The Modular Square for Quantum Groups

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Abstract

This is mainly an expository paper with results gathered from [12], [5], [6] and [11].

We define the modular square and the modular characteristic homomorphisms for algebraic quantum groups. We recall Hopf cyclic cohomology and methods developed for computing it, which includes a Künneth exact sequence for Hopf cyclic cohomology. Focus is on the category of Hopf algebras in involution associated to compact quantum groups. We review the computations of the Hopf cyclic cohomology and the modular characteristic homomorphisms for the compact quantum group $SU_q(2)$.

We define a cocyclic object which yields twisted cyclic cohomology. This cohomology is shown to have a Chern character as well, and a beautiful connection to differential calculi for compact quantum groups is recalled. As an entirely new result, we show that characteristic homomorphisms can be defined with target, twisted cyclic cohomology, without involving the modular square, and which to some degree are better behaved than the modular characteristic homomorphism.

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1 Introduction

We define Hopf cyclic cohomology in the sense of A. Connes and H. Moscovici. By the general machine recalled in Section 2, all that is needed to get a full-fledged cyclic cohomology theory with the long exact IBS-sequence intact, is a cocyclic object. Such an object can be canonically associated to a Hopf algebra $(\mathcal{H}, \Delta)$ with a modular pair (δ, σ) in involution. The latter consists of a character δ and a group-like element σ satisfying certain conditions. It is shown that the corresponding Hopf Hochschild cohomology $HH^*_{(\delta,\sigma)}(\mathcal{H})$ can be regarded as a derived functor in the category of bicomodules over Hopf algebras.

In the following section, we study the category of Hopf algebras with modular pairs in involution. We notice that it is closed under formation of opposites, duals and tensor products. We state a Künneth theorem for the Hopf Hochschild cohomology, and a Künneth exact sequence for the Hopf cyclic cohomology. We focus then on various examples of Hopf algebras with modular pairs in involution associated to compact quantum groups, and compute them in the case of compact quantum groups coming from certain Lie groups. We consider $SU_q(2)$ and compute the Hopf cohomologies given by the Hopf algebra $(U_q(\mathrm{sl}_2), \Delta)$ with a canonical modular pair in involution. To demonstrate the power of our Künneth theorems, we include also results for the tensor product Hopf algebra $(\tilde{\mathcal{H}}_q, \tilde{\Delta})$ of $(U_q(\mathrm{sl}_2), \Delta)$ with its opposite.

Next we define the *modular square* for a Hopf algebra with a canonical modular pair in involution associated to a compact quantum group. The modular square is an intricate construction defined in order to obtain the *modular characteristic homomorphism* from Hopf cyclic cohomology to ordinary cyclic cohomology.

To explain what this roughly means, consider the Hopf algebra $(\mathcal{A}, \Phi)$ of regular functions on a compact quantum group. Let $(\mathcal{H}, \Delta)$ denote a Hopf subalgebra of its maximal dual Hopf algebra. Let $\{f_z\}$ denote the family governing the modular properties of the Haar state h. Suppose $f_1 \in \mathcal{H}$, which is indeed the case when $\mathcal{H} = U_q(\mathrm{sl}_2)$. Then the tensor product Hopf algebra $(\tilde{\mathcal{H}}, \tilde{\Delta})$ of $(\mathcal{H}, \Delta)$ and its opposite Hopf algebra $(\tilde{\mathcal{H}}_{op}, \tilde{\Delta})$, is endowed with a modular pair $(\delta, \sigma) = (I \otimes I, f_1 \otimes f_1)$ in involution. Next a canonical $\tilde{\mathcal{H}}$-action of $(\tilde{\mathcal{H}}, \tilde{\Delta})$ on $\mathcal{A}$ is defined under which h is a δ-invariant σ-trace on $\mathcal{A}$. The δ-invariance property follows from strong left invariance for h, and the σ-trace property is the KMS-condition. This assures the existence of a homomorphism γ_* from $HC^*_{(\delta,\sigma)}(\tilde{\mathcal{H}})$ to ordinary cyclic cohomology $HC^*(\mathcal{A})$ of $\mathcal{A}$, which is called the modular characteristic homomorphism of $(\mathcal{H}, \Delta)$. Thus we are dealing with a theory of characteristic classes for actions of non-unimodular Hopf algebras compatible with the modular theory of weights.

Combining the computation of the Hopf cohomologies for $(U_q(\mathrm{sl}_2), \Delta)$ with the computations in [20] of the ordinary Hochschild and cyclic cohomology of $\mathcal{A}_q$, gives the result that the modular characteristic homomorphism of $(U_q(\mathrm{sl}_2), \Delta)$ is, unfortunately, ZERO.

We are then led to consider the maximal dual Hopf $*$-algebra $(\mathcal{A}_q^o, \Delta)$ of $(\mathcal{A}_q, \Phi)$. A stronger result would be to show that the modular characteristic homomorphism of $(\mathcal{A}_q^o, \Delta)$ is zero, in which case it would be zero for any Hopf subalgebra of $(\mathcal{A}_q^o, \Delta)$. We devote a section to this end, and show that under a certain assumption on $SU_q(2)$, which we unfortunately have not been able to check, this stronger result holds. We prove a general duality theorem to the effect that $HC^*_{(I,f_1)}(\mathcal{A}^o)$ can be computed from a finitely generated free (or projective) resolution of $\mathcal{A}$ as an $\mathcal{A}$-bimodule, provided $\mathcal{A}^o \otimes \mathcal{A}^o$ is injective as an $\mathcal{A}$-bimodule. Assuming this assumption holds for $SU_q(2)$, we use an explicit resolution of $\mathcal{A}_q$ given by T. Masuda, Y. Nakagami and J. Watanabe to compute $HC^*_{(I,f_1)}(\mathcal{A}^o_q)$. A fine study of the B-operator yields as a consequence that the modular characteristic homomorphism of $(\mathcal{A}^o_q, \Delta)$ is again zero in Hochschild cohomology, cyclic cohomology and periodic cyclic cohomology.

We make an effort to exhibit generators for the various Hopf cohomologies, both for $(U_q(\mathrm{sl}_2), \Delta)$ and for $(\mathcal{A}^o_q, \Delta)$, in the respective defining complexes. Their representatives involve elements in $U_q(\mathrm{sl}_2)$ and, in the latter case, also an element $H \in \mathcal{A}^o_q$, which plays the role of the Cartan element for quantum $SU(2)$ and does not belong to $U_q(\mathrm{sl}_2) \subset \mathcal{A}^o_q$. We also write down the generators for their corresponding modular squares, a procedure which requires explicit formulas for the Alexander-Whitney map and the shuffle map implementing the Künneth isomorphisms.

In a separate section we define the modular square for the more general case of algebraic quantum groups, and discuss generalizations to the locally compact case as well. Loosely speaking, the modular square construction associates to a Hopf algebra A a modular pair in involution on its dual Hopf algebra A°, an action of this dual algebra A° on A itself and a linear functional on A that is 'invariant' with respect to this action.

The construction presented in this section was hinted at in [3] (worked out in [10]) and formally generalized in [5], thereby giving a good indication how the modular square should be build up in the general framework of locally compact quantum groups.

An intermediate step between Hopf algebras and locally compact quantum groups are the algebraic quantum groups discussed in [14] (section 6). Thanks to the algebraic nature of the latter objects (see [23]), combined with their analytic properties (see [9]), the modular square construction presented in [5] can be easily made rigorous for these algebraic quantum groups and that is precisely what we are going to do in this section. We present the Künneth theorems also in this setting, and define the modular characteristic homomorphism.

Thereafter we include an alternative approach to Hopf symmetry and the modular square construction by recalling the twisted cyclic cohomology $HC^*(\mathcal{A}, \theta)$ associated to an algebra $\mathcal{A}$ with an automorphism θ. It generalizes ordinary cyclic cohomology in the sense that $HC^*(\mathcal{A}, \iota) = HC^*(\mathcal{A})$. Again this cohomology is introduced by defining a cocyclic object, which assures a satisfactory cohomology theory. We show how the Chern character can be extended to this setting, and we state a beautiful one-to-one correspondence between twisted cyclic cocycles and differential calculi with twisted graded traces [11]. In the case of a compact quantum group this correspondence preserves invariance, in that invariant cocycles correspond to covariant calculi and invariant twisted graded traces.

In Section A we prove that there exist characteristic homomorphisms

$$\gamma^l_* : HC^*_{(I,f_1)}(\mathcal{A}^o) \longrightarrow HC^*(\mathcal{A}, \theta_l)$$

$$\gamma^r_* : HC^*_{(I,f_1)}(\mathcal{A}^o) \longrightarrow HC^*(\mathcal{A}, \theta_r)$$

defined using h and canonical (left- and right-) $\mathcal{A}^o$-actions. Here $\theta_l(a) = (f_{-1} \otimes \iota)\Phi(a)$ and $\theta_r(a) = (\iota \otimes f_{-1})\Phi(a)$, for $a \in \mathcal{A}$. These characteristic homomorphisms have certain advantages compared to the modular characteristic homomorphism.

In Section B we spell out in what sense the Hopf algebra $(\tilde{\mathcal{H}}, \tilde{\Delta})$ can be identified with one of Majid's bicrossproducts known as the Mirror product $\mathcal{H} \bowtie \mathcal{H}_{\mathrm{op}}$ [18].

2 Hopf Cyclic Cohomology

In this section we consider cyclic cohomology from the point of view of cocyclic objects. We introduce the cocyclic object that defines the cyclic cohomology $HC^*_{(\delta,\sigma)}(\mathcal{H})$ of a Hopf algebra $(\mathcal{H}, \Delta)$ with a modular pair (δ, σ) in involution. Also we include a brief reminder of the Chern character of Connes.

Let $\mathbb{N}_0$ denote the set of non-negative integers. The *cyclic category* Λ is defined [2] [15] [19] to be the category with objects $[n]$ for $n \in \mathbb{N}_0$, and morphisms universally generated by the elements

$$\delta_i^n : [n-1] \to [n], \qquad \sigma_i^n : [n+1] \to [n], \qquad \tau_n : [n] \to [n],$$

where $i \in \{0, 1, \ldots, n\}$, satisfying the following relations:

$$\begin{aligned}
\delta_j^n \delta_i^{n-1} &= \delta_i^n \delta_{j-1}^{n-1} && \text{for } i < j \\
\sigma_j^n \sigma_i^{n+1} &= \sigma_i^n \sigma_{j+1}^{n+1} && \text{for } i \leq j \\
\sigma_j^n \delta_i^{n+1} &= \delta_i^n \sigma_{j-1}^{n-1} && \text{for } i < j \\
\sigma_j^n \delta_i^{n+1} &= 1_n && \text{for } i = j \text{ or } i = j+1 \\
\sigma_j^n \delta_i^{n+1} &= \delta_{i-1}^n \sigma_j^{n-1} && \text{for } i > j+1
\end{aligned}$$

and

$$\begin{aligned}
\tau_n \delta_i^n &= \delta_{i-1}^n \tau_{n-1} && \text{for } i \geq 1 \text{ and } \tau_n \delta_0^n = \delta_n^n \\
\tau_n \sigma_i^n &= \sigma_{i-1}^n \tau_{n+1} && \text{for } i \geq 1 \text{ and } \tau_n \sigma_0^n = \sigma_n^n \tau_{n+1}^2
\end{aligned}$$

and

$$\tau_n^{n+1} = 1_n.$$

Here 1_n denotes the identity morphism from $[n]$ to $[n]$. Of course, in the relations above, n has to be taken greater than zero for some of the expressions to make sense, since δ_i^0 and σ_i^{-1} and τ_{-1} are not defined. The morphisms δ_i^n and σ_i^n are referred to as *coface maps* and *codegeneracy maps*, respectively, and generate a subcategory Δ of Λ called the *cosimplicial* category. The morphism τ_n is referred to as the *cyclicity map*.

One can alternatively define the category Λ in terms of its *cyclic covering* $E\Lambda$. The category $E\Lambda$ has one object $(\mathbb{Z}, n)$, for each $n \in \mathbb{N}_0$, and morphisms $f : (\mathbb{Z}, n) \to (\mathbb{Z}, m)$ are given by non-decreasing maps $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(x+n) = f(x) + m$, for $x \in \mathbb{Z}$. Then $\Lambda \cong E\Lambda/\mathbb{Z}$ with respect to the obvious action of $\mathbb{Z}$ by translation.

A *cocyclic object* of any category $\mathcal{C}$ is a covariant functor from Λ to $\mathcal{C}$. By the universal property of the cyclic category Λ, a cocyclic object of a category $\mathcal{C}$ is equivalently described by a quadruple (C^n, d_i^n, s_i^n, t_n), where d_i^n, s_i^n and t_n are morphisms of $\mathcal{C}$ with objects C^n satisfying the same relations when substituted for δ_i^n, σ_i^n and τ_n.

Consider a cocyclic object (C^n, d_i^n, s_i^n, t_n) of an abelian category $\mathcal{C}$. Define morphisms $b_n, b'_n : C^{n-1} \to C^n$ and $\lambda_n, N^n : C^n \to C^n$ by:

$$\begin{aligned}
b_n &= \sum_{i=0}^{n} (-1)^i d_i^n \quad \text{and} \quad b'_n = \sum_{i=0}^{n-1} (-1)^i d_i^n \\
\lambda_n &= (-1)^n t_n \quad \text{and} \quad N^n = \sum_{i=0}^{n} \lambda_n^i.
\end{aligned}$$

It is easy to check (by using the defining relations for the morphisms δ_i^n in the category Λ) that (C, b) and (C, b') are complexes, i.e. $b_{n+1}b_n = b'_{n+1}b'_n = 0$, for any $n \in \mathbb{N}_0$. The cohomology of (C, b) is by definition the *Hochschild cohomology* $HH^*(C)$ of the cosimplicial object (C^n, d_i^n, s_i^n). The complex (C, b') is, however, acyclic with contracting homotopy $s_{n+1} = \sigma_n^n \tau_{n+1}$. Again, using the relations between the τ's and δ's in Λ, one can check that $(1_n - \lambda_n)b_n = b'_n(1_{n-1} - \lambda_{n-1})$, for any $n \in \mathbb{N}_0$. Therefore we get a subcomplex (C_λ, b) of (C, b) with objects $C_\lambda^n = \ker(1_n - \lambda_n)$ consisting of the cyclic n-cochains. The cohomology of this subcomplex is denoted by $H_\lambda^*(C)$.

Further, it can be shown (by using the relations for the δ's and τ's in Λ) that $bN = Nb'$ and $(1 - \lambda)N = N(1 - \lambda) = 0$. From this one obtains a *bicomplex*:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
b\uparrow & & -b'\uparrow & & b\uparrow & & -b'\uparrow & & \\
C^2 & \overset{1-\lambda}{\rightarrow} & C^2 & \overset{N}{\rightarrow} & C^2 & \overset{1-\lambda}{\rightarrow} & C^2 & \overset{N}{\rightarrow} & \cdots \\
b\uparrow & & -b'\uparrow & & b\uparrow & & -b'\uparrow & & \\
C^1 & \overset{1-\lambda}{\rightarrow} & C^1 & \overset{N}{\rightarrow} & C^1 & \overset{1-\lambda}{\rightarrow} & C^1 & \overset{N}{\rightarrow} & \cdots \\
b\uparrow & & -b'\uparrow & & b\uparrow & & -b'\uparrow & & \\
C^0 & \overset{1-\lambda}{\rightarrow} & C^0 & \overset{N}{\rightarrow} & C^0 & \overset{1-\lambda}{\rightarrow} & C^0 & \overset{N}{\rightarrow} & \cdots
\end{array}$$

We denote this bicomplex by (C^{mn}, b, b'), where $C^{mn} = C^n$, for all $m, n \in \mathbb{N}_0$. The cohomology of the total complex of (C^{mn}, b, b') is, by definition, the *cyclic cohomology* $HC^*(C)$ of the cocyclic object (C^n, d_i^n, s_i^n, t_n). The cohomology of the total complex of (C^{mn}, b, b'), where we instead let $m \in \mathbb{Z}$, is by definition, the *periodic cyclic cohomology* $HP^*(C)$ of the cocyclic object (C^n, d_i^n, s_i^n, t_n). From the bicomplex construction the long exact IBS-sequence

$$\cdots \longrightarrow HC^n \xrightarrow{I} HH^n \xrightarrow{B} HC^{n-1} \xrightarrow{S} HC^{n+1} \xrightarrow{I} HH^{n+1} \xrightarrow{B} HC^n \longrightarrow \cdots$$

of Connes follows easily. Here we have used the abbreviations HC^* for $HC^*(C)$ and HH^* for $HH^*(C)$. Recall that the B-operator is given by $B = Ns(1 - \lambda)$, or more explicitly

$$B_n = N^n \sigma_n^n \tau_{n+1}(1_{n+1} - \lambda_{n+1}) : C^{n+1} \to C^n,$$

for $n \in \mathbb{N}_0$. We denote the *mixed complex* of C. Kassel associated to a cocyclic object (C^n, d_i^n, s_i^n, t_n) by (C, b, B) [2] [24]. It is known that $H_\lambda^*(C) \cong HC^*(C)$ whenever the ground field contains the rationals.

Now Λ is a small category, which means that the collection $\mathrm{Mor}(\Lambda) = \{f \in \mathrm{Hom}(n, m) \mid n, m \in \mathbb{N}_0\}$ of all its morphisms is a set. Write $C(\Lambda)$ for the vector space with basis $\mathrm{Mor}(\Lambda)$. On $C(\Lambda)$ define an algebra structure with product $(af)(bg)$ to be $abfg$, if the composition fg makes sense, and to be zero otherwise, where $a, b \in \mathbb{C}$ and $f, g \in \mathrm{Mor}(\Lambda)$.

A cocyclic object (C^n, d_i^n, s_i^n, t_n) in the category of vector spaces yields in an obvious manner a $C(\Lambda)$-module $C^\natural$ whose underlying vector space is the direct sum $\bigoplus_{n \in \mathbb{N}_0} C^n$. We refer to $C(\Lambda)$-modules simply as Λ-*modules*. Associated to the Λ-module $C^\natural$ given by a cocyclic object, there are linear maps $b = \oplus_{n \in \mathbb{N}_0} b_n$ and $b' = \oplus_{n \in \mathbb{N}_0} b'_n$ and $\lambda = \oplus_{n \in \mathbb{N}_0} \lambda_n$ and $N = \oplus_{n \in \mathbb{N}_0} N^n$. We then get the following nice interpretation of the cyclic cohomology groups $HC^*(C)$ of a cocyclic object in the category of vector spaces, in terms of a derived functor over the category of Λ-modules [5], i.e.,

$$HC^*(C) \cong \mathrm{Ext}^*_\Lambda(\mathbb{C}^\natural, C^\natural).$$

Similarly, cyclic objects - defined as contavariant functors from Λ to any abelian category - will give rise to homology groups. And for a cyclic object in the category of vector spaces, the cyclic homology groups $HC_*(C)$ can be interpreted as a derived functor over the category of Λ-modules, i.e.,

$$HC_*(C) \cong \mathrm{Tor}^\Lambda_*(C^\natural, \mathbb{C}^\natural).$$

Suppose that $\mathcal{A}$ is a unital algebra. We recall the basic example of a cocyclic object, namely the one which gives rise to ordinary Hochschild cohomology $HH^*(\mathcal{A})$ and ordinary cyclic cohomology $HC^*(\mathcal{A})$ of $\mathcal{A}$. Let $n \in \mathbb{N}_0$, and denote by C^n the vector space of n-cochains, that is, the set of multilinear maps from the vector space $\mathcal{A}^{n+1}$ to $\mathbb{C}$. Let $n \in \mathbb{N}_0$ and $i \in \{0, 1, \ldots, n\}$. Define linear maps

$$d_i^n : C^{n-1} \to C^n, \qquad s_i^n : C^{n+1} \to C^n, \qquad t_n : C^n \to C^n$$

as follows:

$$\begin{aligned}
(d_i^n\varphi)(a^0, a^1, \ldots, a^n) &= \varphi(a^0, \ldots, a^i a^{i+1}, \ldots, a^n) \quad \text{for } i \leq n-1 \\
(d_n^n\varphi)(a^0, a^1, \ldots, a^n) &= \varphi(a^n a^0, a^1, \ldots, a^{n-1}) \\
(s_i^n\varphi)(a^0, a^1, \ldots, a^n) &= \varphi(a^0, \ldots, a^i, I, a^{i+1}, \ldots, a^n) \\
(t_n\varphi)(a^0, a^1, \ldots, a^n) &= \varphi(a^n, a^0, a^1, \ldots, a^{n-1}),
\end{aligned}$$

where I is the unit of $\mathcal{A}$, $a^0, \ldots, a^n \in \mathcal{A}$ and φ is a cochain. Note that C^{-1} and d_i^0 are not defined, so the range of n must be restricted accordingly for some formulas to make sense. It is straightforward to check that (C^n, d_i^n, s_i^n, t_n) is a cocyclic object of the abelian category of vector spaces. In Section 8 we shall consider a twisted version of this cocyclic object.

Before we consider the cocyclic object given by a Hopf algebra, let us recall the definition of the Chern character [2] in noncommutative geometry, which manifests a profound link between ordinary cyclic cohomology and K-theory. Let φ be an n-dimensional cyclic cocycle, so $\varphi \in C^n$, $\lambda_n(\varphi) = \varphi$ and $b_{n+1}(\varphi) = 0$, and denote by $[\varphi]$ the corresponding equivalence class in $HC^n(\mathcal{A}) = H^n_\lambda(\mathcal{A})$. Let $n, N \in \mathbb{N}_0$ and consider the standard trace Tr on $M_N(\mathbb{C})$. Denote by $\varphi \otimes \mathrm{Tr}$ the $(n+1)$-multilinear functional on $M_N(\mathcal{A}) = \mathcal{A} \otimes M_N(\mathbb{C})$ given by

$$(\varphi \otimes \mathrm{Tr})(a^0 \otimes x^0, a^1 \otimes x^1, \ldots, a^n \otimes x^n) = \varphi(a^0, a^1, \ldots, a^n)\mathrm{Tr}(x^0 x^1 \cdots x^n),$$

for all $a^0, \ldots, a^n \in \mathcal{A}$ and $x^0, \ldots, x^n \in M_N(\mathbb{C})$.

Consider first the case when $n = 2m$ is even. It can be shown that the scalar $(\varphi \otimes \mathrm{Tr})(E, E, \ldots, E)$ is invariant under homotopy for idempotents $E^2 = E \in M_N(\mathcal{A})$. Thus φ provides in this way a numerical invariant for the K-group $K_0(\mathcal{A})$. Moreover, the scalar is independent of what representative φ in $[\varphi]$ is employed. Thus we obtain a bilinear pairing

$$\langle\cdot,\cdot\rangle : HC^{\mathrm{even}}(\mathcal{A}) \times K_0(\mathcal{A}) \to \mathbb{C}$$

given by

$$\langle[\varphi],[E]\rangle = \frac{1}{m!}(\varphi\otimes \mathrm{Tr})(E,E,\ldots,E),$$

for all $[\varphi] \in HC^{2m}(\mathcal{A})$ and projections $E \in M_N(\mathcal{A})$. Here $HC^{\mathrm{even}}(\mathcal{A})$ denotes the even cyclic cohomology of $\mathcal{A}$.

Consider next the case when n is odd. For u an invertible element of the unital algebra $M_N(\mathcal{A})$ and φ a cyclic n-cocycle of $\mathcal{A}$, the formula

$$\langle[\varphi],[u]\rangle = \frac{1}{\sqrt{2i}}2^{-n}\Gamma(\frac{n}{2}+1)^{-1}(\varphi\otimes \mathrm{Tr})(u^{-1}-1,u-1,u^{-1}-1,\ldots,u-1)$$

defines similarly a bilinear pairing

$$\langle\cdot,\cdot\rangle : HC^{\mathrm{odd}}(\mathcal{A}) \times K_1(\mathcal{A}) \to \mathbb{C}.$$

Here, $HC^{\mathrm{odd}}(\mathcal{A})$ denotes the odd cyclic cohomology of $\mathcal{A}$, Γ is the gamma function and $[u]$ denotes the K-theory class of the invertible $u \in M_N(\mathcal{A})$.

Let $(\mathcal{H},\Delta)$ be a Hopf algebra with comultiplication Δ, coinverse (antipode) S, counit ε, unit I and multiplication $m : \mathcal{H}\otimes\mathcal{H} \to \mathcal{H}$. If U and V are vector spaces, we denote the flip from $U \otimes V$ to $V \otimes U$ by F. Our basic reference for Hopf algebras is [1].

A *modular pair* for $(\mathcal{H},\Delta)$ consists of a pair (δ,σ), where δ is a unital, multiplicative, linear functional on $\mathcal{H}$ and $\sigma \in \mathcal{H}$ satisfies $\Delta(\sigma) = \sigma\otimes\sigma$ and $\delta(\sigma) = 1$. Thus (ε, I) is an example of a modular pair for $(\mathcal{H},\Delta)$. Now $\hat{\delta} : \mathcal{H} \to \mathcal{H}$ given by $\hat{\delta} = (\delta\otimes\iota)\Delta$ is a unital, multiplicative, linear map with inverse $\hat{\delta}^{-1} = (\delta S\otimes\iota)\Delta$. We associate to the modular pair (δ,σ) a *twisted coinverse* $S_\delta : \mathcal{H} \to \mathcal{H}$ defined by $S_\delta = (\delta \otimes S)\Delta = S\circ\hat{\delta}$, see [3] [6]. Note that if S is invertible, then S_δ is invertible with inverse $S_\delta^{-1} = (S^{-1}\otimes\delta)\Delta$. We say that the modular pair (δ,σ) is *in involution* if $S_\delta^2(x) = \sigma x\sigma^{-1}$, for all $x \in \mathcal{H}$.

Let $n \in \mathbb{N}_0$ and $i \in \{0,1,\ldots,n\}$. Define linear maps

$$\delta_i^n : \mathcal{H}^{\otimes(n-1)} \to \mathcal{H}^{\otimes n}, \qquad \sigma_i^n : \mathcal{H}^{\otimes(n+1)} \to \mathcal{H}^{\otimes n}, \qquad \tau_n : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$$

as follows:

$$\begin{aligned}
\delta_0^n(x^1\otimes\ldots\otimes x^{n-1}) &= I\otimes x^1\otimes\ldots\otimes x^{n-1}\\
\delta_i^n(x^1\otimes\ldots\otimes x^{n-1}) &= x^1\otimes\ldots\otimes\Delta x^i\otimes\ldots\otimes x^{n-1} \quad \text{for } i\in\{1,\ldots,n-1\}\\
\delta_n^n(x^1\otimes\ldots\otimes x^{n-1}) &= x^1\otimes\ldots\otimes x^{n-1}\otimes\sigma\\
\sigma_i^n(x^1\otimes\ldots\otimes x^{n+1}) &= x^1\otimes\ldots\otimes\varepsilon(x^{i+1})\otimes\ldots\otimes x^{n+1}\\
\tau_n(x^1\otimes\ldots\otimes x^n) &= (\Delta^{n-1}S_\delta(x^1))\cdot(x^2\otimes\ldots\otimes x^n\otimes\sigma),
\end{aligned}$$

where $x^1,\ldots,x^{n+1} \in \mathcal{H}$. Here $\mathcal{H}^{\otimes(-1)}$ and δ_i^0 are not defined. By $\mathcal{H}^{\otimes 0}$ we mean $\mathbb{C}$. We define $\delta_0^1,\delta_1^1 : \mathbb{C} \to \mathcal{H}$, $\tau_0 : \mathbb{C} \to \mathbb{C}$ and $\sigma_0^0 : \mathcal{H} \to \mathbb{C}$ by setting $\delta_0^1(1) = I$, $\delta_1^1(1) = \sigma$, $\tau_0(1) = 1$ and $\sigma_0^0 = \varepsilon$. The map Δ^n is defined inductively by setting $\Delta^n = (\iota\otimes\Delta)\Delta^{n-1}$ for $n \geq 2$. One can check directly [4] that $(\mathcal{H}^{\otimes n}, \delta_i^n, \sigma_i^n, \tau_n)$ is a cocyclic object of the abelian category of vector spaces if and only if the modular pair (δ, σ) is in involution. Hereafter we assume that this is the case. Following Connes and Moscovici we denote the Hochschild cohomology and the cyclic cohomology of this cocyclic object by $HH^*_{(\delta,\sigma)}(\mathcal{H})$ and $HC^*_{(\delta,\sigma)}(\mathcal{H})$, respectively.

Now $\mathbb{C}$ is an $\mathcal{H}$-bicomodule with respect to $\alpha : \mathbb{C} \to \mathcal{H}\otimes\mathbb{C}\otimes\mathcal{H}$ given by $\alpha(1) = I\otimes 1\otimes\sigma$, and $\mathcal{H}$ is an $\mathcal{H}$-bicomodule with respect to $\alpha : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ given by $\alpha = (\Delta \otimes \iota)\Delta$. Since the face maps and degeneracy maps of our Hopf cyclic object are intimately linked to those defining the cobar resolution, we obtain the following useful result.

Proposition 2.1. *([12]) Let $(\mathcal{H}, \Delta)$ be a Hopf algebra with a modular pair (δ, σ) in involution. Then*

$$HH^*_{(\delta,\sigma)}(\mathcal{H}) \cong \mathrm{Ext}^*_{\mathcal{H}-\mathrm{bicom}}(\mathbb{C}, \mathcal{H}).$$

Alternatively, if we regard $\mathbb{C}$ as an $\mathcal{H}$-comodule by $\beta_1(1) = I \otimes 1$ and denote the $\mathcal{H}$-comodule $\mathbb{C}$ with respect to $\beta_2(1) = \sigma \otimes 1$ by $\mathbb{C}_\sigma$, we get the following interpretation.

Proposition 2.2. *([6]) Let $(\mathcal{H}, \Delta)$ be a Hopf algebra with a modular pair (δ, σ) in involution. Then*

$$HH^*_{(\delta,\sigma)}(\mathcal{H}) \cong \mathrm{Cotor}^*_{\mathcal{H}-\mathrm{com}}(\mathbb{C}, \mathbb{C}_\sigma).$$

3 The Category of Hopf Algebras with Modular Pairs

In this section we study the category of Hopf algebras with modular pairs in involution, and recall various basic and motivating examples.

Suppose $(\mathcal{H}_i, \Delta_i)$ are two Hopf algebras with modular pairs (δ_i, σ_i) in involution for $i = 1, 2$, and let $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$ be a morphism of Hopf algebras. We say φ is a *morphism* in the category of Hopf algebras with modular pairs in involution if $\delta_2\varphi = \delta_1$ and $\sigma_2 = \varphi(\sigma_1)$. Consider the tensor product Hopf algebra $(\mathcal{H}_1 \otimes \mathcal{H}_2, \Delta_1 \times \Delta_2)$ of $(\mathcal{H}_1, \Delta_1)$ and $(\mathcal{H}_2, \Delta_2)$ with comultiplication $\Delta_1 \times \Delta_2 = (\iota\otimes F\otimes\iota)(\Delta_1\otimes\Delta_2)$. Obviously, the pair $(\delta_1\otimes\delta_2, \sigma_1\otimes\sigma_2)$ is a modular pair in involution for $(\mathcal{H}_1 \otimes \mathcal{H}_2, \Delta_1 \times \Delta_2)$. The following Künneth theorem for this setting can now be stated.

Theorem 3.1. *([12]) Consider two Hopf algebras $(\mathcal{H}_i, \Delta_i)$ with modular pairs (δ_i, σ_i) in involution. Then*

$$HH^*_{(\delta_1\otimes\delta_2,\sigma_1\otimes\sigma_2)}(\mathcal{H}_1 \otimes \mathcal{H}_2) \cong HH^*_{(\delta_1,\sigma_1)}(\mathcal{H}_1) \otimes HH^*_{(\delta_2,\sigma_2)}(\mathcal{H}_2),$$

and, moreover, the isomorphism is implemented by the shuffle map sh (going from left to right) with the Alexander-Whitney map AW as its inverse.

In this context we also have the following Künneth exact sequence.

Theorem 3.2. *([12]) Consider two Hopf algebras $(\mathcal{H}_i, \Delta_i)$ with modular pairs (δ_i, σ_i) in involution. Then there is a canonical long exact sequence*

$$\cdots \xrightarrow{i} HC^{n-1}_{(\delta_1\otimes\delta_2,\sigma_1\otimes\sigma_2)}(\mathcal{H}_1 \otimes \mathcal{H}_2) \xrightarrow{\partial}$$

$$\bigoplus_{p+q=n-2} HC^p_{(\delta_1,\sigma_1)}(\mathcal{H}_1) \otimes HC^q_{(\delta_2,\sigma_2)}(\mathcal{H}_2) \xrightarrow{S\otimes 1-1\otimes S}$$

$$\bigoplus_{r+s=n} HC^r_{(\delta_1,\sigma_1)}(\mathcal{H}_1) \otimes HC^s_{(\delta_2,\sigma_2)}(\mathcal{H}_2) \xrightarrow{i} HC^n_{(\delta_1\otimes\delta_2,\sigma_1\otimes\sigma_2)}(\mathcal{H}_1 \otimes \mathcal{H}_2) \xrightarrow{\partial} \cdots$$

Suppose that $(\mathcal{H}, \Delta)$ is a Hopf algebra with coinverse S. Denote by $\mathcal{H}_{\mathrm{op}}$ the algebra $\mathcal{H}$ with the opposite multiplication mF, and denote by Δ_{op} the opposite comultiplication $F\Delta$. Let $\sigma \in \mathcal{H}$ and let δ be a linear functional on $\mathcal{H}$. Then the following four statements are equivalent ([12]):

1. (δ, σ) is a modular pair in involution for $(\mathcal{H}, \Delta)$;
2. $(\delta S, \sigma)$ is a modular pair in involution for $(\mathcal{H}_{\mathrm{op}}, \Delta)$;
3. (δ, σ^{-1}) is a modular pair in involution for $(\mathcal{H}, \Delta_{\mathrm{op}})$;
4. $(\delta S, \sigma^{-1})$ is a modular pair in involution for $(\mathcal{H}_{\mathrm{op}}, \Delta_{\mathrm{op}})$.

The next result shows that the category of Hopf algebras with modular pairs in involution is also closed under formation of duals.

Proposition 3.3. *([12]) Suppose $\langle \cdot, \cdot \rangle : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{C}$ is a non-degenerate dual pairing between two Hopf algebras $(\mathcal{H}_i, \Delta_i)$ with coinverses S_i. Consider two elements $\delta \in \mathcal{H}_2$ and $\sigma \in \mathcal{H}_1$. Then $(\langle \cdot, \delta \rangle, \sigma)$ is a modular pair in involution for $(\mathcal{H}_1, \Delta_1)$ if and only if $(\langle S_1(\sigma), \cdot \rangle, S_2(\delta))$ is a modular pair in involution for $(\mathcal{H}_2, \Delta_2)$.*

Example 3.4. ([4][6]) Let $(\mathcal{A}, \Phi)$ be the Hopf $*$-algebra associated to a compact quantum group in the sense of S.L. Woronowicz [25] [26]. Let $\{f_z\}_{z \in \mathbb{C}}$ denote the family of functionals on $\mathcal{A}$ describing the modular properties of the Haar state, Proposition A.1, and let I denote the unit of the algebra $\mathcal{A}$. Then (f_{-1}, I) is a modular pair in involution for $(\mathcal{A}, \Phi)$. The twisted coinverse is given by $\kappa_\delta = (f_{-1} \otimes \kappa)\Phi$, where κ is the coinverse of $(\mathcal{A}, \Phi)$. If Γ is a discrete group and $\mathcal{A}$ is the group algebra $\mathbb{C}[\Gamma]$ with $\Phi(g) = g \otimes g$ for $g \in \Gamma \subset \mathcal{A}$, then $f_z = \varepsilon$ for all $z \in \mathbb{C}$, so (ε, I) is a modular pair in involution for $(\mathbb{C}[\Gamma], \Phi)$.

However, this example is not terribly interesting due to the following result, which follows easily from the existence of the Haar state for $(\mathcal{A}, \Phi)$.

Proposition 3.5. *([7]) Let $(\mathcal{A}, \Phi)$ be the Hopf $*$-algebra associated to a compact quantum group. Then*

1. $HH^n_{(f_{-1}, I)}(\mathcal{A}) \cong 0$ *if* $n \neq 0$, *and* $HH^0_{(f_{-1}, I)}(\mathcal{A}) \cong \mathbb{C}$.
2. $HC^n_{(f_{-1}, I)}(\mathcal{A}) \cong \mathbb{C}$ *if n is even, and* $HC^n_{(f_{-1}, I)}(\mathcal{A}) \cong 0$ *if n is odd.*
3. $HP^0_{(f_{-1}, I)}(\mathcal{A}) \cong \mathbb{C}$ *and* $HP^1_{(f_{-1}, I)}(\mathcal{A}) \cong 0$.

The following proposition shows that in the absence of a Haar state, something interesting can nevertheless be obtained from the Hopf algebra of polynomial functions on a Lie group.

Proposition 3.6. *([5][3]) Let $(\mathcal{H}(G), \Phi)$ be the Hopf algebra of polynomial functions on a simply connected affine nilpotent group G, with Lie algebra $\mathfrak{g}$. Then*

$$HP^*_{(\varepsilon, I)}(\mathcal{H}(G)) \cong \bigoplus_{i = * \bmod 2} H^i(\mathfrak{g}, \mathbb{C}),$$

where $H^*(\mathfrak{g}, \mathbb{C})$ is the Lie algebra cohomology.

For compact quantum groups the dual setting turns out to be more interesting.

Example 3.7. ([4]) Denote by $(\mathcal{A}^o, \Delta)$ the maximal Hopf $*$-algebra dual to $(\mathcal{A}, \Phi)$ [1]. It has product $xy = (x \otimes y)\Phi$ and coproduct $\Delta(x)(a \otimes b) = x(ab)$, for $x, y \in \mathcal{A}^o$ and $a, b \in \mathcal{A}$. Thus $S^2(x) = f_1 x f_{-1}$, where S is the coinverse of $(\mathcal{A}^o, \Delta)$ given by $S(x) = x\kappa$. Note that $f_1^* = f_1$, where $x^*(a) = \overline{x(\kappa(a)^*)}$ is the $*$-operation for $(\mathcal{A}^o, \Delta)$. Suppose that $(\mathcal{H}, \Delta)$ is a Hopf $*$-subalgebra of $(\mathcal{A}^o, \Delta)$ such that $f_1 \in \mathcal{H}$. Regard the unit of $\mathcal{A}$ as the linear functional on $\mathcal{H}$ given by $I(x) = x(I)$, for $x \in \mathcal{H}$, so I is just the counit of $(\mathcal{A}^o, \Delta)$. Then (I, f_1) is a modular pair in involution for $(\mathcal{H}, \Delta)$. Note that $S_I = S$.

If $\mathcal{A}$ is the algebra of regular functions on a compact Lie group G with pointwise operations and Φ is the transpose of the group multiplication on G, then one can consider $\mathcal{H} = U(\mathfrak{g})$, where $U(\mathfrak{g})$ is the (complex) universal enveloping algebra of the associated Lie algebra $\mathfrak{g}$. The inclusion $U(\mathfrak{g}) \subset \mathcal{A}^o$ is given by $X(a) = X(a)(e)$, for $a \in \mathcal{A}$. Here e is the unit of G and X on the right hand side is regarded as a left invariant differential operator on G. Then $(U(\mathfrak{g}), \Delta)$ is a Hopf $*$-subalgebra of $(\mathcal{A}^o, \Delta)$ with $f_z = \varepsilon$ for $z \in \mathbb{C}$, and where $\Delta(X) = X \otimes \varepsilon + \varepsilon \otimes X$ for $X \in \mathfrak{g}$. Hence (I, ε) is a modular pair in involution for $(U(\mathfrak{g}), \Delta)$.

In view of Example 3.7 the following uniqueness result is worthwhile recalling. Let $\operatorname{Rep}\mathcal{B}$ denote the category of finite-dimensional $*$-representations of the unital $*$-algebra $\mathcal{B}$. We say that an element of $\mathcal{B}$ is *positive* if it can be written as a finite sum of elements of the form b^*b, where $b \in \mathcal{B}$. As usual Tr will denote the operator trace.

Proposition 3.8. *([21]) Suppose* $(\mathcal{H}, \Delta)$ *is a Hopf* $*$*-algebra with coinverse* S *and that* $\operatorname{Rep}\mathcal{H}$ *separates the elements of* $\mathcal{H}$*. Then there is at most one positive invertible* $f \in \mathcal{H}$ *such that* $S^2(x) = fxf^{-1}$*, for all* $x \in \mathcal{H}$*, and* $\operatorname{Tr}\pi(f) = \operatorname{Tr}\pi(f^{-1})$*, for all* $\pi \in \operatorname{Rep}\mathcal{H}$*. Moreover, the element* f *is necessarily group-like, and whenever* $(\mathcal{H}, \Delta)$ *is cocommutative, it equals the unit.*

Remark 3.9. Let $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{A} \to \mathbb{C}$ denote the dual pairing $\langle x, a\rangle = x(a)$ between $(\mathcal{A}, \Phi)$ and $(\mathcal{H}, \Delta)$ considered in Example 3.4 and Example 3.7. Then $(f_{-1}, I) = (\langle S(f_1), \cdot\rangle, \kappa(I))$ is a modular pair in involution for $(\mathcal{A}, \Phi)$ dual (in the sense of Proposition 3.3) to the modular pair $(I, f_1) = (\langle\cdot, I\rangle, f_1)$ in involution for $(\mathcal{H}, \Delta)$.

In Example 3.7 we saw that (I, ε) is a modular pair in involution for $(U(\mathfrak{g}), \Delta)$. We can provide other types of modular pairs in involution in this case. Namely, let δ be a character of $\mathfrak{g}$, which means that $\delta : \mathfrak{g} \to \mathbb{C}$ is linear and $\delta([\mathfrak{g}, \mathfrak{g}]) = 0$. Such a character has a unique extension to a unital multiplicative functional on $U(\mathfrak{g})$, which we also denote by δ. Since the twisted coinverse S_δ in this case is given by $S_\delta(X) = -X + \delta(X)$, for $X \in \mathfrak{g}$, we see that (δ, ε) is a modular pair in involution for $(U(\mathfrak{g}), \Delta)$ generalizing that of (I, ε). Let $\mathbb{C}_\delta$ denote the $\mathfrak{g}$-module $\mathbb{C}$ with action induced by δ, and let $H_*(\mathfrak{g}, \mathbb{C}_\delta)$ denote the corresponding Lie algebra homology. By definition it is the homology of the Chevalley-Eilenberg complex

$$\Lambda^0(\mathfrak{g}) \xleftarrow{d_{Lie}} \Lambda^1(\mathfrak{g}) \xleftarrow{d_{Lie}} \Lambda^2(\mathfrak{g}) \xleftarrow{d_{Lie}} \Lambda^3(\mathfrak{g}) \xleftarrow{d_{Lie}} \cdots,$$

where $\Lambda(\mathfrak{g}) = \oplus_i \Lambda^i(\mathfrak{g})$ is the exterior algebra over $\mathfrak{g}$ and

$$d_{Lie}(X^1 \wedge ... \wedge X^n) = \sum_{i=1}^n (-1)^{i+1}\delta(X^i) X^1 \wedge \cdots \wedge \hat{X^i} \wedge \cdots \wedge X^n$$

$$+ \sum_{i<j} (-1)^{i+j} [X^i, X^j] \wedge X^1 \wedge \cdots \wedge \hat{X^i} \wedge \cdots \wedge \hat{X^j} \wedge \cdots \wedge X^n,$$

for $X^k \in \mathfrak{g}$. This complex can be regarded as a mixed complex $(\Lambda^n(\mathfrak{g}), b, B)$ with $b = 0$ and $B = d_{\mathrm{Lie}}$.

It can be shown that the linear map $A : \Lambda^n(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes n}$ given by

$$A(X^1 \wedge ... \wedge X^n) = \frac{1}{n!} \sum_{\sigma} \mathrm{sign}(\sigma) X^{\sigma(1)} \otimes \cdots \otimes X^{\sigma(n)},$$

for $X^k \in \mathfrak{g}$, is a quasi-isomorphism from the mixed complex $(\Lambda^n(\mathfrak{g}), b, B)$ to the mixed complex associated to the Hopf algebra $(U(\mathfrak{g}), \Delta)$ with the modular pair (δ, ε) in involution. Hence we get the following result.

Proposition 3.10. *([4][5]) Let $\mathfrak{g}$ be a Lie algebra with character δ. Then*

$$HP^*_{(\delta,\varepsilon)}(U(\mathfrak{g})) \cong \bigoplus_{i = * \bmod 2} H_i(\mathfrak{g}, \mathbb{C}_\delta)$$

and

$$HH^n_{(\delta,\varepsilon)}(U(\mathfrak{g})) \cong \Lambda^n(\mathfrak{g}),$$

for all $n \in \mathbb{N}_0$.

The Hopf algebra $(U(\mathfrak{g}), \Delta)$ is cocommutative. It is desirable to compute the Hopf cyclic cohomology given by a Hopf algebra of the type considered in Example 3.7 for a genuine quantum group.

Example 3.11. In this example we consider the quantum $SU_q(2)$, where the parameter $q \in \langle 0, 1 \rangle$. The Hopf $*$-algebra associated to $SU_q(2)$ is denoted by $(\mathcal{A}_q, \Phi)$. The fundamental unitary corepresentation U of $(\mathcal{A}_q, \Phi)$ is given by

$$U = (U_{ij}) = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix},$$

where α and γ are the well known generators of the unital $*$-algebra $\mathcal{A}_q$ introduced by Woronowicz [14]. The multiplicative functional f_1 is uniquely determined by

$$f_1(\alpha) = q^{-1} \ , \quad f_1(\alpha^*) = q \ , \quad f_1(\gamma) = f_1(\gamma^*) = 0 \ , \quad f_1(I) = 1.$$

Consider the universal algebra $U_q(\mathrm{sl}_2)$ with unit ε and generators e, f, k, k^{-1} satisfying the relations:

$$kk^{-1} = k^{-1}k = \varepsilon \ , \quad ke = qek \ , \quad kf = q^{-1}fk \ , \quad ef - fe = \frac{1}{q - q^{-1}}(k^2 - k^{-2}).$$

It is a Hopf $*$-algebra with comultiplication $\Delta : U_q(\mathrm{sl}_2) \to U_q(\mathrm{sl}_2) \otimes U_q(\mathrm{sl}_2)$ and $*$-operation uniquely determined by

$$\Delta(k) = k \otimes k \ , \quad \Delta(e) = e \otimes k^{-1} + k \otimes e \ , \quad \Delta(f) = f \otimes k^{-1} + k \otimes f$$

and

$$k^* = k \ , \quad (k^{-1})^* = k^{-1} \ , \quad e^* = f \ , \quad f^* = e.$$

We shall regard the Hopf $*$-algebra $(U_q(\mathrm{sl}_2), \Phi)$ as a Hopf $*$-subalgebra of $(\mathcal{A}_q^o, \Delta)$ such that

$$k(U_{11}) = q^{1/2} \ , \quad k(U_{22}) = q^{-1/2} \ , \quad k^{-1}(U_{11}) = q^{-1/2} \ , \quad k^{-1}(U_{22}) = q^{1/2} \ ,$$

$$k(U_{ij}) = 0 \ , \ \text{for } i \neq j \ , \quad e(U_{ij}) = \delta_{i1}\delta_{j2} \ , \quad f(U_{ij}) = \delta_{i2}\delta_{j1} \ .$$

Under this identification we have $k^{-2} = f_1$.

Recall that the coboundary operator $b_n : (\mathcal{A}_q^o)^{\otimes(n-1)} \to (\mathcal{A}_q^o)^{\otimes n}$ is given by $b_n = \sum_{i=0}^n (-1)^i \delta_i^n$, for $n \in \mathbb{N}_0$. Thus $b_1(1) = \varepsilon - f_1 \neq 0$, so $HH^0_{(I,f_1)}(\mathcal{A}_q^o) = \mathrm{Ker}(b_1) \cong 0$ and therefore also $HH^0_{(I,f_1)}(U_q(\mathrm{sl}_2)) \cong 0$. Next, we get $b_2(x) = \varepsilon \otimes x - \Delta(x) + x \otimes f_1$, for all $x \in \mathcal{A}_q^o$, so

$$\mathrm{Ker}(b_2) = \{x \in \mathcal{A}_q^o \mid \Delta(x) = x \otimes f_1 + \varepsilon \otimes x\}.$$

Clearly, the elements $\{f_1 - \varepsilon, ek^{-1}, fk^{-1}\}$ are linearly independent and belong to $\mathrm{Ker}(b_2)$. It is not hard to show, for instance by using methods from differential calculi for quantum groups [12], that these elements generate the vector space $\mathrm{Ker}(b_2)$. Since $f_1 - \varepsilon$ is a generator for $\mathrm{Im}(b_1)$, we thus see that the two elements $\{[ek^{-1}], [fk^{-1}]\}$ are generators for $HH^1_{(I,f_1)}(\mathcal{A}_q^o)$.

To compute $HC^1_{(I,f_1)}(\mathcal{A}_q^o)$, first note that $\lambda_1 : \mathcal{A}_q^o \to \mathcal{A}_q^o$ is given by $\lambda_1(x) = -\tau_1(x) = -S_I(x)f_1 = -S(x)f_1$, for all $x \in \mathcal{A}_q^o$. Thus for any $x \in \mathrm{Ker}(b_2)$, we have $\lambda_1(x) = x$ if and only if $x(I) = 0$. Since $(ek^{-1})(I) = (fk^{-1})(I) = 0$, we therefore see that the two elements $\{[ek^{-1} \oplus 0], [fk^{-1} \oplus 0]\}$ are generators in the total complex for $HC^1_{(I,f_1)}(\mathcal{A}_q^o)$ (with $ek^{-1} \oplus 0$ and $fk^{-1} \oplus 0$ in the total complex of the bicomplex defining cyclic cohomology).

Since these generators also belong to $U_q(\mathrm{sl}_2)$, the same is true for the groups $HH^1_{(I,f_1)}(U_q(sl_2))$ and $HC^1_{(I,f_1)}(U_q(sl_2))$. We have shown part of the following result due to M. Crainic [6]. The computation of the full Hopf Hochschild cohomology uses essentially Proposition 2.2 and Proposition 2.1. The result for Hopf cyclic cohomology then follows from the long exact IBS-sequence, whereas the Hopf periodic cyclic cohomology is obtained as the limit of Hopf cyclic cohomology.

Proposition 3.12. *([6]) Consider the Hopf algebra $(U_q(\mathrm{sl}_2), \Delta)$ with the modular pair (I, f_1) in involution. Then:*

1. *$HH^1_{(I,f_1)}(U_q(\mathrm{sl}_2)) \cong \mathbb{C}^2$ with generators $\{[ek^{-1}], [fk^{-1}]\}$.*
2. *$HH^n_{(I,f_1)}(U_q(\mathrm{sl}_2)) \cong 0$, for $n \neq 1$.*
3. *$HC^{2n}_{(I,f_1)}(U_q(\mathrm{sl}_2)) \cong 0$, for $n \in \mathbb{N}_0$.*
4. *$HC^{2n+1}_{(I,f_1)}(U_q(\mathrm{sl}_2)) \cong \mathbb{C}^2$ with generators $\{S^n[ek^{-1} \oplus 0], S^n[fk^{-1} \oplus 0]\}$, for $n \in \mathbb{N}_0$.*
5. *$HP^0_{(I,f_1)}(U_q(\mathrm{sl}_2)) \cong 0$.*
6. *$HP^1_{(I,f_1)}(U_q(\mathrm{sl}_2)) \cong \mathbb{C}^2$.*

Remark 3.13. This should be compared with the classical case ($q \to 1$), where by Proposition 3.10 we have:

1. $HH^3_{(I,\varepsilon)}(U(\mathrm{sl}_2)) \cong HH^0_{(I,\varepsilon)}(U(\mathrm{sl}_2)) \cong \Lambda^0(\mathrm{sl}_2) \cong \mathbb{C}$.
2. $HH^2_{(I,\varepsilon)}(U(\mathrm{sl}_2)) \cong HH^1_{(I,\varepsilon)}(U(\mathrm{sl}_2)) \cong \Lambda^1(\mathrm{sl}_2) \cong \mathbb{C}^3$.
3. $HH^n_{(I,\varepsilon)}(U(\mathrm{sl}_2)) \cong 0$, for $n \geq 4$.

Example 3.14. Here we combine the results in Example 3.4 with the general discussion preceding it, to construct Hopf $*$-algebras with modular pairs in involution which will play an important role in the next section. Let $(\mathcal{H}, \Delta)$ be a Hopf subalgebra of $(\mathcal{A}^o, \Delta)$ such that $f_1 \in \mathcal{H}$. Let $\tilde{H} = \mathcal{H} \otimes \mathcal{H}_{\mathrm{op}}$ and $\tilde{\Delta} = \Delta \times \Delta$. Then $(\tilde{\mathcal{H}}, \tilde{\Delta})$ is a Hopf algebra with a modular pair $(\delta, \sigma) = (I \otimes I, f_1 \otimes f_1)$ in involution. Note that its coinverse is $S \otimes S^{-1}$, its counit is $I \otimes I$ and its unit is $\varepsilon \otimes \varepsilon$, where ε is the counit of $(\mathcal{A}, \Phi)$.

To compute the cohomologies of such tensor products, we state a useful corollary of Theorem 3.1 and Theorem 3.2.

Corollary 3.15. *([12]) Let $(\mathcal{H}, \Delta)$ be a Hopf subalgebra of $(\mathcal{A}^o, \Delta)$ such that $f_1 \in \mathcal{H}$. Consider the Hopf algebra $(\tilde{\mathcal{H}}, \tilde{\Delta})$ with a modular pair $(\delta, \sigma) = (I \otimes I, f_1 \otimes f_1)$ in involution. Then*

$$HH^n_{(\delta,\sigma)}(\tilde{\mathcal{H}}) \cong \bigoplus_{i+j=n} HH^i_{(I,f_1)}(\mathcal{H}) \otimes HH^j_{(I,f_1)}(\mathcal{H}),$$

for $n \in \mathbb{N}_0$. Moreover, the isomorphism is implemented by the shuffle map sh (going from left to right) with the Alexander-Whitney map AW as its inverse. Concerning Hopf cyclic cohomology, there exists a canonical long exact sequence

$$\cdots \xrightarrow{i} HC^{n-1}_{(\delta,\sigma)}(\tilde{\mathcal{H}}) \xrightarrow{\partial} \bigoplus_{p+q=n-2} HC^p_{(I,f_1)}(\mathcal{H}) \otimes HC^q_{(I,f_1)}(\mathcal{H}) \overset{S\otimes 1-1\otimes S}{\longrightarrow}$$

$$\bigoplus_{r+s=n} HC^r_{(I,f_1)}(\mathcal{H}) \otimes HC^s_{(I,f_1)}(\mathcal{H}) \xrightarrow{i} HC^n_{(\delta,\sigma)}(\tilde{\mathcal{H}}) \xrightarrow{\partial} \cdots$$

Combining Theorem 3.12 and Corollary 3.15 we immediately get the following result.

Theorem 3.16. *([12]) Let $(\mathcal{H}_q, \Delta) = (U_q(\mathrm{sl}_2), \Delta)$, so $(\tilde{\mathcal{H}}_q, \tilde{\Delta})$ is a Hopf $*$-algebra with a modular pair $(\delta, \sigma) = (I \otimes I, f_1 \otimes f_1)$ in involution. Then:*

1. $HH^2_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q) \cong \mathbb{C}^4$.
2. $HH^n_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q) \cong 0$, *for* $n \neq 2$.
3. $HC^0_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q) \cong 0$.
4. $HC^{2n}_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q) \cong \mathbb{C}^4$, *for* $n \in \mathbb{N}$.
5. $HC^{2n+1}_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q) \cong 0$, *for* $n \in \mathbb{N}_0$.
6. $HP^0_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q) \cong \mathbb{C}^4$.
7. $HP^1_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q) \cong 0$.

To find representatives for the generators of $HH^2_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q)$ we use the Alexander-Whitney map

$$AW_{i,j} : HH^i_{(I,f_1)}(U_q(\mathrm{sl}_2)) \otimes HH^j_{(I,f_1)}(U_q(\mathrm{sl}_2)) \longrightarrow HH^n_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q),$$

where $i + j = n \in \mathbb{N}_0$. We can thus exhibit the following generators (see Section 6) of $HH^2_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q)$ in $(\tilde{\mathcal{H}}_q^{\otimes n}, b)$:

$$[ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}],\ [ek^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}],$$

$$[fk^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}],\ [fk^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}].$$

Proposition 3.17. *([12]) Let $n \in \mathbb{N}$. The following four elements*

1. $S^{n-1}[(ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}) \oplus (-ek^{-1} \otimes ek^{-1}) \oplus 0]$,
2. $S^{n-1}[(ek^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}) \oplus (-ek^{-1} \otimes fk^{-1}) \oplus 0]$,
3. $S^{n-1}[(fk^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}) \oplus (-fk^{-1} \otimes ek^{-1}) \oplus 0]$,
4. $S^{n-1}[(fk^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}) \oplus (-fk^{-1} \otimes fk^{-1}) \oplus 0]$,

are generators of $HC^{2n}_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q)$.

4 The Modular Square for Compact Quantum Groups

Let $(\mathcal{H}, \Delta)$ be a Hopf algebra with a modular pair (δ, σ) in involution and let $\mathcal{A}$ be an arbitrary algebra. We use the Sweedler notation, so $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, for $x \in \mathcal{H}$. Recall that by an *$\mathcal{H}$-action* of $(\mathcal{H}, \Delta)$ on $\mathcal{A}$ we mean a linear map $\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$ such that $y \triangleright (x \triangleright a) = (yx) \triangleright a$ and $I \triangleright a = a$ and

$$x \triangleright (ab) = \sum (x_{(1)} \triangleright a)(x_{(2)} \triangleright b),$$

for $a, b \in \mathcal{A}$ and $x, y \in \mathcal{H}$ (where we have used the convention $\triangleright(x \otimes a) = x \triangleright a$).

Now suppose $\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$ is an $\mathcal{H}$-action of $(\mathcal{H}, \Delta)$ on $\mathcal{A}$. A linear functional τ on $\mathcal{A}$ is said to be a *σ-trace* under $\triangleright$ if $\tau(ab) = \tau(b(\sigma \triangleright a))$ holds for all $a, b \in \mathcal{A}$. A linear functional τ on $\mathcal{A}$ is said to be *δ-invariant* under $\triangleright$ if $\tau((x \triangleright a)b) = \tau(a(S_\delta(x) \triangleright b))$ holds for all $a, b \in \mathcal{A}$ and $x \in \mathcal{H}$.

The proof of the following lemma is quite similar to that of Theorem 8.2.

Lemma 4.1. *([5]) Suppose τ is a δ-invariant σ-trace for an $\mathcal{H}$-action $\triangleright : \mathcal{H} \otimes \mathcal{A} \to \mathcal{A}$ of a Hopf algebra $(\mathcal{H}, \Delta)$ with a modular pair (δ, σ) in involution. Consider the ordinary cocyclic object (C^n, d_i^n, s_i^n, t_n) associated to the algebra $\mathcal{A}$, and the cocyclic object $(\mathcal{H}^{\otimes n}, \delta_i^n, \sigma_i^n, \tau_n)$ associated to the Hopf algebra $(\mathcal{H}, \Delta)$ with the modular pair (δ, σ) in involution. Let $n \in \mathbb{N}$. Define a linear map*

$$\gamma_n : \mathcal{H}^{\otimes n} \to C^n\ ;\ x^1 \otimes \cdots \otimes x^n \mapsto \gamma_n(x^1 \otimes \cdots \otimes x^n)$$

by

$$\gamma_n(x^1 \otimes \cdots \otimes x^n)(a^0, \ldots, a^n) = \tau(a^0(x^1 \triangleright a^1) \cdots (x^n \triangleright a^n)),$$

for $a^0, \ldots, a^n \in \mathcal{A}$ and $x^1, \ldots, x^n \in \mathcal{H}$. Define $\gamma_0 : \mathcal{H}^{\otimes 0} = \mathbb{C} \to C^0 = \mathcal{A}'$ by $\gamma_0(1) = \tau$, for $1 \in \mathbb{C}$. Then

$$\gamma_n \delta_i^n = d_i^n \gamma_{n-1}, \quad \gamma_n \sigma_i^n = s_i^n \gamma_{n+1}, \quad \gamma_n \tau_n = t_n \gamma_n,$$

for any $n \in \mathbb{N}_0$ and $i \in \{0, \ldots, n\}$.

The significance of Lemma 4.1 is that γ_n induces a homomorphism from $HH^n_{(\delta,\sigma)}(\mathcal{H})$ to $HH^n(\mathcal{A})$ and from $HC^n_{(\delta,\sigma)}(\mathcal{H})$ to $HC^n(\mathcal{A})$ and from $HP^n_{(\delta,\sigma)}(\mathcal{H})$ to $HP^n(\mathcal{A})$, for $n \in \mathbb{N}_0$. We denote also these maps by γ_n.

In terms of Λ-modules, the lemma says that the linear map

$$\gamma : \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}^{\otimes n} \longrightarrow \bigoplus_{n\in\mathbb{N}_0} C^n$$

defined by $\gamma = \oplus_{n\in\mathbb{N}_0}\gamma_n$, is a Λ-module morphism from $\mathcal{H}^{\natural}_{(\delta,\sigma)}$ to $C^{\natural}$.

Let $(\mathcal{K},\Delta)$ be any Hopf subalgebra of $(\mathcal{H},\Delta)$ such that $\sigma\in\mathcal{K}$, and let $\gamma' : \mathcal{K}^{\natural}_{(\delta,\sigma)} \to C^{\natural}$ denote the Λ-module morphism given by Lemma 4.1. Clearly, the inclusion $j : \mathcal{K} \to \mathcal{H}$ induces an inclusion j of associated Λ-modules such that $\gamma' = \gamma j$, and thus also a homomorphism

$$j : HH^*_{(\delta,\sigma)}(\mathcal{K}) \to HH^*_{(\delta,\sigma)}(\mathcal{H})$$

on the level of Hopf Hochschild cohomology such that $\gamma' = \gamma j$. The same is true on the level of Hopf cyclic cohomology and Hopf periodic cyclic cohomology.

Consider now a Hopf $*$-subalgebra $(\mathcal{H},\Delta)$ of the maximal Hopf $*$-algebra $(\mathcal{A}^o,\Delta)$ dual to the Hopf $*$-algebra $(\mathcal{A},\Phi)$ associated to a compact quantum group with Haar state h. Assume that $f_1\in\mathcal{H}$. Define $x*a$ and $a*x$ in $\mathcal{A}$, for $x\in\mathcal{A}^o$ and $a\in\mathcal{A}$, by $x*a = (\iota\otimes x)\Phi(a)$ and $a*x = (x\otimes\iota)\Phi(a)$. Then $x\otimes a\mapsto x*a$, for $a\in\mathcal{A}$ and $x\in\mathcal{H}$, is an $\mathcal{H}$-action of $(\mathcal{H},\Delta)$ on $\mathcal{A}$. Similarly, the map $x\otimes a\mapsto a*x$, for $a\in\mathcal{A}$ and $x\in\mathcal{H}_{op}$, is an $\mathcal{H}_{op}$-action of $(\mathcal{H}_{op},\Delta)$ on $\mathcal{A}$. Now consider the Hopf algebra $(\tilde{\mathcal{H}},\tilde{\Delta})$ from Example 3.14. Define a linear map $\triangleright : \tilde{\mathcal{H}}\otimes\mathcal{A}\to\mathcal{A}$ by $(x\otimes y)\triangleright a = x*a*y$, for $a\in\mathcal{A}$, $x\in\mathcal{H}$ and $y\in\mathcal{H}_{op}$. It is easily seen to be an $\tilde{\mathcal{H}}$-action of $(\tilde{\mathcal{H}},\tilde{\Delta})$ on $\mathcal{A}$.

Theorem 4.2. *([5][12]) Let notation be as in the preceeding paragraph. Then the Haar state h of $(\mathcal{A},\Phi)$ is a δ-invariant σ-trace for the $\tilde{\mathcal{H}}$-action $\triangleright$ on $\mathcal{A}$ of the Hopf algebra $(\tilde{\mathcal{H}},\tilde{\Delta})$ with modular pair $(\delta,\sigma) = (I\otimes I, f_1\otimes f_1)$ in involution.*

A detailed proof of this theorem will be given in a later section, in the more general setting of algebraic quantum groups. The data used in the statement of Theorem 4.2 is referred to as the *modular square* associated to the Hopf algebra $(\mathcal{H},\Delta)$. By Lemma 4.1, it immediately yields the following main result.

Theorem 4.3. *([5][12]) Consider a Hopf $*$-subalgebra $(\mathcal{H},\Delta)$ of the maximal Hopf $*$-algebra $(\mathcal{A}^o,\Delta)$ dual to the Hopf $*$-algebra $(\mathcal{A},\Phi)$ associated to a compact quantum group with Haar state h. Assume that $f_1\in\mathcal{H}$. Let $\tilde{\mathcal{H}}^{\natural}_{(\delta,\sigma)}$ denote the Λ-module canonically associated to the modular square of $(\mathcal{H},\Delta)$. Then the formulas $\gamma_0(1) = h$ and*

$$\gamma_n((x^1\otimes y^1)\otimes\cdots\otimes(x^n\otimes y^n))(a^0,\ldots,a^n) = h(a^0(x^1*a^1*y^1)\cdots(x^n*a^n*y^n)),$$

for $a^0,\ldots,a^n\in\mathcal{A}$ and $x^1,y^1,\ldots,x^n,y^n\in\mathcal{H}$ and $n\in\mathbb{N}$, yield a Λ-module morphism $\gamma : \tilde{\mathcal{H}}^{\natural}_{(\delta,\sigma)} \to C^{\natural}$, which in turn induces characteristic homomorphisms in cyclic cohomology

$$\gamma_* : HC^*_{(\delta,\sigma)}(\tilde{\mathcal{H}}) \to HC^*(\mathcal{A}),$$

where $(\delta,\sigma) = (I\otimes I, f_1\otimes f_1)$ is a modular pair in involution for $(\tilde{\mathcal{H}},\tilde{\Delta}) = (\mathcal{H}\otimes\mathcal{H}_{op},\Delta\times\Delta)$. Characteristic homomorphisms are similarly induced on the level of Hochschild cohomology and periodic cyclic cohomology.

The map γ from Theorem 4.3 is called the *modular characteristic homomorphism* and in general it is neither injective nor surjective. In fact, in the case of $SU_q(2)$, we shall presently see that it is zero.

5 The Modular Characteristic Homomorphism for $SU_q(2)$

First recall [20] that $HH^n(\mathcal{A}_q)$ is infinite dimensional for $n \in \{0,1\}$, and zero otherwise. Since

$$\gamma_* : HH^*_{(\delta,\sigma)}(\tilde{\mathcal{H}}_q) \to HH^*(\mathcal{A}_q),$$

it is by Theorem 3.16 already evident that γ_* is zero on the level of Hochschild cohomology.

Next recall [20] the following result for ordinary cyclic cohomology.

PROPOSITION 5.1. *Consider the algebra $\mathcal{A}_q$ of regular functions associated to $SU_q(2)$. Then:*

1. $HC^0(\mathcal{A}_q) \cong \mathbb{C}[\tau_{\mathrm{even}}] \oplus \mathrm{Ker}S$, *where*

$$\mathrm{Ker}S \cong (\oplus_{l>0}\mathbb{C}[\tau_x^l]) \oplus (\oplus_{l>0}\mathbb{C}[\tau_y^l]) \oplus (\oplus_{l>0}\mathbb{C}[\tau_u^l]) \oplus (\oplus_{l>0}\mathbb{C}[\tau_v^l]).$$

2. $HC^{2n+1}(\mathcal{A}_q) \cong \mathbb{C}S^n[\tau_{\mathrm{odd}}]$, *for* $n \in \mathbb{N}_0$.
3. $HC^{2n}(\mathcal{A}_q) \cong \mathbb{C}S^n[\tau_{\mathrm{even}}]$, *for* $n \in \mathbb{N}$.

In [20] the generators τ_x^l, τ_y^l, τ_u^l, τ_v^l, τ_{even} and τ_{odd} are written down explicitly in terms of the Poincarè-Birkhoff-Witt-type linear basis

$$\{\alpha^r\gamma^m(\gamma^*)^n \mid m,n,r \in \mathbb{N}_0\} \cup \{(\alpha^*)^s\gamma^m(\gamma^*)^n \mid m,n \in \mathbb{N}_0\ ,\ s \in \mathbb{N}\}$$

for $\mathcal{A}_q$. In that paper $\alpha = y$, $\gamma = u$, $\alpha^* = x$, $\gamma^* = -q^{-1}v$ and $q = \mu$. Also recall that τ_{even} is the Chern character of the 1-summable even Fredholm module which generates the K-homology group $K^0(A_q) \cong \mathbb{Z}$, where A_q is the C^*- algebra envelope of $\mathcal{A}_q$. And τ_{odd} is the Chern character of the 1-summable odd Fredholm module which generates the K-homology group $K^1(A_q) \cong \mathbb{Z}$. It would be desirable to hit at least these two generators with the modular characteristic homomorphism. A direct check could in principle be carried out using the explicit expressions for the generators involved together with a formula for the Haar functional in terms of the Poincarè-Birkhoff-Witt-type basis. However, this can be avoided by introducing the following element.

There exist a functional $H \in \mathcal{A}_q^o$ such that

$$\Delta(H) = H \otimes \varepsilon + \varepsilon \otimes H,$$

which is uniquely determined by $H(U_{ij}) = 0$, for $i \neq j$, and by $H(U_{11}) = -1$, $H(U_{22}) = 1$ and $H(I) = 0$. It is easily checked that $e^{hH} = k^2$, where $h = -\ln(q) > 0$, so $H \notin U_q(\mathrm{sl}_2)$. Thus we have the following familiar identities

$$Hk = kH\ ,\quad Hk^{-1} = k^{-1}H\ ,\quad He - eH = -2e\ ,\quad Hf - fH = 2f.$$

Consider the cocyclic object $((\mathcal{A}_q^o)^{\otimes n}, \delta_i^n, \sigma_i^n, \tau_n)$ associated to the Hopf $*$-algebra $(\mathcal{A}_q^o, \Delta)$ with the modular pair (I, f_1) in involution. Recall that

$$B_1 = N^1\sigma_1^1\tau_2(1_2 - \lambda_2) = N^1(\sigma_1^1\tau_2 - \tau_1\sigma_0^1),$$

so

$$B_1(x \otimes y) = S(x)y - S(y)f_1x - x(I)S(y)f_1 + x(I)y\ ,$$

for all $x, y \in \mathcal{A}_q^o$. Recall also that the coboundary operator $b_3 : \mathcal{A}_q^o \otimes \mathcal{A}_q^o \to \mathcal{A}_q^o \otimes \mathcal{A}_q^o \otimes \mathcal{A}_q^o$ is given by

$$b_3(x \otimes y) = \varepsilon \otimes x \otimes y - \Delta(x) \otimes y + x \otimes \Delta(y) - x \otimes y \otimes f_1,$$

for all $x, y \in \mathcal{A}_q^o$. The following lemma follows by direct verification.

LEMMA 5.2. *([12]) Let notation be as in the above paragraph and consider $B_1 : \mathcal{A}_q^o \otimes \mathcal{A}_q^o \to \mathcal{A}_q^o$. Then:*

1. $b_3(H \otimes ek^{-1}) = b_3(H \otimes fk^{-1}) = 0$.
2. $B_1(H \otimes ek^{-1}) = 2ek^{-1}$.
3. $B_1(H \otimes fk^{-1}) = -2fk^{-1}$.

This lemma tells us that the induced map

$$IB_1 : HH^2_{(I,f_1)}(\mathcal{A}_q^o) \to HH^1_{(I,f_1)}(\mathcal{A}_q^o)$$

from the long exact IBS-sequence is surjective. Moreover, the equivalence classes $\{[H \otimes ek^{-1}], [H \otimes fk^{-1}]\}$ are linearly independent in $HH^2_{(I,f_1)}(\mathcal{A}_q^o)$, because $B_1([H \otimes ek^{-1}]) = 2[ek^{-1} \oplus 0]$ and $B_1([H \otimes fk^{-1}]) = -2[fk^{-1} \oplus 0]$.

A somewhat complicated diagram chase involving the homomorphism

$$j : HC^*_{(I,f_1)}(\tilde{\mathcal{H}}_q) \to HC^*_{(I,f_1)}(\tilde{\mathcal{A}}_q^o)$$

such that $\gamma j = \gamma'$ (from the discussion following Lemma 4.1), Theorem 3.16 and Proposition 5.1, now yields the following surprising result.

THEOREM 5.3. *([12]) The modular characteristic homomorphisms in Hochschild cohomology, cyclic cohomology and periodic cyclic cohomology of the Hopf algebra $(U_q(\mathrm{sl}_2), \Delta)$ with the modular pair (I, f_1) in involution, are all ZERO.*

6 Additional Considerations

In this section we expand on results for $SU_q(2)$ briefly presented in [12]. Due to the presence of the element $H \in \mathcal{A}_q^o$ it is evident that $HH^*_{(I,f_1)}(\mathcal{A}_q^o) \ncong HH^*_{(I,f_1)}(\mathcal{H}_q)$, so in principle the modular characteristic homomorphism γ of the Hopf $*$-algebra $(\mathcal{A}_q^o, \Delta)$ could be non-zero on the level of cyclic cohomology and periodic cyclic cohomology, although we have already seen that the restricted modular characteristic homomorphism γ' of the Hopf $*$-algebra $(U_q(\mathrm{sl}_2), \Delta)$ is zero. In this section we prove that provided a certain plausible assumption (to be explained below in Theorem 6.1) on $\mathcal{A}_q^o$ holds, which we unfortunately have not been able to check, then $\gamma = 0$ and therefore $\gamma' = \gamma j = 0$. By the same argument, the modular characteristic homomorphism of any other Hopf subalgebra of $(\mathcal{A}_q^o, \Delta)$ containing f_1 will then also be zero. Of course, $\gamma = 0$ in Hochschild cohomology, so we need only focus on the case of cyclic cohomology.

We need a theorem showing how $HH^*_{(I,f_1)}(\mathcal{A}^o)$ can be computed from a resolution of $\mathcal{A}$. Throughout this discussion $(\mathcal{A}, \Phi)$ denotes a compact quantum group with maximal dual Hopf algebra $(\mathcal{A}^o, \Delta)$ and modular element $f_1 \in \mathcal{A}^o$, and $\mathcal{B}$ denotes the algebra $\mathcal{A} \otimes \mathcal{A}_{op}$.

Proposition 2.1 tells us that $HH^*_{(I,f_1)}(\mathcal{A}^o)$ can be computed from any injective resolution of $\mathcal{A}^o$ as an $\mathcal{A}^o$-bicomodule. Using this we shall give a formula for computing $HH^*_{(I,f_1)}(\mathcal{A}^o)$ from a given finitely generated free resolution

$$\cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} \mathcal{A} \longrightarrow 0$$

of $\mathcal{A}$ as a left $\mathcal{B}$-module (or as an $\mathcal{A}$-bimodule). We assume that $M_0 = \mathcal{B}$ and that $d_0 : M_0 \to \mathcal{A}$ is the algebra multiplication.

Suppose M_{nL} is a finite dimensional vector space, that $M_n \cong \mathcal{B} \otimes M_{nL}$ as left $\mathcal{B}$-modules, so that $d_n : M_n \to M_{n-1}$ needs only be specified on elements of the type $1_{\mathcal{B}} \otimes z$, where $1_{\mathcal{B}} = I \otimes I$ is the unit of $\mathcal{B}$ and $z \in M_{nL}$. Let $\{e^n_k\}$ be a linear basis of M_{nL}, and let $\{e^{n*}_k\}$ be a basis for the dual vector space M^o_{nL} of M_{nL} such that $e^{n*}_k(e^n_l) = \delta_{kl}$, for all k, l. We shall regard $\mathcal{A}^o \otimes \mathcal{A}^o$ as an $\mathcal{A}$-bimodule with respect to the action $a(\xi \otimes \eta)b = a\xi \otimes \eta b$, where $\xi, \eta \in \mathcal{A}^o$ and $a, b \in \mathcal{A}$ and where by $a\xi, \eta b \in \mathcal{A}^o$ we mean $a\xi(c) = \xi(ca)$ and $\eta b(c) = \eta(bc)$ for $c \in \mathcal{A}$.

Theorem 6.1. *([12]) Let notation be as in the preceeding paragraph. Assume that (M_*, d) is a finitely generated free resolution of $\mathcal{A}$ as a left $\mathcal{B}$-module. Suppose the $\mathcal{A}$-bimodule $\mathcal{A}^o \otimes \mathcal{A}^o$ defined above is injective. Then the linear map $\delta_n : M^o_{(n-1)L} \to M^o_{nL}$ given by*

$$\delta_n(y) = \sum_k (\varepsilon \otimes f_1 \otimes y) d_n(1_{\mathcal{B}} \otimes e^n_k)\ e^{n*}_k,$$

for all $y \in M^o_{(n-1)L}$ and $n \in \mathbb{N}$, defines a complex (M^o_{*L}, δ) such that its cohomology coincides with $HH^*_{(I,f_1)}(\mathcal{A}^o)$. Hence

$$\dim(HH^n_{(I,f_1)}(\mathcal{A}^o)) = \dim(M^o_{nL}) - \dim(\mathrm{Im}\delta_n) - \dim(\mathrm{Im}\delta_{n+1}),$$

for all $n \in \mathbb{N}_0$.

Remark 6.2. Since $\mathcal{A}^o \otimes \mathcal{A}^o \cong (\mathcal{A} \otimes \mathcal{A})^o$ where the latter is defined with respect to $(\mathcal{A} \otimes \mathcal{A}, \Phi \times \Phi)$, our assumption reduces to requiring that the left $\mathcal{A}$-module $\mathcal{A}^o$ is injective. Also $\mathcal{A}^o$ is a left $\mathcal{A}$-submodule of the injective left $\mathcal{A}$-module $\mathcal{A}'$. However, injectivity is unfortunately not necessarily inherited to submodules.

Let us now return to the case $SU_q(2)$.

In the rest of this section we assume that the $\mathcal{A}_q$-bimodule $\mathcal{A}^o_q \otimes \mathcal{A}^o_q$ defined above is injective.

In [20] an explicit finitely generated free resolution

$$\cdots \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{m} \mathcal{A}_q \longrightarrow 0$$

of $\mathcal{A}_q$ as a left $\mathcal{B}$-module (here $M_0 = \mathcal{B}$ and m is the multiplication) was constructed by hand, to compute $HH^*(\mathcal{A}_q)$. We will now use this resolution together with Theorem 6.1 to write down the complex (M^o_{*L}, δ), which we will then use to compute $HH^*_{(I,f_1)}(\mathcal{A}^o_q)$.

Below we follow closely the notation and conventions in [20] which was used for specifying the left $\mathcal{B}$-modules M_n and the differentials $d_n : M_n \to M_{n-1}$, for $n \in \mathbb{N}$. Define the vector spaces M_{nL} as the linear spans of the $\mathcal{B}$-bases $\{e^n_k\}$ for M_n, given by $e^0_0 = I$ and

$$e^1_1 = e_x, \quad e^1_2 = e_y, \quad e^1_3 = e_u, \quad e^1_4 = e_v$$

and

$$
\begin{array}{llll}
e_1^2 = \vartheta_T^{(1)}, & e_2^2 = \vartheta_S^{(1)}, & e_3^2 = e_u \wedge e_x, & e_4^2 = e_u \wedge e_y \\
e_5^2 = e_v \wedge e_x, & e_6^2 = e_v \wedge e_y, & e_7^2 = e_v \wedge e_u &
\end{array}
$$

and

$$
\begin{array}{llll}
e_1^3 = e_x \wedge \vartheta_S^{(1)}, & e_2^3 = e_y \wedge \vartheta_T^{(1)}, & e_3^3 = e_u \wedge \vartheta_T^{(1)}, & e_4^3 = e_u \wedge \vartheta_S^{(1)}, \\
e_5^3 = e_v \wedge \vartheta_T^{(1)}, & e_6^3 = e_v \wedge \vartheta_S^{(1)}, & e_7^3 = e_v \wedge e_u \wedge e_x, & e_8^3 = e_v \wedge e_u \wedge e_y,
\end{array}
$$

whereas $\{e_k^{2p+4}\}$ is listed as follows

$$
\begin{array}{lll}
e_1^{2p+4} = \vartheta_T^{(p+2)}, & e_2^{2p+4} = \vartheta_S^{(p+2)}, & e_3^{2p+4} = e_u \wedge e_x \wedge \vartheta_S^{(p+1)}, \\
e_4^{2p+4} = e_u \wedge e_y \wedge \vartheta_T^{(p+1)}, & e_5^{2p+4} = e_v \wedge e_x \wedge \vartheta_S^{(p+1)}, & e_6^{2p+4} = e_v \wedge e_y \wedge \vartheta_T^{(p+1)}, \\
e_7^{2p+4} = e_v \wedge e_u \wedge \vartheta_T^{(p+1)}, & e_8^{2p+4} = e_v \wedge e_u \wedge \vartheta_S^{(p+1)}, &
\end{array}
$$

for $p \geq 0$, and finally $\{e_k^{2p+3}\}$ is listed as follows

$$
\begin{array}{lll}
e_1^{2p+3} = e_x \wedge \vartheta_S^{(p+1)}, & e_2^{2p+3} = e_y \wedge \vartheta_T^{(p+1)}, & e_3^{2p+3} = e_u \wedge \vartheta_T^{(p+1)}, \\
e_4^{2p+3} = e_u \wedge \vartheta_S^{(p+1)}, & e_5^{2p+3} = e_v \wedge \vartheta_T^{(p+1)}, & e_6^{2p+3} = e_v \wedge \vartheta_S^{(p+1)}, \\
e_7^{2p+3} = e_v \wedge e_u \wedge e_x \wedge \vartheta_S^{(p)}, & e_8^{2p+3} = e_v \wedge e_u \wedge e_y \wedge \vartheta_T^{(p)}, &
\end{array}
$$

for $p > 0$.

Denote the elements e_k^{n*} of the dual basis $\{e_k^{n*}\} \subset M_{nL}^o$ of $\{e_k^n\}$ by v_k^n. Using the formula

$$\delta_n(v_i^{n-1}) = \sum_k (\varepsilon \otimes f_1 \otimes v_i^{n-1}) d_n(1_{\mathcal{B}} \otimes e_k^n) v_k^n,$$

for $n \in \mathbb{N}$ from Theorem 6.1 for the maps $\delta_n : M_{(n-1)L}^o \to M_{nL}^o$, and the formulas for $d_n : M_n \to M_{n-1}$ given on pp. 160-163 in [20], we can then obtain the explicit expressions for δ_n written below.

To show how they come about, consider for example the linear map $\delta_1 : M_{0L}^o \to M_{1L}^o$, which can be calculated as follows

$$
\begin{aligned}
\delta_1(v_0^0) &= \sum_{k=1}^{4} (\varepsilon \otimes f_1 \otimes v_0^0) d_1(1_{\mathcal{B}} \otimes e_k^1) v_k^1 \\
&= (\varepsilon \otimes f_1) d_1(1_{\mathcal{B}} \otimes e_x) v_1^1 + (\varepsilon \otimes f_1) d_1(1_{\mathcal{B}} \otimes e_y) v_2^1 \\
&\quad + (\varepsilon \otimes f_1) d_1(1_{\mathcal{B}} \otimes e_u) v_3^1 + (\varepsilon \otimes f_1) d_1(1_{\mathcal{B}} \otimes e_v) v_4^1 \\
&= (\varepsilon \otimes f_1)(x \otimes I - I \otimes x) v_1^1 + (\varepsilon \otimes f_1)(y \otimes I - I \otimes y) v_2^1 \\
&\quad + (\varepsilon \otimes f_1)(u \otimes I - I \otimes u) v_3^1 + (\varepsilon \otimes f_1)(v \otimes I - I \otimes v) v_4^1 \\
&= (1-q) v_1^1 + (1-q^{-1}) v_2^1.
\end{aligned}
$$

Similarly, consider $\delta_2 : M_{1L}^o \to M_{2L}^o$, which can be calculated, for any number $i \in$

$\{1, \ldots, 4\}$, as follows

$$
\begin{aligned}
\delta_2(v_i^1) &= \sum_{k=1}^{7} (\varepsilon \otimes f_1 \otimes v_i^1) d_2(1_{\mathcal{B}} \otimes e_k^2) v_k^2 \\
&= (\varepsilon \otimes f_1 \otimes v_i^1)(I \otimes y \otimes e_1^1 + x \otimes I \otimes e_2^1 - qu \otimes I \otimes e_4^1 - I \otimes q^{-1} v \otimes e_3^1) v_1^2 \\
&\quad + (\varepsilon \otimes f_1 \otimes v_i^1)(y \otimes I \otimes e_1^1 + I \otimes x \otimes e_2^1 - qu \otimes I \otimes e_4^1 - I \otimes qv \otimes e_3^1) v_2^2 \\
&\quad + (\varepsilon \otimes f_1 \otimes v_i^1)((u \otimes I - I \otimes qu) \otimes e_1^1 + (qx \otimes I - I \otimes x) \otimes e_3^1)) v_3^2 \\
&\quad + (\varepsilon \otimes f_1 \otimes v_i^1)((qu \otimes I - I \otimes u) \otimes e_2^1 + (y \otimes I - I \otimes qy) \otimes e_3^1)) v_4^2 \\
&\quad + (\varepsilon \otimes f_1 \otimes v_i^1)((v \otimes I - I \otimes qv) \otimes e_1^1 + (qx \otimes I - I \otimes x) \otimes e_4^1)) v_5^2 \\
&\quad + (\varepsilon \otimes f_1 \otimes v_i^1)((qv \otimes I - I \otimes v) \otimes e_2^1 + (y \otimes I - I \otimes qy) \otimes e_3^1)) v_6^2 \\
&\quad + (\varepsilon \otimes f_1 \otimes v_i^1)((v \otimes I - I \otimes v) \otimes e_3^1 + (u \otimes I - I \otimes u) \otimes e_4^1)) v_7^2,
\end{aligned}
$$

so for instance,

$$
\begin{aligned}
\delta_2(v_1^1) &= (\varepsilon \otimes f_1)(I \otimes y) v_1^2 + (\varepsilon \otimes f_1)(y \otimes I) v_2^2 \\
&\quad + (\varepsilon \otimes f_1)(u \otimes I - I \otimes qu) v_3^2 + (\varepsilon \otimes f_1)(v \otimes I - I \otimes qv) v_5^2 \\
&= q^{-1} v_1^2 + v_2^2.
\end{aligned}
$$

We trust that the reader can now easily verify the following result:

$$\delta_1(v_0^0) = (1-q) v_1^1 + (1-q^{-1}) v_2^1$$

and

$$
\begin{aligned}
&\delta_2(v_1^1) = q^{-1} v_1^2 + v_2^2, && \delta_2(v_2^1) = v_1^2 + q v_2^2 \\
&\delta_2(v_3^1) = 0, && \delta_2(v_4^1) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\delta_3(v_1^2) = -q v_1^3 + v_2^3, && \delta_3(v_2^2) = v_1^3 - q^{-1} v_2^3 \\
&\delta_3(v_3^2) = -q^{-1} v_3^3 - q^{-1} v_4^3, && \delta_3(v_4^2) = -v_3^3 - v_4^3 \\
&\delta_3(v_5^2) = -q^{-1} v_5^3 - q^{-1} v_6^3, && \delta_3(v_6^2) = -v_5^3 - v_6^3 \\
&\delta_3(v_7^2) = (q^2 - q) v_7^3 + (1-q) v_8^3
\end{aligned}
$$

and

$$
\begin{aligned}
&\delta_4(v_1^3) = q^{-1} v_1^4 + v_2^4, && \delta_4(v_2^3) = v_1^4 + q v_2^4 \\
&\delta_4(v_3^3) = q v_3^4 - v_4^4, && \delta_4(v_4^3) = -q v_3^4 + v_4^4 \\
&\delta_4(v_5^3) = q v_5^4 - v_6^4, && \delta_4(v_6^3) = -q v_5^4 + v_6^4 \\
&\delta_4(v_7^3) = q^{-1} v_7^4 + q^{-2} v_8^4, && \delta_4(v_8^3) = v_7^4 + q^{-1} v_8^4
\end{aligned}
$$

and for $p \in \mathbb{N}$, we get

$$
\begin{aligned}
&\delta_{2p+3}(v_1^{2p+2}) = -q v_1^{2p+3} + v_2^{2p+3}, && \delta_{2p+3}(v_2^{2p+2}) = v_1^{2p+3} - q^{-1} v_2^{2p+3} \\
&\delta_{2p+3}(v_3^{2p+2}) = -q^{-1} v_3^{2p+3} - q^{-1} v_4^{2p+3}, && \delta_{2p+3}(v_4^{2p+2}) = -v_3^{2p+3} - v_4^{2p+3} \\
&\delta_{2p+3}(v_5^{2p+2}) = -q^{-1} v_5^{2p+3} - q^{-1} v_6^{2p+3}, && \delta_{2p+3}(v_6^{2p+2}) = -v_5^{2p+3} - v_6^{2p+3} \\
&\delta_{2p+3}(v_7^{2p+2}) = -q v_7^{2p+3} + v_8^{2p+3}, && \delta_{2p+3}(v_8^{2p+2}) = q^2 v_7^{2p+3} - q v_8^{2p+3}
\end{aligned}
$$

and finally

$$
\begin{aligned}
\delta_{2p+4}(v_1^{2p+3}) &= q^{-1}v_1^{2p+4} + v_2^{2p+4}, & \delta_{2p+4}(v_2^{2p+3}) &= v_1^{2p+4} + qv_2^{2p+4}\\
\delta_{2p+4}(v_3^{2p+3}) &= qv_3^{2p+4} - v_4^{2p+4}, & \delta_{2p+4}(v_4^{2p+3}) &= -qv_3^{2p+4} + v_4^{2p+4}\\
\delta_{2p+4}(v_5^{2p+3}) &= qv_5^{2p+4} - v_6^{2p+4}, & \delta_{2p+4}(v_6^{2p+3}) &= -qv_5^{2p+4} + v_6^{2p+4}\\
\delta_{2p+4}(v_7^{2p+3}) &= q^{-1}v_7^{2p+4} + q^{-2}v_8^{2p+4}, & \delta_{2p+4}(v_8^{2p+3}) &= v_7^{2p+4} + q^{-1}v_8^{2p+4}.
\end{aligned}
$$

The result of this is the following result. By 'our assumption' we refer to the one in Theorem 6.1

THEOREM 6.3. *([12]) Provided our assumption holds for $SU_q(2)$, we get:*

1. $HH^1_{(I,f_1)}(\mathcal{A}_q^o) \cong HH^2_{(I,f_1)}(\mathcal{A}_q^o) \cong \mathbb{C}^2$
2. $HH^n_{(I,f_1)}(\mathcal{A}_q^o) \cong 0$, *for all* $n \notin \{1, 2\}$.

PROOF. By Theorem 6.1 we have

$$\dim(HH^n_{(I,f_1)}(\mathcal{A}_q^o)) = \dim(M^o_{nL}) - \dim(\mathrm{Im}\delta_n) - \dim(\mathrm{Im}\delta_{n+1}),$$

for all $n \in \mathbb{N}_0$. Now the dimensions $\dim(M^o_{nL})$ are

$$\dim(M^o_{0L}) = 0, \quad \dim(M^o_{1L}) = 4, \quad \dim(M^o_{2L}) = 7, \quad \dim(M^o_{nL}) = 8,$$

for $n \geq 3$. Whereas by looking at the formulas for δ_n listed above, we see that the dimensions $\dim(\mathrm{Im}\delta_n)$ are

$$\dim(\mathrm{Im}\delta_0) = 0, \quad \dim(\mathrm{Im}\delta_1) = 1, \quad \dim(\mathrm{Im}\delta_2) = 1, \quad \dim(\mathrm{Im}\delta_n) = 4,$$

for $n \geq 3$. Combining these facts yields the result for $HH^*_{(I,f_1)}(\mathcal{A}_q^o)$. □

Thus by Lemma 5.2 the generators for $HH^1_{(I,f_1)}(\mathcal{A}_q^o)$ are $\{[ek^{-1}], [fk^{-1}]\}$, whereas those of $HH^2_{(I,f_1)}(\mathcal{A}_q^o)$ are $\{[H \otimes ek^{-1}], [H \otimes fk^{-1}]\}$.

REMARK 6.4. In the classical limit $q \to 1$, we easily see that the numbers $\dim(\mathrm{Im}\delta_n)$ are

$$\dim(\mathrm{Im}\delta_0) = 0, \quad \dim(\mathrm{Im}\delta_1) = 0, \quad \dim(\mathrm{Im}\delta_2) = 1,$$

$$\dim(\mathrm{Im}\delta_3) = 3, \quad \dim(\mathrm{Im}\delta_n) = 4,$$

for $n \geq 4$. Hence, when $q \to 1$, we get the following results, which should be compared with the remark following Proposition 3.12:

1. $HH^0_{(I,f_1)}(\mathcal{A}_q^o) \cong HH^3_{(I,f_1)}(\mathcal{A}_q^o) \cong \mathbb{C}$
2. $HH^1_{(I,f_1)}(\mathcal{A}_q^o) \cong HH^2_{(I,f_1)}(\mathcal{A}_q^o) \cong \mathbb{C}^3$
3. $HH^n_{(I,f_1)}(\mathcal{A}_q^o) \cong 0$, for all $n \geq 4$.

Remark 6.5. Before we move on let us briefly explain how the oracle could possibly come up with the two generators $\{[H \otimes ek^{-1}]\ ,\ [H \otimes fk^{-1}]\}$ in Lemma 5.2. We know that both the bar resolution $B(\mathcal{A}_q)_* \xrightarrow{\simeq} \mathcal{A}_q$ and resolution $M_* \xrightarrow{\simeq} \mathcal{A}_q$ in [20] are free resolutions of $\mathcal{A}_q$. We can then built up inductively a chain map h_* from $M_* \xrightarrow{\simeq} \mathcal{A}_q$ with boundary d to $B(\mathcal{A}_q)_* \xrightarrow{\simeq} \mathcal{A}_q$ with boundary b, which by the fundamental theorem in Homological algebra automatically will be a quasi-isomorphism.

We start with $h_0 : M_0 = \mathcal{A}_q \otimes \mathcal{A}_q \to \mathcal{A}_q \otimes \mathcal{A}_q = B(\mathcal{A}_q)_0$, which is the identity. The map $h_1 : M_1 = \mathcal{A}_q \otimes M_{1L} \otimes \mathcal{A}_q \to \mathcal{A}_q \otimes \mathcal{A}_q \otimes \mathcal{A}_q$ has to satisfy $b_1 h_1 = h_0 d_1 = d_1$, and since $b_1(a_0 \otimes a_1 \otimes a_2) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2$, for $a_i \in \mathcal{A}_q$, a possible choice for h_1 sends $(e_1^1, \ldots,\ e_4^1)$ to

$$(I \otimes x \otimes I,\ I \otimes y \otimes I,\ I \otimes u \otimes I,\ I \otimes v \otimes I).$$

Similarly, the map $h_2 : M_2 = \mathcal{A}_q \otimes M_{2L} \otimes \mathcal{A}_q \to \mathcal{A}_q \otimes \mathcal{A}_q \otimes \mathcal{A}_q \otimes \mathcal{A}_q$ has to satisfy $b_2 h_2 = h_1 d_2$, and since

$$b_2(a_0 \otimes a_1 \otimes a_2 \otimes a_3) = a_0 a_1 \otimes a_2 \otimes a_3 - a_0 \otimes a_1 a_2 \otimes a_3 + a_0 \otimes a_1 \otimes a_2 a_3\ ,$$

for $a_i \in \mathcal{A}_q$, a possible choice for h_2 is:

$$\begin{aligned}
h_2(e_1^2) &= I \otimes x \otimes y \otimes I - I \otimes q^{-1} u \otimes v \otimes I - I \otimes I \otimes I \otimes I,\\
h_2(e_2^2) &= I \otimes y \otimes x \otimes I - I \otimes qu \otimes v \otimes I - I \otimes I \otimes I \otimes I,\\
h_2(e_3^2) &= I \otimes u \otimes x \otimes I - I \otimes qx \otimes u \otimes I,\\
h_2(e_4^2) &= I \otimes qu \otimes y \otimes I - I \otimes y \otimes u \otimes I,\\
h_2(e_5^2) &= I \otimes v \otimes x \otimes I - I \otimes qx \otimes v \otimes I,\\
h_2(e_6^2) &= I \otimes qv \otimes y \otimes I - I \otimes y \otimes v \otimes I,\\
h_2(e_7^2) &= I \otimes v \otimes u \otimes I - I \otimes u \otimes v \otimes I,
\end{aligned}$$

where we have invoked the relations of the generators $u, v, x, y \in \mathcal{A}_q$.

Tracing this chain map through the proof of Theorem 6.1, our assumption on $SU_q(2)$ thus gives us a quasi-isomorphism h_*^o from (M_{*L}^o, δ) to the chain complex for $(\mathcal{A}_q^o)^\natural_{(I,f_1)}$. Now two convenient generators for $HH^2_{(I,f_1)}(\mathcal{A}_q^o) \cong \mathbb{C}^2$ in (M_{*L}^o, δ) are $\{[qv_3^2 - v_4^2], [qv_5^2 - v_6^2]\}$. It is easy to see that $h_2^o(qv_3^2 - v_4^2) \in \mathcal{A}_q^o \otimes \mathcal{A}_q^o$ is then a linear map such that

$$\begin{aligned}
x \otimes y - q^{-1} u \otimes v - I \otimes I &\mapsto 0,\\
y \otimes x - qu \otimes v - I \otimes I &\mapsto 0,\\
u \otimes x - qx \otimes u &\mapsto q,\\
qu \otimes y - y \otimes u &\mapsto -1,\\
v \otimes x - qx \otimes v &\mapsto 0,\\
qv \otimes y - y \otimes v &\mapsto 0,\\
v \otimes u - u \otimes v &\mapsto 0.
\end{aligned}$$

Staring at these formulas suggests the (admittingly still ad hoc) candidate $-q^{1/2} H \otimes fk^{-1}$ as the obvious linear map satisfying these evaluation requirements. By its nature it belongs to $\mathrm{Ker}(b_3)$. Similar considerations connect the two elements $qv_5^2 - v_6^2$ and $-q^{-1/2} H \otimes ek^{-1}$.

The method of inductively establishing a chain map also gives a hint to how the (seemingly ad hoc) free resolution of $\mathcal{A}_q$ in [20] was constructed.

Using Theorem 6.3, the IBS-sequence and Lemma 5.2, one can now check that $HC^1_{(I,f_1)}(\mathcal{A}^o_q) \cong \mathbb{C}^2$ is the only non-vanishing component in cyclic cohomology, with generators $\{[ek^{-1}\oplus 0],[fk^{-1}\oplus 0]\}$, so $HP^*_{(I,f_1)}(\mathcal{A}^o_q)\cong 0$. The non-zero B-operator $IB_1 : HH^2_{(I,f_1)}(\mathcal{A}^o_q) \to HH^1_{(I,f_1)}(\mathcal{A}^o_q)$ kills the periodic cyclic cohomology classes for $U_q(\mathrm{sl}_2)$ found in Proposition 3.12.

Together with Corollary 3.15, we therefore see that $HH^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q)$ is $\mathbb{C}^4$ for $n\in\{2,4\}$ and $\mathbb{C}^8$ for $n=3$ and otherwise zero. Using the Alexander-Whitney map and the shuffle map, we can further find explicit generators for these cohomologies in the respective original complexes.

Lemma 6.6. *([12]) Let $n\in\mathbb{N}_0$. The Alexander-Whitney map $AW_n=\oplus_{i+j=n}AW_{ij}$, which induces the isomorphism*

$$AW_n : \bigoplus_{i+j=n} HH^i_{(I,f_1)}(\mathcal{A}^o_q)\otimes HH^j_{(I,f_1)}(\mathcal{A}^o_q) \longrightarrow HH^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q),$$

is in low degrees given by:

1. $AW_{11}(x_0\otimes y_0) = x_0\otimes\varepsilon\otimes f_1\otimes y_0$.
2. $AW_{12}(x_0\otimes y_0\otimes y_1) = x_0\otimes\varepsilon\otimes f_1\otimes y_0\otimes f_1\otimes y_1$.
3. $AW_{21}(x_0\otimes x_1\otimes y_0) = x_0\otimes\varepsilon\otimes x_1\otimes\varepsilon\otimes f_1\otimes y_0$.
4. $AW_{22}(x_0\otimes x_1\otimes y_0\otimes y_1) = x_0\otimes\varepsilon\otimes x_1\otimes\varepsilon\otimes f_1\otimes y_0\otimes f_1\otimes y_1$,

for $x_i, y_i\in\mathcal{A}^o_q$.

The shuffle map $\mathrm{sh}_n=\oplus_{i+j=n}\mathrm{sh}_{ij}$, which induces an isomorphism

$$\mathrm{sh}_n \ : \ HH^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q) \ \longrightarrow \ \bigoplus_{i+j=n} HH^i_{(I,f_1)}(\mathcal{A}^o_q)\otimes HH^j_{(I,f_1)}(\mathcal{A}^o_q)$$

is in low degrees given by:

1. $\mathrm{sh}_{11}(x_0\otimes y_0\otimes x_1\otimes y_1) = x_0I(x_1)\otimes I(y_0)y_1 - I(x_0)x_1\otimes y_0I(y_1)$,
2. $$\mathrm{sh}_{12}(x_0\otimes y_0\otimes x_1\otimes y_1\otimes x_2\otimes y_2) = x_0I(x_1)I(x_2)\otimes I(y_0)y_1\otimes y_2$$ $$-I(x_0)x_1I(x_2)\otimes y_0I(y_1)\otimes y_2 + I(x_0)I(x_1)x_2\otimes y_0\otimes y_1I(y_2),$$
3. $$\mathrm{sh}_{21}(x_0\otimes y_0\otimes x_1\otimes y_1\otimes x_2\otimes y_2) = x_0\otimes x_1I(x_2)\otimes I(y_0)I(y_1)\otimes y_2$$ $$-x_0I(x_1)\otimes x_2\otimes I(y_0)y_1I(y_2) + I(x_0)x_1\otimes x_2\otimes y_0I(y_1)I(y_2),$$

for all $x_i, y_i\in\mathcal{A}^o_q$.

Proposition 6.7. *([12]) We have the following generators of $HH^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q)$:*

1. *$n=2$:* $[ek^{-1}\otimes\varepsilon\otimes f_1\otimes ek^{-1}]$, $[ek^{-1}\otimes\varepsilon\otimes f_1\otimes fk^{-1}]$,

$$[fk^{-1}\otimes\varepsilon\otimes f_1\otimes ek^{-1}],\ [fk^{-1}\otimes\varepsilon\otimes f_1\otimes fk^{-1}].$$

2. $n = 3$*:* $[ek^{-1} \otimes \varepsilon \otimes f_1 \otimes H \otimes f_1 \otimes ek^{-1}], [ek^{-1} \otimes \varepsilon \otimes f_1 \otimes H \otimes f_1 \otimes fk^{-1}],$

$$[fk^{-1} \otimes \varepsilon \otimes f_1 \otimes H \otimes f_1 \otimes ek^{-1}], [fk^{-1} \otimes \varepsilon \otimes f_1 \otimes H \otimes f_1 \otimes fk^{-1}],$$

$$[H \otimes \varepsilon \otimes ek^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}], [H \otimes \varepsilon \otimes ek^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}],$$
$$[H \otimes \varepsilon \otimes fk^{-1} \otimes \varepsilon \otimes f_1 \otimes ek^{-1}], [H \otimes \varepsilon \otimes fk^{-1} \otimes \varepsilon \otimes f_1 \otimes fk^{-1}],$$

3. $n = 4$*:* $[H \otimes \varepsilon \otimes ek^{-1} \otimes \varepsilon \otimes f_1 \otimes H \otimes f_1 \otimes ek^{-1}],$

$$[H \otimes \varepsilon \otimes ek^{-1} \otimes \varepsilon \otimes f_1 \otimes H \otimes f_1 \otimes fk^{-1}], [H \otimes \varepsilon \otimes fk^{-1} \otimes \varepsilon \otimes f_1 \otimes H \otimes f_1 \otimes ek^{-1}],$$

$$[H \otimes \varepsilon \otimes fk^{-1} \otimes \varepsilon \otimes f_1 \otimes H \otimes f_1 \otimes fk^{-1}].$$

Again by Corollary 3.15 we see that $HC^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q)$ is $\mathbb{C}^4$ for $n \in \{2,3\}$ and otherwise zero, so $HP^*_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q) \cong 0$. To illustrate the role of the Leibniz rule for the B-operator in the Künneth exact sequence in Corollary 3.15 it is perhaps more instructive to present the generators for $HC^*_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q)$ as below, rather than applying the Alexander-Whitney map.

Remark 6.8. Let B denote the linear map $IB_1 : HH^2_{(I,f_1)}(\mathcal{A}_q) \to HH^1_{(I,f_1)}(\mathcal{A}_q)$. Consider $\xi = [H \otimes ek^{-1}]$ and $\eta = [H \otimes fk^{-1}]$ in $HH^2_{(I,f_1)}(\mathcal{A}_q)$. Then by Corollary 3.15, the vector space
$$HH^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q) \cong \bigoplus_{n=i+j} HH^i_{(I,f_1)}(\mathcal{A}_q) \otimes HH^j_{(I,f_1)}(\mathcal{A}_q)$$
has the following generators:

1. $\{\xi \otimes \xi, \xi \otimes \eta, \eta \otimes \xi, \eta \otimes \eta\}$, for $n = 4$.
2. $\{B(\xi) \otimes \xi, B(\xi) \otimes \eta, B(\eta) \otimes \xi, B(\eta) \otimes \eta, \xi \otimes B(\xi), \xi \otimes B(\eta), \eta \otimes B(\xi), \eta \otimes B(\eta)\}$, for $n = 3$.
3. $\{B(\xi) \otimes B(\xi), B(\xi) \otimes B(\eta), B(\eta) \otimes B(\xi), B(\eta) \otimes B(\eta)\}$, for $n = 2$.

This is a consequence of the following Leibniz rule
$$IB_n = \bigoplus_{n+1=i+j} (IB_{i-1} \otimes 1_j \bigoplus (-1)^i 1_i \otimes IB_{j-1})$$
for $IB_n : HH^{n+1}_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q) \to HH^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q)$, where $n \in \mathbb{N}_0$, with respect to the corresponding operators for $HH^*_{(I,f_1)}(\mathcal{A}_q)$.

We conclude this discussion with the following results.

Theorem 6.9. *([12]) Provided our assumption on* $SU_q(2)$ *holds, we have:*

1. $HH^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q) \cong \mathbb{C}^4$*, for* $n \in \{2,4\}$.

2. $HH^3_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q) \cong \mathbb{C}^8$.

3. $HH^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}^o_q) \cong 0$*, for* $n \notin \{2,3,4\}$.

4. $HC^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}_q^o) \cong \mathbb{C}^4$*, for* $n \in \{2,3\}$*.*

5. $HC^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}_q^o) \cong 0$*, for* $n \notin \{2,3\}$*.*

6. $HP^*_{(\delta,\sigma)}(\tilde{\mathcal{A}}_q^o) \cong 0$*.*

Moreover, explicit representatives of the generators in the respectively defining complexes can be written down in each case (see the results prior to this).

THEOREM 6.10. *([12]) Provided our assumption on* $SU_q(2)$ *holds, the modular characteristic homomorphisms in Hochschild cohomology, cyclic cohomology and periodic cyclic cohomology of the Hopf algebra* $(\mathcal{A}_q^o, \Delta)$ *with the modular pair* (I, f_1) *in involution, are all ZERO.*

PROOF. By Proposition 5.1 the operator $S : HC^n(\mathcal{A}_q) \to HC^{n+2}(\mathcal{A}_q)$ is an isomorphism for $n \geq 1$. By Theorem 6.9 we see that for any $[x] \in HC^n_{(\delta,\sigma)}(\tilde{\mathcal{A}}_q^o)$, with $n \geq 2$, we have

$$S\gamma_n([x]) = \gamma_{n+2}S([x]) = \gamma_{n+2}(0) = 0.$$

By injectivity of $S : HC^n(\mathcal{A}_q) \to HC^{n+2}(\mathcal{A}_q)$, for $n \geq 2$, we therefore have $\gamma_n([x]) = 0$. Since clearly $\gamma_n = 0$, for $n \in \{0,1\}$, we thus conclude that $\gamma = 0$ in the case of cyclic cohomology. □

7 The Modular Square for Algebraic Quantum Groups

Fix an algebraic quantum group (A, Δ) in the sense of definition 6.13 in [14]. First we recall the most important objects naturally associated to (A, Δ):

(i) The counit and antipode of (A, Δ) will be denoted by ε and S respectively. We will denote the unitary antipode of (A, Δ) by R, its scaling group by τ (see [14] Propositions 6.16 and 6.17). Recall that both these objects provide the polar decomposition of the antipode:

$$S = R\,\tau_{-\frac{i}{2}} = \tau_{-\frac{i}{2}}\,R\ .$$

(ii) Let us also fix a non-zero, positive left invariant linear functional φ on A and define a non-zero, positive right invariant linear functional ψ on A by $\psi = \varphi R$.

Denote the modular automorphism groups of φ by σ (see [14] Proposition 6.15). Recall that

$$\varphi(ab) = \varphi(b\,\sigma_{-i}(a)) \qquad \text{and} \qquad \varphi\sigma_z = \varphi$$

for all $a, b \in A$ and $z \in \mathbb{C}$.

(iii) The modular element of (A, Δ) will be denoted by δ (see [14] Prop. 6.8). Remember that δ belongs to $M(A)$ and that $\varphi(S(a)) = \varphi(a\delta)$ for all $a \in A$. As we mentioned after Prop. 6.17, we can define any complex power $\delta^z \in M(A)$ $(z \in \mathbb{C})$. Then $\psi(a) = \varphi(\delta^{\frac{1}{2}}a\delta^{\frac{1}{2}})$ for all $a \in A$.

(iv) The scaling constant $\nu > 0$ of (A, Δ) was defined in [14], Prop. 6.16. Thus, $\varphi\tau_z = \nu^{-z}\,\varphi$ for all $z \in \mathbb{C}$. Let us also mention that $\sigma_z(\delta^r) = \nu^{rz}\,\delta^r$ for all $z \in \mathbb{C}$ and $r \in \mathbb{R}$.

As in [14] Prop. 6.10 and Thm. 6.11, we let $(\hat{A},\hat{\Delta})$ denote the dual algebraic quantum group of (A,Δ). Remember that $M(\hat{A})$ can be realized as a subspace of the dual A' (see the discussion before Thm. 6.11). Hence, the counit ε is the unit of $M(\hat{A})$.

Denoting the modular element of $(\hat{A},\hat{\Delta})$ by $\hat{\delta}$, one can show that $\hat{\delta}^z = \varepsilon\,\sigma_{iz}$ for all $z \in \mathbb{C}$ (see Prop. 5.14 of [9]).

In order to perform the modular square construction, we need a Hopf algebra that is paired with (A,Δ). But in general, $(M(\hat{A}),\hat{\Delta})$ is not a Hopf algebra since it can happen that $\hat{\Delta}(M(\hat{A})) \not\subseteq M(\hat{A}) \otimes M(\hat{A})$. (where we used the canonical extension of $\hat{\Delta}$ to $M(\hat{A})$).

Therefore we suppose that we have a unital subalgebra $\hat{\mathcal{H}}$ of $M(\hat{A})$ at our disposal such that

1. $\hat{\Delta}(\hat{\mathcal{H}}) \subseteq \hat{\mathcal{H}} \otimes \hat{\mathcal{H}}$,
2. $\hat{S}(\hat{\mathcal{H}}) = \hat{\mathcal{H}}$ (where we use the canonical extension of the antipode $\hat{S}$ of $(\hat{A},\hat{\Delta})$),
3. $\hat{\delta}^{\frac{1}{2}}$, $\hat{\delta}^{-\frac{1}{2}}$ belong to $\hat{\mathcal{H}}$.

So $(\hat{\mathcal{H}},\hat{\Delta})$ is a Hopf algebra consisting of linear functionals on A (in the classical case, $\hat{\mathcal{H}}$ could be the universal enveloping algebra of the Lie algebra associated to the Lie group of which A is the appropriate function algebra).

In order to obtain a good modular pair in action, it turns out that not $\hat{\mathcal{H}}$ is the Hopf algebra we should focus on, but rather the Hopf algebra $(\tilde{\mathcal{H}},\tilde{\Delta})$ defined by

$$\tilde{\mathcal{H}} = \hat{\mathcal{H}} \otimes \hat{\mathcal{H}}^{\mathrm{op}} \qquad \text{and} \qquad \tilde{\Delta} = (\iota \otimes \chi \otimes \iota)(\Delta \otimes \Delta)\ ,$$

where χ denotes the flip map. Hence, the antipode $\tilde{S}$ of $(\tilde{\mathcal{H}},\tilde{\Delta})$ is given by $\tilde{S} = \hat{S} \otimes \hat{S}^{-1}$.

Next we introduce the modular pair for $(\tilde{\mathcal{H}},\tilde{\Delta})$.

, (1) Set $\tilde{\alpha} = \hat{\delta}^{-\frac{1}{2}} \otimes \hat{\delta}^{-\frac{1}{2}} \in \tilde{\mathcal{H}}$. We know that $\hat{\delta}^{-\frac{1}{2}}$ is group-like in $M(\hat{A})$. Hence, $\tilde{\alpha}$ is group-like in $\tilde{\mathcal{H}}$.

, (2) Since $\Delta(\delta^{\frac{1}{2}}) = \delta^{\frac{1}{2}} \otimes \delta^{\frac{1}{2}}$, we can define non-zero characters $\theta, \theta^{-1} : \hat{A} \to \mathbb{C}$ such that

$$\theta(\omega) = \omega(\delta^{\frac{1}{2}}) \qquad \text{and} \qquad \theta^{-1}(\omega) = \omega(\delta^{-\frac{1}{2}})$$

for all $\omega \in \hat{A}$. The characters θ and θ^{-1} extend uniquely to characters on $M(\hat{A})$. For instance, $\theta(\omega a) = \omega(a\delta^{\frac{1}{2}})$ for all $a \in A$ and $\omega \in A'$. These extensions restrict to characters on $\hat{\mathcal{H}}$ which we denote by the same symbols. Define the character $\tilde{\theta} : \tilde{\mathcal{H}} \to \mathbb{C}$ as $\tilde{\theta} = \theta \otimes \theta^{-1}$.

Let $\tilde{S}_{\tilde{\theta}}$ be the twisted antipode on $(\tilde{\mathcal{H}},\tilde{\Delta})$, i.e. $\tilde{S}_{\tilde{\theta}} = (\tilde{\theta} \otimes \tilde{S})\tilde{\Delta}$. In terms of the twisted antipodes $\hat{S}_\theta = (\theta \otimes \hat{S})\hat{\Delta}$ on $(\hat{\mathcal{H}},\hat{\Delta})$ and $\hat{S}^{-1}_{\theta^{-1}} = (\theta^{-1} \otimes \hat{S}^{-1})\hat{\Delta}$ on $(\hat{\mathcal{H}}^{\mathrm{op}},\hat{\Delta})$, the twisted antipode $\tilde{S}_{\tilde{\theta}}$ is given by

$$\tilde{S}_{\tilde{\theta}} = \hat{S}_\theta \otimes \hat{S}^{-1}_{\theta^{-1}}\ . \tag{7.1}$$

and an easy calculation reveals that

$$\hat{S}_\theta(\omega)(a) = \omega(\delta^{\frac{1}{2}} S(a)) \qquad \text{and} \qquad \hat{S}^{-1}_{\theta^{-1}}(\omega)(a) = \omega(\delta^{-\frac{1}{2}} S^{-1}(a)) \tag{7.2}$$

for all $\omega \in \hat{\mathcal{H}}$ and $a \in A$.

It is not so difficult to check that

PROPOSITION 7.1. *The couple $(\tilde{\theta},\tilde{\alpha})$ is a modular pair in involution for $(\tilde{\mathcal{H}},\tilde{\Delta})$.*

PROOF.

(1) We have that

$$\tilde{\theta}(\tilde{\alpha}) = \theta(\hat{\delta}^{-\frac{1}{2}})\,\theta^{-1}(\hat{\delta}^{-\frac{1}{2}}) = \hat{\delta}^{-\frac{1}{2}}(\delta^{\frac{1}{2}})\,\hat{\delta}^{-\frac{1}{2}}(\delta^{-\frac{1}{2}}) = (\varepsilon\,\sigma_{-\frac{i}{2}})(\delta^{\frac{1}{2}})\,(\varepsilon\,\sigma_{-\frac{i}{2}})(\delta^{-\frac{1}{2}})\ ,$$

which by the fact that $\delta^{\frac{1}{2}}$ and $\delta^{-\frac{1}{2}}$ are group-like, implies that

$$\tilde{\theta}(\tilde{\alpha}) = (\nu^{-\frac{i}{4}}\,\varepsilon(\delta^{\frac{1}{2}}))\,(\nu^{\frac{i}{4}}\,\varepsilon(\delta^{-\frac{1}{2}})) = 1\ .$$

(2) Choose $\omega\in\hat{\mathcal{H}}$, $\eta\in\hat{\mathcal{H}}^{\mathrm{op}}$ and $a,b\in A$. Then Eqs. (7.1) and (7.2) imply that

$$\begin{aligned}\tilde{S}^2_{\tilde{\theta}}(\omega\otimes\eta)(a\otimes b) &= \hat{S}^2_{\theta}(\omega)(a)\,(\hat{S}^{-1}_{\theta^{-1}})^2(\eta)(b)\\ &= \omega(\delta^{\frac{1}{2}}S^2(a)\delta^{-\frac{1}{2}})\,\eta(\delta^{-\frac{1}{2}}S^{-2}(b)\delta^{\frac{1}{2}}) = \omega(\delta^{\frac{1}{2}}\tau_{-i}(a)\delta^{-\frac{1}{2}})\,\eta(\delta^{-\frac{1}{2}}\tau_i(b)\delta^{\frac{1}{2}})\ .\end{aligned}$$

Hence, Lem. 5.15 of [9] guarantees that

$$\begin{aligned}\tilde{S}^2_{\tilde{\theta}}(\omega\otimes\eta)(a\otimes b) &= \omega\big(\,(\varepsilon\sigma_{-\frac{i}{2}}\otimes\iota\otimes\varepsilon\sigma_{\frac{i}{2}})\Delta^{(2)}(a)\,\big)\ \eta\big(\,(\varepsilon\sigma_{\frac{i}{2}}\otimes\iota\otimes\varepsilon\sigma_{-\frac{i}{2}})\Delta^{(2)}(b)\,\big)\\ &= \omega\big(\,(\hat{\delta}^{-\frac{1}{2}}\otimes\iota\otimes\hat{\delta}^{\frac{1}{2}})\Delta^{(2)}(a)\,\big)\ \eta\big(\,(\hat{\delta}^{\frac{1}{2}}\otimes\iota\otimes\hat{\delta}^{-\frac{1}{2}})\Delta^{(2)}(b)\,\big)\\ &= (\hat{\delta}^{-\frac{1}{2}}\,\omega\,\hat{\delta}^{\frac{1}{2}})(a)\ (\hat{\delta}^{-\frac{1}{2}}\,\omega\,\hat{\delta}^{\frac{1}{2}})(b)\ ,\end{aligned}$$

(remember that we use the opposite product on $\hat{\mathcal{H}}^{\mathrm{op}}$). It follows that $\tilde{S}^2_{\tilde{\theta}}(\omega\otimes\eta) = \tilde{\alpha}\,(\omega\otimes\eta)\,\tilde{\alpha}^{-1}$.

$\sqcup$
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In the next step, we let $\tilde{\mathcal{H}}$ act on A. Since $\hat{\mathcal{H}}\subseteq M(\hat{A})$, the discussion before Thm. 6.11 in [14] guarantees that $(\omega\otimes\iota)\Delta(a)$ and $(\iota\otimes\omega)\Delta(a)$ belong to A for all $a\in A$ and $\omega\in\tilde{\mathcal{H}}$. Thus, we can define left Hopf actions

$$\begin{aligned}\hat{\mathcal{H}}\times A\to A:(\omega,a)\mapsto\omega*a &= (\iota\otimes\omega)\Delta(a)\quad\text{and}\\ \hat{\mathcal{H}}^{\mathrm{op}}\times A\to A:(\omega,a)\mapsto a*\omega &= (\omega\otimes\iota)\Delta(a)\ .\end{aligned}\tag{7.3}$$

We leave it to the reader to check that these are really left Hopf actions and that they commute. For $\omega\in\hat{\mathcal{H}}$, $\eta\in\hat{\mathcal{H}}^{\mathrm{op}}$ and $a\in A$, we set

$$\omega*a*\eta = (\omega*a)*\eta = \omega*(a*\eta)\ .$$

So we can define a left Hopf action $\triangleright:\tilde{\mathcal{H}}\times A\to A$ so that

$$(\omega\otimes\eta)\triangleright a = \omega*a*\eta$$

for all $\omega\in\hat{\mathcal{H}}$, $\eta\in\hat{\mathcal{H}}^{\mathrm{op}}$ and $a\in A$.

In order to define the γ-map that connects the Hopf cyclic cohomology of the Hopf algebra $\hat{\mathcal{H}}$ and the ordinary cyclic cohomology of the algebra A, one needs a $\tilde{\theta}$-invariant $\tilde{\alpha}$-trace. To this end, Connes and Moscovici introduced in [5] the mid-weight $\rho : A \to \mathbb{C}$ defined by

$$\rho(a) = \varphi(\delta^{\frac{1}{4}} a \delta^{\frac{1}{4}}) = \psi(\delta^{-\frac{1}{4}} a \delta^{-\frac{1}{4}})$$

for all $a \in A$ (recall that $\psi = \varphi(\delta^{\frac{1}{2}} \,.\, \delta^{\frac{1}{2}})$.

Proposition 7.2. *The linear functional ρ is a $\tilde{\theta}$-invariant $\tilde{\alpha}$-trace.*

Proof.

(1) ρ is $\tilde{\theta}$-invariant:

Choose $\eta \in \hat{\mathcal{H}}^{\mathrm{op}}$ and $a, b \in A$. Then, since $\sigma_{-i}(\delta^{\frac{1}{4}}) = \nu^{-\frac{i}{4}} \, \delta^{\frac{1}{4}}$,

$$\begin{aligned} \rho((a * \eta)\, b) &= \varphi\big(\delta^{\frac{1}{4}}\, (a * \eta)\, b\, \delta^{\frac{1}{4}}\big) \\ &= \nu^{-\frac{i}{4}}\ \varphi\big((\eta \otimes \iota)(\Delta(a))\, b\, \delta^{\frac{1}{2}}\,\big) = \nu^{-\frac{i}{4}}\ \eta\big(\,(\iota \otimes \varphi)(\Delta(a)(1 \otimes b\, \delta^{\frac{1}{2}}))\,\big)\ . \end{aligned}$$

The proof of Prop. 3.11 of [23] shows that

$$S\big(\,(\iota \otimes \varphi)(\Delta(p)(1 \otimes q))\,\big) = (\iota \otimes \varphi)((1 \otimes p)\Delta(q))$$

for all $p, q \in A$. Thus,

$$\begin{aligned} \rho\big((a * \eta)\, b\big) &= \nu^{-\frac{i}{4}}\ \eta\big(\,S^{-1}(\,(\iota \otimes \varphi)((1 \otimes a)\Delta(b\delta^{\frac{1}{2}}))\,)\,\big) \\ &= \nu^{-\frac{i}{4}}\ \eta\big(\,S^{-1}(\,(\iota \otimes \varphi)((1 \otimes a)\Delta(b)(1 \otimes \delta^{\frac{1}{2}}))\, \delta^{\frac{1}{2}}\,)\,\big) \\ &= \eta\big(\,\delta^{-\frac{1}{2}}\, S^{-1}(\,(\iota \otimes \varphi)((1 \otimes \delta^{\frac{1}{4}})(1 \otimes a)\Delta(b)(1 \otimes \delta^{\frac{1}{4}}))\,)\,\big) \\ &= \eta\big(\,\delta^{-\frac{1}{2}}\, S^{-1}(\,(\iota \otimes \rho)((1 \otimes a)\Delta(b))\,)\,\big) = \hat{S}^{-1}_{\theta^{-1}}(\eta)\big(\,(\iota \otimes \rho)((1 \otimes a)\Delta(b))\,\big) \\ &= \rho\big(a\, (b * \hat{S}^{-1}_{\theta^{-1}}(\eta))\big)\ . \end{aligned}$$

Using the fact that $\rho = \psi(\delta^{-\frac{1}{4}} \,.\, \delta^{-\frac{1}{4}})$, one proves in a similar fashion that $\rho\big((\omega * a)\, b\big) = \rho\big(a\, (\hat{S}_{\theta}(\omega) * b)\big)$ for all $a, b \in A$ and $\omega \in \hat{\mathcal{H}}$. Combining these two results and Eq. (7.1), one sees that ρ is $\tilde{\theta}$-invariant, i.e.

$$\rho\big((\varphi \triangleright a)\, b\big) = \rho\big(a\, (\tilde{S}_{\tilde{\theta}}(\varphi) \triangleright b)\big)$$

for all $a, b \in A$ and $\varphi \in \tilde{\mathcal{H}}$.

(2) ρ is an $\tilde{\alpha}$-trace:

Choose $a, b \in A$. Then,

$$\begin{aligned} \rho(ab) &= \varphi(\delta^{\frac{1}{4}} ab\, \delta^{\frac{1}{4}}) = \varphi(b\delta^{\frac{1}{4}} \sigma_{-i}(\delta^{\frac{1}{4}} a)) = \nu^{-\frac{1}{4}}\ \varphi(b\, \delta^{\frac{1}{2}} \sigma_{-i}(a)) \\ &= \nu^{-\frac{1}{4}}\ \varphi\big(b\, \delta^{\frac{1}{2}} \sigma_{-i}(a) \delta^{-\frac{1}{2}}\, \delta^{\frac{1}{4}}\, \delta^{\frac{1}{4}}\big) = \varphi\big(\delta^{\frac{1}{4}} b\, \delta^{\frac{1}{2}} \sigma_{-i}(a) \delta^{-\frac{1}{2}}\, \delta^{\frac{1}{4}}\,\big) \\ &= \rho\big(b\, \delta^{\frac{1}{2}} \sigma_{-i}(a) \delta^{-\frac{1}{2}}\,\big) \end{aligned}$$

According to Lem. 5.15 of [9], we have that

$$\delta^{\frac{1}{2}} \sigma_{-i}(a) \delta^{-\frac{1}{2}} = (\varepsilon\sigma_{-\frac{i}{2}} \otimes \iota \otimes \varepsilon\sigma_{-\frac{i}{2}})\Delta^{(2)}(a) = (\hat{\delta}^{-\frac{1}{2}} \otimes \iota \otimes \hat{\delta}^{-\frac{1}{2}})\Delta^{(2)}(a) = (\hat{\delta}^{-\frac{1}{2}} \otimes \hat{\delta}^{-\frac{1}{2}}) \triangleright a = \tilde{\alpha} \triangleright a\ .$$

Hence, $\rho(ab) = \rho\big(b\, (\tilde{\alpha} \triangleright a)\big)$, meaning that ρ is an $\tilde{\alpha}$-trace.

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We end this section by commenting on the case of locally compact quantum groups in a formal way. In this case we start of with a locally compact quantum group (A,Δ) as in [14] Def. 8.1, which we assume to act on the GNS-space H_φ of its left Haar weight φ. Let $W \in B(H_\varphi \otimes H_\varphi)$ be the multiplicative unitary as defined before Thm. 8.3 in [14] and recall from Prop. 8.4 in [14] that A is the norm closure of the algebra $B := \{ (\iota\otimes\omega)(W) \mid \omega \in B(H_\varphi)_* \}$.

Also, the dual quantum group $(\hat{A},\hat{\Delta})$ of (A,Δ) is defined such that $\hat{A}$ is the norm closure of the algebra $\{ (\omega\otimes\iota)(W) \mid \omega \in B(H_\varphi)_* \}$ and

$$\hat{\Delta}(x) = W(x\otimes 1)W^* \text{ for all } x \in \hat{A}\ . \tag{7.4}$$

So we see that $\hat{A}$ can be identified with a subspace of the dual B' such that

$$\langle x, (\iota\otimes\omega)(W)\rangle = \omega(x) \tag{7.5}$$

for $x \in \hat{A}$ and $\omega \in B(H_\varphi)_*$. In a formal sense, one obtains a dual pairing between $\hat{A}$ and B this way.

Regarding $\hat{A}$ as sitting inside B' this way, referring to Eq. (7.3) and using the fact that $(\Delta\otimes\iota)(W) = W_{13}W_{23}$ (see [14] Thm. 8.3 and Prop. 8.4), we should define left actions of $\hat{A}$ and $\hat{A}^{\mathrm{op}}$ on B in the following way:

$$\hat{A}\times B \to B : (x,(\iota\otimes\omega)(W)) \mapsto x * (\iota\otimes\omega)(W) = (\iota\otimes\omega)(W(1\otimes x))$$

and

$$\hat{A}^{\mathrm{op}}\times B \to B : (x,(\iota\otimes\omega)(W)) \mapsto (\iota\otimes\omega)(W) * x = (\iota\otimes\omega)((1\otimes x)W)\ . \tag{7.6}$$

In a lot of specific examples it happens to be the case that there is a Hopf algebra $\hat{\mathcal{H}}$ around that can play the role of the Hopf algebra in the construction of the modular square (as far as we know, the possible presence and the desirable properties of such a Hopf algebra has not been seriously studied yet in a more general theoretical framework). However, the precise relation between this Hopf algebra $\hat{\mathcal{H}}$ and the actual C*-algebra $\hat{A}$ or von Neumann $\hat{A}''$ can be cumbersome to describe. Let us discuss this a little bit further.

In a lot of cases, there is some dense subspace D of H_φ present together with a number of linear operators $T_1,\ldots,T_n$ on D (in order not to cloud the discussion, we ignore the important *-operation) such that $\hat{\mathcal{H}}$ is the algebra of linear operators on D that is generated by $T_1,\ldots,T_n$.

However, the linear operators $T_1,\ldots,T_n$ that generate this Hopf algebra $\hat{\mathcal{H}}$ are not really the ones that are of fundamental importance to $\hat{A}$. The important ones are most of the time (possibly unbounded) closed linear operators $\bar{T}_1,\ldots,\bar{T}_n$ in H_φ that are extensions of $T_1,\ldots,T_n$ respectively and are affiliated to the C*-algebra $\hat{A}$ or to the von Neumann algebra $\hat{A}''$. It also can happen that the proper extension $\bar{T}_i$ is not merely the closure of T_i but that a more refined description of the extension $\bar{T}_i$ is needed (comparable to the theory of self-adjoint extensions of linear operators that are not essentially self-adjoint).

In accordance with Eq. (7.4), the comultiplication on $\hat{A}$ and $\hat{\mathcal{H}}$ should be related through the fact that $\Delta(T_i)$ is a restriction of $W(\bar{T}_i \otimes 1)W^*$.

It can happen that the C*-algebra $\hat{A}$ is generated by $\bar{T}_1, \ldots, \bar{T}_n$ in the sense of Def. 1.24 in [14] (e.g. quantum $E(2)$, quantum $az+b$). But there also instances (e.g. quantum $ax+b$, quantum $\widetilde{SU}(1,1)$) where one needs to add an extra bounded linear operator S on D so that $\hat{A}$ is generated by $\bar{T}_1, \ldots, \bar{T}_n$ and $\bar{S}$ (where $\bar{S}$ is the unique bounded linear operator on H_φ extending S) and for which there does not exist an element X in the algebraic tensor product $\hat{\mathcal{H}}_* \otimes \hat{\mathcal{H}}_*$ (where $\hat{\mathcal{H}}_*$ denotes the algebra of linear operators on D generated by $\hat{\mathcal{H}}$ and S) so that $W(1 \otimes \bar{S})W^*$ is an extension of X. In short, we can not extend the Hopf algebra structure on $\hat{\mathcal{H}}$ to an appropriate Hopf algebra structure on $\hat{\mathcal{H}}_*$. We refer to [14] section **??** (ref: part of S.L. Woronowicz concerning examples) for more details on this subject.

Let us end this discussion by saying that a reason for not working with the 'Hopf algebra generated by' the extensions $\bar{T}_1, \ldots, \bar{T}_n$ lies in the fact that for such unbounded operators, addition and multiplication are tricky and sometimes ill-defined operations.

But let us ignore this important issue for the rest of this section and focus on $\hat{\mathcal{H}}$. Recall that in the case of algebraic quantum groups we used the inherent pairing between A and $\hat{\mathcal{H}}$ to define actions of $\hat{\mathcal{H}}$ on A (see Eq. (7.3)). So we would like to use the formula in Eq. (7.5) to define a pairing between $\hat{\mathcal{H}}$ and B. Due to the unboundedness of the elements in $\hat{\mathcal{H}}$, the expression in Eq. (7.5) does not make sense for every $\omega \in B(H_\varphi)_*$ and $x \in \hat{\mathcal{H}}$. But if ω is the vector functional $\omega_{v,w}$ for $v \in D$ and $w \in H_\varphi$ (see the comments before Def. 1.29), we can interpret $\omega(x)$ to be equal to $\langle x(v), w\rangle$. So we should look for an appropriate subspace $E \subseteq B(H_\varphi)_*$ for which $\omega(x)$ can be well defined, a technical issue we will not further go into. But the vector functionals introduced above indicate that E can be chosen to be dense in $B(H_\varphi)_*$.

This way, Eq. (7.5) will provide us with a pairing between $\hat{\mathcal{H}}$ and $B_0 := \{\, (\iota \otimes \omega)(W) \mid \omega \in E \,\} \subseteq B$. The above discussion is only intended to give a first impression of the technical subtleties involved. It turns out that further restrictions on E (and hence on B_0) are needed in order to

1. well define the left actions of $\hat{\mathcal{H}}$ and $\hat{\mathcal{H}}^{\mathrm{op}}$ on B_0 inspired by formula (7.6),
2. be able to apply the mid-weight on elements of B_0 and products thereof.

The data used in the statement of Proposition 7.1 and Proposition 7.2 is referred to as the *modular square* associated to the Hopf algebra $(\hat{\mathcal{H}}, \hat{\Delta})$. By Lemma 4.1, these propositions immediately yield the following important result.

THEOREM 7.3. *([5]) Consider a Hopf algebra $(\hat{\mathcal{H}}, \hat{\Delta})$ with $\hat{\mathcal{H}} \subset M(\hat{A})$ as a unital algebra, with coproduct given by restricting $\hat{\Delta} : \hat{A} \to M(\hat{A} \times \hat{A})$ to $\hat{\mathcal{H}}$, and such that $\hat{\delta}^{\frac{1}{2}}, \hat{\delta}^{-\frac{1}{2}} \in \hat{\mathcal{H}}$. Let $\tilde{\mathcal{H}}^\natural_{(\tilde{\theta},\tilde{\alpha})}$ denote the Λ-module canonically associated to the modular square of $(\hat{\mathcal{H}}, \hat{\Delta})$. Then the formulas $\gamma_0(1) = \rho$ and*

$$\gamma_n((x^1 \otimes y^1) \otimes \cdots \otimes (x^n \otimes y^n))(a^0, \ldots, a^n) = \rho(a^0(x^1 * a^1 * y^1) \cdots (x^n * a^n * y^n)),$$

for $a^0, \ldots, a^n \in A$ and $x^1, y^1, \ldots, x^n, y^n \in \hat{\mathcal{H}}$ and $n \in \mathbb{N}$, yield a Λ-module morphism $\gamma : \tilde{\mathcal{H}}^\natural_{(\tilde{\theta},\tilde{\alpha})} \to C^\natural$, which in turn induces characteristic homomorphisms in cyclic cohomology

$$\gamma_* : HC^*_{(\tilde{\theta},\tilde{\alpha})}(\tilde{\mathcal{H}}) \to HC^*(A).$$

Characteristic homomorphisms are similarly induced on the level of Hochschild cohomology and periodic cyclic cohomology.

The map γ from Theorem 7.3 is called the *modular characteristic homomorphism* associated to $(\hat{\mathcal{H}}, \hat{\Delta})$.

To compute the cyclic cohomology of the modular square, we include a convenient corollary of Theorem 3.1 and Theorem 3.2.

COROLLARY 7.4. *Consider a Hopf $*$-algebra $(\hat{\mathcal{H}}, \hat{\Delta})$ with $\hat{\mathcal{H}} \subset M(\hat{A})$ as a unital $*$-algebra, with coproduct given by restricting $\hat{\Delta} : \hat{A} \to M(\hat{A} \times \hat{A})$ to $\hat{\mathcal{H}}$, and such that $\hat{\delta}^{\frac{1}{2}}, \hat{\delta}^{-\frac{1}{2}} \in \hat{\mathcal{H}}$. Let $(\tilde{\mathcal{H}}, \tilde{\Delta})$ denote its modular square with modular pair $(\tilde{\theta}, \tilde{\alpha}) = (\theta \otimes \theta^{-1}, \hat{\delta}^{-\frac{1}{2}} \otimes \hat{\delta}^{-\frac{1}{2}})$ in involution. Then*

$$HH^n_{(\tilde{\theta},\tilde{\alpha})}(\tilde{\mathcal{H}}) \cong \bigoplus_{i+j=n} HH^i_{(\theta,\hat{\delta}^{-\frac{1}{2}})}(\hat{\mathcal{H}}) \otimes HH^j_{(\theta,\hat{\delta}^{-\frac{1}{2}})}(\hat{H}),$$

for $n \in \mathbb{N}_0$. Moreover, the isomorphism is implemented by the shuffle map sh (going from left to right) with the Alexander-Whitney map AW as its inverse. There exists a canonical long exact sequence

$$\cdots \xrightarrow{i} HC^{n-1}_{(\tilde{\theta},\tilde{\alpha})}(\tilde{\mathcal{H}}) \xrightarrow{\partial} \bigoplus_{p+q=n-2} HC^p_{(\theta,\hat{\delta}^{-\frac{1}{2}})}(\hat{\mathcal{H}}) \otimes HC^q_{(\theta,\hat{\delta}^{-\frac{1}{2}})}(\hat{\mathcal{H}}) \xrightarrow{S\otimes 1 - 1\otimes S}$$

$$\bigoplus_{r+s=n} HC^r_{(\theta,\hat{\delta}^{-\frac{1}{2}})}(\hat{\mathcal{H}}) \otimes HC^s_{(\theta,\hat{\delta}^{-\frac{1}{2}})}(\hat{\mathcal{H}}) \xrightarrow{i} HC^n_{(\tilde{\theta},\tilde{\alpha})}(\tilde{\mathcal{H}}) \xrightarrow{\partial} \cdots$$

Notation. We recover Theorem 4.3 and Corollary 3.15 in the compact case from Theorem 7.3 and Corollary 7.4, respectively, by first observing that $(A, \Delta) = (\mathcal{A}, \Phi)$ and $M(\hat{A}) \cong \mathcal{A}'$, so $(\hat{\mathcal{H}}, \hat{\Delta})$ is a Hopf subalgebra of $(\mathcal{A}^o, \Delta)$. Now since (A, Φ) is a compact quantum group, we have $\nu = 1$, $\delta = I$ and $\hat{\delta} = f_{-2}$, so $\rho = h$ and $(\tilde{\theta}, \tilde{\alpha}) = (I \otimes I, f_1 \otimes f_1)$.

8 Twisted Cyclic Cohomology

Suppose $\mathcal{A}$ is a unital algebra and let $\theta : \mathcal{A} \to \mathcal{A}$ be an automorphism of $\mathcal{A}$. We now define a new example of a cocyclic object, namely the one which gives rise to *twisted* (see [11] and [7]) cyclic cohomology $HC^*(\mathcal{A}, \theta)$ of $\mathcal{A}$. Fix $n \in \mathbb{N}_0$. Let C^n denote (as in the case of ordinary cyclic cohomology) the vector space of n-cochains, that is, the set of multilinear maps from the vector space $\mathcal{A}^{n+1}$ to $\mathbb{C}$. Let $i \in \{0, 1, \ldots, n\}$. Define linear maps

$$d^n_i : C^{n-1} \to C^n, \quad s^n_i : C^{n+1} \to C^n, \quad t_n : C^n \to C^n$$

as follows:

$$\begin{aligned}
(d^n_i\varphi)(a^0, a^1, \ldots, a^n) &= \varphi(a^0, \ldots, a^i a^{i+1}, \ldots, a^n) \quad \text{for } i \leq n-1 \\
(d^n_n\varphi)(a^0, a^1, \ldots, a^n) &= \varphi(\theta(a^n)a^0, a^1, \ldots, a^{n-1}) \\
(s^n_i\varphi)(a^0, a^1, \ldots, a^n) &= \varphi(a^0, \ldots, a^i, I, a^{i+1}, \ldots, a^n) \\
(t_n\varphi)(a^0, a^1, \ldots, a^n) &= \varphi(\theta(a^n), a^0, a^1, \ldots, a^{n-1}),
\end{aligned}$$

where I is the unit of $\mathcal{A}$, $a^0,\dots,a^n \in \mathcal{A}$ and φ is a cochain. As before C^{-1} and d_i^0 are not defined, so the range of n must be restricted accordingly for some formulas to make sense. Note that $(t_n^{n+1}\varphi)(a^0,\dots,a^n) = \varphi(\theta(a^0),\dots,\theta(a^n))$, for any $\varphi \in C^n$ and $a^i \in \mathcal{A}$. Define a subspace $C^n(\theta)$ of C^n by

$$C^n(\theta) = \{\varphi \in C^n \mid t_n^{n+1}\varphi = \varphi\}.$$

It is straightforward to check that $d_i^n(C^{n-1}(\theta)) \subset C^n(\theta)$, $s_i^n(C^{n+1}(\theta)) \subset C^n(\theta)$ and $t_n(C^n(\theta)) \subset C^n(\theta)$. Thus $(C^n(\theta), d_i^n, s_i^n, t_n)$ is a cocyclic object of the abelian category of vector spaces. We denote the Hochschild cohomology, the cyclic cohomology and the periodic cyclic cohomology canonically associated to this cocyclic object by $HH^*(\mathcal{A},\theta)$, $HC^*(\mathcal{A},\theta)$ and $HP^*(\mathcal{A},\theta)$, respectively.

Dually, define the invariant subspace $C_n(\theta)$ of $C_n = \mathcal{A}^{\otimes(n+1)}$ by

$$C_n(\theta) = \{x \in \mathcal{A}^{\otimes(n+1)} \mid \theta^{\otimes(n+1)}(x) = x\ \}.$$

Let $i \in \{0,1,\dots,n\}$, and define linear maps

$$d_n^i : C_{n-1}(\theta) \to C_n(\theta), \qquad s_n^i : C_{n+1}(\theta) \to C_n(\theta), \qquad t^n : C_n(\theta) \to C_n(\theta)$$

as follows:

$$\begin{aligned}
d_n^i(a^0 \otimes a^1 \otimes ... \otimes a^n) &= a^0 \otimes ... \otimes a^i a^{i+1} \otimes ... \otimes a^n \quad \text{for } i \le n-1\\
d_n^n(a^0 \otimes a^1 \otimes ... \otimes a^n) &= \theta(a^n)a^0 \otimes a^1 \otimes ... \otimes a^{n-1}\\
s_n^i(a^0 \otimes a^1 \otimes ... \otimes a^n) &= a^0 \otimes ... \otimes a^i \otimes I \otimes a^{i+1} \otimes ... \otimes a^n\\
t^n(a^0 \otimes a^1 \otimes ... \otimes a^n) &= \theta(a^n) \otimes a^0 \otimes a^1 \otimes ... \otimes a^{n-1},
\end{aligned}$$

where I is the unit of $\mathcal{A}$ and $a^0,\dots,a^n \in \mathcal{A}$. Note that $C_{-1}(\theta)$ and d_0^i are not defined, so the range of n must be restricted accordingly for some formulas to make sense. Then $(C_n(\theta), d_i^n, s_i^n, t_n)$ is a cyclic object of the abelian category of vector spaces. Let $C_\theta^\natural$ denote the corresponding Λ-module. We denote the Hochschild homology, the cyclic homology and the periodic cyclic homology canonically associated to $C_\theta^\natural$ by $HH_*(\mathcal{A},\theta)$, $HC_*(\mathcal{A},\theta)$ and $HP_*(\mathcal{A},\theta)$, respectively.

Of course, these definitions extend to yield cyclic and cocyclic objects over arbitrary fields, rings or modules. Note that when $\theta = \iota$, then the twisted cohomology (homology) groups coincide with the ordinary cohomology (homology) groups of $\mathcal{A}$.

Notice that $C^n(\theta)$ is isomorphic (as a vector space) to the annihilator

$$(\mathrm{Im}(\theta^{\otimes(n+1)} - \iota^{\otimes(n+1)}))^\perp$$

in $\mathcal{A}^{\otimes(n+1)}$ of the image of the linear map $\theta^{\otimes(n+1)} - \iota^{\otimes(n+1)}$, and, moreover, that

$$\mathrm{Im}(\theta^{\otimes(n+1)} - \iota^{\otimes(n+1)}) = \mathcal{A}^{\otimes(n+1)}/C_n(\theta),$$

for all $n \in \mathbb{N}_0$. When $\theta = \iota$, then $C_n(\theta) = \mathcal{A}^{\otimes(n+1)}$ and $C^n(\theta) = C_n(\theta)'$.

We also have a Chern character in the twisted case generalizing the ordinary one. Recall that $C_0(\theta)$ is the unital fix point algebra $\mathcal{A}_\theta$ in $\mathcal{A}$ of θ, which obviously coincides with $\mathcal{A}$ if and only if $\theta = \iota$. Let $C^\natural(\mathcal{A}_\theta)$ denote the ordinary Λ-module of the algebra $\mathcal{A}_\theta$ (to distinguish it from the Λ-modules $C^\natural_\iota = C^\natural$ and $C^\natural_\theta$ associated to $(\mathcal{A}, \iota)$ and $(\mathcal{A}, \theta)$, respectively).

It is readily seen that the map

$$p : (\mathcal{A}_\theta, \iota) \mapsto (\mathcal{A}, \theta)$$

of algebras with automorphisms induces a Λ-module morphism $p : C^\natural(\mathcal{A}_\theta) \to C^\natural_\theta$, and thus group homomorphisms

$$p_* : HC_*(\mathcal{A}_\theta) \to HC_*(\mathcal{A}, \theta)$$

$$p^* : HC^*(\mathcal{A}, \theta) \to HC^*(\mathcal{A}_\theta)$$

in cyclic homology and cyclic cohomology. Of course the same holds for Hochschild (co-) homology and periodic cyclic (co-) homology.

Let

$$\langle \cdot, \cdot \rangle : HC^*(\mathcal{A}_\theta) \times K_*(\mathcal{A}_\theta) \to \mathbb{C}$$

denote the ordinary bilinear pairing between ordinary (even or odd) cyclic cohomology and K-theory associated to the algebra $\mathcal{A}_\theta$. Then we have the following bilinear pairing

$$HC^*(\mathcal{A}, \theta) \times K_*(\mathcal{A}_\theta) \xrightarrow{p^* \times \iota} HC^*(\mathcal{A}_\theta) \times K_*(\mathcal{A}_\theta) \xrightarrow{\langle \cdot, \cdot \rangle} \mathbb{C}.$$

Similarly, using the Chern character

$$\mathrm{ch}_* : K_*(\mathcal{A}_\theta) \to HC_*(\mathcal{A}_\theta)$$

from K-theory to (even or odd) cyclic homology associated to the algebra $\mathcal{A}_\theta$, we get the following Chern character

$$K_*(\mathcal{A}_\theta) \xrightarrow{\mathrm{ch}_*} HC_*(\mathcal{A}_\theta) \xrightarrow{p_*} HC_*(\mathcal{A}, \theta)$$

for twisted cyclic homology.

There exists a beautiful relation between twisted cyclic cocycles and differential calculi of quantum groups [27][8], which we shall briefly sketch (**see also the proceeding of Gerard Murphy**). Henceforth let $(\mathcal{A}, \Phi)$ denote the Hopf $*$-algebra associated to a compact quantum group.

Theorem 8.1. *([11]) Let notation be as above and let $Z^n_\lambda(\mathcal{A}, \theta)$ denote the vector space of θ-twisted n-cyclic cocycles of $\mathcal{A}$. For any $\varphi \in Z^n_\lambda(\mathcal{A}, \theta)$ there exists a canonical triple $(\Omega, d, \int)$, where (Ω, d) is an n-dimensional differential calculus and $\int$ is a closed faithful θ-twisted graded trace on (Ω, d) (in the sense of [11]). The assignment*

$$\varphi \in Z^n_\lambda(\mathcal{A}, \theta) \longrightarrow (\Omega, d, \int)$$

is bijective. Moreover, with respect to the natural notions of invariance for φ and $\int$, we have for this correspondence that:

1. φ *is left invariant if and only if* (Ω, d) *is left covariant.*

2. φ *is right invariant if and only if* (Ω, d) *is right covariant.*

3. φ *is bi-invariant if and only if* (Ω, d) *is bicovariant.*

Finite dimensional covariant differential calculi come with a unique (up to a scalar) closed faithful twisted graded trace exactly when there is only one (up to a scalar) top invariant volume form [11]. In the bicovariant case the twist is then usually given by $\theta_D(a) = f_{-1} * a * f_{-1}$, for $a \in \mathcal{A}$, which can be shown to be the case for the $4D_+$-calculus of Woronowicz associated to $SU_q(2)$.

In the twisted world we can define characteristic homomorphisms without involving the modular square.

THEOREM 8.2. *Let* θ_l *and* θ_r *denote the automorphisms of* $\mathcal{A}$ *defined for* $a \in \mathcal{A}$, *by* $\theta_l(a) = a * f_{-1}$ *and* $\theta_r(a) = f_{-1} * a$, *respectively, so* $\theta_l\theta_r = \theta_r\theta_l = \theta_D$. *Denote the corresponding* Λ*-modules by* $C_l^\natural$ *and* $C_r^\natural$. *Consider a Hopf subalgebra* $(\mathcal{H}, \Delta)$ *of* $(\mathcal{A}^o, \Delta)$ *with the modular pair* (I, f_1) *in involution. Define linear maps* $\gamma_n^{l,r} : \mathcal{H}^{\otimes n} \to C^n$ *by*

$$\gamma_n^l(x^1 \otimes \cdots \otimes x^n)(a^0, \ldots, a^n) = h(a^0(x^1 * a^1) \cdots (x^n * a^n))$$

and

$$\gamma_n^r(x^1 \otimes \cdots \otimes x^n)(a^0, \ldots, a^n) = h(a^0(a^1 * x^1) \cdots (a^n * x^n)),$$

for $a^0, \ldots, a^n \in \mathcal{A}$, $x^1, \ldots, x^n \in \mathcal{H}$ *and* $n \in \mathbb{N}$. *For* $n = 0$, *define linear maps*

$$\gamma_n^{l,r} : \mathcal{H}^{\otimes 0} = \mathbb{C} \to C^0 = \mathcal{A}'$$

by $\gamma_0^l(1) = \gamma_0^r(1) = h$, *where* h *is the Haar state of* $(\mathcal{A}, \Phi)$. *Then* $\gamma_n^{l,r}(\mathcal{H}^{\otimes n}) \subset C^n(\theta_{l,r})$, *for* $n \in \mathbb{N}_0$. *Moreover, the linear maps*

$$\gamma^{l,r} = \oplus_{n=0}^{\infty} \gamma_n^{l,r} : \mathcal{H}^\natural_{(I,f_1)} \to C^\natural_{l,r}$$

are injective Λ*-module morphisms. Hence* γ^l *and* γ^r *induce characteristic homomorphisms*

$$\gamma^{l,r} : HC^*_{(I,f_1)}(\mathcal{H}) \to HC^*(\mathcal{A}, \theta_{l,r})$$

of the corresponding cyclic cohomology groups. The same holds for Hochschild cohomology and periodic cyclic cohomology.

REMARK 8.3. The map γ^l does not in general map into the Λ-module underlying the ordinary cyclic cohomology $HC^*(\mathcal{A})$ because h is not a δ-invariant σ-trace with respect to the modular pair $(\delta, \sigma) = (I, f_1)$ in involution and the $\mathcal{H}$-action $\triangleright_l : x \otimes a \to x * a$, where $x \in \mathcal{H}$ and $a \in \mathcal{A}$. As we have seen, γ^l maps instead into the Λ-module $C_l^\natural$ underlying the twisted cyclic cohomology $HC^*(\mathcal{A}, \theta_l)$ of $\mathcal{A}$, and in general, there exists no Λ-module morphism between $C^\natural$ and $C_l^\natural$, nor any interesting homomorphism between their corresponding cohomology groups. Similar remarks hold for γ^r with the $\mathcal{H}_{op}$-action $\triangleright_r : x \otimes a \to a * x$, where $x \in \mathcal{H}_{op}$ and $a \in \mathcal{A}$.

We have seen that to get a characteristic homomorphism into $HC^*(\mathcal{A})$, one needs to involve both $\triangleright_l$ and $\triangleright_r$ and form the *modular square* $(\tilde{\mathcal{H}}, \tilde{\Delta})$ of $(\mathcal{H}, \Delta)$. The $\tilde{\mathcal{H}}$-action on $\mathcal{A}$ is given by

$$x \otimes y \mapsto x \triangleright_l (y \triangleright_r a) = y \triangleright_r (x \triangleright_l a),$$

for $x \in \mathcal{H}$, $y \in \mathcal{H}_{op}$ and $a \in \mathcal{A}$. However, as we have seen, the γ-map obtained this way is not injective on the level of Λ-modules, whereas both $\gamma^{l,r}$ are.

Also there is another important feature of the maps $\gamma^{l,r}$. The image $\gamma^l(\mathcal{H}^\natural_{(I,f_1)})$ consists entirely of left-invariant cochains. Similarly, the image $\gamma^r(\mathcal{H}^\natural_{(I,f_1)})$ consists entirely of right-invariant cochains. The characteristic homomorphism associated to the modular square, on the other hand, has an image which consists of cochains that are neither left- nor right-invariant.

Finally, it follows from [11], that a left-invariant θ_l-twisted cyclic cocycle will necessary belong to $\gamma^l(\mathcal{H}^\natural_{(I,f_1)})$. Similarly, a right-invariant θ_r-twisted cyclic cocycle will necessary belong to $\gamma^r(\mathcal{H}^\natural_{(I,f_1)})$. This suggests that a characterization of the images of $\gamma^{l,r}(HC^*_{(I,f_1)}(\mathcal{H}))$ in $HC^*(\mathcal{A}, \theta_{l,r})$ can be made. There is work in progress discussing these matters, where the case of $SU_q(2)$ is considered in great detail.

A Proof of Theorem 8.2

In this section we include a detailed proof of Theorem 8.2, which has not been written down elsewhere. To this end we need the following result [25][26].

Proposition A.1. *Let $(\mathcal{A}, \Phi)$ denote the Hopf $*$-algebra associated to a compact quantum group. Then there exists a canonical family $\{f_z\}_{z\in\mathbb{C}}$ of unital multiplicative linear functionals on $\mathcal{A}$ such that:*

1. $f_0 = \varepsilon$ *and* $f_z f_w = f_{z+w}$ *for all* $z, w \in \mathbb{C}$.
2. $f_z(\kappa(a)) = f_{-z}(a)$ *and* $\overline{f_z(a)} = f_{-\overline{z}}(a^*)$ *for all* $a \in \mathcal{A}$ *and* $z \in \mathbb{C}$.
3. $\kappa^2(a) = f_{-1} * a * f_1$ *for all* $a \in \mathcal{A}$.
4. $h(ab) = h(bf_1 * a * f_1)$ *for all* $a, b \in \mathcal{A}$.

We also need the following result [12], which follows from Proposition A.1.

Proposition A.2. *([12]) Let $x, y \in \mathcal{H} \subset \mathcal{A}^o$ and $a, b \in \mathcal{A}$. Then:*

1. $\kappa((h \otimes \iota)((a \otimes I)\Phi(b))) = (h \otimes \iota)(\Phi(a)(b \otimes I))$.
2. $\kappa((\iota \otimes h)(\Phi(a)(I \otimes b))) = (\iota \otimes h)((I \otimes a)\Phi(b))$.
3. $h((y * a)b) = h(a(S(y) * b))$.
4. $h((a * y)b) = h(a(b * S^{-1}(y)))$.

Proof of Theorem 8.2 We consider first the map γ^l corresponding to the left action $\triangleright_l$. Let $n \in \mathbb{N}_0$. Now $\gamma^l_n(\mathcal{H}^{\otimes n}) \subset C^n(\theta_l)$ if and only if $t_n^{n+1}\gamma^l_n = \gamma^l_n$, which we may express as

$$h(\theta_l(a^0)(x^1 * \theta_l(a^1)) \cdots (x^n * \theta_l(a^n))) = h(a^0(x^1 * a^1) \cdots (x^n * a^n)),$$

for $a^0,\dots,a^n \in \mathcal{A}$ and $x^1,\dots,x^n \in \mathcal{H}$. But

$$h(\theta_l(a^0)(x^1 * \theta_l(a^1))\cdots(x^n * \theta_l(a^n))) = h(a^0(x^1 * a^1)\cdots(x^n * a^n))$$

holds for any twist given by $a \to a * \xi$, where $a \in \mathcal{A}$, as long as $\xi \in \mathcal{A}'$ satisfies $\xi(I) = 1$.

To see that γ^l is injective, consider first $n \in \mathbb{N}$ and pick a typical element $\sum_i x_i^1 \otimes \cdots \otimes x_i^n$ of $\mathcal{H}^{\otimes n}$ such that $\gamma_n^l(\sum_i x_i^1 \otimes \cdots \otimes x_i^n) = 0$. Then by faithfulness of h on $\mathcal{A}$, we conclude that

$$\sum_i (x_i^1 * a^1)\cdots(x_i^n * a^n) = 0,$$

for any $a^1,\dots,a^n \in \mathcal{A}$. By applying the counit ε of $(\mathcal{A},\Phi)$ to this expression, we get

$$(\sum_i x_i^1 \otimes \cdots \otimes x_i^n)(a^1,\dots,a^n) = \sum_i x_i^1(a^1)\cdots x_i^n(a^n) = 0,$$

for any $a^1,\dots,a^n \in \mathcal{A}$. Thus $\sum_i x_i^1 \otimes \cdots \otimes x_i^n = 0$ and γ_n^l is injective. Since $\gamma_0^l(c) = ch$, for any $c \in \mathbb{C}$, obviously γ_0^l is injective. Thus γ^l is injective.

To show that γ^l is a Λ-module morphism we need to check the conditions

$$\gamma_n^l \delta_i^n = d_i^n \gamma_{n-1}^l, \quad \gamma_n^l \sigma_i^n = s_i^n \gamma_{n+1}^l, \quad \gamma_n^l \tau_n = t_n \gamma_n^l,$$

for all $n \in \mathbb{N}_0$ and $i \in \{0,\dots,n\}$. Recall that d_i^0 and δ_i^0 are not defined.

Consider first the condition $\gamma_n^l \delta_i^n = d_i^n \gamma_{n-1}^l$, where we restrict to $n \geq 2$ and $i \in \{1,\dots,n-1\}$. Let $a^0,\dots,a^n \in \mathcal{A}$ and $x^1,\dots,x^{n-1} \in \mathcal{H}$. Then

$$\begin{aligned}
&(d_i^n \gamma_{n-1}^l)(x^1 \otimes \cdots \otimes x^{n-1})(a^0,\dots,a^n) \\
&= \gamma_{n-1}^l(x^1 \otimes \cdots \otimes x^{n-1})(a^0,\dots,a^i a^{i+1},\dots,a^n) \\
&= h(a^0(x^1 * a^1)\cdots(x^{i-1} * a^{i-1})(x^i * (a^i a^{i+1}))(x^{i+1} * a^{i+2})\cdots(x^{n-1} * a^n)) \\
&= h(a^0(x^1 * a^1)\cdots(x^{i-1} * a^{i-1})(\sum (x^i_{(1)} * a^i)(x^i_{(2)} * a^{i+1}))(x^{i+1} * a^{i+2})\cdots(x^{n-1} * a^n)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&(d_i^n \gamma_{n-1}^l)(x^1 \otimes \cdots \otimes x^{n-1})(a^0,\dots,a^n) \\
&= \gamma_n^l(x^1 \otimes \cdots \otimes x^{i-1} \otimes \Delta(x^i) \otimes x^{i+1} \otimes \cdots \otimes x^{n-1})(a^0,\dots,a^n)
\end{aligned}$$

and thus

$$\begin{aligned}
&(d_i^n \gamma_{n-1}^l)(x^1 \otimes \cdots \otimes x^{n-1}) \\
&= \gamma_n^l(x^1 \otimes \cdots \otimes x^{i-1} \otimes \Delta(x^i) \otimes x^{i+1} \otimes \cdots \otimes x^{n-1}) = (\gamma_n^l \delta_i^n)(x^1 \otimes \cdots \otimes x^{n-1}),
\end{aligned}$$

so $\gamma_n^l \delta_i^n = d_i^n \gamma_{n-1}^l$, for $n \geq 2$ and $i \in \{1,\dots,n-1\}$.

Next consider the condition $\gamma_n^l \delta_n^n = d_n^n \gamma_{n-1}^l$. We have

$$\begin{aligned}
&(d_n^n \gamma_{n-1}^l)(x^1 \otimes \cdots \otimes x^{n-1})(a^0,a^1,\dots,a^n) \\
&= \gamma_{n-1}^l(x^1 \otimes \cdots \otimes x^{n-1})(\theta_l(a^n)a^0,a^1,\dots,a^{n-1}) \\
&= h((a^n * f_{-1})a^0(x^1 * a^1)\cdots(x^{n-1} * a^{n-1})).
\end{aligned}$$

Using Proposition A.1, we thus get

$$
\begin{aligned}
&(d_n^n\gamma_{n-1}^l)(x^1\otimes\cdots\otimes x^{n-1})(a^0,a^1,\ldots,a^n)\\
&= h(a^0(x^1*a^1)\cdots(x^{n-1}*a^{n-1})(f_1*a^n))\\
&= \gamma_n^l(x^1\otimes\cdots\otimes x^{n-1}\otimes f_1)(a^0,\ldots,a^n)\\
&= \gamma_n^l(\delta_n^n(x^1\otimes\cdots\otimes x^{n-1}))(a^0,\ldots,a^n),
\end{aligned}
$$

for all $x^1,\ldots,x^{n-1}\in\mathcal{H}$ and $a^0,\ldots,a^n\in\mathcal{A}$, so $\gamma_n^l\delta_n^n = d_n^n\gamma_{n-1}^l$.

Next consider the condition $\gamma_n^l\delta_0^n = d_0^n\gamma_{n-1}^l$. We have

$$
\begin{aligned}
&(d_0^n\gamma_{n-1}^l)(x^1\otimes\cdots\otimes x^{n-1})(a^0,\ldots,a^n)\\
&= \gamma_{n-1}^l(x^1\otimes\cdots\otimes x^{n-1})(a^0a^1,a^2,\ldots,a^n)\\
&= h(a^0a^1(x^1*a^2)\cdots(x^{n-1}*a^n)) = h(a^0(\varepsilon*a^1)(x^1*a^2)\cdots(x^{n-1}*a^n))\\
&= \gamma_n^l(\varepsilon\otimes x^1\otimes\cdots\otimes x^{n-1})(a^0,\ldots,a^n) = \gamma_n^l(\delta_0^n(x^1\otimes\cdots\otimes x^{n-1}))(a^0,\ldots,a^n),
\end{aligned}
$$

for all $x^1,\ldots,x^{n-1}\mathcal{H}$ and $a^0,\ldots,a^n\mathcal{A}$, so $\gamma_n^l\delta_0^n = d_0^n\gamma_{n-1}^l$.

We now look at the case when $n=1$. To see that $\gamma_1^l\delta_i^1 = d_i^1\gamma_0^l$, for $i=0$ and $i=1$, observe that

$$\gamma_1^l(\delta_i^1(1))(a^0,a^1) = h(a^0\delta_i^1(1)*a^1),$$

whereas

$$d_i^1(\gamma_0^l(1))(a^0,a^1) = d_i^1(h)(a^0,a^1),$$

for all $a^0,a^1\in\mathcal{A}$. Since $\delta_0^1(1)=\varepsilon$, we therefore get

$$\gamma_1^l(\delta_0^1(1))(a^0,a^1) = h(a^0\varepsilon*a^1) = h(a^0a^1) = d_0^1(h)(a^0,a^1),$$

for all $a^0,a^1\in\mathcal{A}$, as wanted. Since $\delta_1^1(1)=f_1$, we get

$$
\begin{aligned}
\gamma_1^l(\delta_1^1(1))(a^0,a^1) &= h(a^0(f_1*a^1)) = h((a^1*f_{-1})a^0)\\
&= h(\theta_l(a^1)a^0) = d_1^1(h)(a^0,a^1),
\end{aligned}
$$

for all $a^0,a^1\in\mathcal{A}$, also as wanted.

Consider now the condition $\gamma_n^l\sigma_i^n = s_i^n\gamma_{n+1}^l$, for $n\in\mathbb{N}_0$ and $i\in\{0,\ldots,n\}$. We have

$$
\begin{aligned}
&(s_i^n\gamma_{n+1}^l)(x^1\otimes\cdots\otimes x^{n+1})(a^0,\ldots,a^n)\\
&= \gamma_{n+1}^l(x^1\otimes\cdots\otimes x^{n+1})(a^0,\ldots,a^i,I,a^{i+1},\ldots,a^n)\\
&= h(a^0(x^1*a^1)\cdots(x^i*a^i)(x^{i+1}*I)(x^{i+2}*a^{i+1})\cdots(x^{n+1}*a^n))\\
&= h(a^0(x^1*a^1)\cdots(x^i*a^i)I(x^{i+1})(x^{i+2}*a^{i+1})\cdots(x^{n+1}*a^n))\\
&= \gamma_n^l(x^1\otimes\cdots\otimes x^i\otimes I(x^{i+1})\otimes x^{i+2}\otimes\cdots\otimes x^{n+1})(a^0,\ldots,a^n),\\
&= \gamma_n^l(\sigma_i^n(x^1\otimes\cdots\otimes x^{n+1}))(a^0,\ldots,a^n),
\end{aligned}
$$

for all $x^1,\ldots,x^{n+1}\in\mathcal{H}$ and $a^0,\ldots,a^n\in\mathcal{A}$, so $\gamma_n^l\sigma_i^n = s_i^n\gamma_{n+1}^l$.

Consider the condition $\gamma^l_n \tau_n = t_n \gamma^l_n$, for $n \geq 2$. We have

$$(t_n\gamma^l_n)(x^1 \otimes \cdots \otimes x^n)(a^0, \ldots, a^n) = \gamma^l_n(x^1 \otimes \cdots \otimes x^n)(\theta_l(a^n), a^0, \ldots, a^{n-1})$$
$$= h((a^n * f_{-1})(x^1 * a^0) \cdots (x^n * a^{n-1})) = h((x^1 * a^0) \cdots (x^n * a^{n-1})(f_1 * a^n))$$
$$= h(a^0 S(x^1) * ((x^2 * a^1) \cdots (x^n * a^{n-1})(f_1 * a^n))),$$

where we in the last step have used Proposition A.2. Say

$$\Delta^{n-1}(S(x^1)) = \sum_i y^2_i \otimes \cdots \otimes y^n_i \otimes z_i.$$

Then

$$(t_n\gamma^l_n)(x^1 \otimes \cdots \otimes x^n)(a^0, \ldots, a^n)$$
$$= \sum_i h(a^0(y^2_i * (x^2 * a^1)) \cdots (y^n_i * (x^n * a^{n-1}))(z_i * (f_1 * a^n)))$$
$$= \sum_i h(a^0((y^2_i x^2) * a^1) \cdots ((y^n_i x^n) * a^{n-1})((z_i f_1) * a^n))$$
$$= \gamma^l_n(\sum_i y^2_i x^2 \otimes \cdots \otimes y^n_i x^n \otimes z_i f_1)(a^0, \ldots, a^n),$$

for all $a^0, \ldots, a^n \in \mathcal{A}$, so

$$(t_n\gamma^l_n)(x^1 \otimes \cdots \otimes x^n) = \gamma^l_n(\sum_i y^2_i x^2 \otimes \cdots \otimes y^n_i x^n \otimes z_i f_1)$$
$$= \gamma^l_n((\sum_i y^2_i \otimes \cdots \otimes y^n_i \otimes z_i)(x^2 \otimes \cdots \otimes x^n \otimes f_1))$$
$$= \gamma^l_n(\Delta^{n-1}(S(x^1)) \cdot (x^2 \otimes \cdots \otimes x^n \otimes f_1)) = \gamma^l_n(\tau_n(x^1 \otimes \cdots \otimes x^n)),$$

for all $x^1, \ldots, x^n \in \mathcal{H}$. Therefore $\gamma^l_n \tau_n = t_n \gamma^l_n$, for $n \geq 2$.

Also, we have $\gamma^l_1 \tau_1 = t_1 \gamma^l_1$ because

$$(t_1\gamma^l_1)(x^1)(a^0, a^1) = \gamma^l_1(x^1)((a^1 * f_{-1})a^0) = h((a^1 * f_{-1})(x^1 * a^0))$$
$$= h((x^1 * a^0)(f_1 * a^1)) = h(a^0 S(x^1) * (f_1 * a^1)) = h(a^0(S(x^1)f_1) * a^1)$$
$$= \gamma^l_1(S(x^1)f_1)(a^0, a^1) = \gamma^l_1(\tau_1(x^1))(a^0, a^1),$$

for all $x^1 \in \mathcal{H}$ and $a^0, a^1 \in \mathcal{A}$.

The condition $\gamma^l_0 \tau_0 = t_0 \gamma^l_0$ is true because $(\gamma^l_0 \tau_0)(1) = \gamma^l_0(1) = h$ and

$$(t_0\gamma^l_0)(1)(a^0) = t_0(h)(a^0) = h\theta_l(a^0) = h(a^0),$$

for all $a^0 \in \mathcal{A}$. This finally shows that all the conditions involving the basic operations for the Λ-modules are satisfied, and thus γ^l is a Λ-module map. The statements regarding γ^r corresponding to the $\mathcal{H}_{op}$-action $\triangleright_r$ are proved in a similar fashion. In order to check the condition $\gamma^r_n \tau_n = t_n \gamma^r_n$, one needs to use the formula $S^2(x) = f_1 x f_{-1}$, for all $x \in \mathcal{H}$, in conjunction with Proposition A.2 involving $\triangleright_r$. This completes the proof.

B The Bicrossproduct Hopf Algebra $\mathcal{H} \bowtie \mathcal{H}_{\text{op}}$

The tensor product Hopf algebra $(\tilde{\mathcal{H}}, \tilde{\Delta})$, from Theorem 4.2, is a special case of a left-right bicrossproduct $\mathcal{H} \bowtie \mathcal{H}_{\text{op}}$ [17]. In this section we define what is mean by this. We also discuss the Hopf algebra $\mathcal{H}(G)$ appearing in the calculations of the transverse index theorem of Connes and Moscovici [3].

Let $(\mathcal{H}, \Delta)$ be a Hopf algebra with counit ε, coinverse S, multiplication $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ and unit I. Let $\text{ad} : \mathcal{H}_{\text{op}} \otimes \mathcal{H} \to \mathcal{H}$ be the left action of $(\mathcal{H}_{\text{op}}, \Delta)$ on the algebra $\mathcal{H}$ defined by $\text{ad}(x \otimes y) = \sum S(x_{(1)}) y x_{(2)}$, where $x, y \in \mathcal{H}$. Note that $\text{ad}(x \otimes I) = \varepsilon(x) I$, for $x \in \mathcal{H}$. Hence $\mathcal{H}$ is a left $\mathcal{H}_{\text{op}}$-module algebra with respect to the action ad [17].

According to Proposition 1.6.6 in [17], one may thus form a left cross product algebra $\mathcal{H} \bowtie_{\text{left}} \mathcal{H}_{\text{op}}$ built on the vector space $\mathcal{H} \otimes \mathcal{H}_{\text{op}}$ with product m_l given by

$$m_l((x \otimes y) \otimes (x' \otimes z)) = \sum x \ \text{ad}(y_{(1)} \otimes x') \otimes y_{(2)} z,$$

where $x, x' \in \mathcal{H}$ and $y, z \in \mathcal{H}_{\text{op}}$. The unit element I_l of the algebra $\mathcal{H} \bowtie_{\text{left}} \mathcal{H}_{\text{op}}$ is $I \otimes I$.

Define a linear map $\text{ad}_c : \mathcal{H}_{\text{op}} \to \mathcal{H}_{\text{op}} \otimes \mathcal{H}$ by

$$\text{ad}_c = (\iota \otimes m)(\iota \otimes S \otimes \iota)(F \otimes \iota)(\Delta \otimes \iota)\Delta.$$

This is a coaction of $(\mathcal{H}, \Delta)$ on $\mathcal{H}_{\text{op}}$, that is, the identities $(\iota \otimes \Delta)\text{ad}_c = (\text{ad}_c \otimes \iota)\text{ad}_c$ and $(\iota \otimes \varepsilon)\text{ad}_c = \iota$ hold. Also $\mathcal{H}_{\text{op}}$ is a right $\mathcal{H}$-comodule coalgebra under the coaction ad_c, that is, the additional identities $(\Delta \otimes \iota)\text{ad}_c = (\iota \otimes \iota \otimes m)(\iota \otimes F \otimes \iota)(\text{ad}_c \otimes \text{ad}_c)\Delta$ and $(\varepsilon \otimes \iota)\text{ad}_c = \iota$ hold.

According to Proposition 1.6.16 in [17], one may thus form a right cross coproduct coalgebra $\mathcal{H} \bowtie_{\text{right}} \mathcal{H}_{\text{op}}$ built on the vector space $\mathcal{H} \otimes \mathcal{H}_{\text{op}}$ with coproduct Δ_r given by

$$\Delta_r(x \otimes y) = \sum x_{(1)} \otimes y_{(1)}^{(1)} \otimes x_{(2)} y_{(1)}^{(2)} \otimes y_{(2)},$$

where $x \in \mathcal{H}$ and $y \in \mathcal{H}_{\text{op}}$ with $\text{ad}_c(y_{(1)}) = \sum y_{(1)}^{(1)} \otimes y_{(1)}^{2)}$. Note also that the counit ε_r of $\mathcal{H} \bowtie_{\text{right}} \mathcal{H}_{\text{op}}$ is given by $\varepsilon_r(x \otimes y) = \varepsilon(x)\varepsilon(y)$.

The maps ad and ad_c satisfy the compatibility conditions of Theorem 6.2.2 in [17], and therefore the product m_l from $\mathcal{H} \bowtie_{\text{left}} \mathcal{H}_{\text{op}}$ and the coproduct Δ_r from $\mathcal{H} \bowtie_{\text{right}} \mathcal{H}_{\text{op}}$ turn the vector space $\mathcal{H} \otimes \mathcal{H}_{\text{op}}$ into a Hopf algebra $\mathcal{H} \bowtie \mathcal{H}_{\text{op}}$ called a *left-right bicrossproduct Hopf algebra*. The coinverse S_r of $\mathcal{H} \bowtie \mathcal{H}_{\text{op}}$ is given by

$$S_r(x \otimes y) = \sum (I \otimes S^{-1}(y^{(1)}))(S(xy^{(2)}) \otimes I),$$

where $x \in \mathcal{H}$ and $y \in \mathcal{H}_{\text{op}}$ with $\text{ad}_c(y) = \sum y^{(1)} \otimes y^{(2)}$. Thus, the coinverse S_r may alternatively be written $S_r = (S \otimes S^{-1})(m \otimes \iota)(\iota \otimes F)(\iota \otimes \text{ad}_c)$.

For the sake of convenience, we spell out the product m_l and coproduct Δ_r of $\mathcal{H} \bowtie \mathcal{H}_{\text{op}}$. Recall that as a vector space $\mathcal{H} \bowtie \mathcal{H}_{\text{op}}$ is just $\mathcal{H} \otimes \mathcal{H}_{\text{op}}$. Suppose $x, z \in \mathcal{H}$ and $y, w \in \mathcal{H}_{\text{op}}$. Then

$$\begin{aligned} m_l((x \otimes y) \otimes (z \otimes w)) &= \sum x S(y_{(1)}) z y_{(2)} \otimes y_{(3)} w, \\ \Delta_r(x \otimes y) &= \sum x_{(1)} \otimes y_{(2)} \otimes x_{(2)} S(y_{(1)}) y_{(3)} \otimes y_{(4)} \end{aligned}$$

and

$$S_r(x \otimes y) = \sum S(xS(y_{(1)})y_{(3)}) \otimes S^{-1}(y_{(2)}).$$

The Hopf algebra $\mathcal{H} \bowtie \mathcal{H}_{\mathrm{op}}$ has a simpler description [17], namely, the map $E : \mathcal{H} \otimes \mathcal{H}_{\mathrm{op}} \to \mathcal{H} \bowtie \mathcal{H}_{\mathrm{op}}$ defined by $E(x \otimes y) = (x \otimes I)\Delta(y)$, where $x \in \mathcal{H}$ and $y \in \mathcal{H}_{\mathrm{op}}$, is a Hopf algebra isomorphism from the tensor product Hopf algebra $(\tilde{\mathcal{H}}, \tilde{\Delta})$ to $\mathcal{H} \bowtie \mathcal{H}_{\mathrm{op}}$ with inverse $E^{-1}(x \otimes y) = (x \otimes I)(S \otimes \iota)\Delta(y)$, and it restricts to embeddings of $(\mathcal{H}, \Delta)$ and $(\mathcal{H}_{\mathrm{op}}, \Delta)$ into $\mathcal{H} \bowtie \mathcal{H}_{\mathrm{op}}$.

Suppose, for $i = 1, 2$, G_i are finite subgroups of a group G such that any $g \in G$ admits a decomposition as $g = ka$ for unique $k \in G_1$ and $a \in G_2$. Let $\pi_i : G \to G_i$ be the canonical epimorphisms. Define a right action $\alpha : G_2 \times G_1 \to G_2$ of G_1 on G_2 by $\alpha(a, k) = \alpha_k(a) = \pi_2(ak)$ for $a \in G_2$ and $k \in G_1$. Similarly define a left action $\beta : G_2 \times G_1 \to G_1$ of G_2 on G_1 by $\beta(a, k) = \beta_a(k) = \pi_1(ak)$ for $a \in G_2$ and $k \in G_1$.

With respect to the actions α and β, the groups (G_2, G_1) form a right-left matched pair [17], that is, the actions α and β satisfy the following compatibility conditions:

$$\alpha_k(1) = 1, \quad \alpha_k(a_1 a_2) = \alpha_{\beta_{a_2}(k)}(a_1)\alpha_k(a_2),$$

and

$$\beta_a(1) = 1, \quad \beta_a(k_1 k_2) = \beta_a(k_1)\beta_{\alpha_{k_1}(a)}(k_2),$$

where $k, k_i \in G_1$ and $a, a_i \in G_2$.

According to Example 6.2.11 in [17], the induced action $\tilde{\alpha}$ of $\mathbb{C}[G_2]$ on $\mathbf{C}(G_1)$ and the induced coaction $\tilde{\beta}$ of $\mathbf{C}(G_1)$ on $\mathbb{C}[G_2]$, give a left-right bicrossproduct Hopf algebra $\mathbf{C}(G_1) \bowtie \mathbb{C}[G_2]$ as described in Theorem 6.2.2 in [17]. Let $\delta_s \in \mathbf{C}(G_1)$ and $\varepsilon_u \in \mathbb{C}[G_2]$, where $s \in G_1$ and $u \in G_2$, be given by $\delta_s(t) = 1$, if $t \in G_1$ equals s and is zero otherwise, whereas $\varepsilon_u(v) = 1$, if $v \in G_2$ equals u and is zero otherwise. Thus $\{\delta_s \otimes \varepsilon_u \mid s \in G_1, \ u \in G_2 \}$ is a basis of the vector space $\mathbf{C}(G_1) \otimes \mathbb{C}[G_2]$. The product m of $\mathbf{C}(G_1) \bowtie \mathbb{C}[G_2]$ is now explicitly given by

$$m((\delta_s \otimes \varepsilon_u) \otimes (\delta_t \otimes \varepsilon_v)) = \delta_{\alpha_u(s),t}(\delta_s \otimes \varepsilon_{uv})$$

and the coproduct Δ of $\mathbf{C}(G_1) \bowtie \mathbb{C}[G_2]$ is given by

$$\Delta(\delta_s \otimes \varepsilon_u) = \sum_{ab=s} \delta_a \otimes \varepsilon_{\beta_b(u)} \otimes \delta_b \otimes \varepsilon_u,$$

where $s, t \in G_1$ and $u, v \in G_2$. Here $\delta_{s,t}$ is the number which is one if $s = t$ and is zero otherwise.

The left-right bicrossproduct Hopf algebra $\mathbf{C}(G_1) \bowtie \mathbb{C}[G_2]$ defined above is isomorphic to the Hopf algebra $\mathcal{H}(G)$ defined in Section 5 in [3]. In the latter paper, the notation $a \cdot k = \alpha_k(a)$ and $a(k) = \beta_a(k)$ and $X_k = \varepsilon_k$ and $\varepsilon_a = \delta_a$ for $k \in G_1$ and $a \in G_2$, is used, and the same product but the opposite coproduct is defined.

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