ALGEBRAIC K-THEORY OF THE FIRST MORAVA K-THEORY

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1. Introduction

In this paper we continue the investigation from [AR02] and [Aus10] of the algebraic K-theory of topological K-theory and related S-algebras. Let ℓ_p be the p-complete Adams summand of connective complex K-theory, and let $\ell/p = k(1)$ be the first connective Morava K-theory. It has a unique S-algebra structure [Ang], and we show in Section 2 that ℓ/p is an ℓ_p-algebra (in uncountably many ways), so that $K(\ell/p)$ is a $K(\ell_p)$-module spectrum.

Let $V(1) = S/(p, v_1)$ be the type 2 Smith–Toda complex. It is a homotopy commutative ring spectrum for $p \geq 5$, with a preferred periodic class $v_2 \in V(1)_*$. We write $V(1)_*(X) = \pi_*(V(1) \wedge X)$ for the $V(1)$-homotopy of a spectrum X. Multiplication by v_2 makes $V(1)_*(X)$ a $P(v_2)$-module, where $P(v_2)$ denotes the polynomial algebra over $\mathbb{F}_p$ generated by v_2.

We computed the $V(1)$-homotopy of $K(\ell_p)$ in [AR02], showing that it is essentially a free $P(v_2)$-module on $(4p+4)$ generators. In particular, there are preferred classes $\lambda_1, \lambda_2 \in V(1)_*K(\ell_p)$ generating an exterior subalgebra $E(\lambda_1, \lambda_2)$. Hence $V(1)_*K(\ell/p)$ is an $E(\lambda_1, \lambda_2) \otimes P(v_2)$-module. The following is our main result, corresponding to Theorem 7.10 in the body of the paper.

Theorem 1.1. *Let $p \geq 5$ be a prime and let $\ell/p = k(1)$ be the first connective Morava K-theory spectrum. There is an isomorphism of $E(\lambda_1, \lambda_2) \otimes P(v_2)$-modules*

$$\begin{aligned} V(1)_*K(\ell/p) \cong{} & P(v_2) \otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{1, \partial\lambda_2, \lambda_2, \partial v_2\} \\ & \oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ & \oplus P(v_2) \otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\}\,. \end{aligned}$$

Here $|\lambda_1| = |\bar\epsilon_1| = 2p-1$, $|\lambda_2| = 2p^2-1$, $|v_2| = 2p^2-2$, $|\operatorname{dlog} v_1| = 1$, $|\partial| = -1$ and $|t| = -2$. This is a free $P(v_2)$-module of rank $(2p^2 - 2p + 8)$ and of zero Euler characteristic.

We prove this theorem by means of the cyclotomic trace map [BHM93] to topological cyclic homology $TC(\ell/p)$. Along the way we evaluate $V(1)_*THH(\ell/p)$, where THH denotes topological Hochschild homology, as well as $V(1)_*TC(\ell/p)$, see Proposition 4.6 and Theorem 7.8.

Let L_p be the p-complete Adams summand of periodic complex K-theory, and let $L/p = K(1)$ be the first periodic Morava K-theory. The localization cofiber sequence $K(\mathbb{Z}) \to K(ku) \to K(KU)$ of Blumberg and Mandell [BM08] has the mod p Adams analogue

$$K(\mathbb{Z}/p) \to K(\ell/p) \to K(L/p)\,.$$

Using Quillen's computation [Qui72] of $K(\mathbb{Z}/p)$, we obtain the following consequence:

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Corollary 1.2. *Let $p \geq 5$ be a prime and let $L/p = K(1)$ be the first Morava K-theory spectrum. There is an isomorphism of $E(\lambda_1, \lambda_2) \otimes P(v_2^{\pm 1})$-modules*

$$V(1)_* K(L/p)[v_2^{-1}] \cong V(1)_* K(\ell/p)[v_2^{-1}]\,.$$

If the relation $\lambda_2 = v_2 \operatorname{dlog} v_1$ holds in $V(1)_* K(L/p)$, then there is an isomorphism of $E(\operatorname{dlog} v_1, \lambda_1) \otimes P(v_2)$-modules

$$\begin{aligned} V(1)_* K(L/p) \cong\ & P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{1, \partial\lambda_2, \operatorname{dlog} v_1, \partial v_2\} \\ & \oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ & \oplus P(v_2) \otimes E(\bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^{dp} v_2 \operatorname{dlog} v_1 \mid 0 < d < p\}\,, \end{aligned}$$

where the degrees of the generators are as in Theorem 1.1. This is a free $P(v_2)$-module of rank $(2p^2 - 2p + 8)$ and of zero Euler characteristic.

Our far-reaching aim is to conceptually understand the algebraic K-theory of ℓ_p and other commutative S-algebras in terms of localization and Galois descent, in the same way as we understand the algebraic K-theory of rings of integers in (local) number fields or more general regular rings. The first task is to relate $K(\ell_p)$ to the algebraic K-theory of its "residue fields" and "fraction field", for which we expect a description in terms of Galois cohomology to exist, starting with the Galois theory for commutative S-algebras developed by the second author [Rog08]. The residue rings of ℓ_p appear to be ℓ/p, $H\mathbb{Z}_p$ and $H\mathbb{Z}/p$, while the fraction field $f\!f(\ell_p)$ appears to be a localization of L_p away from L/p, less drastic than the algebraic localization $L_p[p^{-1}] = L\mathbb{Q}_p$. So far we do not have a proper definition of this S-algebraic fraction field, but by analogy with the localization sequence above, we expect that its algebraic K-theory appears in a localization cofiber sequence

$$K(L/p) \to K(L_p) \to K(f\!f(\ell_p))\,,$$

where the transfer map on the left is a $K(L_p)$-module map. Taking this as a *preliminary definition* of the symbol $K(f\!f(\ell_p))$, we can use our computations to evaluate its $V(1)$-homotopy:

Theorem 1.3. *Let $p \geq 5$ be a prime, and define $K(f\!f(\ell_p))$ as the homotopy cofiber above. There is an isomorphism of $P(v_2^{\pm 1})$-modules*

$$V(1)_* K(f\!f(\ell_p))[v_2^{-1}] \cong P(v_2^{\pm 1}) \otimes \Lambda_*$$

where

$$\begin{aligned} \Lambda_* \cong\ & E(\partial v_2, \operatorname{dlog} p, \operatorname{dlog} v_1) \\ & \oplus E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d \lambda_1 \mid 0 < d < p\} \\ & \oplus E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \operatorname{dlog} p \mid 0 < d < p^2 - p, p \nmid d\} \\ & \oplus E(\operatorname{dlog} p) \otimes \mathbb{F}_p\{t^{dp} \lambda_2 \mid 0 < d < p\}\,. \end{aligned}$$

Here $|\operatorname{dlog} p| = 1$, and the degrees of the other classes are as in Theorem 1.1. The localization homomorphism

$$V(1)_* K(f\!f(\ell_p)) \to V(1)_* K(f\!f(\ell_p))[v_2^{-1}]$$

is an isomorphism in degrees $* \geq 2p$.

In particular, the homotopy cofiber $K(f\!f(\ell_p))$ cannot be equivalent to the $K(\mathbb{Q}_p)$-module $K(L\mathbb{Q}_p)$, since $V(1)_* K(\mathbb{Q}_p)$ is a torsion $P(v_2)$-module.

We may now conjecturally interpret $V(1)_*K(\mathit{ff}(\ell_p))[v_2^{-1}]$ in terms of Galois descent. Indeed, the second author conjectured that if Ω_1 is an S-algebraic "separable closure" of $\mathit{ff}(\ell_p)$, then there is a homotopy equivalence

$$L_{K(2)}K(\Omega_1) \simeq E_2\,.$$

Here E_2 is Morava's second E-theory [GH04], with coefficients $(E_2)_* = \mathbb{W}(\mathbb{F}_{p^2})[[u_1]][u^{\pm 1}]$, and $L_{K(2)}$ denotes Bousfield localization with respect to the second Morava K-theory $K(2)$, with coefficients $K(2)_* = \mathbb{F}_p[v_2^{\pm 1}]$. The v_2-periodic $V(1)$-homotopy groups of $K(\Omega_1)$ will then be given by

$$V(1)_*K(\Omega_1)[v_2^{-1}] \cong \mathbb{F}_{p^2}[u^{\pm 1}].$$

We would expect to have a corresponding Galois descent spectral sequence

$$E^2_{s,t} = H^{-s}_{Gal}(\mathit{ff}(\ell_p); \mathbb{F}_{p^2}(t/2)) \Longrightarrow V(1)_{s+t}K(\mathit{ff}(\ell_p))[v_2^{-1}]\,.$$

If this spectral sequence collapses at E^2 when $p \geq 5$, as is the case for p-adic number fields when $p \geq 3$, we get a conjectural description of the Galois cohomology of $\mathit{ff}(\ell_p)$ with coefficients in $\mathbb{F}_{p^2}(t/2)$, for all even t. Promisingly, this fits very well with the example of the Galois cohomology of $\mathbb{Q}_p$ with coefficients in $\mathbb{F}_p(t/2)$, with the difference that the absolute Galois group of $\mathit{ff}(\ell_p)$ has p-cohomological dimension 3 instead of 2. Also, by analogy with Tate–Poitou duality [Tat63] in the Galois cohomology of local number fields, there appears to be a perfect arithmetic duality pairing in the conjectural Galois cohomology of $\mathit{ff}(\ell_p)$, with fundamental class dual to $\partial v_2 \cdot \mathrm{dlog}\, p \cdot \mathrm{dlog}\, v_1$ in $H^3_{Gal}(\mathit{ff}(\ell_p); \mathbb{F}_{p^2}(2))$. This indicates that $\mathit{ff}(\ell_p)$ ought to be a form of S-algebraic two-dimensional local field, mixing three different residue characteristics. We elaborate more on this in [AR].

The paper is organized as follows. In Section 2 we fix our notations, show that ℓ/p admits the structure of an associative ℓ_p-algebra, and give a similar discussion for ku/p and the periodic versions L/p and KU/p. Section 3 contains the computation of the mod p homology of $THH(\ell/p)$, and in Section 4 we evaluate its $V(1)$-homotopy. In Section 5 we show that the C_{p^n}-fixed points and C_{p^n}-homotopy fixed points of $THH(\ell/p)$ are closely related, and use this to inductively determine their $V(1)$-homotopy in Section 6. Finally, in Section 7 we achieve the computation of $TC(\ell/p)$ and $K(\ell/p)$ in $V(1)$-homotopy.

2. Base change squares of S-algebras

We fix some notations. Let p be a prime, even or odd for now. Write $X_{(p)}$ and X_p for the p-localization and the p-completion, respectively, of any spectrum or abelian group X. Let ku and KU be the connective and the periodic complex K-theory spectra, with homotopy rings $ku_* = \mathbb{Z}[u]$ and $KU_* = \mathbb{Z}[u^{\pm 1}]$, where $|u| = 2$. Let $\ell = BP\langle 1\rangle$ and $L = E(1)$ be the p-local Adams summands, with $\ell_* = \mathbb{Z}_{(p)}[v_1]$ and $L_* = \mathbb{Z}_{(p)}[v_1^{\pm 1}]$, where $|v_1| = 2p-2$. The inclusion $\ell \to ku_{(p)}$ maps v_1 to u^{p-1}. Alternate notations in the p-complete cases are $KU_p = E_1$ and $L_p = \widehat{E(1)}$. These ring spectra are all commutative S-algebras, in the sense that each admits a unique E_∞ ring spectrum structure. See [BR05] for proofs of uniqueness in the periodic cases.

Let ku/p and KU/p be the connective and periodic mod p complex K-theory spectra, with coefficients $(ku/p)_* = \mathbb{Z}/p[u]$ and $(KU/p)_* = \mathbb{Z}/p[u^{\pm 1}]$. These are 2-periodic versions of the first Morava K-theory spectra $\ell/p = k(1)$ and $L/p = K(1)$, with $(\ell/p)_* = \mathbb{Z}/p[v_1]$ and $(L/p)_* = \mathbb{Z}/p[v_1^{\pm 1}]$. Each of these can be constructed as the cofiber of the multiplication by p map, as a module over the corresponding commutative S-algebra. For example, there is a cofiber sequence of ku-modules $ku \xrightarrow{p} ku \xrightarrow{i} ku/p$.

Let HR be the Eilenberg–Mac Lane spectrum of a ring R. When R is associative, HR admits a unique associative S-algebra structure, and when R is commutative, HR admits a unique commutative S-algebra structure. The zeroth Postnikov section defines unique maps of commutative S-algebras $\pi\colon ku \to H\mathbb{Z}$ and $\pi\colon \ell \to H\mathbb{Z}_{(p)}$, which can be followed by unique commutative S-algebra maps to $H\mathbb{Z}/p$.

The ku-module spectrum ku/p does not admit the structure of a commutative ku-algebra. It cannot even be an E_2 or H_2 ring spectrum, since the homomorphism induced in mod p homology by the resulting map $\pi\colon ku/p \to H\mathbb{Z}/p$ of H_2 ring spectra would not commute with the homology operation $Q^1(\bar\tau_0) = \bar\tau_1$ in the target $H_*(H\mathbb{Z}/p; \mathbb{F}_p)$ [BMMS86, III.2.3]. Similar remarks apply for KU/p, ℓ/p and L/p. Associative algebra structures, or A_∞ ring spectrum structures, are easier to come by. The following result is a direct application of the methods of [Laz01, §§9–11]. We adapt the notation of [BJ02, §3] to provide some details in our case.

Proposition 2.1. *The ku-module spectrum ku/p admits the structure of an associative ku-algebra, but the structure is not unique. Similar statements hold for KU/p as a KU-algebra, ℓ/p as an ℓ-algebra and L/p as an L-algebra.*

Proof. We construct ku/p as the (homotopy) limit of its Postnikov tower of associative ku-algebras $P^{2m-2} = ku/(p, u^m)$, with coefficient rings $ku/(p, u^m)_* = ku_*/(p, u^m)$ for $m \geq 1$. To start the induction, $P^0 = H\mathbb{Z}/p$ is a ku-algebra via $i \circ \pi\colon ku \to H\mathbb{Z} \to H\mathbb{Z}/p$. Assume inductively for $m \geq 1$ that $P = P^{2m-2}$ has been constructed. We will define P^{2m} by a (homotopy) pullback diagram

$$\begin{tikzcd}
P^{2m} \arrow[r] \arrow[d] & P \arrow[d, "in_1"] \\
P \arrow[r, "d"] & P \vee \Sigma^{2m+1} H\mathbb{Z}/p
\end{tikzcd}$$

in the category of associative ku-algebras. Here

$$d \in \mathrm{ADer}^{2m+1}_{ku}(P, H\mathbb{Z}/p) \cong THH^{2m+2}_{ku}(P, H\mathbb{Z}/p)$$

is an associative ku-algebra derivation of P with values in $\Sigma^{2m+1}H\mathbb{Z}/p$, and the group of such can be identified with the indicated topological Hochschild cohomology group of P over ku. We recall that these are the homotopy groups (cohomologically graded) of the function spectrum $F_{P \wedge_{ku} P^{op}}(P, H\mathbb{Z}/p)$. The composite map $pr_2 \circ d\colon P \to \Sigma^{2m+1}H\mathbb{Z}/p$ of ku-modules, where pr_2 projects onto the second wedge summand, is restricted to equal the ku-module Postnikov k-invariant of ku/p in

$$H^{2m+1}_{ku}(P; \mathbb{Z}/p) = \pi_0 F_{ku}(P, \Sigma^{2m+1}H\mathbb{Z}/p)\,.$$

We compute that $\pi_*(P \wedge_{ku} P^{op}) = ku_*/(p, u^m) \otimes E(\tau_0, \tau_{1,m})$, where $|\tau_0| = 1$, $|\tau_{1,m}| = 2m+1$ and $E(-)$ denotes the exterior algebra on the given generators. (For $p = 2$, the use of the opposite product is essential here [Ang08, §3].) The function spectrum description of topological Hochschild cohomology leads to the spectral sequence

$$\begin{aligned}
E_2^{**} &= \mathrm{Ext}^{**}_{\pi_*(P \wedge_{ku} P^{op})}(\pi_*(P), \mathbb{Z}/p) \\
&\cong \mathbb{Z}/p[y_0, y_{1,m}] \\
&\Longrightarrow THH^*_{ku}(P, H\mathbb{Z}/p)\,,
\end{aligned}$$

where y_0 and $y_{1,m}$ have cohomological bidegrees $(1,1)$ and $(1, 2m+1)$, respectively. The spectral sequence collapses at $E_2 = E_\infty$, since it is concentrated in even total degrees. In

particular,
$$\mathrm{ADer}^{2m+1}_{ku}(P, H\mathbb{Z}/p) \cong \mathbb{F}_p\{y_{1,m}, y_0^{m+1}\}\,.$$
Additively, $H^{2m+1}_{ku}(P;\mathbb{Z}/p) \cong \mathbb{F}_p\{Q_{1,m}\}$ is generated by a class dual to $\tau_{1,m}$, which is the image of $y_{1,m}$ under left composition with pr_2. It equals the ku-module k-invariant of ku/p. Thus there are precisely p choices $d = y_{1,m} + \alpha y_0^{m+1}$, with $\alpha \in \mathbb{F}_p$, for how to extend any given associative ku-algebra structure on $P = P^{2m-2}$ to one on $P^{2m} = ku/(p, u^{m+1})$. In the limit, we find that there are an uncountable number of associative ku-algebra structures on $ku/p = \mathrm{holim}_m\, P^{2m}$, each indexed by a sequence of choices $\alpha \in \mathbb{F}_p$ for all $m \geq 1$.

The periodic spectrum KU/p can be obtained from ku/p by Bousfield KU-localization in the category of ku-modules [EKMM97, VIII.4], which makes it an associative KU-algebra. The classification of periodic S-algebra structures is the same as in the connective case, since the original ku-algebra structure on ku/p can be recovered from that on KU/p by a functorial passage to the connective cover. To construct ℓ/p as an associative ℓ-algebra, or L/p as an associative L-algebra, replace u by v_1 in these arguments. □

By varying the ground S-algebra, we obtain the same conclusions about ku/p as a $ku_{(p)}$-algebra or ku_p-algebra, and about ℓ/p as an ℓ_p-algebra.

For each choice of ku-algebra structure on ku/p, the zeroth Postnikov section $\pi\colon ku/p \to H\mathbb{Z}/p$ is a ku-algebra map, with the unique ku-algebra structure on the target. Hence there is a commutative square of associative ku-algebras

$$\begin{array}{ccc} ku & \xrightarrow{i} & ku/p \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ H\mathbb{Z} & \xrightarrow{i} & H\mathbb{Z}/p \end{array}$$

and similarly in the p-local and p-complete cases. In view of the weak equivalence $H\mathbb{Z}\wedge_{ku} ku/p \simeq H\mathbb{Z}/p$, this square expresses the associative $H\mathbb{Z}$-algebra $H\mathbb{Z}/p$ as the base change of the associative ku-algebra ku/p along $\pi\colon ku \to H\mathbb{Z}$. Likewise, there is a commutative square of associative ℓ_p-algebras

$$\begin{array}{ccc} \ell_p & \xrightarrow{i} & \ell/p \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ H\mathbb{Z}_p & \xrightarrow{i} & H\mathbb{Z}/p \end{array} \tag{2.2}$$

that expresses $H\mathbb{Z}/p$ as the base change of ℓ/p along $\ell_p \to H\mathbb{Z}_p$, and similarly in the p-local case. By omission of structure, these squares are also diagrams of S-algebras and S-algebra maps.

3. Topological Hochschild homology

We shall compute the $V(1)$-homotopy of the topological Hochschild homology $THH(-)$ and topological cyclic homology $TC(-)$ of the S-algebras in diagram (2.2), for primes $p \geq 5$. Passing to connective covers, this also computes the $V(1)$-homotopy of the algebraic K-theory spectra appearing in that square. With these coefficients, or more generally, after p-adic completion, the functors THH and TC are insensitive to p-completion in the argument, so we shall simplify the notation slightly by working with the associative

S-algebras ℓ and $H\mathbb{Z}_{(p)}$ in place of ℓ_p and $H\mathbb{Z}_p$. For ordinary rings R we almost always shorten notations like $THH(HR)$ to $THH(R)$.

The computations follow the strategy of [Bök], [BM94], [BM95] and [HM97] for $H\mathbb{Z}/p$ and $H\mathbb{Z}$, and of [MS93] and [AR02] for ℓ. See also [AR05, §§4–7] for further discussion of the THH-part of such computations. In this section we shall compute the mod p homology of the topological Hochschild homology of ℓ/p as a module over the corresponding homology for ℓ, for any odd prime p.

We write $E(x) = \mathbb{F}_p[x]/(x^2)$ for the exterior algebra, $P(x) = \mathbb{F}_p[x]$ for the polynomial algebra and $P(x^{\pm 1}) = \mathbb{F}_p[x, x^{-1}]$ for the Laurent polynomial algebra on one generator x, and similarly for a list of generators. We will also write $\Gamma(x) = \mathbb{F}_p\{\gamma_i(x) \mid i \geq 0\}$ for the divided power algebra, with $\gamma_i(x) \cdot \gamma_j(x) = (i,j)\gamma_{i+j}(x)$, where $(i,j) = (i+j)!/i!j!$ is the binomial coefficient. We use the obvious abbreviations $\gamma_0(x) = 1$ and $\gamma_1(x) = x$. Finally, we write $P_h(x) = \mathbb{F}_p[x]/(x^h)$ for the truncated polynomial algebra of height h, and recall the isomorphism $\Gamma(x) \cong P_p(\gamma_{p^e}(x) \mid e \geq 0)$ in characteristic p.

We write $H_*(-)$ for homology with mod p coefficients. It takes values in A_*-comodules, where A_* is the dual Steenrod algebra [Mil58]. Explicitly (for p odd),

$$A_* = P(\bar\xi_k \mid k \geq 1) \otimes E(\bar\tau_k \mid k \geq 0)$$

with coproduct

$$\psi(\bar\xi_k) = \sum_{i+j=k} \bar\xi_i \otimes \bar\xi_j^{p^i}$$

and

$$\psi(\bar\tau_k) = 1 \otimes \bar\tau_k + \sum_{i+j=k} \bar\tau_i \otimes \bar\xi_j^{p^i} \,.$$

Here $\bar\xi_0 = 1$, $\bar\xi_k = \chi(\xi_k)$ has degree $2(p^k - 1)$ and $\bar\tau_k = \chi(\tau_k)$ has degree $2p^k - 1$, where χ is the canonical conjugation [MM65]. Then the zeroth Postnikov sections induce identifications

$$\begin{aligned} H_*(H\mathbb{Z}_{(p)}) &= P(\bar\xi_k \mid k \geq 1) \otimes E(\bar\tau_k \mid k \geq 1) \\ H_*(\ell) &= P(\bar\xi_k \mid k \geq 1) \otimes E(\bar\tau_k \mid k \geq 2) \\ H_*(\ell/p) &= P(\bar\xi_k \mid k \geq 1) \otimes E(\bar\tau_0, \bar\tau_k \mid k \geq 2) \end{aligned}$$

as A_*-comodule subalgebras of $H_*(H\mathbb{Z}/p) = A_*$. We often make use of the following A_*-comodule coactions

$$\begin{aligned} \nu(\bar\tau_0) &= 1 \otimes \bar\tau_0 + \bar\tau_0 \otimes 1 \\ \nu(\bar\xi_1) &= 1 \otimes \bar\xi_1 + \bar\xi_1 \otimes 1 \\ \nu(\bar\tau_1) &= 1 \otimes \bar\tau_1 + \bar\tau_0 \otimes \bar\xi_1 + \bar\tau_1 \otimes 1 \\ \nu(\bar\xi_2) &= 1 \otimes \bar\xi_2 + \bar\xi_1 \otimes \bar\xi_1^p + \bar\xi_2 \otimes 1 \\ \nu(\bar\tau_2) &= 1 \otimes \bar\tau_2 + \bar\tau_0 \otimes \bar\xi_2 + \bar\tau_1 \otimes \bar\xi_1^p + \bar\tau_2 \otimes 1 \,. \end{aligned}$$

The Bökstedt spectral sequences

$$E^2_{**}(B) = HH_*(H_*(B)) \Longrightarrow H_*(THH(B))$$

for the commutative S-algebras $B = H\mathbb{Z}/p$, $H\mathbb{Z}_{(p)}$ and ℓ begin

$$\begin{aligned} E^2_{**}(\mathbb{Z}/p) &= A_* \otimes E(\sigma\bar\xi_k \mid k \geq 1) \otimes \Gamma(\sigma\bar\tau_k \mid k \geq 0) \\ E^2_{**}(\mathbb{Z}_{(p)}) &= H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar\xi_k \mid k \geq 1) \otimes \Gamma(\sigma\bar\tau_k \mid k \geq 1) \\ E^2_{**}(\ell) &= H_*(\ell) \otimes E(\sigma\bar\xi_k \mid k \geq 1) \otimes \Gamma(\sigma\bar\tau_k \mid k \geq 2) \,. \end{aligned}$$

They are (graded) commutative A_*-comodule algebra spectral sequences, and there are differentials

$$d^{p-1}(\gamma_j \sigma\bar\tau_k) = \sigma\bar\xi_{k+1} \cdot \gamma_{j-p}\sigma\bar\tau_k$$

for $j \ge p$ and $k \ge 0$, see [Bök], [Hun96] or [Aus05, 4.3], leaving

$$\begin{aligned} E^\infty_{**}(\mathbb{Z}/p) &= A_* \otimes P_p(\sigma\bar\tau_k \mid k \ge 0) \\ E^\infty_{**}(\mathbb{Z}_{(p)}) &= H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar\xi_1) \otimes P_p(\sigma\bar\tau_k \mid k \ge 1) \\ E^\infty_{**}(\ell) &= H_*(\ell) \otimes E(\sigma\bar\xi_1, \sigma\bar\xi_2) \otimes P_p(\sigma\bar\tau_k \mid k \ge 2)\,. \end{aligned}$$

The inclusion of 0-simplices $\eta\colon B \to THH(B)$ is split for commutative B by the augmentation $\epsilon\colon THH(B) \to B$. Thus there are unique representatives in Bökstedt filtration 1, with zero augmentation, for each of the classes σx. They correspond to $1 \otimes x - x \otimes 1$ in the Hochschild complex, or just $1 \otimes x$ in the normalized Hochschild complex. There are multiplicative extensions $(\sigma\bar\tau_k)^p = \sigma\bar\tau_{k+1}$ for $k \ge 0$, see [AR05, 5.9], so

$$\begin{aligned} H_*(THH(\mathbb{Z}/p)) &= A_* \otimes P(\sigma\bar\tau_0) \\ H_*(THH(\mathbb{Z}_{(p)})) &= H_*(H\mathbb{Z}_{(p)}) \otimes E(\sigma\bar\xi_1) \otimes P(\sigma\bar\tau_1) \\ H_*(THH(\ell)) &= H_*(\ell) \otimes E(\sigma\bar\xi_1, \sigma\bar\xi_2) \otimes P(\sigma\bar\tau_2) \end{aligned} \tag{3.1}$$

as A_*-comodule algebras. The A_*-comodule coactions are given by

$$\begin{aligned} \nu(\sigma\bar\tau_0) &= 1 \otimes \sigma\bar\tau_0 \\ \nu(\sigma\bar\xi_1) &= 1 \otimes \sigma\bar\xi_1 \\ \nu(\sigma\bar\tau_1) &= 1 \otimes \sigma\bar\tau_1 + \bar\tau_0 \otimes \sigma\bar\xi_1 \\ \nu(\sigma\bar\xi_2) &= 1 \otimes \sigma\bar\xi_2 \\ \nu(\sigma\bar\tau_2) &= 1 \otimes \sigma\bar\tau_2 + \bar\tau_0 \otimes \sigma\bar\xi_2\,. \end{aligned} \tag{3.2}$$

The natural map $\pi_*\colon THH(\ell) \to THH(\mathbb{Z}_{(p)})$ induced by $\pi\colon \ell \to \mathbb{Z}_{(p)}$ takes $\sigma\bar\xi_2$ to 0 and $\sigma\bar\tau_2$ to $(\sigma\bar\tau_1)^p$. The natural map $i_*\colon THH(\mathbb{Z}_{(p)}) \to THH(\mathbb{Z}/p)$ induced by $i\colon \mathbb{Z}_{(p)} \to \mathbb{Z}/p$ takes $\sigma\bar\xi_1$ to 0 and $\sigma\bar\tau_1$ to $(\sigma\bar\tau_0)^p$.

The Bökstedt spectral sequence for the associative S-algebra $B = \ell/p$ begins

$$E^2_{**}(\ell/p) = H_*(\ell/p) \otimes E(\sigma\bar\xi_k \mid k \ge 1) \otimes \Gamma(\sigma\bar\tau_0, \sigma\bar\tau_k \mid k \ge 2)\,.$$

It is an A_*-comodule module spectral sequence over the Bökstedt spectral sequence for ℓ, since the ℓ-algebra multiplication $\ell \wedge \ell/p \to \ell/p$ is a map of associative S-algebras. However, it is not itself an algebra spectral sequence, since the product on ℓ/p is not commutative enough to induce a natural product structure on $THH(\ell/p)$. Nonetheless, we will use the algebra structure present at the E^2-term to help in naming classes.

The map $\pi\colon \ell/p \to H\mathbb{Z}/p$ induces an injection of Bökstedt spectral sequence E^2-terms, so there are differentials generated algebraically by

$$d^{p-1}(\gamma_j \sigma\bar\tau_k) = \sigma\bar\xi_{k+1} \cdot \gamma_{j-p}\sigma\bar\tau_k$$

for $j \ge p$, $k = 0$ or $k \ge 2$, leaving

$$E^\infty_{**}(\ell/p) = H_*(\ell/p) \otimes E(\sigma\bar\xi_2) \otimes P_p(\sigma\bar\tau_0, \sigma\bar\tau_k \mid k \ge 2) \tag{3.3}$$

as an A_*-comodule module over $E^\infty_{**}(\ell)$. In order to obtain $H_*(THH(\ell/p))$, we need to resolve the A_*-comodule and $H_*(THH(\ell))$-module extensions. This is achieved in Lemma 3.6 below.

The natural map $\pi_*\colon E^\infty_{**}(\ell/p) \to E^\infty_{**}(\mathbb{Z}/p)$ is an isomorphism in total degrees $\le (2p-2)$ and injective in total degrees $\le (2p^2-2)$. The first class in the kernel is $\sigma\bar\xi_2$. Hence there are unique classes

$$1\ ,\ \bar\tau_0\ ,\ \sigma\bar\tau_0\ ,\ \bar\tau_0\sigma\bar\tau_0\ ,\ \dots\ ,\ (\sigma\bar\tau_0)^{p-1}$$

in degrees $0 \le * \le 2p-2$ of $H_*(THH(\ell/p))$, mapping to classes with the same names in $H_*(THH(\mathbb{Z}/p))$. More concisely, these are the monomials $\bar\tau_0^\delta(\sigma\bar\tau_0)^i$ for $0 \le \delta \le 1$ and $0 \le i \le p-1$, except that the degree $(2p-1)$ case $(\delta, i) = (1, p-1)$ is omitted. The A_*-comodule coaction on these classes is given by the same formulas in $H_*(THH(\ell/p))$ as in $H_*(THH(\mathbb{Z}/p))$, cf. (3.2).

There is also a class $\bar\xi_1$ in degree $(2p-2)$ of $H_*(THH(\ell/p))$ mapping to a class with the same name, and same A_*-coaction, in $H_*(THH(\mathbb{Z}/p))$.

In degree $(2p-1)$, π_* is a map of extensions from

$$0 \to \mathbb{F}_p\{\bar\xi_1\bar\tau_0\} \to H_{2p-1}(THH(\ell/p)) \to \mathbb{F}_p\{\bar\tau_0(\sigma\bar\tau_0)^{p-1}\} \to 0$$

to

$$0 \to \mathbb{F}_p\{\bar\tau_1, \bar\xi_1\bar\tau_0\} \to H_{2p-1}(THH(\mathbb{Z}/p)) \to \mathbb{F}_p\{\bar\tau_0(\sigma\bar\tau_0)^{p-1}\} \to 0\,.$$

The latter extension is canonically split by the augmentation $\epsilon\colon THH(\mathbb{Z}/p) \to H\mathbb{Z}/p$, which uses the commutativity of the S-algebra $H\mathbb{Z}/p$.

In degree $2p$, the map π_* goes from

$$H_{2p}(THH(\ell/p)) = \mathbb{F}_p\{\bar\xi_1\sigma\bar\tau_0\}$$

to

$$0 \to \mathbb{F}_p\{\bar\tau_0\bar\tau_1\} \to H_{2p}(THH(\mathbb{Z}/p)) \to \mathbb{F}_p\{\sigma\bar\tau_1, \bar\xi_1\sigma\bar\tau_0\} \to 0\,.$$

Again the latter extension is canonically split.

Lemma 3.4. *There is a unique class y in $H_{2p-1}(THH(\ell/p))$ that is represented by $\bar\tau_0(\sigma\bar\tau_0)^{p-1}$ in $E^\infty_{p-1,p}(\ell/p)$ and maps by π_* to $\bar\tau_0(\sigma\bar\tau_0)^{p-1} - \bar\tau_1$ in $H_*(THH(\mathbb{Z}/p))$.*

Proof. This follows from naturality of the suspension operator σ and the multiplicative relation $(\sigma\bar\tau_0)^p = \sigma\bar\tau_1$ in $H_*(THH(\mathbb{Z}/p))$. A class y in $H_{2p-1}(THH(\ell/p))$ represented by $\bar\tau_0(\sigma\bar\tau_0)^{p-1}$ is determined modulo $\bar\xi_1\bar\tau_0$. Its image in $H_{2p-1}(THH(\mathbb{Z}/p))$ thus has the form $\alpha\bar\tau_1 + \bar\tau_0(\sigma\bar\tau_0)^{p-1}$ modulo $\bar\xi_1\bar\tau_0$, for some $\alpha \in \mathbb{F}_p$. The suspension σy lies in $H_{2p}(THH(\ell/p)) = \mathbb{F}_p\{\bar\xi_1\sigma\bar\tau_0\}$, so its image in $H_{2p}(THH(\mathbb{Z}/p))$ is 0 modulo $\bar\tau_0\bar\tau_1$ and $\bar\xi_1\sigma\bar\tau_0$. It is also the suspension of $\alpha\bar\tau_1 + \bar\tau_0(\sigma\bar\tau_0)^{p-1}$ modulo $\bar\xi_1\bar\tau_0$, which equals $\sigma(\alpha\bar\tau_1) + (\sigma\bar\tau_0)^p = (\alpha+1)\sigma\bar\tau_1$. In particular, the coefficient $(\alpha+1)$ of $\sigma\bar\tau_1$ is 0, so $\alpha = -1$. □

Remark 3.5. For $p = 2$ this can alternatively be read off from the explicit form [Wür91] of the commutator for the product μ in ℓ/p. The coequalizer C of the two maps

$$\ell/p \wedge \ell/p \underset{\mu\tau}{\overset{\mu}{\rightrightarrows}} \ell/p$$

maps to (the 1-skeleton of) $THH(\ell/p)$. The commutator $\mu - \mu\tau$ factors as

$$\ell/p \wedge \ell/p \xrightarrow{\beta\wedge\beta} \Sigma\ell/p \wedge \Sigma\ell/p \xrightarrow{\mu} \Sigma^2\ell/p \xrightarrow{v_1} \ell/p$$

where β is the mod p Bockstein associated to the cofiber sequence $\ell \xrightarrow{p} \ell \xrightarrow{i} \ell/p$ and the cofiber of v_1 is $H\mathbb{Z}/p$. We get a map of cofiber sequences

$$\begin{tikzcd} \ell/p \wedge \ell/p \arrow[r, "\mu-\mu\tau"] \arrow[d, "\mu(\beta\wedge\beta)"'] & \ell/p \arrow[r] \arrow[d, equal] & C \arrow[d] \\ \Sigma^2\ell/p \arrow[r, "v_1"] & \ell/p \arrow[r] & H\mathbb{Z}/p\,, \end{tikzcd}$$

so there is a class in $H_3(C)$ that maps to $\bar\xi_1 \otimes \bar\xi_1$ in $H_2(\ell/p \wedge \ell/p)$ and to $\bar\xi_1\sigma\bar\xi_1$ in $H_3(THH(\ell/p))$, which also maps to $\bar\xi_2$ in the cofiber of v_1, i.e., whose A_*-coaction contains the term $\bar\xi_2 \otimes 1$. (The classes $\bar\tau_0$ and $\bar\tau_1$ go by the names $\bar\xi_1$ and $\bar\xi_2$ at $p = 2$.)

For odd primes there is a similar interpretation of how the non-commutativity of the product on ℓ/p provides an obstruction to splitting off the 0-simplices from the $(p-1)$-skeleton of $THH(\ell/p)$, where the cyclic permutation of the p factors in the $(p-1)$-simplex $\bar\tau_0(\sigma\bar\tau_0)^{p-1}$, represented by the Hochschild cycle $\bar\tau_0 \otimes \cdots \otimes \bar\tau_0$, plays a similar role to the twist map τ above.

Let

$$H_*(THH(\ell))/(\sigma\bar\xi_1) \cong H_*(\ell) \otimes E(\sigma\bar\xi_2) \otimes P(\sigma\bar\tau_2)$$

denote the quotient algebra of $H_*(THH(\ell))$ by the ideal generated by $\sigma\bar\xi_1$.

Lemma 3.6. *There is an isomorphism of $H_*(THH(\ell))$-modules*

$$H_*(THH(\ell/p)) \cong H_*(THH(\ell))/(\sigma\bar\xi_1) \otimes \mathbb{F}_p\{1, \bar\tau_0, \sigma\bar\tau_0, \bar\tau_0\sigma\bar\tau_0, \dots, (\sigma\bar\tau_0)^{p-1}, y\}\,.$$

Hence $H_*(THH(\ell/p))$ is a free module of rank $2p$ over $H_*(THH(\ell))/(\sigma\bar\xi_1)$, generated by classes

$$1\ ,\ \bar\tau_0\ ,\ \sigma\bar\tau_0\ ,\ \bar\tau_0\sigma\bar\tau_0\ ,\ \dots\ ,\ (\sigma\bar\tau_0)^{p-1}\ ,\ y$$

in degrees 0 through $2p-1$. These generators are represented in $E^\infty_{**}(\ell/p)$ by the classes

$$1\ ,\ \bar\tau_0\ ,\ \sigma\bar\tau_0\ ,\ \bar\tau_0\sigma\bar\tau_0\ ,\ \dots\ ,\ (\sigma\bar\tau_0)^{p-1}\ ,\ \bar\tau_0(\sigma\bar\tau_0)^{p-1}\,,$$

and map under π_* to classes with the same names in $H_*(THH(\mathbb{Z}/p))$, except for y, which maps to

$$\bar\tau_0(\sigma\bar\tau_0)^{p-1} - \bar\tau_1\,.$$

The A_*-comodule coactions are given by

$$\nu((\sigma\bar\tau_0)^i) = 1 \otimes (\sigma\bar\tau_0)^i$$

for $0 \le i \le p-1$,

$$\nu(\bar\tau_0(\sigma\bar\tau_0)^i) = 1 \otimes \bar\tau_0(\sigma\bar\tau_0)^i + \bar\tau_0 \otimes (\sigma\bar\tau_0)^i$$

for $0 \le i \le p-2$, and

$$\nu(y) = 1 \otimes y + \bar\tau_0 \otimes (\sigma\bar\tau_0)^{p-1} - \bar\tau_0 \otimes \bar\xi_1 - \bar\tau_1 \otimes 1\,.$$

Proof. $H_*(\ell/p)$ is freely generated as a module over $H_*(\ell)$ by 1 and $\bar\tau_0$, and the classes $\sigma\bar\xi_2$ and $\sigma\bar\tau_2$ in $H_*(THH(\ell))$ induce multiplication by the same symbols in $E^\infty_{**}(\ell/p)$, as given in (3.3). This generates all of $E^\infty_{**}(\ell/p)$ from the $2p$ classes $\bar\tau_0^\delta(\sigma\bar\tau_0)^i$ for $0 \le \delta \le 1$ and $0 \le i \le p-1$.

We claim that multiplication by $\sigma\bar\xi_1$ acts trivially on $H_*(THH(\ell/p))$. It suffices to verify this on the module generators $\bar\tau_0^\delta(\sigma\bar\tau_0)^i$, for which the product with $\sigma\bar\xi_1$ remains in the range of degrees where the map to $H_*(THH(\mathbb{Z}/p))$ is injective. The action of $\sigma\bar\xi_1$ is trivial on $H_*(THH(\mathbb{Z}/p))$, since $d^{p-1}(\gamma_p\sigma\bar\tau_0) = \sigma\bar\xi_1$ and $\epsilon(\sigma\bar\xi_1) = 0$, and this implies the claim.

The A_*-comodule coaction on each module generator, including y, is determined by that on its image under π_*. In the latter case, the thing to check is that

$$\begin{aligned}(1 \otimes \pi_*)(\nu(y)) &= \nu(\pi_*(y)) = \nu(\bar\tau_0(\sigma\bar\tau_0)^{p-1} - \bar\tau_1) \\ &= 1 \otimes \bar\tau_0(\sigma\bar\tau_0)^{p-1} + \bar\tau_0 \otimes (\sigma\bar\tau_0)^{p-1} - 1 \otimes \bar\tau_1 - \bar\tau_0 \otimes \bar\xi_1 - \bar\tau_1 \otimes 1\end{aligned}$$

equals

$$(1 \otimes \pi_*)(1 \otimes y + \bar\tau_0 \otimes (\sigma\bar\tau_0)^{p-1} - \bar\tau_0 \otimes \bar\xi_1 - \bar\tau_1 \otimes 1)\,.$$

□

We note that these results do not visibly depend on the particular choice of ℓ-algebra structure on ℓ/p.

4. Passage to $V(1)$-homotopy

For $p \geq 5$ the Smith–Toda complex $V(1) = S \cup_p e^1 \cup_{\alpha_1} e^{2p-1} \cup_p e^{2p}$ is a homotopy commutative ring spectrum [Smi70], [Oka84]. It is defined as the mapping cone of the Adams self-map $v_1 \colon \Sigma^{2p-2}V(0) \to V(0)$ of the mod p Moore spectrum $V(0) = S \cup_p e^1$. Hence there is a cofiber sequence

$$\Sigma^{2p-2}V(0) \xrightarrow{v_1} V(0) \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^{2p-1}V(0)\,.$$

The composite map $\beta_{1,1} = i_1 j_1 \colon V(1) \to \Sigma^{2p-1}V(1)$ defines the primary v_1-Bockstein homomorphism, acting naturally on $V(1)_*(X)$.

In this section we compute $V(1)_*THH(\ell/p)$ as a module over $V(1)_*THH(\ell)$, for any prime $p \geq 5$. The unique ring spectrum map from $V(1)$ to $H\mathbb{Z}/p$ induces the identification

$$H_*(V(1)) = E(\tau_0, \tau_1)$$

(no conjugations) as A_*-comodule subalgebras of A_*. Here

$$\begin{aligned}\nu(\tau_0) &= 1 \otimes \tau_0 + \tau_0 \otimes 1 \\ \nu(\tau_1) &= 1 \otimes \tau_1 + \xi_1 \otimes \tau_0 + \tau_1 \otimes 1\,.\end{aligned}$$

For each ℓ-algebra B, $V(1) \wedge THH(B)$ is a module spectrum over $V(1) \wedge THH(\ell)$ and thus over $V(1) \wedge \ell \simeq H\mathbb{Z}/p$, so $H_*(V(1) \wedge THH(B))$ is a sum of copies of A_* as an A_*-comodule. In particular, $V(1)_*THH(B) = \pi_*(V(1) \wedge THH(B))$ is naturally identified with the subgroup of A_*-comodule primitives in

$$H_*(V(1) \wedge THH(B)) \cong H_*(V(1)) \otimes H_*(THH(B))$$

with the diagonal A_*-comodule coaction. We write $v \wedge x$ for the image of $v \otimes x$ under this identification, with $v \in H_*(V(1))$ and $x \in H_*(THH(B))$. Let

$$\begin{aligned}\epsilon_0 &= 1 \wedge \bar\tau_0 + \tau_0 \wedge 1 \\ \epsilon_1 &= 1 \wedge \bar\tau_1 + \tau_0 \wedge \bar\xi_1 + \tau_1 \wedge 1 \\ \lambda_1 &= 1 \wedge \sigma\bar\xi_1 \\ \lambda_2 &= 1 \wedge \sigma\bar\xi_2 \\ \mu_0 &= 1 \wedge \sigma\bar\tau_0 \\ \mu_1 &= 1 \wedge \sigma\bar\tau_1 + \tau_0 \wedge \sigma\bar\xi_1 \\ \mu_2 &= 1 \wedge \sigma\bar\tau_2 + \tau_0 \wedge \sigma\bar\xi_2\,.\end{aligned} \tag{4.1}$$

These are all A_*-comodule primitive, where defined. By a dimension count,

$$\begin{aligned} V(1)_*THH(\mathbb{Z}/p) &= E(\epsilon_0,\epsilon_1)\otimes P(\mu_0) \\ V(1)_*THH(\mathbb{Z}_{(p)}) &= E(\epsilon_1)\otimes E(\lambda_1)\otimes P(\mu_1) \\ V(1)_*THH(\ell) &= E(\lambda_1,\lambda_2)\otimes P(\mu_2) \end{aligned} \tag{4.2}$$

as commutative $\mathbb{F}_p$-algebras. The map $\pi\colon \ell\to H\mathbb{Z}_{(p)}$ takes λ_2 to 0 and μ_2 to μ_1^p. The map $i : H\mathbb{Z}_{(p)}\to H\mathbb{Z}/p$ takes λ_1 to 0 and μ_1 to μ_0^p. Note that $\mu_2\in V(1)_{2p^2}THH(\ell)$ was simply denoted μ in [AR02].

In degrees $\le (2p-2)$ of $H_*(V(1)\wedge THH(\ell/p))$ the classes

$$\mu_0^i := 1\wedge(\sigma\bar\tau_0)^i \tag{4.3}$$

for $0\le i\le p-1$ and

$$\epsilon_0\mu_0^i := 1\wedge\bar\tau_0(\sigma\bar\tau_0)^i + \tau_0\wedge(\sigma\bar\tau_0)^i \tag{4.4}$$

for $0\le i\le p-2$ are A_*-comodule primitive, hence lift uniquely to $V(1)_*THH(\ell/p)$. These map to the classes $\epsilon_0^\delta\mu_0^i$ in $V(1)_*THH(\mathbb{Z}/p)$ for $0\le\delta\le 1$ and $0\le i\le p-1$, except that the degree bound excludes the top case of $\epsilon_0\mu_0^{p-1}$.

In degree $(2p-1)$ of $H_*(V(1)\wedge THH(\ell/p))$ we have generators $1\wedge\bar\xi_1\bar\tau_0$, $\tau_0\wedge(\sigma\bar\tau_0)^{p-1}$, $\tau_0\wedge\bar\xi_1$, $\tau_1\wedge 1$ and $1\wedge y$. These have coactions

$$\begin{aligned} \nu(1\wedge\bar\xi_1\bar\tau_0) &= 1\otimes 1\wedge\bar\xi_1\bar\tau_0 + \bar\tau_0\otimes 1\wedge\bar\xi_1 + \bar\xi_1\otimes 1\wedge\bar\tau_0 + \bar\xi_1\bar\tau_0\otimes 1\wedge 1 \\ \nu(\tau_0\wedge(\sigma\bar\tau_0)^{p-1}) &= 1\otimes\tau_0\wedge(\sigma\bar\tau_0)^{p-1} + \tau_0\otimes 1\wedge(\sigma\bar\tau_0)^{p-1} \\ \nu(\tau_0\wedge\bar\xi_1) &= 1\otimes\tau_0\wedge\bar\xi_1 + \tau_0\otimes 1\wedge\bar\xi_1 + \bar\xi_1\otimes\tau_0\wedge 1 + \bar\xi_1\tau_0\otimes 1\wedge 1 \\ \nu(\tau_1\wedge 1) &= 1\otimes\tau_1\wedge 1 + \xi_1\otimes\tau_0\wedge 1 + \tau_1\otimes 1\wedge 1 \end{aligned}$$

and

$$\nu(1\wedge y) = 1\otimes 1\wedge y + \bar\tau_0\otimes 1\wedge(\sigma\bar\tau_0)^{p-1} - \bar\tau_0\otimes 1\wedge\bar\xi_1 - \bar\tau_1\otimes 1\wedge 1\,.$$

Hence the sum

$$\bar\epsilon_1 := 1\wedge y + \tau_0\wedge(\sigma\bar\tau_0)^{p-1} - \tau_0\wedge\bar\xi_1 - \tau_1\wedge 1 \tag{4.5}$$

is A_*-comodule primitive. Its image under π_* in $H_*(V(1)\wedge THH(\mathbb{Z}/p))$ is

$$\epsilon_0\mu_0^{p-1} - \epsilon_1 = 1\wedge\bar\tau_0(\sigma\bar\tau_0)^{p-1} + \tau_0\wedge(\sigma\bar\tau_0)^{p-1} - 1\wedge\bar\tau_1 - \tau_0\wedge\bar\xi_1 - \tau_1\wedge 1\,.$$

Let

$$V(1)_*THH(\ell)/(\lambda_1)\cong E(\lambda_2)\otimes P(\mu_2)$$

be the quotient algebra of $V(1)_*THH(\ell)$ by the ideal generated by λ_1.

Proposition 4.6. *There is an isomorphism of $V(1)_*THH(\ell)$-modules*

$$V(1)_*THH(\ell/p) = V(1)_*THH(\ell)/(\lambda_1)\otimes\mathbb{F}_p\{1,\epsilon_0,\mu_0,\epsilon_0\mu_0,\dots,\mu_0^{p-1},\bar\epsilon_1\}\,,$$

where the classes μ_0^i, $\epsilon_0\mu_0^i$ and $\bar\epsilon_1$ are defined in (4.3), (4.4) and (4.5) above. Multiplication by λ_1 is 0, so this is a free module on the $2p$ generators

$$1\ ,\ \epsilon_0\ ,\ \mu_0\ ,\ \epsilon_0\mu_0\ ,\ \dots\ ,\ \mu_0^{p-1}\ ,\ \bar\epsilon_1$$

over $V(1)_*THH(\ell)/(\lambda_1)$. The map π_* to $V(1)_*THH(\mathbb{Z}/p)$ takes $\epsilon_0^\delta\mu_0^i$ in degree $0\le\delta+2i\le 2p-2$ to $\epsilon_0^\delta\mu_0^i$, and takes $\bar\epsilon_1$ in degree $(2p-1)$ to $\epsilon_0\mu_0^{p-1}-\epsilon_1$.

Proof. Additively, this follows by another dimension count. The multiplication by λ_1 is 0 for degree and filtration reasons: λ_1 has Bökstedt filtration 1 and cannot map to $\bar\epsilon_1$ in Bökstedt filtration $(p-1)$. Similarly in higher degrees. □

5. The C_p-Tate construction

Let $C = C_{p^n}$ denote the cyclic group of order p^n, considered as a closed subgroup of the circle group S^1. For each spectrum X with C-action, $X_{hC} = EC_+ \wedge_C X$ and $X^{hC} = F(EC_+, X)^C$ denote its homotopy orbit and homotopy fixed point spectra, as usual. We now write $X^{tC} = [\widetilde{EC} \wedge F(EC_+, X)]^C$ for the C-Tate construction on X, which was denoted $t_C(X)^C$ in [GM95] and $\hat{\mathbb{H}}(C, X)$ in [AR02]. There are C-homotopy fixed point and C-Tate spectral sequences in $V(1)$-homotopy for X, with

$$E^2_{s,t}(C, X) = H^{-s}_{gp}(C; V(1)_t(X)) \Longrightarrow V(1)_{s+t}(X^{hC})$$

and

$$\hat{E}^2_{s,t}(C, X) = \hat{H}^{-s}_{gp}(C; V(1)_t(X)) \Longrightarrow V(1)_{s+t}(X^{tC}) \,.$$

We write $H^*_{gp}(C_{p^n}; \mathbb{F}_p) = E(u_n) \otimes P(t)$ and $\hat{H}^*_{gp}(C_{p^n}; \mathbb{F}_p) = E(u_n) \otimes P(t^{\pm 1})$ with u_n in degree 1 and t in degree 2. So u_n, t and $x \in V(1)_t(X)$ have bidegree $(-1, 0)$, $(-2, 0)$ and $(0, t)$ in either spectral sequence, respectively. See [HM03, §4.3] for proofs of the multiplicative properties of these spectral sequences.

We are principally interested in the case when $X = THH(B)$, with the S^1-action given by the cyclic structure. It is a cyclotomic spectrum, in the sense of [HM97], leading to the commutative diagram

$$\begin{CD}
THH(B)_{hC_{p^n}} @>{N}>> THH(B)^{C_{p^n}} @>{R}>> THH(B)^{C_{p^{n-1}}} \\
@| @VV{\Gamma_n}V @VV{\hat{\Gamma}_n}V \\
THH(B)_{hC_{p^n}} @>{N^h}>> THH(B)^{hC_{p^n}} @>{R^h}>> THH(B)^{tC_{p^n}}
\end{CD}$$

of horizontal cofiber sequences. We abbreviate $\hat{E}^2_{**}(C, THH(B))$ to $\hat{E}^2_{**}(C, B)$, etc. When B is a commutative S-algebra, this is a commutative algebra spectral sequence, and when B is an associative A-algebra, with A commutative, then $\hat{E}^*(C, B)$ is a module spectral sequence over $\hat{E}^*(C, A)$. The map R^h corresponds to the inclusion $E^2_{**}(C, B) \to \hat{E}^2_{**}(C, B)$ from the second quadrant to the upper half-plane, for connective B.

In this section we compute $V(1)_* THH(\ell/p)^{tC_p}$ by means of the C_p-Tate spectral sequence in $V(1)$-homotopy for $THH(\ell/p)$. In Propositions 5.8 and 5.9 we show that the comparison map $\hat{\Gamma}_1 \colon V(1)_* THH(\ell/p) \to V(1)_* THH(\ell/p)^{tC_p}$ is $(2p-2)$-coconnected and can be identified with the algebraic localization homomorphism that inverts μ_2.

First we recall the structure of the C_p-Tate spectral sequence for $THH(\mathbb{Z}/p)$, with $V(0)$- and $V(1)$-coefficients. We have $V(0)_* THH(\mathbb{Z}/p) = E(\epsilon_0) \otimes P(\mu_0)$, and with an obvious notation the E^2-terms are

$$\begin{aligned}
\hat{E}^2_{**}(C_p, \mathbb{Z}/p; V(0)) &= E(u_1) \otimes P(t^{\pm 1}) \otimes E(\epsilon_0) \otimes P(\mu_0) \\
\hat{E}^2_{**}(C_p, \mathbb{Z}/p) &= E(u_1) \otimes P(t^{\pm 1}) \otimes E(\epsilon_0, \epsilon_1) \otimes P(\mu_0) \,.
\end{aligned}$$

In each C-Tate spectral sequence we have a first differential

$$d^2(x) = t \cdot \sigma x \,,$$

see e.g. [Rog98, 3.3]. We easily deduce $\sigma\epsilon_0 = \mu_0$ and $\sigma\epsilon_1 = \mu_0^p$ from (4.1), so

$$\begin{aligned}
\hat{E}^3_{**}(C_p, \mathbb{Z}/p; V(0)) &= E(u_1) \otimes P(t^{\pm 1}) \\
\hat{E}^3_{**}(C_p, \mathbb{Z}/p) &= E(u_1) \otimes P(t^{\pm 1}) \otimes E(\epsilon_0 \mu_0^{p-1} - \epsilon_1) \,.
\end{aligned}$$

Thus the $V(0)$-homotopy spectral sequence collapses at $\hat{E}^3 = \hat{E}^\infty$. By naturality with respect to the map $i_1 \colon V(0) \to V(1)$, all the classes on the horizontal axis of $\hat{E}^3(C_p, \mathbb{Z}/p)$ are infinite cycles, so also the latter spectral sequence collapses at $\hat{E}^3_{**}(C_p, \mathbb{Z}/p)$.

We know from [HM97, Prop. 5.3] that the comparison map

$$\hat{\Gamma}_1 : V(0)_* THH(\mathbb{Z}/p) \to V(0)_* THH(\mathbb{Z}/p)^{tC_p}$$

takes $\epsilon_0^\delta \mu_0^i$ to $(u_1 t^{-1})^\delta t^{-i}$, for all $0 \le \delta \le 1$, $i \ge 0$. In particular, the integral map $\hat{\Gamma}_1 \colon \pi_* THH(\mathbb{Z}/p) \to \pi_* THH(\mathbb{Z}/p)^{tC_p}$ is (-2)-coconnected, meaning that it induces an injection in degree (-2) and an isomorphism in all higher degrees. From this we can deduce the following behavior of the comparison map $\hat{\Gamma}_1$ in $V(1)$-homotopy.

Lemma 5.1. *The map*

$$\hat{\Gamma}_1 \colon V(1)_* THH(\mathbb{Z}/p) \to V(1)_* THH(\mathbb{Z}/p)^{tC_p}$$

takes the classes $\epsilon_0^\delta \mu_0^i$ from $V(0)_* THH(\mathbb{Z}/p)$, for $0 \le \delta \le 1$ and $i \ge 0$, to classes represented in $\hat{E}^\infty_{**}(C_p, \mathbb{Z}/p)$ by $(u_1 t^{-1})^\delta t^{-i}$ (on the horizontal axis).

Furthermore, it takes the class $\epsilon_0 \mu_0^{p-1} - \epsilon_1$ in degree $(2p-1)$ to a class represented by $\epsilon_0 \mu_0^{p-1} - \epsilon_1$ (on the vertical axis).

Proof. The classes $\epsilon_0^\delta \mu_0^i$ are in the image from $V(0)$-homotopy, and we recalled above that they are detected by $(u_1 t^{-1})^\delta t^{-i}$ in the $V(0)$-homotopy C_p-Tate spectral sequence for $THH(\mathbb{Z}/p)$. By naturality along $i_1 \colon V(0) \to V(1)$, they are detected by the same (nonzero) classes in the $V(1)$-homotopy spectral sequence $\hat{E}^\infty_{**}(C_p, \mathbb{Z}/p)$.

To find the representative for $\hat{\Gamma}_1(\epsilon_0 \mu_0^{p-1} - \epsilon_1)$ in degree $(2p-1)$, we appeal to the cyclotomic trace map from algebraic K-theory, or more precisely, to the commutative diagram

(5.2)
$$\begin{tikzcd}
& K(B) \arrow[dl, "tr"'] \arrow[d, "tr_1"] \arrow[dr, "tr"] & \\
THH(B) & THH(B)^{C_p} \arrow[l, "F"'] \arrow[r, "R"] \arrow[d, "\Gamma_1"] & THH(B) \arrow[d, "\hat{\Gamma}_1"] \\
& THH(B)^{hC_p} \arrow[ul] \arrow[r, "R^h"] & THH(B)^{tC_p} .
\end{tikzcd}$$

The Bökstedt trace map $tr \colon K(B) \to THH(B)$ admits a preferred lift tr_n through each fixed point spectrum $THH(B)^{C_{p^n}}$, which homotopy equalizes the iterated restriction and Frobenius maps R^n and F^n to $THH(B)$, see [BHM93, 2.5]. In particular, the circle action and the σ-operator act trivially on classes in the image of tr.

In the case $B = H\mathbb{Z}/p$ we know that $K(\mathbb{Z}/p)_p \simeq H\mathbb{Z}_p$, so $V(1)_* K(\mathbb{Z}/p) = E(\bar{\epsilon}_1)$, where the v_1-Bockstein of $\bar{\epsilon}_1$ is -1. The Bökstedt trace image $tr(\bar{\epsilon}_1) \in V(1)_* THH(\mathbb{Z}/p)$ lies in $\mathbb{F}_p\{\epsilon_1, \epsilon_0 \mu_0^{p-1}\}$, has v_1-Bockstein $tr(-1) = -1$ and suspends by σ to 0. Hence

$$tr(\bar{\epsilon}_1) = \epsilon_0 \mu_0^{p-1} - \epsilon_1 .$$

As we recalled above, the map $\hat{\Gamma}_1 \colon \pi_* THH(\mathbb{Z}/p) \to \pi_* THH(\mathbb{Z}/p)^{tC_p}$ is (-2)-coconnected, so the corresponding map in $V(1)$-homotopy is at least $(2p-2)$-coconnected. Thus it takes $\epsilon_0 \mu_0^{p-1} - \epsilon_1$ to a nonzero class in $V(1)_* THH(\mathbb{Z}/p)^{tC_p}$, represented somewhere in total degree $(2p-1)$ of $\hat{E}^\infty_{**}(C_p, \mathbb{Z}/p)$, in the lower right hand corner of the diagram.

Going down the middle of the diagram, we reach a class $(\Gamma_1 \circ tr_1)(\bar\epsilon_1)$, represented in total degree $(2p-1)$ of the left half-plane C_p-homotopy fixed point spectral sequence $E^\infty_{**}(C_p, \mathbb{Z}/p)$. Its image under the edge homomorphism to $V(1)_*THH(\mathbb{Z}/p)$ equals $(F \circ tr_1)(\bar\epsilon_1) = tr(\bar\epsilon_1)$, hence $(\Gamma_1 \circ tr_1)(\bar\epsilon_1)$ is represented by $\epsilon_0\mu_0^{p-1} - \epsilon_1$ in $E^\infty_{0,2p-1}(C_p, \mathbb{Z}/p)$. Its image under R^h in the C_p-Tate spectral sequence is the generator of $\hat{E}^\infty_{0,2p-1}(C_p, \mathbb{Z}/p) = \mathbb{F}_p\{\epsilon_0\mu_0^{p-1} - \epsilon_1\}$, hence that generator is the E^∞-representative of $\hat\Gamma_1(\epsilon_0\mu_0^{p-1} - \epsilon_1)$. □

We can lift the algebraic K-theory class $\bar\epsilon_1$ to ℓ/p.

Definition 5.3. The $(2p-2)$-connected map $\pi \colon \ell/p \to H\mathbb{Z}/p$ induces a $(2p-1)$-connected map $V(1)_*K(\ell/p) \to V(1)_*K(\mathbb{Z}/p) = E(\bar\epsilon_1)$, by [BM94, 10.9]. We can therefore choose a class
$$\bar\epsilon_1^K \in V(1)_{2p-1}K(\ell/p)$$
that maps to the generator $\bar\epsilon_1$ in $V(1)_{2p-1}K(\mathbb{Z}/p) \cong \mathbb{Z}/p$.

Lemma 5.4. *The Bökstedt trace $tr \colon V(1)_*K(\ell/p) \to V(1)_*THH(\ell/p)$ takes $\bar\epsilon_1^K$ to $\bar\epsilon_1$.*

Proof. In the commutative square
$$\begin{tikzcd} V(1)_*K(\ell/p) \arrow[r, "tr"] \arrow[d, "\pi_*"] & V(1)_*THH(\ell/p) \arrow[d, "\pi_*"] \\ V(1)_*K(\mathbb{Z}/p) \arrow[r, "tr"] & V(1)_*THH(\mathbb{Z}/p) \end{tikzcd}$$
the trace image $tr(\bar\epsilon_1^K)$ in $V(1)_*THH(\ell/p)$ must map under π_* to $tr(\bar\epsilon_1) = \epsilon_0\mu_0^{p-1} - \epsilon_1$ in $V(1)_*THH(\mathbb{Z}/p)$, which by Proposition 4.6 characterizes it as being equal to the class $\bar\epsilon_1$. Hence $tr(\bar\epsilon_1^K) = \bar\epsilon_1$. □

Next we turn to the C_p-Tate spectral sequence $\hat{E}^*(C_p, \ell/p)$ in $V(1)$-homotopy for $THH(\ell/p)$. Its E^2-term is
$$\hat{E}^2_{**}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm 1}) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar\epsilon_1\} \otimes E(\lambda_2) \otimes P(\mu_2)\,.$$
We have $d^2(x) = t \cdot \sigma x$, where
$$\sigma(\epsilon_0^\delta \mu_0^{i-1}) = \begin{cases} \mu_0^i & \text{for } \delta = 1,\ 0 < i < p, \\ 0 & \text{otherwise} \end{cases}$$
is readily deduced from (4.1), and $\sigma(\bar\epsilon_1) = 0$ since $\bar\epsilon_1$ is in the image of tr. Thus
$$\hat{E}^3_{**}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm 1}) \otimes E(\bar\epsilon_1) \otimes E(\lambda_2) \otimes P(t\mu_2)\,. \tag{5.5}$$
We prefer to use $t\mu_2$ rather than μ_2 as a generator, since it represents multiplication by v_2 in all module spectral sequences over $E^*(S^1, \ell)$, by [AR02, 4.8].

To proceed, we shall use that $\hat{E}^*(C_p, \ell/p)$ is a module over the spectral sequence for $THH(\ell)$. We therefore recall the structure of the latter spectral sequence, from [AR02, 5.5]. It begins
$$\hat{E}^2_{**}(C_p, \ell) = E(u_1) \otimes P(t^{\pm 1}) \otimes E(\lambda_1, \lambda_2) \otimes P(\mu_2)\,.$$
The classes λ_1, λ_2 and $t\mu_2$ are infinite cycles, and the differentials
$$d^{2p}(t^{1-p}) = t\lambda_1$$
$$d^{2p^2}(t^{p-p^2}) = t^p\lambda_2$$
$$d^{2p^2+1}(u_1 t^{-p^2}) = t\mu_2$$

(up to units in $\mathbb{F}_p$, which we will always suppress) leave the terms

$$\begin{aligned}\hat{E}^{2p+1}_{**}(C_p,\ell) &= E(u_1,\lambda_1,\lambda_2)\otimes P(t^{\pm p}, t\mu_2)\\ \hat{E}^{2p^2+1}_{**}(C_p,\ell) &= E(u_1,\lambda_1,\lambda_2)\otimes P(t^{\pm p^2}, t\mu_2)\\ \hat{E}^{2p^2+2}_{**}(C_p,\ell) &= E(\lambda_1,\lambda_2)\otimes P(t^{\pm p^2})\end{aligned}$$

with $\hat{E}^{2p^2+2}=\hat{E}^\infty$, converging to $V(1)_*THH(\ell)^{tC_p}$. The comparison map $\hat{\Gamma}_1$ takes λ_1, λ_2 and μ_2 to λ_1, λ_2 and t^{-p^2}, respectively, inducing the algebraic localization map and identification

$$\hat{\Gamma}_1\colon V(1)_*THH(\ell)\to V(1)_*THH(\ell)[\mu_2^{-1}]\cong V(1)_*THH(\ell)^{tC_p}\,.$$

Lemma 5.6. *In $\hat{E}^*(C_p,\ell/p)$, the class u_1t^{-p} supports the nonzero differential*

$$d^{2p^2}(u_1t^{-p}) = u_1t^{p^2-p}\lambda_2\,,$$

and does not survive to the E^∞-term.

Proof. In $\hat{E}^*(C_p,\ell)$, there is such a nonzero differential. By naturality along $i\colon \ell\to\ell/p$, it follows that there is also such a differential in $\hat{E}^*(C_p,\ell/p)$. It remains to argue that the target is nonzero. Considering the E^3-term in (5.5), the only possible source of a previous differential hitting $u_1t^{p^2-p}\lambda_2$ is $\bar{\epsilon}_1$. But $\bar{\epsilon}_1$ is in an even column and $u_1t^{p^2-p}\lambda_2$ is in an odd column. By naturality with respect to the Frobenius (group restriction) map from the S^1-Tate spectral sequence to the C_p-Tate spectral sequence, which takes $\hat{E}^2_{**}(S^1,B)$ isomorphically to the even columns of $\hat{E}^2_{**}(C_p,B)$, any such differential from an even to an odd column must be zero. □

To determine the map $\hat{\Gamma}_1$ we use naturality with respect to the map $\pi\colon \ell/p\to H\mathbb{Z}/p$.

Lemma 5.7. *The classes $1,\epsilon_0,\mu_0,\epsilon_0\mu_0,\dots,\mu_0^{p-1}$ and $\bar{\epsilon}_1$ in $V(1)_*THH(\ell/p)$ map under $\hat{\Gamma}_1$ to classes in $V(1)_*THH(\ell/p)^{tC_p}$ that are represented in $\hat{E}^\infty_{**}(C_p,\ell/p)$ by the permanent cycles $(u_1t^{-1})^\delta t^{-i}$ (on the horizontal axis) in degrees $\le (2p-2)$, and by the permanent cycle $\bar{\epsilon}_1$ (on the vertical axis) in degree $(2p-1)$.*

Proof. In the commutative square

$$\begin{array}{ccc} V(1)_*THH(\ell/p) & \xrightarrow{\hat{\Gamma}_1} & V(1)_*THH(\ell/p)^{tC_p}\\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi_*}\\ V(1)_*THH(\mathbb{Z}/p) & \xrightarrow{\hat{\Gamma}_1} & V(1)_*THH(\mathbb{Z}/p)^{tC_p}\end{array}$$

the classes $\epsilon_0^\delta\mu_0^i$ in the upper left hand corner map to classes in the lower right hand corner that are represented by $(u_1t^{-1})^\delta t^{-i}$ in degrees $\le (2p-2)$, and $\bar{\epsilon}_1$ maps to $\epsilon_0\mu_0^{p-1}-\epsilon_1$ in degree $(2p-1)$. This follows by combining Proposition 4.6 and Lemma 5.1.

The first $(2p-1)$ of these are represented in maximal filtration (on the horizontal axis), so their images in the upper right hand corner must be represented by permanent cycles $(u_1t^{-1})^\delta t^{-i}$ in the Tate spectral sequence $\hat{E}^\infty_{**}(C_p,\ell/p)$.

The image of the last class, $\bar{\epsilon}_1$, in the upper right hand corner could either be represented by $\bar{\epsilon}_1$ in bidegree $(0,2p-1)$ or by u_1t^{-p} in bidegree $(2p-1,0)$. However, the last class supports a differential $d^{2p^2}(u_1t^{-p})=u_1t^{p^2-p}\lambda_2$, by Lemma 5.6 above. This only leaves the other possibility, that $\hat{\Gamma}_1(\bar{\epsilon}_1)$ is represented by $\bar{\epsilon}_1$ in $\hat{E}^\infty_{**}(C_p,\ell/p)$. □

We proceed to determine the differential structure in $\hat{E}^*(C_p, \ell/p)$, making use of the permanent cycles identified above.

Proposition 5.8. *The C_p-Tate spectral sequence in $V(1)$-homotopy for $THH(\ell/p)$ has*

$$\hat{E}^3_{**}(C_p, \ell/p) = E(u_1, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm 1}, t\mu_2)\,.$$

It has differentials generated by

$$d^{2p^2-2p+2}(t^{p-p^2} \cdot t^{-i}\bar{\epsilon}_1) = t\mu_2 \cdot t^{-i}$$

for $0 < i < p$, $d^{2p^2}(t^{p-p^2}) = t^p\lambda_2$ and $d^{2p^2+1}(u_1 t^{-p^2}) = t\mu_2$. The following terms are

$$\begin{aligned}
\hat{E}^{2p^2-2p+3}_{**}(C_p, \ell/p) &= E(u_1, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p}) \\
&\quad \oplus E(u_1, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p}, t\mu_2) \\
\hat{E}^{2p^2+1}_{**}(C_p, \ell/p) &= E(u_1, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\
&\quad \oplus E(u_1, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^2}, t\mu_2) \\
\hat{E}^{2p^2+2}_{**}(C_p, \ell/p) &= E(u_1, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\
&\quad \oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^2})\,.
\end{aligned}$$

The last term can be rewritten as

$$\hat{E}^\infty(C_p, \ell/p) = \big(E(u_1) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \oplus E(\bar{\epsilon}_1)\big) \otimes E(\lambda_2) \otimes P(t^{\pm p^2})\,.$$

Proof. We have already identified the E^2- and E^3-terms above. The E^3-term (5.5) is generated over $\hat{E}^3(C_p, \ell)$ by an $\mathbb{F}_p$-basis for $E(\bar{\epsilon}_1)$, so the next possible differential is induced by $d^{2p}(t^{1-p}) = t\lambda_1$. But multiplication by λ_1 is trivial in $V(1)_*THH(\ell/p)$, by Proposition 4.6, so $\hat{E}^3(C_p, \ell/p) = \hat{E}^{2p+1}(C_p, \ell/p)$. This term is generated over $\hat{E}^{2p+1}(C_p, \ell)$ by $P_p(t^{-1}) \otimes E(\bar{\epsilon}_1)$. Here $1, t^{-1}, \dots, t^{1-p}$ and $\bar{\epsilon}_1$ are permanent cycles, by Lemma 5.7. Any d^r-differential before d^{2p^2} must therefore originate on a class $t^{-i}\bar{\epsilon}_1$ for $0 < i < p$, and be of even length r, since these classes lie in even columns. For bidegree reasons, the first possibility is $r = 2p^2 - 2p + 2$, so $\hat{E}^3(C_p, \ell/p) = \hat{E}^{2p^2-2p+2}(C_p, \ell/p)$.

Multiplication by v_2 acts trivially on $V(1)_*THH(\ell)$ and $V(1)_*THH(\ell)^{tC_p}$ for degree reasons, and therefore also on $V(1)_*THH(\ell/p)$ and $V(1)_*THH(\ell/p)^{tC_p}$ by the module structure. The class v_2 maps to $t\mu_2$ in the S^1-Tate spectral sequence for ℓ, as recalled above, so multiplication by v_2 is represented by multiplication by $t\mu_2$ in the C_p-Tate spectral sequence for ℓ/p. Applied to the permanent cycles $(u_1 t^{-1})^\delta t^{-i}$ in degrees $\leq (2p-2)$, this implies that the products

$$t\mu_2 \cdot (u_1 t^{-1})^\delta t^{-i}$$

must be infinite cycles representing zero, i.e., they must be hit by differentials. In the cases $\delta = 1$, $0 \leq i \leq p-2$, these classes in odd columns cannot be hit by differentials of odd length, such as d^{2p^2+1}, so the only possibility is

$$d^{2p^2-2p+2}(t^{p-p^2} \cdot (u_1 t^{-1}) t^{-i}\bar{\epsilon}_1) = t\mu_2 \cdot (u_1 t^{-1}) t^{-i}$$

for $0 \leq i \leq p-2$. By the module structure (consider multiplication by u_1) it follows that

$$d^{2p^2-2p+2}(t^{p-p^2} \cdot t^{-i}\bar{\epsilon}_1) = t\mu_2 \cdot t^{-i}$$

for $0 < i < p$. Hence we can compute from (5.5) that

$$\begin{aligned}\hat{E}^{2p^2-2p+3}_{**}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm p}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\ \oplus E(u_1) \otimes P(t^{\pm p}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2) \otimes P(t\mu_2)\,.\end{aligned}$$

This is generated over $\hat{E}^{2p+1}(C_p, \ell)$ by the permanent cycles $1, t^{-1}, \dots, t^{1-p}$ and $\bar{\epsilon}_1$, so the next differential is induced by $d^{2p^2}(t^{p-p^2}) = t^p\lambda_2$. This leaves

$$\begin{aligned}\hat{E}^{2p^2+1}_{**}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm p^2}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\ \oplus E(u_1) \otimes P(t^{\pm p^2}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2) \otimes P(t\mu_2)\,.\end{aligned}$$

Finally, $d^{2p^2+1}(u_1 t^{-p^2}) = t\mu_2$ applies, and leaves

$$\begin{aligned}\hat{E}^{2p^2+2}_{**}(C_p, \ell/p) = E(u_1) \otimes P(t^{\pm p^2}) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes E(\lambda_2) \\ \oplus P(t^{\pm p^2}) \otimes E(\bar{\epsilon}_1) \otimes E(\lambda_2)\,.\end{aligned}$$

For bidegree reasons, $\hat{E}^{2p^2+2} = \hat{E}^{\infty}$. □

Proposition 5.9. *The comparison map $\hat{\Gamma}_1$ takes the classes $\epsilon_0^\delta \mu_0^i$, $\bar{\epsilon}_1$, λ_2 and μ_2 in $V(1)_*THH(\ell/p)$ to classes in $V(1)_*THH(\ell/p)^{tC_p}$ represented by $(u_1 t^{-1})^\delta t^{-i}$, $\bar{\epsilon}_1$, λ_2 and t^{-p^2} in $\hat{E}^{\infty}_{**}(C_p, \ell/p)$, respectively. Thus*

$$V(1)_*THH(\ell/p)^{tC_p} \cong \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar{\epsilon}_1\} \otimes E(\lambda_2) \otimes P(\mu_2^{\pm 1})$$

and $\hat{\Gamma}_1$ factors as the algebraic localization map and identification

$$\hat{\Gamma}_1 \colon V(1)_*THH(\ell/p) \to V(1)_*THH(\ell/p)[\mu_2^{-1}] \cong V(1)_*THH(\ell/p)^{tC_p}\,.$$

In particular, this map is $(2p-2)$-coconnected.

Proof. The action of the map $\hat{\Gamma}_1$ on the classes $1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}$ and $\bar{\epsilon}_1$ was given in Lemma 5.7, and the action on the classes λ_2 and μ_2 was already recalled from [AR02]. The structure of $V(1)_*THH(\ell/p)^{tC_p}$ is then immediate from the E^∞-term in Proposition 5.8. The top class not in the image of $\hat{\Gamma}_1$ is $\bar{\epsilon}_1\lambda_2\mu_2^{-1}$, in degree $(2p-2)$. □

Recall that

$$TF(B) = \operatorname*{holim}_{n,F} THH(B)^{C_{p^n}}$$
$$TR(B) = \operatorname*{holim}_{n,R} THH(B)^{C_{p^n}}$$

are defined as the homotopy limits over the Frobenius and the restriction maps

$$F, R \colon THH(B)^{C_{p^n}} \to THH(B)^{C_{p^{n-1}}}\,,$$

respectively.

Corollary 5.10. *The comparison maps*

$$\Gamma_n \colon THH(\ell/p)^{C_{p^n}} \to THH(\ell/p)^{hC_{p^n}}$$
$$\hat{\Gamma}_n \colon THH(\ell/p)^{C_{p^{n-1}}} \to THH(\ell/p)^{tC_{p^n}}$$

for $n \geq 1$, and

$$\Gamma \colon TF(\ell/p) \to THH(\ell/p)^{hS^1}$$
$$\hat{\Gamma} \colon TF(\ell/p) \to THH(\ell/p)^{tS^1}$$

all induce $(2p-2)$-coconnected maps on $V(1)$-homotopy.

Proof. This follows from a theorem of Tsalidis [Tsa98] and Proposition 5.9 above, just like in [AR02, 5.7]. See also [BBLNR]. □

6. Higher fixed points

Let $n \geq 1$. Write $v_p(i)$ for the p-adic valuation of i. Define a numerical function $\rho(-)$ by

$$\rho(2k-1) = (p^{2k+1}+1)/(p+1) = p^{2k} - p^{2k-1} + \cdots - p + 1$$
$$\rho(2k) = (p^{2k+2}-p^2)/(p^2-1) = p^{2k} + p^{2k-2} + \cdots + p^2$$

for $k \geq 0$, so $\rho(-1) = 1$ and $\rho(0) = 0$. For even arguments, $\rho(2k) = r(2k)$ as defined in [AR02, 2.5].

In all of the following spectral sequences we know that λ_2, $t\mu_2$ and $\bar\epsilon_1$ are infinite cycles. For λ_2 and $\bar\epsilon_1$ this follows from the C_{p^n}-fixed point analogue of diagram (5.2), by [AR02, 2.8] and Lemma 5.4. For $t\mu_2$ it follows from [AR02, 4.8], by naturality.

Theorem 6.1. *The C_{p^n}-Tate spectral sequence in $V(1)$-homotopy for $THH(\ell/p)$ begins*

$$\hat E^2_{**}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar\epsilon_1\} \otimes P(t^{\pm 1}, \mu_2)$$

and converges to $V(1)_* THH(\ell/p)^{tC_{p^n}}$. It is a module spectral sequence over the algebra spectral sequence $\hat E^*(C_{p^n}, \ell)$ converging to $V(1)_* THH(\ell)^{tC_{p^n}}$.

There is an initial d^2-differential generated by

$$d^2(\epsilon_0\mu_0^{i-1}) = t\mu_0^i$$

for $0 < i < p$. Next, there are $2n$ families of even length differentials generated by

$$d^{2\rho(2k-1)}(t^{p^{2k-1}-p^{2k}+i} \cdot \bar\epsilon_1) = (t\mu_2)^{\rho(2k-3)} \cdot t^i$$

for $v_p(i) = 2k-2$, for each $k = 1, \dots, n$, and

$$d^{2\rho(2k)}(t^{p^{2k-1}-p^{2k}}) = \lambda_2 \cdot t^{p^{2k-1}} \cdot (t\mu_2)^{\rho(2k-2)}$$

for each $k = 1, \dots, n$. Finally, there is a differential of odd length generated by

$$d^{2\rho(2n)+1}(u_n \cdot t^{-p^{2n}}) = (t\mu_2)^{\rho(2n-2)+1}.$$

We shall prove Theorem 6.1 by induction on n. The base case $n = 1$ is covered by Proposition 5.8. We can therefore assume that Theorem 6.1 holds for some fixed $n \geq 1$. First we make the following deduction.

Corollary 6.2. *The initial differential in the C_{p^n}-Tate spectral sequence in $V(1)$-homotopy for $THH(\ell/p)$ leaves*

$$\hat E^3_{**}(C_{p^n}, \ell/p) = E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm 1}, t\mu_2).$$

The next $2n$ families of differentials leave the intermediate terms

$$\hat E^{2\rho(1)+1}_{**}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p})$$
$$\oplus\, E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p}, t\mu_2)$$

(for $m = 1$),

$$\begin{aligned}\hat{E}^{2\rho(2m-1)+1}_{**}(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\oplus \bigoplus_{k=2}^{m} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ &\oplus \bigoplus_{k=2}^{m-1} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ &\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2)\end{aligned}$$

for $m = 2, \dots, n$, and

$$\begin{aligned}\hat{E}^{2\rho(2m)+1}_{**}(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\oplus \bigoplus_{k=2}^{m} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ &\oplus \bigoplus_{k=2}^{m} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ &\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m}}, t\mu_2)\end{aligned}$$

for $m = 1, \dots, n$. The final differential leaves the $E^{2\rho(2n)+2} = E^\infty$-term, equal to

$$\begin{aligned}\hat{E}^{\infty}_{**}(C_{p^n}, \ell/p) &= E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ &\oplus \bigoplus_{k=2}^{n} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ &\oplus \bigoplus_{k=2}^{n} E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ &\oplus E(\bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}) \otimes P_{\rho(2n-2)+1}(t\mu_2)\,.\end{aligned}$$

Proof. The statements about the E^3-, $E^{2\rho(1)+1}$- and $E^{2\rho(2)+1}$-terms are clear from Proposition 5.8. For each $m = 2, \dots, n$ we proceed by a secondary induction. The differential

$$d^{2\rho(2m-1)}(t^{p^{2m-1}-p^{2m}+i} \cdot \bar{\epsilon}_1) = (t\mu_2)^{\rho(2m-3)} \cdot t^i$$

for $v_p(i) = 2m-2$ is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-2}}, t\mu_2)$$

of the $E^{2\rho(2m-2)+1} = E^{2\rho(2m-1)}$-term, with homology

$$\begin{aligned}&E(u_n, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2m-2\} \otimes P_{\rho(2m-3)}(t\mu_2) \\ &\oplus E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2)\,.\end{aligned}$$

This gives the stated $E^{2\rho(2m-1)+1}$-term. Similarly, the differential

$$d^{2\rho(2m)}(t^{p^{2m-1}-p^{2m}}) = \lambda_2 \cdot t^{p^{2m-1}} \cdot (t\mu_2)^{\rho(2m-2)}$$

is non-trivial only on the summand

$$E(u_n, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2m-1}}, t\mu_2)$$

of the $E^{2\rho(2m-1)+1} = E^{2\rho(2m)}$-term, with homology

$$\begin{aligned} &E(u_n, \bar\epsilon_1) \otimes \mathbb{F}_p\{t^j\lambda_2 \mid v_p(j) = 2m-1\} \otimes P_{\rho(2m-2)}(t\mu_2) \\ &\quad \oplus E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^{2m}}, t\mu_2)\,. \end{aligned}$$

This gives the stated $E^{2\rho(2m)+1}$-term. The final differential

$$d^{2\rho(2n)+1}(u_n \cdot t^{-p^{2n}}) = (t\mu_2)^{\rho(2n-2)+1}$$

is non-trivial only on the summand

$$E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)$$

of the $E^{2\rho(2n)+1}$-term, with homology

$$E(\bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^{2n}}) \otimes P_{\rho(2n-2)+1}(t\mu_2)\,.$$

This gives the stated $E^{2\rho(2n)+2}$-term. At this stage there is no room for any further differentials, since the spectral sequence is concentrated in a narrower horizontal band than the vertical height of the following differentials. □

Next we compare the C_{p^n}-Tate spectral sequence with the C_{p^n}-homotopy spectral sequence obtained by restricting the E^2-term to the second quadrant ($s \le 0$, $t \ge 0$). It is algebraically easier to handle the latter after inverting μ_2, which can be interpreted as comparing $THH(\ell/p)$ with its C_p-Tate construction.

In general, there is a commutative diagram

(6.3)
$$\begin{array}{ccccc} THH(B)^{C_{p^n}} & \xrightarrow{R} & THH(B)^{C_{p^{n-1}}} & \xrightarrow{\Gamma_{n-1}} & THH(B)^{hC_{p^{n-1}}} \\ \downarrow{\scriptstyle \Gamma_n} & & \downarrow{\scriptstyle \hat\Gamma_n} & & \downarrow{\scriptstyle \hat\Gamma_1^{hC_{p^{n-1}}}} \\ THH(B)^{hC_{p^n}} & \xrightarrow{R^h} & THH(B)^{tC_{p^n}} & \xrightarrow{G_{n-1}} & (THH(B)^{tC_p})^{hC_{p^{n-1}}} \end{array}$$

where G_{n-1} is the comparison map from the $C_{p^{n-1}}$-fixed points to the $C_{p^{n-1}}$-homotopy fixed points of $THH(B)^{tC_p}$, in view of the identification

$$(THH(B)^{tC_p})^{C_{p^{n-1}}} = THH(B)^{tC_{p^n}}\,.$$

We are of course considering the case $B = \ell/p$. In $V(1)$-homotopy all four maps with labels containing Γ are $(2p-2)$-coconnected, by Corollary 5.10, so G_{n-1} is at least $(2p-1)$-coconnected. (We shall see in Lemma 6.11 that $V(1)_*G_{n-1}$ is an isomorphism in all degrees.) By Proposition 5.9 the map $\hat\Gamma_1$ precisely inverts μ_2, so the E^2-term of the C_{p^n}-homotopy fixed point spectral sequence in $V(1)$-homotopy for $THH(\ell/p)^{tC_p}$ is obtained by inverting μ_2 in $E^2_{**}(C_{p^n}, \ell/p)$. We denote it by $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$, even though in later terms only a power of μ_2 is present.

Theorem 6.4. *The C_{p^n}-homotopy fixed point spectral sequence $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$ in $V(1)$-homotopy for $THH(\ell/p)^{tC_p}$ begins*

$$\mu_2^{-1}E^2_{**}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{1, \epsilon_0, \mu_0, \epsilon_0\mu_0, \dots, \mu_0^{p-1}, \bar\epsilon_1\} \otimes P(t, \mu_2^{\pm 1})$$

and converges to $V(1)_*(THH(\ell/p)^{tC_p})^{hC_{p^n}}$, which receives a $(2p-2)$-coconnected map $(\hat\Gamma_1)^{hC_{p^n}}$ from $V(1)_*THH(\ell/p)^{hC_{p^n}}$. There is an initial d^2-differential generated by

$$d^2(\epsilon_0\mu_0^{i-1}) = t\mu_0^i$$

for $0 < i < p$. Next, there are $2n$ families of even length differentials generated by

$$d^{2\rho(2k-1)}(\mu_2^{p^{2k}-p^{2k-1}+j} \cdot \bar\epsilon_1) = (t\mu_2)^{\rho(2k-1)} \cdot \mu_2^j$$

for $v_p(j) = 2k-2$, for each $k = 1, \dots, n$, and

$$d^{2\rho(2k)}(\mu_2^{p^{2k}-p^{2k-1}}) = \lambda_2 \cdot \mu_2^{-p^{2k-1}} \cdot (t\mu_2)^{\rho(2k)}$$

for each $k = 1, \dots, n$. Finally, there is a differential of odd length generated by

$$d^{2\rho(2n)+1}(u_n \cdot \mu_2^{p^{2n}}) = (t\mu_2)^{\rho(2n)+1}\,.$$

Proof. The differential pattern follows from Theorem 6.1 by naturality with respect to the maps of spectral sequences

$$\mu_2^{-1}E^*(C_{p^n}, \ell/p) \xleftarrow{\hat\Gamma_1^{hC_{p^n}}} E^*(C_{p^n}, \ell/p) \xrightarrow{R^h} \hat E^*(C_{p^n}, \ell/p)$$

induced by $\hat\Gamma_1^{hC_{p^n}}$ and R^h. The first inverts μ_2 and the second inverts t, at the level of E^2-terms. We are also using that $t\mu_2$, the image of v_2, multiplies as an infinite cycle in all of these spectral sequences. □

Corollary 6.5. *The initial differential in the C_{p^n}-homotopy fixed point spectral sequence in $V(1)$-homotopy for $THH(\ell/p)^{tC_p}$ leaves*

$$\begin{aligned}\mu_2^{-1}E^3_{**}(C_{p^n}, \ell/p) = {} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ & \oplus E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm 1}, t\mu_2)\,.\end{aligned}$$

The next $2n$ families of differentials leave the intermediate terms

$$\begin{aligned}\mu_2^{-1}E^{2\rho(2m-1)+1}_{**}(C_{p^n}, \ell/p) = {} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ & \oplus \bigoplus_{k=1}^{m} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\ & \oplus \bigoplus_{k=1}^{m-1} E(u_n, \bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^j \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\ & \oplus E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-1}}, t\mu_2)\end{aligned}$$

and

$$\begin{aligned}\mu_2^{-1}E^{2\rho(2m)+1}_{**}(C_{p^n}, \ell/p) = {} & E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ & \oplus \bigoplus_{k=1}^{m} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\ & \oplus \bigoplus_{k=1}^{m} E(u_n, \bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^j \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\ & \oplus E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m}}, t\mu_2)\end{aligned}$$

for $m = 1, \dots, n$. The final differential leaves the $E^{2\rho(2n)+2} = E^\infty$-term, equal to

$$\begin{aligned}
\mu_2^{-1} E^\infty_{**}(C_{p^n}, \ell/p) = E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\
\oplus \bigoplus_{k=1}^{n} E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\
\oplus \bigoplus_{k=1}^{n} E(u_n, \bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\
\oplus E(\bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}) \otimes P_{\rho(2n)+1}(t\mu_2) \,.
\end{aligned}$$

Proof. The computation of the E^3-term from the E^2-term is straightforward. The rest of the proof goes by a secondary induction on $m = 1, \dots, n$, very much like the proof of Corollary 6.2. The differential

$$d^{2\rho(2m-1)}(\mu_2^{p^{2m}-p^{2m-1}+j} \cdot \bar\epsilon_1) = (t\mu_2)^{\rho(2m-1)} \cdot \mu_2^j$$

for $v_p(j) = 2m-2$ is non-trivial only on the summand

$$E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-2}}, t\mu_2)$$

of the $E^3 = E^{2\rho(1)}$-term (for $m = 1$), resp. the $E^{2\rho(2m-2)+1} = E^{2\rho(2m-1)}$-term (for $m = 2, \dots, n$). Its homology is

$$\begin{aligned}
E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2m-2\} \otimes P_{\rho(2m-1)}(t\mu_2) \\
\oplus E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-1}}, t\mu_2) \,,
\end{aligned}$$

which gives the stated $E^{2\rho(2m-1)+1}$-term. The differential

$$d^{2\rho(2m)}(\mu_2^{p^{2m}-p^{2m-1}}) = \lambda_2 \cdot \mu_2^{-p^{2m-1}} \cdot (t\mu_2)^{\rho(2m)}$$

is non-trivial only on the summand

$$E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m-1}}, t\mu_2)$$

of the $E^{2\rho(2m-1)+1} = E^{2\rho(2m)}$-term, leaving

$$\begin{aligned}
E(u_n, \bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid v_p(j) = 2m-1\} \otimes P_{\rho(2m)}(t\mu_2) \\
\oplus E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2m}}, t\mu_2) \,.
\end{aligned}$$

This gives the stated $E^{2\rho(2m)+1}$-term. The final differential

$$d^{2\rho(2n)+1}(u_n \cdot \mu_2^{p^{2n}}) = (t\mu_2)^{\rho(2n)+1}$$

is non-trivial only on the summand

$$E(u_n, \bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}, t\mu_2)$$

of the $E^{2\rho(2n)+1}$-term, with homology

$$E(\bar\epsilon_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}) \otimes P_{\rho(2n)+1}(t\mu_2) \,.$$

This gives the stated $E^{2\rho(2n)+2}$-term. There is no room for any further differentials, since the spectral sequence is concentrated in a narrower vertical band than the horizontal width of the following differentials, so $E^{2\rho(2n)+2} = E^\infty$. $\square$

Proof of Theorem 6.1. To make the inductive step to $C_{p^{n+1}}$, we use that the first d^r-differential of odd length in $\hat{E}^*(C_{p^n}, \ell/p)$ occurs for $r = r_0 = 2\rho(2n) + 1$. It follows from [AR02, 5.2] that the terms $\hat{E}^r(C_{p^n}, \ell/p)$ and $\hat{E}^r(C_{p^{n+1}}, \ell/p)$ are isomorphic for $r \le 2\rho(2n)+1$, via the Frobenius map (taking t^i to t^i) in even columns and the Verschiebung map (taking $u_n t^i$ to $u_{n+1} t^i$) in odd columns. Furthermore, the differential $d^{2\rho(2n)+1}$ is zero in the latter spectral sequence. This proves the part of Theorem 6.1 for $n+1$ that concerns the differentials leading up to the term

$$\begin{aligned}
\hat{E}^{2\rho(2n)+2}(C_{p^{n+1}}, \ell/p) &= E(u_{n+1}, \lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\
&\oplus \bigoplus_{k=2}^{n} E(u_{n+1}, \lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\
&\oplus \bigoplus_{k=2}^{n} E(u_{n+1}, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{t^j\lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\
&\oplus E(u_{n+1}, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)\,.
\end{aligned} \tag{6.6}$$

Next we use the following commutative diagram, where we abbreviate $THH(B)$ to $T(B)$:

$$\begin{array}{ccccccc}
(T(B)^{tC_p})^{hC_{p^n}} & \xleftarrow{\hat{\Gamma}_1^{hC_{p^n}}} & T(B)^{hC_{p^n}} & \xleftarrow{\Gamma_n} & T(B)^{C_{p^n}} & \xrightarrow{\hat{\Gamma}_{n+1}} & T(B)^{tC_{p^{n+1}}} \\
\downarrow F & & \downarrow F & & \downarrow F & & \downarrow F \\
T(B)^{tC_p} & \xleftarrow{\hat{\Gamma}_1} & T(B) & = & T(B) & \xrightarrow{\hat{\Gamma}_1} & T(B)^{tC_p}
\end{array} \tag{6.7}$$

The horizontal maps all induce $(2p-2)$-coconnected maps in $V(1)$-homotopy for $B = \ell/p$. Here F is the Frobenius map, forgetting part of the equivariance. Thus the map $\hat{\Gamma}_{n+1}$ to the right induces an isomorphism of $E(\lambda_2) \otimes P(v_2)$-modules in all degrees $* > (2p-2)$ from $V(1)_*THH(\ell/p)^{C_{p^n}}$, implicitly identified to the left with the abutment of $\mu_2^{-1}E^*(C_{p^n}, \ell/p)$, to $V(1)_*THH(\ell/p)^{tC_{p^{n+1}}}$, which is the abutment of $\hat{E}^*(C_{p^{n+1}}, \ell/p)$. The diagram above ensures that the isomorphism induced by $\hat{\Gamma}_{n+1}$ is compatible with the one induced by $\hat{\Gamma}_1$. By Proposition 5.9 it takes $\bar{\epsilon}_1$, λ_2 and μ_2 to $\bar{\epsilon}_1$, λ_2 and t^{-p^2}, respectively, and similarly for monomials in these classes.

We focus on the summand

$$E(u_n, \lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2n-2\} \otimes P_{\rho(2n-1)}(t\mu_2)$$

in $\mu_2^{-1}E^\infty_{**}(C_{p^n}, \ell/p)$, abutting to $V(1)_*THH(\ell/p)^{C_{p^n}}$ in degrees $> (2p-2)$. In the $P(v_2)$-module structure on the abutment, each class μ_2^j with $v_p(j) = 2n-2$, $j > 0$, generates a copy of $P_{\rho(2n-1)}(v_2)$, since there are no permanent cycles in the same total degree as $y = (t\mu_2)^{\rho(2n-1)} \cdot \mu_2^j$ that have lower (= more negative) homotopy fixed point filtration. See Lemma 6.8 below for the elementary verification. The $P(v_2)$-module isomorphism induced by $\hat{\Gamma}_{n+1}$ must take this to a copy of $P_{\rho(2n-1)}(v_2)$ in $V(1)_*THH(\ell/p)^{tC_{p^{n+1}}}$, generated by $t^{-p^2 j}$.

Writing $i = -p^2 j$, we deduce that for $v_p(i) = 2n$, $i < 0$, the infinite cycle $z = (t\mu_2)^{\rho(2n-1)} \cdot t^i$ must represent zero in the abutment, and must therefore be hit by a differential $z = d^r(x)$ in the $C_{p^{n+1}}$-Tate spectral sequence. Here $r \ge 2\rho(2n) + 2$.

Since z generates a free copy of $P(t\mu_2)$ in the $E^{2\rho(2n)+2}$-term displayed in (6.6), and d^r is $P(t\mu_2)$-linear, the class x cannot be annihilated by any power of $t\mu_2$. This means that x must be contained in the summand

$$E(u_{n+1}, \bar{\epsilon}_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)$$

of $\hat{E}^{2\rho(2n)+2}_{**}(C_{p^{n+1}}, \ell/p)$. By an elementary check of bidegrees, see Lemma 6.9 below, the only possibility is that x has vertical degree $(2p-1)$, so that we have differentials

$$d^{2\rho(2n+1)}(t^{p^{2n+1}-p^{2n+2}+i} \cdot \bar{\epsilon}_1) = (t\mu_2)^{\rho(2n-1)} \cdot t^i$$

for all $i < 0$ with $v_p(i) = 2n$. The cases $i > 0$ follow by the module structure over the $C_{p^{n+1}}$-Tate spectral sequence for ℓ. The remaining two differentials,

$$d^{2\rho(2n+2)}(t^{p^{2n+1}-p^{2n+2}}) = \lambda_2 \cdot t^{p^{2n+1}} \cdot (t\mu_2)^{\rho(2n)}$$

and

$$d^{2\rho(2n+2)+1}(u_{n+1} \cdot t^{-p^{2n+2}}) = (t\mu_2)^{\rho(2n)+1}$$

are also present in the $C_{p^{n+1}}$-Tate spectral sequence for ℓ, see [AR02, 6.1], hence follow in the present case by the module structure. With this we have established the complete differential pattern asserted by Theorem 6.1. □

Lemma 6.8. *For $v_p(j) = 2n-2$, $n \geq 1$, there are no classes in $\mu_2^{-1}E^\infty_{**}(C_{p^n}, \ell/p)$ in the same total degree as $y = (t\mu_2)^{\rho(2n-1)} \cdot \mu_2^j$ that have lower homotopy fixed point filtration.*

Proof. The total degree of y is $2(p^{2n+2} - p^{2n+1} + p - 1) + 2p^2 j \equiv (2p-2) \mod 2p^{2n}$, which is even.

Looking at the formula for $\mu_2^{-1}E^\infty_{**}(C_{p^n}, \ell/p)$ in Corollary 6.5, the classes of lower filtration than y all lie in the terms

$$E(u_n, \bar{\epsilon}_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^i \mid v_p(i) = 2n-1\} \otimes P_{\rho(2n)}(t\mu_2)$$

and

$$E(\bar{\epsilon}_1, \lambda_2) \otimes P(\mu_2^{\pm p^{2n}}) \otimes P_{\rho(2n)+1}(t\mu_2)\,.$$

Those in even total degree and of lower filtration than y are

$$u_n\lambda_2 \cdot \mu_2^i(t\mu_2)^e, \quad \bar{\epsilon}_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$$

with $v_p(i) = 2n-1$, $\rho(2n-1) < e < \rho(2n)$, and

$$\mu_2^i(t\mu_2)^e, \quad \bar{\epsilon}_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$$

with $v_p(i) \geq 2n$, $\rho(2n-1) < e \leq \rho(2n)$.

The total degree of $u_n\lambda_2 \cdot \mu_2^i(t\mu_2)^e$ for $v_p(i) = 2n-1$ is $(-1)+(2p^2-1)+2p^2 i+(2p^2-2)e \equiv (2p^2-2)(e+1) \mod 2p^{2n}$. For this to agree with the total degree of y, we must have $(2p-2) \equiv (2p^2-2)(e+1) \mod 2p^{2n}$, so $(e+1) \equiv 1/(1+p) \mod p^{2n}$ and $e \equiv \rho(2n-1)-1 \mod p^{2n}$. There is no such e with $\rho(2n-1) < e < \rho(2n)$.

The total degree of $\bar{\epsilon}_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$ for $v_p(i) = 2n-1$ is $(2p-1) + (2p^2-1) + 2p^2 i + (2p^2-2)e \equiv 2p + (2p^2-2)(e+1) \mod 2p^{2n}$. To agree with that of y, we must have $(2p-2) \equiv 2p + (2p^2-2)(e+1) \mod 2p^{2n}$, so $(e+1) \equiv 1/(1-p^2) \mod p^{2n}$ and $e \equiv \rho(2n) \mod p^{2n}$. There is no such e with $\rho(2n-1) < e < \rho(2n)$.

The total degree of $\mu_2^i(t\mu_2)^e$ for $v_p(i) \geq 2n$ is $2p^2 i + (2p^2-2)e \equiv (2p^2-2)e \mod 2p^{2n}$. To agree with that of y, we must have $(2p-2) \equiv (2p^2-2)e \mod 2p^{2n}$, so $e \equiv 1/(1+p) \equiv \rho(2n-1) \mod p^{2n}$. There is no such e with $\rho(2n-1) < e \leq \rho(2n)$.

The total degree of $\bar\epsilon_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$ for $v_p(i) \ge 2n$ is $(2p-1)+(2p^2-1)+2p^2i+(2p^2-2)e$. To agree modulo $2p^{2n}$ with that of y, we must have $e \equiv \rho(2n) \mod p^{2n}$. The only such e with $\rho(2n-1) < e \le \rho(2n)$ is $e = \rho(2n)$. But in that case, the total degree of $\bar\epsilon_1\lambda_2 \cdot \mu_2^i(t\mu_2)^e$ is $2p + 2p^2i + (2p^2-2)(\rho(2n)+1) = 2(p^{2n+2}+p-1) + 2p^2i$. To be equal to that of y, we must have $2p^2i + 2p^{2n+1} = 2p^2j$, which is impossible for $v_p(i) \ge 2n$ and $v_p(j) = 2n-2$. □

Lemma 6.9. *For $v_p(i) = 2n$, $n \ge 1$ and $z = (t\mu_2)^{\rho(2n-1)} \cdot t^i$, the only class in*

$$E(u_{n+1}, \bar\epsilon_1, \lambda_2) \otimes P(t^{\pm p^{2n}}, t\mu_2)$$

that can support a differential $d^r(x) = z$ for $r \ge 2\rho(2n)+2$ is (a unit times)

$$x = t^{p^{2n+1}-p^{2n+2}+i} \cdot \bar\epsilon_1 \,.$$

Proof. The class z has total degree $(2p^2-2)\rho(2n-1)-2i = 2p^{2n+2}-2p^{2n+1}+2p-2-2i \equiv (2p-2) \mod 2p^{2n}$, which is even, and vertical degree $2p^2\rho(2n-1)$. Hence x has odd total degree, and vertical degree at most $2p^2\rho(2n-1)-2\rho(2n)-1 = 2p^{2n+2}-2p^{2n+1}-\cdots-2p^3-1$. This leaves the possibilities

$$u_{n+1} \cdot t^j(t\mu_2)^e, \quad \bar\epsilon_1 \cdot t^j(t\mu_2)^e, \quad \lambda_2 \cdot t^j(t\mu_2)^e$$

with $v_p(j) \ge 2n$ and $0 \le e < p^{2n} - p^{2n-1} - \cdots - p = \rho(2n-1) - \rho(2n-2) - 1$, and

$$u_{n+1}\bar\epsilon_1\lambda_2 \cdot t^j(t\mu_2)^e$$

with $v_p(j) \ge 2n$ and $0 \le e < p^{2n} - p^{2n-1} - \cdots - p - 1 = \rho(2n-1) - \rho(2n-2) - 2$.

The total degree of x must be one more than the total degree of z, hence is congruent to $(2p-1)$ modulo $2p^{2n}$.

The total degree of $u_{n+1} \cdot t^j(t\mu_2)^e$ is $-1-2j+(2p^2-2)e \equiv -1+(2p^2-2)e \mod 2p^{2n}$. To have $(2p-1) \equiv -1 + (2p^2-2)e \mod 2p^{2n}$ we must have $e \equiv -p/(1-p^2) \equiv p^{2n} - p^{2n-1} - \cdots - p \mod p^{2n}$, which does not happen for e in the allowable range.

The total degree of $\lambda_2 \cdot t^j(t\mu_2)^e$ is $(2p^2-1) - 2j + (2p^2-2)e \equiv (2p^2-1) + (2p^2-2)e \mod 2p^{2n}$. To have $(2p-1) \equiv (2p^2-1) + (2p^2-2)e \mod 2p^{2n}$ we must have $e \equiv -p/(1+p) \equiv \rho(2n-1) - 1 \mod p^{2n}$, which does not happen.

The total degree of $u_{n+1}\bar\epsilon_1\lambda_2 \cdot t^j(t\mu_2)^e$ is $-1 + (2p-1) + (2p^2-1) - 2j + (2p^2-2)e \equiv (2p-1) + (2p^2-2)(e+1) \mod 2p^{2n}$. To have $(2p-1) \equiv (2p-1) + (2p^2-2)(e+1) \mod 2p^{2n}$ we must have $(e+1) \equiv 0 \mod p^{2n}$, so $e \equiv p^{2n}-1 \mod p^{2n}$, which does not happen.

The total degree of $\bar\epsilon_1 \cdot t^j(t\mu_2)^e$ is $(2p-1) - 2j + (2p^2-2)e \equiv (2p-1) + (2p^2-2)e \mod 2p^{2n}$. To have $(2p-1) \equiv (2p-1) + (2p^2-2)e \mod 2p^{2n}$, we must have $e \equiv 0 \mod p^{2n}$, so $e = 0$ is the only possibility in the allowable range. In that case, a check of total degrees shows that we must have $j = p^{2n+1} - p^{2n+2} + i$. □

Corollary 6.10. *$V(1)_*THH(\ell/p)^{C_{p^n}}$ is finite in each degree.*

Proof. This is clear by inspection of the E^∞-term in Corollary 6.2. □

Lemma 6.11. *The map G_n induces an isomorphism*

$$V(1)_*THH(\ell/p)^{tC_{p^{n+1}}} \xrightarrow{\cong} V(1)_*(THH(\ell/p)^{tC_p})^{hC_{p^n}}$$

in all degrees. In the limit over the Frobenius maps F, there is a map G inducing an isomorphism

$$V(1)_*THH(\ell/p)^{tS^1} \xrightarrow{\cong} V(1)_*(THH(\ell/p)^{tC_p})^{hS^1} \,.$$

Proof. As remarked after diagram (6.3), G_n induces an isomorphism in $V(1)$-homotopy above degree $(2p-2)$. The permanent cycle $t^{-p^{2n+2}}$ in $\hat{E}^\infty_{**}(C_{p^{n+1}}, \ell)$ acts invertibly on $\hat{E}^\infty_{**}(C_{p^{n+1}}, \ell/p)$, and its image $G_n(t^{-p^{2n+2}}) = \mu_2^{p^{2n}}$ in $\mu_2^{-1} E^\infty_{**}(C_{p^n}, \ell)$ acts invertibly on $\mu_2^{-1} E^\infty_{**}(C_{p^n}, \ell/p)$. Therefore the module action derived from the ℓ-algebra structure on ℓ/p ensures that G_n induces isomorphisms in $V(1)$-homotopy in all degrees. □

Theorem 6.12. *(a) The associated graded of $V(1)_* THH(\ell/p)^{tS^1}$ for the S^1-Tate spectral sequence is*

$$\begin{aligned}\hat{E}^\infty_{**}(S^1, \ell/p) = {} & E(\lambda_2) \otimes \mathbb{F}_p\{t^{-i} \mid 0 < i < p\} \otimes P(t^{\pm p^2}) \\ & \oplus \bigoplus_{k \geq 2} E(\lambda_2) \otimes \mathbb{F}_p\{t^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ & \oplus \bigoplus_{k \geq 2} E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^j \lambda_2 \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k-2)}(t\mu_2) \\ & \oplus E(\bar\epsilon_1, \lambda_2) \otimes P(t\mu_2)\,.\end{aligned}$$

(b) The associated graded of $V(1)_* THH(\ell/p)^{hS^1}$ for the S^1-homotopy fixed point spectral sequence maps by a $(2p-2)$-coconnected map to

$$\begin{aligned}\mu_2^{-1} E^\infty_{**}(S^1, \ell/p) = {} & E(\lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2^{\pm 1}) \\ & \oplus \bigoplus_{k \geq 1} E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^j \mid v_p(j) = 2k-2\} \otimes P_{\rho(2k-1)}(t\mu_2) \\ & \oplus \bigoplus_{k \geq 1} E(\bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2 \mu_2^j \mid v_p(j) = 2k-1\} \otimes P_{\rho(2k)}(t\mu_2) \\ & \oplus E(\bar\epsilon_1, \lambda_2) \otimes P(t\mu_2)\,.\end{aligned}$$

(c) The isomorphism from (a) to (b) induced by G takes t^{-i} to μ_0^i for $0 < i < p$ and t^i to μ_2^j for $i + p^2 j = 0$. Furthermore, it takes multiples by $\bar\epsilon_1$, λ_2 or $t\mu_2$ in the source to the same multiples in the target.

Proof. Claims (a) and (b) follow by passage to the limit over n from Corollaries 6.2 and 6.5. Claim (c) follows by passage to the same limit from the formulas for the isomorphism induced by $\hat\Gamma_{n+1}$, which were given below diagram (6.7). □

7. Topological cyclic homology

By definition, there is a fiber sequence

$$TC(B) \xrightarrow{\pi} TF(B) \xrightarrow{R-1} TF(B)$$

inducing a long exact sequence

$$\dots \xrightarrow{\partial} V(1)_* TC(B) \xrightarrow{\pi} V(1)_* TF(B) \xrightarrow{R-1} V(1)_* TF(B) \xrightarrow{\partial} \dots \tag{7.1}$$

in $V(1)$-homotopy. By Corollary 5.10, there are $(2p-2)$-coconnected maps Γ and $\hat\Gamma$ from $V(1)_* TF(\ell/p)$ to $V(1)_* THH(\ell/p)^{hS^1}$ and $V(1)_* THH(\ell/p)^{tS^1}$, respectively. We model $V(1)_* TF(\ell/p)$ in degrees $> (2p-2)$ by the map $\hat\Gamma$ to the S^1-Tate construction. Then, by diagram (6.3), R is modeled in the same range of degrees by the chain of maps below.

$$\begin{tikzcd}
V(1)_*THH(B)^{tS^1} \arrow[dr, "G"'] & V(1)_*THH(B)^{hS^1} \arrow[r, "R^h"] \arrow[d, "(\hat\Gamma_1)^{hS^1}"] & V(1)_*THH(B)^{tS^1} \\
& V(1)_*(THH(B)^{tC_p})^{hS^1} &
\end{tikzcd}$$

Here R^h induces a map of spectral sequences

$$E^*(R^h)\colon E^*(S^1, B) \to \hat{E}^*(S^1, B)\,,$$

which at the E^2-term equals the inclusion that algebraically inverts t. When $B = \ell/p$, the left hand map G is an isomorphism by Lemma 6.11, and the middle (wrong-way) map is $(2p-2)$-coconnected.

Proposition 7.2. *In degrees $> (2p-2)$, the homomorphism*

$$E^\infty(R^h)\colon E^\infty(S^1, \ell/p) \to \hat{E}^\infty(S^1, \ell/p)$$

maps

(a) $E(\bar\epsilon_1, \lambda_2) \otimes P(t\mu_2)$ identically to the same expression;

(b) $E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^{-j}\} \otimes P_{\rho(2k-1)}(t\mu_2)$ surjectively onto

$$E(\lambda_2) \otimes \mathbb{F}_p\{t^j\} \otimes P_{\rho(2k-3)}(t\mu_2)$$

for each $k \geq 2$, $j = dp^{2k-2}$, $0 < d < p^2 - p$ and $p \nmid d$;

(c) $E(\bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^{-j}\} \otimes P_{\rho(2k)}(t\mu_2)$ surjectively onto

$$E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^j\lambda_2\} \otimes P_{\rho(2k-2)}(t\mu_2)$$

for each $k \geq 2$, $j = dp^{2k-1}$ and $0 < d < p$;

(d) the remaining terms to zero.

Proof. Consider the summands of $E^\infty(S^1, \ell/p)$ and $\hat{E}^\infty(S^1, \ell/p)$, as given in Theorem 6.12. Clearly, the first term $E(\lambda_2) \otimes \mathbb{F}_p\{\mu_0^i \mid 0 < i < p\} \otimes P(\mu_2)$ goes to zero (these classes are hit by d^2-differentials), and the last term $E(\bar\epsilon_1, \lambda_2) \otimes P(t\mu_2)$ maps identically to the same term. This proves (a) and part of (d).

For each $k \geq 1$ and $j = dp^{2k-2}$ with $p \nmid d$, the term $E(\lambda_2) \otimes \mathbb{F}_p\{\mu_2^{-j}\} \otimes P_{\rho(2k-1)}(t\mu_2)$ maps to the term $E(\lambda_2) \otimes \mathbb{F}_p\{t^j\} \otimes P_{\rho(2k-3)}(t\mu_2)$, except that the target is zero for $k = 1$. In symbols, the element $\lambda_2^\delta \mu_2^{-j}(t\mu_2)^i$ maps to the element $\lambda_2^\delta t^j (t\mu_2)^{i-j}$. If $d < 0$, then the t-exponent in the target is bounded above by $dp^{2k-2} + \rho(2k-3) < 0$, so the target lives in the right half-plane and is essentially not hit by the source, which lives in the left half-plane. If $d > p^2 - p$, then the total degree in the source is bounded above by $(2p^2-1) - 2dp^{2k} + \rho(2k-1)(2p^2-2) < 2p-2$, so the source lives in total degree $< (2p-2)$ and will be disregarded. If $0 < d < p^2 - p$, then $\rho(2k-1) - dp^{2k-2} > \rho(2k-3)$ and $-dp^{2k-2} < 0$, so the source surjects onto the target. This proves (b) and part of (d).

Lastly, for each $k \geq 1$ and $j = dp^{2k-1}$ with $p \nmid d$, the term $E(\bar\epsilon_1) \otimes \mathbb{F}_p\{\lambda_2\mu_2^{-j}\} \otimes P_{\rho(2k)}(t\mu_2)$ maps to the term $E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^j\lambda_2\} \otimes P_{\rho(2k-2)}(t\mu_2)$. The target is zero for $k = 1$. If $d < 0$, then $dp^{2k-1} + \rho(2k-2) < 0$ so the target lives in the right half-plane. If $d > p$, then $(2p-1) + (2p^2-1) - 2dp^{2k+1} + \rho(2k)(2p^2-2) < 2p-2$, so the source lives in total degree $< (2p-2)$. If $0 < d < p$, then $\rho(2k) - dp^{2k-1} > \rho(2k-2)$ and $-dp^{2k-1} < 0$, so the source surjects onto the target. This proves (c) and the remaining part of (d). $\square$

Definition 7.3. Let

$$\begin{aligned} A &= E(\bar\epsilon_1, \lambda_2) \otimes P(t\mu_2) \\ B_k &= E(\lambda_2) \otimes \mathbb{F}_p\{t^{dp^{2k-2}} \mid 0 < d < p^2 - p, p \nmid d\} \otimes P_{\rho(2k-3)}(t\mu_2) \\ C_k &= E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp^{2k-1}}\lambda_2 \mid 0 < d < p\} \otimes P_{\rho(2k-2)}(t\mu_2) \end{aligned}$$

for $k \geq 2$ and let D be the span of the remaining monomials in $\hat{E}^\infty(S^1, \ell/p)$. Let $B = \bigoplus_{k\geq 2} B_k$ and $C = \bigoplus_{k\geq 2} C_k$. Then $\hat{E}^\infty(S^1, \ell/p) = A \oplus B \oplus C \oplus D$.

Proposition 7.4. *In degrees $> (2p-2)$, there are closed subgroups $\widetilde{A} = E(\bar\epsilon_1, \lambda_2) \otimes P(v_2)$, $\widetilde{B}_k$, $\widetilde{C}_k$ and $\widetilde{D}$ in $V(1)_*TF(\ell/p)$, represented by A, B_k, C_k and D in $\hat{E}^\infty(S^1, \ell/p)$, respectively, such that the homomorphism induced by the restriction map R*

(a) is the identity on $\widetilde{A}$;
(b) maps $\widetilde{B}_{k+1}$ surjectively onto $\widetilde{B}_k$ for all $k \geq 2$;
(c) maps $\widetilde{C}_{k+1}$ surjectively onto $\widetilde{C}_k$ for all $k \geq 2$;
(d) is zero on $\widetilde{B}_2$, $\widetilde{C}_2$ and $\widetilde{D}$.

In these degrees, $V(1)_*TF(\ell/p) \cong \widetilde{A} \oplus \widetilde{B} \oplus \widetilde{C} \oplus \widetilde{D}$, where $\widetilde{B} = \prod_{k\geq 2} \widetilde{B}_k$ and $\widetilde{C} = \prod_{k\geq 2} \widetilde{C}_k$.

Proof. In terms of the model $THH(\ell/p)^{tS^1}$ for $TF(\ell/p)$, the restriction map R is given in these degrees as the composite of the isomorphism G, computed in Theorem 6.12(c), and the map $\hat{E}^\infty(R^h)$, computed in Proposition 7.2. This gives the desired formulas at the level of E^∞-terms. The rest of the argument is the same as that for Theorem 7.7 of [AR02], using Corollary 6.10 to control the topologies, and will be omitted. □

Remark 7.5. Here we have followed the basic computational strategy of [BM94], [BM95] and [AR02]. It would be interesting to have a more concrete construction of the lifts $\widetilde{B}_k$, $\widetilde{C}_k$ and $\widetilde{D}$, in terms of de Rham–Witt operators R, F, V and $d = \sigma$, like in the algebraic case of [HM97] and [HM03].

Proposition 7.6. *In degrees $> (2p-2)$ there are isomorphisms*

$$\begin{aligned} \ker(R-1) &\cong \widetilde{A} \oplus \lim_k \widetilde{B}_k \oplus \lim_k \widetilde{C}_k \\ &\cong E(\bar\epsilon_1, \lambda_2) \otimes P(v_2) \\ &\qquad \oplus E(\lambda_2) \otimes \mathbb{F}_p\{t^d \mid 0 < d < p^2 - p, p \nmid d\} \otimes P(v_2) \\ &\qquad \oplus E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \otimes P(v_2) \end{aligned}$$

and $\operatorname{cok}(R-1) \cong \widetilde{A} = E(\bar\epsilon_1, \lambda_2) \otimes P(v_2)$. *Hence there is an isomorphism*

$$\begin{aligned} V(1)_*TC(\ell/p) &\cong E(\partial, \bar\epsilon_1, \lambda_2) \otimes P(v_2) \\ &\qquad \oplus E(\lambda_2) \otimes \mathbb{F}_p\{t^d \mid 0 < d < p^2 - p, p \nmid d\} \otimes P(v_2) \\ &\qquad \oplus E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\} \otimes P(v_2) \end{aligned}$$

in these degrees, where ∂ has degree -1 and represents the image of 1 *under the connecting map ∂ in* (7.1).

Proof. By Proposition 7.4, the homomorphism $R-1$ is zero on $\widetilde{A}$ and an isomorphism on $\widetilde{D}$. Furthermore, there is an exact sequence

$$0 \to \lim_k \widetilde{B}_k \to \prod_{k\geq 2} \widetilde{B}_k \xrightarrow{R-1} \prod_{k\geq 2} \widetilde{B}_k \to \lim_k{}^1 \widetilde{B}_k \to 0$$

and similarly for the C's. The derived limit on the right vanishes since each $\widetilde{B}_{k+1}$ surjects onto $\widetilde{B}_k$.

Multiplication by $t\mu_2$ in each B_k is realized by multiplication by v_2 in $\widetilde{B}_k$. Each $\widetilde{B}_k$ is a sum of $2(p-1)^2$ cyclic $P(v_2)$-modules, and since $\rho(2k-3)$ grows to infinity with k their limit is a free $P(v_2)$-module of the same rank, with the indicated generators t^d and $t^d\lambda_2$ for $0 < d < p^2-p$, $p \nmid d$. The argument for the C's is practically the same.

The long exact sequence (7.1) yields the short exact sequence

$$0 \to \Sigma^{-1} \operatorname{cok}(R-1) \xrightarrow{\partial} V(1)_* TC(\ell/p) \xrightarrow{\pi} \ker(R-1) \to 0\,,$$

from which the formula for the middle term follows. □

Remark 7.7. A more obvious set of $E(\lambda_2)\otimes P(v_2)$-module generators for $\lim_k \widetilde{B}_k$ would be the classes t^{dp^2} in $B_2 \cong \widetilde{B}_2$, for $0 < d < p^2-p$, $p \nmid d$. Under the canonical map $TF(\ell/p) \to THH(\ell/p)^{C_p}$, modeled here by $THH(\ell/p)^{tS^1} \to (THH(\ell/p)^{tC_p})^{hC_p}$, these map to the classes μ_2^{-d}. Since we are only concerned with degrees $> (2p-2)$ we may equally well use their v_2-power multiplies $(t\mu_2)^d \cdot \mu_2^{-d} = t^d$ as generators, with the advantage that these are in the image of the localization map $THH(\ell/p)^{hC_p} \to (THH(\ell/p)^{tC_p})^{hC_p}$. Hence the class denoted t^d in $\lim_k \widetilde{B}_k$ is chosen so as to map under $TF(\ell/p) \to THH(\ell/p)^{hC_p}$ to t^d in $E^\infty_{**}(C_p;\ell/p)$. Similarly, the class denoted $t^{dp}\lambda_2$ in $\lim_k \widetilde{C}_k$ is chosen so as to map to $t^{dp}\lambda_2$ in $E^\infty_{**}(C_p;\ell/p)$.

The map $\pi\colon \ell/p \to \mathbb{Z}/p$ is $(2p-2)$-connected, hence induces $(2p-1)$-connected maps $\pi_*\colon K(\ell/p) \to K(\mathbb{Z}/p)$ and $\pi_* : V(1)_*TC(\ell/p) \to V(1)_*TC(\mathbb{Z}/p)$, by [BM94, 10.9] and [Dun97]. Here $TC(\mathbb{Z}/p) \simeq H\mathbb{Z}_p \vee \Sigma^{-1}H\mathbb{Z}_p$ and $V(1)_*TC(\mathbb{Z}/p) \cong E(\partial, \bar\epsilon_1)$, so we can recover $V(1)_*TC(\ell/p)$ in degrees $\le (2p-2)$ from this map.

Theorem 7.8. *There is an isomorphism of $E(\lambda_1,\lambda_2)\otimes P(v_2)$-modules*

$$\begin{aligned} V(1)_*TC(\ell/p) \cong{}& P(v_2)\otimes E(\partial,\bar\epsilon_1,\lambda_2) \\ &\oplus P(v_2)\otimes E(\operatorname{dlog} v_1)\otimes \mathbb{F}_p\{t^d v_2 \mid 0<d<p^2-p, p\nmid d\} \\ &\oplus P(v_2)\otimes E(\bar\epsilon_1)\otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0<d<p\} \end{aligned}$$

where $v_2\cdot \operatorname{dlog} v_1 = \lambda_2$. The degrees are $|\partial| = -1$, $|\bar\epsilon_1| = |\lambda_1| = 2p-1$, $|\lambda_2| = 2p^2-1$ and $|v_2| = 2p^2-2$. The formal multipliers have degrees $|t| = -2$ and $|\operatorname{dlog} v_1| = 1$.

The notation $\operatorname{dlog} v_1$ for the multiplier $v_2^{-1}\lambda_2$ is suggested by the relation $v_1\cdot \operatorname{dlog} p = \lambda_1$ in $V(0)_*TC(\mathbb{Z}_{(p)}|\mathbb{Q})$.

Proof. Only the additive generators t^d for $0 < d < p^2-p$, $p \nmid d$ from Proposition 7.6 do not appear in $V(1)_*TC(\ell/p)$, but their multiples by λ_2 and positive powers of v_2 do. This leads to the given formula, where $\operatorname{dlog} v_1 \cdot t^d v_2$ must be read as $t^d\lambda_2$. □

By [HM97] the cyclotomic trace map of [BHM93] induces cofiber sequences

$$K(B_p)_p \xrightarrow{trc} TC(B)_p \xrightarrow{g} \Sigma^{-1}H\mathbb{Z}_p \tag{7.9}$$

for each connective S-algebra B with $\pi_0(B_p) = \mathbb{Z}_p$ or $\mathbb{Z}/p$, and thus long exact sequences

$$\cdots \to V(1)_*K(B_p) \xrightarrow{trc} V(1)_*TC(B) \xrightarrow{g} \Sigma^{-1}E(\bar\epsilon_1) \to \dots\,.$$

This uses the identifications $W(\mathbb{Z}_p)_F \cong W(\mathbb{Z}/p)_F \cong \mathbb{Z}_p$ of Frobenius coinvariants of Witt rings, and applies in particular for $B = H\mathbb{Z}_{(p)}$, $H\mathbb{Z}/p$, ℓ and ℓ/p.

Theorem 7.10. *There is an isomorphism of $E(\lambda_1, \lambda_2) \otimes P(v_2)$-modules*

$$\begin{aligned} V(1)_*K(\ell/p) \cong P(v_2) &\otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{1, \partial\lambda_2, \lambda_2, \partial v_2\} \\ &\oplus P(v_2) \otimes E(\operatorname{dlog} v_1) \otimes \mathbb{F}_p\{t^d v_2 \mid 0 < d < p^2 - p, p \nmid d\} \\ &\oplus P(v_2) \otimes E(\bar\epsilon_1) \otimes \mathbb{F}_p\{t^{dp}\lambda_2 \mid 0 < d < p\}\,. \end{aligned}$$

This is a free $P(v_2)$-module of rank $(2p^2 - 2p + 8)$ and of zero Euler characteristic.

Proof. In the case $B = \mathbb{Z}/p$, $K(\mathbb{Z}/p)_p \simeq H\mathbb{Z}_p$ and the map g is split surjective up to homotopy. So the induced homomorphism to $V(1)_*\Sigma^{-1}H\mathbb{Z}_p = \Sigma^{-1}E(\bar\epsilon_1)$ is surjective. Since $\pi\colon \ell/p \to \mathbb{Z}/p$ induces a $(2p-1)$-connected map in topological cyclic homology, and $\Sigma^{-1}E(\bar\epsilon_1)$ is concentrated in degrees $\le (2p-2)$, it follows by naturality that also in the case $B = \ell/p$ the map g induces a surjection in $V(1)$-homotopy. The kernel of the surjection $P(v_2) \otimes E(\partial, \bar\epsilon_1, \lambda_2) \to \Sigma^{-1}E(\bar\epsilon_1)$ gives the first row in the asserted formula. □

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