete calculation of the abutment of this spectral sequence in Theorem 9.5, by methods completely independent of the conjectural Galois descent properties. We are therefore entitled to make an educated guess at the structure of the E^2-term of this spectral sequence, which in turn makes strong suggestions about the form of the Galois module $V(1)_* K(\Omega_1)[v_2^{-1}]$, and about the absolute Galois group $G_{\mathit{ff}(L_p)}$.

We therefore study the spectral sequence

$$E^2_{s,t} = H^{-s}_{cont}(G_{\mathit{ff}(L_p)}; V(1)_t K(\Omega_1)[v_2^{-1}]) \Longrightarrow V(1)_{s+t} K(\mathit{ff}(L_p))[v_2^{-1}]\,, \tag{10.2}$$

which we assume to be an algebra spectral sequence, and one that collapses at the E^2-term for $p \geq 5$, as is the case for the Galois descent spectral sequence (10.1) for any local number field, as well as for any global number field when $p \geq 3$. The abutment $V(1)_* K(\mathit{ff}(L_p))[v_2^{-1}]$ is isomorphic to $P(v_2^{\pm 1})$ tensored with Λ_*, as given in Theorem 9.4. Hence the $E^2 = E^\infty$-term of (10.2) is the associated graded for a "Galois" filtration of this $P(v_2^{\pm 1})$-module. Since v_2 is invertible, it must be represented in Galois filtration $s = 0$. By consideration of localization sequences in motivic cohomology, and the relation of motivic cohomology to Galois cohomology, we can partially justify why the other algebra generators ∂v_2, $\operatorname{dlog} p$, $\operatorname{dlog} v_1$, $t^d\lambda_1$, $t^d v_2 \operatorname{dlog} p$ and $t^{dp} v_2 \operatorname{dlog} v_1$ must all be represented in Galois filtration $s = -1$. It follows that the entire spectral sequence is concentrated in the four columns $-3 \le s \le 0$, with the product $\partial v_2 \cdot \operatorname{dlog} p \cdot \operatorname{dlog} v_1$ as the only generator in filtration $s = -3$.

We display the placement in the E^2-term of the $P(v_2^{\pm 1})$-module generators for $V(1)_* K(\mathit{ff}(L_p))[v_2^{-1}]$, or equivalently the $\mathbb{F}_p$-basis for Λ_*, in Figure 10.3 below. To make room on the page, we write $\frac{dp}{p}$ and $\frac{dv_1}{v_1}$ for $\operatorname{dlog} p$ and $\operatorname{dlog} v_1$, respectively. Multiple generators in the same bidegree are separated by colons. The symbol (v_2) indicates the placement of this v_2-multiple of 1. Also to save space, the chart is drawn for $p = 3$, even if we are really assuming $p \ge 5$.

$\partial v_2 \frac{dp}{p}\frac{dv_1}{v_1}$	$\cdot$	$\cdot$	$\cdot$	$2p^2+2$
$\cdot$	$\partial v_2 \frac{dp}{p} : \partial v_2 \frac{dv_1}{v_1} : t v_2 \frac{dp}{p}\frac{dv_1}{v_1}$	$\cdot$	$\cdot$	$2p^2$
$\cdot$	$t^2 v_2 \frac{dp}{p}\frac{dv_1}{v_1}$	$\partial v_2 : t v_2 \frac{dp}{p}$	(v_2)	$2p^2-2$
$\cdot$	$t^p v_2 \frac{dp}{p}\frac{dv_1}{v_1}$	$t^2 v_2 \frac{dp}{p}$	$\cdot$	
$\cdot$	$t^{p+1} v_2 \frac{dp}{p}\frac{dv_1}{v_1}$	$t^p v_2 \frac{dv_1}{v_1}$	$\cdot$	
$\cdot$	$t^{p+2} v_2 \frac{dp}{p}\frac{dv_1}{v_1}$	$t^{p+1} v_2 \frac{dp}{p}$	$\cdot$	
$\cdot$	$t^{2p} v_2 \frac{dp}{p}\frac{dv_1}{v_1}$	$t^{p+2} v_2 \frac{dp}{p}$	$\cdot$	
$\cdot$	$t\lambda_1 \frac{dv_1}{v_1}$	$t^{2p} v_2 \frac{dv_1}{v_1}$	$\cdot$	$2p$
$\cdot$	$\frac{dp}{p}\frac{dv_1}{v_1} : t^2\lambda_1 \frac{dv_1}{v_1}$	$t\lambda_1$	$\cdot$	$2p-2$
$\cdot$	$\cdot$	$\frac{dp}{p} : \frac{dv_1}{v_1} : t^2\lambda_1$	$\cdot$	2
$\cdot$	$\cdot$	$\cdot$	1	0
-3	-2	-1	0	$s \backslash t$

FIGURE 10.3: $P(v_2^{\pm 1})$-BASIS FOR $E^2_{s,t} \Longrightarrow V(1)_{s+t} K(\mathit{ff}(L_p))[v_2^{-1}]$

From these considerations, it is apparent that $V(1)_t K(\Omega_1)[v_2^{-1}]$ will be nontrivial for each even t, and zero for each odd t. Let us briefly write M_t for this Galois module. The presence of nonzero products by $\operatorname{dlog} p$ or $\operatorname{dlog} v_1$ from $E^2_{s,t}$ to $E^2_{s-1,t+2}$, for each even t and some s, implies that the pairing

$$M_t \otimes M_2 \to M_{t+2}$$

(tensor product over M_0, here and below) is also nontrivial for each even t, since the product in the spectral sequence should be induced by the group cohomology cup product and this pairing of the coefficients. This can most simply be the case if these pairings induce an isomorphism

$$M_2^{\otimes(t/2)} \xrightarrow{\cong} M_t$$

for all even t, which we now assume. Since $E^2_{0,t} = 0$ for $0 < t < 2p^2 - 2$, the Galois action on $M_2^{\otimes(t/2)}$ should have no invariants for these t, but $M_2^{\otimes(p^2-1)}$ should have trivial action. This indicates that the M_0-linear automorphism group of M_2 should have exponent $(p^2 - 1)$, which excludes $\mathbb{F}_p$, but strongly suggests the minimal example $M_2 = \mathbb{F}_{p^2}(1)$, with some Galois automorphism of Ω_1 over $ff(L_p)$ acting by multiplication by a generator of the group of units $\mathbb{F}_{p^2}^\times \cong \mathbb{Z}/(p^2-1)$. Then $M_0 = \mathbb{F}_{p^2}(0)$ with the trivial Galois action, and letting $u \in M_2 = V(1)_2 K(\Omega_1)[v_2^{-1}]$ be a generator for M_2 as an M_0-module, we see that u^{p^2-1} generates $M_{2p^2-2} = M_0\{v_2\}$, hence equals a unit multiple of v_2. Thus we deduce that the coefficient module in (10.2) can most simply be

$$V(1)_* K(\Omega_1)[v_2^{-1}] = M_* = \mathbb{F}_{p^2}[u^{\pm 1}] = \pi_*(K_2)$$

where K_2 is the 2-periodic form of the Morava K-theory spectrum $K(2)$, related to the (elliptic) Lubin–Tate spectrum E_2 with $\pi_* E_2 = \mathbb{W}(\mathbb{F}_{p^2})[[u_1]][u^{\pm 1}]$ by $K_2 \simeq V(1) \wedge E_2$. Here $K(2) \to K_2$ takes v_2 to u^{p^2-1}, and v_1 acts on $V(0) \wedge E_2$ by multiplication by $u_1 u^{p-1}$. In other words, $\pi_*(K_2) \cong \pi_*(E_2)/(p, u_1) = \pi_*(E_2)/(p, v_1)$. We are therefore led to the formula

$$V(1)_* K(\Omega_1)[v_2^{-1}] \cong V(1)_* E_2 \,.$$

By [HoSt99, 7.2], the left hand side is the $V(1)$-homotopy of $L_{K(2)} K(\Omega_1)$, and E_2 is $K(2)$-local.

Conjecture 10.4. *$K(\Omega_1)$ is a connective form of E_2, in the sense that there is an equivalence $L_{K(2)} K(\Omega_1) \simeq E_2$.*

This would generalize Suslin's theorem, which can be formulated as an equivalence $L_{K(1)} K(\bar{F}) \simeq KU_p = E_1$. Assuming this conjecture, we write $\mathbb{F}_{p^2}(t/2)$ for the Galois module $V(1)_t K(\Omega_1)[v_2^{-1}] = V(1)_t E_2 = \pi_t(K_2)$, which is $\mathbb{F}_{p^2}\{u^{t/2}\}$ for t even, and zero for t odd. Then the Galois descent spectral sequence takes the following form.

Conjecture 10.5. *For $K(1)$-local fraction fields $ff(A)$ contained in the separable closure Ω_1 of $ff(J_p) = ff(L_{K(1)} S)$ there is a natural algebra spectral sequence*

$$E^2_{s,t} = H^{-s}_{Gal}(ff(A); \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t} K(ff(A))[v_2^{-1}] \,,$$

where $H^r_{Gal}(ff(A); M) = H^r_{cont}(G_{ff(A)}; M)$ *is the continuous group cohomology of the absolute Galois group of* $ff(A)$*.*

The apparent self-duality of the $\mathbb{F}_p$-algebra Λ_* displayed in Figure 10.3 suggests that there is such a self-duality in the Galois cohomology of most or all $K(1)$-local fraction fields contained in Ω_1. Recall [Se97, §II.5] that for each p-local number field F there is a canonical isomorphism $H^2_{Gal}(F, \mathbb{F}_p(1)) \cong \mathbb{F}_p$, and that by the Tate–Poitou arithmetic duality theorem the cup product

$$H^r_{Gal}(F; \mathbb{F}_p(i)) \otimes H^{2-r}_{Gal}(F; \mathbb{F}_p(1-i)) \xrightarrow{\cup} H^2_{Gal}(F; \mathbb{F}_p(1)) \cong \mathbb{F}_p$$

is a perfect pairing for each r and i.

Conjecture 10.6. *For each finite* $K(1)$*-local Galois extension* $ff(A)$ *of* $ff(L_p)$ *there is a canonical isomorphism*

$$H^3_{Gal}(ff(A); \mathbb{F}_{p^2}(2)) \cong \mathbb{F}_p$$

and the cup product

$$H^r_{Gal}(ff(A); \mathbb{F}_{p^2}(i)) \otimes H^{3-r}_{Gal}(ff(A); \mathbb{F}_{p^2}(2-i)) \xrightarrow{\cup} H^3_{Gal}(ff(A); \mathbb{F}_{p^2}(2)) \cong \mathbb{F}_p$$

is a perfect pairing for each r *and* i*.*

Remark 10.7. Assuming these conjectures, it would be interesting to interpret the Galois module $\mathbb{F}_{p^2}(1) \cong V(1)_2K(\Omega_1)[v_2^{-1}]$ directly in terms of $K(\Omega_1)$, perhaps as suitable torsion points of a derived algebraic or formal group over a Galois extension $ff(A)$ of $ff(L_p)$. In particular, it would be illuminating to find a (minimal) such extension for which

$$V(1)_2K(ff(A))[v_2^{-1}] \to V(1)_2K(\Omega_1)[v_2^{-1}]$$

is surjective. This would play the role of the p-th cyclotomic field $\mathbb{Q}(\zeta_p)$ in the case of mod p algebraic K-theory of number fields, since there is a Bott element $\beta \in V(0)_2K(\mathbb{Q}(\zeta_p))[v_1^{-1}]$ mapping to $u \in V(0)_2K(\bar{\mathbb{Q}})[v_1^{-1}] \cong \mathbb{F}_p(1)$, up to a unit in $\mathbb{F}_p$. We recall that the choice of a p-th root of unity ζ_p determines a group homomorphism $C_p \to GL_1(\mathbb{Q}(\zeta_p))$, a map of E_∞ spaces

$$BC_p \to BGL_1(\mathbb{Q}(\zeta_p)) \to \Omega^\infty_\otimes K(\mathbb{Q}(\zeta_p)),$$

and a map of commutative S-algebras $S[BC_p] \to K(\mathbb{Q}(\zeta_p))$. The Bott class $\beta \in V(0)_2(BC_p)$ satisfies $\beta^p = v_1\beta$, as can be seen by mapping along $V(0)_*(BC_p) \to K(1)_*(BC_p)$, and its image in $V(0)_2K(\mathbb{Q}(\zeta_p))$ satisfies $\beta^{p-1} = v_1 = u^{p-1}$.

The calculations of [Au:tcku, 1.1] show that a generator in the $(p+1)$-st tensor power $\mathbb{F}_{p^2}(p+1) \cong V(1)_{2p+2}K(\Omega_1)[v_2^{-1}]$ of the Galois module $\mathbb{F}_{p^2}(1)$ is realized by the unramified Δ-Galois extension $L_p \to KU_p$, and its fraction field analogue $ff(L_p) \to ff(KU_p)$. For there is a higher Bott element $b \in V(1)_{2p+2}K(ku)$ that satisfies $b^{p-1} = -v_2$, whose image in $V(1)_{2p+2}K(\Omega_1)[v_2^{-1}] \cong \mathbb{F}_{p^2}\{u^{p+1}\}$ must have the form wu^{p+1}, for some element $w \in \mathbb{F}_{p^2}$ with $w^{p-1} = -1$. Hence $ku_p \to \Omega_1$ realizes the $\mathbb{F}_p$-subalgebra generated by wu^{p+1} and $u^{\pm(p^2-1)}$ in $\mathbb{F}_{p^2}[u^{\pm 1}]$. The higher Bott element is constructed from the map

$$K(\mathbb{Z}, 2) \simeq BU(1) \to BU_\otimes \to GL_1(ku)$$

that interprets the left hand abelian group as the v_1-torsion points in the homotopy units of ku. More precisely, $BU_\otimes \simeq BU(1) \times BSU_\otimes$, and p-locally $BSU \simeq BSU_\otimes$ [AP76] is v_1-torsion free, so $K(\mathbb{Z}, 2)$ is the v_1-torsion in $BU_\otimes$. The delooped E_∞ map $K(\mathbb{Z}, 3) \to BGL_1(ku) \to \Omega^\infty_\otimes(ku)$ is adjoint to a map $S[K(\mathbb{Z}, 3)] \to K(ku)$ of commutative S-algebras. The Bott element $\beta \in V(1)_{2p+2}(K(\mathbb{Z}, 3))$ satisfies $\beta^p = -v_2\beta$, as can be detected in $K(2)_*(K(\mathbb{Z}, 3))$ using [RaWi80, 9.2, 12.1], and its image b in $V(1)_{2p+2}K(ku)$ satisfies $b^{p-1} = -v_2 = -u^{p^2-1}$. A generator δ of Δ multiplies b by a generator of $\mathbb{F}_p^\times \cong \mathbb{Z}/(p-1)$, so an extension of δ to a Galois automorphism of Ω_1 must multiply u by a generator of $\mathbb{F}_{p^2}^\times \cong \mathbb{Z}/(p^2-1)$. Hence the absolute Galois group of $\mathit{ff}(L_p)$ will act faithfully on $\mathbb{F}_{p^2}(1)$.

Now suppose that there is a homotopy-commutative H-space X with

$$K(2)_*(X) \cong K(2)_*[x]/(x^{p^2} = v_2^j x)\,,$$

realizing one of the finite irreducible commutative Morava–Hopf algebras [SW98, 1.4]. Here $|x| = 2j$, and we assume that $(j, p+1) = 1$. Let ΩX be the loop group of X, and let $S[\Omega X]$ be the spherical monoid ring. As usual, there will be a homotopy commutative S-algebra map

$$S[X] \to K(S[\Omega X]) = A(X)\,,$$

where $A(X)$ is as in [Wa85]. The unit map $S \to E(2)$ induces a map $V(1)_*(X) \to K(2)_*(X)$, and we assume that x lifts to a class in $V(1)_{2j}(X)$, with image $\xi \in V(1)_{2j}K(S[\Omega X])$, still satisfying $\xi^{p^2} = v_2^j\xi$. If one can extend $S[\Omega X]$ to a $K(1)$-local Galois extension $\mathit{ff}(A)$ of $\mathit{ff}(J_p)$ or $\mathit{ff}(L_p)$, then the image $\xi \in V(1)_{2j}K(\mathit{ff}(A))$ will map to $u^j \in V(1)_{2j}K(\Omega_1)[v_2^{-1}] \cong \mathbb{F}_{p^2}(j)$, up to a unit in $\mathbb{F}_{p^2}$. If we also arrange that $\mathit{ff}(A)$ contains $\mathit{ff}(KU_p)$, then $\mathit{ff}(A)$ will realize all of $\mathbb{F}_{p^2}[u^{\pm1}]$, since this is generated as an $\mathbb{F}_p$-algebra by the classes $b \mapsto wu^{p+1}$, $\xi \mapsto u^j$ and $v_2^{\pm1} \mapsto u^{\pm(p^2-1)}$.

Remark 10.8. The optimistic reader can now extend these conjectures for all $n \geq 1$ to Galois extensions in the $K(n)$-local category, as seen by the v_{n+1}-periodic algebraic K-theory functor $V(n)_*K(-)[v_{n+1}^{-1}]$, where the Smith–Toda complex $V(n)$ might be replaced with any fixed type $(n+1)$ finite complex (to ensure that it exists). The extension $J_p \to L_p$ is best replaced with the $K(n)$-local pro-Galois extension $L_{K(n)}S \to E_n$ with Galois group $\mathbb{G}_n$ constructed in [DH04], see [Rog08, 5.4] and [AuR08]. We suggest writing Ω_n for the $K(n)$-local separable closure of the fraction field of $L_{K(n)}S$ or E_n. Then Conjecture 10.4 may be extended to the prediction

$$L_{K(n+1)}K(\Omega_n) \simeq E_{n+1}\,,$$

and similarly for Conjecture 10.5. If correct, then $e_{n+1} = K(\Omega_n)_p$ is a good connective form of the p-complete commutative S-algebra E_{n+1}.

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