The Adams Spectral Sequence

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Foreword

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1 Stable homotopy theory

1.1 Vector fields on spheres

Many topological problems can be formulated as questions about the existence or enumeration of continuous maps with suitable properties. To answer these questions one needs tools to help determine when such maps exist or how many there are.

An interesting example is the vector fields problem on spheres. Let $S^n \subset \mathbb{R}^{n+1}$ be the unit sphere in $(n+1)$-space. At each point $p \in S^n$ there is an n-dimensional tangent space T_pS^n, consisting of the vectors $v \in \mathbb{R}^{n+1}$ with $p \perp v$. These combine to the total space of the tangent bundle $\pi\colon TS^n \to S^n$ the n-sphere. A vector field on the sphere is a section in the tangent bundle, i.e., a map $X\colon S^n \to TS^n$ with $\pi \circ X = id$. It associates to each point $p \in S^n$ a tangent vector $X(p) \in T_pS^n$ at that point.

If $n = 2e-1$ is odd, there is an everywhere nonzero vector field on S^n. Identifying $\mathbb{R}^{n+1} = \mathbb{R}^{2e}$ with $\mathbb{C}^e$, one such field is given in terms of the complex multiplication by $X(p) = ip$. In coordinates, the tangent vector at $p = (x_1, x_2, \dots, x_{2e-1}, x_{2e}) \in S^n$ is $X(p) = (-x_2, x_1, \dots, -x_{2e}, x_{2e-1})$. On the other hand, if n is even there is no everywhere nonzero vector field on S^n. One proof uses that the Euler characteristic of S^n, which is 2 for n even, can be written as a sum over the zeros of any (reasonably nice) vector field, and such a sum would be 0 if the vector field had no zeros. Similarly, if $n = 4e-1$ is congruent to 3 mod 4, there are three everywhere linearly independent vector fields on S^n. Identifying

$\mathbb{R}^{n+1} = \mathbb{R}^{4e}$ with $\mathbb{H}^e$, these can be given in terms of the quaternionic multiplication by $X_1(p) = ip$, $X_2(p) = jp$ and $X_3(p) = kp$, where $\mathbb{H} = \mathbb{R}\{1, i, j, k\}$ and $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$ and $ki = j = -ik$. On the other hand, if $n \equiv 1 \mod 4$ there is no pair of everywhere independent vector fields on S^n. Continuing, if $n = 8e - 1 \equiv 7 \mod 8$, then there are 7 independent vector fields on S^n, given in terms of the octonionic multiplication on $\mathbb{R}^{n+1} = \mathbb{O}^e$. When $n \equiv 3 \mod 8$ there is no quadruple of independent vector fields. After this, the pattern changes. There is no division algebra structure on $\mathbb{R}^{16}$, and the maximum number of independent vector fields on S^{15} is 8, not 15.

The vector fields on spheres problem is then this: What is the maximal number m of vector fields $X_1, \dots, X_m$ on the n-dimensional sphere S^n such that $X_1(p), \dots, X_m(p) \in T_pS^n$ are linearly independent for each $p \in S^n$? By an application of the Gram–Schmidt process, any m-tuple of everywhere linearly independent vector fields can be converted into an m-tuple of everywhere orthonormal vector fields. The problem may therefore be reformulated as: What is the maximal number of everywhere orthonormal vector fields on the n-sphere? Another reformulation is: What is the maximal dimension of a trivial subbundle $\epsilon^m \subset \tau_{S^n}$ of the tangent bundle of S^n?

An orthonormal m-tuple of vectors $v_1, \dots, v_m$ in T_pS^n, together with the point $p \in S^n$, constitute an orthonormal $(m+1)$-tuple $(v_1, \dots, v_m, p)$ in $\mathbb{R}^{n+1}$, and conversely. Any such orthonormal $(m+1)$-tuple, also known as an $(m+1)$-frame, can be completed to an orthonormal basis $(w_1, \dots, w_k, v_1, \dots, v_m, p)$ by prepending k more vectors, where $k = n - m$ is the complementary dimension of ϵ^m in τ_{S^n}. The vectors in such an orthonormal basis constitute the column vectors of a matrix in $O(n+1)$, the Lie group of $(n+1) \times (n+1)$ orthogonal matrices, and the different choices of completing vectors $w_1, \dots, w_k$ correspond to an orbit for the right action of the subgroup $O(k) \subset O(n+1)$, placed in the upper left hand corner. The space of $(m+1)$-frames $(v_1, \dots, v_m, p)$ in $\mathbb{R}^{n+1}$ is therefore the homogeneous space $O(n+1)/O(k)$, also known as a Stiefel manifold. As special cases we have $O(n+1)/O(n) \cong S^n$ and $O(n+1)/O(n-1) \subset TS^n$ is the subspace of unit tangent vectors. The map taking $(v_1, \dots, v_n, p)$ to $p \in S^n$ corresponds to the map $\pi\colon O(n+1)/O(k) \to O(n+1)/O(n) \cong S^n$, induced by the inclusion $O(k) \subset O(n)$. An m-tuple of everywhere orthonormal vector fields $X_1, \dots, X_m$ on S^n now defines a map $\sigma\colon S^n \to O(n+1)/O(k)$ taking p to the $(m+1)$-frame $(X_1(p), \dots, X_m(p), p)$, with the property that $\pi \circ \sigma = id$. The vector fields problem is thus: Given n, what is the maximal m, or the minimal $k = n - m$, such that there is a map $\sigma\colon S^n \to O(n+1)/O(k)$ with $\pi \circ \sigma = id$?

The map $\pi\colon O(n+1)/O(k) \to S^n$ is a fiber bundle (over a numerable base), which means that it has the homotopy lifting property. This means that if there exists a map $\sigma'\colon S^n \to O(n+1)/O(k)$ with $\pi \circ \sigma'$ homotopic to the identity map, then the homotopy can be lifted to a homotopy from σ' to a map $\sigma\colon S^n \to O(n+1)/O(k)$ with $\pi \circ \sigma$ equal to the identity. This means that the vector fields problem is a question about homotopy classes of maps, rather than about individual maps, and this makes it a problem in homotopy theory, rather than general topology.

1.2 Homology and homotopy

Let X be a topological space, with a chosen base point $x_0 \in X$. Give S^n the base point $s_0 = (1, 0, \dots, 0) \in \mathbb{R}^{n+1}$, for $n \geq 0$. The n-th homotopy group $\pi_n(X) = [S^n, X]$ is the set of homotopy classes of base-point preserving maps $f\colon S^n \to X$. It is a group for $n \geq 1$, and an abelian group for $n \geq 2$. We usually omit x_0 from the notation. We say that X is n-connected, for $n \geq 0$, if $\pi_i(X) = 0$ for all $0 \leq i \leq n$. A base-point preserving map $f\colon X \to Y$ is n-connected if $f_*\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for $0 \leq i < n$ and a surjection for $i = n$. It is a weak homotopy equivalence if $f_*\colon \pi_i(X) \to \pi_i(Y)$ is an isomorphism for all $i \geq 0$. The Hurewicz homomorphism $h_n\colon \pi_n(X) \to H_n(X)$ (integer coefficients) takes the homotopy class $[f]$ of a map $f\colon S^n \to X$ to the image $f_*[S^n]$ of the fundamental class $[S^n] \in H_n(S^n)$.

Lemma 1.1 (Poincaré). *Let X be a 0-connected space. The homomorphism $h_1\colon \pi_1(X) \to H_1(X)$ is surjective with kernel the commutator subgroup of $\pi_1(X)$, inducing an isomorphism $\pi_1(X)_{ab} \cong H_1(X)$.*

Theorem 1.2 (Hurewicz). *Let X be an $(n-1)$-connected space, for some $n \geq 2$. Then the homomorphism $h_n\colon \pi_n(X) \to H_n(X)$ is an isomorphism.*

See Hatcher (2002) Theorem 4.32. ((Also state relative version, for maps of 1-connected spaces.))

Corollary 1.3. *Let X be a 1-connected space, with $H_i(X) = 0$ for all $2 \leq i \leq n$. Then X is n-connected.*

Let $\iota\colon A \subset X$ be a cofibration, so that $X \cup_A CA \to X/A$ is a homotopy equivalence. For example, A might be a subcomplex of a CW complex X. There is a long exact sequence in homology

$$\cdots \to H_i(A) \to H_i(X) \to \tilde{H}_i(X/A) \xrightarrow{\partial} H_{i-1}(A) \to \dots$$

for all i (with arbitrary coefficients). There is a corresponding diagram in homotopy, but only in a restricted range. Let $a_0 \in A \subset X$. Using relative homotopy groups, there is a long exact sequence

$$\cdots \to \pi_i(A) \to \pi_i(X) \to \pi_i(X, A) \xrightarrow{\partial} \pi_{i-1}(A) \to \dots .$$

Theorem 1.4 (Homotopy excision)**.** *If A is $(m-1)$-connected and $\iota\colon A \to X$ is n-connected with $m, n \geq 1$, then $\pi_i(X, A) \to \pi_i(X/A)$ is an isomorphism for $i < m+n$ and a surjection for $i = m+n$. Hence there is an exact sequence*

$$\pi_{m+n-1}(A) \to \cdots \to \pi_i(A) \to \pi_i(X) \to \pi_i(X/A) \xrightarrow{\partial} \pi_{i-1}(A) \to \dots .$$

See Hatcher (2002) Theorem 4.23.

Dually, let $\pi\colon E \to B$ be a fibration, so that $F = \pi^{-1}(b_0) \to E \times_B PB$ is a homotopy equivalence. For example, $E \to B$ might be a numerable fiber bundle. There is a long exact sequence in homotopy

$$\cdots \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \xrightarrow{\partial} \pi_{i-1}(F) \to \dots$$

for all i. There is a corresponding diagram in homology, but only in a restricted range. Using relative homology groups, there is a long exact sequence

$$\cdots \to H_i(F) \to H_i(E) \to H_i(E, F) \xrightarrow{\partial} H_{i-1}(F) \to \dots$$

(with arbitrary coefficients).

Theorem 1.5 (Serre homology sequence)**.** *If B is $(m-1)$-connected and F is $(n-1)$-connected, with $m, n \geq 1$, then $H_i(E, F) \to H_i(B)$ is an isomorphism for $i < m+n$ and a surjection for $i = m+n$. Hence there is an exact sequence*

$$H_{m+n-1}(F) \to \cdots \to H_i(F) \to H_i(E) \to H_i(B) \xrightarrow{\partial} H_{i-1}(F) \to \dots .$$

This is an easy application of the Serre spectral sequence.

1.3 Stunted projective spaces

The n-sphere S^n is $(n-1)$-connected, and $h_n\colon \pi_n(S^n) \to H_n(S^n) \cong \mathbb{Z}$ is an isomorphism. The vector field problem for S^n asks what is the minimal $k \leq n$ such that $\pi_*\colon \pi_n(O(n+1)/O(k)) \to \pi_n(S^n) \cong \mathbb{Z}$ is surjective. The maximal number of orthonormal vector fields on S^n is then $m = n - k$.

Lemma 1.6. *The Stiefel manifold $O(n+1)/O(k)$ is $(k-1)$-connected.*

Proof. This can be seen by induction on $m = n - k \geq 0$, using the fiber sequences $O(k+m)/O(k) \to O(k+m+1)/O(k) \to S^{k+m}$. Here $O(k+m)/O(k)$ is $(k-1)$-connected by inductive hypothesis and S^{k+m} is $(k+m-1)$-connected, so $O(k+m+1)/O(k)$ is $(k-1)$-connected by the long exact sequence in homotopy. $\square$

Let $\mathbb{R}P^n$ be the projective n-space of lines through the origin in $\mathbb{R}^{n+1}$. Each such line L determines an orthogonal splitting $\mathbb{R}^{n+1} \cong L \oplus L^\perp$ and an orthonormal reflection $r_L\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that reverses L and fixes $L^\perp$. This defines a map $r^n\colon \mathbb{R}P^n \to O(n+1)$, taking L to the matrix representing r_L. If $L \subset \mathbb{R}^k$ represents a point in $\mathbb{R}P^{k-1}$ then $L^\perp$ contains $\{0\} \times \mathbb{R}^{m+1} \subset \mathbb{R}^{n+1}$, so r_L lies in the subgroup $O(k)$. Hence the composite map $\mathbb{R}P^n \to O(n+1) \to O(n+1)/O(k)$ factors through the quotient space $\mathbb{R}P^n/\mathbb{R}P^{k-1} = \mathbb{R}P^n_k$, known as a stunted projective space.

$$\begin{tikzcd}
\mathbb{R}P^{k-1} \arrow[r, tail] \arrow[d, "r^{k-1}"] & \mathbb{R}P^n \arrow[r] \arrow[d, "r^n"] & \mathbb{R}P^n_k \arrow[d, "r^n_k"] \\
O(k) \arrow[r] & O(n+1) \arrow[r, two heads] & O(n+1)/O(k)
\end{tikzcd}$$

The usual CW structure on $\mathbb{R}P^n$, with one i-cell for each $0 \le i \le n$, contains $\mathbb{R}P^{k-1}$ as its $(k-1)$-skeleton and induces a CW structure on $\mathbb{R}P^n_k$, with one i-cell for each $k \le i \le n$. For $k = n$, the identifications $\mathbb{R}P^n_n \cong O(n+1)/O(n) \cong S^n$ are compatible. The stunted projective spaces are "smaller" than the Stiefel manifolds, hence may be easier to analyze. Still, they are large enough to have the same homotopy groups, in a useful range of dimensions:

Lemma 1.7. *The map $r^n_k\colon \mathbb{R}P^n_k \to O(n+1)/O(k)$ is $2k$-connected.*

Proof. Proof by induction on $m = n - k \ge 0$. For $m = 0$ the map $\mathbb{R}P^k_k \to O(k+1)/O(k)$ is a homeomorphism. For $m > 0$ we use the diagram

$$\begin{tikzcd}
\mathbb{R}P^{k+m-1}_k \arrow[r, tail] \arrow[d, "r^{k+m-1}_k"] & \mathbb{R}P^{k+m}_k \arrow[r, "p"] \arrow[d, "r^{k+m}_k"] & S^{k+m} \arrow[d, "="] \\
O(k+m)/O(k) \arrow[r] & O(k+m+1)/O(k) \arrow[r, two heads, "\pi"] & S^{k+m}
\end{tikzcd}$$

where the upper row is a cofiber sequence, and the lower row is a fiber sequence.

Since $O(k+m)/O(k)$ is $(k-1)$-connected and S^{k+m} is $(k+m-1)$-connected, the homomorphism $H_i(O(k+m+1)/O(k), O(k+m)/O(k)) \to H_i(S^{k+m})$ is an isomorphism for $i \le 2k$ by Serre's homology sequence. Hence $H_i(\mathbb{R}P^{k+m}_k, \mathbb{R}P^{k+m-1}_k) \to H_i(O(k+m+1)/O(k), O(k+m)/O(k))$ is also an isomorphism for $i \le 2k$. By inductive hypothesis, $H_i(\mathbb{R}P^{k+m-1}_k) \to H_i(O(k+m)/O(k))$ is an isomorphism for $i < 2k$ and surjective for $i = 2k$, which implies that $H_i(\mathbb{R}P^{k+m}_k) \to H_i(O(k+m+1)/O(k))$ has the same property. ((Deduce that $\mathbb{R}P^{k+m}_k \to O(k+m+1)/O(k)$ is $2k$-connected.)) $\square$

Hence, as long as $n \le 2k$ the problem of finding a section σ for the fiber bundle projection $\pi\colon O(n+1)/O(k) \to S^n$ is equivalent to that of finding a section up to homotopy for the pinch map $p\colon \mathbb{R}P^n_k \to S^n$, i.e., deciding whether $p_*\colon \pi_n(\mathbb{R}P^n_k) \to \pi_n(S^n)$ is surjective.

$$\begin{tikzcd}
S^n \arrow[r, dashed, "s"] \arrow[dr, dashed, "\sigma"'] & \mathbb{R}P^n_k \arrow[d, "r^n_k"] \arrow[dr, "p"] & \\
& O(n+1)/O(k) \arrow[r, two heads, "\pi"'] & S^n
\end{tikzcd}$$

Except in a few cases, namely $n = 1$, 3, 7 and 15 ((check)) it turns out that the minimal k such that p_* is surjective satisfies $n \le 2k - 2$, so that the fact that $\pi_n(\mathbb{R}P^n_{k-1}) \to \pi_n(S^n)$ is not surjective implies that $\pi_n(O(n+1)/O(k-1)) \to \pi_n(S^n)$ is not surjective either.

The pinch map p fits in a Puppe cofiber sequence

$$\begin{tikzcd}
S^{n-1} \arrow[r, "\phi"] & \mathbb{R}P^{n-1}_k \arrow[r, tail] & \mathbb{R}P^n_k \arrow[r, "p"] & S^n \arrow[r, "\Sigma\phi"] & \Sigma\mathbb{R}P^{n-1}_k \\
& & & S^n \arrow[u, "="'] \arrow[ul, dashed, "s"] &
\end{tikzcd}$$

where ϕ is the attaching map for the top n-cell in $\mathbb{R}P^n_k$, and Σ denotes suspension. If the maps p and $\Sigma\phi$ had formed a homotopy fiber sequence, then p would admit a section up to homotopy s if and only if $\Sigma\phi$ were null-homotopic. However, p and ϕ form a (homotopy) cofiber sequence, and that is in general something different from a homotopy fiber sequence. Fortunately, in the cases n less than approximately $2k$ the difference is negligible. This leads us to concentrate on the homotopy groups in dimensions below $2k$ for $(k-1)$-connected spaces, and the extent to which homotopy cofiber sequences and homotopy fiber sequences agree in this range. This is the subject of stable homotopy theory.

1.4 The stable category

The suspension ΣX is the smash product $X \wedge S^1 = (X \times S^1)/(X \times \{s_0\} \cup \{x_0\} \times S^1)$, based at the image of (x_0, s_0). There is a homeomorphism $\Sigma S^n \cong S^{n+1}$, and a suspension homomorphism $E\colon \pi_n(X) \to \pi_{n+1}(\Sigma X)$ ('E' for 'Einhängung') taking the homotopy class of $f\colon S^n \to X$ to that of $\Sigma f\colon S^{n+1} \cong \Sigma S^n \to \Sigma X$.

Theorem 1.8 (Freudenthal suspension). *Let X be $(k-1)$-connected, with $k \geq 1$. The homomorphism $E\colon \pi_n(X) \to \pi_{n+1}(\Sigma X)$ is an isomorphism for $n < 2k-1$ and is surjective for $n = 2k-1$.*

This follows from homotopy excision for the cofibration $X \to CX$, with $CX/X \cong \Sigma X$.

Let $\pi_n^S(X) = \operatorname{colim}_i \pi_{n+i}(\Sigma^i X)$ be the n-th stable homotopy group of X. When X is $(k-1)$-connected the stabilization homomorphism $\pi_n(X) \to \pi_n^S(X)$ is an isomorphism for $n < 2k-1$ and surjective for $n = 2k-1$.

In the special case $X = S^0$ we call $\pi_n^S = \pi_n^S(S^0) = \operatorname{colim}_i \pi_{n+i}(S^i)$ the n-th stable stem. The homomorphism $\pi_{n+i}(S^i) \to \pi_n^S$ is surjective for $i = n+1$ and an isomorphism for $i > n+1$. In particular, $\pi_n^S = 0$ for $n < 0$, while $\pi_0^S \cong \mathbb{Z}$.

Corollary 1.9. *Let X be a CW complex of dimension d and Y a $(k-1)$-connected space. The suspension homomorphism $E\colon [X, Y] \to [\Sigma X, \Sigma Y]$ is bijective if $d < 2k-1$ and surjective if $d = 2k-1$.*

This follows from Freudenthal's theorem by induction over the cells of X.

Let $\{X, Y\} = \operatorname{colim}_i [\Sigma^i X, \Sigma^i Y]$ be the group of stable homotopy classes of maps $X \to Y$. When $[X, Y] \to \{X, Y\}$ is an isomorphism we say that X and Y are in the stable range. With notations as above, $\Sigma^i X$ is a CW complex of dimension $d+i$ and $\Sigma^i Y$ is $(k+i-1)$-connected, so $\Sigma^i X$ and $\Sigma^i Y$ are in the stable range if $(d+i) < 2(k+i)-1$, which holds for $i > d-2k+1$, i.e., for all sufficiently large i.

The homotopy category $\mathscr{F}$ of finite based CW complexes has morphism sets $\mathscr{F}(X, Y) = [X, Y]$. It maps to the stable homotopy category $\mathscr{F}[\Sigma^{-1}]$ of finite based CW complexes, with morphisms sets $\{X, Y\}$. The suspension induces a full and faithful functor from this category to itself, since $E\colon \{X, Y\} \to \{\Sigma X, \Sigma Y\}$ is always an isomorphism, but it is not an equivalence of categories, because not every object is isomorphic to a suspension. This can be arranged by formally adjoining desuspensions $\Sigma^{-n} X$ for all n, leading to the Spanier–Whitehead stable category $\mathscr{SW}$. However, this category does still not have (weak) colimits. This can be arranged by considering formal sequences of desuspensions

$$X_0 \to \cdots \to \Sigma^{-n} X_n \to \Sigma^{-n-1} X_{n+1} \to \dots ,$$

which is more commonly encoded by a sequence of spaces $\{n \mapsto X_n\}$ and structure maps $\Sigma X_n \to X_{n+1}$, leading to the notion of a (sequential) spectrum. Boardman's stable category $\mathscr{B}$ is the homotopy category of spectra, with morphism groups $\mathscr{B}(\mathbf{X}, \mathbf{Y}) = [\mathbf{X}, \mathbf{Y}]$ given by homotopy classes of maps between spectra $\mathbf{X}$ and $\mathbf{Y}$, and contains $\mathscr{SW}$ as a full subcategory. This stable category $\mathscr{B}$ has "better" formal properties than the unstable homotopy category $\mathscr{F}$. In particular it is a triangulated category, so that cofiber sequences and fiber sequences agree (up to a sign in the connecting maps), finite coproducts are isomorphic to finite products, etc.

Given a diagram in $\mathscr{F}$, we can view it as a diagram in $\mathscr{B}$ by applying the suspension spectrum functor, taking a based space X to the spectrum $\Sigma^\infty X = \{n \mapsto \Sigma^n X\}$ with identity maps as structure maps. We refer to the result as a stable diagram.

The sphere spectrum $\mathbf{S} = \Sigma^\infty S^0$ is the suspension spectrum on the 0-sphere. There is an n-sphere spectrum $\mathbf{S}^n$ for each integer n, having S^n as 0-th space if $n \geq 0$, and having S^0 as $(-n)$-th space if $n \leq 0$. The homotopy groups of a spectrum $\mathbf{X}$ are given by the stable morphism groups $\pi_n(\mathbf{X}) = [\mathbf{S}^n, \mathbf{X}]$, so that $\pi_n(\Sigma^\infty X) = \pi_n^S(X)$ for a space X.

Let $\mathbf{X}$ and $\mathbf{Y}$ be finite CW spectra. These sit in cofiber sequences $\mathbf{S}^{m-1} \to \mathbf{X}' \to \mathbf{X} \to \mathbf{S}^m$ and $\mathbf{S}^{n-1} \to \mathbf{Y}' \to \mathbf{Y} \to \mathbf{S}^n$ for smaller such spectra $\mathbf{X}'$ and $\mathbf{Y}'$. The stable morphism group $[\mathbf{X}, \mathbf{Y}]$ sits in an exact sequence

$$[\Sigma \mathbf{X}', \mathbf{Y}] \to [\mathbf{S}^m, \mathbf{Y}] \to [\mathbf{X}, \mathbf{Y}] \to [\mathbf{X}', \mathbf{Y}] \to [\mathbf{S}^{m-1}, \mathbf{Y}] ,$$

hence is in principle determined by the groups $[\mathbf{S}^m, \mathbf{Y}] = \pi_m(\mathbf{Y})$. These in turn sit in exact sequences

$$\pi_m(\mathbf{S}^{n-1}) \to \pi_m(\mathbf{Y}') \to \pi_m(\mathbf{Y}) \to \pi_m(\mathbf{S}^n) \to \pi_m(\Sigma \mathbf{Y}')$$

(since a stable cofiber sequence is a stable fiber sequence), hence are in principle determined by the groups $\pi_m(\mathbf{S}^n) \cong \pi_{m-n}^S$, i.e., the stable homotopy groups of spheres. Cells, or cones on spheres, are the basic building blocks for CW complexes, and in the stable category, stable maps between spheres are the basic building instructions for CW spectra. (This is less pronounced in the unstable category $\mathscr{F}$, since $\pi_m(Y)$ is not so directly determined by $\pi_m(Y')$ and $\pi_m(S^n)$.)

Whenever it is clear that we are working with stable diagrams, we shall omit the boldface notation for spectra and the Σ^∞ notation for suspension spectra.

1.5 Thom spectra

When $n \leq 2k-2$, the stabilization homomorphism $\pi_n(\mathbb{R}P^n_k) \to \pi^S_n(\mathbb{R}P^n_k)$ is an isomorphism, as is the homomorphism $\pi_n(S^n) \to \pi^S_n(S^n) \cong \pi^S_0$, so the question if $p_*\colon \pi_n(\mathbb{R}P^n_k) \to \pi_n(S^n)$ is surjective is equivalent to the stable question if $p_*\colon \pi^S_n(\mathbb{R}P^n_k) \to \pi^S_n(S^n)$ is surjective. In other words, does the pinch map $p\colon \mathbb{R}P^n_k \to S^n$ admit a stable section, so that the top cell on $\mathbb{R}P^n_k$ splits off? If so, we say that $\mathbb{R}P^n_k$ is stably coreducible.

This is equivalent to the question if the attaching map $\phi\colon S^{n-1} \to \mathbb{R}P^{n-1}_k$ is stably null-homotopic. In terms of the stable diagram

$$\begin{tikzcd} & S^{n-1} \arrow[dl, dashed] \arrow[d, "q"] \arrow[dr, "\phi"] & \\ \mathbb{R}P^{k-1} \arrow[r, tail] & \mathbb{R}P^{n-1} \arrow[r] & \mathbb{R}P^{n-1}_k \end{tikzcd}$$

(the lower row is a cofiber sequence, hence stably a fiber sequence) this is the question how far back the attaching map q of the top cell in $\mathbb{R}P^n$ pulls back. In other words, what is the minimal k such that $q\colon S^{n-1} \to \mathbb{R}P^{n-1}$ can be compressed into the $(k-1)$-skeleton, as a stable map?

Boardman's stable category admits function spectra, in the sense that given two spectra X and Y there is a natural function spectrum $F(X,Y)$ with suitable properties. For example, $\pi_n F(X,Y) = [\Sigma^n X, Y]$. Let $DX = F(X,S)$ be the functional dual of X. For example, $DS^n = S^{-n}$. The rule $X \to DX$ induces a contravariant endofunctor $D\colon \mathscr{B}^{op} \to \mathscr{B}$. There is a natural map $\rho\colon X \to DDX$, which is an equivalence if X is a finite CW spectrum, in which case we call DX the Spanier–Whitehead dual of X. When restricted to finite CW spectra, D is a contravariant equivalence of categories.

The question if the map $p\colon \mathbb{R}P^n_k \to S^n$ admits a stable section is thus equivalent to the question if the dual map $Dp\colon DS^n \to D(\mathbb{R}P^n_k)$ admits a stable retraction.

((Discuss Thom complexes and Thom spectra.))

Lemma 1.10. *There is a homeomorphism $\mathbb{R}P^{k+m}_k \cong Th(k\gamma^1_m)$ where γ^1_m is the tautological line bundle over $\mathbb{R}P^m$.*

Proof. The normal bundle of S^m in S^{k+m} is trivial, and covers the bundle $k\gamma^1_m$ over $\mathbb{R}P^m$. It embeds as the complement $S^{k+m} \setminus S^{k-1}$, and has one-point compactification S^{k+m}/S^{k-1}. Identifying antipodal points, the quotient space $\mathbb{R}P^{k+m}/\mathbb{R}P^{k-1} = \mathbb{R}P^{k+m}_k$ maps homeomorphically to $Th(k\gamma^1_m)$. □

Theorem 1.11 (Atiyah duality). *Let M be a closed manifold, with tangent bundle τ_M and virtual normal bundle $\nu_M = -\tau_M$. Then $D(M_+) \cong Th(\nu_M)$.*

Lemma 1.12. *$\tau_{\mathbb{R}P^m} \oplus \epsilon^1 \cong (m+1)\gamma^1_m$, so $\nu_{\mathbb{R}P^m} \cong \epsilon^1 - (m+1)\gamma^1_m$ and $D(\mathbb{R}P^{k+m}_k) \cong Th(\epsilon^1 - (k+m+1)\gamma^1_m) \cong \Sigma\mathbb{R}P^{-k-1}_{-k-m-1}$.*

The question of stable coreducibility of $\mathbb{R}P^n_k$ is thus equivalent to the question of stable reducibility of $Th(-(n+1)\gamma^1_m) \cong \mathbb{R}P^{-k-1}_{-n-1}$, i.e., whether the inclusion $i\colon S^{-n-1} \to \mathbb{R}P^{-k-1}_{-n-1}$ of the bottom cell admits a stable retraction up to homotopy.

If $(n+1)\gamma^1_m \cong \epsilon^{n+1}$ as vector bundles over $\mathbb{R}P^m$, or more generally, if the sphere bundle $S((n+1)\gamma^1_m)$ is fiber homotopy trivial over $\mathbb{R}P^m$, then $Th(-(n+1)\gamma^1_m) \simeq Th(-(\epsilon^{n+1})) \cong \Sigma^{-(n+1)}\mathbb{R}P^m_+$, and the bottom cell does indeed split off.

((Concerned with the additive order of $\epsilon^1 - \gamma^1_m$ in $\widetilde{KO}(\mathbb{R}P^m) \cong \mathbb{Z}/2^{\phi(m)}$, where $\phi(m) = \#\{1 \leq i \leq m \mid i \equiv 1,2,4,8 \mod 8\}$, or perhaps in the isomorphic image $JO(\mathbb{R}P^m)$. Computation with Atiyah–Hirzebruch spectral sequence. Adams conjecture?))

Theorem 1.13 (Adams). *$\mathbb{R}P^n_k$ is stably coreducible (if and) only if $n+1 \equiv 0 \mod 2^{\phi(m)}$, where $n = k+m$. The maximal m with this property is $8c + 2^d - 1$, when $n+1 = 2^a \cdot b$ and $a = 4c+d$, with b odd and $0 \leq d \leq 3$.*

By inspection, $n \geq 2m+2$ except for $n = 1, 3, 7, 15$, which is equivalent to the stability condition $n \leq 2k-2$. Hence $8c + 2^d - 1$ is also the maximal number of everywhere linearly independent vector fields on S^n. ((Separate check for $n = 15$, using Toda's work.))

2 Spectral sequences

2.1 Exhaustive complete Hausdorff filtrations

Consider a filtered space or spectrum X, i.e., a diagram

$$\dots \to X_{s-1} \xrightarrow{i} X_s \to \dots \to X$$

with $s \in \mathbb{Z}$. For example, we might have a map $f\colon X \to Y$ and $X_s = f^{-1}(Y^{(s)})$, where $Y^{(s)}$ is the s-skeleton of a CW complex Y. Applying homology we get a diagram

$$\begin{tikzcd} \dots \arrow[r] & H_*(X_{s-1}) \arrow[r, "i_*"] & H_*(X_s) \arrow[r] \arrow[d, "j_*"] & \dots \arrow[r] & H_*(X) \\ & & H_*(X_s, X_{s-1}) \arrow[ul, dashed, "\partial"] & & \end{tikzcd}$$

where ∂ has degree -1. We would like to use knowledge of the graded groups $H_*(X_s, X_{s-1})$ for all s to obtain knowledge of the graded group $H_*(X)$. There is an induced increasing filtration

$$\dots \subset F_{s-1} \subset F_s \subset \dots \subset H_*(X)$$

where $F_s = F_s H_*(X) = \operatorname{im}(H_*(X_s) \to H_*(X))$. There is a short exact sequence, or extension,

$$0 \to F_{s-1} \to F_s \to F_s/F_{s-1} \to 0$$

for each s. If we have inductively determined the subgroup $F_{s-1}H_*(X)$, and somehow know the quotient group F_s/F_{s-1}, then it is an algebraic extension problem to determine the total group F_s. For this to be useful in determining $H_*(X)$, we must at least assume that the filtration $\{F_s\}_s$ exhausts $H_*(X)$, i.e., that

$$H_*(X) = \operatorname*{colim}_s F_s = \bigcup_s F_s\,.$$

Furthermore, we apparently need to start the induction somewhere.

The reader who is unfamiliar with limits may prefer to assume that the filtration is bounded, in the sense that there is a natural number N such that $H_*(X_s) = 0$ for $s < -N$ and $H_*(X_s) = H_*(X)$ for $s \ge N$. Then F_s/F_{s-1} is only nonzero for $-N \le s \le N$. We can start the induction with $F_{-N-1} = 0$, and it stops after a finite number of steps at $F_N = H_*(X)$.

However, there is a refined approach to this that is a little better. Fix a filtration degree k, until further notice, and consider the problem of determining the quotients $H_*(X)/F_k$ in place of $H_*(X)$. There is an extension

$$0 \to F_{s-1}/F_k \to F_s/F_k \to F_s/F_{s-1} \to 0$$

for each $s > k$. We know that $F_{s-1}/F_k = 0$ for $s = k+1$, and this starts the induction. If we know $F_sH_*(X)/F_{s-1}$ for each $s > k$ and can resolve each extension problem, then we can determine F_s/F_k for each s, hence also

$$H_*(X)/F_k = \operatorname*{colim}_s F_s/F_k\,.$$

There is an exact sequence

$$0 \to \lim_k F_k \to H_*(X) \to \lim_k H_*(X)/F_k \to \operatorname*{Rlim}_k F_k \to 0\,,$$

where $\lim_k F_k = \bigcap_k F_k$ is the limit, and $\operatorname{Rlim}_k F_k$ is the right derived limit, also known as $\lim^1$, of the sequence

$$\dots \to F_{k-1} \to F_k \to \dots\,.$$

These graded groups are the kernel and cokernel, respectively, of the homomorphism

$$1 - i\colon \prod_k F_k \to \prod_k F_k$$

where 1 is the identity and i is the identification $\prod_k F_k = \prod_k F_{k-1}$ combined with the product of the homomorphisms $F_{k-1} \to F_k$. It is known that $\operatorname{Rlim}_k F_k = 0$ if each homomorphism $F_{k-1} \to F_k$ is surjective, or if each group F_k is finite. (The Mittag–Leffler condition also ensures the vanishing of Rlim.)

If $\lim_k F_k = 0$ we say that the filtration $\{F_s\}_s$ is Hausdorff. If $\operatorname{Rlim}_k F_k = 0$ we say that it is complete. The terminology can be justified by thinking of the filtration as a neighborhood basis around 0 and considering the associated linear topology on $H_*(X)$. If $\{F_s\}_s$ is both complete and Hausdorff, then

$$H_*(X) \cong \lim_k H_*(X)/F_k$$

and we can recover the abutment $H_*(X)$ from the quotients $H_*(X)/F_k$, as desired.

Lemma 2.1. *Let $\{F_s\}_s$ be an exhaustive complete Hausdorff filtration of $H_*(X)$. Then $H_*(X) \cong \lim_k \operatorname{colim}_s F_s/F_k$.*

2.2 Spectral sequences of homological type

Definition 2.2. A spectral sequence of homological type is a sequence of bigraded abelian groups $E^r_{*,*} = \{E^r_{s,t}\}_{s,t}$, differentials $d^r \colon E^r_{*,*} \to E^r_{*,*}$ of bidegree $(-r, r-1)$, and isomorphisms $E^{r+1}_{s,t} \cong H_{s,t}(E^r_{*,*}, d^r)$ for all $r \geq 1$. We call $E^r_{*,*}$ the E^r-term, d^r the d^r-differential, s the filtration degree and $s+t$ the total degree of the spectral sequence. Sometimes only the terms for $r \geq 2$ are specified.

Making the bigrading explicit, the components of the d^r-differential are homomorphisms $d^r_{s,t} \colon E^r_{s,t} \to E^r_{s-r,t_{\omega_n}+r-1}$. Note that the differential reduces the total degree by 1. The condition to be a differential is that $d^r \circ d^r = 0$, so that $\operatorname{im} d^r_{s+r,t-r+1} \subset \ker d^r_{s,t} \subset E^r_{s,t}$. The homology group $H_{s,t}(E^r_{*,*}, d^r)$ is the quotient group $\ker d^r_{s,t} / \operatorname{im} d^r_{s+r,t-r+1}$, which is required to be isomorphic to $E^{r+1}_{s,t}$. In this sense the E^r-term and the d^r-differential determine the E^{r+1}-term.

Fix a bidegree (s,t) and consider the sequence of groups $\{E^r_{s,t}\}$ for $r \geq 1$. If there is a natural number N such that $d^r_{s,t} = 0$ for all $r \geq N$, then there is a sequence of surjective homomorphisms $E^N_{s,t} \to \dots \to E^r_{s,t} \to \dots$ for $r \geq N$. We then let $E^\infty_{s,t} = \operatorname{colim}_r E^r_{s,t}$. ((On the other hand, if there is an integer N such that $d^r_{s+t,t-r+1} = 0$ for all $r \geq N$, then there is a sequence of injective homomorphisms $\dots \subset E^r_{s,t} \subset \dots \subset E^N_{s,t}$ for $r \geq N$. In that case we let $E^\infty_{s,t} = \lim_r E^r_{s,t}$.))

Definition 2.3. A spectral sequence $\{E^r_{*,*}, d^r\}_r$ converges strongly to a graded abelian group G_* if there is an exhaustive complete Hausdorff filtration $\dots \subset F_{s-1}G_* \subset F_sG_* \subset \dots$ of G_*, and isomorphisms

$$E^\infty_{s,t} \cong F_sG_{s+t}/F_{s-1}G_{s+t}$$

for all s and t. We call G_* the abutment of the spectral sequence.

If one can resolve the extension questions of how to recover F_sG_*/F_kG_* from $F_{s-1}G_*/F_kG_*$ and $E^\infty_{s,*}$, then strong convergence suffices to recover the abutment G_* as $\lim_k \operatorname{colim}_s F_sG_*/F_kG_*$.

Definition 2.4. If there is a natural number N such that $d^r = 0$ for all $r \geq N$ (in all bidegrees (s,t)), then there are isomorphisms $E^r_{*,*} \cong E^{r+1}_{*,*} \cong \dots \cong E^\infty_{*,*}$ for all $r \geq N$. In this case we say that the spectral sequence collapses at the E^N-term.

In many cases one can prove that a spectral sequence collapses at an E^N-term by an appeal to the internal grading t. One needs to check that for each bidegree (s,t) where $E^N_{s,t}$ is nonzero, all of the groups $E^N_{s-r,t+r-1}$ are zero for $r \geq N$. Since d^r has bidegree $(-r, r+1)$, this will imply that $d^r_{s,t} = 0$. In this case, we may say that the spectral sequence collapses at the E^N-term for bidegree reasons.

Definition 2.5. A morphism from a spectral sequence $\{E^r_{*,*}\}_r$ to a spectral sequence $\{'E^r_{*,*}\}_r$ is a sequence of bidegree-preserving homomorphisms

$$f^r \colon E^r_{*,*} \longrightarrow {}'E^r_{*,*}$$

such that the diagrams

$$\begin{tikzcd} E^r_{*,*} \arrow[r, "f^r"] \arrow[d, "d^r"'] & {}'E^r_{*,*} \arrow[d, "d^r"] \\ E^r_{*,*} \arrow[r, "f^r"] & {}'E^r_{*,*} \end{tikzcd}$$

and

$$\begin{tikzcd} H_{*,*}(E^r) \arrow[r, "f^r_*"] \arrow[d, "\cong"'] & H_{*,*}('E^r) \arrow[d, "\cong"] \\ E^{r+1}_{*,*} \arrow[r, "f^{r+1}"] & {}'E^{r+1}_{*,*} \end{tikzcd}$$

commute. In other words, f^r is a chain map from $(E^r_{*,*}, d^r)$ to $('E^r_{*,*}, d^r)$, and induces f^{r+1} on passage to homology.

A morphism $\{f^r\}_r$ of spectral sequences induces a homomorphism $f^\infty\colon E^\infty_{*,*} \to {}'E^\infty_{*,*}$ of E^∞-terms, when they are defined as discussed above.

Proposition 2.6. *Let $\{f^r\colon E^r_{*,*} \to {}'E^r_{*,*}\}_r$ be a morphism of spectral sequences. If there is a natural number N such that f^N is an isomorphism, then f^r is an isomorphism for all $r \geq N$, including $r = \infty$.*

Proof. If f^r is an isomorphism, then so is the homomorphism f^r_* induced on homology, so f^{r+1} is an isomorphism. Proceed by induction, starting at $r = N$. Pass to (co-)limits to get to $r = \infty$. □

Definition 2.7. A morphism $\{f^r\colon E^r_{*,*} \to {}'E^r_{*,*}\}_r$ of spectral sequences converges to a homomorphism $f\colon G_* \to G'_*$ if f restricts to homomorphisms $F_sG_* \to F_sG'_*$ for all s and the induced homomorphisms $F_sG_*/F_{s-1}G_* \to F_sG'_*/F_{s-1}G'_*$ agree with the homomorphisms $f^\infty\colon E^\infty_{s,*} \to {}'E^\infty_{s,*}$ under the isomorphisms $F_sG_*/F_{s-1}G_* \cong E^\infty_{s,*}$ and $F_sG'_*/F_{s-1}G'_* \cong {}'E^\infty_{s,*}$, for all s.

Proposition 2.8. *Let $\{f^r\colon E^r_{*,*} \to {}'E^r_{*,*}\}_r$ be a morphism of spectral sequences, converging strongly to a homomorphism $f\colon G_* \to G'_*$. If $f^\infty\colon E^\infty_{*,*} \to {}'E^\infty_{*,*}$ is an isomorphism, then so is $f\colon G_* \to G'_*$.*

Proof. We use the map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & F_{s-1}G_*/F_{s-r}G_* \arrow[r] \arrow[d] & F_sG_*/F_{s-r}G_* \arrow[r] \arrow[d] & F_sG_*/F_{s-1}G_* \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_{s-1}G'_*/F_{s-r}G'_* \arrow[r] & F_sG'_*/F_{s-r}G'_* \arrow[r] & F_sG'_*/F_{s-1}G'_* \arrow[r] & 0 \end{tikzcd}$$

to prove, by induction on r, that $F_sG_*/F_{s-r}G_* \to F_sG'_*/F_{s-r}G'_*$ is an isomorphism for all $r \geq 1$ and all s. Passing to limits over r, we get an isomorphism $F_sG_* \to F_sG'_*$ for all s. Passing to colimits over s we get the isomorphism $f\colon G_* \to G'_*$. □

2.3 Cycles and boundaries

Recall the diagram

$$\begin{tikzcd} \dots \arrow[r] & H_*(X_{s-1}) \arrow[r, "i_*"] & H_*(X_s) \arrow[r] \arrow[d, "j_*"] & \dots \arrow[r] & H_*(X) \\ & & H_*(X_s, X_{s-1}) \arrow[ul, dashed, "\partial"] & & \end{tikzcd}$$

where the triangle is a rolled-up long exact sequence. The homomorphism $H_*(X_s) \to H_*(X)$ induces an isomorphism

$$F_s = \operatorname{im}(H_*(X_s) \to H_*(X)) \cong \frac{H_*(X_s)}{\ker(H_*(X_s) \to H_*(X))}\,.$$

The image of i_* maps onto F_{s-1}, so there is a quotient isomorphism

$$F_s/F_{s-1} \cong \frac{H_*(X_s)}{\ker(H_*(X_s) \to H_*(X)) + \operatorname{im} i_*} .$$

The homomorphism j_* induces isomorphisms $H_*(X_s)/\operatorname{im} i_* \cong \operatorname{im} j_* \cong \ker \partial$, and there is a quotient isomorphism

$$\frac{H_*(X_s)}{\ker(H_*(X_s) \to H_*(X)) + \operatorname{im} i_*} \cong \frac{\ker \partial}{j_*(\ker(H_*(X_s) \to H_*(X)))} .$$

Lemma 2.9. *There is a natural isomorphism*

$$F_s/F_{s-1} \cong Z_s/B_s$$

where $Z_s = \ker \partial$, $B_s = j_*(\ker(H_*(X_s) \to H_*(X)))$, *and* $B_s \subset Z_s \subset H_*(X_s, X_{s-1})$.

The task of a spectral sequence is to start with the groups $H_*(X_s, X_{s-1})$ and to determine the cycle and boundary subgroups Z_s and B_s, or more precisely, the quotient groups $Z_s/B_s \cong F_s/F_{s-1}$. The starting groups will be the E^1-term, $E^1_{s,*} = H_*(X_s, X_{s-1})$, while the quotient groups will be the E^∞-term $Z_s/B_s = E^\infty_{s,*}$. The passage from E^1 to E^∞ can be done in steps, by weakening the condition that an element in $Z_s = \ker \partial$ must map to 0 under ∂, and strengthening the condition that an element in $\ker(H_*(X_s) \to H_*(X))$ goes to 0 in $H_*(X)$. The intermediate steps give the E^r-terms in the spectral sequence.

Regarding the cycles, we let $r \geq 1$ and consider the diagram:

$$\begin{tikzcd}
\dots \arrow[r, "i_*"] & H_*(X_{s-r}) \arrow[r, "i_*^{r-1}"] & H_*(X_{s-1}) \arrow[r, "i_*"] & H_*(X_s) \arrow[d, "j_*"] \\
& & & H_*(X_s, X_{s-1}) \arrow[ul, dashed, "\partial"]
\end{tikzcd}$$

Let

$$Z^r_s = \partial^{-1}(\operatorname{im} i_*^{r-1} \colon H_{*-1}(X_{s-r}) \to H_{*-1}(X_{s-1}))$$

be the r-th cycles in $H_*(X_s, X_{s-1})$. Then

$$Z_s = \ker \partial \subset Z^\infty_s \subset \dots \subset Z^r_s \subset \dots \subset Z^1_s = H_*(X_s, X_{s-1})$$

where $Z^\infty_s = \lim_r Z^r_s = \bigcap_r Z^r_s$ is the (graded abelian) group of infinite cycles.

There is a subtle point about limits and images here. If the intersection

$$\bigcap_r \operatorname{im} i_*^{r-1} \colon H_{*-1}(X_{s-r}) \to H_{*-1}(X_{s-1})$$

is zero, then $Z_s = Z^\infty_s$, so that we can obtain $Z_s = \ker \partial$ as the limit over r of the cycle groups Z^r_s. This is certainly the case if there is an integer N such that $H_{*-1}(X_s) = 0$ for $s < -N$, but it is not, in general, enough to assume that $\lim_s H_{*-1}(X_s) = 0$. We shall soon return to this in greater generality.

Regarding the boundaries, we let $r \geq 1$ and consider the diagram:

$$\begin{tikzcd}
H_*(X_{s-1}) \arrow[r, "i_*"] & H_*(X_s) \arrow[r, "i_*^{r-1}"] \arrow[d, "j_*"] & H_*(X_{s+r-1}) \arrow[r, "i_*"] & \dots \\
& H_*(X_s, X_{s-1}) \arrow[ul, dashed, "\partial"] & &
\end{tikzcd}$$

Let

$$B^r_s = j_*(\ker i_*^{r-1} \colon H_*(X_s) \to H_*(X_{s+r-1}))$$

be the r-th boundaries in $H_*(X_s, X_{s-1})$. Then

$$0 = B^1_s \subset \dots \subset B^r_s \subset \dots \subset B^\infty_s \subset B_s = j_*(\ker(H_*(X_s) \to H_*(X)))$$

where $B_s^\infty = \operatorname{colim}_r B_s^r = \bigcup_r B_s^r$ is the (graded abelian) group of infinite boundaries.

The interaction between colimits and kernels is less subtle. If the union

$$\bigcup_r \ker i_*^{r-1} \colon H_*(X_s) \to H_*(X_{s+r-1})$$

equals $\ker(H_*(X_s) \to H_*(X))$, then $B_s^\infty = B_s$, so that we can obtain $B_s = j_*(\ker(H_*(X_s) \to H_*(X)))$ as the colimit over r of the boundary groups B_s^r. In this case it suffices to assume that $\operatorname{colim}_s H_*(X_s) \cong H_*(X)$. This is a reasonable assumption, which also implies that the filtration $\{F_s\}_s$ of $H_*(X)$ is exhaustive.

We now have a doubly infinite filtration

$$0 = B_s^1 \subset \dots \subset B_s^r \subset \dots \subset B_s^\infty \subset B_s \subset Z_s \subset Z_s^\infty \subset \dots \subset Z_s^r \subset \dots \subset Z_s^1 = H_*(X_s, X_{s-1})$$

and in favorable cases (this is the subject of convergence), $B_s^\infty = B_s$ and $Z_s = Z_s^\infty$. We define the E^r-term

$$E_s^r = Z_s^r / B_s^r$$

to be given by the r-th cycles modulo the r-th boundaries, for $1 \le r \le \infty$. Then $E_s^1 \cong H_*(X_s, X_{s-1})$ and, assuming convergence, $E_s^\infty \cong F_s/F_{s-1}$. The wonderful algebraic fact is that there is a differential $d^r \colon E_s^r \to E_{s-r}^r$ of degree $(r-1)$ that makes the collection $\{E_s^r, d^r\}_r$ a spectral sequence, so that there are isomorphisms $H_s(E_*^r, d^r) \cong E_s^{r+1}$ for all finite $r \ge 1$.

Theorem 2.10. *Suppose that $H_*(X_s) = 0$ for $s < 0$ and that $\operatorname{colim}_s H_*(X_s) \cong H_*(X)$. Then there is a spectral sequence of homological type, with $E_{s,t}^1 = H_{s+t}(X_s, X_{s-1})$ and $d^1 \colon E_{s,t}^1 \to E_{s-1,t}^1$ given by the composite homomorphism*

$$H_{s+t}(X_s, X_{s-1}) \xrightarrow{\partial} H_{s+t-1}(X_{s-1}) \xrightarrow{j_*} H_{s+t-1}(X_{s-1}, X_{s-2}),$$

converging strongly to $H_*(X)$.

2.4 Unrolled exact couples

Following Massey and Boardman, we extract the essential algebraic structure from the discussion above.

Definition 2.11. An unrolled exact couple (of homological type) is a diagram

$$\begin{tikzcd}
\dots \arrow[r] & A_{s-2} \arrow[r, "i"] & A_{s-1} \arrow[r, "i"] \arrow[d, "j"] & A_s \arrow[r, "i"] \arrow[d, "j"] & A_{s+1} \arrow[r] \arrow[d, "j"] & \dots \\
& \dots & E_{s-1} \arrow[ul, "\partial"] & E_s \arrow[ul, "\partial"] & E_{s+1} \arrow[ul, "\partial"] & \dots
\end{tikzcd}$$

of graded abelian groups and homomorphisms, in which each triangle

$$\dots \longrightarrow A_{s-1} \xrightarrow{i} A_s \xrightarrow{j} E_s \xrightarrow{\partial} A_{s-1} \longrightarrow \dots$$

is a long exact sequence. Usually i and j will be of degree 0 and ∂ of degree -1.

For $r \ge 1$, let

$$Z_s^r = \partial^{-1}(\operatorname{im} i^{r-1} \colon A_{s-r} \to A_{s-1})$$

be the r-th cycle subgroup of E_s, let

$$B_s^r = j(\ker i^{r-1} \colon A_s \to A_{s+r-1})$$

be the r-th boundary subgroup of E_s, and let

$$E_s^r = Z_s^r / B_s^r$$

be the component of the E^r-term in filtration degree s. Let

$$d_s^r \colon E_s^r \longrightarrow E_{s-r}^r$$

be the r-th differential, given by $d_s^r([x]) = [j(y)]$, where $x \in Z_s^r$, $y \in A_{s-r}$ and $\partial(x) = i^{r-1}(y)$.

Proposition 2.12. *d^r is well-defined, $\ker d^r_s \cong Z^{r+1}_s/B^r_s$ and $\operatorname{im} d^r_{s+r} \cong B^{r+1}_s/B^r_s$, so $H_s(E^r_*, d^r) \cong E^{r+1}_s$. Hence $\{E^r, d^r\}_r$ is a spectral sequence of homological type.*

Proof. (Straightforward.) □

Definition 2.13. Let $G = \operatorname{colim}_s A_s$ be the direct limit. Let $F_s = \operatorname{im}(A_s \to G)$, so that there is an increasing, exhaustive filtration $\cdots \subset F_{s-1} \subset F_s \subset \cdots \subset G$.

Theorem 2.14 (Cartan–Eilenberg(?))**.** *Suppose that $A_s = 0$ for $s < 0$, so that $E^1_s = 0$ for $s < 0$, and all but finitely many differentials leaving any fixed bidegree are zero. Then $\{F_s\}_s$ is trivially a complete Hausdorff filtration, and there are isomorphisms $E^\infty_s \cong F_s/F_{s-1}$, so that the spectral sequence $\{E^r, d^r\}_r$ converges strongly to the colimit G.*

2.5 Spectral sequences of cohomological type

If we apply cohomology, in place of homology, to the filtered spectrum X, we get a diagram

$$\begin{tikzcd}
H^*(X) \arrow[r] & \dots \arrow[r] & H^*(X_s) \arrow[r, "i^*"] & H^*(X_{s-1}) \arrow[r] \arrow[dl, dashed, "\delta"] & \dots \\
 & & H^*(X_s, X_{s-1}) \arrow[u, "j^*"] & &
\end{tikzcd}$$

where δ has cohomological degree $+1$. This leads to an unrolled exact couple and a spectral sequence, where we may be able to recover $H^*(X)$ as the limit group $\lim_s H^*(X_s)$ under the assumption that $\operatorname{colim}_s H^*(X_s) = 0$.

We shall instead focus on spectral sequences that converge to the colimit groups. By passing to relative cohomology groups, we can transform the diagram above as follows:

$$\begin{tikzcd}
\dots \arrow[r] & H^*(X, X_s) \arrow[r, "j^*"] & H^*(X, X_{s-1}) \arrow[r] \arrow[d, "i^*"] & \dots \arrow[r] & H^*(X) \\
 & & H^*(X_s, X_{s-1}) \arrow[ul, dashed, "\delta"] & &
\end{tikzcd}$$

This leads to an unrolled exact couple and a spectral sequence, with $A_{-s} = H^*(X, X_{s-1})$ and $E_{-s} = H^*(X_s, X_{s-1})$, so that $i = j^*$, $j = i^*$ have degree 0 and $\partial = \delta$ has (cohomological) degree $+1$. Note that the E^1-term, given by the relative groups $H^*(X_s, X_{s-1})$, is the same as before. The sign change in the filtration grading is undesirable. We therefore convert to a cohomological indexing, by letting $A^s = A_{-s}$ and $E^s = E_{-s}$. In the example above we would then have $A^s = H^*(X, X_{s-1})$ and $E^s = H^*(X_s, X_{s-1})$.

If there is an integer N such that $H^*(X, X_s) = 0$ for $s > N$, or more subtle limiting conditions are satisfied (see the subsection on conditional convergence), then the associated spectral sequence will converge to $\operatorname{colim}_s H^*(X, X_s)$. If $\operatorname{colim}_s H^*(X_s) = 0$ then this is isomorphic to the desired abutment group $H^*(X)$.

We shall mostly be interested in filtered spectra where $X_s = Y$ for all $s \geq 0$, so that the E^1-term is concentrated in the region where $s \leq 0$. In this case is is also convenient to convert to a cohomological indexing, by letting $Y^s = X_{-s}$, so that we have a tower

$$\cdots \to Y^{s+1} \to Y^s \to \cdots \to Y^1 \to Y^0 = Y$$

of spectra. Let K^s be the mapping cone (homotopy cofiber) of the map $i\colon Y^{s+1} \to Y^s$, so that there is a cofiber sequence

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1}$$

for each $s \geq 0$. We may apply any generalized homology theory to this diagram, such as the (stable) homotopy groups of spectra. This leads to an unrolled exact couple

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_*(Y^{s+1}) \arrow[r, "i_*"] & \pi_*(Y^s) \arrow[r] \arrow[d, "j_*"] & \dots \arrow[r] & \pi_*(Y^1) \arrow[r, "i_*"] & \pi_*(Y^0) \arrow[r, equal] \arrow[d, "j_*"] & \pi_*(Y) \\
 & \dots & \pi_*(K^s) \arrow[ul, dashed, "\partial"] & \dots & & \pi_*(K^0) \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

where i_* and j_* have degree zero and ∂ has (homotopical) degree -1. We have $A^s = \pi_*(Y^s)$ and $E^s = \pi_*(K^s)$.

Definition 2.15. A spectral sequence of cohomological type is a sequence of bigraded abelian groups $E_r^{*,*} = \{E_r^{s,t}\}_{s,t}$, differentials $d_r \colon E_r^{*,*} \to E_r^{*,*}$ of bidegree $(r, -r+1)$, and isomorphisms $E_{r+1}^{s,t} \cong H^{s,t}(E_r^{*,*}, d_r)$ for all $r \geq 1$. We call $E_r^{*,*}$ the E_r-term, d_r the d_r-differential, s the filtration degree and $s+t$ the total degree of the spectral sequence.

Definition 2.16. A spectral sequence of Adams type is a sequence of bigraded abelian groups $E_r^{*,*} = \{E_r^{s,t}\}_{s,t}$, differentials $d_r \colon E_r^{*,*} \to E_r^{*,*}$ of bidegree $(r, r-1)$, and isomorphisms $E_{r+1}^{s,t} \cong H^{s,t}(E_r^{*,*}, d_r)$ for all $r \geq 1$. We call $E_r^{*,*}$ the E_r-term, d_r the d_r-differential, s the filtration degree and $t-s$ the total degree of the spectral sequence.

Definition 2.17. An unrolled exact couple (of cohomological type, resp. of Adams type) is a diagram

$$\begin{tikzcd}
\dots \arrow[r] & A^{s+2} \arrow[r, "i"] & A^{s+1} \arrow[r, "i"] \arrow[d, "j"] & A^s \arrow[r, "i"] \arrow[d, "j"] & A^{s-1} \arrow[r] \arrow[d, "j"] & \dots \\
& \dots & E^{s+1} \arrow[ul, "\partial"] & E^s \arrow[ul, "\partial"] & E^{s-1} \arrow[ul, "\partial"] & \dots
\end{tikzcd}$$

of graded abelian groups and homomorphisms, in which each triangle

$$\dots \longrightarrow A^{s+1} \xrightarrow{i} A^s \xrightarrow{j} E^s \xrightarrow{\partial} A^{s+1} \longrightarrow \dots$$

is a long exact sequence. The respective bidegrees of i, j and ∂ are $(-1,1)$, $(0,0)$ and $(1,0)$ in the cohomological case and $(-1,-1)$, $(0,0)$ and $(1,0)$ in the Adams case.

For $r \geq 1$ let

$$Z_r^s = \partial^{-1}(\operatorname{im} i^{r-1} \colon A^{s+r} \to A^{s+1})$$

be the r-th (co-)cycle subgroup of E^s, let

$$B_r^s = j(\ker i^{r-1} \colon A^s \to A^{s-r+1})$$

be the r-th (co-)boundary subgroup, and let

$$E_r^s = Z_r^s / B_r^s$$

be the filtration degree s component of the E_r-term. Note that $Z_1^s = E^s$ and $B_1^s = 0$ so $E_1^s = E^s$. Let

$$d_r^s \colon E_r^s \longrightarrow E_r^{s+r}$$

be the r-th differential, satisfying $d_r^s([x]) = [j(y)]$, where $x \in Z_r^s$, $y \in A^{s+r}$ and $\partial(x) = i^{r-1}(y)$. Then d_r has bidegree $(r, -r+1)$ in the cohomological case and bidegree $(r, r-1)$ in the Adams case.

Proposition 2.18. *d_r is well-defined, $\ker d_r^s \cong Z_{r+1}^s / B_r^s$ and $\operatorname{im} d_r^s = B_{r+1}^s / B_r^s$, so $H^s(E_r^*, d_r) \cong E_{r+1}^s$. Hence $\{E_r, d_r\}_r$ is a spectral sequence of cohomological type, resp. of Adams type.*

Proposition 2.19. *Consider a tower of spectra*

$$\begin{tikzcd}
\dots \arrow[r] & Y^{s+1} \arrow[r, "i"] & Y^s \arrow[r] \arrow[d, "j"] & \dots \arrow[r] & Y^1 \arrow[r, "i"] & Y^0 \arrow[r, equal] \arrow[d, "j"] & Y \\
\dots & & K^s \arrow[ul, dashed, "\partial"] & \dots & & K^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

where K^s is the mapping cone of $i \colon Y^{s+1} \to Y^s$, and $\partial \colon K^s \to \Sigma Y^{s+1}$ is the cofiber map. Applying homotopy one obtains an unrolled exact couple of Adams type, giving rise to a spectral sequence of Adams type with E_1-term

$$E_1^{s,t} = \pi_{t-s}(K^s)$$

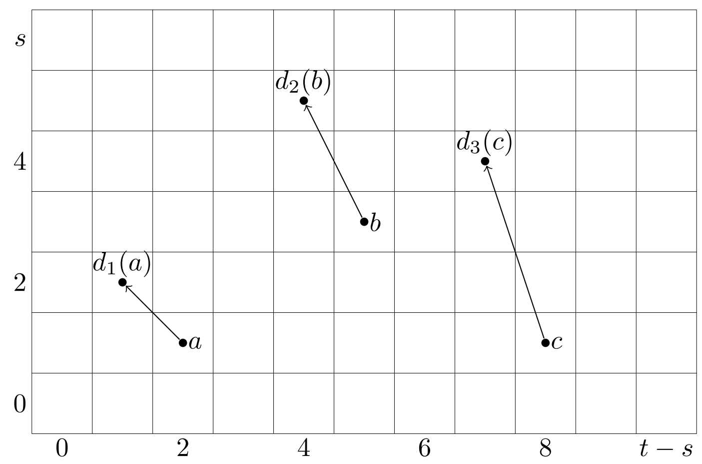

Figure 1: Adams type differentials

for $s \geq 0$, and d^1-differential $d_1^{s,t} \colon E_1^{s,t} \to E_1^{s+1,t}$ given by the composite

$$\pi_{t-s}(K^s) \xrightarrow{\partial} \pi_{t-s-1}(Y^{s+1}) \xrightarrow{j_*} \pi_{t-s-1}(K^{s+1}) \,.$$

If the images $F^s = \operatorname{im}(\pi_*(Y^s) \to \pi_*(Y))$ define a complete Hausdorff filtration of $\operatorname{colim}_s \pi_*(Y_s) = \pi_*(Y)$, *meaning that* $\lim_s F^s = 0$ *and* $\operatorname{Rlim}_s F^s = 0$*, and there are isomorphisms $E_\infty^s \cong F^s/F^{s+1}$ for all $s \geq 0$, then the spectral sequence converges strongly to $\pi_*(Y)$.*

For spectral sequences of Adams type, it is traditional to display the E_r-terms in a coordinate system with the total degree $t-s$ on the horizontal axis, and the filtration degree s on the vertical axis, thus using $(t-s, s)$-coordinates, rather than (s,t)-coordinates. The d_r-differentials change $(t-s, s)$ by $(-1, r)$, mapping one unit to the left and r units upwards.

The groups $E_\infty^{s,t} = E_\infty^{s,s+n}$ contributing to the homotopy group $\pi_n(Y)$ in the abutment are precisely those that sit in the column $t-s=n$, for each integer n.

2.6 Conditional convergence

Following Boardman, we address the issue of convergence for spectral sequences of cohomological type, or of Adams type. For simplicity, we concentrate on the case when $E_1^s = 0$ for $s < 0$, so that all but finitely many differentials entering any fixed bidegree are zero.

Definition 2.20. Consider an unrolled exact couple (of cohomological type, or Adams type)

$$\begin{tikzcd} \dots \arrow[r] & A^{s+1} \arrow[r, "i"] & A^s \arrow[r] \arrow[d, "j"] \arrow[l, phantom] & \dots \arrow[r] & A^1 \arrow[r, "i"] & A^0 \arrow[r, equal] \arrow[d, "j"] & G \\ \dots & & E^s \arrow[ul, "\partial"] & \dots & & E^0 \arrow[ul, "\partial"] & \end{tikzcd}$$

with $A^0 = A^s = G$ and $E^s = 0$ for all $s < 0$. We say that the resulting spectral sequence converges conditionally (to $G = \operatorname{colim}_s A_s$) if $\lim_s A^s = 0$ and $\operatorname{Rlim}_s A^s = 0$. Note that conditional convergence is a condition on the groups A^s in the unrolled exact couple, not on the filtration groups $F^s = \operatorname{im}(A^s \to G)$.

Definition 2.21. Let $Z_\infty^s = \lim_r Z_r^s = \bigcap_r Z_r^s$ be the infinite cycles in E^s, let $B_\infty^s = \operatorname{colim}_r B_r^s = \bigcup_r B_r^s$ be the infinite boundaries, and let $E_\infty^s = Z_\infty^s / B_\infty^s$ be the filtration s component of the E_∞-term.

As in the homological case we have inclusions $Z^s = \ker \partial \subset Z_\infty^s$ and $B_\infty^s \subset B^s = j_*(\ker(A^s \to G))$. We also have isomorphisms $F^s/F^{s+1} \cong Z^s/B^s$. We have assumed that $E^s = 0$ for $s < 0$, so $B_r^s = B_\infty^s = B^s$ for all $r > s$. To establish strong convergence, we therefore need to know that $Z^s = Z_\infty^s$ and that $\{F^s\}_s$ is a complete Hausdorff filtration. The E_∞-term is the limit of the sequence of inclusions

$$E_\infty^s = \lim_r E_r^s \subset \dots \subset E_{r+1}^s \subset E_r^s \subset \dots$$

where $r > s$. The following derived limit group measures the difference between conditional convergence and strong convergence.

Definition 2.22. Let $RE^s_\infty = \mathrm{Rlim}_r E^s_r$ be the derived E_∞-term.

Lemma 2.23. *If there is a natural number N such that $E^*_N = E^*_\infty$ (the spectral sequence collapses at the E_N-term), or such that $E^{s,t}_N$ is finite in each bidegree (s,t), then $RE_\infty = 0$.*

Consider an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & A^{s+1} \arrow[r, "i"] & A^s \arrow[r] \arrow[d, "j"] & \dots \arrow[r] & A^1 \arrow[r, "i"] & A^0 \arrow[r, equal] \arrow[d, "j"] & G \\
\dots & & E^s \arrow[ul, "\partial"] & \dots & & E^0 \arrow[ul, "\partial"] &
\end{tikzcd}$$

Theorem 2.24 (Boardman). *Suppose that (a) $A^0 = A^s$ for $s < 0$, so that $E^s = 0$ for $s < 0$ and all but finitely many differentials entering any fixed bidegree are zero, (b) The spectral sequence is conditionally convergent, so that $\lim_s A^s = 0$ and $\mathrm{Rlim}_s A^s = 0$, and (c) $RE_\infty = 0$. Then the spectral sequence converges strongly to $A^0 = G$. In other words, the subgroups $F^s = \mathrm{im}(A^s \to G)$ form an exhaustive complete Hausdorff filtration of G, and there are isomorphisms $F^s/F^{s+1} \cong E^s_\infty$.*

This is part of Boardman's Theorem 7.3, which builds on his Lemmas 5.6 and 5.9. We omit the proof.

Consider a tower of spectra

$$\begin{tikzcd}
\dots \arrow[r] & Y^{s+1} \arrow[r, "i"] & Y^s \arrow[r] \arrow[d, "j"] & \dots \arrow[r] & Y^1 \arrow[r, "i"] & Y^0 \arrow[r, equal] \arrow[d, "j"] & Y \\
\dots & & K^s \arrow[ul, dashed, "\partial"] & \dots & & K^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

where K^s is the mapping cone of $i\colon Y^{s+1} \to Y^s$, and $\partial\colon K^s \to \Sigma Y^{s+1}$ is the cofiber map.

Definition 2.25. The homotopy limit of the tower Y^s is the homotopy fiber

$$\operatorname*{holim}_s Y^s \longrightarrow \prod_s Y^s \xrightarrow{1-i} \prod_s Y^s$$

where 1 is the identity map and i is the composite of the identification $\prod_s Y^s \cong \prod_s Y^{s+1}$ and the product of the maps $i\colon Y^{s+1} \to Y^s$.

Proposition 2.26 (Milnor). *There is a short exact sequence*

$$0 \to \operatorname*{Rlim}_s \pi_{n+1}(Y^s) \longrightarrow \pi_n(\operatorname*{holim}_s Y^s) \longrightarrow \lim_s \pi_n(Y^s) \to 0$$

for each integer n.

Consider the unrolled exact couple with $A^s = \pi_*(Y^s)$ and $E^s = \pi_*(K^s)$ associated to a tower of spectra as above. The following two conditions ensure strong convergence to $\pi_*(Y)$.

Corollary 2.27. *The associated spectral sequence is conditionally convergent if and only if* $\mathrm{holim}_s Y^s$ *is contractible. If $\pi_n(K^s)$ is a finite group, for each s and n, then $RE_\infty = 0$. If both conditions hold then the spectral sequence is strongly convergent.*

Proof. Conditional convergence means that $A^\infty = \lim_s \pi_*(Y^s)$ and $RA^\infty = \mathrm{Rlim}_s \pi_*(Y^s)$ both vanish. By Milnor's lim-Rlim sequence this is equivalent to the vanishing of $\pi_*(\mathrm{holim}_s Y^s)$. We have $E^s = \pi_*(K^s)$, so if each $\pi_n(K^s)$ is finite then E_1 is finite in each bidegree, which implies that $RE_\infty = 0$. $\square$

3 The Steenrod algebra

3.1 Steenrod operations

We start at the prime $p = 2$. For brevity, we write $H_*(X)$ for $H_*(X; \mathbb{F}_2)$ and $H^*(X)$ for $H^*(X; \mathbb{F}_2)$.

Theorem 3.1 (Steenrod, Cartan)**.** *(a) For each pair of integers $i, n \geq 0$ there is a natural transformation $Sq^i \colon \tilde{H}^n(X) \longrightarrow \tilde{H}^{n+i}(X)$ of functors from based spaces to abelian groups.*

(b) $Sq^0 = 1$ is the identity.

(c) If $n = |x|$ then $Sq^n(x) = x^2$ is the cup square.

(d) If $i > |x|$ then $Sq^i(x) = 0$.

(e) (Cartan formula) $Sq^k(xy) = \sum_{i=0}^{k} Sq^i(x) Sq^{k-i}(y)$.

We call Sq^i the i-th Steenrod (reduced) squaring operation. Naturality means that for each base-point preserving map $f \colon X \to Y$ we have $f^* Sq^i(x) = Sq^i(f^* x)$, and Sq^i is a homomorphism. The Cartan formula can be rewritten as $Sq^k(xy) = \sum_{i+j=k} Sq^i(x) Sq^j(y)$, with the convention that $Sq^i(x) = 0$ for $i < 0$, or in terms of the smash product $\wedge \colon \tilde{H}^n(X) \otimes \tilde{H}^m(Y) \to \tilde{H}^{n+m}(X \wedge Y)$ as $Sq^k(x \wedge y) = \sum_{i+j=k} Sq^i(x) \wedge Sq^j(y)$.

The properties in the theorem can be taken as axioms, and imply the following results. Recall that the Bockstein homomorphism of the coefficient sequence $\mathbb{F}_2 \to \mathbb{Z}/4 \to \mathbb{F}_2$ is the connecting homomorphism $\beta \colon \tilde{H}^n(X) \to \tilde{H}^{n+1}(X)$ in the long exact sequence associated to the short exact sequence $0 \to C^*(X; \mathbb{F}_2) \to C^*(X; \mathbb{Z}/4) \to C^*(X; \mathbb{F}_2) \to 0$ of cochain complexes. Let $\Sigma \colon \tilde{H}^n(X) \to \tilde{H}^{n+1}(X)$ be the suspension isomorphism.

Theorem 3.2. *(a) $Sq^1 = \beta$ is the Bockstein homomorphism.*

(b) (Adem relations) If $a < 2b$ then

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j \,.$$

(c) $Sq^i(\Sigma x) = \Sigma Sq^i(x)$.

With the convention that $\binom{n}{k} = 0$ for $k < 0$, the summation limits $j \geq 0$ and $j \leq [a/2]$ can be ignored. Notice that $Sq^1 Sq^b = Sq^{b+1}$ for b even, and $Sq^1 Sq^b = 0$ for b odd. Note also that $Sq^{2b-1} Sq^b = 0$ for all *b.* The Adem relations in degrees ≤ 11 are:

$$
\begin{array}{ll}
Sq^1Sq^1 = 0 & Sq^1Sq^8 = Sq^9 \\
Sq^1Sq^2 = Sq^3 & Sq^2Sq^7 = Sq^9 + Sq^8Sq^1 \\
Sq^1Sq^3 = 0 & Sq^3Sq^6 = 0 \\
Sq^2Sq^2 = Sq^3Sq^1 & Sq^4Sq^5 = Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 \\
Sq^1Sq^4 = Sq^5 & Sq^5Sq^4 = Sq^7Sq^2 \\
Sq^2Sq^3 = Sq^5 + Sq^4Sq^1 & Sq^1Sq^9 = 0 \\
Sq^3Sq^2 = 0 & Sq^2Sq^8 = Sq^{10} + Sq^9Sq^1 \\
Sq^1Sq^5 = 0 & Sq^3Sq^7 = Sq^9Sq^1 \\
Sq^2Sq^4 = Sq^6 + Sq^5Sq^1 & Sq^4Sq^6 = Sq^{10} + Sq^8Sq^2 \\
Sq^3Sq^3 = Sq^5Sq^1 & Sq^5Sq^5 = Sq^9Sq^1 \\
Sq^1Sq^6 = Sq^7 & Sq^6Sq^4 = Sq^7Sq^3 \\
Sq^2Sq^5 = Sq^6Sq^1 & Sq^1Sq^{10} = Sq^{11} \\
Sq^3Sq^4 = Sq^7 & Sq^2Sq^9 = Sq^{10}Sq^1 \\
Sq^4Sq^3 = Sq^5Sq^2 & Sq^3Sq^8 = Sq^{11} \\
Sq^1Sq^7 = 0 & Sq^4Sq^7 = Sq^{11} + Sq^9Sq^2 \\
Sq^2Sq^6 = Sq^7Sq^1 & Sq^5Sq^6 = Sq^{11} + Sq^9Sq^2 \\
Sq^3Sq^5 = Sq^7Sq^1 & Sq^6Sq^5 = Sq^9Sq^2 + Sq^8Sq^3 \\
Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2 & Sq^7Sq^4 = 0 \\
Sq^5Sq^3 = 0 &
\end{array}
$$

To prove (a) one considers the case $X = \mathbb{R}P^2$. To prove (b) one considers $X = (\mathbb{R}P^\infty)^r$ for large r, as we will outline below. To prove (c) one uses the smash product form of the Cartan formula for $Y = S^1$.

Now let $p > 2$ be an odd prime.

Theorem 3.3 (Steenrod, Cartan)**.** *(a) For each pair of integers $i, n \geq 0$ there is a natural transformation $P^i\colon \tilde{H}^n(X;\mathbb{F}_p) \longrightarrow \tilde{H}^{n+2i(p-1)}(X;\mathbb{F}_p)$ of functors from based spaces to abelian groups.*

(b) $P^0 = 1$ is the identity.

(c) If $2k = |x|$ then $P^k(x) = x^p$ is the cup p-th power.

(d) If $2k > |x|$ then $P^k(x) = 0$.

(e) (Cartan formula) $P^k(xy) = \sum_{i=0}^k P^i(x)P^{k-i}(y)$.

Let $\beta\colon \tilde{H}^n(X;\mathbb{F}_p) \to \tilde{H}^{n+1}(X;\mathbb{F}_p)$ be the Bockstein homomorphism associated to the coefficient sequence $\mathbb{F}_p \to \mathbb{Z}/p^2 \to \mathbb{F}_p$.

Theorem 3.4. *(a) (Adem relations) If $a < pb$ then*

$$P^aP^b = \sum_{j=0}^{[a/p]} (-1)^{a+j} \binom{(p-1)(b-j)-1}{a-pj} P^{a+b-j}P^j \,.$$

(b) If $a \leq pb$ then

$$P^a\beta P^b = \sum_{j=0}^{[a/p]}(-1)^{a+j}\binom{(p-1)(b-j)}{a-pj}\beta P^{a+b-j}P^j$$
$$- \sum_{j=0}^{[(a-1)/p]}(-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj-1}P^{a+b-j}\beta P^j$$

(c) $P^i(\Sigma x) = \Sigma P^i(x)$ and $\beta(\Sigma x) = -\Sigma\beta(x)$.

The first few p-primary Adem relations (for $b = 1$) are

$$P^aP^1 = (-1)^a\binom{p-2}{a}P^{a+1}$$
$$P^a\beta P^1 = (-1)^a\binom{p-1}{a}\beta P^{a+1} - (-1)^a\binom{p-2}{a-1}P^{a+1}\beta$$

for $0 < a < p$, which imply that $(P^1)^p = 0$, and $P^p\beta P^1 = \beta P^pP^1$.

3.2 Construction of the reduced squares

We follow Steenrod–Epstein, Chapter VII and Hatcher, Section 4.L.

Definition 3.5. Let $H_n = K(\mathbb{F}_2, n)$ be an Eilenberg–Mac Lane complex of type $(\mathbb{F}_2, n)$, i.e., a space with $\pi_i(H_n) = 0$ for $i \neq n$ and $\pi_n(H_n) \cong \mathbb{F}_2$. Such spaces exist, and are uniquely determined up to weak homotopy equivalence. There is a universal class $\iota_n \in \tilde{H}^n(H_n)$ that corresponds to the identity homomorphism $\mathbb{F}_2 \to \mathbb{F}_2$ under the isomorphisms $H^n(H_n) \cong \mathrm{Hom}(H_n(H_n), \mathbb{F}_2) \cong \mathrm{Hom}(\pi_n(H_n), \mathbb{F}_2) \cong \mathrm{Hom}(\mathbb{F}_2, \mathbb{F}_2)$.

Note that $H_1 \simeq \mathbb{R}P^\infty$.

Theorem 3.6 (Eilenberg–Mac Lane)**.** *There is a natural isomorphism $[X, H_n] \cong \tilde{H}^n(X)$ taking the homotopy class of a base-point preserving map $f\colon X \to H_n$ to the image $f^*(\iota_n)$ of the universal class.*

See Hatcher (2002) Theorem 4.57.

The smash product $\iota_n \wedge \iota_n \in \tilde{H}^{2n}(H_n \wedge H_n)$ is represented by a map $\phi\colon H_n \wedge H_n \to H_{2n}$. By homotopy commutativity, there is a homotopy $I_+ \wedge H_n \wedge H_n \to H_{2n}$ from ϕ to $\phi\gamma$, where $\gamma\colon H_n \wedge H_n \to H_n \wedge H_n$ is the twist map. Thinking of the interval I as the upper half of a circle S^1, this homotopy can be thought of as a C_2-equivariant map $S^1_+ \wedge H_n \wedge H_n \to H_{2n}$ where $C_2 = \{\pm 1\}$ acts antipodally on S^1 and by the twist on $H_n \wedge H_n$. Equivalently, it corresponds to a map $\phi_1\colon S^1_+ \wedge_{C_2} H_n \wedge H_n \to H_{2n}$. This map ϕ_1 extends (uniquely, up to homotopy) to a map

$$\Phi\colon S^\infty_+ \wedge_{C_2} H_n \wedge H_n \to H_{2n}\,,$$

where S^∞ has the antipodal action. We call $S^\infty_+ \wedge_{C_2} H_n \wedge H_n$ the quadratic construction on H_n.

There is a diagonal map $\Delta\colon H_n \to H_n \wedge H_n$, and an induced map

$$\nabla = 1 \wedge \Delta\colon \mathbb{R}P^\infty_+ \wedge H_n \to S^\infty_+ \wedge_{C_2} H_n \wedge H_n\,,$$

where $\mathbb{R}P^\infty = S^n/C_2$. The composite map $\Phi\nabla\colon \mathbb{R}P^\infty_+ \wedge H_n \longrightarrow H_{2n}$ induces a map $(\Phi\nabla)^*$ in cohomology, taking the universal class ι_{2n} to an element in degree $2n$ of $\tilde{H}^*(\mathbb{R}P^\infty_+ \wedge H_n) \cong H^*(\mathbb{R}P^\infty) \otimes \tilde{H}^*(H_n)$. Writing $H^*(\mathbb{R}P^\infty) = P(u) = \mathbb{F}_2[u]$ with $|u| = 1$, we can write $(\Phi\nabla)^*(\iota_{2n})$ as a sum of terms

$$(\Phi\nabla)^*(\iota_{2n}) = \sum_{i=0}^{n} u^{n-i} \otimes Sq^i(\iota_n)$$

where $Sq^i(\iota_n) \in \tilde{H}^{n+i}(H_n)$. More generally, for any class $x \in \tilde{H}^n(X)$ represented by a map $f\colon X \to H_n$ we have a commutative diagram

$$\begin{tikzcd}
\mathbb{R}P^\infty_+ \wedge X \arrow[r, "\nabla"] \arrow[d, "1\wedge f"'] & S^\infty_+ \wedge_{C_2} X \wedge X \arrow[d, "1\wedge f\wedge f"] & \\
\mathbb{R}P^\infty_+ \wedge H_n \arrow[r, "\nabla"] \arrow[rr, bend right, "\Phi\nabla"'] & S^\infty_+ \wedge_{C_2} H_n \wedge H_n \arrow[r, "\Phi"] & H_{2n}
\end{tikzcd}$$

In terms of the isomorphism $\tilde{H}^*(\mathbb{R}P^\infty \wedge X) \cong P(u) \otimes \tilde{H}^*(X)$ we can define classes $Sq^i(x) \in \tilde{H}^{n+i}(X)$ by the formula

$$(1 \wedge f)^*(\Phi\nabla)^*(\iota_{2n}) = \sum_i u^{n-i} \otimes Sq^i(x)\,.$$

It is then clear that $f^*Sq^i(\iota_n) = Sq^i(x)$, and naturality follows easily. The restriction of $\Phi\nabla$ to $H_n \cong \mathbb{R}P^0_+ \wedge H_n$ is the diagonal $\Delta\colon H_n \to H_n \wedge H_n$ followed by $\phi\colon H_n \wedge H_n \to H_{2n}$, taking ι_{2n} to ι_n^2, hence $Sq^n(x) = x^2$.

For the Cartan formula, consider the map $\mu\colon H_n \wedge H_m \to H_{n+m}$ representing the smash product $\iota_n \wedge \iota_m$. There is a commutative diagram

$$\begin{tikzcd}
\mathbb{R}P^\infty_+ \wedge H_{n+m} \arrow[r, "\nabla"] & S^\infty_+ \wedge_{C_2} H_{n+m} \wedge H_{n+m} \arrow[r, "\Phi"] & H_{2(n+m)} \\
\mathbb{R}P^\infty_+ \wedge H_n \wedge H_m \arrow[u, "1\wedge\mu"] \arrow[r, "\nabla"] \arrow[d, "\Delta\wedge 1"'] & S^\infty_+ \wedge_{C_2} H_n \wedge H_m \wedge H_n \wedge H_m \arrow[u, "1\wedge\mu\wedge\mu"'] \arrow[d, "\pi"] & \\
\mathbb{R}P^\infty_+ \wedge \mathbb{R}P^\infty_+ \wedge H_n \wedge H_m \arrow[r, "\nabla\wedge\nabla"] & S^\infty_+ \wedge_{C_2} H_n \wedge H_n \wedge S^\infty_+ \wedge_{C_2} H_m \wedge H_m \arrow[r, "\Phi\wedge\Phi"] & H_{2n} \wedge H_{2m} \arrow[uu, "\mu"']
\end{tikzcd}$$

where π is induced by the $(C_2 \to C_2 \times C_2)$-equivariant diagonal embedding $S^\infty_+ \to S^\infty_+ \wedge S^\infty_+$. The right hand rectangle commutes by a check in $H^{2(n+m)}(-)$ of the central term. Granted this, the class $\iota_{2(n+m)}$ at the upper right pulls back to $\iota_{2n} \otimes \iota_{2m}$ at the lower right, and across to $\sum_{i,j} u^{n-i} \otimes u^{m-j} \otimes Sq^i(\iota_n) \otimes Sq^j(\iota_m)$ at the lower left. Pulling up the center left term we obtain $\sum_{i,j} u^{n+m-i-j} \otimes Sq^i(\iota_n) \otimes Sq^j(\iota_m)$. Going the other way around the diagram, we first come to $\sum_k u^{n+m-k} \otimes Sq^k(\iota_{n+m})$, and then to $\sum_k u^{n+m-k} Sq^k(\iota_n \wedge \iota_m)$. Comparing the coefficients of u^{n+m-k} we get $Sq^k(\iota_n \wedge \iota_m) = \sum_{i+j=k} Sq^i(\iota_n) \wedge Sq^j(\iota_m)$. This implies $Sq^k(x \wedge y) = \sum_{i+j=k} Sq^i(x) \wedge Sq^j(y)$ and the Cartan formula by naturality.

The fact that $Sq^0(x) = x$ can be deduced from the case $X = S^1$.

3.3 Admissible monomials

Again, we start with $p = 2$. For $x \in \tilde{H}^*(X)$ let $Sq(x) = \sum_i Sq^i(x)$ be the total squaring operation. Then $Sq(xy) = Sq(x)Sq(y)$ by the Cartan formula.

Lemma 3.7. *The Steenrod operations in $\tilde{H}^*(\mathbb{R}P^\infty_+) = H^*(\mathbb{R}P^\infty) \cong P(x)$, with $|x| = 1$, are given by $Sq^i(x^n) = \binom{n}{i} x^{n+i}$.*

Proof. $Sq(x) = x + x^2 = x(1 + x)$ since $Sq^0(x) = x$ and $Sq^1(x) = x^2$. Hence $Sq(x^n) = Sq(x)^n = x^n(1 + x)^n$. Thus $Sq^i(x^n) = \binom{n}{i} x^{n+i}$ in degree $n + i$. □

Let $(\mathbb{R}P^\infty)^r = \mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty$ be the product of $r \geq 1$ copies of $\mathbb{R}P^\infty$, so that $(\mathbb{R}P^\infty)^r_+ = \mathbb{R}P^\infty_+ \wedge \cdots \wedge \mathbb{R}P^\infty_+$. Then $\tilde{H}^*((\mathbb{R}P^\infty)^r_+) = H^*((\mathbb{R}P^\infty)^r) \cong P(x_1, \dots, x_r)$ with $|x_1| = \cdots = |x_r| = 1$. The Cartan formula implies:

Lemma 3.8. *The Steenrod operations in $H^*((\mathbb{R}P^\infty)^r) = P(x_1, \dots, x_r)$ are given by*

$$Sq^k(x_1^{n_1} \cdots x_r^{n_r}) = \sum_{i_1+\cdots+i_r=k} \binom{n_1}{i_1} \cdots \binom{n_r}{i_r} x_1^{n_1+i_1} \cdots x_r^{n_r+i_r}\,.$$

Using this, it is matter of algebra to check that the Adem relations hold for the Steenrod squares in $P(x_1, \dots, x_r)$, in the sense that for $a < 2b$ the action of $Sq^a \circ Sq^b$ equals the sum over j of the actions of $\binom{b-1-j}{a-2j} Sq^{a+b-j} \circ Sq^j$.

Definition 3.9. Let the mod 2 Steenrod algebra, $\mathscr{A} = \mathscr{A}(2)$, be the graded, unital, associative $\mathbb{F}_2$-algebra generated by the symbols Sq^i for $i \geq 0$, subject to the relation $Sq^0 = 1$ and the Adem relations $Sq^a Sq^b = \sum_j \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$ for all $a < 2b$.

For any based space X, the reduced cohomology $\tilde{H}^*(X)$ is naturally a left module over the Steenrod algebra, i.e., an $\mathscr{A}$-module, with $Sq^I(x) = Sq^{i_1}(\dots Sq^{i_\ell}(x)\dots)$. We write

$$\lambda \colon \mathscr{A} \otimes \tilde{H}^*(X) \longrightarrow \tilde{H}^*(X)$$

for the left module action map.

Definition 3.10. For each sequence $I = (i_1, \dots, i_\ell)$ of non-negative integers, with $\ell \geq 0$, let $Sq^I = Sq^{i_1} \dots Sq^{i_\ell}$ be the product in $\mathscr{A} = \mathscr{A}(2)$. We say that I has length ℓ and degree $i_1 + \dots + i_\ell$. We say that I (or Sq^I) is admissible if $i_s \geq 2i_{s+1}$ for all $1 \leq s < \ell$ and $i_\ell \geq 1$. The empty sequence $I = ()$ is admissible, with length $\ell = 0$, and $Sq^{()} = 1$.

The admissible monomials of degree ≤ 11 are $Sq^{()} = 1$ in degree 0, and:

(1) Sq^1

(2) Sq^2

(3) Sq^3, Sq^2Sq^1

(4) Sq^4, Sq^3Sq^1

(5) Sq^5, Sq^4Sq^1

(6) Sq^6, Sq^5Sq^1, Sq^4Sq^2

(7) Sq^7, Sq^6Sq^1, Sq^5Sq^2, $Sq^4Sq^2Sq^1$

(8) Sq^8, Sq^7Sq^1, Sq^6Sq^2, $Sq^5Sq^2Sq^1$

(9) Sq^9, Sq^8Sq^1, Sq^7Sq^2, Sq^6Sq^3, $Sq^6Sq^2Sq^1$

(10) Sq^{10}, Sq^9Sq^1, Sq^8Sq^2, Sq^7Sq^3, $Sq^7Sq^2Sq^1$, $Sq^6Sq^3Sq^1$

(11) Sq^{11}, $Sq^{10}Sq^1$, Sq^9Sq^2, Sq^8Sq^3, $Sq^8Sq^2Sq^1$, $Sq^7Sq^3Sq^1$

Theorem 3.11. *The admissible monomials form a vector space basis for the Steenrod algebra:*

$$\mathscr{A} = \mathbb{F}_2\{Sq^I \mid I \text{ is admissible}\}.$$

See Steenrod and Epstein (1962) Theorem I.3.1.

The Adem relations imply that any inadmissible Sq^I can be written as a sum of admissible monomials, so the admissible Sq^I generate $\mathscr{A}$. To prove that they are linearly independent, one uses the fact that the Adem relations hold for the Steenrod operations on $H^*((\mathbb{R}P^\infty)^r) = P(x_1, \dots, x_r)$, so that there is a pairing

$$\mathscr{A} \otimes P(x_1, \dots, x_r) \longrightarrow P(x_1, \dots, x_r)$$

making $P(x_1, \dots, x_r)$ a graded, left $\mathscr{A}$-module. The action on the product $w_r = x_1 \cdots x_r \in H^r((\mathbb{R}P^\infty)^r)$ is particularly useful. This is the top Stiefel–Whitney class of the canonical r-dimensional vector bundle over $(\mathbb{R}P^\infty)^r$. It defines a homomorphism

$$\mathscr{A} \longrightarrow P(x_1, \dots, x_r)$$

of degree r, taking Sq^I to $Sq^{i_1}(\dots Sq^{i_\ell}(w_r)\dots)$. It can be checked that this homomorphism takes the admissible monomials Sq^I of degree $\leq r$ to linearly independent elements in $P(x_1, \dots, x_r)$ (in degrees $r \leq * \leq 2r$). Letting r grow to infinity, this implies that the admissible Sq^I are independent.

Corollary 3.12. *The homomorphism $\mathscr{A} \to P(x_1, \dots, x_r)$, taking Sq^I to $Sq^I(w_r)$ for $w_r = x_1 \cdots x_r$, is injective in degrees $\le r$ (in the source).*

Hence, in order to verify a formula in $\mathscr{A}$ in degrees $\le r$, it suffices to establish this formula for the action on w_r in $H^*((\mathbb{R}P^\infty)^r)$. This gives one way to verify the Adem relations.

Definition 3.13. The Steenrod algebra is connected as a graded algebra, in the sense that it is zero in negative degrees and the unit map $\eta\colon \mathbb{F}_2 \to \mathscr{A}$ is an isomorphism in degree zero. Let $\epsilon\colon \mathscr{A} \to \mathbb{F}_2$ be the augmentation, such that $\epsilon\eta = 1$, and let $I(\mathscr{A}) = \ker(\epsilon)$ be the augmentation ideal, i.e., the positive-degree part of $\mathscr{A}$. The decomposable part of $\mathscr{A}$ is the image $I(\mathscr{A})^2$ of $I(\mathscr{A}) \otimes I(\mathscr{A})$ under the algebra multiplication $\phi\colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$, and the vector space $Q(\mathscr{A}) = I(\mathscr{A})/I(\mathscr{A})^2$ is the set of indecomposables in $\mathscr{A}$.

Theorem 3.14. *Sq^k is decomposable if and only if k is not a power of 2. Hence the elements Sq^{2^i} for $i \ge 0$ (i.e., $Sq^1, Sq^2, Sq^4, Sq^8, \dots$) generate $\mathscr{A}$ as an algebra.*

See Steenrod–Epstein (1962) section I.4.

The Adem relation

$$\binom{b-1}{a} Sq^{a+b} = Sq^a Sq^b + \sum_{j=1}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j$$

for $0 < a < 2b$ shows that Sq^{a+b} is decomposable if $\binom{b-1}{a} \equiv 1 \mod 2$. If k is not a power of 2 then $k = a + b$ with $0 < a < 2^i$ and $b = 2^i$. Then $b - 1 = 1 + 2 + \dots + 2^{i-1}$, so $\binom{b-1}{a} \equiv 1 \mod 2$ by the following lemma:

Lemma 3.15. *Let $a = a_0 + a_1 2 + \dots + a_\ell 2^\ell$ and $b = b_0 + b_1 2 + \dots + b_\ell 2^\ell$ with $0 \le a_s, b_s \le 1$. Then*

$$\binom{b}{a} \equiv \prod_{s=0}^{\ell} \binom{b_s}{a_s} \mod 2\,.$$

For the converse, suppose that $Sq^{2^i} = \sum_{j=1}^{2^i-1} m_j Sq^j$ is decomposable, where each $m_j \in I(\mathscr{A})$. Consider the action on x^{2^i} in $H^*(\mathbb{R}P^\infty) = P(x)$. On one hand, $Sq^j(x^{2^i}) = \binom{2^i}{j} x^{j+2^i} = 0$ for $0 < j < 2^i$, while $Sq^{2^i}(x^{2^i}) = x^{2^{i+1}} \ne 0$. This leads to a contradiction.

Now let p be odd.

Definition 3.16. Let the mod p Steenrod algebra, $\mathscr{A} = \mathscr{A}(p)$, be the graded, unital, associative $\mathbb{F}_p$-algebra generated by the symbols P^i of degree $2i(p-1)$ for $i \ge 0$, and β of degree 1, subject to the relations $P^0 = 1$, $\beta^2 = 0$ and the Adem relations.

Definition 3.17. For each sequence $I = (\epsilon_0, i_1, \epsilon_1, \dots, i_\ell, \epsilon_\ell, 0, 0, \dots)$ of non-negative integers, with $\epsilon_s \le 1$, let $P^I = \beta^{\epsilon_0} P^{i_1} \beta^{\epsilon_1} \dots P^{i_\ell} \beta^{\epsilon_\ell}$ be the product in $\mathscr{A}(p)$. We say that I is admissible if $i_s \ge \epsilon_s + p i_{s+1}$ for all $s \ge 1$.

Theorem 3.18. *The admissible monomials P^I form a basis for the Steenrod algebra:*

$$\mathscr{A}(p) = \mathbb{F}_p\{P^I \mid I \text{ admissible}\}\,.$$

See Steenrod and Epstein (1962) Theorem VI.2.5.

Theorem 3.19. *P^k is decomposable if and only if k is not a power of p. Hence the elements β and P^{p^i} for $i \ge 0$ generate $\mathscr{A}(p)$ as an algebra.*

3.4 Eilenberg–Mac Lane spectra

Definition 3.20. Let $H = \{n \mapsto H_n\}$ be the mod 2 Eilenberg–Mac Lane spectrum. The structure maps $\Sigma H_n \to H_{n+1}$ are left adjoint to the homotopy equivalences $H_n \xrightarrow{\simeq} \Omega H_{n+1}$, for all $n \geq 0$.

Proposition 3.21 (Whitehead)**.** *There are natural isomorphisms $H_n(Y) \cong \pi_n(H \wedge Y) = [S^n, H \wedge Y]$ and $H^n(Y) \cong \pi_{-n}F(Y, H) = [Y, \Sigma^n H]$ for all spectra Y and integers n.*

The composite $\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty \to H_1 \wedge \cdots \wedge H_1 \to H_r$ induces a homomorphism in cohomology that takes the universal class $\iota_r \in \tilde{H}^r(H_r)$ to w_r.

Proposition 3.22 (Serre)**.** *The homomorphism*

$$\Sigma^r \mathscr{A} \longrightarrow \tilde{H}^*(H_r)\,,$$

taking $\Sigma^r Sq^I$ to $Sq^I(\iota_r)$, induces an isomorphism in degrees $* \leq 2r$.

Corollary 3.23. *There is an isomorphism*

$$\mathscr{A} \xrightarrow{\cong} H^*(H) = [H, H]_{-*}$$

of graded $\mathbb{F}_2$-algebras, taking each Sq^i to its representing map $H \to \Sigma^i H$.

This shows that the Steenrod operations account for all stable mod 2 cohomology operations. The mod 2 cohomology of any spectrum Y is a left $\mathscr{A}$-module, and the module action map

$$\lambda\colon \mathscr{A} \otimes H^*(Y) \longrightarrow H^*(Y)$$

can be written as the composition pairing

$$[H, H]_* \otimes [Y, H]_* \longrightarrow [Y, H]_*$$

taking $Sq^i\colon H \to \Sigma^i H$ and $x\colon Y \to \Sigma^n H$ to $\Sigma^n(Sq^i) \circ x\colon Y \to \Sigma^{n+i} H$.

The mod 2 reduction h_1 of the Hurewicz homomorphism is the composite

$$\pi_*(Y) \xrightarrow{h} H_*(Y; \mathbb{Z}) \longrightarrow H_*(Y)\,.$$

The adjoint

$$\rho\colon H_*(Y) \longrightarrow \mathrm{Hom}(H^*(Y), \mathbb{F}_2)$$

to the Kronecker pairing is an isomorphism when $H_*(Y)$ is of finite type, i.e., if $H_n(Y) = H_n(Y; \mathbb{F}_2)$ is finite-dimensional (= finite) for each integer n. The composite

$$\rho \circ h_1\colon \pi_*(Y) \longrightarrow \mathrm{Hom}(H^*(Y), \mathbb{F}_2)$$

is the homomorphism taking the homotopy class of a map $f\colon S^n \to Y$ to the induced homomorphism $f^*\colon H^*(Y) \to \tilde{H}^*(S^n) \cong \Sigma^n \mathbb{F}_2$. By naturality of the Steenrod operations, the homomorphism f^* is one of left $\mathscr{A}$-modules, so that $\rho \circ h_1$ factors as a homomorphism

$$d\colon \pi_*(Y) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(H^*(Y), \mathbb{F}_2)$$

followed by the inclusion $\mathrm{Hom}_{\mathscr{A}}(H^*(Y), \mathbb{F}_2) \subset \mathrm{Hom}(H^*(Y), \mathbb{F}_2)$. More generally, there is a homomorphism

$$d\colon [X, Y] \longrightarrow \mathrm{Hom}_{\mathscr{A}}(H^*(Y), H^*(X))$$

(the cohomology d-invariant) taking the homotopy class of $f\colon X \to Y$ to the induced $\mathscr{A}$-module homomorphism $f^*\colon H^*(Y) \to H^*(X)$.

Lemma 3.24. *When $Y = \Sigma^n H$, for any integer n, the homomorphism*

$$d\colon \pi_*(\Sigma^n H) \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(H^*(\Sigma^n H), \mathbb{F}_2)$$

is an isomorphism. More generally, there is an isomorphism

$$d\colon [X, \Sigma^n H] \xrightarrow{\cong} \mathrm{Hom}_{\mathscr{A}}(H^*(\Sigma^n H), H^*(X))$$

for any spectrum X.

Proof. There is a class $\iota_n \in H^n(\Sigma^n H)$, with $\iota_n = \Sigma^n \iota_0$, such that $[f] \mapsto f^*(\iota_n)$ defines an isomorphism $[X, \Sigma^n H] \cong H^n(X)$. Since $H^*(\Sigma^n H) = \Sigma^n \mathscr{A}$ is the free $\mathscr{A}$-module generated by ι_n, the correspondence $f^* \mapsto f^*(\iota_n)$ defines another isomorphism $\operatorname{Hom}_{\mathscr{A}}(H^*(\Sigma^n H), H^*(X)) \cong H^n(X)$. Thus $d\colon [f] \mapsto f^*$ is also an isomorphism. □

Definition 3.25. We say that a spectrum Y is bounded below if $\pi_*(Y)$ is bounded below, i.e., if there exists an integer N such that $\pi_n(Y) = 0$ for $n < N$.

Lemma 3.26. *Suppose that $K = \bigvee_u \Sigma^{n_u} H$ is a wedge sum of suspended Eilenberg–Mac Lane spectra, such that $\{u \mid n_u \le N\}$ is finite for each integer N.*

Then the canonical map $\bigvee_u \Sigma^{n_u} H \to \prod_u \Sigma^{n_u} H$ is a stable equivalence, and

$$d\colon \pi_*(K) \xrightarrow{\cong} \operatorname{Hom}_{\mathscr{A}}(H^*(K), \mathbb{F}_2)$$

is an isomorphism. More generally, there is an isomorphism

$$d\colon [X, K] \xrightarrow{\cong} \operatorname{Hom}_{\mathscr{A}}(H^*(K), H^*(X))$$

for any spectrum X.

Proof. The finiteness hypothesis is equivalent to asking that $\pi_*(K)$ is bounded below and $H_*(K)$ is of finite type. It implies that the canonical map $\bigvee_u \Sigma^{n_u} H \to \prod_u \Sigma^{n_u} H$ is a weak equivalence, since the induced map in homotopy is the isomorphism $\bigoplus_u \Sigma^{n_u} \mathbb{F}_2 \to \prod_u \Sigma^{n_u} \mathbb{F}_2$. We deduce that

$$H^*(K) \cong \prod_u \Sigma^{n_u} \mathscr{A} \cong \bigoplus_u \Sigma^{n_u} \mathscr{A} \cong \bigoplus_u H^*(\Sigma^{n_u} H)$$

is a free $\mathscr{A}$-module, so

$$[X, K] \cong [X, \prod_u \Sigma^{n_u} H] \cong \prod_u [X, \Sigma^{n_u} H]$$

and

$$\operatorname{Hom}_{\mathscr{A}}(H^*(K), H^*(X)) \cong \prod_u \operatorname{Hom}_{\mathscr{A}}(\Sigma^{n_u} \mathscr{A}, H^*(X)) \cong \prod_u \operatorname{Hom}_{\mathscr{A}}(H^*(\Sigma^{n_u} H), H^*(X)) .$$

Hence d for K is the product of the isomorphisms d for the summands/factors $\Sigma^{n_u} H$, and is therefore an isomorphism. □

The pairings $\phi\colon H_m \wedge H_n \to H_{m+n}$ (representing the cup product $\iota_m \cup \iota_n$, or more precisely, its reduced version $\iota_m \wedge \iota_n$) combine to a map $\phi\colon H \wedge H \to H$ of spectra. Together with the unit map $\eta\colon S \to H$ coming from the maps $S^n \to H_n$ (representing the generator of $\tilde{H}^n(S^n)$), these make H a homotopy commutative ring spectrum. In fact it is a homotopy everything ring spectrum, i.e., an E_∞ ring spectrum.

Lemma 3.27. *Let Y be bounded below with $H_*(Y) = \mathbb{F}_2\{\alpha_u\}_u$ of finite type. Let $\{a_u\}_u$ be the dual basis for $H^*(Y)$, with $|a_u| = |\alpha_u| = n_u$. Let $\alpha_u\colon S^{n_u} \to H \wedge Y$ and $a_u\colon Y \to \Sigma^{n_u} H$ be the representing maps. Then the sum of the composites $(\phi \wedge 1)(1 \wedge \alpha_u)\colon \Sigma^{n_u} H \to H \wedge Y$ and the product of the composites $(\phi \wedge 1)(1 \wedge a_u)\colon H \wedge Y \to \Sigma^{n_u} Y$ are stable equivalences*

$$\bigvee_u \Sigma^{n_u} H \xrightarrow{\simeq} H \wedge Y \xrightarrow{\simeq} \prod_u \Sigma^{n_u} H .$$

Corollary 3.28. *Let $j\colon Y \to K$ be a map of spectra, where $K = \bigvee_u \Sigma^{n_u} H$ and $\{u \mid n_u \le N\}$ is finite for each N, and suppose that $j^*\colon H^*(K) \to H^*(Y)$ is surjective. Then a map $f\colon X \to Y$ of spectra induces the zero homomorphism $f^*\colon H^*(Y) \to H^*(X)$ if and only if the composite $jf\colon X \to K$ is null-homotopic.*

Proof. We have an isomorphism $d\colon [X, K] \cong \operatorname{Hom}_{\mathscr{A}}(H^*(K), H^*(X))$ taking jf to $f^* j^*$, and an injective homomorphism $\operatorname{Hom}_{\mathscr{A}}(H^*(Y), H^*(X)) \rightarrowtail \operatorname{Hom}_{\mathscr{A}}(H^*(K), H^*(X))$ taking f^* to $f^* j^*$, so $[jf] = 0$ if and only if $f^* = 0$. □

The corollary tells us that in the diagram

$$X \xrightarrow{f} Y \xrightarrow{j} K$$

the map f induces the zero map in cohomology, if and only if the composite jf is null-homotopic. By the lemma above, the unit map $\eta\colon S \to H$ induces a map $j\colon Y = S \wedge Y \to H \wedge Y \simeq K$, where K has the properties of the corollary when Y is bounded below with $H_*(Y)$ of finite type. Furthermore, the map $j_*\colon H_*(Y) \to H_*(K)$ is split injective, since it is the homomorphism of homotopy groups represented by the map

$$1 \wedge \eta \wedge 1\colon H \wedge Y \cong H \wedge S \wedge Y \longrightarrow H \wedge H \wedge H$$

which admits the retraction $\phi\wedge 1$. By the universal coefficient theorem, $j^*\colon H^*(K) \to H^*(Y)$ is surjective. Hence, under these hypotheses on Y we can use the diagram

$$X \xrightarrow{f} Y \xrightarrow{j} H \wedge Y$$

with $j = \eta \wedge 1$ to interpret the vanishing of f^* in homotopical terms.

4 The Adams spectral sequence

We follow Bruner's Adams spectral sequence primer. We continue working at $p = 2$, using the abbreviations $H_*(Y) = H_*(Y; \mathbb{F}_2)$ and $H^*(Y) = H^*(Y; \mathbb{F}_2)$.

4.1 Adams resolutions

Definition 4.1. Let Y be a spectrum with $\pi_*(Y)$ bounded below and $H_*(Y) = H_*(Y; \mathbb{F}_2)$ of finite type. An Adams resolution of Y is a diagram of spectra

$$\begin{tikzcd}
\dots \arrow[r] & Y^2 \arrow[r, "i"] \arrow[d, "j"] & Y^1 \arrow[r, "i"] \arrow[d, "j"] & Y^0 \arrow[r, equal] \arrow[d, "j"] & Y \\
& K^2 \arrow[ul, dashed, "\partial"] & K^1 \arrow[ul, dashed, "\partial"] & K^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

where $Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1}$ is a cofiber sequence, for each $s \geq 0$, such that (a) each K^s is a wedge sum of suspended mod 2 Eilenberg–Mac Lane spectra that is bounded below and of finite type, and (b) each homomorphism $j^*\colon H^*(K^s) \to H^*(Y^s)$ is surjective.

Writing $K^s \simeq \bigvee_u \Sigma^{n_u} H$, the finiteness condition in (a) is the same as asking that $\{u \mid n_u \leq N\}$ is finite for each integer N. By induction on s it implies that each Y^s is bounded below with $H_*(Y^s)$ of finite type. In view of the long exact sequence

$$\dots \to H^{*-1}(Y^{s+1}) \xrightarrow{\partial^*} H^*(K^s) \xrightarrow{j^*} H^*(Y^s) \xrightarrow{i^*} H^*(Y^{s+1}) \to \dots$$

the condition that j^* is surjective is equivalent to asking that $i^* = 0$ or that ∂^* is injective. ((Also homological interpretation, by the universal coefficient theorem.))

Lemma 4.2. *Adams resolutions exist.*

Proof. Suppose that Y^s has been constructed, with $\pi_*(Y^s)$ bounded below and $H_*(Y^s)$ of finite type. Let $K^s = H \wedge Y^s$ and let $j = 1 \wedge \eta\colon Y^s = S \wedge Y^s \to H \wedge Y^s = K^s$. Then K^s is a wedge sum of Eilenberg–Mac Lane spectra, bounded below and of finite type, and j^* is surjective. Let $Y^{s+1} = \mathrm{hofib}(j\colon Y^s \to K^s)$ be the homotopy fiber. Then $\pi_*(Y^{s+1})$ is bounded below by the long exact sequence in homotopy, and $H_*(Y^{s+1})$ is of finite type by the long exact sequence in mod 2 homology. Continue by induction. $\square$

Let $\bar{H}$ be the cofiber of the unit map $\eta\colon S \to H$, so that there is a cofiber sequence

$$\Sigma^{-1}\bar{H} \longrightarrow S \xrightarrow{\eta} H \longrightarrow \bar{H}$$

The unit map induces the augmentation $\epsilon\colon \mathscr{A} \to \mathbb{F}_2$ in cohomology, so $H^*(\bar{H}) = I(\mathscr{A}) = \ker(\epsilon)$ is the augmentation ideal.

Smashing with Y^s we get the cofiber sequence

$$\Sigma^{-1}\bar{H} \wedge Y^s \xrightarrow{i} Y^s \xrightarrow{j} H \wedge Y^s \xrightarrow{\partial} \bar{H} \wedge Y^s$$

so that the construction in the proof above gives $K^s = H \wedge Y^s$ and $Y^{s+1} = \Sigma^{-1}\bar{H} \wedge Y^s$.

Definition 4.3. The canonical Adams resolution of Y is the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & (\Sigma^{-1}\bar{H})^{\wedge 2} \wedge Y \arrow[rr, "i"] \arrow[d, "j"] & & \Sigma^{-1}\bar{H} \wedge Y \arrow[rr, "i"] \arrow[d, "j"] & & Y \arrow[d, "j"] \\
& H \wedge (\Sigma^{-1}\bar{H})^{\wedge 2} \wedge Y \arrow[ul, dashed, "\partial"] & & H \wedge \Sigma^{-1}\bar{H} \wedge Y \arrow[ull, dashed, "\partial"] & & H \wedge Y \arrow[ull, dashed, "\partial"]
\end{tikzcd}$$

where

$$\begin{aligned}
Y^s &= (\Sigma^{-1}\bar{H})^{\wedge s} \wedge Y \\
K^s &= H \wedge (\Sigma^{-1}\bar{H})^{\wedge s} \wedge Y
\end{aligned}$$

and i, j and ∂ are induced by $\Sigma^{-1}\bar{H} \to S$, $\eta\colon S \to H$ and $H \to \bar{H}$, respectively. We note that the canonical resolution is natural in Y.

Lemma 4.4. *For any Adams resolution, let*

$$\begin{aligned}
P_s &= H^*(\Sigma^s K^s) \\
\partial_s &= \partial^* j^*\colon H^*(\Sigma^s K^s) \to H^*(\Sigma^{s-1} K^{s-1})
\end{aligned}$$

and $\epsilon = j^*\colon H^*(K^0) \to H^*(Y)$. Then the diagram

$$\cdots \to P_s \xrightarrow{\partial_s} P_{s-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(Y) \to 0$$

is a resolution of $H^*(Y)$ by free $\mathscr{A}$-modules, each of which is bounded below of finite type.

The homomorphisms ∂_s and ϵ all preserve the cohomological grading of $H^*(Y)$ and P_s, which is called the internal grading and usually denoted by t.

Proof. By assumption (a) each j^* is surjective, so each i^* is zero and the long exact sequences in cohomology break up into short exact sequences

$$0 \to H^*(\Sigma^{s+1} Y^{s+1}) \xrightarrow{\partial^*} H^*(\Sigma^s K^s) \xrightarrow{j^*} H^*(\Sigma^s Y^s) \to 0$$

for all $s \geq 0$. These splice together to a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[dr, tail, "\partial^*"] & & H^*(\Sigma^2 Y^2) \arrow[dr, tail, "\partial^*"] & & H^*(\Sigma Y^1) \arrow[dr, tail, "\partial^*"] & & H^*(Y) \\
\cdots \arrow[r] & H^*(\Sigma^2 K^s) \arrow[rr, "\partial_2"] \arrow[ur, two heads, "j^*"] & & H^*(\Sigma K^1) \arrow[rr, "\partial_1"] \arrow[ur, two heads, "j^*"] & & H^*(K^0) \arrow[ur, two heads, "j^*"] &
\end{tikzcd}$$

along the lower edge of this diagram of $\mathscr{A}$-modules. By assumption (b), each $H^*(K^s)$ is a free $\mathscr{A}$-module. Hence $\epsilon\colon P_* \to H^*(Y)$ is a free resolution of the $\mathscr{A}$-module $H^*(Y)$. $\square$

The Adams resolution $\{Y^s\}_s$ is called a realization of the free resolution $\{P_s\}_s$ of $H^*(Y)$. The resolution is induced by passage to cohomology from the diagram

$$\dots \xleftarrow{j\partial} \Sigma^2 K^2 \xleftarrow{j\partial} \Sigma K^1 \xleftarrow{j\partial} K^0 \xleftarrow{j} Y$$

where each composite of two maps is null-homotopic. In the case of the canonical resolution this diagram appears as follows:

$$\dots \xleftarrow{j\partial} H \wedge (\bar{H})^{\wedge 2} \wedge Y \xleftarrow{j\partial} H \wedge \bar{H} \wedge Y \xleftarrow{j\partial} H \wedge Y \xleftarrow{j} Y$$

The associated free resolution has the form

$$\dots \to \mathscr{A} \otimes I(\mathscr{A})^{\otimes 2} \otimes H^*(Y) \xrightarrow{\partial_2} \mathscr{A} \otimes I(\mathscr{A}) \otimes H^*(Y) \xrightarrow{\partial_1} \mathscr{A} \otimes H^*(Y) \xrightarrow{\epsilon} H^*(Y) \to 0\,,$$

where $\mathscr{A} = H^*(H)$, and $I(\mathscr{A}) = H^*(\bar{H})$ is the augmentation ideal. We shall return to this complex later, in the context of the bar resolution.

4.2 The Adams E_2-term

We follows Adams (1958), using the spectrum level reformulation that appears in Moss (1968).

Let Y be a spectrum such that $\pi_*(Y)$ is bounded below and $H_*(Y) = H_*(Y;\mathbb{F}_2)$ is of finite type. Consider any Adams resolution

$$\begin{tikzcd}
\dots \arrow[r] & Y^2 \arrow[r, "i"] \arrow[d, "j"] & Y^1 \arrow[r, "i"] \arrow[d, "j"] & Y^0 \arrow[r, equal] \arrow[d, "j"] & Y \\
& K^2 \arrow[ul, dashed, "\partial"] & K^1 \arrow[ul, dashed, "\partial"] & K^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

of Y. Applying homotopy groups, we get an unrolled exact couple of Adams type

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(Y^2) \arrow[r, "i_*"] \arrow[d, "j_*"] & \pi_*(Y^1) \arrow[r, "i_*"] \arrow[d, "j_*"] & \pi_*(Y^0) \arrow[r, equal] \arrow[d, "j_*"] & \pi_*(Y) \\
& \pi_*(K^2) \arrow[ul, "\partial_*"] & \pi_*(K^1) \arrow[ul, "\partial_*"] & \pi_*(K^0) \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

where $A^s = \pi_*(Y^s)$, $E^s = \pi_*(K^s)$ are graded abelian groups, i_* and j_* have degree 0, and ∂_* has degree -1. There is an associated spectral sequence of Adams type

$$\{E_r = E_r^{*,*}, d_r = d_r^{*,*}\}_r$$

with

$$E_1^{s,t} = \pi_{t-s}(K^s)$$

and

$$d_1^{s,t} = (j\partial)_* \colon \pi_{t-s}(K^s) \to \pi_{t-s-1}(K^{s+1})\,.$$

The d_r-differentials have bidegree $(r, r-1)$. This is the Adams spectral sequence of Y, sometimes denotes $\{E_r(Y) = E_r^{*,*}(Y)\}_r$. The expected abutment is the graded abelian group $G = \pi_*(Y)$, filtered by the image groups $F^s = \operatorname{im}(i_*^s \colon \pi_*(Y^s) \to \pi_*(Y))$.

((NOTE: Explain "expected abutment". Do we mean that there are isomorphisms $F^s/F^{s+1} \cong E_\infty^s$, but that the filtration might not be complete Hausdorff and/or exhaustive? If so, discuss this in the section on convergence.))

Definition 4.5. An element in $E_r^{s,t}$ is said to be of filtration s, total degree $t-s$ and internal degree t. An element in $F^s \subset \pi_*(Y)$ is said to be of Adams filtration $\geq s$.

A class in $\pi_*(Y)$ has Adams filtration 0 if it is detected by the d-invariant in $\pi_*(K^0)$, i.e., if it has non-zero mod 2 Hurewicz image. If the Hurewicz image is zero, then the class lifts to $\pi_*(Y^1)$. Then it has Adams filtration 1 if the lift is detected in $\pi_*(K^1)$, i.e., if the lift has non-zero mod 2 Hurewicz image. If also that Hurewicz image is zero, then the class lifts to $\pi_*(Y^2)$. And so on.

Theorem 4.6. *The E_2-term of the Adams spectral sequence of Y is*

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_2)\,.$$

In particular, it is independent of the choice of Adams resolution.

Proof. The Adams E_1-term and d_1-differential is the complex

$$\dots \longleftarrow \pi_*(\Sigma^2 K^2) \xleftarrow{(j\partial)_*} \pi_*(\Sigma K^1) \xleftarrow{(j\partial)_*} \pi_*(K^0) \longleftarrow 0$$

of graded abelian groups. It maps isomorphically, under the d-invariant $\pi_*(K) \to \mathrm{Hom}_{\mathscr{A}}(H^*(K), \mathbb{F}_2)$, to the complex

$$\dots \longleftarrow \mathrm{Hom}_{\mathscr{A}}(H^*(\Sigma^2 K^2), \mathbb{F}_2) \xleftarrow{((j\partial)^*)^*} \mathrm{Hom}_{\mathscr{A}}(H^*(\Sigma K^1), \mathbb{F}_2) \xleftarrow{((j\partial)^*)^*} \mathrm{Hom}_{\mathscr{A}}(H^*(K^0), \mathbb{F}_2) \longleftarrow 0$$

where $((j\partial)^*)^* = \mathrm{Hom}_{\mathscr{A}}((j\partial)^*, 1)$. With the notation of the previous subsection, this complex can be rewritten as

$$\dots \longleftarrow \mathrm{Hom}_{\mathscr{A}}(P_2, \mathbb{F}_2) \xleftarrow{\partial_2^*} \mathrm{Hom}_{\mathscr{A}}(P_1, \mathbb{F}_2) \xleftarrow{\partial_1^*} \mathrm{Hom}_{\mathscr{A}}(P_0, \mathbb{F}_2) \longleftarrow 0\,.$$

This is the complex $\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2)$ obtained by applying the functor $\mathrm{Hom}_{\mathscr{A}}(-, \mathbb{F}_2)$ to the resolution $\epsilon\colon P_* \to H^*(Y)$ of $H^*(Y)$ by free $\mathscr{A}$-modules. Its cohomology groups are by definition, the Ext-groups

$$\mathrm{Ext}_{\mathscr{A}}^{s}(H^*(Y), \mathbb{F}_2) = H^s(\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2))\,.$$

At the same time, the cohomology of the E_1-term of a spectral sequence is the E_2-term. Hence

$$E_2^s \cong \mathrm{Ext}_{\mathscr{A}}^{s}(H^*(Y), \mathbb{F}_2)\,.$$

As regards the internal grading, $E_1^{s,t} = \pi_{t-s}(K^s) \cong \pi_t(\Sigma^s K^s)$ corresponds to the $\mathscr{A}$-module homomorphisms $H^*(\Sigma^s K^s) \to \Sigma^t \mathbb{F}_2$. This is the same as the $\mathscr{A}$-module homomorphisms $H^*(\Sigma^s K^s) \to \mathbb{F}_2$ that lower the cohomological degrees by t. We denote the group of these homomorphisms by $\mathrm{Hom}_{\mathscr{A}}^t(H^*(\Sigma^s K^s), \mathbb{F}_2) = \mathrm{Hom}_{\mathscr{A}}^t(P_s, \mathbb{F}_2)$, and similarly for the derived functors. With these conventions, $E_2^{s,t} \cong \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_2)$, as asserted. □

We are particularly interested in the special case $Y = S$, with $H^*(S) = \mathbb{F}_2$ and $\pi_*(S) = \pi_*^S$ equal to the stable homotopy groups of spheres.

Theorem 4.7. *The Adams spectral sequence for S has E_2-term*

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)\,.$$

On the other hand, we can also generalize (following Brinkmann (1968)). Let X be any spectrum and apply the functor $[X, -]_*$ to an Adams resolution of Y. This yields an unrolled exact couple

$$\begin{tikzcd}
\dots \arrow[r] & {[X, Y^2]_*} \arrow[r, "i_*"] \arrow[d, "j_*"] & {[X, Y^1]_*} \arrow[r, "i_*"] \arrow[d, "j_*"] & {[X, Y^0]_*} \arrow[r, equal] \arrow[d, "j_*"] & {[X, Y]_*} \\
& {[X, K^2]_*} \arrow[ul, "\partial_*"] & {[X, K^1]_*} \arrow[ul, "\partial_*"] & {[X, K^0]_*} \arrow[ul, "\partial_*"] &
\end{tikzcd}$$

where $A^s = [X, Y^s]_*$, $E^s = [X, K^s]_*$ are graded abelian groups, i_* and j_* have degree 0, and ∂_* has degree -1. There is an associated spectral sequence with

$$E_1^{s,t} = [X, K^s]_{t-s}$$

and

$$d_1^{s,t} = (j\partial)_*\colon [X, K^s]_{t-s} \to [X, K^{s+1}]_{t-s-1}\,.$$

The d_r-differentials have bidegree $(r, r-1)$. The expected abutment is the graded abelian group $G = [X, Y]_*$, filtered by the image groups $F^s = \mathrm{im}(i_*^s\colon [X, Y^s]_* \to [X, Y]_*)$.

Theorem 4.8. *The Adams spectral sequence $\{E_r(X, Y) = E_r^{*,*}(X, Y)\}_r$ of maps $X \to Y$, with expected abutment $[X, Y]_*$, has E_2-term*

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X))\,.$$

The proof is the same as for $X = S$, replacing $\mathbb{F}_2$ by $H^*(X)$ in the right hand argument of all $\mathrm{Hom}_{\mathscr{A}}$- and $\mathrm{Ext}_{\mathscr{A}}^s$-groups.

4.3 A minimal resolution

To compute the Adams E_2-term for the sphere spectrum, we need to compute

$$\mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = H^{*,*}(\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2))$$

for any free resolution P_* of $\mathbb{F}_2$. We now construct such a free resolution by hand, in a small range of degrees.

4.3.1 Filtration $s = 0$

We need a surjection $\epsilon\colon P_0 \to \mathbb{F}_2$, so we let $P_0 = \mathscr{A}\{g_{0,0}\}$ be the free $\mathscr{A}$-module on a single generator $g_{0,0}$ in internal degree 0. We will also use the notation $g_{0,0} = 1$. More generally, we will let $g_{s,i}$ denote the i-th generator in filtration degree s, counting from $i = 0$ in some order of non-decreasing internal degrees t.

4.3.2 Filtration $s = 1$

Next, we need a surjection $\partial_1\colon P_1 \to \ker(\epsilon)$, where $\ker(\epsilon) \cong I(\mathscr{A})$. An additive basis for $\ker(\epsilon)$ is given by the admissible monomials $Sq^I g_{0,0} = Sq^I$ for I of length ≥ 1. (We listed these through degree 11 in the subsection on admissible monomials.)

Starting in low degrees, we first need a generator $g_{1,0} = [Sq^1]$ in internal degree 1 that maps to Sq^1. The free summand $\mathscr{A}\{g_{1,0}\}$ that it will generate in P_1 will then map by ∂_1 to all classes of the form $Sq^I \circ Sq^1$, with I admissible. In view of the Adem relation $Sq^1 \circ Sq^1 = 0$, the image consists of all classes Sq^J where $J = (j_1, \dots, j_\ell)$ is admissible and $j_\ell = 1$. See the left hand column in Table 1.

The first class not in the image from $\mathscr{A}\{g_{1,0}\}$ is Sq^2 in internal degree 2, so we must add a second generator $g_{1,1} = [Sq^2]$ to P_1, that maps to Sq^2 under ∂_1. We use the Adem relations to compute the image $Sq^I Sq^2$ of $Sq^I[Sq^2]$. For example, $Sq^4Sq^2Sq^1 \circ Sq^2 = Sq^4Sq^2Sq^3 = Sq^4Sq^5 + Sq^4Sq^4Sq^1 = Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 + Sq^6Sq^2Sq^1$ (where we omitted $Sq^7Sq^1Sq^1 = 0$ at the last step). See the right hand column in Table 1.

The images of $Sq^2[Sq^1]$ and $Sq^1[Sq^2]$ generate $\ker(\epsilon)$ in internal degree 3, and Sq^3Sq^1 is in the image of ∂_1, but the class Sq^4 is not in the image from $\mathscr{A}\{g_{1,0}, g_{1,1}\}$, so we must add a third generator $g_{1,2} = [Sq^4]$ to P_1, mapping to Sq^4 under ∂_1. See the left hand column in Table 2.

All the admissible monomials in degree $1 \leq t \leq 7$ are then in the image of ∂_1, but Sq^8 is not hit. We must therefore add a fourth generator $g_{1,3} = [Sq^8]$ with $\partial_1(g_{1,3}) = Sq^8$. An inspection then reveals that $\partial_1\colon \mathscr{A}\{g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}\} \to \ker(\epsilon)$ is surjective in degrees $t \leq 11$. See the right hand column in Table 2.

In general, we need enough $\mathscr{A}$-module generators $\{g_{1,i}\}_i$ for P_1 to map surjectively to the indecomposables $Q(\mathscr{A}) = I(\mathscr{A})/I(\mathscr{A})^2 \cong \mathbb{F}_2\{Sq^{2^i} \mid i \geq 0\}$. This is necessary, since if $\partial_1\colon P_1 \to \ker(\epsilon) = I(\mathscr{A})$ is surjective, then so is the composite $P_1 \to I(\mathscr{A}) \to Q(\mathscr{A})$. It is also sufficient, since if $P_1 \to I(\mathscr{A})$ is surjective below degree t and $P_1 \to Q(\mathscr{A})$ is surjective in degree t, then all classes in $I(\mathscr{A})^2$ of degree t are in the image of P_1, and any class in $I(\mathscr{A})$ of degree t is congruent modulo $I(\mathscr{A})^2$ to a class in the image of P_1. The full definition of P_1 is therefore $P_1 = \mathscr{A}\{g_{1,i} \mid i \geq 0\}$ with $g_{1,i} = [Sq^{2^i}]$ mapping to $\partial_1(g_{1,i}) = Sq^{2^i}$, for all $i \geq 0$. Below internal degree 16 we thus have an isomorphism $P_1 \cong \mathscr{A}\{g_{1,0}, g_{1,1}, g_{1,2}, g_{1,3}\}$. ((References to Milnor–Moore, Steenrod–Epstein?))

4.3.3 Filtration $s = 2$

To continue, we ignore classes in degree $t > 11$. We need a surjection $\partial_2\colon P_2 \to \ker(\partial_1)$. First we go through the linear algebra exercise of computing an additive basis for $\ker(\partial_1)$. See Table 3.

The class in lowest degree in $\ker(\delta_1)$ is $Sq^1 g_{1,0} = Sq^1[Sq^1]$, which corresponds to the Adem relation $Sq^1Sq^1 = 0$. We put a first generator $g_{2,0}$ of degree 2 in P_2, with $\partial_2(g_{2,0}) = Sq^1[Sq^1]$. See the left hand column of Table 4.

The first class in $\ker(\partial_1)$ that is not in the image of ∂_2 on $\mathscr{A}\{g_{2,0}\}$ is $Sq^3[Sq^1] + Sq^2[Sq^2]$, which corresponds to the Adem relation $Sq^2Sq^2 = Sq^3Sq^1$. We add a second generator $g_{2,1}$ to P_2, in degree 4, with $\partial_2(g_{2,1}) = Sq^3[Sq^1] + Sq^2[Sq^2]$, and compute the value of $\partial_2(Sq^I g_{2,1}) = Sq^I(Sq^3[Sq^1] + Sq^2[Sq^2])$ in $\ker(\partial_1) \subset P_1$ for each admissible I, using the Adem relations. See the right hand column of Table 4.

$$
\begin{array}{rlrl}
g_{1,0} = [Sq^1] & \xmapsto{\partial_1} Sq^1 & & \\
Sq^1[Sq^1] & \longmapsto 0 & g_{1,1} = [Sq^2] & \xmapsto{\partial_1} Sq^2 \\
Sq^2[Sq^1] & \longmapsto Sq^2Sq^1 & Sq^1[Sq^2] & \longmapsto Sq^3 \\
Sq^3[Sq^1] & \longmapsto Sq^3Sq^1 & Sq^2[Sq^2] & \longmapsto Sq^3Sq^1 \\
Sq^2Sq^1[Sq^1] & \longmapsto 0 & & \\
Sq^4[Sq^1] & \longmapsto Sq^4Sq^1 & Sq^3[Sq^2] & \longmapsto 0 \\
Sq^3Sq^1[Sq^1] & \longmapsto 0 & Sq^2Sq^1[Sq^2] & \longmapsto Sq^5 + Sq^4Sq^1 \\
Sq^5[Sq^1] & \longmapsto Sq^5Sq^1 & Sq^4[Sq^2] & \longmapsto Sq^4Sq^2 \\
Sq^4Sq^1[Sq^1] & \longmapsto 0 & Sq^3Sq^1[Sq^2] & \longmapsto Sq^5Sq^1 \\
Sq^6[Sq^1] & \longmapsto Sq^6Sq^1 & Sq^5[Sq^2] & \longmapsto Sq^5Sq^2 \\
Sq^5Sq^1[Sq^1] & \longmapsto 0 & Sq^4Sq^1[Sq^2] & \longmapsto Sq^5Sq^2 \\
Sq^4Sq^2[Sq^1] & \longmapsto Sq^4Sq^2Sq^1 & & \\
Sq^7[Sq^1] & \longmapsto Sq^7Sq^1 & Sq^6[Sq^2] & \longmapsto Sq^6Sq^2 \\
Sq^6Sq^1[Sq^1] & \longmapsto 0 & Sq^5Sq^1[Sq^2] & \longmapsto 0 \\
Sq^5Sq^2[Sq^1] & \longmapsto Sq^5Sq^2Sq^1 & Sq^4Sq^2[Sq^2] & \longmapsto Sq^5Sq^2Sq^1 \\
Sq^4Sq^2Sq^1[Sq^1] & \longmapsto 0 & & \\
Sq^8[Sq^1] & \longmapsto Sq^8Sq^1 & Sq^7[Sq^2] & \longmapsto Sq^7Sq^2 \\
Sq^7Sq^1[Sq^1] & \longmapsto 0 & Sq^6Sq^1[Sq^2] & \longmapsto Sq^6Sq^3 \\
Sq^6Sq^2[Sq^1] & \longmapsto Sq^6Sq^2Sq^1 & Sq^5Sq^2[Sq^2] & \longmapsto 0 \\
Sq^5Sq^2Sq^1[Sq^1] & \longmapsto 0 & Sq^4Sq^2Sq^1[Sq^2] & \longmapsto Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 + Sq^6Sq^2Sq^1 \\
Sq^9[Sq^1] & \longmapsto Sq^9Sq^1 & Sq^8[Sq^2] & \longmapsto Sq^8Sq^2 \\
Sq^8Sq^1[Sq^1] & \longmapsto 0 & Sq^7Sq^1[Sq^2] & \longmapsto Sq^7Sq^3 \\
Sq^7Sq^2[Sq^1] & \longmapsto Sq^7Sq^2Sq^1 & Sq^6Sq^2[Sq^2] & \longmapsto Sq^6Sq^3Sq^1 \\
Sq^6Sq^3[Sq^1] & \longmapsto Sq^6Sq^3Sq^1 & Sq^5Sq^2Sq^1[Sq^2] & \longmapsto Sq^9Sq^1 + Sq^7Sq^2Sq^1 \\
Sq^6Sq^2Sq^1[Sq^1] & \longmapsto 0 & & \\
Sq^{10}[Sq^1] & \longmapsto Sq^{10}Sq^1 & Sq^9[Sq^2] & \longmapsto Sq^9Sq^2 \\
Sq^9Sq^1[Sq^1] & \longmapsto 0 & Sq^8Sq^1[Sq^2] & \longmapsto Sq^8Sq^3 \\
Sq^8Sq^2[Sq^1] & \longmapsto Sq^8Sq^2Sq^1 & Sq^7Sq^2[Sq^2] & \longmapsto Sq^7Sq^3Sq^1 \\
Sq^7Sq^3[Sq^1] & \longmapsto Sq^7Sq^3Sq^1 & Sq^6Sq^3[Sq^2] & \longmapsto 0 \\
Sq^7Sq^2Sq^1[Sq^1] & \longmapsto 0 & Sq^6Sq^2Sq^1[Sq^2] & \longmapsto Sq^9Sq^2 + Sq^8Sq^3 + Sq^7Sq^3Sq^1
\end{array}
$$

Table 1: ∂_1 on $\mathscr{A}\{g_{1,0}, g_{1,1}\} \subset P_1$

$$
\begin{aligned}
g_{1,2} = [Sq^4] &\xmapsto{\partial_1} Sq^4 \\
Sq^1[Sq^4] &\longmapsto Sq^5 \\
Sq^2[Sq^4] &\longmapsto Sq^6 + Sq^5Sq^1 \\
Sq^3[Sq^4] &\longmapsto Sq^7 \\
Sq^2Sq^1[Sq^4] &\longmapsto Sq^6Sq^1 \\
Sq^4[Sq^4] &\longmapsto Sq^7Sq^1 + Sq^6Sq^2 \\
Sq^3Sq^1[Sq^4] &\longmapsto Sq^7Sq^1 \\
Sq^5[Sq^4] &\longmapsto Sq^7Sq^2 \\
Sq^4Sq^1[Sq^4] &\longmapsto Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 \\
Sq^6[Sq^4] &\longmapsto Sq^7Sq^3 \\
Sq^5Sq^1[Sq^4] &\longmapsto Sq^9Sq^1 \\
Sq^4Sq^2[Sq^4] &\longmapsto Sq^{10} + Sq^9Sq^1 + Sq^8Sq^2 + Sq^7Sq^2Sq^1 \\
Sq^7[Sq^4] &\longmapsto 0 \\
Sq^6Sq^1[Sq^4] &\longmapsto Sq^9Sq^2 + Sq^8Sq^3 \\
Sq^5Sq^2[Sq^4] &\longmapsto Sq^{11} + Sq^9Sq^2 \\
Sq^4Sq^2Sq^1[Sq^4] &\longmapsto Sq^{10}Sq^1 + Sq^8Sq^2Sq^1
\end{aligned}
$$

$$
\begin{aligned}
g_{1,3} = [Sq^8] &\xmapsto{\partial_1} Sq^8 \\
Sq^1[Sq^8] &\longmapsto Sq^9 \\
Sq^2[Sq^8] &\longmapsto Sq^{10} + Sq^9Sq^1 \\
Sq^3[Sq^8] &\longmapsto Sq^{11} \\
Sq^2Sq^1[Sq^8] &\longmapsto Sq^{10}Sq^1
\end{aligned}
$$

Table 2: ∂_1 on $\mathscr{A}\{g_{1,2}, g_{1,3}\} \subset P_1$

The lowest degree class not in the image of ∂_2 on $\mathscr{A}\{g_{2,0}, g_{2,1}\} \subset P_2$ is $Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4]$, in degree 5. It corresponds to the Adem relation $Sq^2Sq^3 = Sq^5 + Sq^4Sq^1$, in view of the identities $Sq^1Sq^2 = Sq^3$ and $Sq^1Sq^4 = Sq^5$. We add a third generator $g_{2,2}$ to P_2, with $\partial_2(g_{2,2}) = Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4]$, and compute $\partial_2(Sq^I g_{2,2})$, as before. See Table 5.

The first class in $\ker(\partial_1)$ not in the image of ∂_2 on $\mathscr{A}\{g_{2,0}, g_{2,1}, g_{2,2}\}$ is $Sq^7[Sq^1]+Sq^6[Sq^2]+Sq^4[Sq^4]$. We add a fourth generator $g_{2,3}$ to P_2 in degree 8, corresponding to the Adem relation $Sq^4Sq^4 = Sq^7Sq^1 + Sq^6Sq^2$, and let $\partial_2(g_{2,3}) = Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4]$.

$$
\begin{aligned}
g_{2,3} &\xmapsto{\partial_2} Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4] \\
Sq^1 g_{2,3} &\longmapsto Sq^7[Sq^2] + Sq^5[Sq^4] \\
Sq^2 g_{2,3} &\longmapsto (Sq^9 + Sq^8Sq^1)[Sq^1] + Sq^7Sq^1[Sq^2] + (Sq^6 + Sq^5Sq^1)[Sq^4] \\
Sq^3 g_{2,3} &\longmapsto Sq^9Sq^1[Sq^1] + Sq^7[Sq^4] \\
Sq^2Sq^1 g_{2,3} &\longmapsto (Sq^9 + Sq^8Sq^1)[Sq^2] + Sq^6Sq^1[Sq^4]
\end{aligned}
$$

This still leaves $Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8]$ not in the image of ∂_2, so we add a fifth generator $g_{2,4}$ in degree 9, corresponding to the Adem relation $Sq^4Sq^5 = Sq^9 + Sq^8Sq^1 + Sq^7Sq^2$, and let $\partial_2(g_{2,4}) = Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8]$.

$$
\begin{aligned}
g_{2,4} &\xmapsto{\partial_2} Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8] \\
Sq^1 g_{2,4} &\longmapsto Sq^9[Sq^1] + Sq^5Sq^1[Sq^4] \\
Sq^2 g_{2,4} &\longmapsto (Sq^{10} + Sq^9Sq^1)[Sq^1] + (Sq^9 + Sq^8Sq^1)[Sq^2] + Sq^6Sq^1[Sq^4] + Sq^2Sq^1[Sq^8]
\end{aligned}
$$

Finally we need a sixth generator, $g_{2,5}$ in degree 10, mapping to $Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8]$. It derives from the Adem relations for Sq^2Sq^8 and for Sq^4Sq^6, using the Adem relation for Sq^2Sq^4. ((Can we pick a different generator that corresponds to just a single Adem relation?))

$$
\begin{aligned}
g_{2,5} &\xmapsto{\partial_2} Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8] \\
Sq^1 g_{2,5} &\longmapsto Sq^9[Sq^2] + Sq^5Sq^2[Sq^4] + Sq^3[Sq^8]
\end{aligned}
$$

$Sq^1[Sq^1]$

$Sq^2Sq^1[Sq^1]$

$Sq^3[Sq^1] + Sq^2[Sq^2]$

$Sq^3Sq^1[Sq^1]$

$Sq^3[Sq^2]$

$Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4]$

$Sq^4Sq^1[Sq^1]$

$Sq^5[Sq^1] + Sq^3Sq^1[Sq^2]$

$Sq^5Sq^1[Sq^1]$

$(Sq^5 + Sq^4Sq^1)[Sq^2]$

$Sq^6[Sq^1] + Sq^2Sq^1[Sq^4]$

$Sq^6Sq^1[Sq^1]$

$Sq^4Sq^2Sq^1[Sq^1]$

$Sq^5Sq^1[Sq^2]$

$Sq^5Sq^2[Sq^1] + Sq^4Sq^2[Sq^2]$

$Sq^7[Sq^1] + Sq^6[Sq^2] + Sq^4[Sq^4]$

$Sq^7[Sq^1] + Sq^3Sq^1[Sq^4]$

$Sq^7Sq^1[Sq^1]$

$Sq^5Sq^2Sq^1[Sq^1]$

$Sq^5Sq^2[Sq^2]$

$Sq^7[Sq^2] + Sq^5[Sq^4]$

$Sq^6Sq^2[Sq^1] + Sq^4Sq^2Sq^1[Sq^2] + Sq^4Sq^1[Sq^4]$

$Sq^8[Sq^1] + Sq^7[Sq^2] + Sq^4Sq^1[Sq^4] + Sq^1[Sq^8]$

$Sq^8Sq^1[Sq^1]$

$Sq^6Sq^2Sq^1[Sq^1]$

$Sq^6Sq^3[Sq^1] + Sq^6Sq^2[Sq^2]$

$(Sq^9 + Sq^7Sq^2)[Sq^1] + Sq^5Sq^2Sq^1[Sq^2]$

$Sq^7Sq^1[Sq^2] + Sq^6[Sq^4]$

$Sq^9[Sq^1] + Sq^5Sq^1[Sq^4]$

$Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8]$

$Sq^7Sq^2[Sq^1] + Sq^8[Sq^2] + Sq^4Sq^2[Sq^4] + Sq^2[Sq^8]$

$Sq^9Sq^1[Sq^1]$

$Sq^7Sq^2Sq^1[Sq^1]$

$Sq^6Sq^3Sq^1[Sq^1]$

$Sq^7Sq^3[Sq^1] + Sq^7Sq^2[Sq^2]$

$Sq^6Sq^3[Sq^2]$

$Sq^7Sq^3[Sq^1] + (Sq^9 + Sq^8Sq^1 + Sq^6Sq^2Sq^1)[Sq^2]$

$Sq^7[Sq^4]$

$(Sq^9 + Sq^8Sq^1)[Sq^2] + Sq^6Sq^1[Sq^4]$

$(Sq^{10} + Sq^8Sq^2)[Sq^1] + Sq^4Sq^2Sq^1[Sq^4]$

$Sq^9[Sq^2] + Sq^5Sq^2[Sq^4] + Sq^3[Sq^8]$

$Sq^{10}[Sq^1] + Sq^2Sq^1[Sq^8]$

Table 3: A basis for $\ker(\partial_1)$ in degrees ≤ 11

$$
\begin{array}{rlrl}
g_{2,0} & \xmapsto{\partial_2} Sq^1[Sq^1] & & \\
Sq^1 g_{2,0} & \longmapsto 0 & & \\
Sq^2 g_{2,0} & \longmapsto Sq^2Sq^1[Sq^1] & g_{2,1} & \xmapsto{\partial_2} Sq^3[Sq^1] + Sq^2[Sq^2] \\
Sq^3 g_{2,0} & \longmapsto Sq^3Sq^1[Sq^1] & Sq^1 g_{2,1} & \longmapsto Sq^3[Sq^2] \\
Sq^2Sq^1 g_{2,0} & \longmapsto 0 & & \\
Sq^4 g_{2,0} & \longmapsto Sq^4Sq^1[Sq^1] & Sq^2 g_{2,1} & \longmapsto (Sq^5 + Sq^4Sq^1)[Sq^1] + Sq^3Sq^1[Sq^2] \\
Sq^3Sq^1 g_{2,0} & \longmapsto 0 & & \\
Sq^5 g_{2,0} & \longmapsto Sq^5Sq^1[Sq^1] & Sq^3 g_{2,1} & \longmapsto Sq^5Sq^1[Sq^1] \\
Sq^4Sq^1 g_{2,0} & \longmapsto 0 & Sq^2Sq^1 g_{2,1} & \longmapsto (Sq^5 + Sq^4Sq^1)[Sq^2] \\
Sq^6 g_{2,0} & \longmapsto Sq^6Sq^1[Sq^1] & Sq^4 g_{2,1} & \longmapsto Sq^5Sq^2[Sq^1] + Sq^4Sq^2[Sq^2] \\
Sq^5Sq^1 g_{2,0} & \longmapsto 0 & Sq^3Sq^1 g_{2,1} & \longmapsto Sq^5Sq^1[Sq^2] \\
Sq^4Sq^2 g_{2,0} & \longmapsto Sq^4Sq^2Sq^1[Sq^1] & & \\
Sq^7 g_{2,0} & \longmapsto Sq^7Sq^1[Sq^1] & Sq^5 g_{2,1} & \longmapsto Sq^5Sq^2[Sq^2] \\
Sq^6Sq^1 g_{2,0} & \longmapsto 0 & Sq^4Sq^1 g_{2,1} & \longmapsto Sq^5Sq^2[Sq^2] \\
Sq^5Sq^2 g_{2,0} & \longmapsto Sq^5Sq^2Sq^1[Sq^1] & & \\
Sq^4Sq^2Sq^1 g_{2,0} & \longmapsto 0 & & \\
Sq^8 g_{2,0} & \longmapsto Sq^8Sq^1[Sq^1] & Sq^6 g_{2,1} & \longmapsto Sq^6Sq^3[Sq^1] + Sq^6Sq^2[Sq^2] \\
Sq^7Sq^1 g_{2,0} & \longmapsto 0 & Sq^5Sq^1 g_{2,1} & \longmapsto 0 \\
Sq^6Sq^2 g_{2,0} & \longmapsto Sq^6Sq^2Sq^1[Sq^1] & Sq^4Sq^2 g_{2,1} & \longmapsto (Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 + Sq^6Sq^2Sq^1)[Sq^1] + \\
Sq^5Sq^2Sq^1 g_{2,0} & \longmapsto 0 & & \qquad + Sq^5Sq^2Sq^1[Sq^2] \\
Sq^9 g_{2,0} & \longmapsto Sq^9Sq^1[Sq^1] & Sq^7 g_{2,1} & \longmapsto Sq^7Sq^3[Sq^1] + Sq^7Sq^2[Sq^2] \\
Sq^8Sq^1 g_{2,0} & \longmapsto 0 & Sq^6Sq^1 g_{2,1} & \longmapsto Sq^6Sq^3[Sq^2] \\
Sq^7Sq^2 g_{2,0} & \longmapsto Sq^7Sq^2Sq^1[Sq^1] & Sq^5Sq^2 g_{2,1} & \longmapsto (Sq^9Sq^1 + Sq^7Sq^2Sq^1)[Sq^1] \\
Sq^6Sq^3 g_{2,0} & \longmapsto Sq^6Sq^3Sq^1[Sq^1] & Sq^4Sq^2Sq^1 g_{2,1} & \longmapsto (Sq^9 + Sq^8Sq^1 + Sq^7Sq^2 + Sq^6Sq^2Sq^1)[Sq^2] \\
Sq^6Sq^2Sq^1 g_{2,0} & \longmapsto 0 & &
\end{array}
$$

Table 4: ∂_2 on $\mathscr{A}\{g_{2,0}, g_{2,1}\} \subset P_2$

$$
\begin{aligned}
g_{2,2} &\overset{\partial_2}{\longmapsto} Sq^4[Sq^1] + Sq^2Sq^1[Sq^2] + Sq^1[Sq^4] \\
Sq^1 g_{2,2} &\longmapsto Sq^5[Sq^1] + Sq^3Sq^1[Sq^2] \\
Sq^2 g_{2,2} &\longmapsto (Sq^6 + Sq^5Sq^1)[Sq^1] + Sq^2Sq^1[Sq^4] \\
Sq^3 g_{2,2} &\longmapsto Sq^7[Sq^1] + Sq^3Sq^1[Sq^4] \\
Sq^2Sq^1 g_{2,2} &\longmapsto Sq^6Sq^1[Sq^1] + Sq^5Sq^1[Sq^2] \\
Sq^4 g_{2,2} &\longmapsto (Sq^7Sq^1 + Sq^6Sq^2)[Sq^1] + Sq^4Sq^2Sq^1[Sq^2] + Sq^4Sq^1[Sq^4] \\
Sq^3Sq^1 g_{2,2} &\longmapsto Sq^7Sq^1[Sq^1] \\
Sq^5 g_{2,2} &\longmapsto Sq^7Sq^2[Sq^1] + Sq^5Sq^2Sq^1[Sq^2] + Sq^5Sq^1[Sq^4] \\
Sq^4Sq^1 g_{2,2} &\longmapsto (Sq^9 + Sq^8Sq^1 + Sq^7Sq^2)[Sq^1] + Sq^5Sq^2Sq^1[Sq^2] \\
Sq^6 g_{2,2} &\longmapsto Sq^7Sq^3[Sq^1] + Sq^6Sq^2Sq^1[Sq^2] + Sq^6Sq^1[Sq^4] \\
Sq^5Sq^1 g_{2,2} &\longmapsto Sq^9Sq^1[Sq^1] \\
Sq^4Sq^2 g_{2,2} &\longmapsto (Sq^{10} + Sq^9Sq^1 + Sq^8Sq^2 + Sq^7Sq^2Sq^1)[Sq^1] + Sq^4Sq^2Sq^1[Sq^4]
\end{aligned}
$$

Table 5: ∂_2 on $\mathscr{A}\{g_{2,2}\} \subset P_2$

Now $\partial_2 \colon \mathscr{A}\{g_{2,0}, \dots, g_{2,5}\} \to \ker(\partial_1)$ is surjective in degrees $t \leq 11$. (In fact, it is surjective below internal degree 16.)

4.3.4 Filtration $s = 3$

We carry on to filtration degree $s = 3$, looking for a surjection $\partial_3 \colon P_3 \to \ker(\partial_2)$. First we must compute a basis for $\ker(\partial_2) \subset P_2$, in our range of degrees. The result is displayed in Table 6.

As usual, the lowest degree class is $Sq^1 g_{2,0}$, so we first put a generator $g_{3,0}$ of degree 3 in P_3 with $\partial_3(g_{3,0}) = Sq^1 g_{2,0}$. The extension to $\mathscr{A}\{g_{3,0}\}$ is given in the left hand column of Table 7.

The lowest class not in the image of this extension is $\partial_3(g_{3,1}) = Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2}$ in degree 6. See the right hand column of Table 7.

After this, the only class not in the image of ∂_3 on $\mathscr{A}\{g_{3,0}, g_{3,1}\}$ is $\partial_3(g_{3,2}) = Sq^8 g_{2,0} + (Sq^5 + Sq^4Sq^1)g_{2,2} + Sq^1 g_{2,4}$ in degree 10:

$$
\begin{aligned}
g_{3,2} &\overset{\partial_3}{\longmapsto} Sq^8 g_{2,0} + (Sq^5 + Sq^4Sq^1)g_{2,2} + Sq^1 g_{2,4} \\
Sq^1 g_{3,2} &\longmapsto Sq^9 g_{2,0} + Sq^5Sq^1 g_{2,2}
\end{aligned}
$$

Finally, we need a fourth generator, $g_{3,3}$ in degree 11, with

$$
g_{3,3} \overset{\partial_3}{\longmapsto} Sq^4Sq^2Sq^1 g_{2,0} + Sq^6 g_{2,2} + Sq^2Sq^1 g_{2,3} \,.
$$

(This generator will be particularly interesting when we get to the multiplicative structure in the Adams E_2-term.) Then $\partial_3 \colon \mathscr{A}\{g_{3,0}, \dots, g_{3,3}\} \to \ker(\partial_2)$ is surjective in degrees $t \leq 11$.

4.3.5 Filtration $s = 4$

In degrees ≤ 11 we have an additive basis

$$
\begin{array}{ll}
Sq^1 g_{3,0} & Sq^6Sq^1 g_{3,0} \\
Sq^2Sq^1 g_{3,0} & Sq^4Sq^2Sq^1 g_{3,0} \\
Sq^3Sq^1 g_{3,0} & Sq^7Sq^1 g_{3,0} \\
Sq^4Sq^1 g_{3,0} & Sq^5Sq^2Sq^1 g_{3,0} \\
Sq^5Sq^1 g_{3,0} & Sq^8 g_{3,0} + (Sq^5 + Sq^4Sq^1) g_{3,1} + Sq^1 g_{3,2}
\end{array}
$$

$Sq^1 g_{2,0}$	$Sq^7 Sq^1 g_{2,0}$
$Sq^2 Sq^1 g_{2,0}$	$Sq^5 Sq^2 Sq^1 g_{2,0}$
$Sq^3 Sq^1 g_{2,0}$	$Sq^5 Sq^1 g_{2,1}$
$Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2}$	$Sq^6 Sq^2 g_{2,0} + Sq^4 Sq^2 g_{2,1} + Sq^4 Sq^1 g_{2,2}$
$Sq^4 Sq^1 g_{2,0}$	$Sq^8 g_{2,0} + (Sq^5 + Sq^4 Sq^1) g_{2,2} + Sq^1 g_{2,4}$
$Sq^5 g_{2,0} + Sq^3 g_{2,1}$	$Sq^8 Sq^1 g_{2,0}$
$Sq^5 Sq^1 g_{2,0}$	$Sq^6 Sq^2 Sq^1 g_{2,0}$
$Sq^6 g_{2,0} + Sq^3 Sq^1 g_{2,1} + Sq^2 Sq^1 g_{2,2}$	$(Sq^9 + Sq^7 Sq^2) g_{2,0} + Sq^5 Sq^2 g_{2,1}$
$Sq^6 Sq^1 g_{2,0}$	$Sq^9 g_{2,0} + Sq^5 Sq^1 g_{2,2}$
$Sq^4 Sq^2 Sq^1 g_{2,0}$	$Sq^4 Sq^2 Sq^1 g_{2,0} + Sq^6 g_{2,2} + Sq^2 Sq^1 g_{2,3}$
$(Sq^5 + Sq^4 Sq^1) g_{2,1}$	
$Sq^7 g_{2,0} + Sq^3 Sq^1 g_{2,2}$	

Table 6: A basis for $\ker(\partial_2)$ in degrees ≤ 11

$g_{3,0} \overset{\partial_3}{\longmapsto} Sq^1 g_{2,0}$	
$Sq^1 g_{3,0} \longmapsto 0$	
$Sq^2 g_{3,0} \longmapsto Sq^2 Sq^1 g_{2,0}$	
$Sq^3 g_{3,0} \longmapsto Sq^3 Sq^1 g_{2,0}$	$g_{3,1} \overset{\partial_3}{\longmapsto} Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2}$
$Sq^2 Sq^1 g_{3,0} \longmapsto 0$	
$Sq^4 g_{3,0} \longmapsto Sq^4 Sq^1 g_{2,0}$	$Sq^1 g_{3,1} \longmapsto Sq^5 g_{2,0} + Sq^3 g_{2,1}$
$Sq^3 Sq^1 g_{3,0} \longmapsto 0$	
$Sq^5 g_{3,0} \longmapsto Sq^5 Sq^1 g_{2,0}$	$Sq^2 g_{3,1} \longmapsto (Sq^6 + Sq^5 Sq^1) g_{2,0} + Sq^3 Sq^1 g_{2,1} + Sq^2 Sq^1 g_{2,2}$
$Sq^4 Sq^1 g_{3,0} \longmapsto 0$	
$Sq^6 g_{3,0} \longmapsto Sq^6 Sq^1 g_{2,0}$	$Sq^3 g_{3,1} \longmapsto Sq^7 g_{2,0} + Sq^3 Sq^1 g_{2,2}$
$Sq^5 Sq^1 g_{3,0} \longmapsto 0$	$Sq^2 Sq^1 g_{3,1} \longmapsto Sq^6 Sq^1 g_{2,0} + (Sq^5 + Sq^4 Sq^1) g_{2,1}$
$Sq^4 Sq^2 g_{3,0} \longmapsto Sq^4 Sq^2 Sq^1 g_{2,0}$	
$Sq^7 g_{3,0} \longmapsto Sq^7 Sq^1 g_{2,0}$	$Sq^4 g_{3,1} \longmapsto (Sq^7 Sq^1 + Sq^6 Sq^2) g_{2,0} + Sq^4 Sq^2 g_{2,1} + Sq^4 Sq^1 g_{2,2}$
$Sq^6 Sq^1 g_{3,0} \longmapsto 0$	$Sq^3 Sq^1 g_{3,1} \longmapsto Sq^7 Sq^1 g_{2,0} + Sq^5 Sq^1 g_{2,1}$
$Sq^5 Sq^2 g_{3,0} \longmapsto Sq^5 Sq^2 Sq^1 g_{2,0}$	
$Sq^4 Sq^2 Sq^1 g_{3,0} \longmapsto 0$	
$Sq^8 g_{3,0} \longmapsto Sq^8 Sq^1 g_{2,0}$	$Sq^5 g_{3,1} \longmapsto Sq^7 Sq^2 g_{2,0} + Sq^5 Sq^2 g_{2,1} + Sq^5 Sq^1 g_{2,2}$
$Sq^7 Sq^1 g_{3,0} \longmapsto 0$	$Sq^4 Sq^1 g_{3,1} \longmapsto (Sq^9 + Sq^8 Sq^1 + Sq^7 Sq^2) g_{2,0} + Sq^5 Sq^2 g_{2,1}$
$Sq^6 Sq^2 g_{3,0} \longmapsto Sq^6 Sq^2 Sq^1 g_{2,0}$	
$Sq^5 Sq^2 Sq^1 g_{3,0} \longmapsto 0$	

Table 7: ∂_3 on $\mathscr{A}\{g_{3,0}, g_{3,1}\} \subset P_3$

for $\ker(\partial_3)$, and a surjection $\partial_4 \colon P_4 = \mathscr{A}\{g_{4,0}, g_{4,1}\} \to \ker(\partial_3)$ where

$$\partial_4(g_{4,0}) = Sq^1 g_{3,0}$$

in degree 4, and

$$\partial_4(g_{4,1}) = Sq^8 g_{3,0} + (Sq^5 + Sq^4 Sq^1) g_{3,1} + Sq^1 g_{3,2}$$

in degree 11.

4.3.6 Filtration $s \geq 5$

Things become quite simple from filtration degree $s = 5$ and onwards. In degrees ≤ 11 we have an additive basis

$$\begin{array}{ll}
Sq^1 g_{4,0} & Sq^5 Sq^1 g_{4,0} \\
Sq^2 Sq^1 g_{4,0} & Sq^6 Sq^1 g_{4,0} \\
Sq^3 Sq^1 g_{4,0} & Sq^4 Sq^2 Sq^1 g_{4,0} \\
Sq^4 Sq^1 g_{4,0} &
\end{array}$$

for $\ker(\partial_4)$, and a surjection $\partial_5 \colon P_5 = \mathscr{A}\{g_{5,0}\} \to \ker(\partial_4)$ where $\partial_5(g_{5,0}) = Sq^1 g_{4,0}$ in degree 5. Continuing, we have a surjection $\partial_s \colon P_s = \mathscr{A}\{g_{s,0}\} \to \ker(\partial_{s-1})$ in degrees ≤ 11, where $\partial_s(g_{0,s}) = Sq^1 g_{0,s-1}$ in degree s, for all $5 \leq s \leq 11$.

Definition 4.9. We say that P_* is a minimal resolution when $\mathrm{im}(\partial_{s+1}) \subset I(\mathscr{A}) \cdot P_s$ for all $s \geq 0$. Then $1 \otimes \partial_{s+1} \colon \mathbb{F}_2 \otimes_{\mathscr{A}} P_{s+1} \to \mathbb{F}_2 \otimes_{\mathscr{A}} P_s$ and $\mathrm{Hom}(\partial_{s+1}, 1) \colon \mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \to \mathrm{Hom}_{\mathscr{A}}(P_{s+1}, \mathbb{F}_2)$ are the zero homomorphisms, so that $\mathrm{Tor}_s^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2 \otimes_{\mathscr{A}} P_s$ and $\mathrm{Ext}_{\mathscr{A}}^s(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2)$, for all $s \geq 0$. Equivalently, the number of generators of P_s is minimal in each degree.

Theorem 4.10. *There is a minimal resolution $\epsilon \colon P_* \to \mathbb{F}_2$ with $P_0 = \mathscr{A}\{g_{0,0}\}$ and $P_s = \mathscr{A}\{g_{s,i} \mid i \geq 0\}$, where $\partial_s \colon P_s \to P_{s-1}$ is given in internal degrees $t \leq 11$ by*

$$\begin{aligned}
\partial_1(g_{1,0}) &= Sq^1 g_{0,0} \\
\partial_1(g_{1,1}) &= Sq^2 g_{0,0} \\
\partial_1(g_{1,2}) &= Sq^4 g_{0,0} \\
\partial_1(g_{1,3}) &= Sq^8 g_{0,0} \\
\partial_2(g_{2,0}) &= Sq^1 g_{1,0} \\
\partial_2(g_{2,1}) &= Sq^3 g_{1,0} + Sq^2 g_{1,1} \\
\partial_2(g_{2,2}) &= Sq^4 g_{1,0} + Sq^2 Sq^1 g_{1,1} + Sq^1 g_{1,2} \\
\partial_2(g_{2,3}) &= Sq^7 g_{1,0} + Sq^6 g_{1,1} + Sq^4 g_{1,2} \\
\partial_2(g_{2,4}) &= Sq^8 g_{1,0} + Sq^7 g_{1,1} + Sq^4 Sq^1 g_{1,2} + Sq^1 g_{1,3} \\
\partial_2(g_{2,5}) &= Sq^7 Sq^2 g_{1,0} + Sq^8 g_{1,1} + Sq^4 Sq^2 g_{1,2} + Sq^2 g_{1,3} \\
\partial_3(g_{3,0}) &= Sq^1 g_{2,0} \\
\partial_3(g_{3,1}) &= Sq^4 g_{2,0} + Sq^2 g_{2,1} + Sq^1 g_{2,2} \\
\partial_3(g_{3,2}) &= Sq^8 g_{2,0} + (Sq^5 + Sq^4 Sq^1) g_{2,2} + Sq^1 g_{2,4} \\
\partial_3(g_{3,3}) &= (Sq^7 + Sq^4 Sq^2 Sq^1) g_{2,1} + Sq^6 g_{2,2} + Sq^2 Sq^1 g_{2,3} \\
\partial_4(g_{4,0}) &= Sq^1 g_{3,0} \\
\partial_4(g_{4,1}) &= Sq^8 g_{3,0} + (Sq^5 + Sq^4 Sq^1) g_{3,1} + Sq^1 g_{3,2} \\
\partial_5(g_{5,0}) &= Sq^1 g_{4,0} \\
&\cdots \\
\partial_{11}(g_{11,0}) &= Sq^1 g_{10,0}\,.
\end{aligned}$$

Proof. This summarizes the calculations above. The resolution is minimal, since we only added generators $g_{s,i}$ with $\partial_s(g_{s,i}) \in I(\mathscr{A}) \cdot P_{s-1} = I(\mathscr{A})\{g_{s-1,j}\}_j$. It should be clear that we can continue that way, since $\mathscr{A}$ is connected. If any sum involving $1 \cdot g_{s,n}$ occurs in $\ker(\partial_s)$, then $g_{s,n}$ could be omitted from the basis for P_s and $\partial_s \colon P_s \to \ker(\partial_{s-1})$ would still be surjective. $\square$

Theorem 4.11. *$\mathrm{Ext}^{s,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,i}\}_i$ where $\gamma_{s,i} \colon P_s \to \mathbb{F}_2$ is the $\mathscr{A}$-module homomorphism dual to $g_{s,i}$, for each $s \geq 0$. The bidegrees of the generators in internal degrees $t \leq 11$ are as displayed in the following chart. The horizontal coordinate is the topological degree $t - s$, the vertical coordinate is the cohomological degree s, and the sum of these coordinates is the internal degree t.*

	$\gamma_{11,0}$	·	·	·	·	·	·	·	·	·	·	·
10	$\gamma_{10,0}$		·	·	·	·	·	·	·	·	·	·
	$\gamma_{9,0}$			·	·	·	·	·	·	·	·	·
8	$\gamma_{8,0}$				·	·	·	·	·	·	·	·
	$\gamma_{7,0}$					·	·	·	·	·	·	?
6	$\gamma_{6,0}$						·	·	·	·	?	?
	$\gamma_{5,0}$							·	·	?	?	?
4	$\gamma_{4,0}$							$\gamma_{4,1}$	?	?	?	?
	$\gamma_{3,0}$			$\gamma_{3,1}$				$\gamma_{3,2}$	$\gamma_{3,3}$	?	?	?
2	$\gamma_{2,0}$		$\gamma_{2,1}$	$\gamma_{2,2}$			$\gamma_{2,3}$	$\gamma_{2,4}$	$\gamma_{2,5}$		?	?
	$\gamma_{1,0}$	$\gamma_{1,1}$		$\gamma_{1,2}$				$\gamma_{1,3}$				?
0	$\gamma_{0,0}$											
	0		2		4		6		8		10	

We have not yet computed the groups labeled · or ?, but we will prove below that the groups labeled · are 0. In fact, many of the groups labeled ? are also zero.

Proof. For each $s \geq 0$ we have $\mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(\mathscr{A}\{g_{s,i}\}_i, \mathbb{F}_2) \cong \prod_i \mathbb{F}_2\{\gamma_{s,i}\}$, where $\gamma_{s,i}(g_{s,j}) = \delta_{i,j}$ is 1 if $i = j$ and 0 otherwise. It will be clear later that there are at most finitely many $g_{s,i}$ in a given bidegree, so this product is finite in each degree. Then $\gamma_{s,i} \circ \partial_{s+1} = 0$, so the cocomplex $\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2)$ has trivial coboundary. Hence $\mathrm{Ext}^s_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,i}\}_i$, as claimed. $\square$

Lemma 4.12. *Let $\epsilon \colon P_* \to \mathbb{F}_2$ be a free $\mathscr{A}$-module resolution. Then $\mathrm{Hom}_{\mathscr{A}}(P_s, \mathbb{F}_2) \cong \mathrm{Hom}(\mathbb{F}_2 \otimes_{\mathscr{A}} P_s, \mathbb{F}_2)$, so there is an isomorphism $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{Tor}^{\mathscr{A}}_{s,t}(\mathbb{F}_2, \mathbb{F}_2), \mathbb{F}_2)$.*

4.4 The Hopf–Steenrod invariant

The standard notation for the class $\gamma_{1,i}$, dual to the indecomposable Sq^{2^i}, is h_i. See Adams (1958). The h is for Hopf, since these classes detect the stable maps of spheres with Hopf invariant one.

Lemma 4.13. *$\mathrm{Tor}^{\mathscr{A}}_1(\mathbb{F}_2, \mathbb{F}_2) \cong I(\mathscr{A})/I(\mathscr{A})^2 = Q(\mathscr{A}) \cong \mathbb{F}_2\{Sq^{2^i} \mid i \geq 0\}$ and $\mathrm{Ext}^1_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}(\mathrm{Tor}^{\mathscr{A}}_1(\mathbb{F}_2, \mathbb{F}_2), \mathbb{F}_2) \cong \mathbb{F}_2\{h_i \mid i \geq 0\}$ where h_i has bidegree $(s, t) = (1, 2^i)$ and is dual to Sq^{2^i}, for each $i \geq 0$.*

Proof. There exists a free resolution $\cdots \to P_1 \to P_0 \to \mathbb{F}_2 \to 0$ where $P_0 = \mathscr{A}$ and $P_1 = \mathscr{A}\{g_{1,i}\}_i$ with $\partial_1 \colon g_{1,i} \mapsto Sq^{2^i}$ for all $i \geq 0$. The resolution is exact at P_0 since the Sq^{2^i} generate the left ideal $I(\mathscr{A}) \subset \mathscr{A}$, and it is minimal there since $\partial_1(P_1) \subset I(\mathscr{A})P_0$. It is also minimal at P_1, since the surjection

$P_1 \to I(\mathscr{A})$ induces an isomorphism $\mathbb{F}_2\{g_{1,i}\}_i = \mathbb{F}_2 \otimes_{\mathscr{A}} P_1 = P_1/I(\mathscr{A})P_1 \to I(\mathscr{A})/I(\mathscr{A})^2 = Q(\mathscr{A})$, so that $\partial_2(P_2) = \ker(\partial_1) \subset I(\mathscr{A})P_1$. Hence $\mathrm{Tor}_1^{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2 \otimes_{\mathscr{A}} P_1 \cong Q(\mathscr{A})$ and $\mathrm{Ext}^1_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}_{\mathscr{A}}(P_1, \mathbb{F}_2) \cong \mathbb{F}_2\{h_i\}_i$, as claimed. ((Proof using bar complex?)) $\square$

We shall soon prove that the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(S)^\wedge_2$$

converges to the 2-adic completion of the stable homotopy groups of spheres. The chart in the theorem above displays the E_2-term in the range $t \le 11$. ((EDIT FROM HERE TO TAKE INTO ACCOUNT THE ADAMS VANISHING LINE.)) We will see later that the pattern above the diagonal line, where $s > t-s$, continues. There is an isomorphism $\mathrm{Ext}^{s,s}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_{s,0}\}$ for all $s \ge 0$, while $\mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) = 0$ for $t-s < 0$ and for $0 < t-s < s$. Thus the groups labeled $\cdot$ in the chart are 0. Granting this, the only possible d_r-differentials starting in total degree $t-s \le 6$, for $r \ge 2$, are the ones starting on $\gamma_{1,1} = h_1$ and landing in the group generated by $\gamma_{r+1,0}$.

However, these differentials are all 0, as can be seen either by proving that $\gamma_{s,0}$ detected $2^s \in \pi_0(S)$, or that $\gamma_{1,1}$ detects $\eta \in \pi_1(S)$, or by appealing to multiplicative structure in the spectral sequence. Granting this, we can conclude that $E_2 = E_\infty$ in this range of degrees, so that the groups $\mathbb{F}_2\{\gamma_{s,i}\}$ in one topological degree $n = t-s$, for $s \ge 0$ and $n \le 5$ are the filtration quotients of a complete Hausdorff filtration $\{F^s\}_s$ that exhausts $\pi_n(S)^\wedge_2$.

For $n = 0$, we already know that $\pi_0(S) = \mathbb{Z}$ so $\pi_0(S)^\wedge_2 = \mathbb{Z}_2$. The only possible filtration is the 2-adic one, with $F^s = 2^s\mathbb{Z}_2 \subset \mathbb{Z}_2$ and $F^s/F^{s+1} \cong 2^s\mathbb{Z}_2/2^{s+1}\mathbb{Z}_2 \cong \mathbb{F}_2\{\gamma_{s,0}\}$ for all $s \ge 0$. For $n = 1$ we deduce that $\pi_1(S)^\wedge_2 \cong \mathbb{Z}/2\{\gamma_{1,1}\} = \mathbb{Z}/2\{h_1\}$. In fact $\pi_1(S) = \mathbb{Z}/2\{\eta\}$ is generated by the complex Hopf map $\eta\colon S^1 \to S$. For $n = 2$ we deduce that $\pi_2(S)^\wedge_2 \cong \mathbb{Z}/2\{\gamma_{2,1}\}$. We shall see later that $\pi_2(S) = \mathbb{Z}/2\{\eta^2\}$ is generated by the composite $\eta^2 = \eta \circ \Sigma\eta\colon S^2 \to S$. For $n = 3$ we deduce that $\pi_3(S)^\wedge_2$ is an abelian group of order 8. We shall see later that $\pi_3(S)^\wedge_2 \cong \mathbb{Z}/(8)$ is the 2-Sylow subgroup of $\pi_3(S) \cong \mathbb{Z}/24$, generated by the quaternionic Hopf map $\nu\colon S^3 \to S$. Finally, for now, we conclude that $\pi_4(S)^\wedge_2 = 0$ and $\pi_5(S)^\wedge_2 = 0$, and in fact $\pi_4(S) = \pi_5(S) = 0$. ((EDIT TO HERE.))

Lemma 4.14. *(Hopf, Steenrod) Let $f\colon S^n \to S$ be a map with $0 = f^*\colon H^*(S) \to H^*(S^n)$, and let $C_f = \mathrm{hocofib}(f) = S \cup_f CS^n$ be its mapping cone. Suppose that $Sq^{n+1}\colon H^0(C_f) \to H^{n+1}(C_f)$ is nonzero. Then $n+1 = 2^i$ for some $i \ge 0$ and $[f] \in \pi_n(S)$ is detected in the Adams spectral sequence by $h_i \in E_2^{1,2^i}$.*

Proof. Consider the canonical Adams tower for $Y = S$, with $Y^0 = S$, $K^0 = H$, $Y^1 = \Sigma^{-1}\bar{H}$ and $K^1 = H \wedge \Sigma^{-1}\bar{H}$. The composite $j \circ f$ is null-homotopic, since $d(f) = f^* = 0$, so we have a map of cofiber sequences:

$$\begin{array}{ccccccc}
S^n & \xrightarrow{f} & S & \longrightarrow & C_f & \longrightarrow & S^{n+1} \\
\downarrow{\scriptstyle e} & & \| & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \Sigma e} \\
\Sigma^{-1}\bar{H} & \xrightarrow{i} & S & \xrightarrow{j} & H & \xrightarrow{\partial} & \bar{H} \\
\downarrow{\scriptstyle j} & & & & & & \\
H \wedge \Sigma^{-1}\bar{H} & & & & & &
\end{array}$$

Here $d\colon C_f \to H$ and $e\colon S^n \to \Sigma^{-1}\bar{H}$ are determined by a null-homotopy of f. Applying cohomology to the right hand part of the diagram, we get a map of $\mathscr{A}$-module extensions:

$$\begin{array}{ccccc}
\mathbb{F}_2 & \twoheadleftarrow & H^*(C_f) & \leftarrowtail & \Sigma^{n+1}\mathbb{F}_2 \\
\| & & \uparrow{\scriptstyle d^*} & & \uparrow{\scriptstyle \Sigma e^*} \\
\mathbb{F}_2 & \xleftarrow{j^*} & \mathscr{A} & \xleftarrow{\partial^*} & I(\mathscr{A})
\end{array}$$

Here $d^*(1) = 1$, so by assumption $d^*(Sq^{n+1}) \ne 0$. Hence $\Sigma e^*(Sq^{n+1}) \ne 0$. This is impossible if Sq^{n+1} is decomposable, so we must have $n+1 = 2^i$ for some $i \ge 0$. Then $e^* \ne 0$, which implies that $j \circ e\colon S^n \to H \wedge \Sigma^{-1}\bar{H}$ is essential (= not null-homotopic).

This proves that $[f] \in \pi_n(S)$ lifts to $\pi_n(Y^1)$ but not to $\pi_n(Y^2)$, hence corresponds under the isomorphism $F^1/F^2 \cong E_\infty^{1,*}$ to a nonzero class in $E_\infty^{1,2^i} \subset E_2^{1,2^i} = \mathbb{F}_2\{h_i\}$. The only possibility is that $[f]$ is detected by h_i. □

The class of $\Sigma e^* \circ \partial_1 \colon P_1 \to \Sigma^{n+1}\mathbb{F}_2$ in $\mathrm{Ext}_{\mathscr{A}}^{1,2^i}(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2\{h_i\}$ is called the Hopf–Steenrod invariant, or the cohomology e-invariant, of $[f]$. It is only defined for the $[f]$ with vanishing d-invariant. More generally, we have a diagram

$$\begin{tikzcd} F^2 \arrow[r, rightarrowtail] & F^1 \arrow[r, rightarrowtail] \arrow[d, "e"] & F^0 = [X, Y]_n \arrow[d, "d"] \\ & \mathrm{Ext}_{\mathscr{A}}^{1,n+1}(H^*(X), H^*(Y)) & \mathrm{Hom}_{\mathscr{A}}^n(H^*(X), H^*(Y)) . \end{tikzcd}$$

Theorem 4.15. *The Hopf maps $2\colon S \to S$, $\eta\colon S^1 \to S$, $\nu\colon S^3 \to S$ and $\sigma\colon S^7 \to S$ are detected in the Adams spectral sequence by the classes h_0, h_1, h_2 and h_3, respectively. These are infinite cycles in the spectral sequence.*

Proof. In each case, $f\colon S^n \to S$ is the stable form of a fibration $\Sigma^{n+1}f\colon S^{2n+1} \to S^{n+1}$, with mapping cone a projective plane P^2. Here $H^*(P^2) = P(x)/(x^3) = \mathbb{F}_2\{1, x, x^2\}$, where $|x| = n+1$, by Poincaré duality Hence $Sq^{n+1}(x) = x^2 \neq 0$, and the previous lemma applies. Quite explicitly, $\Sigma C_2 = \mathbb{R}P^2$ has a nonzero Sq^1, $\Sigma^2 C_\eta = \mathbb{C}P^2$ has a nonzero Sq^2, $\Sigma^4 C_\nu = \mathbb{H}P^2$ has a nonzero Sq^4 and $\Sigma^8 C_\sigma = \mathbb{O}P^2$ has a nonzero Sq^8. □

The names η, ν and σ for the Hopf maps detected by h_1, h_2 and h_3 are supposedly unrelated to the correspondence between the initial phonemes in the Greek letters "eta", "nu" and "sigma" and in the first three Japanese numerals "ichi", "ni" and "san". We shall see later that none of the classes h_i for $i \geq 4$ survive to the E_∞-term, so there are no maps $S^n \to S$ with nonzero Hopf–Steenrod invariant for $n \geq 8$.

4.5 Naturality

The essential uniqueness of free resolutions lifts to the level of spectral realizations. Consider diagrams

$$\cdots \to Y^{s+1} \xrightarrow{i} Y^s \to \cdots \to Y^0 = Y$$

and

$$\cdots \to Z^{s+1} \xrightarrow{i} Z^s \to \cdots \to Z^0 = Z$$

with cofibers $K^s = \mathrm{hocofib}(Y^{s+1} \to Y^s)$ and $L^s = \mathrm{hocofib}(Z^{s+1} \to Z^s)$ for all $s \geq 0$. There are associated chain complexes

$$\cdots \to P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(Y) \to 0$$

and

$$\cdots \to Q_2 \xrightarrow{\partial_2} Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\epsilon} H^*(Z) \to 0$$

of $\mathscr{A}$-modules, where $P_s = H^*(\Sigma^s K^s)$, $Q_s = H^*(\Sigma^s L^s)$, $\partial_s = \partial^* j^*$ and $\epsilon = j^*$.

Theorem 4.16. *Suppose that (a) each cofiber L^s is a wedge sum of Eilenberg–Mac Lane spectra that is bounded below and of finite type, and (b) each map $i\colon Y^{s+1} \to Y^s$ induces the zero map on cohomology. (For instance, $\{Y^s\}_s$ and $\{Z^s\}_s$ might be Adams resolutions.) Then each Q_s is a free $\mathscr{A}$-module, and the augmented chain complex $\epsilon\colon P_* \to H^*(Y) \to 0$ is exact.*

Let $f\colon Y \to Z$ be any map. Then there exists a chain map $g_*\colon Q_* \to P_*$ lifting f^*, in the sense that the diagram

$$\begin{tikzcd} \dots \arrow[r] & P_2 \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & H^*(Y) \arrow[r] & 0 \\ \dots \arrow[r] & Q_2 \arrow[r, "\partial_2"] \arrow[u, "g_2"] & Q_1 \arrow[r, "\partial_1"] \arrow[u, "g_1"] & Q_0 \arrow[r, "\epsilon"] \arrow[u, "g_0"] & H^*(Z) \arrow[r] \arrow[u, "f^*"] & 0 \end{tikzcd}$$

commutes. Furthermore, there is a map of resolutions $\{f^s \colon Y^s \to Z^s\}_s$ lifting f and realizing g_*, in the sense that there is a homotopy commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & Y^2 \arrow[r, "i"] \arrow[d, "f^2"] & Y^1 \arrow[r] \arrow[d, "f^1"] & Y \arrow[d, "f"] \\
\cdots \arrow[r] & Z^2 \arrow[r, "i"] & Z^1 \arrow[r] & Z\,,
\end{tikzcd}$$

and given any choice of commuting homotopies, the induced map of homotopy cofibers $g^s \colon K^s \to L^s$ induces $g_s = (\Sigma^s g^s)^* \colon Q_s \to P_s$, for each $s \geq 0$.

If $\bar{g}_* \colon Q_* \to P_*$ is a second chain map lifting f^*, and $\{\bar{f}^s\}_s$ is a map of resolutions lifting f and realizing $\bar{g}_*$, then g_* and $\bar{g}_*$ are chain homotopic, and $\{f^s\}_s$ and $\{\bar{f}^s\}_s$ are homotopic in the sense that the composites $f^s \circ i$ and $\bar{f}^s \circ i \colon Y^{s+1} \to Z^s$ are homotopic for all $s \geq 0$.

Proof. Freeness of each Q_s is clear from the wedge sum decomposition of L^s. Exactness of $\epsilon \colon P_* \to H^*(Y) \to 0$ is clear from the vanishing of i^*. The existence of a chain map g_* lifting f^* is standard homological algebra. We need to construct a diagram

$$\begin{tikzcd}
\cdots \arrow[r, "i"] & Y^2 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd, "f^2"] & & Y^1 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd, "f^1"] & & Y \arrow[dl, "j"] \arrow[dd, "f"] \\
\cdots & & K^1 \arrow[ul, dashed, "\partial"] \arrow[dd, "g^1"] & & K^0 \arrow[ul, dashed, "\partial"] \arrow[dd, "g^0"] & \\
\cdots \arrow[r] & Z^2 \arrow[rr, "i"] \arrow[dl, "j"] & & Z^1 \arrow[rr, "i"] \arrow[dl, "j"] & & Z \arrow[dl, "j"] \\
\cdots & & L^1 \arrow[ul, dashed, "\partial"] & & L^0 \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

of spectra, inducing a commutative diagram

$$\begin{tikzcd}
 & H^*(\Sigma^2 Y^2) \arrow[dr, "\partial^*"] & & H^*(\Sigma Y^1) \arrow[dr, "\partial^*"] & & H^*(Y) \\
\cdots \arrow[ur, "j^*"] \arrow[rr] & & H^*(\Sigma K^1) \arrow[ur, "j^*"] \arrow[rr] & & H^*(K^0) \arrow[ur, "j^*"] & \\
 & H^*(\Sigma^2 Z^2) \arrow[uu, "(\Sigma^2 f^2)^*"] \arrow[dr, "\partial^*"] & & H^*(\Sigma Z^1) \arrow[uu, "(\Sigma f^1)^*"] \arrow[dr, "\partial^*"] & & H^*(Z) \arrow[uu, "f^*"] \\
\cdots \arrow[ur, "j^*"] \arrow[rr] & & H^*(\Sigma L^1) \arrow[uu, "(\Sigma g^1)^*"] \arrow[ur, "j^*"] \arrow[rr] & & H^*(L^0) \arrow[uu, "(g^0)^*"] \arrow[ur, "j^*"] &
\end{tikzcd}$$

of $\mathscr{A}$-modules, with $g_s = (\Sigma^s g^s)^*$.

Inductively, suppose the maps $f = f^0, \dots, f^s$ and $g^0, \dots, g^{s-1}$ are given, for some $s \geq 0$. Then $j^* \circ g_s = (\Sigma^s f^s)^* \circ j^*$, by the assumption that g_0 lifts f^* for $s = 0$, and by the assumption that $\partial^* j^* \circ g_s = g_{s-1} \circ \partial^* j^* = \partial^* (\Sigma^s f^s)^* \circ j^*$ and the injectivity of ∂^* for $s \geq 1$.

We have an isomorphism $[K^s, L^s] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(L^s), H^*(K^s))$, so there is a unique homotopy class of maps $g^s \colon K^s \to L^s$ with $(\Sigma^s g^s)^* = g_s$. Note that $g^s \circ j \colon Y^s \to L^s$ is homotopic to $j \circ f^s \colon Y^s \to L^s$, because of the isomorphism $[Y^s, L^s] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(L^s), H^*(Y^s))$ and the fact that $(g^s \circ j)^* = (j \circ f^s)^*$. (Both isomorphisms follow from hypothesis (a)).

Choosing a commuting homotopy and passing to mapping cones, or appealing to the triangulated structure on the stable category of spectra, we can find a map $f^{s+1} \colon Y^{s+1} \to Z^{s+1}$ making the diagram

$$\begin{tikzcd}
Y^{s+1} \arrow[r, "i"] \arrow[d, "f^{s+1}"'] & Y^s \arrow[r, "j"] \arrow[d, "f^s"'] & K^s \arrow[r, "\partial"] \arrow[d, "g^s"'] & \Sigma Y^{s+1} \arrow[d, "\Sigma f^{s+1}"'] \\
Z^{s+1} \arrow[r, "i"] & Z^s \arrow[r, "j"] & L^s \arrow[r, "\partial"] & \Sigma Z^{s+1}
\end{tikzcd}$$

commute up to homotopy. This completes the inductive step.

The uniqueness of g_* up to chain homotopy, meaning that any other lift $\bar{g}_*$ is chain homotopic to g_*, is standard homological algebra. We prove that $f^s \circ i$ is homotopic to $\bar{f}^s \circ i$ by induction on s. This is clear for $s = 0$, since $f_0 = \bar{f}_0 = f$. Suppose that $i \circ f^s \simeq f^{s-1} \circ i$ is homotopic to $i \circ \bar{f}^s \simeq \bar{f}^{s-1} \circ i \colon Y^s \to Z^{s-1}$, for some $s \geq 1$.

$$\begin{tikzcd}
Y^{s+1} \arrow[r, "i"] & Y^s \arrow[r, "i"] \arrow[d, shift right, "f^s"'] \arrow[d, shift left, "\bar{f}^s"] & Y^{s-1} \arrow[d, shift right, "f^{s-1}"'] \arrow[d, shift left, "\bar{f}^{s-1}"] \\
 & Z^s \arrow[r, "i"] & Z^{s-1} \\
 & & \Sigma^{-1} L^{s-1} \arrow[ul, "\partial"]
\end{tikzcd}$$

Then $i \circ (\bar{f}^s - f^s)$ is null-homotopic, so that $\bar{f}^s - f^s$ factors through a map $h \colon Y^s \to \Sigma^{-1} L^{s-1}$. Then $\bar{f}^s \circ i - f^s \circ i = (\bar{f}^s - f^s) \circ i$ factors through $h \circ i \colon Y^{s+1} \to \Sigma^{-1} L^{s-1}$. This map induces $i^* \circ h^* = 0$ in cohomology, hence is null-homotopic because of the isomorphism $[Y^{s+1}, \Sigma^{-1} L^{s-1}] \cong \mathrm{Hom}_{\mathscr{A}}(H^*(\Sigma^{-1} L^{s-1}), H^*(Y^{s+1}))$. In other words, $f^s \circ i \simeq \bar{f}^s \circ i$. □

Corollary 4.17. *Let $f \colon Y \to Z$ be a map of bounded below spectra with $H_*(Y)$ and $H_*(Z)$ of finite type. Then there is a map*

$$f_* \colon \{E_r(Y), d_r\}_r \longrightarrow \{E_r(Z), d_r\}_r$$

of Adams spectral sequences, given at the E_2-level by the homomorphism

$$(f^*)^* \colon \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Y), \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(Z), \mathbb{F}_2)$$

induced by the $\mathscr{A}$-module homomorphism $f^* \colon H^*(Z) \to H^*(Y)$, with expected abutment the homomorphism

$$f_* \colon \pi_*(Y) \to \pi_*(Z)\,.$$

(Similarly for the Adams spectral sequences converging to $[X, Y]_*$ and $[X, Z]_*$, for any spectrum X.)

Lemma 4.18. *Let $\{Y^s\}_s$ and $\{Z^s\}_s$ be Adams resolutions of a bounded below spectrum Y with $H_*(Y)$ of finite type. Then there is a homotopy equivalence* $\mathrm{holim}_s\, Y^s \simeq \mathrm{holim}_s\, Z^s$.

Proof. There are maps $\{f^s \colon Y^s \to Z^s\}_s$ and $\{\tilde{f}^s \colon Z^s \to Y^s\}_s$ of resolutions covering the identity map of $Y = Y^0 = Z^0$, and homotopies $\tilde{f}^s \circ f^s \circ i \simeq i \colon Y^{s+1} \to Y^s$ and $f^s \circ \tilde{f}^s \circ i \simeq i \colon Z^{s+1} \to Z^s$, for all $s \geq 0$. Hence $\mathrm{holim}_s\, f^s$ and $\mathrm{holim}_s\, \tilde{f}^s$ are homotopy inverses. □

Theorem 4.19. *Let $\{Y^s\}_s$ be an Adams resolution of Y, and let X be any spectrum. (The case $X = S$ is of particular interest.) A class $[f] \in [X, Y]_n$ has Adams filtration $\geq s$, i.e., is in the image F^s of $i^s \colon [X, Y^s]_n \to [X, Y]_n$, if and only if the representing map $f \colon \Sigma^n X \to Y$ can be factored as the composite of s maps*

$$\Sigma^n X = X_s \xrightarrow{z_s} X_{s-1} \xrightarrow{z_{s-1}} \dots \xrightarrow{z_2} X_1 \xrightarrow{z_1} X_0 = Y$$

where $0 = z_u^* \colon H^*(X_{u-1}) \to H^*(X_u)$ for each $1 \leq u \leq s$. In particular, $F^s \subset [X, Y]_*$ is independent of the choice of Adams resolution.

Proof. If $[f]$ has Adams filtration $\geq s$, let $g \colon \Sigma^n X \to Y^s$ be a lift, with $i^s \circ g \simeq f$. Let $X_u = Y^u$ and $z_u = i$ for $0 \leq u \leq s - 1$, and let $z_s = ig$:

$$S^n \xrightarrow{ig} Y^{s-1} \xrightarrow{i} \dots \xrightarrow{i} Y^1 \xrightarrow{i} Y$$

Conversely, given a factorization $f = z_1 \circ \dots \circ z_s$ as above, let $f^0 \colon Y \to Y$ be the identity map. We can inductively find lifts $f^u \colon X_u \to Y^u$ making the diagram

$$\begin{tikzcd}
X_s \arrow[r, "z_s"] \arrow[d, "f^s"'] & X_{s-1} \arrow[r, "z_{s-1}"] \arrow[d, "f^{s-1}"'] & \dots \arrow[r, "z_2"] & X_1 \arrow[r, "z_1"] \arrow[d, "f^1"'] & Y \arrow[d] \\
Y^s \arrow[r, "i"] & Y^{s-1} \arrow[r, "i"] & \dots \arrow[r, "i"] & Y^1 \arrow[r, "i"] & Y
\end{tikzcd}$$

commute, since the obstruction to lifting $f^{u-1} \circ z_u \colon X_u \to Y^{u-1}$ over $i \colon Y^u \to Y^{u-1}$ is the homotopy class of the composite $j \circ f^{u-1} \circ z_u \colon X_u \to K^{u-1}$, which is zero because $z_u^* = 0$. Let $g = f^s \colon \Sigma^n X \to Y^s$. Then $i^s \circ g \simeq f$, and $[f]$ has Adams filtration $\geq s$. □

4.6 Convergence

Definition 4.20. For each natural number m let the mod m Moore spectrum S/m be defined by the cofiber sequence

$$S \xrightarrow{m} S \longrightarrow S/m \longrightarrow S^1$$

where the map m induces multiplication by m in integral (co-)homology. Note that $H_*(S/m;\mathbb{Z}) \cong \mathbb{Z}/m$ is concentrated in degree 0. For any spectrum Y let $Y/m = Y \wedge S/m$, so that there is a cofiber sequence

$$Y \xrightarrow{m} Y \longrightarrow Y/m \longrightarrow \Sigma Y\,.$$

Applying $F(-, Y)$ to the cofiber sequence

$$S^{-1} \longrightarrow S^{-1}/m \longrightarrow S \xrightarrow{m} S$$

leads to the cofiber sequence

$$Y \xrightarrow{m} Y \longrightarrow F(S^{-1}/m, Y) \longrightarrow \Sigma Y$$

and an equivalence $Y/m \simeq F(S^{-1}/m, Y)$.

Definition 4.21. For each prime p there is a horizontal tower of vertical cofiber sequences

$$\begin{tikzcd}
\dots \arrow[r, "p"] & S \arrow[r, "p"] \arrow[d, "p^e"'] & \dots \arrow[r, "p"] & S \arrow[r, "p"] \arrow[d, "p^2"] & S \arrow[d, "p"] \\
\dots \arrow[r, "="] & S \arrow[r, "="] \arrow[d] & \dots \arrow[r, "="] & S \arrow[r, "="] \arrow[d] & S \arrow[d] \\
\dots \arrow[r] & S/p^e \arrow[r] \arrow[d] & \dots \arrow[r] & S/p^2 \arrow[r] \arrow[d] & S/p \arrow[d] \\
\dots \arrow[r, "p"] & S^1 \arrow[r, "p"] & \dots \arrow[r, "p"] & S^1 \arrow[r, "p"] & S^1
\end{tikzcd}$$

We define the p-completion of Y as the homotopy limit $Y_p^\wedge = \operatorname{holim}_e Y/p^e$ of the tower

$$\dots \to Y \wedge S/p^e \to \dots \to Y \wedge S/p^2 \to Y \wedge S/p\,.$$

The maps $S \to S/p^e$ induce the p-completion map $Y \to Y_p^\wedge$.

Dually there is a horizontal sequence of vertical cofiber sequence

$$\begin{tikzcd}
S^{-1} \arrow[r, "p"] \arrow[d] & S^{-1} \arrow[r, "p"] \arrow[d] & \dots \arrow[r, "p"] & S^{-1} \arrow[r, "p"] \arrow[d] & \dots \\
S^{-1}/p \arrow[r] \arrow[d] & S^{-1}/p^2 \arrow[r] \arrow[d] & \dots \arrow[r] & S^{-1}/p^e \arrow[r] \arrow[d] & \dots \\
S \arrow[r, "="] \arrow[d, "p"'] & S \arrow[r, "="] \arrow[d, "p^2"'] & \dots \arrow[r, "="] & S \arrow[r, "="] \arrow[d, "p^e"'] & \dots \\
S \arrow[r, "p"] & S \arrow[r, "p"] & \dots \arrow[r, "p"] & S \arrow[r, "p"] & \dots
\end{tikzcd}$$

Let $S^{-1}/p^\infty = \operatorname{hocolim}_e S^{-1}/p^e$. Note that $H_*(S^{-1}/p^\infty;\mathbb{Z}) \cong \mathbb{Z}/p^\infty \cong \mathbb{Q}/\mathbb{Z}_{(p)} \cong \mathbb{Q}_p/\mathbb{Z}_p$. Applying $F(-, Y)$ we get the tower defining the p-completion, so

$$Y_p^\wedge \simeq F(S^{-1}/p^\infty, Y)\,.$$

The map $S^{-1}/p^\infty \to S$ induces the p-completion map $Y \to Y_p^\wedge$.

((See Bousfield.))

Lemma 4.22. *The p-completion map induces an equivalence $Y/p^e \to (Y_p^\wedge)/p^e$ for each e. Hence it induces an isomorphism $H_*(Y) \cong H_*(Y_p^\wedge)$ in mod p homology (and cohomology).*

Proof. The map $S^{-1}/p^\infty \to S$ induces an equivalence $S^{-1}/p^e \wedge S^{-1}/p^\infty \to S^{-1}/p^e \wedge S = S^{-1}/p^e$, for each e. Apply $F(-, Y)$ to get the first conclusion. Apply integral homology to the equivalence $Y/p \to (Y_p^\wedge)/p$ to get the second conclusion. □

Lemma 4.23. *The p-completion map for $Y_p^\wedge$ is an equivalence $Y_p^\wedge \to (Y_p^\wedge)_p^\wedge$, meaning that p-completion is idempotent.*

Proof. Use that the map $S^{-1}/p^\infty \to S$ induces an equivalence $S^{-1}/p^\infty \wedge S^{-1}/p^\infty \to S^{-1}/p^\infty$, or pass to the limit over e from the previous lemma. □

Lemma 4.24. *Let $\pi_n(Y)_p^\wedge = \lim_e \pi_n(Y)/p^e$ be the algebraic p-completion of $\pi_n(Y)$. There is a short exact sequence*

$$0 \to \pi_n(Y)_p^\wedge \to \lim_e \pi_n(Y/p^e) \to \mathrm{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}(Y)) \to 0$$

and an isomorphism $\mathrm{Rlim}_e\, \pi_{n+1}(Y/p^e) \cong \mathrm{Rlim}_e \mathrm{Hom}(\mathbb{Z}/p^e, \pi_n(Y))$. If $\pi_*(Y)$ is of finite type, i.e., if $\pi_n(Y)$ is finitely generated for each n, then $\pi_n(Y) \otimes \mathbb{Z}_p \cong \pi_n(Y)_p^\wedge \cong \pi_n(Y_p^\wedge)$ for all n.

Proof. ((Straightforward. TBW.)) □

Example 4.25. (a) $H \simeq H_2^\wedge$ and $(H\mathbb{Z})_2^\wedge \simeq (H\mathbb{Z}_{(2)})_2^\wedge \simeq H\mathbb{Z}_2$.

(b) For $Y = H\mathbb{Z}[1/2]$ or $H\mathbb{Q}$ we have $Y/2^e \simeq *$ for all e, so $(H\mathbb{Z}[1/2])_2^\wedge \simeq (H\mathbb{Q})_2^\wedge \simeq *$.

(c) For $Y = H(\mathbb{Z}[1/2]/\mathbb{Z}) = H\mathbb{Z}/2^\infty$ or $H(\mathbb{Q}/\mathbb{Z})$ we have $Y/2^e \simeq \Sigma H\mathbb{Z}/2^e$ for all e, so $H(\mathbb{Z}[1/2]/\mathbb{Z})_2^\wedge = H(\mathbb{Z}/2^\infty)_2^\wedge \simeq H(\mathbb{Q}/\mathbb{Z})_2^\wedge \simeq \Sigma H\mathbb{Z}_2$.

Lemma 4.26. *Let $0 \to \bigoplus_\alpha \mathbb{Z} \to \bigoplus_\beta \mathbb{Z} \to \mathbb{Z}_2 \to 0$ be a short free resolution of $\mathbb{Z}_2$. There is a corresponding cofiber sequence $\bigvee_\alpha S \to \bigvee_\beta S \to S\mathbb{Z}_2$, where $H_*(S\mathbb{Z}_2; \mathbb{Z}) \cong \mathbb{Z}_2$ is concentrated in degree 0. Then $\pi_n(Y \wedge S\mathbb{Z}_2) \simeq \pi_n(Y) \otimes \mathbb{Z}_2$ for all n. In particular, $S_2^\wedge \simeq (S\mathbb{Z}_2)_2^\wedge \simeq S\mathbb{Z}_2$. If $\pi_*(Y)$ is of finite type then the natural map $Y \wedge S\mathbb{Z}_2 \to Y_2^\wedge$ is an equivalence, and $H_*(Y) \to H_*(Y_2^\wedge)$ is an isomorphism.*

Proof. ((Straightforward. TBW.)) □

Let $H\mathbb{Z}$ be the integral Eilenberg–Mac Lane spectrum, with $\pi_0(H\mathbb{Z}) = \mathbb{Z}$ and $\pi_i(H\mathbb{Z}) = 0$ for $i \neq 0$. It is a ring spectrum, with multiplication $\phi\colon H\mathbb{Z} \wedge H\mathbb{Z} \to H\mathbb{Z}$ and unit $\eta\colon S \to H\mathbb{Z}$. (Not to be confused with the Hopf map $\eta\colon S^1 \to S$.) Let $\overline{H\mathbb{Z}} = H\mathbb{Z}/S$ be the cofiber.

Lemma 4.27. $H^*(H\mathbb{Z}) \cong \mathscr{A}/\mathscr{A}\{Sq^1\}$.

Proof. Since the unit map $S \to H\mathbb{Z}$ induces an isomorphism on π_0 and a surjection on π_1, we find that $\overline{H\mathbb{Z}}$ is 1-connected. Hence $H^1(H\mathbb{Z}) \cong H^1(\overline{H\mathbb{Z}}) = 0$.

There is a short exact sequence of $\mathscr{A}$-modules

$$0 \longleftarrow \mathscr{A}/\mathscr{A}\{Sq^1\} \longleftarrow \mathscr{A} \longleftarrow \Sigma\mathscr{A}/\mathscr{A}\{Sq^1\} \longleftarrow 0$$

where the right hand arrow takes $\Sigma 1$ to Sq^1. It is clear that $\Sigma Sq^I \mapsto Sq^I \circ Sq^1$ maps to 0, for admissible I, if and only if $I = (i_1, \dots, i_\ell)$ with $i_\ell = 1$. These Sq^I generate precisely the left ideal $\mathscr{A}\{Sq^1\}$.

There is also a cofiber sequence $H\mathbb{Z} \xrightarrow{2} H\mathbb{Z} \longrightarrow H \longrightarrow \Sigma H\mathbb{Z}$, where $2^* = 0$, so that there is an associated short exact sequence

$$0 \longleftarrow H^*(H\mathbb{Z}) \longleftarrow H^*(H) \longleftarrow \Sigma H^*(H\mathbb{Z}) \longleftarrow 0\,.$$

in cohomology. Let $\mathscr{A} \to H^*(H)$ be the isomorphism taking Sq^I to its value on the generator $1 \in H^0(H)$. The composite $\Sigma A/A\{Sq^1\} \to \mathscr{A} \to H^*(H) \to H^*(H\mathbb{Z})$ is zero, since the source is generated by $\Sigma 1$ in degree 1, and $H^1(H\mathbb{Z}) = 0$. Hence there is a map from the first short exact sequence of $\mathscr{A}$-modules to the second one. By induction, we may assume that the left hand homomorphism $f\colon \mathscr{A}/\mathscr{A}\{Sq^1\} \to H^*(H\mathbb{Z})$ is an isomorphism in degrees $* < t$. Then the right hand homomorphism $\Sigma f\colon \Sigma\mathscr{A}/\mathscr{A}\{Sq^1\} \to \Sigma H^*(H\mathbb{Z})$ is an isomorphism in degrees $* \le t$. Since the middle map is an isomorphism, it follows that the left hand homomorphism is an isomorphism, also in degree t. □

Recall Boardman's notion of conditional convergence, meaning that $\lim_s A^s = 0$ and $\operatorname{Rlim}_s A^s = 0$, and the result that strong convergence follows from conditional convergence and the vanishing of the derived E_∞-term RE_∞. For the spectral sequence associated to an Adams resolution $\{Y^s\}_s$, conditional convergence is equivalent to the contractibility of the homotopy limit $Y^\infty = \operatorname{holim}_s Y^s$, in view of Milnor's short exact sequence

$$0 \to \operatorname*{Rlim}_s \pi_{n+1}(Y^s) \to \pi_n(\operatorname*{holim}_s Y^s) \to \lim_s \pi_n(Y^s) \to 0 \,.$$

As we have seen before, the condition $\operatorname{holim}_s Y^s \simeq *$ is independent of the choice of Adams resolution.

Lemma 4.28. *Let Y be bounded below with $H_*(Y)$ of finite type. Then there is an Adams resolution $\{Z^s\}_s$ of $Z = Y/2$ with* $\operatorname{holim}_s Z^s \simeq *$.

((Enough that $Y/2$ is bounded below with $H_*(Y/2)$ of finite type?))

Proof. The "canonical $H\mathbb{Z}$-based resolution"

$$\begin{tikzcd}
\dots \arrow[r] & (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge 2} \arrow[r, "i"] \arrow[d, "j"] & \Sigma^{-1}\overline{H\mathbb{Z}} \arrow[r, "i"] \arrow[d, "j"] & S \arrow[d, "j"] \\
& H\mathbb{Z} \wedge (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge 2} \wedge \overline{H\mathbb{Z}} & H\mathbb{Z} \wedge \Sigma^{-1}\overline{H\mathbb{Z}} & H\mathbb{Z}
\end{tikzcd}$$

is not an Adams resolution, since $H\mathbb{Z}$ is not a wedge sum of mod 2 Eilenberg–Mac Lane spectra, but the ring spectrum structure ensures that $j = \eta \wedge 1 \colon X \to H\mathbb{Z} \wedge X$ induces a split injection $1 \wedge j \colon H \wedge X \to H \wedge H\mathbb{Z} \wedge X$, so that $j^* \colon H^*(H\mathbb{Z} \wedge X) \to H^*(X)$ is surjective, for each spectrum X.

Smashing this diagram with $Z = Y/2$, we get a diagram

$$\begin{tikzcd}
\dots \arrow[r] & (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge 2} \wedge Y/2 \arrow[r, "i"] \arrow[d, "j"] & \Sigma^{-1}\overline{H\mathbb{Z}} \wedge Y/2 \arrow[r, "i"] \arrow[d, "j"] & Y/2 \arrow[d, "j"] \\
& H \wedge (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge 2} \wedge Y & H \wedge \Sigma^{-1}\overline{H\mathbb{Z}} \wedge Y & H \wedge Y
\end{tikzcd}$$

where we have identified $H\mathbb{Z} \wedge X \wedge Y/2$ with $H \wedge X \wedge Y$, for suitable X. This is the desired Adams resolution, with $Z^s = (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/2$ and cofibers $L^s = H \wedge (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s} \wedge Y$. The maps j are split injective, so each j^* is surjective, as before. Since $(\overline{H\mathbb{Z}})^{\wedge s} \wedge Y$ is bounded below and $H_*((\overline{H\mathbb{Z}})^{\wedge s} \wedge Y) \cong H_*(\overline{H\mathbb{Z}})^{\otimes s} \otimes H_*(Y)$ is of finite type, it follows that each L^s is a wedge sum of suspended mod 2 Eilenberg–Mac Lane spectra, satisfying the finiteness condition required for an Adams resolution.

It remains to show that $\operatorname{holim}_s Z^s \simeq *$. This is true in the strong sense that in each topological degree n, $\pi_n(Z^s) = 0$ for all sufficiently large s. By assumption there is an integer N such that $\pi_n(Y) = 0$ for all $n < N$. We have seen that $\overline{H\mathbb{Z}}$ is 1-connected, so that $(\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s}$ is $(s-1)$-connected. Then $Z^s = (\Sigma^{-1}\overline{H\mathbb{Z}})^{\wedge s} \wedge Y/2$ is $(N+s-1)$-connected. Hence $\pi_n(Z^s) = 0$ for all $n \le N+s-1$, or equivalently, for all $s > n - N$. □

Theorem 4.29. *Let Y be bounded below with $H_*(Y)$ of finite type. Then the Adams spectral sequence*

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(Y^\wedge_2)$$

is strongly convergent. In particular, there is a strongly convergent Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s}(S)^\wedge_2 \,.$$

More generally, the Adams spectral sequence

$$E_2^{s,t} = \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)) \Longrightarrow [X, Y^\wedge_2]_{t-s}$$

is conditionally convergent. It is strongly convergent when $RE_\infty = 0$, which happens, for instance, if $H^*(X)$ is of finite type and bounded above, or if the spectral sequence collapses at a finite stage.

Proof. Let $\{Y^s\}_s$ be an Adams resolution of $Y^0 = Y$, with cofiber sequences

$$Y^{s+1} \xrightarrow{i} Y^s \xrightarrow{j} K^s \xrightarrow{\partial} \Sigma Y^{s+1} .$$

Smashing with $S/2^e$ for each $e \geq 1$, we get a tower of Adams resolutions $\{Y^s/2^e\}_s$ of $Y^0/2^e = Y/2^e$, with cofiber sequences

$$Y^{s+1}/2^e \xrightarrow{i} Y^s/2^e \xrightarrow{j} K^s/2^e \xrightarrow{\partial} \Sigma Y^{s+1}/2^e .$$

(We check that these diagrams satisfy the conditions to be Adams resolutions: Each homomorphism $j^*\colon H^*(K^s/2^e) \to H^*(Y^2/2^e)$ can be rewritten as $j^* \otimes 1\colon H^*(K^s) \otimes H^*(S/2^e) \to H^*(Y^s) \otimes H^*(S/2^e)$, hence remains surjective. Each cofiber $K^s/2^e$ sits in a cofiber sequence

$$K^s \xrightarrow{2^e} K^s \longrightarrow K^s/2^e \longrightarrow \Sigma K^s$$

where 2^e is null-homotopic, so that $K^s/2^e \simeq K^s \vee \Sigma K^s$ is still a suitably finite wedge sum of mod 2 Eilenberg–Mac Lane spectra.) Now pass to the homotopy limit over e of these Adams resolutions. The result is a diagram $\{(Y^s)^\wedge_2\}_s$ of spectra, with cofiber sequences

$$(Y^{s+1})^\wedge_2 \xrightarrow{i} (Y^s)^\wedge_2 \xrightarrow{j} (K^s)^\wedge_2 \xrightarrow{\partial} \Sigma(Y^{s+1})^\wedge_2 .$$

(Cofiber sequences are fiber sequences, up to a sign, hence are preserved by passage to homotopy limits, such as completions.) It is again an Adams resolution, since the completion map $K^s \to (K^s)^\wedge_2$ is an equivalence ($K^s \simeq \bigvee_u \Sigma^{n_u} H \simeq \prod_u \Sigma^{n_u} H$ and $H \to H^\wedge_2$ is easily seen to be an equivalence) and $j\colon (Y^s)^\wedge_2 \to (K^s)^\wedge_2$ induces the "same" map as $j\colon Y^s \to K^s$ in mod 2 cohomology. We get the following vertical maps of Adams resolutions:

$$\begin{tikzcd}
\operatorname{holim}_s Y^s \arrow[rr] \arrow[dd] & & Y^2 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & Y^1 \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & Y \arrow[dl, "j"] \arrow[dd] \\
& K^2 \arrow[dd, "\simeq"] & & K^1 \arrow[dd, "\simeq"] & & K^0 \arrow[dd, "\simeq"] & \\
\operatorname{holim}_s (Y^s)^\wedge_2 \arrow[rr, crossing over] \arrow[dd] & & (Y^2)^\wedge_2 \arrow[rr, "i", crossing over] \arrow[dl, "j"] \arrow[dd] & & (Y^1)^\wedge_2 \arrow[rr, "i", crossing over] \arrow[dl, "j"] \arrow[dd] & & Y^\wedge_2 \arrow[dl, "j"] \arrow[dd] \\
& (K^2)^\wedge_2 \arrow[dd] & & (K^1)^\wedge_2 \arrow[dd] & & (K^0)^\wedge_2 \arrow[dd] & \\
\operatorname{holim}_s Y^s/2^e \arrow[rr, crossing over] & & Y^2/2^e \arrow[rr, "i", crossing over] \arrow[dl, "j"] & & Y^1/2^e \arrow[rr, "i", crossing over] \arrow[dl, "j"] & & Y/2^e \arrow[dl, "j"] \\
& K^2/2^e & & K^1/2^e & & K^0/2^e &
\end{tikzcd}$$

(We omit the maps $\partial\colon K^s \to \Sigma Y^{s+1}$, etc.) By the previous lemma, there exists an Adams resolution $\{Z^s\}_s$ for $Y/2$ with $\operatorname{holim}_s Z^s \simeq *$. Since this homotopy limit is independent of the choice of resolution, we must also have $\operatorname{holim}_s Y^s/2 \simeq *$.

There are cofiber sequences $S/2 \to S/2^{e+1} \to S^e \to \Sigma S/2$, inducing cofiber sequences $Y^s/2 \to Y^s/2^{e+1} \to Y^s/2^e \to \Sigma Y^s/2$ for all s, hence also

$$\operatorname*{holim}_s Y^s/2 \longrightarrow \operatorname*{holim}_s Y^s/2^{e+1} \longrightarrow \operatorname*{holim}_s Y^s/2^e \longrightarrow \Sigma \operatorname*{holim}_s Y^s/2 .$$

We deduce that $\operatorname{holim}_s Y^s/2^e \simeq *$ for all $e \geq 1$, by induction on e. Thus

$$\operatorname*{holim}_s (Y^s)^\wedge_2 = \operatorname*{holim}_s \operatorname*{holim}_e Y^s/2^e \simeq \operatorname*{holim}_e \operatorname*{holim}_s Y^s/2^e \simeq *$$

by the standard exchange of homotopy limits equivalence.

Applying homotopy, we get a map of unrolled exact couples from the one for Y to the one for $Y^\wedge_2$:

$$\begin{tikzcd}
\cdots \arrow[rr] & & \pi_*(Y^2) \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & \pi_*(Y^1) \arrow[rr, "i"] \arrow[dl, "j"] \arrow[dd] & & \pi_*(Y) \arrow[dl, "j"] \arrow[dd] \\
& \pi_*(K^2) \arrow[ul, dashed, "\partial"] \arrow[dd, "\cong"] & & \pi_*(K^1) \arrow[ul, dashed, "\partial"] \arrow[dd, "\cong"] & & \pi_*(K^0) \arrow[ul, dashed, "\partial"] \arrow[dd, "\cong"] & \\
\cdots \arrow[rr] & & \pi_*((Y^2)^\wedge_2) \arrow[rr, "i"] \arrow[dl, "j"] & & \pi_*((Y^1)^\wedge_2) \arrow[rr, "i"] \arrow[dl, "j"] & & \pi_*(Y^\wedge_2) \arrow[dl, "j"] \\
& \pi_*((K^2)^\wedge_2) \arrow[ul, dashed, "\partial"] & & \pi_*((K^1)^\wedge_2) \arrow[ul, dashed, "\partial"] & & \pi_*((K^0)^\wedge_2) \arrow[ul, dashed, "\partial"] &
\end{tikzcd}$$

This induces a map of spectral sequences, from the Adams spectral sequence for Y to the one associated to the lower exact couple. The equivalences $K^s \to (K^s)^\wedge_2$ induce isomorphisms

$$E_1^{s,t} = \pi_{t-s}(K^s) \xrightarrow{\cong} \pi_{t-s}((K^s)^\wedge_2)$$

of E_1-terms between these spectral sequences. By induction on r, it follows that it also induces an isomorphism of E_r-terms, for all $r \ge 1$. Hence we have two different exact couples generating the same spectral sequence. The upper one is the Adams spectral sequence for Y. The lower one is conditionally convergent to $\pi_*(Y^\wedge_2)$, since $\operatorname{holim}_s (Y^s)^\wedge_2 \simeq *$. Hence the Adams spectral sequence for Y, with $E_2^{*,*} = \operatorname{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_2)$, is conditionally convergent to $\pi_*(Y^\wedge_2)$, as asserted. Replacing $\pi_*(-)$ by $[X,-]_*$ we get the same conclusion for the Adams spectral sequence for maps $X \to Y$.

To get strong convergence to $\pi_*(Y^\wedge_2)$ or $[X, Y^\wedge_2]_*$, we need to verify Boardman's criterion $RE_\infty = 0$. In the first case, this follows since $E_2^{s,t}(Y)$ is of finite type, i.e., is finite(-dimensional) in each bidegree (s,t). In fact, this holds already at the E_1-term if we use the canonical Adams resolution for Y, with $\Sigma^s K^s = H \wedge (\bar{H})^{\wedge s} \wedge Y$, since then

$$E_1^{s,t} = \pi_{t-s}(K^s) \cong \pi_t(\Sigma^s K^s) \cong H_t((\bar{H})^{\wedge s} \wedge Y) \cong [H_*(\bar{H})^{\otimes s} \otimes H_*(Y)]_t \,.$$

In the case of a general spectrum X, we have

$$\begin{aligned} E_1^{s,t} &= [X, K^s]_{t-s} \cong [X, \Sigma^s K^s]_t \cong \operatorname{Hom}^t_{\mathscr{A}}(H^*(\Sigma^s K^s), H^*(X)) \\ &\cong \operatorname{Hom}^t_{\mathscr{A}}(\mathscr{A} \otimes I(\mathscr{A})^{\otimes s} \otimes H^*(Y), H^*(X)) \cong \operatorname{Hom}^t(I(\mathscr{A})^{\otimes s} \otimes H^*(Y), H^*(X)) \,. \end{aligned}$$

This group is finite if $H^*(X)$ is of finite type and bounded above, in the sense that there exists an integer N with $H^n(X) = 0$ for $n > N$. For instance, this is the case of X is a finite CW spectrum. $\square$

Proposition 4.30. *Let Y be bounded below with $H_*(Y)$ of finite type. There is a cofiber sequence*

$$\operatorname*{holim}_s Y^s \longrightarrow Y \longrightarrow Y^\wedge_2$$

where $\{Y^s\}_s$ is any Adams resolution of Y.

Proof. We use the notation of the proof above. In view of the equivalences $K^s \simeq (K^s)^\wedge_2$, we get a chain of equivalences

$$\operatorname*{holim}_s \operatorname{hofib}(Y^s \to (Y^s)^\wedge_2) \simeq \operatorname{hofib}(Y^s \to (Y^s)^\wedge_2) \simeq \cdots \simeq \operatorname{hofib}(Y \to Y^\wedge_2)$$

for all s. Passing to homotopy limits, we find that

$$\operatorname*{holim}_s Y^s \simeq \operatorname{hofib}(\operatorname*{holim}_s Y^s \to \operatorname*{holim}_s (Y^s)^\wedge_2) \simeq \operatorname*{holim}_s \operatorname{hofib}(Y^s \to (Y^s)^\wedge_2) \simeq \operatorname{hofib}(Y \to Y^\wedge_2) \,.$$

In other words, the 2-completion $Y \to Y^\wedge_2$ precisely annihilates the obstruction $\operatorname{holim}_s Y^s$ to conditional convergence for the unrolled exact couple associated to the Adams resolution of Y. $\square$

((Mention Bousfield's E-nilpotent completion $Y^\wedge_E = Y/\operatorname{holim}_s Y^s_E$ where $Y^s_E = (\Sigma^{-1}\bar{E})^{\wedge s} \wedge Y$?))

5 Multiplicative structure

5.1 Composition and the Yoneda product

Let X, Y and Z be spectra. We have a composition pairing

$$\circ \colon [Y,Z]_* \otimes [X,Y]_* \longrightarrow [X,Z]_*$$

that takes $g\colon \Sigma^v Y \to Z$ and $f\colon \Sigma^t X \to Y$ to the composite $g \circ \Sigma^v f \colon \Sigma^{v+t} X \to Z$. To simplify the notation we refer to f and g as maps $f\colon X \to Y$ and $g\colon Y \to Z$ of degree t and v, respectively.

Suppose that Y and Z are bounded below, and that $H_*(Y)$ and $H_*(Z)$ are of finite type. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, respectively, with cofibers $Y^s/Y^{s+1} = K^s$ and $Z^u/Z^{u+1} = L^u$. If f and g have Adams filtrations $\ge s$ and $\ge u$, meaning that they factor as $f = i^s \tilde{f}$ and $g = i^u \tilde{g}$ with $\tilde{f}\colon X \to Y^s$ and $\tilde{g}\colon Y \to Z^u$ of degree t and v, respectively, then we can lift $\tilde{g}$ to a map $\{g^s\}_s$ of Adams resolutions

$$\begin{tikzcd}
X \arrow[d, "\tilde{f}"'] & & \\
Y^s \arrow[r, "i"] \arrow[d, "g^s"'] & \dots \arrow[r, "i"] & Y \arrow[d, "\tilde{g}"] \\
Z^{u+s} \arrow[r, "i"] & \dots \arrow[r, "i"] & Z^u \,.
\end{tikzcd}$$

Hence $gf = i^u \tilde{g} i^s \tilde{f} = i^{u+s} g^s f$ factors through $i^{u+s}\colon Z^{u+s} \to Z$, and has Adams filtration $\ge (u+s)$. We thus get a restricted pairing

$$F^u[Y,Z]_* \otimes F^s[X,Y]_* \longrightarrow F^{u+s}[X,Z]_*$$

that induces a pairing

$$F^u/F^{u+1} \otimes F^s/F^{s+1} \longrightarrow F^{u+s}/F^{u+s+1}$$

of filtration subquotients. When the respective spectral sequences converge, we can rewrite this as a pairing

$$E_\infty^{u,*} \otimes E_\infty^{s,*} \longrightarrow E_\infty^{u+s,*}$$

of E_∞-terms. Conversely, this pairing of E_∞-terms will determine the restricted pairings $F^u \otimes F^s \to F^{u+s}$ modulo F^{u+s+1}, i.e., modulo higher Adams filtrations. In this way the pairing of E_∞-terms determines the composition pairing $[Y,Z]_* \otimes [X,Y]_* \to [X,Z]_*$ modulo the Adams filtration.

((Example of this phenomenon: $h_2^3 = h_1^2 h_3$ so $\nu^3 \equiv \eta^2\sigma$ modulo Adams filtration ≥ 4. In fact, $\nu^3 = \eta^2\sigma + \eta\epsilon$.))

Let $P_s = H^*(\Sigma^s K^s)$ and $Q_u = H^*(\Sigma^u L^u)$, so that there are free resolutions

$$\cdots \to P_s \xrightarrow{\partial_s} \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(Y) \to 0$$

and

$$\cdots \to Q_u \xrightarrow{\partial_u} \cdots \to Q_1 \xrightarrow{\partial_1} Q_0 \xrightarrow{\epsilon} H^*(Z) \to 0\,.$$

By definition,

$$\begin{aligned}
\mathrm{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), H^*(Y)) &= H^u(\mathrm{Hom}_{\mathscr{A}}^v(Q_*, H^*(Y))) \\
\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X)) &= H^s(\mathrm{Hom}_{\mathscr{A}}^t(P_*, H^*(X))) \\
\mathrm{Ext}_{\mathscr{A}}^{u+s,v+t}(H^*(Z), H^*(X)) &= H^{u+s}(\mathrm{Hom}_{\mathscr{A}}^{v+t}(Q_*, H^*(X)))\,.
\end{aligned}$$

The (opposite) Yoneda product is a pairing

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), H^*(X)) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(X))\,,$$

and we shall see that the Adams spectral sequence relates the Yoneda product in $E_2 = \mathrm{Ext}_{\mathscr{A}}(-,-)$ to the composition product in homotopy. (This is the opposite of the usual Yoneda pairing, meaning that the two factors in the source have been interchanged. This comes about due to the contravariance of cohomology. Working at odd primes the interchange introduces a sign, which we ignore here.)

Let $f\colon P_s \to \Sigma^t H^*(X)$ and $g\colon Q_u \to \Sigma^v H^*(Y)$ be $\mathscr{A}$-module homomorphisms. To simplify the notation, we will refer to these as homomorphisms $f\colon P_s \to H^*(X)$ and $g\colon Q_u \to H^*(Y)$ of degree t and v, respectively. We also suppose that f and g are cocycles, meaning that $0 = f\partial_{s+1}\colon P_{s+1} \to H^*(X)$ and $0 = g\partial_{u+1}\colon Q_{u+1} \to H^*(Y)$. The cohomology classes $[f]$ and $[g]$ are then elements in $\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), H^*(X))$ and $\mathrm{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), H^*(Y))$, respectively. Then g lifts to a chain map $g_* = \{g_n\colon Q_{u+n} \to P_n\}_n$, where each g_n has degree v, making the diagram

$$\begin{tikzcd}
 & H^*(X) & & & & \\
\dots \arrow[r] & P_s \arrow[u, "f"] \arrow[r, "\partial_s"] & \dots \arrow[r] & P_1 \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & H^*(Y) \\
\dots \arrow[r] & Q_{u+s} \arrow[u, "g_s"] \arrow[r, "\partial_{u+s}"] & \dots \arrow[r] & Q_{u+1} \arrow[u, "g_1"] \arrow[r, "\partial_{u+1}"] & Q_u \arrow[u, "g_0"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commute. The composite $fg_s\colon Q_{u+s} \to H^*(X)$ is then an $\mathscr{A}$-module homomorphism of degree $(v+t)$, and satisfies $fg_s\partial_{u+s+1} = 0$. It is therefore a cocycle in $\mathrm{Hom}_{\mathscr{A}}^{v+t}(H^*(Z), H^*(X))$, and its cohomology class $[fg_s]$ in $\mathrm{Ext}_{\mathscr{A}}^{u+s,v+t}(H^*(Z), H^*(X))$ is by definition the Yoneda product of $[g]$ and $[f]$. It is not hard to check that a different choice of chain map lifting g only changes the cocycle fg_s by a coboundary, i.e., a homomorphism that factors through $\partial_{u+s}\colon Q_{u+s} \to Q_{u+s-1}$, so that its cohomology class is unchanged. Likewise, changing f or g by a coboundary only changes fg_s by a coboundary, so that the Yoneda product is well defined.

Example 5.1. Let $X = Y = Z = S$ and let $P_* = Q_*$ be the minimal resolution of $\mathbb{F}_2$ computed earlier. We can compute the Yoneda product

$$\mathrm{Ext}_{\mathscr{A}}^{u,v}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{u+s,v+t}(\mathbb{F}_2, \mathbb{F}_2)$$

that makes $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ into a bigraded algebra, by choosing cocycle representatives $f\colon P_s \to \mathbb{F}_2$ and $g\colon P_u \to \mathbb{F}_2$, lifting g to a chain map $g_*\colon P_{u+*} \to P_*$, and computing the composite fg_s.

Let $f = \gamma_{1,0} = h_0\colon P_1 \to \mathbb{F}_2$ be dual to $g_{1,0} \in P_1$ and let $g = \gamma_{1,2} = h_2\colon P_1 \to \mathbb{F}_2$ be dual to $g_{1,2} \in P_1$. A lift $g_0\colon P_1 \to P_0$ of g is given by $g_{1,2} \mapsto g_{0,0}$ and $g_{1,i} \mapsto 0$ for $i \neq 2$.

$$\begin{tikzcd}
\mathbb{F}_2 & & \\
P_1 \arrow[u, "f=h_0"] \arrow[r, "\partial_1"] & P_0 \arrow[r, "\epsilon"] & \mathbb{F}_2 \\
P_2 \arrow[u, "g_1"] \arrow[r, "\partial_2"] & P_1 \arrow[u, "g_0"] \arrow[ur, "g=h_2"'] &
\end{tikzcd}$$

The composite $g_0\partial_2\colon P_2 \to P_0$ is then given by $g_{2,0} \mapsto 0$, $g_{2,1} \mapsto 0$, $g_{2,2} \mapsto Sq^1 g_{0,0}$, $g_{2,3} \mapsto Sq^4 g_{0,0}$ etc. A lift $g_1\colon P_2 \to P_1$ is given by $g_{2,0} \mapsto 0$, $g_{2,1} \mapsto 0$, $g_{2,2} \mapsto g_{1,0}$, $g_{2,3} \mapsto g_{1,2}$ etc. Hence $fg_1\colon P_2 \to \mathbb{F}_2$ is given by $g_{2,2} \mapsto 1$ and $g_{2,i} \mapsto 0$ for $i \neq 2$ (for degree reasons), so that $[fg_1] = \gamma_{2,2}$. Thus $h_0 h_2 = \gamma_{2,2}$ in bidegree $(s,t) = (2,4)$ of $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. In hindsight, this it the only possible nonzero value of the product, and it is realized because of the summand $Sq^1 g_{1,2}$ in $\partial_2(g_{2,2})$ and the summand $Sq^4 g_{0,0}$ in $\partial_1(g_{1,2})$, with Sq^1 detecting h_0 and Sq^4 detecting h_2.

Definition 5.2. Consider any two complexes P_* and Q_* of $\mathscr{A}$-modules. Let

$$\mathrm{HOM}_{\mathscr{A}}^{u,v}(Q_*, P_*) = \prod_s \mathrm{Hom}_{\mathscr{A}}^v(Q_{u+s}, P_s)$$

be the abelian group of sequences $\{g_s\colon Q_{u+s} \to P_s\}_s$ of $\mathscr{A}$-module homomorphisms, each of degree v. Thus $\mathrm{HOM}_{\mathscr{A}}^u(Q_*, P_*)$ is a graded abelian group. Let

$$\delta_u\colon\ \mathrm{HOM}_{\mathscr{A}}^u(Q_*, P_*) \to \mathrm{HOM}_{\mathscr{A}}^{u+1}(Q_*, P_*)$$

map $\{g_s\}_s$ to $\{\partial_{s+1} g_{s+1} + g_s \partial_{u+s+1}\}_s$. ((We are working mod 2, so there is no sign.)) Then $\delta_{u+1}\delta_u = 0$, so $\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*)$ is a cocomplex of graded abelian groups.

Lemma 5.3. *The kernel*

$$\ker(\delta_0) \subset \mathrm{HOM}^0_{\mathscr{A}}(Q_*, P_*)$$

consists of the chain maps $g_* \colon Q_* \to P_*$, meaning the sequences $\{g_s \colon Q_s \to P_s\}_s$ of $\mathscr{A}$-module homomorphisms such that $\partial_{s+1} g_{s+1} = g_s \partial_{s+1}$ for all s. The image

$$\mathrm{im}(\delta_{-1}) \subset \ker(\delta_0)$$

consists of the chain maps that are chain homotopic to 0, i.e., those of the form $\{\partial_{s+1} h_{s+1} + h_s \partial_s\}_s$ for some collection of $\mathscr{A}$-module homomorphisms $h_{s+1} \colon Q_s \to P_{s+1}$ for all s. Hence the 0-th cohomology

$$H^0(\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*)) \cong \{g_* \colon Q_* \to P_*\}/(\simeq) = [Q_*, P_*]$$

is the (graded abelian) group of chain homotopy classes of chain maps $Q_* \to P_*$. More generally, $H^u(\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*))$ is the group $[Q_{u+*}, P_*]$ of chain homotopy classes of chain maps $Q_{u+*} \to P_*$.

In the special case when $P_* = H^*(Y)$ is concentrated in filtration $s = 0$, so that $P_0 = H^*(Y)$ and $P_s = 0$ for $s \neq 0$, then $\mathrm{HOM}^{u,v}_{\mathscr{A}}(Q_*, H^*(Y)) \cong \mathrm{Hom}^v_{\mathscr{A}}(Q_u, H^*(Y))$ and $\delta_u = (\partial_{u+1})^*$, so that $H^u(\mathrm{HOM}_{\mathscr{A}}(Q_*, H^*(Y))) \cong H^u(\mathrm{Hom}_{\mathscr{A}}(Q_*, H^*(Y)))$. When Q_* is a free resolution of $H^*(Z)$, this is $\mathrm{Ext}^u_{\mathscr{A}}(H^*(Z), H^*(Y))$.

Proposition 5.4. *Let $\epsilon \colon P_* \to H^*(Y)$ and $\epsilon \colon Q_* \to H^*(Z)$ be free $\mathscr{A}$-module resolutions. Then*

$$\epsilon_* \colon \mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*) \xrightarrow{\simeq} \mathrm{HOM}^*_{\mathscr{A}}(Q_*, H^*(Y)) \cong \mathrm{Hom}_{\mathscr{A}}(Q_*, H^*(Y))$$

is a quasi-isomorphism, in the sense that it induces an isomorphism

$$\epsilon_* \colon H^u(\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*)) \xrightarrow{\cong} \mathrm{Ext}^u_{\mathscr{A}}(H^*(Z), H^*(Y))$$

in cohomology, in each filtration u.

This is standard homological algebra. The first assertion only requires that Q_* is free and $P_* \to H^*(Y)$ is exact, but the identification with the final Ext requires that $Q_* \to H^*(Z)$ is exact.

The composition pairing and the quasi-isomorphism

$$\begin{tikzcd}
\mathrm{HOM}^*_{\mathscr{A}}(Q_*, P_*) \otimes \mathrm{Hom}_{\mathscr{A}}(P_*, H^*(X)) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}_{\mathscr{A}}(Q_*, H^*(X)) \\
\mathrm{Hom}^*_{\mathscr{A}}(Q_*, H^*(Y)) \otimes \mathrm{Hom}_{\mathscr{A}}(P_*, H^*(X)) &
\end{tikzcd}$$

thus induce a pairing and an isomorphism

$$\begin{tikzcd}
H^u(\mathrm{Hom}^*_{\mathscr{A}}(Q_*, P_*)) \otimes \mathrm{Ext}^s_{\mathscr{A}}(H^*(Y), H^*(X)) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^{u+s}_{\mathscr{A}}(H^*(Z), H^*(X)) \\
\mathrm{Ext}^u_{\mathscr{A}}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}^s_{\mathscr{A}}(H^*(Y), H^*(X)) \arrow[ur, dashed] &
\end{tikzcd}$$

in cohomology, and the Yoneda product is given by the dashed arrow. From this description it is easy to see that the Yoneda product is associative and unital.

5.2 Smash product and tensor product

Let T, V, Y and Z be spectra. We have a smash product pairing

$$\wedge \colon [T, Y]_* \otimes [V, Z]_* \longrightarrow [T \wedge V, Y \wedge Z]_*$$

taking $f\colon T \to Y$ and $g\colon V \to Z$ to $f \wedge g\colon T \wedge V \to Y \wedge Z$, and similarly for graded maps. In particular, for $T = V = S$ we have a pairing

$$\wedge\colon \pi_*(Y) \otimes \pi_*(Z) \longrightarrow \pi_*(Y \wedge Z)\,.$$

If Y is a ring spectrum, with unit $\eta\colon S \to Y$ and multiplication $\mu\colon Y \wedge Y \to Y$, we have a unit homomorphism

$$\eta_*\colon \pi_*(S) \longrightarrow \pi_*(Y)$$

and a product

$$\pi_*(Y) \otimes \pi_*(Y) \xrightarrow{\wedge} \pi_*(Y \wedge Y) \xrightarrow{\mu_*} \pi_*(Y)$$

that make $\pi_*(Y)$ an algebra over $\pi_*(S)$. If Y is homotopy commutative, then $\pi_*(Y)$ is a (graded) commutative $\pi_*(S)$-algebra.

When $Y = S$, the smash product $\wedge\colon \pi_*(S) \otimes \pi_*(S) \to \pi_*(S)$ agrees up to sign with the composition product $\circ\colon \pi_*(S) \otimes \pi_*(S) \to \pi_*(S)$. In detail, the smash product of $f\colon S^t \to S$ and $g\colon S^v \to S$ is $f \wedge g\colon S^{t+v} \cong S^t \wedge S^v \to S \wedge S = S$, while the composition product is $f \circ \Sigma^t g\colon S^{v+t} = \Sigma^t S^v \to \Sigma^t S = S^t \to S$. These agree up to the twist equivalence $\gamma\colon S^t \wedge S^v \cong S^v \wedge S^t$, which is a a map of degree $(-1)^{tv}$.

Now suppose that Y and Z are bounded below with $H_*(Y)$ and $H_*(Z)$ of finite type, and let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions. If $f\colon T \to Y$ and $g\colon V \to Z$ have Adams filtrations $\ge s$ and $\ge u$, respectively, then they factor as the composites of s maps

$$T = T_s \to \cdots \to T_0 = Y$$

and u maps

$$V = V_u \to \cdots \to V_0 = Z\,,$$

all inducing zero on cohomology. By the Künneth theorem, the smash product $f \wedge g$ then factors as the composite of $(s+u)$ cohomologically trivial maps

$$T \wedge V = T_s \wedge V_u \to \cdots \to T_0 \wedge V_u \to \cdots \to T_0 \wedge V_0 = Y \wedge Z\,.$$

Hence we get a restricted pairing

$$F^s[T, Y]_* \otimes F^u[V, Z]_* \longrightarrow F^{s+u}[T \wedge V, Y \wedge Z]_*$$

that descends to a pairing

$$F^s/F^{s+1} \otimes F^u/F^{u+1} \longrightarrow F^{s+u}/F^{s+u+1}$$

of filtration quotients.

((TODO: Discuss tensor product pairing of complexes and Ext, and compare with the Yoneda pairing.))

The Yoneda pairing

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$$

agrees with the tensor product pairing

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)\,,$$

so the two multiplicative structures on the Adams spectral sequence for S agree. ((Give proof?))

5.3 Pairings of spectral sequences

Definition 5.5. Let $\{'E_r\}_r$, $\{''E_r\}_r$ and $\{E_r\}_r$ be three spectral sequence. A pairing of these spectral sequences is a sequence of homomorphisms

$$\phi_r\colon {}'E_r^{*,*} \otimes {}''E_r^{*,*} \longrightarrow E_r^{*,*}$$

((for $r \ge 1$)) such that the Leibniz rule

$$d_r(\phi_r(x \otimes y)) = \phi_r(d_r(x) \otimes y) + (-1)^n \phi_r(x \otimes d_r(y))$$

holds, where $n = |x|$ is the total degree of x, and

$$\phi_{r+1}([x] \otimes [y]) = [\phi_r(x \otimes y)]$$

where $[x] \in {}'E^{*,*}_{r+1}$ is the homology class of a d_r-cycle $x \in {}'E^{*,*}_r$, and similarly for $[y]$ and the right hand side. In other words, the diagrams

$$\begin{tikzcd}
{}'E^{*,*}_r \otimes {}''E^{*,*}_r \arrow[r, "\phi_r"] \arrow[d, "d_r\otimes 1 \pm 1 \otimes d_r"'] & E^{*,*}_r \arrow[d, "d_r"] \\
{}'E^{*,*}_r \otimes {}''E^{*,*}_r \arrow[r, "\phi_r"] & E^{*,*}_r
\end{tikzcd}$$

and

$$\begin{tikzcd}
H^{*,*}({}'E_r) \otimes H^{*,*}({}''E_r) \arrow[r] \arrow[d, "\cong"'] & H^{*,*}({}'E_r \otimes {}''E_r) \arrow[r, "(\phi_r)_*"] & H^{*,*}(E_r) \arrow[d, "\cong"] \\
{}'E^{*,*}_{r+1} \otimes {}''E^{*,*}_{r+1} \arrow[rr, "\phi_{r+1}"] & & E^{*,*}_{r+1}
\end{tikzcd}$$

commute.

A spectral sequence pairing $\{\phi_r\}_r$ induces a pairing

$$\phi_\infty \colon {}'E^{*,*}_\infty \otimes {}''E^{*,*}_\infty \longrightarrow E^{*,*}_\infty$$

of E_∞-terms. ((Clear if each spectral sequence vanishes in negative filtrations, so that in each bidegree (s,t) the E_r-terms eventually form a descending sequence, with intersection equal to the E_∞-term.))

When the Künneth homomorphism $H^{*,*}({}'E_r) \otimes H^{*,*}({}''E_r) \to H^{*,*}({}'E_r \otimes {}''E_r)$ is an isomorphism, for all r, one can readily define a tensor product spectral sequence $\{{}'E_r \otimes {}''E_r\}_r$, and the pairing of spectral sequences is the same as a morphism $\{{}'E_r \otimes {}''E_r\}_r \to \{E_r\}_r$ of spectral sequences.

Definition 5.6. Suppose that the spectral sequences above converge to the graded abelian groups G', G'' and G, respectively, in the sense that there are filtrations $\{{}'F^s\}_s$, $\{{}''F^s\}_s$ and $\{F^s\}_s$ of these groups, and isomorphisms ${}'F^s/{}'F^{s+1} \cong {}'E^s_\infty$, ${}''F^s/{}''F^{s+1} \cong {}''E^s_\infty$ and $F^s/F^{s+1} \cong E^s_\infty$, for all s.

A pairing $\{\phi_r\}_r$ of spectral sequences, as above, converges to a pairing $\phi \colon G' \otimes G'' \to G$ if the latter pairing restricts to homomorphisms $\phi \colon {}'F^u \otimes {}''F^s \to F^{u+s}$ for all u and s, and if the induced homomorphisms $\phi \colon {}'F^u/{}'F^{u+1} \otimes {}''F^s/{}''F^{s+1} \to F^{u+s}/F^{u+s+1}$ agree with the limit $\phi_\infty \colon {}'E^u_\infty \otimes {}''E^s_\infty \to E^{u+s}_\infty$ of the pairings ϕ_r.

In other words, the diagram

$$\begin{tikzcd}
{}'E^u_\infty \otimes {}''E^s_\infty \arrow[d, "\phi_\infty"'] & {}'F^u/{}'F^{u+1} \otimes {}''F^s/{}''F^{s+1} \arrow[l, "\cong"'] \arrow[d, "\phi"'] & {}'F^u \otimes {}''F^s \arrow[l, two heads] \arrow[r] \arrow[d, "\phi"'] & G' \otimes G'' \arrow[d, "\phi"'] \\
E^{u+s}_\infty & F^{u+s}/F^{u+s+1} \arrow[l, "\cong"'] & F^{u+s} \arrow[l, two heads] \arrow[r, tail] & G
\end{tikzcd}$$

commutes. ((Consequences?))

Definition 5.7. An algebra spectral sequence is a spectral sequence $\{E_r\}_r$ with a spectral sequence pairing $\{\phi_r \colon E_r \otimes E_r \to E_r\}_r$ that is associative and unital. It is commutative if the pairing satisfies $\phi_r(y \otimes x) = (-1)^{mn}\phi_r(x \otimes y)$ for all x, y and r, where $n = |x|$ and $m = |y|$ are the total degrees. ((Elaborate?))

5.4 The composition pairing

Adams (1958) defined a join pairing in his spectral sequence for S, which is stably equivalent to a smash product pairing in that spectral sequence. We shall return to those pairings later, but first look at the case of composition pairings, since these are most closely related to the Yoneda product. ((We may also need to look at this for Moss' later theorem on Toda brackets and Massey products.))

Theorem 5.8 (Moss (1968)). *Let X, Y and Z be spectra, with Y and Z bounded below and $H_*(Y)$ and $H_*(Z)$ of finite type. There is a pairing of spectral sequences*

$$E_r^{*,*}(Y,Z) \otimes E_r^{*,*}(X,Y) \longrightarrow E_r^{*,*}(X,Z)$$

which agrees for $r = 2$ with the Yoneda pairing

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(Y)) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), H^*(X)) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), H^*(X))$$

and which converges to the composition pairing

$$[Y, Z_2^\wedge]_* \otimes [X, Y_2^\wedge]_* \longrightarrow [X, Z_2^\wedge]_* .$$

The pairing is associative and unital.

Here is a version of Moss' original proof.

Proof. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, respectively, with cofibers $K^s = Y^s/Y^{s+1}$ and $L^u = Z^u/Z^{u+1}$.

In the unrolled exact couple for X mapping to $\{Y^s\}_s$, we can write

$$\begin{aligned} Z_r^s(X,Y) &= \mathrm{im}([X, Y^s/Y^{s+r}]_* \to [X, K^s]_*) \\ B_r^s(X,Y) &= \mathrm{im}([X, \Sigma^{-1}(Y^{s-r+1}/Y^s)]_* \to [X, K^s]_*) \end{aligned}$$

as subgroups of $E_1^s(X,Y) = [X, K^s]_*$. The homomorphisms are induced by the maps $Y^s/Y^{s+r} \to Y^s/Y^{s+1} = K^s$ and $\Sigma^{-1}(Y^{s-r+1}/Y^s) \to Y^s \to K^s$. Similarly,

$$\begin{aligned} Z_r^u(Y,Z) &= \mathrm{im}([Y, Z^u/Z^{u+r}]_* \to [Y, L^u]_*) \\ B_r^u(Y,Z) &= \mathrm{im}([Y, \Sigma^{-1}(Z^{u-r+1}/Z^u)]_* \to [Y, L^u]_*) \end{aligned}$$

as subgroups of $E_1^u(Y,Z) = [Y, L^u]_*$.

We would like to define a pairing

$$Z_1^u(Y,Z) \otimes Z_1^s(X,Y) \longrightarrow Z_1^{u+s}(X,Z)$$

that takes $Z_r^u(Y,Z) \otimes Z_r^s(X,Y)$ into $Z_r^{u+s}(X,Z)$, and satisfies $d_r(xy) = d_r(x)y + xd_r(y)$. ((Cope with indeterminacy!))

This implies that the pairing takes $Z_r^u(Y,Z) \otimes B_r^s(X,Y)$ and $B_r^u(Y,Z) \otimes Z_r^s(X,Y)$ into $B_r^{u+s}(X,Z)$, so that there is an induced pairing $E_r^u(Y,Z) \otimes E_r^s(X,Y) \to E_r^{u+s}(X,Z)$. It follows that d_r satisfies the Leibniz rule, and the pairing of E_r-terms induces the pairing of E_{r+1}-terms upon passage to homology.

We must also check that the pairing of E_2-terms agrees with the Yoneda product, and that the limit pairing of E_∞-terms is compatible with the composition product.

Let $f\colon X \to K^s$ and $g\colon Y \to L^u$ be maps of degree t and v, respectively, that admit lifts $\tilde{f}\colon X \to Y^s/Y^{s+r}$ and $\tilde{g}\colon Y \to Z^u/Z^{u+r}$ across the maps $Y^s/Y^{s+r} \to K^s$ and $Z^u/Z^{u+r} \to L^u$.

There is a map of Adams resolutions $\{i^r\colon Z^{n+r} \to Z^n\}_n$, giving a vertical map of cofiber sequences

$$\begin{tikzcd} Z^{n+1+r} \arrow[r, "i"] \arrow[d, "i^r"'] & Z^{n+r} \arrow[r, "j"] \arrow[d, "i^r"'] & L^{n+r} \arrow[r, "\partial"] \arrow[d] & \Sigma Z^{n+1+r} \arrow[d, "\Sigma i^r"'] \\ Z^{n+1} \arrow[r, "i"] & Z^n \arrow[r, "j"] & L^n \arrow[r, "\partial"] & \Sigma Z^{n+1} \end{tikzcd}$$

for each n. It factors through the cofiber sequence $Z^{n+1} \to Z^{n+1} \to * \to \Sigma Z^{n+1}$, since $r \geq 1$, so the map $L^{n+r} \to L^n$ is null-homotopic. Hence its cofiber splits as $L^n/L^{n+r} \simeq L^n \vee \Sigma L^{n+r}$. ((At least we can choose commuting homotopies in this way. Different null-homotopies could give different splittings.)) Passing to vertical cofibers we get an Adams resolution $\{Z^n/Z^{n+r}\}_n$ of Z/Z^r with cofibers $L^n/L^{n+r} \simeq L^n \vee \Sigma L^{n+r} \simeq L^n \times \Sigma L^{n+r}$.

The map $\tilde{g}\colon Y \to Z^u/Z^{u+r}$ now lifts to a map $\{\tilde{g}^n\colon Y^n \to Z^{u+n}/Z^{u+n+r}\}_n$ of Adams resolutions. Let $\left[\begin{smallmatrix}\lambda^n\\ \delta^n\end{smallmatrix}\right]\colon K^n \to L^{u+n}/L^{u+n+r} \simeq L^{u+n} \vee \Sigma L^{u+n+r}$ be the corresponding map of cofibers.

$$\begin{tikzcd}
\cdots \arrow[rr, "i"] & & Y^n \arrow[rr, "i^n"] \arrow[dl, "j"] \arrow[dd, "\tilde{g}^n"] & & Y \arrow[dl, "j"] \arrow[dd, "\tilde{g}"] \\
& K^n \arrow[dd, "\left[\begin{smallmatrix}\lambda^n\\ \delta^n\end{smallmatrix}\right]"] & & K^0 \arrow[dd, "\left[\begin{smallmatrix}\lambda^0\\ \delta^0\end{smallmatrix}\right]"] & \\
\cdots \arrow[rr, "i"] & & Z^{u+n}/Z^{u+n+r} \arrow[rr, "i^n"] \arrow[dl, "j"] & & Z^u/Z^{u+r} \arrow[dl, "j"] \\
& L^{u+n}/L^{u+n+r} & & L^u/L^{u+r} &
\end{tikzcd}$$

Then $\lambda^0 j\colon Y \to K^0 \to L^u$ equals $g\colon Y \to Z^u/Z^{u+r} \to L^u$, while $\delta^0 j\colon Y \to \Sigma L^{u+r}$ represents $d_r(g)$.

Starting with $\{Y^s\}_s$ in place of $\{Z^u\}_u$, we get an Adams resolution $\{Y^n/Y^{n+r}\}_n$ of Y/Y^r with cofibers $K^n/K^{n+r} \simeq K^n \vee \Sigma K^{n+r}$. We can define a map of cofibers

$$K^n \vee \Sigma K^{n+r} \simeq K^n/K^{n+r} \longrightarrow L^{u+n}/L^{u+n+r} \simeq L^{u+n} \vee \Sigma L^{u+n+r}$$

by the matrix

$$\begin{bmatrix} \lambda^n & 0 \\ \delta^n & \Sigma\lambda^{n+r} \end{bmatrix}.$$

In other words, on K^n it agrees with the cofiber map $\left[\begin{smallmatrix}\lambda^n\\ \delta^n\end{smallmatrix}\right]$ in the map of Adams resolutions lifting $\tilde{g}$, while on ΣK^{n+r} it agrees with the suspended cofiber map $\Sigma\left[\begin{smallmatrix}\lambda^{n+r}\\ \delta^{n+r}\end{smallmatrix}\right]$, but projected away from the summand $\Sigma^2 L^{u+n+2r}$. We claim that there are maps $\theta^n\colon Y^n/Y^{n+r} \to Z^{u+n}/Z^{u+n+r}$ making the diagram

$$\begin{tikzcd}
& Y^n/Y^{n+r} \arrow[r, "j"] \arrow[d, "\theta^n"'] & K^n/K^{n+r} \arrow[d, "\left[\begin{smallmatrix}\lambda^n & 0\\ \delta^n & \Sigma\lambda^{n+r}\end{smallmatrix}\right]"] & \\
Z^{u+n+1}/Z^{u+n+1+r} \arrow[r, "i"] & Z^{u+n}/Z^{u+n+r} \arrow[r, "j"] & L^{u+n}/L^{u+n+r} \arrow[r, "\partial"] & \Sigma(Z^{u+n+1}/Z^{u+n+1+r})
\end{tikzcd}$$

commute. ((Do they extend to a map of Adams resolutions lifting $Y/Y^r \to Z^u/Z^{u+r}$?)) To prove this, one checks that $\partial \circ \left[\begin{smallmatrix}\lambda^n & 0\\ \delta^n & \Sigma\lambda^{n+r}\end{smallmatrix}\right] \circ j$ is null-homotopic.

The pairing of r-th cycles now takes $g \in Z_r^u(Y, Z)$ and $f \in Z_r^s(X, Y)$ to the composite

$$g \cdot f\colon X \xrightarrow{\tilde{f}} Y^s/Y^{s+r} \xrightarrow{\theta^s} Z^{u+s}/Z^{u+s+r} \longrightarrow L^{u+s}$$

in $Z_r^{u+s}(X, Z) \subset [X, L^{u+s}]_*$.

It equals the composite

$$X \xrightarrow{f} K^s \xrightarrow{\lambda^s} L^{u+s},$$

and the explicit lift $\theta^s \circ \tilde{f}$ through Z^{u+s}/Z^{u+s+r} tells us that $d_r(g \cdot f)$ is represented by the composite

$$\delta^s f + \Sigma\lambda^{s+r} d_r(f).$$

((Relate this to $d_r(g) \cdot f + g \cdot d_r(f)$.))

((ETC)) □

5.5 The smash product pairing

Let Y and Z be spectra. We have a smash product pairing

$$\wedge\colon \pi_*(Y) \otimes \pi_*(Z) \longrightarrow \pi_*(Y \wedge Z)$$

that takes $f\colon S^t \to Y$ and $g\colon S^v \to Z$ to the smash product $f \wedge g\colon S^{t+v} \cong S^t \wedge S^v \to Y \wedge Z$.

Suppose that Y and Z are bounded below, and that $H_*(Y)$ and $H_*(Z)$ are of finite type. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, respectively, with cofibers $Y^s/Y^{s+1} = K^s$ and $Z^u/Z^{u+1} = L^u$. Let $P_s = H^*(\Sigma^s K^s)$ and $Q_u = H^*(\Sigma^u L^u)$ be the $\mathscr{A}$-modules that appear in the usual free resolutions $\epsilon\colon P_* \to H^*(Y)$ and $\epsilon\colon Q_* \to H^*(Z)$.

Let $W = Y \wedge Z$ be the smash product. Then W is bounded below and $H_*(W) \cong H_*(Y) \otimes H_*(Z)$ is of finite type. We shall construct an Adams resolution $\{W^n\}_n$ of W by geometrically mixing the Adams resolutions for Y and Z.

Traditionally, this is done by first replacing Y, Z and their Adams resolutions by homotopy equivalent spectra, so that each Y^s and Z^u is a CW spectrum, and each map $i\colon Y^{s+1} \to Y^s$ and $i\colon Z^{u+1} \to Z^u$ is the inclusion of a CW subspectrum. Then $Y^s \wedge Z^u$ is a CW subspectrum of $Y \wedge Z$, and one can form the union of these subspectra for all $s + u = n$. Hence one defines

$$W^n = \bigcup_{s+u=n} Y^s \wedge W^u \,.$$

Then W^{n+1} is a CW subspectrum of W^n, and

$$W^n/W^{n+1} \cong \bigvee_{s+u=n} K^s \wedge L^u \,.$$

Lemma 5.9. *The diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & W^2 \arrow[r, "i"] \arrow[d, "j"] & W^1 \arrow[r, "i"] \arrow[d, "j"] & W \arrow[d, "j"] \\
& W^2/W^3 \arrow[ul, dashed, "\partial"] & W^1/W^2 \arrow[ul, dashed, "\partial"] & W/W^1 \arrow[ul, dashed, "\partial"]
\end{tikzcd}$$

is an Adams resolution of $W = Y \wedge Z$. The associated free resolution $R_* \to H^*(W)$ is the tensor product of the free resolutions $P_* \to H^*(Y)$ and $Q_* \to H^*(Z)$.

Proof. Since each K^s is a wedge sum of suspended copies of H, of finite type, and each L^u is of finite type, we know that W^n/W^{n+1} is a wedge sum of suspended copies of H, of finite type. Let

$$R_n = H^*(\Sigma^n(W^n/W^{n+1})) \cong \bigoplus_{s+u=n} P_s \otimes Q_u \,.$$

This is a free $\mathscr{A}$-module of finite type, by its geometric origin as the cohomology of W^n/W^{n+1}. (We shall discuss the $\mathscr{A}$-module structure on a tensor product of $\mathscr{A}$-modules later.) The composite $W^{n-1}/W^n \to \Sigma W^n \to \Sigma(W^n/W^{n+1})$ splits as the direct sum of the maps $j\partial \wedge 1\colon K^{s-1} \wedge L^u \to \Sigma K^s \wedge L^u \cong \Sigma(K^s \wedge L^u)$ and $1 \wedge j\partial\colon K^s \wedge L^{u-1} \to K^s \wedge \Sigma L^u \cong \Sigma(K^s \wedge L^u)$. Hence the boundary map $\partial_n\colon R_n \to R_{n-1}$ is given by the usual formula

$$\partial_n(x \otimes y) = \partial_n(x) \otimes y + x \otimes \partial_n(y)$$

(we work at $p = 2$, hence there is no sign), so that $R_* = P_* \otimes Q_*$ is the tensor product of the two resolutions. By the Künneth theorem, the homology of R_* is the tensor product of the homologies of P_* and Q_*, so $\epsilon\colon R_* \to H^*(Y) \otimes H^*(Z) \cong H^*(Y \wedge Z)$ is a free resolution.

In particular, $j\colon W^0 = Y \wedge Z \to K^0 \wedge L^0$ induces a surjection j^* in cohomology. It follows that $\partial\colon W/W^1 \to \Sigma W^1$ induces an injection ∂^* in cohomology, with image in $R_0 = H^*(W/W^1)$ equal to the kernel of $j^* = \epsilon$. This equal the image of $\partial_1 = \partial^* j^* : R_1 \to R_0$, by exactness at R_0 of the free resolution, which implies that j^*, induced by $j\colon W^1 \to W^1/W^2$, is surjective. Suppose inductively that $j\colon W^{n-1} \to W^{n-1}/W^n$ induces a surjection j^* in cohomology, for some $n \geq 2$. Then $\partial\colon W^{n-1}/W^n \to \Sigma W^n$ induces an injection ∂^* in cohomology. The image of ∂^* equals the kernel of j^*, hence lies in the kernel of $\partial_{n-1} = \partial^* j^*\colon R_{n-1} \to R_{n-2}$. This equals the image of $\partial_n = \partial^* j^*\colon R_n \to R_{n-1}$, by exactness at R_{n-1}, which implies that j^*, induced by $j\colon W^n \to W^n/W^{n+1}$, is surjective. $\square$

Granting a little more technology, the substitution by CW spectra can be replaced by the passage to a homotopy colimit. For a fixed $n \geq 0$, one considers the diagram of all spectra $Y^s \wedge Z^u$ for $s + u \geq n$, and forms the homotopy colimit

$$W^n = \operatorname*{hocolim}_{s+u \geq n} Y^s \wedge Z^u \,.$$

There is a natural diagram

$$\cdots \to W^2 \xrightarrow{i} W^1 \xrightarrow{i} W^0 \simeq Y \wedge Z$$

and an identification

$$W^n/W^{n+1} \cong \bigvee_{s+u=n} \operatorname{hocofib}(Y^{s+1} \to Y^s) \wedge \operatorname{hocofib}(Z^{u+1} \to Z^u)$$

where $\operatorname{hocofib}(Y^{s+1} \to Y^s) \simeq K^s$ denotes the mapping cone of the given map, etc. The proof of the lemma goes through in the same way with these conventions.

There is a natural tensor product pairing

$$\operatorname{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_2) \otimes \operatorname{Ext}_{\mathscr{A}}^{*,*}(H^*(Z), \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{*,*}(H^*(Y \wedge Z), \mathbb{F}_2)$$

induced by passage to cohomology from the pairing

$$\operatorname{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2) \otimes \operatorname{Hom}_{\mathscr{A}}(Q_*, \mathbb{F}_2) \longrightarrow \operatorname{Hom}_{\mathscr{A}}(P_* \otimes Q_*, \mathbb{F}_2)$$

that takes $f \colon P_s \to \Sigma^t \mathbb{F}_2$ and $g \colon Q_u \to \Sigma^v \mathbb{F}_2$ to the projection $P_* \otimes Q_* \to P_s \otimes Q_u$, followed by $f \otimes g \colon P_s \otimes Q_u \to \mathbb{F}_2$. ((Compare this to the Yoneda pairing when $Y = Z = S$.))

The following theorem is similar to that proved in §4 of Adams (1958).

Theorem 5.10. *There is a natural pairing*

$$E_r^{s,t}(Y) \otimes E_r^{u,v}(Z) \longrightarrow E_r^{s+u,t+v}(Y \wedge Z)$$

of Adams spectral sequences, given at the E_2-term by the tensor product pairing

$$\operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(Y), \mathbb{F}_2) \otimes \operatorname{Ext}_{\mathscr{A}}^{u,v}(H^*(Z), \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s+u,t+v}(H^*(Y \wedge Z), \mathbb{F}_2)$$

and converging to the smash product pairing

$$\pi_{t-s}(Y_2^\wedge) \otimes \pi_{v-u}(Z_2^\wedge) \longrightarrow \pi_{t-s+v-u}((Y \wedge Z)_2^\wedge)\,.$$

((Discuss the role of completion in the pairing?))

Proof. Recall that $E_r^s = Z_r^s/B_r^s$, where

$$Z_r^s = \partial^{-1} \operatorname{im}(i_*^{r-1} \colon \pi_*(Y^{s+r}) \to \pi_*(Y^{s+1}))$$

and

$$B_r^s = j \ker(i_*^{r-1} \colon \pi_*(Y^s) \to \pi_*(Y^{s+r-1}))$$

are subgroups of $E_s^1 = \pi_*(K^s)$. For the purpose of this proof, it is convenient to rewrite these groups as

$$Z_r^s = \operatorname{im}(\pi_*(Y^s/Y^{s+r}) \to \pi_*(K^s))$$

and

$$B_r^s = \operatorname{im}(\pi_*(\Sigma^{-1}(Y^{s-r+1}/Y^s)) \to \pi_*(K^s))\,.$$

These formulas can be obtained by chases in the diagrams

$$\begin{tikzcd}
Y^{s+r} \arrow[r, "i^r"] \arrow[d] & Y^s \arrow[r] \arrow[d, "j"] & Y^s/Y^{s+r} \arrow[d] \\
* \arrow[r] \arrow[d] & K^s \arrow[r, "="] \arrow[d, "\partial"] & K^s \arrow[d] \\
\Sigma Y^{s+r} \arrow[r, "\Sigma i^{r-1}"] & \Sigma Y^{s+1} \arrow[r] & \Sigma(Y^{s+1}/Y^{s+r})
\end{tikzcd}$$

and

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \Sigma^{-1}(Y^{s-r+1}/Y^s) \arrow[r, "="] \arrow[d] & \Sigma^{-1}(Y^{s-r+1}/Y^s) \arrow[d] \\
Y^{s+1} \arrow[r, "i"] \arrow[d, "="'] & Y^s \arrow[r, "j"] \arrow[d, "i^{r-1}"] & K^s \arrow[d] \\
Y^{s+1} \arrow[r, "i^r"] & Y^{s-r+1} \arrow[r] & Y^{s-r+1}/Y^{s+1}
\end{tikzcd}$$

of horizontal and vertical cofiber sequences.

The differential $d_r^s \colon E_r^s \to E_r^{s+r}$ is determined by the homomorphism $\delta \colon \pi_*(Y^s/Y^{s+r}) \to Z_r^{s+r}$ induced by $Y^s/Y^{s+r} \to \Sigma K^{s+r}$ and the surjection $\pi \colon \pi_*(Y^s/Y^{s+r}) \to Z_r^s$ induced by $Y^s/Y^{s+r} \to K^s$:

$$\begin{tikzcd}
E_1^s & Z_r^s \arrow[l, tail] \arrow[d, two heads] & \pi_*(Y^s/Y^{s+r}) \arrow[l, two heads, "\pi"'] \arrow[r, "\delta"] & Z_r^{s+t} \arrow[r, tail] \arrow[d, two heads] & E_1^{s+r} \\
& E_r^s \arrow[rr, "d_r^s"] & & E_r^{s+r} &
\end{tikzcd}$$

It follows that $B_{r+1}^{s+r}/B_r^{s+r} \subset E_r^{s+r}$ equals the image of d_r^s.

So far we have discussed the Adams spectral sequence for a single spectrum Y. We now relate the Adams spectral sequences for Y, Z and $W = Y \wedge Z$, where W has the Adams resolution obtained from given Adams resolutions of Y and Z.

There is a preferred inclusion $Y^s \wedge Z^u \to W^n$ for all $s, u \geq 0$ and $n = s + u$. It restricts to inclusions $Y^{s+r} \wedge Z^u \to W^{n+r}$ and $Y^s \wedge Z^{u+r} \to W^{n+r}$, that agree on $Y^{s+r} \wedge Z^{u+r}$. Hence we have a main commutative diagram

$$\begin{tikzcd}
U \arrow[r] \arrow[d] & Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r} \arrow[r] \arrow[d] \arrow[rr, bend left, "a_r"] & Y^{s+1} \wedge Z^u \cup Y^s \wedge Z^{u+1} \arrow[r, "a_1"] \arrow[d] & Y^s \wedge Z^u \arrow[r] \arrow[d] & Y \wedge Z \arrow[d, "\simeq"] \\
W^{n+r+1} \arrow[r, "i"] & W^{n+r} \arrow[r, "i^{r-1}"] \arrow[rr, bend right, "i^r"'] & W^{n+1} \arrow[r, "i"] & W^n \arrow[r] & W
\end{tikzcd}$$

where $Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r}$ denotes the pushout of $Y^{s+r} \wedge Z^u$ and $Y^s \wedge Z^{u+r}$ along $Y^{s+r} \wedge Z^{u+r}$, and U is brief notation for a similar union of $Y^{s+r+1} \wedge Z^u$, $Y^{s+r} \wedge Z^{u+1}$, $Y^{s+1} \wedge Z^u$ and $Y^s \wedge Z^{u+1}$.

Passing to horizontal cofibers for the middle part of the diagram, we get a commutative diagram

$$\begin{tikzcd}
Y^s \wedge Z^u \arrow[r] \arrow[d] & Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[r] \arrow[d] & K^s \wedge L^u \arrow[d] \\
W^n \arrow[r] & W^n/W^{n+r} \arrow[r] & W^n/W^{n+1}
\end{tikzcd} \tag{1}$$

where the maps in the upper row are smash products of the standard maps $Y^s \to Y^s/Y^{s+r}$, $Y^s/Y^{s+r} \to K^s$, etc. The vertical map $K^s \wedge L^u \to W^n/W^{n+1}$ agrees with the inclusion of a summand in $W^n/W^{n+1} \cong \bigvee_{s+u=n} K^s \wedge L^u$. Hence it induces a pairing

$$\phi_1 \colon E_1^s(Y) \otimes E_1^u(Z) \longrightarrow E_1^n(W)$$

that corresponds to the previously discussed pairing

$$\mathrm{Hom}_{\mathscr{A}}(P_*, \mathbb{F}_2) \otimes \mathrm{Hom}_{\mathscr{A}}(Q_*, \mathbb{F}_2) \longrightarrow \mathrm{Hom}_{\mathscr{A}}(P_* \otimes Q_*, \mathbb{F}_2)$$

under the d-invariant isomorphisms $\pi_{t-s}(K^s) \cong \mathrm{Hom}_{\mathscr{A}}^t(P_s, \mathbb{F}_2)$, etc.

Passing to horizontal cofibers further to the left in the main diagram, we get a commutative diagram

$$\begin{tikzcd}
Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[r] \arrow[d] & \Sigma(Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r}) \arrow[r] \arrow[d] & \Sigma(K^{s+r} \wedge L^u \vee K^s \wedge L^{u+r}) \arrow[d] \\
W^n/W^{n+r} \arrow[r] & \Sigma W^{n+r} \arrow[r] & \Sigma(W^{n+r}/W^{n+r+1})
\end{tikzcd} \tag{2}$$

where the composite map in the upper row is the wedge sum of the smash product of the standard maps $Y^s/Y^{s+r} \to \Sigma K^{s+r}$ and $Z^u/Z^{u+r} \to L^u$, and the smash product of the standard maps $Y^s/Y^{s+r} \to K^s$ and $Z^u/Z^{u+r} \to \Sigma L^{u+r}$. The right hand vertical map is the suspension of the wedge sum of the pairings $K^{s+r} \wedge L^u \to W^{n+r}/W^{n+r+1}$ and $K^s \wedge L^{u+r} \to W^{n+r}/W^{n+r+1}$.

We now claim that (a) $\phi_1 = \tilde{\phi}_1$ restricts to a pairing

$$\tilde{\phi}_r \colon Z_r^s(Y) \otimes Z_r^u(Z) \longrightarrow Z_r^n(W)\,,$$

(b) $\tilde{\phi}_r$ descends to a pairing

$$\phi_r \colon E_r^s(Y) \otimes E_r^u(Z) \longrightarrow E_r^n(W)$$

and (c) ϕ_r satisfies the Leibniz rule

$$d_r(\phi_r(y \otimes z)) = \phi_r(d_r(y) \otimes z) + \phi_r(y \otimes d_r(z))\,.$$

Here $r \geq 1$ and $n = s + u$.

Assuming these claims, which are similar to the conditions of Lemma 2.2 of Moss (1968), we can easily finish the proof of the theorem. The pairings $(\phi_r)_*$ and ϕ_{r+1} agree, under the identification $H^s(E_r^*, d_r) \cong E_{r+1}^s$, since they are both induced by a passage to quotients from $\tilde{\phi}_{r+1}$. Hence the sequence $\{\phi_r\}_r$ qualifies as a pairing of spectral sequences. In particular, $\phi_2 = (\phi_1)_*$ is the tensor product pairing of Ext-groups. This spectral sequence pairing converges to the smash product pairing in homotopy, since the pairing of E_∞-terms is induced by the pairing

$$\pi_*(Y^s) \otimes \pi_*(Z^u) \longrightarrow \pi_*(Y^s \wedge Z^u) \longrightarrow \pi_*(W^n)$$

via the surjections $\pi_*(Y^s) \to E_\infty^s$, etc., and the pairing of filtration quotients is induced by the same pairing via the surjections $\pi_*(Y^s) \to F^s \to F^s/F^{s+1}$, etc. These surjections have the same kernel, so the induced pairings of quotients are compatible under the identifications $F^s/F^{s+1} \cong E_\infty^s$.

It remains to prove the three parts of the claim.

(a) Applying $\pi_*(-)$ to the right hand square in diagram (1), we get the outer rectangle of the following map of pairings:

$$\begin{tikzcd}
\pi_*(Y^s/Y^{s+r}) \otimes \pi_*(Z^u/Z^{u+r}) \arrow[r, two heads] \arrow[d] & Z_r^s(Y) \otimes Z_r^u(Z) \arrow[r, tail] \arrow[d, dashed, "\tilde{\phi}_r"'] & E_1^s(Y) \otimes E_1^u(Z) \arrow[d, "\phi_1"] \\
\pi_*(W^n/W^{n+r}) \arrow[r, two heads, "\pi"] & Z_r^n(W) \arrow[r, tail] & E_1^n(W)
\end{tikzcd}$$

In view of the description of $Z_r^n(W)$ as the image of $\pi_*(W^n/W^{n+r}) \to \pi_*(W^n/W^{n+1}) = E_1^n(W)$, and similarly for Y and Z, it follows that there is a unique pairing $\tilde{\phi}_r$ that makes the whole diagram commute.

(b) To check that $\tilde{\phi}_r$ descends to a pairing $\phi_r \colon E_r^s(Y) \otimes E_r^u(Z) \to E_r^n(W)$, we use the diagram

$$\begin{tikzcd}
E_r^s(Y) \otimes E_r^u(Z) \arrow[d, dashed, "\phi_r"'] & Z_r^s(Y) \otimes Z_r^u(Z) \arrow[l, two heads] \arrow[r, tail] \arrow[d, "\tilde{\phi}_r"'] & Z_{r-1}^s(Y) \otimes Z_{r-1}^u(Z) \arrow[r, two heads] \arrow[d, "\tilde{\phi}_{r-1}"'] & E_{r-1}^s(Y) \otimes E_{r-1}^u(Z) \arrow[d, "\phi_{r-1}"'] \\
E_r^n(W) & Z_r^n(W) \arrow[l, two heads] \arrow[r, tail] & Z_{r-1}^n(W) \arrow[r, two heads] & E_{r-1}^n(W)\,.
\end{tikzcd}$$

There is only something to prove for $r \geq 2$. We assume, by induction on r, that the Leibniz rule in (c) holds for d_{r-1} and ϕ_{r-1}.

Given $y \in B_r^s(Y) \subset Z_r^s(Y)$ and $z \in Z_r^u(Z)$ we must show that $\tilde{\phi}_r(y \otimes z) \in B_r^n(W) \subset Z_r^n(W)$. The image of y in $E_{r-1}^s(Y)$ has the form $[y] = d_{r-1}(x)$ for some $x \in E_{r-1}^{s-r+1}(Y)$, and the image of z in

$E^u_{r-1}(Z)$ satisfies $d_{r-1}([z]) = 0$. Then $d_{r-1}(\phi_{r-1}(x \otimes [z])) = \phi_{r-1}(d_{r-1}(x) \otimes [z]) + \phi_{r-1}(x \otimes d_{r-1}([z])) = \phi_{r-1}([y] \otimes [z]) + 0 = [\tilde{\phi}_r(y \otimes z)]$. Hence $\tilde{\phi}_r(y \otimes z)$ is congruent modulo $B^n_{r-1}(W)$ to a class in $B^n_r(W)$, as we asserted. The same argument shows that $\tilde{\phi}_r$ maps $Z^s_r(Y) \otimes B^u_r(Z)$ into $B^n_r(W)$. Hence $\tilde{\phi}_r$ descends to ϕ_r, and this uniquely determines ϕ_r.

(c) Applying $\pi_*(-)$ to the outer rectangle in diagram (2), we get the outer rectangle of the following map of pairings:

$$\begin{tikzcd}
\pi_*(Y^s/Y^{s+r}) \otimes \pi_*(Z^u/Z^{u+r}) \arrow[r, "\left[\begin{smallmatrix} \delta\otimes\pi \\ \pi\otimes\delta \end{smallmatrix}\right]"] \arrow[d] & \begin{matrix} Z^{s+r}_r(Y) \otimes Z^u_r(Z) \\ \oplus \\ Z^s_r(Y) \otimes Z^{u+r}_r(Z) \end{matrix} \arrow[r, tail] \arrow[d, "{[\tilde{\phi}_r\ \tilde{\phi}_r]}"'] & \begin{matrix} E^{s+r}_1(Y) \otimes E^u_1(Z) \\ \oplus \\ E^s_1(Y) \otimes E^{u+r}_1(Z) \end{matrix} \arrow[d, "{[\phi_1\ \phi_1]}"] \\
\pi_*(W^n/W^{n+r}) \arrow[r, "\delta"] & Z^{n+r}_r(W) \arrow[r, tail] & E^{n+r}_1(W)
\end{tikzcd}$$

Since the pairings $\tilde{\phi}_r$ have been defined to make the right hand square commute, the whole diagram commutes.

Combining parts of four of these diagrams, we have the commutative sprawl:

$$\begin{tikzcd}
E^s_r(Y) \otimes E^u_r(Z) \arrow[dd, "\left[\begin{smallmatrix} d^s_r\otimes 1 \\ 1\otimes d^u_r \end{smallmatrix}\right]"'] \arrow[rrr, bend left, "\phi_r"] & Z^s_r(Y) \otimes Z^u_r(Z) \arrow[l, two heads] \arrow[r, "\tilde{\phi}_r"] & Z^n_r(W) \arrow[r, two heads] & E^n_r(W) \arrow[dd, "d^n_r"] \\
 & \pi_*(Y^s/Y^{s+r}) \otimes \pi_*(Z^u/Z^{u+r}) \arrow[u, two heads, "\pi\otimes\pi"'] \arrow[r] \arrow[d, "\left[\begin{smallmatrix} \delta\otimes\pi \\ \pi\otimes\delta \end{smallmatrix}\right]"'] & \pi_*(W^n/W^{n+r}) \arrow[u, two heads, "\pi"'] \arrow[d, "\delta"] & \\
\begin{matrix} E^{s+r}_r(Y) \otimes E^u_r(Z) \\ \oplus \\ E^s_r(Y) \otimes E^{u+r}_r(Z) \end{matrix} \arrow[rrr, bend right, "{[\phi_r\ \phi_r]}"'] & \begin{matrix} Z^{s+r}_r(Y) \otimes Z^u_r(Z) \\ \oplus \\ Z^s_r(Y) \otimes Z^{u+r}_r(Z) \end{matrix} \arrow[l, two heads] \arrow[r, "{[\tilde{\phi}_r\ \tilde{\phi}_r]}"] & Z^{n+r}_r(W) \arrow[r, two heads] & E^{n+r}_r(W)
\end{tikzcd}$$

Going around the outer boundary of the diagram we see that $d^n_r(\phi_r(y \otimes z)) = \phi_r(d^s_r(y) \otimes z) + \phi_r(y \otimes d^u_r(z))$, proving the Leibniz rule. $\square$

Remark 5.11. If $y \in \pi_*(K^s)$ and $z \in \pi_*(L^u)$ lift to $\tilde{y} \in \pi_*(Y^s/Y^{s+r})$ and $\tilde{z} \in \pi_*(Z^u/Z^{u+r})$, respectively, with images $\delta y \in \pi_*(\Sigma K^{s+r})$ and $\delta z \in \pi_*(\Sigma L^{u+r})$, then $y \wedge z \in \pi_*(K^s \wedge L^u)$ lifts to $\tilde{y} \wedge \tilde{z} \in \pi_*(Y^s/Y^{s+r} \wedge Z^u/Z^{u+r})$.

$$\begin{tikzcd}
\Sigma K^{s+r} & Y^s/Y^{s+r} \arrow[l] \arrow[r] & K^s
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma K^{s+r} \wedge L^u & & K^s \wedge L^u & L^u \\
 & Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[ul] \arrow[ur] \arrow[dr] & & Z^u/Z^{u+r} \arrow[u] \arrow[d] \\
 & & \Sigma K^s \wedge L^{u+r} & \Sigma L^{u+r}
\end{tikzcd}$$

The maps $Y^s \wedge Z^u \to W^{s+u} = W^n$ induce a commutative diagram

$$\begin{tikzcd}
\Sigma K^{s+r} \wedge L^u \vee \Sigma K^s \wedge L^{u+r} \arrow[d] & Y^s/Y^{s+r} \wedge Z^u/Z^{u+r} \arrow[l] \arrow[r] \arrow[d] & K^s \wedge L^u \arrow[d] \\
\Sigma(W^{n+r}/W^{n+r+1}) & W^n/W^{n+r} \arrow[l] \arrow[r] & W^n/W^{n+1}
\end{tikzcd}$$

and $\tilde{y} \wedge \tilde{z}$ maps to a lift $\tilde{y} \cdot \tilde{z}$ in $\pi_*(W^n/W^{n+r})$ of the image $y \cdot z$ of $y \wedge z$ in W^n/W^{n+1}. Hence $\delta(y \cdot z)$ is the image $\delta y \cdot z + y \cdot \delta z$ of $\delta y \wedge z + y \wedge \delta z$ in $\pi_*(\Sigma K^{s+r} \wedge L^u \vee \Sigma K^s \wedge L^{u+r})$. The key point is that, even if $Y^s/Y^{s+r} \wedge Z^u/Z^{u+r}$ is attached to all of $Y^{s+r} \wedge Z^u \cup Y^s \wedge Z^{u+r}$ in $Y^s \wedge Z^u$, the composite map to $W^{n+r} \to W^{n+r}/W^{n+r+1}$ factors through the quotient $K^{s+r} \wedge L^u \vee K^s \wedge L^{u+r}$, making the left hand square above commute. The bookkeeping shows that δy represents $d_r([y])$, and so on, so that $\delta(y \cdot z) = \delta y \cdot z + y \cdot \delta z$ implies the Leibniz rule for d_r.

Corollary 5.12. *Suppose that Y is a ring spectrum, with multiplication $\mu\colon Y \wedge Y \to Y$ and unit $\eta\colon S \to Y$. Then there is a natural pairing*

$$E_r^{*,*}(Y) \otimes E_r^{*,*} \longrightarrow E_r^{*,*}(Y)\,,$$

given at the E_2-term by the composite

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_2) \otimes \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y \wedge Y), \mathbb{F}_2) \xrightarrow{\mu_*} \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_2)\,,$$

and a unit map

$$E_r^{*,*}(S) \xrightarrow{\eta_*} E_r^{*,*}(Y)\,,$$

given at the E_2-term by

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\eta_*} \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(Y), \mathbb{F}_2)\,,$$

that make the Adams spectral sequence $E_r^{*,*}(Y)$ an algebra spectral sequence over $E_r^{*,*}(S)$. If Y is homotopy commutative, then it is a commutative algebra spectral sequence.

5.6 The composition pairing, revisited

Here is a geometric proof of Moss' theorem on the composition pairing, close to the one for the smash product pairing.

Proof. Let $\{Y^s\}_s$ and $\{Z^u\}_u$ be Adams resolutions of Y and Z, with cofibers $Y^s/Y^{s+1} = K^s$ and $Z^u/Z^{u+1} = L^u$, respectively. Let $P_s = H_*(\Sigma^s K^s)$ and $Q_u = H_*(\Sigma^u L^u)$, as usual.

Consider the homotopy limit of mapping spectra

$$M^u = \operatorname*{holim}_{n \le u+s} F(Y^s, Z^n)\,.$$

Restriction from $n \le u+s+1$ to $n \le u+s$ gives a map $i\colon M^{u+1} \to M^u$. Its homotopy fiber is the product over s of the iterated homotopy fiber in the square

$$\begin{tikzcd}
F(Y^s, Z^{u+s+1}) \arrow[r] \arrow[d] & F(Y^s, Z^{u+s}) \arrow[d] \\
F(Y^{s+1}, Z^{u+s+1}) \arrow[r] & F(Y^{s+1}, Z^{u+s})\,,
\end{tikzcd}$$

which is equivalent to $F(K^s, L^{u+s})$. Hence we get a tower

$$\begin{tikzcd}
\dots \arrow[r] & M^{u+1} \arrow[r] & M^u \arrow[r] \arrow[d] & \dots \arrow[r] & M^1 \arrow[r] & M^0 \arrow[d] \\
& & \prod_s F(K^s, L^{u+s}) \arrow[ul, dashed] & & & \prod_s F(K^s, L^s)\,. \arrow[ul, dashed]
\end{tikzcd}$$

Restriction to $(s, n) = (0, u)$ defines a map to the tower

$$\begin{tikzcd}
\dots \arrow[r] & F(Y, Z^{u+1}) \arrow[r] & F(Y, Z^u) \arrow[r] \arrow[d] & \dots \arrow[r] & F(Y, Z^1) \arrow[r] & F(Y, Z) \arrow[d] \\
& & F(Y, L^u) \arrow[ul, dashed] & & & F(Y, L^0)\,. \arrow[ul, dashed]
\end{tikzcd}$$

Applying homotopy we get a map of unrolled exact couples, from

$$\begin{tikzcd}
\dots \arrow[r] & \pi_*(M^{u+1}) \arrow[r] & \pi_*(M^u) \arrow[r] \arrow[d] & \dots \arrow[r] & \pi_*(M^1) \arrow[r] & \pi_*(M^0) \arrow[d] \\
& & \prod_s [K^s, L^{u+s}]_* \arrow[ul, dashed] & & & \prod_s [K^s, L^s]_* \arrow[ul, dashed]
\end{tikzcd}$$

to the one generating the Adams spectral sequence $\{E_r^{*,*}(Y,Z)\}_r$. Let $\{'E_r^{u,*}\}_r$ be the spectral sequence generated by the unrolled exact couple just displayed. The map $'E_1^{u,*} \to E_1^{u,*}(Y,Z)$ of E_1-terms can be identified, using the d-invariant isomorphisms

$$\prod_s [K^s, L^{u+s}]_* \cong \prod_s \mathrm{Hom}^*_{\mathscr{A}}(Q_{u+s}, P_s) = \mathrm{HOM}^{u,*}_{\mathscr{A}}(Q_*, P_*)$$
$$[Y, L^u]_* \cong \mathrm{Hom}^*_{\mathscr{A}}(Q_u, H^*(Y)) \,,$$

with the quasi-isomorphism

$$\epsilon_* \colon \mathrm{HOM}^{u,*}_{\mathscr{A}}(Q_*, P_*) \longrightarrow \mathrm{Hom}^*_{\mathscr{A}}(Q_u, H^*(Y))$$

induced by $\epsilon \colon P_* \to H^*(Y)$. Hence the map of E_2-terms is an isomorphism, identifying $'E_2^{u,*}$ with the Adams E_2-term

$$E_2^{u,*}(Y,Z) = \mathrm{Ext}^{u,*}_{\mathscr{A}}(H^*(Z), H^*(X)) \,.$$

We shall define a pairing of spectral sequences

$$\phi_r \colon {'E_r^{u,*}} \otimes E_r^{s,*}(X,Y) \longrightarrow E_r^{u+s,*}(X,Z)$$

for $r \ge 1$, which agrees with the composition pairing

$$\mathrm{HOM}^{u,*}_{\mathscr{A}}(Q_*, P_*) \otimes \mathrm{Hom}_{\mathscr{A}}(P_s, H^*(X)) \to \mathrm{Hom}_{\mathscr{A}}(Q_{u+s}, H^*(X))$$

for $r = 1$. For $r \ge 2$ the source is isomorphic to

$$E_r^{u,*}(Y,Z) \otimes E_r^{s,*}(X,Y)$$

via $\epsilon_* \otimes 1$, which yields Moss' pairing and the compatibility with the Yoneda product for $r = 2$.

The pairing $\phi_1 \colon {'E_1^{u,*}} \otimes E_1^{s,*}(X,Y) \to E_1^{u+s,*}(X,Z)$ is the composition pairing

$$\prod_s [K^s, L^{u+s}]_* \otimes [X, K^s]_* \longrightarrow [X, L^{u+s}]_*$$

that takes $(g^s)_s \otimes f$ to $g^s f$. We show that it restricts to a pairing $\tilde{\phi}_r \colon {'Z_r^{u,*}} \otimes Z_r^{s,*}(X,Y) \to Z_r^{u+s,*}(X,Z)$ of r-th cycles, that descends to a pairing $\phi_r \colon {'E_r^{u,*}} \otimes E_r^{s,*}(X,Y) \to E_r^{u+s,*}(X,Z)$ satisfying the Leibniz rule, for each $r \ge 1$.

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We shall use the identifications

$$\begin{aligned}
{'Z_r^{u,*}} &= \mathrm{im}(\pi_*(M^u/M^{u+r}) \to \pi_*(M^u/M^{u+1})) \\
Z_r^{s,*}(X,Y) &= \mathrm{im}([X, Y^s/Y^{s+r}]_* \to [X, K^s]_*) \\
Z_r^{s,*}(X,Z) &= \mathrm{im}([X, Z^{u+s}/Z^{u+s+r}]_* \to [X, L^{u+s}]_*)
\end{aligned}$$

where $M^u/M^{u+1} = \prod_s F(K^s, L^{u+s})$.

Consider the commutative square

$$\begin{tikzcd}
F(Y^s, Z^{u+s+r}) \arrow[r] \arrow[d] & F(Y^s, Z^{u+s}) \arrow[d] \\
F(Y^{s+r}, Z^{u+s+r}) \arrow[r] & F(Y^{s+r}, Z^{u+s}) \,.
\end{tikzcd}$$

There are restriction maps from M^{u+r} to the upper left hand corner, and from M^u to the homotopy pullback of the rest of the square. Hence there is a map of homotopy fibers from $\Sigma^{-1}(M^u/M^{u+1})$ to $F(Y^s/Y^{s+r}, \Sigma^{-1}(Z^{u+s}/Z^{u+s+r}))$, giving a map

$$M^u/M^{u+r} \longrightarrow F(Y^s/Y^{s+r}, Z^{u+s}/Z^{u+s+r})$$

and an adjoint pairing

$$M^u/M^{u+r} \wedge Y^s/Y^{s+r} \longrightarrow Z^{u+s}/Z^{u+s+r}$$

compatible with the pairing $M^u/M^{u+1} \wedge K^s \to L^{u+s}$ for $r = 1$. This leads to the commutative diagram

$$\begin{tikzcd}
\pi_*(M^u/M^{u+r}) \otimes [X, Y^s/Y^{s+r}]_* \arrow[r] \arrow[d] & {[X, Z^{u+s}/Z^{u+s+r}]_*} \arrow[d] \\
\prod_s [K^s, L^{u+s}]_* \otimes [X, K^s]_* \arrow[r, "\phi_1"] & {[X, L^{u+s}]_*} \,.
\end{tikzcd}$$

The induced pairing of vertical images is ϕ_r.
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6 Calculations

6.1 The minimal resolution, revisited

Recall the minimal resolution $\epsilon\colon P_* \to \mathbb{F}_2$.

Lemma 6.1. *The product $h_i \cdot \gamma_{s,n}$ contains the summand $\gamma_{s+1,m}$ if and only if $\partial_{s+1}(g_{s+1,m}) = \sum_j a_j g_{s,j}$ contains the summand $Sq^{2^i} g_{s,n}$.*

Proof. Let $\gamma_{s,n}\colon P_s \to \mathbb{F}_2$ be dual to the generator $g_{s,n} \in P_s$, and let $h_i = \gamma_{1,i}\colon P_1 \to \mathbb{F}_2$ be dual to $g_{1,i} = [Sq^{2^i}]$.

$$\begin{tikzcd}
P_{s+1} \arrow[r, "\gamma_1"] \arrow[d, "\partial_{s+1}"'] & P_1 \arrow[d, "\partial_1"'] \arrow[dr, "h_i"] & \\
P_s \arrow[r, "\gamma_0"] \arrow[dr, "\gamma_{s,n}"'] & P_0 \arrow[d, "\epsilon"] & \mathbb{F}_2 \\
& \mathbb{F}_2 &
\end{tikzcd}$$

We lift $\gamma_{s,n}$ to $\gamma_0\colon P_s \to P_0$ mapping $g_{s,n} \mapsto g_{0,0}$ and $g_{s,j} \mapsto 0$ for $j \neq n$. Then $\gamma_0 \circ \partial_{s+1}$ sends $g_{s+1,m}$ to $a_n g_{0,0}$. To lift γ_0 to $\gamma_1\colon P_{s+1} \to P_1$ we write $a_n = \sum_k b_k Sq^{2^k}$, with each $b_k \in \mathscr{A}$. Then we may take $\gamma_1(g_{s+1,m}) = \sum_k b_k g_{1,k}$, since $\partial_1(g_{1,k}) = Sq^{2^k} g_{0,0}$. The coefficient of $g_{s+1,m}$ in the Yoneda product $h_i \cdot \gamma_{s,n}$ is then given by the value of $h_i \circ \gamma_1$ on $g_{s+1,m}$, which equals $h_i(\sum_k b_k g_{1,k}) = \epsilon(b_i)$. Hence $\gamma_{s+1,m}$ occurs as a summand in $h_i \cdot \gamma_{s,n}$ if and only if Sq^{2^i} occurs as a summand in $a_n = \sum_k b_k Sq^{2^k}$. This is equivalent to the condition that Sq^{2^i} occurs as a summand when a_n is written as a sum of admissible monomials. □

Proposition 6.2. *The Yoneda products in $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ in internal degrees $t \leq 11$ are given by:*

$\cdot$	$\gamma_{0,0}$	$\gamma_{1,0}$	$\gamma_{1,1}$	$\gamma_{1,2}$	$\gamma_{1,3}$	$\gamma_{2,0}$	$\gamma_{2,1}$	$\gamma_{2,2}$	$\gamma_{2,3}$	$\gamma_{2,4}$	$\gamma_{2,5}$	$\gamma_{3,0}$	$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{s,0}$
h_0	$\gamma_{1,0}$	$\gamma_{2,0}$	0	$\gamma_{2,2}$	$\gamma_{2,4}$	$\gamma_{3,0}$	0	$\gamma_{3,1}$	0	$\gamma_{3,2}$	0	$\gamma_{4,0}$	0	$\gamma_{4,1}$	$\gamma_{s+1,0}$
h_1	$\gamma_{1,1}$	0	$\gamma_{2,1}$	0	$\gamma_{2,5}$	0	$\gamma_{3,1}$	0	0	0	?	0	0	?	0
h_2	$\gamma_{1,2}$	$\gamma_{2,2}$	0	$\gamma_{2,3}$	?	$\gamma_{3,1}$	0	0	?	?	?	0	0	?	0
h_3	$\gamma_{1,3}$	$\gamma_{2,4}$	$\gamma_{2,5}$	?	?	$\gamma_{3,2}$	?	?	?	?	?	$\gamma_{4,1}$	?	?	?

for $5 \leq s \leq 10$.

Proof. This can be read off from the minimal resolution $\epsilon\colon P_* \to \mathbb{F}_2$, using the lemma above. □

Remark 6.3. The remaining summands, like $Sq^3 g_{1,0}$ in $\partial_2(g_{2,1})$ and $Sq^2 Sq^1 g_{1,1}$ in $\partial_2(g_{2,2})$, contribute to higher compositions like Massey products, like $h_1^2 \in \langle h_0, h_1, h_0\rangle$ and $h_0h_2 \in \langle h_1, h_0, h_1\rangle$, which imply $\eta^2 \in \langle 2, \eta, 2\rangle$ and $2\nu \in \langle \eta, 2, \eta\rangle$, respectively.

Definition 6.4. Let $c_0 \in \mathrm{Ext}_{\mathscr{A}}^{3,11}(\mathbb{F}_2, \mathbb{F}_2)$ be the class of the cocycle $\gamma_{3,3} \colon P_3 \to \mathbb{F}_2$ of degree 11, dual to $g_{3,3}$.

Corollary 6.5. *The algebra unit is* $1 = \gamma_{0,0}$. *The classes* $h_0 = \gamma_{1,0}$, $h_1 = \gamma_{1,1}$, $h_2 = \gamma_{1,2}$, $h_3 = \gamma_{1,3}$ *and* $c_0 = \gamma_{3,3}$ *are indecomposable. The remaining additive generators in internal degree* $t \le 11$ *are decomposable. These algebra generators commute with one another, so the Yoneda product is commutative (in this range). The decomposable generators have the following presentations:*

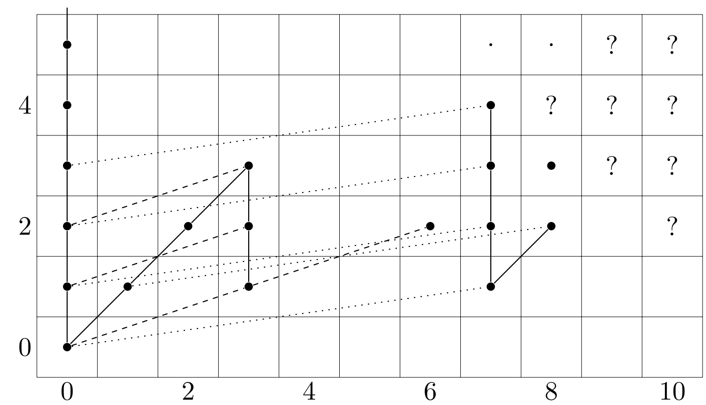
$$\gamma_{2,0} = h_0^2 \qquad \gamma_{3,0} = h_0^3$$
$$\gamma_{2,1} = h_1^2 \qquad \gamma_{3,1} = h_1^3 = h_0^2 h_2$$
$$\gamma_{2,2} = h_0h_2 \qquad \gamma_{3,2} = h_0^2 h_3$$
$$\gamma_{2,3} = h_2^2 \qquad \gamma_{4,1} = h_0^3 h_3$$
$$\gamma_{2,4} = h_0h_3 \qquad \gamma_{s,0} = h_0^s$$
$$\gamma_{2,5} = h_1h_3$$

for $s \ge 5$. *The relations* $h_0h_1 = 0$, $h_1h_2 = 0$, $h_1^3 = h_0^2h_2$ *and* $h_0h_2^2 = 0$ *are satisfied, and generate all other relations for* $s \le 3$ *and* $t \le 11$.

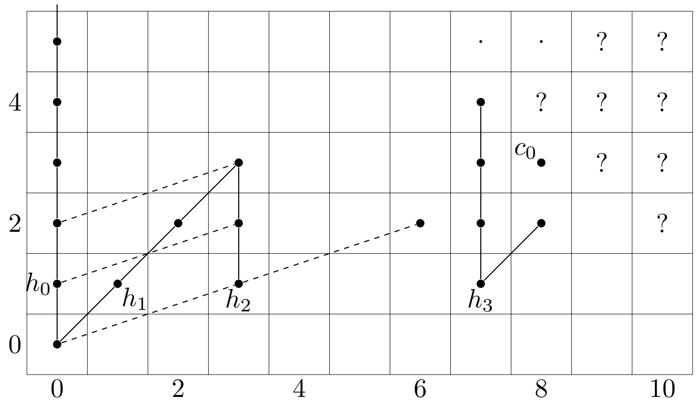
We redraw the Adams E_2-term with these standard names for the generators, in the usual chart with the topological degree $t - s$ on the horizontal axis and the filtration degree s on the vertical axis. (The class labeled h_1^3 could equally well have been called $h_0^2h_2$.)

	h_0^5							·	·	?	?
4	h_0^4							$h_0^3h_3$	?	?	?
	h_0^3			h_1^3				$h_0^2h_3$	c_0	?	?
2	h_0^2		h_1^2	h_0h_2			h_2^2	h_0h_3	h_1h_3		?
	h_0	h_1		h_2				h_3			
0	1										
	0		2		4		6		8		10

Another way to draw the chart is to use a $\bullet$ for each additive generator, a vertical line connecting x to h_0x, a line of slope 1 connecting x to h_1x, a (dashed) line of slope 1/3 connecting x to h_2x, and a (dotted) line of slope 1/7 connecting x to h_3x.



Here is the same chart without the h_3-multiplications, which tend to clutter the picture, but with labels for the indecomposables.



The reader might contemplate the relations $h_i h_{i+1} = 0$, $h_{i+1}^3 = h_i^2 h_{i+2}$ and $h_i h_{i+2}^2 = 0$, in view of this diagram.

Let us take for granted Adams' vanishing result, in the form that the groups $E_2^{s,t} = 0$ for $1 \le t-s \le 7$ and $s \ge 5$. Then:

Lemma 6.6. $E_2^{s,t} = E_\infty^{s,t}$ *for* $t \le 11$.

Proof. Since the h_i for $0 \le i \le 3$ represent homotopy classes, they are infinite cycles, meaning that $d_r(h_i) = 0$ for all $r \ge 2$. By the Leibniz rule, it follows that $d_r(x) = 0$ for each x in the subalgebra generated by these classes. The only remaining additive generator is c_0, but $d_r(c_0)$ lands in Adams' vanishing range, for all $r \ge 2$. □

Theorem 6.7. *(a)* $\pi_0(S)_2^\wedge \cong \mathbb{Z}_2$ *is generated by the identity map* $\iota\colon S \to S$*, represented by* $1 \in E_\infty^{0,0}$*. The class of* $2^s\iota$ *is represented by* $h_0^s \in E_\infty^{s,s}$*, for all* $s \ge 0$*.*

(b) $\pi_1(S)_2^\wedge \cong \mathbb{Z}/2$ *is generated by the complex Hopf map* $\eta\colon S^1 \to S$*, represented by* $h_1 \in E_\infty^{1,2}$*.*

(c) $\pi_2(S)_2^\wedge \cong \mathbb{Z}/2$ *is generated by* η^2*, represented by* $h_1^2 \in E_\infty^{2,4}$*.*

(d) $\pi_3(S)_2^\wedge \cong \mathbb{Z}/8$ *is generated by the quaternionic Hopf map* $\nu\colon S^3 \to S$*, represented by* $h_2 \in E_\infty^{1,4}$*. The class* 2ν *is represented by* $h_0h_2 \in E_\infty^{2,5}$*, and the class* $4\nu = \eta^3$ *is represented by* $h_0^2h_2 = h_1^3$ *in* $E_\infty^{3,6}$*.*

(e) $\pi_4(S)_2^\wedge = 0$*.*

(f) $\pi_5(S)_2^\wedge = 0$*.*

(g) $\pi_6(S)_2^\wedge \cong \mathbb{Z}/2$ *is generated by* ν^2*, represented by* $h_2^2 \in E_\infty^{2,8}$*.*

(h) $\pi_7(S)_2^\wedge \cong \mathbb{Z}/16$ *is generated by the octonionic Hopf map* $\sigma\colon S^7 \to S$*, represented by* $h_3 \in E_\infty^{1,8}$*. The classes* $2^k\sigma$ *are represented by* $h_0^k h_3 \in E_\infty^{k+1,k+8}$*, for* $0 \le k \le 3$*.*

This gives the additive structure of $\pi_*(S)_2^\wedge$ for $* \le 7$. We can also determine the multiplicative structure.

Proposition 6.8. $2\eta = 0$*,* $\eta^3 = 4\nu$*,* $\eta\nu = 0$*,* $2\nu^2 = 0$*.*

Proof. These follow from the relations $h_0h_1 = 0$, $h_1^3 = h_0^2h_2$, $h_1h_2 = 0$ and $h_0h_2^2 = 0$ in $\mathrm{Ext}_{\mathscr{A}}$, together with the fact that there are no classes of higher Adams filtration, in these cases. □

Remark 6.9. By associativity, it is clear that $\eta \cdot \nu^2 = \eta\nu \cdot \nu = 0$. On the other hand, the vanishing of $h_1 \cdot h_2^2$ in $\mathrm{Ext}_{\mathscr{A}}^{3,10}(\mathbb{F}_2, \mathbb{F}_2)$ only tells us that $\eta \cdot \nu^2$ is 0 modulo classes of Adams filtration $s \ge 4$. There is one such class, namely 8σ represented by $h_0^3h_3$, but the factorization of ν^2 tells us that $\eta \cdot \nu^2$ is not equal to 8σ, but is 0.

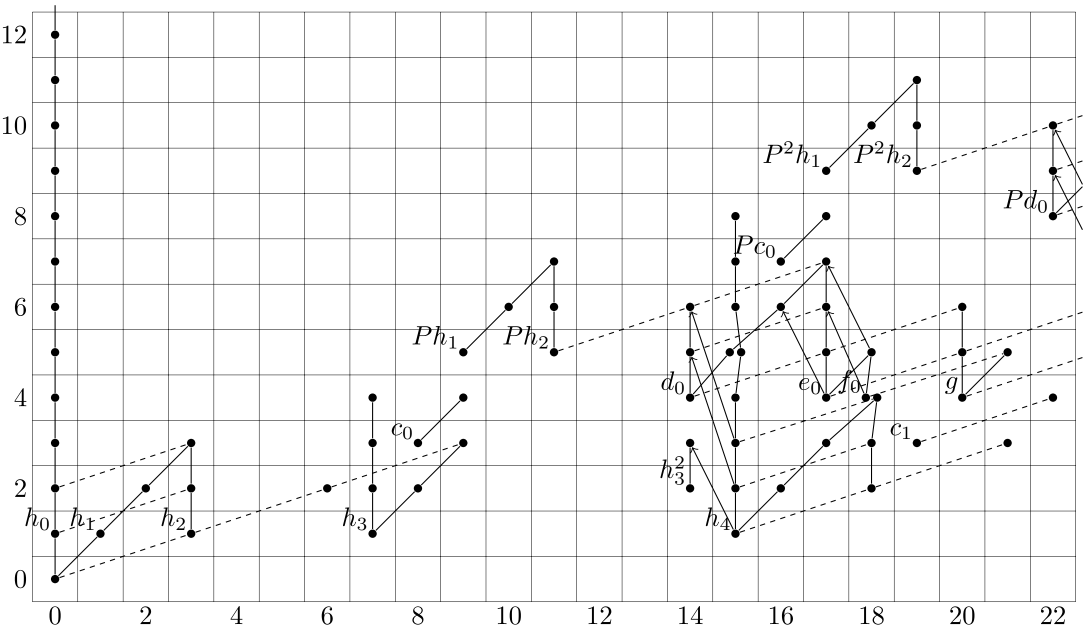


Figure 2: Adams spectral sequence for S, in degrees $0 \le * \le 22$

6.2 The Toda–Mimura range

Toda (1962) calculated $\pi_{n+k}(S^k)$ for all $n \le 19$, Mimura and Toda (1963) extended this to $n = 20$, and Mimura (1965) carried on to $n = 21$ and $n = 22$. For k large, these computations determine the stable homotopy groups $\pi_n(S)$ for $n \le 22$. ((Maybe better to continue to $n \le 23$, to see $\nu\bar\kappa$.))

The Adams E_2-term in this range was originally computed by hand (by Adams (1961) for $t - s \le 17$ and Liulevicius (unpublished) for $t - s \le 23$), then by the May spectral sequence (by May (1964) for $t-s \le 42$ and Tangora (1970) for $t-s \le 70$), but can now quickly be obtained by machine computation. Bruner's `ext`-program yields the chart in Figure 2. The larger chart in Figure 3 was created by Christian Nassau (2001).

((Show hidden extensions: η times ρ is represented by Pc_0, η times $\eta\bar\kappa$ is represented by Pd_0, 2 times $2\nu\bar\kappa$ equals ν times $4\bar\kappa$ and is represented by h_1Pd_0, ν times ν^2 differs from $\eta^2\sigma$ by $\eta\epsilon$.))

With the exception of f_0, each labeled class is the unique nonzero class in its bidegree. The class f_0 is, for now, only defined modulo the decomposable class $h_1^3h_4 = h_0^2h_2h_4$. (A definite choice can be made using Steenrod operations in Ext.)

In addition to the h_0-, h_1- and h_2-multiplications shown, and the product $h_3 \cdot h_3 = h_3^2$ in $E_2^{2,16}$, there are the following nonzero h_3-multiplications:

$$\begin{aligned}
h_3 \cdot Ph_1 &= h_1^2 d_0 \\
h_3 \cdot h_1Ph_1 &= h_2^2Ph_2 = h_1^3d_0 = h_0^3e_0 \\
h_3 \cdot h_3^2 &= h_2^2h_4 \\
h_3 \cdot e_0 &= h_1h_4c_0 \\
h_3 \cdot P^2h_1 &= h_1^2Pd_0 \\
h_3 \cdot h_1P^2h_1 &= h_2^2P^2h_2 = h_1^3Pd_0 = h_0^3Pe_0
\end{aligned}$$

The last three of these land outside the displayed range of topological degrees. We omit to list the h_i-multiplications for $i \ge 4$. ((The multiplicative structure also includes relations like $c_0^2 = h_1^2d_0$.))

The evolution of the Adams spectral sequence in this range is as follows.

Theorem 6.10. *The algebra indecomposables in topological degree $t - s \le 22$ of the Adams E_2-term are h_0, h_1, h_2, h_3 and h_4 in filtration $s = 1$, c_0 and c_1 in filtration $s = 3$, d_0, e_0, f_0 and $g = g_1$ in filtration*

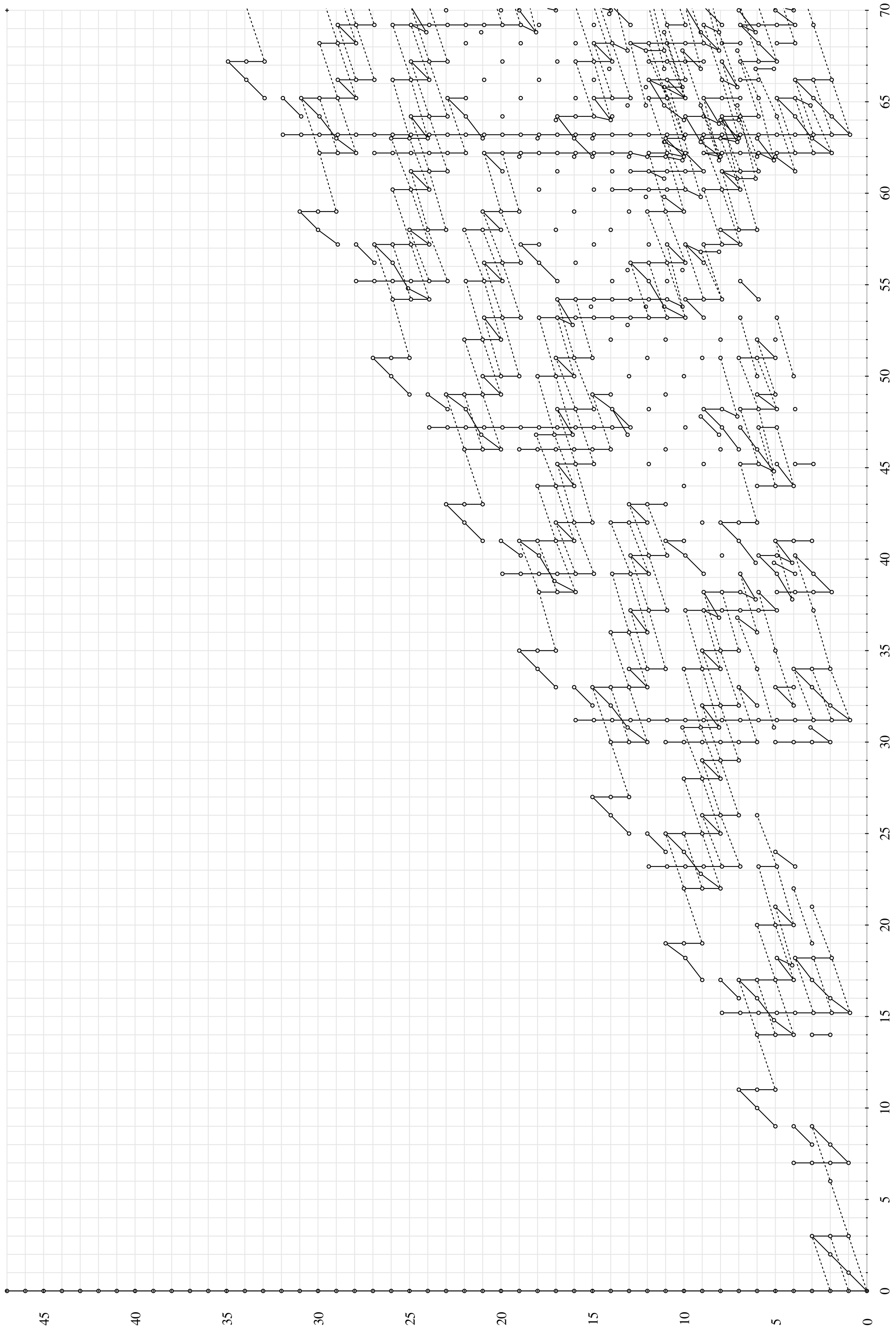


Figure 3: Ext over $\mathscr{A}$ by Christian Nassau (2001)

$s = 4$, Ph_1 *and* Ph_2 *in filtration* $s = 5$, Pc_0 *in filtration* $s = 7$, Pd_0 *in filtration* $s = 8$, *and* P^2h_1 *and* P^2h_2 *in filtration* $s = 9$.

The classes h_0, h_1, h_2, h_3, c_0, c_1, d_0, g, Ph_1, Ph_2, Pc_0, Pd_0, P^2h_1 *and* P^2h_2 *are infinite cycles.*

The nonzero d_2*-differentials affecting this range are:*

$$\begin{aligned} h_4 &\overset{d_2}{\longmapsto} h_0h_3^2 \\ e_0 &\longmapsto h_1^2d_0 \\ f_0 &\longmapsto h_0h_2d_0 = h_0^2e_0 \\ h_1e_0 = h_0f_0 &\longmapsto h_1^3d_0 = h_0^3e_0 \\ i &\longmapsto h_0Pd_0 \\ h_0i &\longmapsto h_0^2Pd_0 \end{aligned}$$

The list of algebra indecomposables of the E_3*-term is as for the* E_2*-term, with* h_4, e_0 *and* f_0 *deleted, but with* h_0h_4, h_1h_4 *and* h_2h_4 *added. The classes* h_1h_4 *and* h_2h_4 *are infinite cycles.*

The nonzero d_3*-differentials are:*

$$\begin{aligned} h_0h_4 &\overset{d_3}{\longmapsto} h_0d_0 \\ h_0^2h_4 &\longmapsto h_0^2d_0 \end{aligned}$$

The list of algebra indecomposables of the E_4*-term is as for the* E_3*-term, with* h_0h_4 *deleted, but with* $h_0^3h_4$ *added. There are no further differentials, so that* $E_4 = E_\infty$ *in this range of topological degrees.*

Sketch proof. Use graded commutativity of $\pi_*(S)$ to see that $2\sigma^2 = 0$, but $h_0h_3^2 \neq 0$ in $E_2^{3,17}$. Since $h_0h_3^2$ is an infinite cycle, it must be a boundary, so $d_2(h_4) = h_0h_3^2$.

Using the homotopy-everything structure on S, one gets a differential $d_2(f_0) = h_0^2e_0$, which implies that $d_2(h_0f_0) = h_0^3e_0$ and $d_2(e_0) = h_1^2d_0$.

Using the J-homomorphism, we known that $\pi_{15}(S)_2^\wedge$ contains $\mathbb{Z}/32$ as a direct summand. We know that $d_2(h_0h_4) = h_0^2h_3^2 = 0$. If also $d_3(h_0h_4) = 0$, then $\pi_{15}(S)_2^\wedge$ would instead contain a copy of $\mathbb{Z}/64$ (unless $d_6(h_1h_4) = h_0^7h_4$). Deduce that $d_3(h_0h_4) = h_0d_0$. □

Toda (1962) uses the following notation.

Definition 6.11. Let $\epsilon \in \pi_8(S)_2^\wedge$ be the unique class represented by $c_0 \in E_\infty^{3,11}$. Then $\eta\epsilon \in \pi_9(S)_2^\wedge$ is represented by $h_1c_0 \in E_\infty^{4,13}$. ((Claim: $\nu^3 = \eta^2\sigma + \eta\epsilon$.))

Let $\mu = \mu_9 \in \pi_9(S)_2^\wedge$ be the unique class represented by $Ph_1 \in E_\infty^{5,14}$. Then $\eta\mu = \mu_{10} \in \pi_{10}(S)_2^\wedge$ is the unique class represented by $h_1Ph_1 \in E_\infty^{6,16}$.

Let $\zeta \in \pi_{11}(S)_2^\wedge$ be a class represented by $Ph_2 \in E_\infty^{5,16}$. It is determined up to an odd multiple. Then $4\zeta = \eta^2\mu$.

The class $\sigma^2 = \theta_3$ in $\pi_{14}(S)_2^\wedge$ is decomposable. It is represented by $h_3^2 \in E_\infty^{2,16}$.

Let $\kappa \in \pi_{14}(S)_2^\wedge$ be the unique class represented by $d_0 \in E_\infty^{4,18}$. ((Then $\eta\kappa \in \pi_{15}(S)_2^\wedge$ is represented by h_1d_0, and $\nu\kappa \in \pi_{17}(S)_2^\wedge$ is represented by h_2d_0, while $\eta^2\kappa = 0$.))

Let $\rho \in \pi_{15}(S)_2^\wedge$ be a class represented by $h_0^3h_4$. It is determined up to an odd multiple. ((There is a hidden multiplicative extension: $\eta\rho$ is represented by Pc_0.))

Let $\eta^* = \eta_4 \in \pi_{16}(S)_2^\wedge$ be a class represented by h_1h_4. ((This only defines it modulo $\eta\rho$.))

Let $\nu^* \in \pi_{18}(S)_2^\wedge$ be a class represented by h_2h_4. ((This only defines it up to an odd multiple, and modulo $\eta\bar{\mu} = \mu_{18}$. Compare σ^3 to $\nu\nu^*$?))

Let $\bar{\mu} = \mu_{17} \in \pi_{17}(S)_2^\wedge$ be the unique class represented by $P^2h_1 \in E_\infty^{9,26}$. Then $\eta\bar{\mu} = \mu_{18} \in \pi_{18}(S)_2^\wedge$ is the unique class represented by $h_1P^2h_1 \in E_\infty^{10,28}$.

((Define $\bar{\sigma}$, $\bar{\zeta}$.))

Definition 6.12. It is traditional to write θ_j for a class in $\pi_{2^{j+1}-2}(S)$ represented by h_j^2 in $E_\infty^{2^{j+1},2}$, if such a class exists, and to write η_j for a class in $\pi_{2^j}(S)$ represented by $h_1h_j \in E_\infty^{2^j+2,2}$.

Remark 6.13. The classes θ_j are realized for $0 \leq j \leq 3$ by $2^2 = 4$, η^2, ν^2 and σ^2. It follows from the computations of Mahowald and Tangora (1967) that h_4^2 is an infinite cycle, so that $\theta_4 \in \pi_{30}(S)$ exists. It was proved by Barratt, Jones and Mahowald (1984) that h_5^2 is an infinite cycle, so that $\theta_5 \in \pi_{62}(S)$

exists. It is an open problem whether $\theta_6 \in \pi_{126}(S)$ exists. Hill, Hopkins and Ravenel (2009, to appear) showed that θ_j does not exist for $j \geq 7$.

Mahowald (Topology, 1977) proved that the η_j exist (so that $h_1 h_j$ is an infinite cycle) for all $j \geq 3$.

It is known (Mahowald and Tangora (1967), plus later calculations) that the only other classes in filtration $s = 2$ that survive to the E_∞-term are h_0h_2, h_0h_3 and h_2h_4, representing 2ν, 2σ and ν^* in $\pi_*(S)$.

Theorem 6.14. *(a) $\pi_8(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ is generated by $\eta\sigma$ and ϵ, represented by $h_1h_3 \in E_\infty^{2,10}$ and $c_0 \in E_\infty^{3,11}$, respectively.*

(b) $\pi_9(S)^\wedge_2 \cong (\mathbb{Z}/2)^3$ is generated by $\eta^2\sigma$, $\eta\epsilon$ and μ, represented by $h_1^2h_3 \in E_\infty^{3,12}$, $h_1c_0 \in E_\infty^{4,13}$ and $Ph_1 \in E_\infty^{5,14}$, respectively.

(c) $\pi_{10}(S)^\wedge_2 \cong \mathbb{Z}/2$ is generated by $\eta\mu$, represented by $h_1Ph_1 \in E_\infty^{6,16}$.

(d) $\pi_{11}(S)^\wedge_2 \cong \mathbb{Z}/8$ is generated by ζ, represented by $Ph_2 \in E_\infty^{5,16}$. The class 2ζ is represented by $h_0Ph_2 \in E_\infty^{6,17}$, and the class $4\zeta = \eta^2\rho$ is represented by $h_0^2Ph_2 = h_1^2Ph_1 \in E_\infty^{7,18}$.

(e) $\pi_{12}(S)^\wedge_2 = 0$.

(f) $\pi_{13}(S)^\wedge_2 = 0$.

(g) $\pi_{14}(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ is generated by σ^2 and κ, represented by $h_3^2 \in E_\infty^{2,16}$ and $d_0 \in E_\infty^{4,18}$, respectively.

(h) $\pi_{15}(S)^\wedge_2 \cong \mathbb{Z}/32 \oplus \mathbb{Z}/2$ is generated by ρ and $\eta\kappa$, represented by $h_0^3h_3 \in E_\infty^{4,19}$ and $h_1d_0 \in E_\infty^{5,20}$, respectively. The classes $2^k\rho$ are represented by $h_0^{k+3}h_3 \in E_\infty^{k+4,k+19}$ for $0 \leq k \leq 4$.

(i) $\pi_{16}(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ is generated by $\eta^* = \eta_4$ and $\eta\rho$, represented by $h_1h_4 \in E_\infty^{2,18}$ and $Pc_0 \in E_\infty^{7,23}$, respectively. ((Note the filtration shift in $\eta \cdot \rho$.))

(j) $\pi_{17}(S)^\wedge_2 \cong (\mathbb{Z}/2)^4$ is generated by $\eta\eta^*$, $\nu\kappa$, $\eta^2\rho$ and $\bar{\mu} = \mu_{17}$, represented by $h_1^2h_4 \in E_\infty^{3,20}$, $h_2d_0 \in E_\infty^{5,22}$, $h_1Pc_0 \in E_\infty^{7,24}$ and $P^2h_1 \in E_\infty^{9,26}$, respectively.

(k) $\pi_{18}(S)^\wedge_2 \cong \mathbb{Z}/8 \oplus \mathbb{Z}/2$ is generated by ν^* and $\eta\bar{\mu} = \mu_{18}$, represented by $h_2h_4 \in E_\infty^{2,20}$ and $h_1P^2h_1 \in E_\infty^{10,28}$, respectively.

(l) $\pi_{19}(S)^\wedge_2 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/8$ is generated by $\bar{\sigma}$ and $\bar{\zeta}$, represented by $c_1 \in E_\infty^{3,22}$ and $P^2h_2 \in E_\infty^{9,28}$, respectively.

(m) $\pi_{20}(S)^\wedge_2 \cong \mathbb{Z}/8$ is generated by $\bar{\kappa}$, represented by $g \in E_\infty^{4,24}$. The class $2\bar{\kappa}$ is represented by $h_0g \in E_\infty^{5,25}$, and the class $4\bar{\kappa} = \nu^2\kappa$ is represented by $h_0^2g = h_2^2d_0 \in E_\infty^{6,26}$.

(n) $\pi_{21}(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ ((?)) is generated by $\nu\nu^*$ and $\eta\bar{\kappa}$, represented by $h_2^2h_4 \in E_\infty^{3,24}$ and $h_1g \in E_\infty^{5,26}$, respectively.

(o) $\pi_{22}(S)^\wedge_2 \cong (\mathbb{Z}/2)^2$ ((?)) is generated by $\nu\bar{\sigma}$ and $\eta^2\bar{\kappa}$, represented by $h_2c_1 \in E_\infty^{4,26}$ and $Pd_0 \in E_\infty^{8,30}$, respectively. ((Note the filtration shift in $\eta \cdot \eta\bar{\kappa}$.))

((Discuss additive splittings, by $2\eta = 0$ and associativity, and multiplicative extensions.))

Remark 6.15. There are Steenrod operations Sq^i in $E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, taking $E_2^{s,t}$ to $E_2^{s+i,2t}$. In particular $Sq^0 \colon E_2^{s,t} \to E_2^{s,2t}$ is multiplicative, and maps h_i to h_{i+1} for $i \geq 0$. A sequence of elements

$$x, Sq^0(x), Sq^0(Sq^0(x)), \dots$$

is called a Sq^0-family. In the Sq^0-family $h_0, h_1, h_2, \dots$ the first four classes detect 2ι, η, ν and σ, but h_4 and all later terms are killed by the Adams differentials $d_2(h_i) = h_0h_{i-1}^2$ for $i \geq 4$.

In the Sq^0-family $h_0^2, h_1^2, h_2^2, \dots$ the first six classes detect 4ι, η^2, ν^2, σ^2, θ_4 and θ_5, but h_7^2 and all later terms are killed by (unknown) differentials. The status of h_6^2 is unknown. In the family $h_0h_2, h_1h_3, h_2h_4, \dots$ the first three classes detect 2ν, $\eta\sigma$ and ν^*, but h_3h_5 and all later terms support differentials. In the family $h_0h_3, h_1h_4, h_2h_5, \dots$ the first two classes detect 2σ and η^*, but h_2h_5 and all later terms support differentials. For each $i \geq 4$, only the term h_1h_{i+1} survives in the family

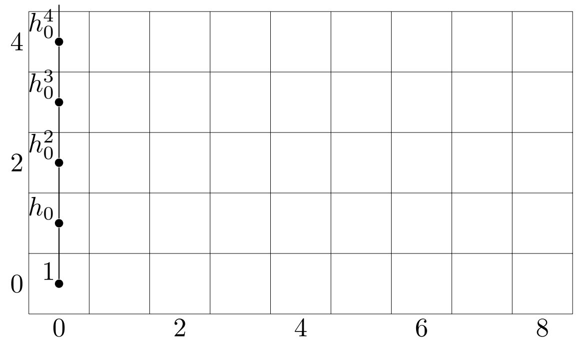


Figure 4: Adams spectral sequence for $H\mathbb{Z}$

$h_0h_i, h_1h_{i+1}, h_2h_{i+2}, \ldots$, detecting η_{i+1}. The classes $c_0, c_1, c_2, \ldots$ also form a Sq^0-family. The first two classes detect ϵ and $\bar{\sigma}$, but there are differentials $d_2(c_i) = h_0 f_{i-1}$ for $i \geq 2$.

These results leads to the conjecture, called the "New Doomsday Conjecture" by Minami, and the "Finiteness Conjecture" by Bruner, saying that only a finite number of terms in each Sq^0-family detects nonzero homotopy classes. ((References?))

6.3 Adams vanishing

Lemma 6.16 (Change of rings)**.** *Let A be any algebra, let $B \subset A$ be a subalgebra such that A is flat as a right B-module, let M be any left B-module and let N be any left A-module. There is a natural isomorphism*

$$\mathrm{Ext}_A^{s,t}(A \otimes_B M, N) \cong \mathrm{Ext}_B^{s,t}(M, N)\,.$$

Proof. Let $P_* \to M$ be a B-free resolution. Then $A \otimes_B P_* \to A \otimes_B M$ is an $\mathscr{A}$-free resolution. The isomorphism $\mathrm{Hom}_A(A \otimes_B P_*, N) \cong \mathrm{Hom}_B(P_*, N)$ induces the asserted isomorphism upon passage to cohomology. □

((TODO: Discuss compatibility of multiplicative structure(s) in Ext_A and Ext_B.))

Definition 6.17. Let A be an algebra and let $B \subset A$ be an augmented subalgebra, with augmentation ideal $I(B) = \ker(\epsilon)$. Let

$$A//B = A \otimes_B \mathbb{F}_2 \cong A/A \cdot I(B)\,.$$

If B is normal in A, meaning that $I(B) \cdot A = A \cdot I(B)$, then $A//B$ is a quotient algebra of A.

Recall that we write $P(x) = \mathbb{F}_2[x]$ and $E(x) = P(x)/(x^2)$ for the polynomial algebra and the exterior algebra, respectively, on a generator x. Let $A(0) = E(0) = E(Sq^1) \subset \mathscr{A}$ be the subalgebra generated by Sq^1. There are isomorphisms $H^*(H\mathbb{Z}) \cong \mathscr{A}/\mathscr{A}Sq^1 \cong \mathscr{A} \otimes_{A(0)} \mathbb{F}_2 = \mathscr{A}//A(0)$.

Proposition 6.18. *The Adams spectral sequence for $H\mathbb{Z}$ collapses at the E_2-term*

$$E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(H\mathbb{Z}), \mathbb{F}_2) \cong \mathrm{Ext}_{A(0)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_0)$$

where $h_0 \in E_2^{1,1}$, and converges strongly to $\pi_*(H\mathbb{Z}_2)$. The class of $2^s \in \pi_0(H\mathbb{Z}_2) = \mathbb{Z}_2$ is represented by $h_0^s \in E_\infty^{s,s}$, for each $s \geq 0$.

Proof. The Steenrod algebra $\mathscr{A}$ is free as a right $A(0)$-module, generated by the admissible monomials Sq^I for which $I = (i_1, \ldots, i_\ell)$ and $i_\ell \geq 2$. (This includes the monomial $1 = Sq^{()}$.)

There is a minimal, free $A(0)$-module resolution P_* of $\mathbb{F}_2$ with $P_s = A(0)\{g_s\} = \mathbb{F}_2\{g_s, Sq^1 g_s\}$ for each $s \geq 0$, and $\partial_s(g_s) = Sq^1 g_{s-1}$ for each $s \geq 1$. Then $\mathrm{Ext}_{A(0)}^s(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Hom}_{A(0)}(P_s, \mathbb{F}_2) \cong \mathbb{F}_2\{\gamma_s\}$ is generated by the dual of g_s. It lifts to a chain map $\tilde{\gamma}_s \colon P_{*+s} \to P_*$ that takes g_{n+s} to g_n for each $n \geq 0$. These satisfy $\tilde{\gamma}_u \circ \tilde{\gamma}_s = \tilde{\gamma}_{u+s}$ under composition, so $\gamma_u \cdot \gamma_s = \gamma_{u+s}$ in the Yoneda product. Let $h_0 = \gamma_1$ be dual to g_1, in internal degree 1. Then $\gamma_s = h_0^s$ and we have proved that $\mathrm{Ext}_{A(0)}^*(\mathbb{F}_2, \mathbb{F}_2) \cong \mathbb{F}_2\{h_0^s \mid s \geq 0\} = P(h_0)$. □

The cofiber sequence

$$S \xrightarrow{\eta} H\mathbb{Z} \to \overline{H\mathbb{Z}}$$

induces a short exact sequence

$$0 \leftarrow \mathbb{F}_2 \xleftarrow{\eta^*} \mathscr{A}//A(0) \longleftarrow I(\mathscr{A}/\mathscr{A}Sq^1) \leftarrow 0$$

in cohomology, and a long exact sequence

$$\mathrm{Ext}_{\mathscr{A}}^{s-1,t}(I(\mathscr{A}/\mathscr{A}Sq^1),\mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \xrightarrow{\eta_*} \mathrm{Ext}_{A(0)}^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t}(I(\mathscr{A}/\mathscr{A}Sq^1),\mathbb{F}_2)$$

of Adams E_2-terms. The map η_* is an isomorphism for $t-s=0$, so the connecting homomorphism δ is an isomorphism for $t-s \neq 0$.

Lemma 6.19. *$I(\mathscr{A}/\mathscr{A}Sq^1)$ is free as a left $A(0)$-module, generated by the admissible Sq^I for which $I=(i_1,\dots,i_\ell)$, i_1 is even and $i_\ell \geq 2$. (This excludes the monomial $1 = Sq^{()}$.) The first few basis elements are*

$$Sq^2, Sq^4, Sq^6, Sq^4Sq^2, Sq^8, Sq^6Sq^2, Sq^6Sq^3, Sq^{10}, Sq^8Sq^2, Sq^8Sq^3, \dots.$$

Proof. When Sq^I ranges over the admissible monomials with i_1 even and $i_\ell \geq 2$, then Sq^I and Sq^1Sq^I range over the admissible monomials with $i_\ell \geq 2$. The only exception occurs for $I=()$. □

Proposition 6.20. *Let M be an $\mathscr{A}$-module that is free as an $A(0)$-module, and concentrated in degrees $* \geq 0$. Let*

$$\epsilon(s) = \begin{cases} 0 & \text{for } s \equiv 0 \mod 4, \\ 1 & \text{for } s \equiv 1 \mod 4, \\ 2 & \text{for } s \equiv 2,3 \mod 4. \end{cases}$$

Then

$$Ext_{\mathscr{A}}^{s,t}(M,\mathbb{F}_2) = 0$$

for $t-s < 2s - \epsilon(s)$.

Proof. First consider the case $M = A(0)$, with the unique $\mathscr{A}$-module structure realized by $H^*(S/2)$. There is a minimal free $\mathscr{A}$-module resolution

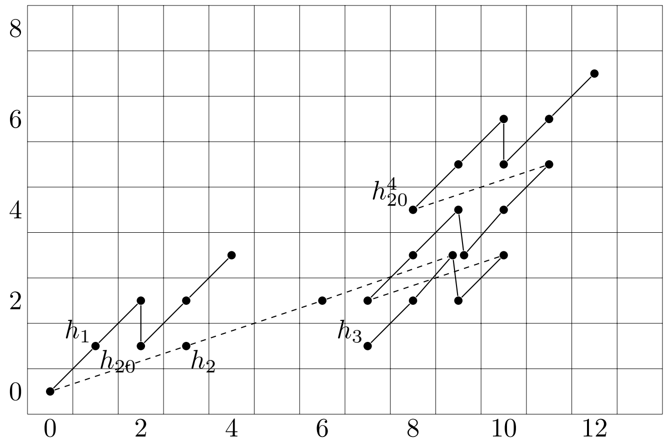
$$P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} A(0) \to 0$$

with $P_0 = \mathscr{A}\{1\}$, P_1 concentrated in degrees $t \geq 2$, P_2 concentrated in degrees $t \geq 4$, P_3 concentrated in degrees $t \geq 7$, and $\Sigma^{12}K = \ker(\partial_3)$ concentrated in degrees $t \geq 12$.

This can be proved by direct calculation, or by using our previous Ext-calculations for the sphere spectrum, the cofiber sequence $S \xrightarrow{2} S \longrightarrow S/2 \longrightarrow \Sigma S = S^1$, the induced extension $0 \leftarrow \mathbb{F}_2 \leftarrow A(0) \leftarrow \Sigma\mathbb{F}_2 \leftarrow 0$ of $\mathscr{A}$-modules, and the associated long exact sequence

$$\dots \to \mathrm{Ext}_{\mathscr{A}}^{s-1,t-1}(\mathbb{F}_2,\mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t}(A(0),\mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t-1}(\mathbb{F}_2,\mathbb{F}_2) \to \dots$$

in Ext. Here each connecting map δ is given by the Yoneda product with h_0, which is the class in $\mathrm{Ext}_{\mathscr{A}}^{1,1}(\mathbb{F}_2,\mathbb{F}_2)$ of the extension above. This leads to the additive structure of the following Adams chart for $\mathrm{Ext}_{\mathscr{A}}^{*,*}(A(0), \mathbb{F}_2)$:



This proves the claim for $M = A(0)$ and $0 \le s < 4$.

Next, consider an extension $0 \to M' \to M \to M'' \to 0$ of $A(0)$-free $\mathscr{A}$-modules, all concentrated in degrees $* \ge 0$, and suppose that the claim holds for M' and M''. Then the claim follows for M, in view of the long exact sequence

$$\dots \to \mathrm{Ext}_{\mathscr{A}}^{s,t}(M'', \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t}(M', \mathbb{F}_2) \to \dots .$$

The claim for general $A(0)$-free M and $0 \le s < 4$ then follows.

Since $A(0)$ and each P_s is $A(0)$-free, it follows that $\Sigma^{12}K = \ker(\partial_3)$ is $A(0)$-free, and concentrated in degrees $* \ge 12$. Thinking of P_{*+4} as a resolution of $\Sigma^{12}K$, we get an isomorphism

$$\mathrm{Ext}_{\mathscr{A}}^{s,t}(K, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathscr{A}}^{s+4,t+12}(\mathbb{F}_2, \mathbb{F}_2)$$

for all $s \ge 0$. Hence the claim for $A(0)$ and $4 \le s < 8$ follows from the one for K and $0 \le s < 4$. The general claim for $A(0)$-free M and $4 \le s < 8$ then follows as above. Continuing this way, the general claim follows for all $s \ge 0$. $\square$

Corollary 6.21. $\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = 0$ *for* $0 < t - s < 2s - \epsilon$*, where* $\epsilon = 1$ *for* $s \equiv 1 \mod 4$*,* $\epsilon = 2$ *for* $s \equiv 2 \mod 4$ *and* $\epsilon = 3$ *for* $s \equiv 0, 3 \mod 4$*.*

Proof. This follows from the isomorphisms

$$\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{\mathscr{A}}^{s-1,t}(I(\mathscr{A}/\mathscr{A}Sq^1), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathscr{A}}^{s-1,t-2}(M, \mathbb{F}_2)$$

for $t - s > 0$, where $\Sigma^2 M = I(\mathscr{A}/\mathscr{A}Sq^1)$, and the proposition as applied to M. $\square$

This result is not quite optimal for $s \equiv 0 \mod 4$. Adams (1966) works a little harder to prove the optimal vanishing range:

Theorem 6.22 (Adams vanishing)**.** $\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) = 0$ *for* $0 < t - s < 2s - \epsilon$*, where* $\epsilon = 1$ *for* $s \equiv 0, 1 \mod 4$*,* $\epsilon = 2$ *for* $s \equiv 2 \mod 4$ *and* $\epsilon = 3$ *for* $s \equiv 3 \mod 4$*.*

((ETC: Approximation for Ext over $A(n) \subset \mathscr{A}$.))

6.4 Topological K-theory

Definition 6.23. Let ku and ko be the complex and real connective K-theory spectra, with underlying infinite loop spaces $\Omega^\infty ku = \mathbb{Z} \times BU$ and $\Omega^\infty ko = \mathbb{Z} \times BO$, respectively. These are the connective covers of the complex and real topological K-theory spectra, KU and KO, respectively.

Definition 6.24. Let bu and bsu be the 1- and 3-connected connected covers of ku, respectively, with $\Omega^\infty bu = BU$ and $\Omega^\infty bsu = BSU$. Let bo, bso and $bspin$ be the 0-, 1- and 3-connected covers of ko, respectively, with $\Omega^\infty bo = BO$, $\Omega^\infty bso = BSO$ and $\Omega^\infty bspin = BSpin$. We may also use the notations $u = \Sigma^{-1}bu$, $su = \Sigma^{-1}bsu$, $o = \Sigma^{-1}bo$, $so = \Sigma^{-1}bso$ and $spin = \Sigma^{-1}bspin$, for the desuspended spectra with infinite loop spaces U, SU, O, SO and $Spin$, respectively.

Remark 6.25. This is the notation used by Adams and May. Mahowald and Ravenel write bu and bo for "our" ku and ko.

Definition 6.26. Let $Q_1 = [Sq^1, Sq^2] = Sq^3 + Sq^2Sq^1$. Let $E(1) = E(Sq^1, Q_1) \subset \mathscr{A}$ be the subalgebra of $\mathscr{A}$ generated by Sq^1 and Q_1, and let $A(1) = \langle Sq^1, Sq^2 \rangle \subset \mathscr{A}$ be the subalgebra generated by Sq^1 and Sq^2. Here is an additive basis for $A(1)$, with the action by Sq^1 and Sq^2 indicated by arrows:

$$\begin{tikzcd}
1 \arrow[rr] \arrow[dr] & & Sq^2 \arrow[r] \arrow[drr] & Sq^3 \arrow[rr] & & Sq^2Sq^3 \arrow[dr] & \\
& Sq^1 \arrow[rr] & & Sq^2Sq^1 \arrow[r] & Sq^3Sq^1 \arrow[rr] & & Sq^1Sq^5
\end{tikzcd}$$

For typographical reasons, we write Sq^2Sq^3 in place of its admissible expansion $Sq^5 + Sq^4Sq^1$. Note that $E(1)//A(0) \cong E(Q_1)$, $A(1)//E(Q_1) \cong E(Sq^1, Sq^2)$ and $A(1)//E(1) \cong E(Sq^2)$.

Proposition 6.27 (Stong)**.** *There are $\mathscr{A}$-module isomorphisms*

$$H^*(ku) \cong \mathscr{A}//E(1) = \mathscr{A}/\mathscr{A}\{Sq^1, Q_1\} = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^3\}$$

and

$$H^*(ko) \cong \mathscr{A}//A(1) = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\}\,.$$

Proof. By complex Bott periodicity, there is a cofiber sequence

$$\Sigma^2 ku \xrightarrow{\beta} ku \to H\mathbb{Z} \to \Sigma^3 ku\,.$$

Here $\Sigma^2 ku = bu$ is the connected cover of ku. The left hand map is a composite

$$\Sigma^2 ku = ku \wedge S^2 \xrightarrow{1 \wedge u} ku \wedge ku \xrightarrow{\phi} ku$$

where $u \in \pi_2(ku)$ is a generator and ϕ is the ring spectrum product. It is known that the mod 2 Hurewicz image of u is zero, so $\beta^* = 0$, and there is a short exact sequence of $\mathscr{A}$-modules

$$0 \leftarrow H^*(ku) \leftarrow H^*(H\mathbb{Z}) \leftarrow \Sigma^3 H^*(ku) \leftarrow 0\,.$$

The short exact sequence of $E(1)$-modules

$$0 \leftarrow \mathbb{F}_2 \leftarrow E(1)//A(0) \leftarrow \Sigma^3 \mathbb{F}_2 \leftarrow 0$$

can be induced up to a short exact sequence

$$0 \leftarrow \mathscr{A}//E(1) \leftarrow \mathscr{A}//A(0) \leftarrow \Sigma^3 \mathscr{A}//E(1) \leftarrow 0\,,$$

since $\mathscr{A}$ is free as a right $E(1)$-module.

The composite $H\mathbb{Z} \to \Sigma^3 ku \to \Sigma^3 H\mathbb{Z}$ is known to take $\Sigma^3 1$ to Q_1 in cohomology, so $ku \to H\mathbb{Z}$ takes Q_1 to 0 in cohomology. Hence there is a map of short exact sequences

$$\begin{tikzcd}
0 & \mathscr{A}//E(1) \arrow[l] \arrow[d] & \mathscr{A}//A(0) \arrow[l] \arrow[d, "\cong"'] & \Sigma^3 \mathscr{A}//E(1) \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & H^*(ku) \arrow[l] & H^*(H\mathbb{Z}) \arrow[l] & \Sigma^3 H^*(ku) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

We know that the middle map is an isomorphism, and the right hand map is the triple suspension of the left hand map. It follows by induction on the internal degree that the latter two maps are also isomorphisms.

By real Bott periodicity, there is a cofiber sequence

$$\Sigma ko \xrightarrow{\eta} ko \to ku \to \Sigma^2 ko\,.$$

The left hand map is a composite

$$\Sigma ko = ko \wedge S^1 \xrightarrow{1\wedge\eta} ko \wedge ko \xrightarrow{\phi} ko$$

where $\eta \in \pi_1(ko)$ is the image of $\eta \in \pi_1(S)$, and ϕ is the ring spectrum product. The mod 2 Hurewicz image of η is zero, so $\eta^* = 0$, and there is a short exact sequence of $\mathscr{A}$-modules

$$0 \leftarrow H^*(ko) \leftarrow H^*(ku) \leftarrow \Sigma^2 H^*(ko) \leftarrow 0\,.$$

The short exact sequence of $A(1)$-modules

$$0 \leftarrow \mathbb{F}_2 \leftarrow A(1)//E(1) \leftarrow \Sigma^2\mathbb{F}_2 \leftarrow 0$$

can be induced up to a short exact sequence

$$0 \leftarrow \mathscr{A}//A(1) \leftarrow \mathscr{A}//E(1) \leftarrow \Sigma^2\mathscr{A}//A(1) \leftarrow 0\,,$$

since $\mathscr{A}$ is free as a right $A(1)$-module.

The composite $ku \to \Sigma^2 ko \to \Sigma^2 ku$ takes $\Sigma^2 1$ to Sq^2 in cohomology, so $ko \to ku$ takes Sq^2 to 0 in cohomology. Hence there is a map of short exact sequences

$$\begin{tikzcd}
0 & \mathscr{A}//A(1) \arrow[l] \arrow[d] & \mathscr{A}//E(1) \arrow[l] \arrow[d, "\cong"'] & \Sigma^2\mathscr{A}//A(1) \arrow[l] \arrow[d] & 0 \arrow[l] \\
0 & H^*(ko) \arrow[l] & H^*(ku) \arrow[l] & \Sigma^2 H^*(ko) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

We know that the middle map is an isomorphism, and the right hand map is the double suspension of the left hand map. It follows by induction on the internal degree that the latter two maps are also isomorphisms. □

Proposition 6.28. *The Adams spectral sequence for ku collapses at the E_2-term*

$$E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(ku), \mathbb{F}_2) \cong \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_0, h_{20})$$

where $h_0 \in E_2^{1,1}$ and $h_{20} \in E_2^{1,3}$, and converges strongly to $\pi_*(ku_2^\wedge) = \mathbb{Z}_2[u]$. The class of $2 \in \pi_0(ku_2^\wedge)$ is represented by h_0, and the class of $u \in \pi_2(ku_2^\wedge)$ is represented by h_{20}.

Proof. We use the change of rings isomorphism $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//E(1), \mathbb{F}_2) \cong \mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. ((Must justify that $\mathscr{A}$ is right free, thus flat, over $E(1)$.)) There is a Künneth isomorphism

$$\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{E(Sq^1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \otimes \mathrm{Ext}_{E(Q_1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$$

and $\mathrm{Ext}_{E(Q_1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_{20})$ with h_{20} dual to Q_1, by the same argument we used to show that $\mathrm{Ext}_{E(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_0)$ with h_0 dual to Sq^1. (Another name for h_{20} is v_1.) The spectral sequence is concentrated in even columns, hence collapses for bidegree reasons. □

Proposition 6.29. *The Adams spectral sequence for ko collapses at the E_2-term*

$$\begin{aligned}
E_2^{*,*} &= \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(ko), \mathbb{F}_2) \cong \mathrm{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \\
&\cong P(h_0, h_1, v, w_1)/(h_0h_1, h_1^3, h_1v, v^2 = h_0^2 w_1)
\end{aligned}$$

where $h_0 \in E_2^{1,1}$, $h_1 \in E_2^{1,2}$, $v \in E_2^{3,7}$ and $w_1 \in E_2^{4,12}$, and converges strongly to

$$\pi_*(ko_2^\wedge) = \mathbb{Z}_2[\eta, \alpha, \beta]/(2\eta, \eta^3, \eta\alpha, \alpha^2 = 4\beta)\,.$$

The classes 2, η, α and β are represented by h_0, h_1, v and w_1, respectively.

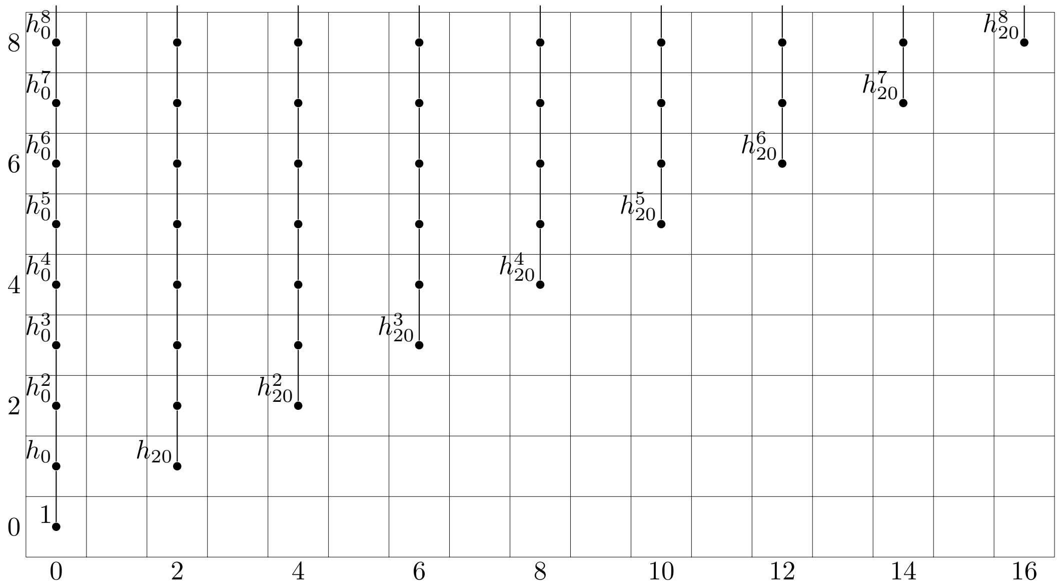


Figure 5: Adams spectral sequence for ku

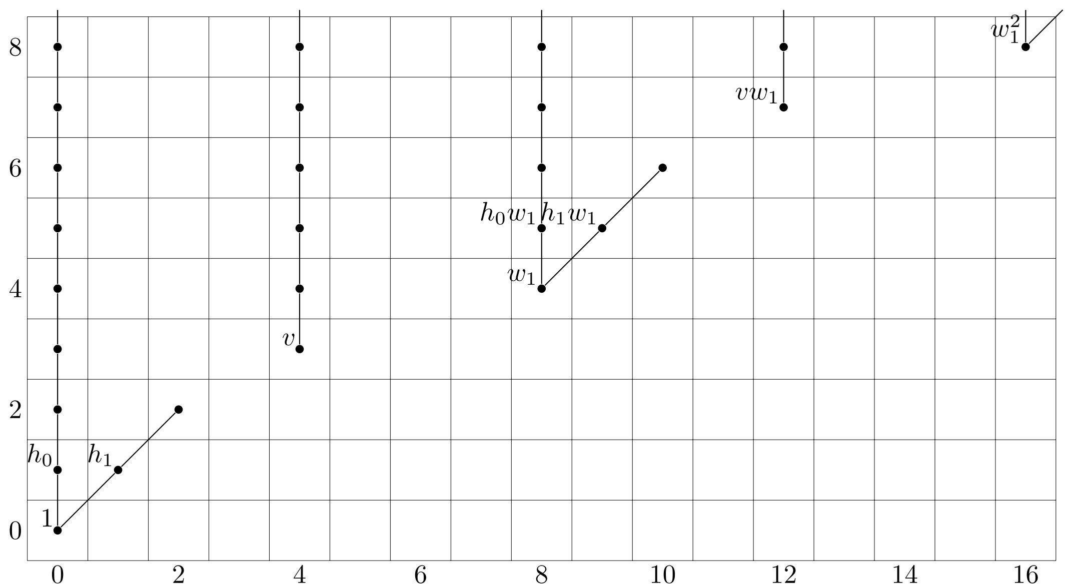


Figure 6: Adams spectral sequence for ko

Hence

$$\pi_n(ku_2^\wedge) \cong \begin{cases} \mathbb{Z}_2\{u^i\} & \text{for } n = 2i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\pi_n(ko_2^\wedge) \cong \begin{cases} \mathbb{Z}_2\{\beta^i\} & \text{for } n = 8i \\ \mathbb{Z}/2\{\eta\beta^i\} & \text{for } n = 8i+1 \\ \mathbb{Z}/2\{\eta^2\beta^i\} & \text{for } n = 8i+2 \\ \mathbb{Z}_2\{\alpha\beta^i\} & \text{for } n = 8i+4 \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq 0$.

((The complexification map $c\colon ko \to ku$ induces $h_0 \mapsto h_0$, $h_1 \mapsto 0$, $v \mapsto h_0 h_{20}^2$ and $w_1 \mapsto h_{20}^4$ in Ext, and similarly in homotopy.))

Remark 6.30. To compute $\operatorname{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, we can use the Cartan–Eilenberg spectral sequence (1956, Theorem XVI.6.1). If A is a connected graded algebra, $B \subset A$ is a normal subalgebra, and A is projective as a right B-module, then this is an algebra spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_{A//B}^p(\mathbb{F}_2, \operatorname{Ext}_B^q(\mathbb{F}_2, \mathbb{F}_2)) \Longrightarrow \operatorname{Ext}_A^{p+q}(\mathbb{F}_2, \mathbb{F}_2)$$

of cohomological type. In the special case when $A = \mathbb{F}_2[G]$ is a group algebra, and $B = \mathbb{F}_2[N]$ is the group algebra of a normal subgroup, we have $B//A = \mathbb{F}_2[G/N]$ and the Cartan–Eilenberg spectral sequence agrees with the Lyndon–Hochschild–Serre spectral sequence

$$E_2^{p,q} = H_{gp}^p(G/N; H_{gp}^q(N; \mathbb{F}_2)) \Longrightarrow H_{gp}^{p+q}(G; \mathbb{F}_2)\,.$$

This is again a special case of the Serre spectral sequence in mod 2 singular cohomology, for the fibration $BN \to BG \to B(G/N)$.

First proof. We use the change of rings isomorphism $\operatorname{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//A(1), \mathbb{F}_2) \cong \operatorname{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$. ((Must justify that $\mathscr{A}$ is right free, thus flat, over $A(1)$.)) The subalgebra $E(Q_1) \subset A(1)$ is normal, with quotient $A(1)//E(Q_1) \cong E(Sq^1, Sq^2)$. Hence there is a Cartan–Eilenberg spectral sequence

$$E_2^{*,*} = \operatorname{Ext}_{E(Sq^1,Sq^2)}^*(\mathbb{F}_2, \operatorname{Ext}_{E(Q_1)}^*(\mathbb{F}_2, \mathbb{F}_2)) \Longrightarrow \operatorname{Ext}_{A(1)}^*(\mathbb{F}_2, \mathbb{F}_2)\,.$$

Here $\operatorname{Ext}_{E(Q_1)}^*(\mathbb{F}_2, \mathbb{F}_2) \cong P(h_{20})$. The module action of $E(Sq^1, Sq^2)$ on $P(h_{20})$ is (necessarily) trivial, so

$$E_2^{*,*} \cong P(h_0, h_1) \otimes P(h_{20})$$

with $h_0 \in E_2^{1,0}$ dual to Sq^1, $h_1 \in E_2^{1,0}$ dual to Sq^2, and $h_{20} \in E_2^{0,1}$ dual to Q_1. (We are ignoring the internal degrees here.) There is a d_2-differential $d_2(h_{20}) = h_0 h_1$, corresponding to the fact that the generator $Q_1 \in E(Q_1)$ becomes decomposable in $A(1)$. This leaves the E_3-term

$$E_3^{*,*} \cong P(h_0, h_1)/(h_0 h_1) \otimes P(h_{20}^2)\,.$$

There is a further d_3-differential $d_2(h_{20}^2) = h_1^3$. This leaves the E_3-term

$$E_4^{*,*} \cong \big(P(h_0, h_1)/(h_0 h_1, h_1^3) \oplus P(h_0)\{h_0 h_{20}^2\}\big) \otimes P(h_{20}^4)\,.$$

The spectral sequence collapses at this stage, for bidegree reasons: A d_5-differential on h_{20}^4 could only hit h_0^5, but the internal degrees do not match. ((No additive or multiplicative extensions.)) □

Second proof. One might also consider the Cartan–Eilenberg spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_{E(Sq^2)}^p(\mathbb{F}_2, \operatorname{Ext}_{E(1)}^q(\mathbb{F}_2, \mathbb{F}_2)) \Longrightarrow \operatorname{Ext}_{A(1)}^{p+q}(\mathbb{F}_2, \mathbb{F}_2)$$

associated to the isomorphism $A(1)//E(1) \cong E(Sq^2)$, but in this case the $E(Sq^2)$-module action on $\operatorname{Ext}_{E(1)}^*(\mathbb{F}_2, \mathbb{F}_2) = P(h_0, h_{20})$ is non-trivial, being given by $Sq^2 \cdot h_{20} = h_0$. With the usual periodic resolution for Ext over $E(Sq^2)$, this gives a d_1-differential $d_1(h_{20}) = h_0 h_1$, so that

$$E_2^{*,*} = P(h_0, h_1)/(h_0 h_1) \otimes P(h_{20}^2)\,.$$

Again there is a d_3-differential $d_3(h_{20}^2) = h_1^3$, leaving

$$E_4^{*,*} = E_\infty^{*,*} = \big(P(h_0, h_1)/(h_0h_1, h_1^3) \oplus P(h_0)\{h_0h_{20}^2\}\big) \otimes P(h_{20}^4)\,.$$

Note that in this case h_0, h_1 and h_{20} have bigradings $(p, q) = (0, 1)$, $(1, 0)$ and $(0, 1)$, respectively. $\square$

Third proof. For a proof without the Cartan–Eilenberg spectra sequence, we may construct a minimal resolution of $\mathbb{F}_2$ by "almost free" $A(1)$-modules. Some interesting examples of indecomposable modules appear along the way. There is an exact sequence

$$0 \to \Sigma^{12}\mathbb{F}_2 \to \Sigma^7 A(1)//A(0) \xrightarrow{\partial_3} \Sigma^4 A(1) \xrightarrow{\partial_2} \Sigma^2 A(1) \xrightarrow{\partial_1} A(1)//A(0) \xrightarrow{\epsilon} \mathbb{F}_2 \to 0$$

of $A(1)$-modules. The kernel of the augmentation ϵ from $A(1)//A(0) = A(1)/A(1)Sq^1$:

$$\begin{tikzcd} 1 \arrow[r] & Sq^2 \arrow[r] & Sq^3 \arrow[r] & Sq^2Sq^3 \end{tikzcd}$$

is the "question mark module"

$$\begin{tikzcd} Sq^2 \arrow[r] & Sq^3 \arrow[r, bend right] & Sq^2Sq^3 \end{tikzcd}$$

which is isomorphic to $\Sigma^2(A(1)/A(1)Sq^2)$. Here $1 \otimes \epsilon\colon \mathscr{A}//A(0) \to \mathscr{A}//A(1)$ is induced by the zeroth Postnikov section $ko \to H\mathbb{Z}$, with homotopy fiber bo, so $\Sigma H^*(bo) \cong \mathscr{A} \otimes_{A(1)} \ker(\epsilon)$ and $H^*(bo) \cong \Sigma(\mathscr{A}/\mathscr{A}Sq^2)$.

The kernel of $\partial_1\colon \Sigma^2 A(1) \to \ker(\epsilon)$, taking $\Sigma^2 1$ to Sq^2, is the double suspension of the "joker module"

$$\begin{tikzcd} Sq^2 \arrow[r] \arrow[rr, bend left] & Sq^3 \arrow[rr, bend right] & Sq^3Sq^1 \arrow[rr, bend left] & Sq^2Sq^3 \arrow[r] & Sq^1Sq^5 \end{tikzcd}$$

which is isomorphic to $\Sigma^4(\mathscr{A}/\mathscr{A}Sq^3)$. Here $1 \otimes \partial_1\colon \Sigma^2\mathscr{A} \to \mathscr{A} \otimes_{A(1)} \ker(\epsilon)$ is induced by the Postnikov section $bo \to \Sigma H$, with homotopy fiber bso, so $\Sigma^2 H^*(bso) \cong \mathscr{A} \otimes_{A(1)} \ker(\partial_1)$ and $H^*(bso) \cong \Sigma^2(\mathscr{A}/\mathscr{A}Sq^3)$.

The kernel of $\partial_2\colon \Sigma^4 A(1) \to \ker(\partial_1)$, taking $\Sigma^4 1$ to $\Sigma^2 Sq^2$, is the fourfold suspension of the "inverted question mark module"

$$\begin{tikzcd} Sq^3 \arrow[r, bend left] & Sq^2Sq^3 \arrow[r] & Sq^1Sq^5 \end{tikzcd}$$

which is isomorphic to $\Sigma^3(\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\})$. Here $1 \otimes \partial_2\colon \Sigma^4\mathscr{A} \to \mathscr{A} \otimes_{A(1)} \ker(\partial_1)$ is induced by the Postnikov section $bso \to \Sigma^2 H$, with homotopy fiber $bspin \cong \Sigma^4 ksp$, so $\Sigma^3 H^*(bspin) \cong \mathscr{A} \otimes_{A(1)} \ker(\partial_2)$ and $H^*(bspin) \cong \Sigma^4(\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\})$.

The kernel of $\partial_3\colon \Sigma^7 A(1)//A(0) \to \ker(\partial_2)$, taking $\Sigma^7 1$ to $\Sigma^4 Sq^3$, is the sevenfold suspension of the trivial module

$$Sq^2Sq^3$$

which is isomorphic to $\Sigma^5\mathbb{F}_2$. Here $1 \otimes \partial_3\colon \Sigma^7\mathscr{A}//A(0) \to \mathscr{A} \otimes_{A(1)} \ker(\partial_2)$ is induced by the Postnikov section $bspin \to \Sigma^4 H\mathbb{Z}$, with homotopy fiber $\Sigma^8 ko$, so $\Sigma^4 H^*(\Sigma^8 ko) \cong \mathscr{A} \otimes_{A(1)} \ker(\partial_3)$ and $H^*(\Sigma^8 ko) \cong \Sigma^8(\mathscr{A}//A(1))$, which we already knew.

From the exact sequence of $A(1)$-modules, we get short exact sequences

$$0 \to \mathrm{Ext}_{A(1)}^{s-1,t}(\ker(\epsilon), \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}_{A(1)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{A(0)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \to 0$$

$$0 \to \mathrm{Ext}_{A(1)}^{s-2,t}(\ker(\partial_1), \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}_{A(1)}^{s-1,t}(\ker(\epsilon), \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathbb{F}_2}^{s-1,t}(\Sigma^2\mathbb{F}_2, \mathbb{F}_2) \to 0$$

$$0 \to \mathrm{Ext}_{A(1)}^{s-3,t}(\ker(\partial_2), \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}_{A(1)}^{s-2,t}(\ker(\partial_1), \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathbb{F}_2}^{s-2,t}(\Sigma^4\mathbb{F}_2, \mathbb{F}_2) \to 0$$

$$0 \to \mathrm{Ext}_{A(1)}^{s-4,t}(\Sigma^{12}\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\delta} \mathrm{Ext}_{A(1)}^{s-3,t}(\ker(\partial_2), \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{A(0)}^{s-3,t}(\Sigma^7\mathbb{F}_2, \mathbb{F}_2) \to 0$$

This determines $\mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$. $\square$

Corollary 6.31. *There are $\mathscr{A}$-module isomorphisms:*

$$\begin{aligned} H^*(bo) &\cong \Sigma(\mathscr{A}/\mathscr{A}Sq^2) \\ H^*(bso) &\cong \Sigma^2(\mathscr{A}/\mathscr{A}Sq^3) \\ H^*(bspin) &\cong \Sigma^4(\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\}) \end{aligned}$$

((Also $k(1) = ku/2$, $ko/2$.))

7 The dual Steenrod algebra

7.1 Hopf algebras

Let G be a topological group with $H_*(G)$ of finite type. Then the cohomology cross product

$$H^*(G) \otimes H^*(G) \xrightarrow{\times} H^*(G \times G)$$

is an isomorphism. The (cocommutative) diagonal map $\Delta\colon G \to G \times G$, and the augmentation $G \to *$ induce a pairing

$$\phi\colon H^*(G) \otimes H^*(G) \cong H^*(G \times G) \xrightarrow{\Delta^*} H^*(G)$$

and a unit map

$$\eta\colon \mathbb{F}_p \longrightarrow H^*(G)$$

that make $H^*(G)$ a (graded commutative) algebra. The group multiplication $m\colon G \times G \to G$ and the inclusion $\{e\} \to G$ induce homomorphisms

$$\psi\colon H^*(G) \xrightarrow{m^*} H^*(G \times G) \cong H^*(G) \otimes H^*(G)$$

and

$$\epsilon\colon H^*(G) \longrightarrow \mathbb{F}_p$$

that make $H^*(G)$ a commutative Hopf algebra, and the group inverse $i\colon G \to G$ induces a homomorphism

$$\chi\colon H^*(G) \xrightarrow{i^*} H^*(G)$$

that makes $H^*(G)$ a commutative Hopf algebra with conjugation, according to the following definitions. It is connected if and only if G is path connected as a topological space.

Dually, the Pontryagin product $\phi = m_*\colon H_*(G) \otimes H_*(G) \to H_*(G)$, unit inclusion $\eta\colon \mathbb{F}_p \to H_*(G)$, diagonal coproduct $\psi = \Delta_*\colon H_*(G) \to H_*(G) \otimes H_*(G)$, augmentation $\epsilon\colon H_*(G) \to \mathbb{F}_p$ and conjugation $\chi = i_*\colon H_*(G) \to H_*(G)$ make $H_*(G)$ a cocommutative Hopf algebra with conjugation.

Let k be any field, and write $\otimes$ for $\otimes_k$.

Definition 7.1. A k-algebra is a graded k-module A equipped with homomorphisms $\phi\colon A \otimes A \to A$ and $\eta\colon k \to A$, such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\phi\otimes 1"] \arrow[d, "1\otimes\phi"'] & A \otimes A \arrow[d, "\phi"] \\ A \otimes A \arrow[r, "\phi"] & A \end{tikzcd}$$

(associativity) and

$$\begin{tikzcd} k \otimes A \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\phi"] & A \otimes k \arrow[l, "1\otimes\eta"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

(unitality) commute. It is commutative if the diagram

$$\begin{tikzcd} A \otimes A \arrow[rr, "\gamma", "\cong"'] \arrow[dr, "\phi"'] & & A \otimes A \arrow[dl, "\phi"] \\ & A & \end{tikzcd}$$

commutes, where $\gamma(a \otimes b) = (-1)^{|a||b|} b \otimes a$. A k-algebra homomorphism $f\colon A \to B$ is a degree-preserving k-module homomorphism such that the diagram

$$\begin{tikzcd} A \otimes A \arrow[r, "\phi"] \arrow[d, "f \otimes f"'] & A \arrow[d, "f"'] & k \arrow[l, "\eta"'] \arrow[d, "="] \\ B \otimes B \arrow[r, "\phi"] & B & k \arrow[l, "\eta"'] \end{tikzcd}$$

commutes.

Definition 7.2. A k-coalgebra is a graded k-module A equipped with homomorphisms $\psi\colon A \to A \otimes A$ and $\epsilon\colon A \to k$, such that the diagrams

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\psi"'] & A \otimes A \arrow[d, "1 \otimes \psi"] \\ A \otimes A \arrow[r, "\psi \otimes 1"] & A \otimes A \otimes A \end{tikzcd}$$

(coassociativity) and

$$\begin{tikzcd} & A \arrow[dl, "\cong"'] \arrow[d, "\psi"'] \arrow[dr, "\cong"] & \\ k \otimes A & A \otimes A \arrow[l, "\epsilon \otimes 1"] \arrow[r, "1 \otimes \epsilon"'] & A \otimes k \end{tikzcd}$$

(counitality) commute. It is cocommutative if the diagram

$$\begin{tikzcd} & A \arrow[dl, "\psi"'] \arrow[dr, "\psi"] & \\ A \otimes A \arrow[rr, "\gamma", "\cong"'] & & A \otimes A \end{tikzcd}$$

commutes. A k-coalgebra homomorphism $f\colon A \to B$ is a degree-preserving k-module homomorphism such that the diagram

$$\begin{tikzcd} k \arrow[d, "="'] & A \arrow[l, "\epsilon"'] \arrow[r, "\psi"] \arrow[d, "f"'] & A \otimes A \arrow[d, "f \otimes f"] \\ k & B \arrow[l, "\epsilon"'] \arrow[r, "\psi"] & B \otimes B \end{tikzcd}$$

commutes.

Definition 7.3. A k-algebra A is connected if the underlying graded k-module is zero in negative degrees and $\eta\colon k \to A$ is an isomorphism in degree 0. A k-coalgebra A is connected if it is zero in negative degrees and $\epsilon\colon A \to k$ is an isomorphism in degree 0.

Definition 7.4. An augmented k-algebra is a k-algebra A with a k-algebra homomorphism $\epsilon\colon A \to k$. Let $I(A) = \ker(\epsilon)$ be the augmentation ideal, and let

$$Q(A) = I(A)/I(A)^2 = k \otimes_A I(A)$$

be the indecomposable quotient module.

$$\begin{tikzcd}
I(A)\otimes I(A) \arrow[r] \arrow[d] & I(A) \arrow[r, two heads] \arrow[d, tail] & Q(A) \\
A\otimes A \arrow[r, "\phi"] & A \arrow[d, two heads, "\epsilon"] & \\
& k &
\end{tikzcd}$$

A homomorphism of augmented algebras is an algebra homomorphism that commutes with the augmentations.

(We make sense of the tensor product over A in the next subsection.)

Proposition 7.5 (Milnor–Moore)**.** *Let $f\colon A\to B$ be a homomorphism of augmented algebras, with B connected. Then f is surjective if and only if $Q(f)\colon Q(A)\to Q(B)$ is surjective.*

Definition 7.6. A coaugmented k-coalgebra is a k-coalgebra A with a k-coalgebra homomorphism $\eta\colon k\to A$. Let $J(A)=\mathrm{cok}(\eta)$ be the coaugmentation coideal, and let

$$P(A)=\{x\in A\mid \psi(x)=x\otimes 1+1\otimes x\}=k\,\square_A\,J(A)$$

be the submodule of primitives.

$$\begin{tikzcd}
J(A)\otimes J(A) & J(A) \arrow[l] & P(A) \arrow[l, tail] \\
A\otimes A \arrow[u] & A \arrow[l, "\psi"'] \arrow[u, two heads] & \\
& k \arrow[u, tail, "\eta"'] &
\end{tikzcd}$$

A homomorphism of coaugmented coalgebras is a coalgebra homomorphism that commutes with the coaugmentations.

(We make sense of the cotensor products under A in the next subsection.)

Proposition 7.7 (Milnor–Moore)**.** *Let $f\colon A\to B$ be a homomorphism of coaugmented coalgebras, with A connected. Then f is injective if and only if $P(f)\colon P(A)\to P(B)$ is injective.*

Definition 7.8. A Hopf algebra (over k) is a k-algebra structure (ϕ,η) and a k-coalgebra structure (ψ,ϵ) on the same graded k-module A, such that ψ and ϵ are algebra homomorphisms and ϕ and η are coalgebra homomorphisms. This means that the diagrams

$$\begin{tikzcd}
A\otimes A \arrow[r, "\phi"] \arrow[d, "\psi\otimes\psi"'] & A \arrow[r, "\psi"] & A\otimes A \\
A\otimes A\otimes A\otimes A \arrow[rr, "\cong", "1\otimes\gamma\otimes 1"'] & & A\otimes A\otimes A\otimes A \arrow[u, "\phi\otimes\phi"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
A\otimes A \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\phi"'] & k\otimes k \arrow[d, "\cong"] \\
A \arrow[r, "\epsilon"] & k
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, "\eta"] \arrow[d, "\cong"'] & A \arrow[d, "\psi"] \\
k\otimes k \arrow[r, "\eta\otimes\eta"] & A\otimes A
\end{tikzcd}$$

commute. A homomorphism if Hopf algebras is an algebra homomorphism that is simultaneously a coalgebra homomorphism.

Definition 7.9. A Hopf algebra with conjugation is a Hopf algebra A with a homomorphism $\chi\colon A \to A$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\epsilon"] \arrow[d, "\psi"'] & k \arrow[r, "\eta"] & A \\ A \otimes A \arrow[rr, "1\otimes\chi"] & & A \otimes A \arrow[u, "\psi"'] \end{tikzcd}$$

commutes. A homomorphism of Hopf algebras with conjugation is a Hopf algebra homomorphism that commutes with the conjugation.

Definition 7.10. Let A be a k-algebra, and let $B \subset A$ be a subalgebra with an augmentation $\epsilon\colon B \to k$, making k a B-module. Then we let

$$A//B = A \otimes_B k = A/A \cdot I(B)$$

and

$$B\backslash\backslash A = k \otimes_B A = A/I(B) \cdot A\,.$$

If $A \cdot I(B) = I(B) \cdot A$ we say that B is normal in A. Then $A//B$ is a k-algebra, and the canonical map $A \to A//B$ is an algebra homomorphism.

Theorem 7.11 (Milnor–Moore)**.** *Let A be a connected Hopf algebra and $B \subset A$ a Hopf subalgebra. Then there is an isomorphism $A \cong A//B \otimes B$ of right B-modules, and an isomorphism $A \cong B \otimes B\backslash\backslash A$ of left B-modules, so A is free as a left B-module and as a right B-module.*

This is part of Theorem 4.4 in Milnor–Moore (1965). More concretely, let $i\colon B \to A$ be the inclusion and let $s\colon A//B \to A$ be any k-linear section to the projection $A \to A//B$. Then the composite

$$A//B \otimes B \xrightarrow{s\otimes i} A \otimes A \xrightarrow{\phi} A$$

is an isomorphism of right B-modules. It is not usually true that A is free as a B-B-bimodule.

7.2 Actions and coactions

Definition 7.12. Let A be a k-algebra. A left A-module is a graded k-module M with a pairing $\lambda\colon A \otimes M \to M$ such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "1\otimes\lambda"] \arrow[d, "\phi\otimes 1"'] & A \otimes M \arrow[d, "\lambda"] \\ A \otimes M \arrow[r, "\lambda"] & M \end{tikzcd} \qquad \begin{tikzcd} k \otimes M \arrow[r, "\eta\otimes 1"] \arrow[dr, "\cong"'] & A \otimes M \arrow[d, "\lambda"] \\ & M \end{tikzcd}$$

commute. A right A-module is a graded k-module N with a pairing $\rho\colon N \otimes A \to N$ such that the diagrams

$$\begin{tikzcd} N \otimes A \otimes A \arrow[r, "\rho\otimes 1"] \arrow[d, "1\otimes\phi"'] & N \otimes A \arrow[d, "\rho"] \\ N \otimes A \arrow[r, "\rho"] & N \end{tikzcd} \qquad \begin{tikzcd} N \otimes k \arrow[r, "1\otimes\eta"] \arrow[dr, "\cong"'] & N \otimes A \arrow[d, "\rho"] \\ & N \end{tikzcd}$$

commute. The tensor product $N \otimes_A M$ is the coequalizer in the diagram

$$\begin{tikzcd} N \otimes A \otimes M \arrow[r, shift left, "1\otimes\lambda"] \arrow[r, shift right, "\rho\otimes 1"'] & N \otimes M \arrow[r, two heads] & N \otimes_A M \end{tikzcd}$$

Definition 7.13. Let A be a k-coalgebra. A left A-comodule is a graded k-module M with a pairing $\lambda\colon M \to A \otimes M$ such that the diagrams

$$\begin{tikzcd} M \arrow[r, "\lambda"] \arrow[d, "\lambda"'] & A \otimes M \arrow[d, "1\otimes\lambda"] \\ A \otimes M \arrow[r, "\psi\otimes 1"'] & A \otimes A \otimes M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[d, "\lambda"'] \arrow[dr, "\cong"] & \\ A \otimes M \arrow[r, "\epsilon\otimes 1"'] & k \otimes M \end{tikzcd}$$

commute. A right A-comodule is a graded k-module N with a pairing $\rho\colon N \to N \otimes A$ such that the diagrams

$$\begin{tikzcd} N \arrow[r, "\rho"] \arrow[d, "\rho"'] & N \otimes A \arrow[d, "\rho\otimes 1"] \\ N \otimes A \arrow[r, "1\otimes\psi"'] & N \otimes A \otimes A \end{tikzcd} \qquad \begin{tikzcd} N \arrow[d, "\rho"'] \arrow[dr, "\cong"] & \\ N \otimes A \arrow[r, "1\otimes\epsilon"'] & N \otimes k \end{tikzcd}$$

commute. The cotensor product $N \,\square_A\, M$ is the equalizer in the diagram

$$N \,\square_A\, M \rightarrowtail N \otimes M \underset{\rho\otimes 1}{\overset{1\otimes\lambda}{\rightrightarrows}} N \otimes A \otimes M$$

Lemma 7.14. *Let M be a left A-module, with action $a \cdot m = \lambda(a \otimes m)$ for $a \in A$ and $m \in M$. Then the linear dual $M^* = \mathrm{Hom}(M, k)$ is a right A-module, with action $\mu \cdot a = \rho(\mu \otimes a)$ given by $\mu \cdot a\colon m \mapsto \mu(a \cdot m)$, for $\mu\colon M \to k$ in M^*. Likewise, if N is a right A-module then N^* is a left A-module.*

Proof. $\mu \cdot a\colon m \mapsto \mu(a \cdot m)$, so $(\mu \cdot a) \cdot b\colon m \mapsto (\mu \cdot a)(b \cdot m) = \mu(a \cdot b \cdot m) = \mu(ab \cdot m)$ equals $\mu \cdot ab$. □

Lemma 7.15. *Let A be a k-algebra, bounded below and of finite type. Then $A^* = \mathrm{Hom}(A, k)$ is a k-coalgebra with coproduct $\psi = \phi^*\colon A^* \to (A \otimes A)^* \cong A^* \otimes A^*$ and counit $\epsilon = \eta^*\colon A^* \to k$. Conversely, if A is a k-coalgebra then A^* is a k-algebra. If A was bounded below and of finite type, then so is A^*, and $A \cong (A^*)^*$.*

Lemma 7.16. *Let A be an augmented k-algebra, bounded below and of finite type. Then A^* is a coaugmented k-coalgebra, $J(A^*) \cong I(A)^*$ and $P(A^*) \cong Q(A)^*$.*

Lemma 7.17. *Let A be a k-algebra, M a left A-module and N a right A-module, all bounded below and of finite type. Then M^* is a left A^*-comodule with coaction $\lambda = \lambda^*\colon M^* \to (A \otimes M)^* \cong A^* \otimes M^*$, and N^* is a right A^*-comodule with coaction $\rho = \rho^*\colon N^* \to (N \otimes A)^* \cong N^* \otimes A^*$.*

Conversely, let A be a k-coalgebra, M a left A-comodule and N a right A-comodule. Then M^* is a left A^*-module with action $\lambda\colon A^* \otimes M^* \to (A \otimes M)^* \to M^*$, and N^* is a right A^*-module with action $\rho\colon N^* \otimes A^* \to (N \otimes A)^* \to N^*$.

Definition 7.18. Let A be an augmented k-algebra and let M be a left A-module. The A-module indecomposables in M is the quotient k-module $k \otimes_A M = M/I(A) \cdot M$.

Definition 7.19. Let A be a coaugmented k-coalgebra and let M be a left A-comodule. The A-comodule primitives in M is the k-submodule $k \,\square_A\, M = \{m \in M \mid \lambda(m) = 1 \otimes m\}$.

Lemma 7.20. *Let A be an augmented k-algebra and M left A-module, both bounded below and of finite type. Let M^* be the dual left A^*-comodule. Then there are natural isomorphisms*

$$\mathrm{Hom}_A(M, k) \cong \mathrm{Hom}(k \otimes_A M, k) \cong k \,\square_{A^*}\, M^*$$

that are compatible with the inclusions into $\mathrm{Hom}(M, k) = M^*$.

See Boardman (1982) for more on left/right algebra/coalgebra actions/coactions.

Definition 7.21. Let A be a Hopf algebra, and let M and N be left A-modules. Then $M \otimes N$ is a left A-module, with the action $\lambda\colon A \otimes M \otimes N$ defined as the composite

$$A \otimes M \otimes N \xrightarrow{\psi\otimes 1\otimes 1} A \otimes A \otimes M \otimes N \xrightarrow[\cong]{1\otimes\gamma\otimes 1} A \otimes M \otimes A \otimes N \xrightarrow{\lambda\otimes\lambda} M \otimes N\,.$$

Likewise for right A-modules.

Conversely, let M and N be left A-comodules. Then $M \otimes N$ is a left A-comodule, with the coaction $\lambda\colon M \otimes N \to A \otimes M \otimes N$ defined as the composite

$$M \otimes N \xrightarrow{\lambda\otimes\lambda} A \otimes M \otimes A \otimes N \xrightarrow[\cong]{1\otimes\gamma\otimes 1} A \otimes A \otimes M \otimes N \xrightarrow{\phi\otimes 1\otimes 1} A \otimes M \otimes N\,.$$

Likewise for right A-comodules.

7.3 The coproduct

Let Y and Z be spectra. If Y and Z are bounded below with $H_*(Y)$ and $H_*(Z)$ of finite type, then the cohomology smash product

$$H^*(Y) \otimes H^*(Z) \xrightarrow{\wedge} H^*(Y \wedge Z)$$

is an isomorphism. The Cartan formula

$$Sq^k(y \wedge z) = \sum_{i+j=k} Sq^i(y) \wedge Sq^j(z)$$

implies the more general formula

$$Sq^K(y \wedge z) = \sum_{I+J=K} Sq^I(y) \wedge Sq^J(z)$$

for sequences $K = (k_1, \dots, k_\ell)$ of non-negative integers, where the sum is over pairs of sequences $I = (i_i, \dots, i_\ell)$ and $J = (j_1, \dots, j_\ell)$ of non-negative integers, such that $k_u = i_u + j_u$ for all $1 \le u \le \ell$. Milnor proved that the rule

$$Sq^K \longmapsto \sum_{I+J=K} Sq^I \otimes Sq^J$$

respects the Adem relations, in the sense that it gives a well-defined algebra homomorphism

$$\psi \colon \mathscr{A} \longrightarrow \mathscr{A} \otimes \mathscr{A} \,.$$

Since $\mathscr{A}$ is connected, there is a unique homomorphism

$$\chi \colon \mathscr{A} \longrightarrow \mathscr{A}$$

with $\chi(1) = 1$ and $\sum a'\chi(a'') = 0$ for all $a \in I(\mathscr{A})$ with $\psi(a) = \sum a' \otimes a''$. Then $\chi(ab) = \chi(b)\chi(a)$ and χ^2 is the identity.

Theorem 7.22 (Milnor (1958))**.** *The Steenrod algebra $\mathscr{A}$, with the composition coproduct ϕ, the coproduct ψ and the conjugation χ, is a cocommutative Hopf algebra with conjugation.*

Definition 7.23. Let the dual Steenrod algebra $\mathscr{A}_* = \mathrm{Hom}(\mathscr{A}, \mathbb{F}_2)$ be the linear dual of the Steenrod algebra. Since $\mathscr{A}$ is of finite type, there is a natural isomorphism $\mathscr{A} \cong \mathrm{Hom}(\mathscr{A}_*, \mathbb{F}_2)$. The algebra structure maps $\phi \colon \mathscr{A} \otimes \mathscr{A} \to \mathscr{A}$ and $\eta \colon \mathbb{F}_2 \to \mathscr{A}$ dualize to coalgebra structure maps $\psi \colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$ and $\epsilon \colon \mathscr{A}_* \to \mathbb{F}_2$. The cocommutative coalgebra structure maps $\psi \colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ and $\epsilon \colon \mathscr{A} \to \mathbb{F}_2$ dualize to commutative algebra structure maps $\phi \colon \mathscr{A}_* \otimes \mathscr{A}_* \to \mathscr{A}_*$ and $\eta \colon \mathbb{F}_2 \to \mathscr{A}_*$. The conjugation $\chi \colon \mathscr{A} \to \mathscr{A}$ dualizes to a conjugation $\chi \colon \mathscr{A}_* \to \mathscr{A}_*$. With these structure maps, $\mathscr{A}_*$ is a commutative Hopf algebra.

Remark 7.24. The isomorphism $\mathscr{A} \cong H^*(H)$ is dual to an isomorphism $\mathscr{A}_* \cong H_*(H)$. This may justify why we write $\mathscr{A}_*$ instead of $\mathscr{A}^*$ for the dual Steenrod algebra, thinking of the star as a homological grading rather than as the symbol for dualization. The ring spectrum product $\mu \colon H \wedge H \to H$ induces the product $\phi \colon \mathscr{A}_* \otimes \mathscr{A}_* \cong H_*(H) \otimes H_*(H) \cong H_*(H \wedge H) \to H_*(H) \cong \mathscr{A}_*$ in homology, and the counit $\epsilon \colon \mathscr{A}_* = \pi_*(H \wedge H) \to \pi_*(H) = \mathbb{F}_2$ in homotopy. The ring spectrum unit $\eta \colon S \to H$ induces a map $H \cong S \wedge H \to H \wedge H$ that induces the coproduct $\psi \colon \mathscr{A}_* = H_*(H) \to H_*(H \wedge H) \cong H_*(H) \otimes H_*(H) \cong \mathscr{A}_* \otimes \mathscr{A}_*$ in homology. The two maps $H \cong S \wedge H \to H \wedge H$ and $H \cong H \wedge S \to H \wedge H$ both induce the unit $\eta \colon \mathbb{F}_2 \to \mathscr{A}_*$ in homotopy. The twist map $\gamma \colon H \wedge H \to H \wedge H$ induces the conjugation $\chi \colon \mathscr{A}_* \to \mathscr{A}_*$. ((Reference?))

By definition, $\psi \colon \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$ makes the diagram

$$\begin{tikzcd}
\mathscr{A} \otimes \mathscr{A} \otimes H^*(Y) \otimes H^*(Z) \arrow[d, "1\otimes\gamma\otimes1"', "\cong"] & \mathscr{A} \otimes H^*(Y) \otimes H^*(Z) \arrow[l, "\psi\otimes1\otimes1"'] \arrow[r, "1\otimes\wedge"] \arrow[d, dashed, "\lambda"'] & \mathscr{A} \otimes H^*(Y \wedge Z) \arrow[d, "\lambda"] \\
\mathscr{A} \otimes H^*(Y) \otimes \mathscr{A} \otimes H^*(Z) \arrow[r, "\lambda\otimes\lambda"] & H^*(Y) \otimes H^*(Z) \arrow[r, "\wedge"] & H^*(Y \wedge Z)
\end{tikzcd}$$

commute, where $\lambda\colon \mathscr{A} \otimes H^*(Y) \to H^*(Y)$ denotes the left $\mathscr{A}$-module action. We defined the $\mathscr{A}$-module action on the tensor product $H^*(Y) \otimes H^*(Z)$ by the dashed composite in this diagram, so that the Künneth homomorphism $\wedge$ is an $\mathscr{A}$-module homomorphism.

By the Hom-tensor adjunction, the diagram can be reformulated as follows:

$$\begin{tikzcd}
\mathrm{Hom}(\mathscr{A}, H^*(Y)) \otimes \mathrm{Hom}(\mathscr{A}, H^*(Z)) \arrow[d, "\otimes"'] & H^*(Y) \otimes H^*(Z) \arrow[l, "\tilde\lambda\otimes\tilde\lambda"'] \arrow[r, "\wedge"] \arrow[d, dashed, "\tilde\lambda"'] & H^*(Y \wedge Z) \arrow[d, "\tilde\lambda"] \\
\mathrm{Hom}(\mathscr{A} \otimes \mathscr{A}, H^*(Y) \otimes H^*(Z)) \arrow[r, "\psi^*"] & \mathrm{Hom}(\mathscr{A}, H^*(Y) \otimes H^*(Z)) \arrow[r, "\wedge_*"] & \mathrm{Hom}(\mathscr{A}, H^*(Y \wedge Z))
\end{tikzcd}$$

where $\tilde\lambda\colon H^*(Y) \to \mathrm{Hom}(\mathscr{A}, H^*(Y))$ takes y to the homomorphism $a \mapsto a(y)$, etc. If we add the assumption that $H^*(Y)$ is bounded above, so that $H_*(Y)$ is (totally) finite, then there is a natural isomorphism

$$H^*(Y) \otimes \mathscr{A}_* \cong \mathrm{Hom}(\mathscr{A}, H^*(Y))$$

taking $y \otimes \alpha$ to $a \mapsto \alpha(a)y$, with $y \in H^*(Y)$, $\alpha \in \mathscr{A}_*$ and $a \in \mathscr{A}$. We also assume that $H_*(Z)$ is (totally) finite. Then we can rewrite the diagram as:

$$\begin{tikzcd}
H^*(Y) \otimes \mathscr{A}_* \otimes H^*(Z) \otimes \mathscr{A}_* \arrow[d, "1\otimes\gamma\otimes1"', "\cong"] & H^*(Y) \otimes H^*(Z) \arrow[l, "\rho\otimes\rho"'] \arrow[r, "\wedge"] \arrow[d, dashed, "\rho"'] & H^*(Y \wedge Z) \arrow[d, "\rho"] \\
H^*(Y) \otimes H^*(Z) \otimes \mathscr{A}_* \otimes \mathscr{A}_* \arrow[r, "1\otimes1\otimes\phi"] & H^*(Y) \otimes H^*(Z) \otimes \mathscr{A}_* \arrow[r, "\wedge\otimes1"] & H^*(Y \wedge Z) \otimes \mathscr{A}_*
\end{tikzcd}$$

where ϕ is the algebra structure on $\mathscr{A}_*$, dual to the coproduct ψ on $\mathscr{A}$, and $\rho\colon H^*(Y) \to H^*(Y) \otimes \mathscr{A}_*$ is the right $\mathscr{A}_*$-comodule coaction on $H^*(Y)$, corresponding to $\tilde\lambda$ via the isomorphism above. We defined the $\mathscr{A}_*$-coaction on the tensor product $H^*(Y) \otimes H^*(Z)$ by the dashed composite. Hence the Künneth morphism $\wedge$ is an $\mathscr{A}_*$-comodule homomorphism.

Proposition 7.25 (Milnor)**.** *Let X be a space with $H_*(X)$ (totally) finite. The right $\mathscr{A}$-comodule coaction*

$$\rho\colon H^*(X) \to H^*(X) \otimes \mathscr{A}_*$$

is an algebra homomorphism, where $H^*(X)$ has the cup product and $\mathscr{A}_*$ has the product dual to the coproduct ψ on $\mathscr{A}$.

Proof. Let $Y = Z = \Sigma^\infty(X_+)$. Then the diagonal $\Delta\colon X \to X \times X$ induces the commutative diagram

$$\begin{tikzcd}
H^*(X) \otimes \mathscr{A}_* \otimes H^*(X) \otimes \mathscr{A}_* \arrow[d, "1\otimes\gamma\otimes1"', "\cong"] & H^*(X) \otimes H^*(X) \arrow[l, "\rho\otimes\rho"'] \arrow[r, "\cup"] \arrow[d, dashed, "\rho"'] & H^*(X) \arrow[d, "\rho"] \\
H^*(X) \otimes H^*(X) \otimes \mathscr{A}_* \otimes \mathscr{A}_* \arrow[r, "1\otimes1\otimes\phi"] & H^*(X) \otimes H^*(X) \otimes \mathscr{A}_* \arrow[r, "\cup\otimes1"] & H^*(X) \otimes \mathscr{A}_*
\end{tikzcd}$$

which says that the cup product $\cup$ is an $\mathscr{A}_*$-comodule homomorphism, or equivalently, that the coaction ρ is an algebra homomorphism. $\square$

This results encodes the Cartan formula for the Steenrod algebra action on the cohomology of a product of spaces, in terms of the coaction of the dual Steenrod algebra, in a very convenient form.

7.4 The Milnor generators

Without appealing to the conjugation χ, we have the following four left and right actions and coactions on the homology and cohomology of a space X with $H_*(X)$ finite:

$$\begin{aligned}
\lambda\colon \mathscr{A} \otimes H^*(X) &\longrightarrow H^*(X) \\
\rho\colon H_*(X) \otimes \mathscr{A} &\longrightarrow H_*(X) \\
\rho\colon H^*(X) &\longrightarrow H^*(X) \otimes \mathscr{A}_* \\
\lambda\colon H_*(X) &\longrightarrow \mathscr{A}_* \otimes H_*(X)
\end{aligned}$$

We specialize to the test object $X = \mathbb{R}P^N \subset \mathbb{R}P^\infty = H_1$, with $H^*(X) = P(x)/(x^{N+1})$ and $H_*(X) = \mathbb{F}_2\{\gamma_j \mid 0 \le j \le N\}$, where x^j is dual to γ_j. We are interested in the limit as $N \to \infty$, when $\lim_N H^*(\mathbb{R}P^N) = P(x)$ and $\operatorname{colim}_N H_*(\mathbb{R}P^N) = \mathbb{F}_2\{\gamma_j \mid j \ge 0\}$. The limiting right coaction

$$\rho\colon P(x) \longrightarrow P(x) \mathbin{\widehat{\otimes}} \mathscr{A}_*$$

was just seen to be an algebra homomorphism, hence is determined by the single value

$$\rho(x) = \sum_{j \ge 1} x^j \otimes \alpha_j$$

where $\alpha_j \in \mathscr{A}_*$ has degree $(j-1)$, for each $j \ge 1$.

Lemma 7.26. *There are well-defined classes $\xi_i \in \mathscr{A}_*$ such that*

$$\rho(x) = \sum_{i \ge 0} x^{2^i} \otimes \xi_i \,.$$

Here $\xi_0 = 1$, and ξ_i has degree $2^i - 1$, for each $i \ge 0$.

Proof. There is a pairing $m\colon \mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$ that represents the tensor product of real line bundles, or comes from the loop structure on $H_1 \simeq \Omega H_2$. It induces a homomorphism

$$m^*\colon P(x) = H^*(\mathbb{R}P^\infty) \to H^*(\mathbb{R}P^\infty \times \mathbb{R}P^\infty) = P(x_1, x_2)$$

with $m^*(x) = x_1 + x_2$, where $x_1 = x \times 1$ and $x_2 = 1 \times x$. By naturality of the right $\mathscr{A}_*$-coaction ρ, we have that

$$m^*(\rho(x)) = \sum_{j \ge 1} (x_1 + x_2)^j \otimes \alpha_j$$

is equal to

$$\rho(m^*(x)) = \rho(x_1 + x_2) = \rho(x_1) + \rho(x_2) = \sum_{j \ge 1} x_1^j \otimes \alpha_j + \sum_{j \ge 1} x_2^j \otimes \alpha_j$$

in $P(x_1, x_2) \mathbin{\widehat{\otimes}} \mathscr{A}_*$. The product formula for binomial coefficients mod 2 implies that $(x_1 + x_2)^j \ne x_1^j + x_2^j$ for all j not of the form $j = 2^i$, $i \ge 0$, hence $\alpha_j = 0$ for all such j. We let $\xi_i = \alpha_{2^i}$ for $i \ge 0$. Counitality of the coaction implies that $\xi_0 = 1$. □

Let $P(\xi_i \mid i \ge 1) = P(\xi_1, \xi_2, \xi_3, \dots)$ be the polynomial algebra generated by the classes ξ_i for $i \ge 0$, only subject to the relation $\xi_0 = 1$.

Theorem 7.27 (Milnor)**.** *The canonical homomorphism*

$$P(\xi_i \mid i \ge 1) \xrightarrow{\cong} \mathscr{A}_*$$

is an algebra isomorphism.

See Milnor (1958) Theorem 2 or Steenrod–Epstein (1962) Theorem 2.2 for the proof. Surjectivity of $P(\xi_i \mid i \ge 1) \to \mathscr{A}_*$ follows by the detection results for $\mathscr{A}$. A count of dimensions then proves isomorphism.

Theorem 7.28 (Milnor)**.** *The Hopf algebra coproduct $\psi\colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$ is given by*

$$\psi(\xi_k) = \sum_{i+j=k} \xi_i^{2^j} \otimes \xi_j$$

where $i, j \ge 0$ and $\xi_0 = 1$. Hence the conjugation $\chi\colon \mathscr{A}_* \to \mathscr{A}_*$ is determined by

$$\sum_{i+j=k} \xi_i^{2^j} \chi(\xi_j) = 0$$

for all $k \ge 1$.

Proof. The coassociativity of the right coaction tells us that

$$(\rho \otimes 1)\rho(x) = (\rho \otimes 1)(\sum_{j\ge 0} x^{2^j} \otimes \xi_j) = \sum_{j\ge 0} \rho(x)^{2^j} \otimes \xi_j = \sum_{i,j\ge 0} x^{2^{i+j}} \otimes \xi_i^{2^j} \otimes \xi_j$$

is equal to

$$(1 \otimes \psi)\rho(x) = \sum_{k\ge 0} x^{2^k} \otimes \psi(\xi_k)\,.$$

□

These formulas for the coproduct in $\mathscr{A}_*$ are often more manageable than the Adem relations for the product in $\mathscr{A}$. Here is list of $\psi(\xi_k)$ and $\chi(\xi_k)$ for small k:

$$\begin{aligned}
\psi(\xi_1) &= \xi_1 \otimes 1 + 1 \otimes \xi_1 \\
\psi(\xi_2) &= \xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2 \\
\psi(\xi_3) &= \xi_3 \otimes 1 + \xi_2^2 \otimes \xi_1 + \xi_1^4 \otimes \xi_2 + 1 \otimes \xi_3 \\
\psi(\xi_4) &= \xi_4 \otimes 1 + \xi_3^2 \otimes \xi_1 + \xi_2^4 \otimes \xi_2 + \xi_1^8 \otimes \xi_3 + 1 \otimes \xi_4
\end{aligned}$$

$$\begin{aligned}
\chi(\xi_1) &= \xi_1 \\
\chi(\xi_2) &= \xi_2 + \xi_1^3 \\
\chi(\xi_3) &= \xi_3 + \xi_1\xi_2^2 + \xi_1^4\xi_2 + \xi_1^7 \\
\chi(\xi_4) &= \xi_4 + \xi_1\xi_3^2 + \xi_1^8\xi_3 + \xi_2^5 + \xi_1^3\xi_2^4 + \xi_1^9\xi_2^2 + \xi_1^{12}\xi_2 + \xi_1^{15}
\end{aligned}$$

We note that $\xi_1^{2^i}$ is primitive for each $i \ge 0$, and that $\chi(\xi_k) \equiv \xi_k$ modulo decomposables.

We now make the Milnor classes $\xi_i \in \mathscr{A}_*$ a little more explicit. Dualizing the formula for $\rho(x)$, the right action

$$\rho\colon H_*(\mathbb{R}P^\infty) \otimes \mathscr{A} \longrightarrow H_*(\mathbb{R}P^\infty)$$

is given in total degree 1 by

$$\gamma_j \otimes a \longmapsto \begin{cases} \langle a, \xi_i\rangle \gamma_1 & \text{for } j = 2^i \\ 0 & \text{otherwise.} \end{cases}$$

Here $a \in \mathscr{A}$ has degree $(j-1)$ and $\langle -, - \rangle \colon \mathscr{A} \otimes \mathscr{A}_* \to \mathbb{F}_2$ is the evaluation pairing. Likewise, the left action

$$\lambda\colon \mathscr{A} \otimes P(x) \longrightarrow P(x)$$

is given on $\mathscr{A} \otimes \mathbb{F}_2\{x\}$ by

$$a \otimes x \longmapsto a(x) = \sum_{i\ge 0} \langle a, \xi_i \rangle x^{2^i}\,.$$

Lemma 7.29. *For admissible sequences I,*

$$Sq^I(x) = \begin{cases} x^{2^i} & \textit{for } I = (2^{i-1}, 2^{i-2}, \dots, 2, 1),\ i \ge 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Hence

$$\langle Sq^I, \xi_i \rangle = \begin{cases} 1 & \textit{for } I = (2^{i-1}, 2^{i-2}, \dots, 2, 1) \\ 0 & \textit{otherwise.} \end{cases}$$

In other words, ξ_i is dual to $Sq^{2^{i-1}} Sq^{2^{i-2}} \dots Sq^2 Sq^1$ when we give $\mathscr{A}$ the admissible basis.

The identification of $\mathbb{R}P^\infty$ with the first space H_1 in the Eilenberg–Mac Lane spectrum H leads to a stable map $f\colon \Sigma^\infty H_1 \to \Sigma H$. The induced $\mathscr{A}$-module homomorphism

$$f^*\colon \Sigma\mathscr{A} = H^*(\Sigma H) \longrightarrow \tilde{H}^*(H_1) \subset P(x)$$

takes the generator $\Sigma 1$ to x, hence agrees with the $\mathscr{A}$-module homomorphism $\mathscr{A} \otimes \mathbb{F}_2\{x\} \to P(x)$ taking $a \otimes x$ to

$$a(x) = \sum_{i \geq 0} \langle a, \xi_i \rangle x^{2^i} \,,$$

via the isomorphism $\Sigma\mathscr{A} \cong \mathscr{A} \otimes \mathbb{F}_2\{x\}$. Dually, it follows that the $\mathscr{A}_*$-comodule homomorphism

$$f_* \colon \tilde{H}_*(H_1) \longrightarrow H_*(\Sigma H) \cong \Sigma\mathscr{A}_*$$

is the linear dual mapping

$$\gamma_j \longmapsto \begin{cases} \Sigma\xi_i & \text{for } j = 2^i,\ i \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 7.30. *The map $f \colon \Sigma^\infty \mathbb{R}P^\infty \to \Sigma H$ induces a homomorphism $\tilde{H}_{*+1}(\mathbb{R}P^\infty) \to \mathscr{A}_*$ taking $\gamma_j \in \tilde{H}_j(\mathbb{R}P^\infty)$ to ξ_i if $j = 2^i$, $i \geq 0$, and to 0 otherwise.*

Definition 7.31. The dual Steenrod algebra $\mathscr{A}_* \cong P(\xi_k \mid k \geq 1)$ has a basis $\{\xi^R\}_R$ given by the monomials

$$\xi^R = \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_\ell^{r_\ell}$$

where $R = (r_1, \dots, r_\ell)$ ranges over all finite sequences of non-negative integers, with $r_\ell \geq 1$ if $\ell \geq 1$. The Milnor basis $\{Sq^R\}_R$ for the Steenrod algebra $\mathscr{A}$ is the dual basis, defined so that

$$\langle Sq^R, \xi^S \rangle = \begin{cases} 1 & \text{for } R = S \\ 0 & \text{otherwise.} \end{cases}$$

Hence $|Sq^R| = |\xi^R| = \sum_{u=1}^{\ell} r_u(2^u - 1)$. The coproduct is given by $\psi(Sq^T) = \sum_{R+S=T} \psi^R \otimes \psi^S$.

Remark 7.32. One should not confuse the notations Sq^I and Sq^R. We let I, J and K range over admissible sequences, and let Sq^I, Sq^J and Sq^K denote the corresponding admissible composites of Steenrod squares. We let R, S and T range over finite sequences of non-negative integers, and let Sq^R, Sq^S and Sq^T denote the corresponding elements in the Milnor basis.

Example 7.33. It is clear that $Sq^{()} = 1$, $Sq^{(1)} = Sq^1$ and $Sq^{(2)} = Sq^2$. In degree 3, we have $\langle Sq^3, \xi_2 \rangle = 0$, $\langle Sq^2 Sq^1, \xi_2 \rangle = 1$, $\langle Sq^3, \xi_1^3 \rangle = 1$ and $\langle Sq^2 Sq^1, \xi_1^3 \rangle = 1$. For example,

$$\begin{aligned} \langle Sq^2 Sq^1, \xi_1^3 \rangle &= \langle Sq^2 Sq^1, \phi(\xi_1 \otimes \xi_1^2) \rangle = \langle \psi(Sq^2 Sq^1), \xi_1 \otimes \xi_1^2 \rangle \\ &= \langle (Sq^2 \otimes 1 + Sq^1 \otimes Sq^1 + 1 \otimes Sq^2)(Sq^1 \otimes 1 + 1 \otimes Sq^1), \xi_1 \otimes \xi_1^2 \rangle \\ &= \langle Sq^1 \otimes (Sq^2 + Sq^1 Sq^1), \xi_1 \otimes \xi_1^2 \rangle = \langle Sq^1, \xi_1 \rangle \langle Sq^2, \xi_1^2 \rangle = 1 \,. \end{aligned}$$

Hence $Sq^{(3)} = Sq^3$ and $Sq^{(0,1)} = Sq^3 + Sq^2 Sq^1 = Q_1$.

Lemma 7.34. *The Milnor basis element $Sq^{(r)}$ equals the Steenrod operation Sq^r, for each $r \geq 1$.*

Proof. Let $S = (s_1, \dots, s_\ell)$ be a finite sequence of non-negative integers, with $s_\ell \geq 1$. We must prove that $\langle Sq^r, \xi^S \rangle$ equals 1 for $S = (r)$ and 0 otherwise. Let Φ be the $\sum_{u=1}^{\ell} s_u$-fold product on $\mathscr{A}_*$, and let Ψ be the $\sum_{u=1}^{\ell} s_u$-fold coproduct on $\mathscr{A}$. Writing $\xi^S = \Phi(\xi_a \otimes \cdots \otimes \xi_\ell)$ with $a \leq \cdots \leq \ell$, we must compute $\langle Sq^r, \xi^S \rangle = \langle Sq^r, \Phi(\xi_a \otimes \cdots \otimes \xi_\ell) \rangle = \langle \Psi(Sq^r), \xi_a \otimes \cdots \otimes \xi_\ell \rangle$. Here $\Psi(Sq^r)$ is a sum of tensor products of factors of the form Sq^j. We have $\langle Sq^{2^i - 1}, \xi_i \rangle$ equals 1 for $i = 1$ and 0 for $i \geq 2$. Hence $\langle \Psi(Sq^r), \xi_a \otimes \cdots \otimes \xi_\ell \rangle = 0$ if $\ell \geq 2$. Furthermore, $\langle \Psi(Sq^r), \xi_a \otimes \cdots \otimes \xi_\ell \rangle = 1$ if $S = (r)$ and $a = \cdots = \ell = 1$, since $\Psi(Sq^r)$ contains the summand $Sq^1 \otimes \cdots \otimes Sq^1$ that evaluates to 1 on $\xi_1 \otimes \cdots \otimes \xi_1$. $\square$

Theorem 7.35 (Milnor). *For each infinite matrix of non-negative integers (almost all zero)*

$$X = \begin{bmatrix} * & x_{01} & x_{02} & \cdots \\ x_{10} & x_{11} & x_{12} & \cdots \\ x_{20} & x_{21} & x_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

let $R(X) = (r_1, r_2, \dots)$, $S(X) = (s_1, s_2, \dots)$ and $T(X) = (t_1, t_2, \dots)$ be given by the sums

$$\begin{aligned} r_i &= \sum_j 2^j x_{ij} && \text{(weighted row sum)}, \\ s_j &= \sum_i x_{ij} && \text{(column sum)}, \\ t_k &= \sum_{i+j=k} x_{ij} && \text{(diagonal sum)}. \end{aligned}$$

Then

$$Sq^R \cdot Sq^S = \sum_X b(X) Sq^T$$

where X ranges over the matrices with $R(X) = R$ and $S(X) = S$, with $T = T(X)$ and

$$b(X) = \prod_k t_k! / \prod_{i,j} x_{ij}! \,.$$

See Milnor (1958) Theorem 4b. To prove this, one must count how often $\xi^R \otimes \xi^S \in \mathscr{A}_* \otimes \mathscr{A}_*$ occurs as a summand in $\psi(\xi^T) = \psi(\xi_1)^{t_1} \cdot \dots \cdot \psi(\xi_\ell)^{t_\ell}$.

Example 7.36. Let $k \ge 2$, $R = (2^k)$ and $S = (0, \dots, 0, 1)$ with $(k-1)$ zeroes. Then $Sq^R \cdot Sq^S$ is a sum of terms $b(X)Sq^T$, where X ranges over the matrices (x_{ij}) with $x_{00} = 0$, $\sum_j 2^j x_{1j} = 2^k$, $\sum_j 2^j x_{ij} = 0$ for $i \ge 2$, $\sum_i x_{ik} = 1$ and $\sum_i x_{ij} = 0$ for $1 \le j \le k-1$ and for $j \ge k+1$. There are only two possible matrices X, namely X' with $x'_{1k} = 1$ and the remaining terms zero, and X'' with $x''_{0k} = 1$, $x''_{10} = 2^k$ and the remaining terms zero. The corresponding sequences are $T' = T(X') = (0, \dots, 0, 1)$ with k zeroes, and $T'' = T(X'') = (2^k, 0, \dots, 0, 1)$ with $(k-2)$ zeroes. The coefficients $b(X')$ and $b(X'')$ are 1, so

$$Sq^{(2^k)} \cdot Sq^{(0,\dots,0,1)} = Sq^{(0,\dots,0,0,1)} + Sq^{(2^k,0,\dots,0,1)} \,.$$

On the other hand, $Sq^S \cdot Sq^R$ is the sum of a single term $b(X)Sq^T$, where X has $x_{01} = 2^k$, $x_{k0} = 1$ and the remaining terms are zero. Again $b(X) = 1$, so

$$Sq^{(0,\dots,0,1)} \cdot Sq^{(2^k)} = Sq^{(2^k,0,\dots,0,1)} \,.$$

Hence the commutator

$$[Sq^{(2^k)}, Sq^{(0,\dots,0,1)}] = Sq^{(2^k)} \cdot Sq^{(0,\dots,0,1)} + Sq^{(0,\dots,0,1)} \cdot Sq^{(2^k)}$$

($(k-1)$ zeroes each time) equals the Milnor element $Sq^{(0,\dots,0,0,1)}$, now with k zeroes.

7.5 Subalgebras of the Steenrod algebra

Definition 7.37. A Hopf ideal in a Hopf algebra A is a two-sided ideal $I \subset A$ such that $\psi(I) \subset A \otimes I + I \otimes A$ and $\epsilon(I) = 0$:

$$\begin{tikzcd} 0 \arrow[d] & I \arrow[l] \arrow[r] \arrow[d, tail] & A \otimes I + I \otimes A \arrow[d, tail] \\ k & A \arrow[l, "\epsilon"'] \arrow[r, "\psi"] \arrow[d, two heads] & A \otimes A \arrow[d, two heads] \\ k \arrow[u, leftarrow, "="'] & A/I \arrow[l, "\bar\epsilon"'] \arrow[r, "\bar\psi"] & A/I \otimes A/I \end{tikzcd}$$

Then ψ and ϵ induce a coproduct $\bar\psi\colon A/I \to A/I \otimes A/I$ and a counit $\bar\epsilon\colon A/I \to k$ that make A/I a Hopf algebra, and the canonical surjection $A \to A/I$ is a Hopf algebra homomorphism. Dually, $(A/I)^* \to A^*$ is a Hopf subalgebra.

Definition 7.38. For each $k \ge 0$, let $Q_k = Sq^{(0,\dots,0,1)}$ (k zeroes) denote the Milnor basis element in $\mathscr{A}$ that is dual to ξ_{k+1}, in degree $2^{k+1} - 1$.

These classes are known as the Milnor primitives; see the next lemma. By the sample calculation above, these classes can also be recursively defined by $Q_0 = Sq^1$ and $[Sq^{2^k}, Q_{k-1}] = Q_k$ for all $k \geq 1$. The first few Milnor primitives are:

$$\begin{aligned}
Q_0 &= Sq^1 \\
Q_1 &= Sq^{(0,1)} = Sq^3 + Sq^2Sq^1 \\
Q_2 &= Sq^{(0,0,1)} = Sq^7 + Sq^6Sq^1 + Sq^5Sq^2 + Sq^4Sq^2Sq^1 \\
Q_3 &= Sq^{(0,0,0,1)}
\end{aligned}$$

Lemma 7.39. *The Q_k are primitive elements, and they generate an exterior Hopf subalgebra*

$$E = E(Q_k \mid k \geq 0) \subset \mathscr{A}$$

of the Steenrod algebra. In symbols, $\psi(Q_k) = Q_k \otimes 1 + 1 \otimes Q_k$, $Q_k^2 = 0$ and $Q_iQ_j = Q_jQ_i$ for all $i, j, k \geq 0$. The conjugation is trivial: $\chi(Q_k) = Q_k$.

Proof. First note that if $A = E(\xi)$ is the primitively generated exterior algebra on one generator, viewed as a bicommutative Hopf algebra, then the dual Hopf algebra $A^* = E(Q)$ is also a primitively generated exterior algebra, with 1 and Q dual to 1 and ξ, respectively.

Now consider the quotient algebra $E_* = \mathscr{A}_*/(\xi_k^2 \mid k \geq 1)$ of the dual Steenrod algebra. The ideal $J = (\xi_k^2 \mid k \geq 1) \subset \mathscr{A}_*$ is a Hopf ideal, since $\psi(\xi_k^2) = \sum_{i+j=k} \xi_i^{2^{j+1}} \otimes \xi_j^2$ lies in $\mathscr{A}_* \otimes J + J \otimes \mathscr{A}_*$, and $\epsilon(\xi_k^2) = 0$. Hence $\mathscr{A}_* \to E_*$ is a Hopf algebra surjection. The generators ξ_k are primitive in E_*, since

$$\psi(\xi_k) \equiv \xi_k \otimes 1 + 1 \otimes \xi_k$$

modulo $A \otimes J + J \otimes A$. It follows that $\chi(\xi_k) \equiv \xi_k$ modulo J. Hence $E_* = E(\xi_k \mid k \geq 1) = \bigotimes_{k\geq 1} E(\xi_k)$ is a primitively generated exterior Hopf algebra.

Passing to duals, we have a Hopf algebra injection $E = (E_*)^* \to \mathscr{A}$. Here $E = E(Q_k \mid k \geq 0) = \bigotimes_{k\geq 0} E(Q_k)$ is also primitively generated, with Q_k dual to ξ_{k+1} in the monomial basis for E_*. Since I is generated by monomials, it follows that the inclusion maps $Q_k \in E$ to $Q_k \in \mathscr{A}$. Hence the Q_k are primitive in $\mathscr{A}$. □

Lemma 7.40. *$Q(\mathscr{A}) \cong \mathbb{F}_2\{Sq^{2^i} \mid i \geq 0\}$, $P(\mathscr{A}_*) \cong \mathbb{F}_2\{\xi_1^{2^i} \mid i \geq 0\}$, $Q(\mathscr{A}_*) \cong \mathbb{F}_2\{\xi_{i+1} \mid i \geq 0\}$ and $P(\mathscr{A}) \cong \mathbb{F}_2\{Q_i \mid i \geq 0\}$.*

Definition 7.41. For each $n \geq 0$, let $E(n) = E(Q_0, \dots, Q_n) \subset \mathscr{A}$ be the exterior subalgebra generated by the Milnor primitives $Q_0, \dots, Q_n$. It is a Hopf subalgebra with conjugation. The dual of $E(n)$ is the quotient Hopf algebra $E(n)_* = \mathscr{A}_*/J(n)$ of $\mathscr{A}_*$ by the Hopf ideal

$$J(n) = (\xi_1^2, \dots, \xi_{n+1}^2, \xi_k \mid k \geq n+2)\,.$$

Definition 7.42. For each $n \geq 0$, let $A(n) = \langle Sq^1, \dots, Sq^{2^n} \rangle \subset \mathscr{A}$ be the subalgebra generated by the Steenrod squares $Sq^1, \dots, Sq^{2^n}$. It is a Hopf subalgebra with conjugation.

Lemma 7.43. *The dual of $A(n)$ is the quotient Hopf algebra $A(n)_* = \mathscr{A}_*/I(n)$ of $\mathscr{A}_*$ by the Hopf ideal*

$$I(n) = (\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_n^{2^2}, \xi_{n+1}^2, \xi_k \mid k \geq n+2)\,.$$

Proof. The ideal $I(n)$ is generated by the classes $\xi_s^{2^t}$ with $s \geq 1$ and $s + t \geq n + 2$. It is a Hopf ideal since

$$\psi(\xi_s^{2^t}) = \sum_{i+j=s} \xi_i^{2^{j+t}} \otimes \xi_j^{2^t}$$

is a sum of terms in $\mathscr{A} \otimes I(n)$ (for $i = 0$) and in $I(n) \otimes \mathscr{A}$ (for $1 \leq i \leq s$). Hence $\mathscr{A}_*/I(n)$ is a finite commutative Hopf algebra, and the dual is a finite cocommutative Hopf subalgebra of $\mathscr{A}$.

We claim that $Sq^k \in A(n)$ for all $0 \leq k < 2^{n+1}$. Equivalently, we must prove that $\langle Sq^k, \xi \rangle = 0$ for all $\xi \in I(n)$. By induction, we may assume that this holds for all smaller values of k. The ideal $I(n)$ is additively generated by products $\xi_s^{2^t} \cdot \xi^R$ with $s \geq 1$ and $s + t \geq n + 2$, and

$$\langle Sq^k, \xi_s^{2^t} \cdot \xi^R \rangle = \langle Sq^k, \phi(\xi_s^{2^t} \otimes \xi^R) \rangle = \langle \psi(Sq^k), \xi_s^{2^t} \otimes \xi^R \rangle = \sum_{i+j=k} \langle Sq^i, \xi_s^{2^t} \rangle \langle Sq^j, \xi^R \rangle\,.$$

By the inductive hypothesis, this equals $\langle Sq^k, \xi_s^{2^t}\rangle \cdot \langle 1, \xi^R\rangle$, which is 0 for $k < 2^{n+1}$ since $|\xi_s^{2^t}| \ge 2^{n+1}$ when $s \ge 1$ and $s+t \ge n+2$. ((It remains to prove that the Sq^k for $k \le 2^n$, or for $k < 2^{n+1}$, generate all of the dual of $A(n)_*$.)) $\square$

Corollary 7.44. *$\mathscr{A} = \operatorname{colim}_{n\ge 0} A(n)$ is a countable union of finite algebras. Hence each element in positive degree of $\mathscr{A}$ is nilpotent.*

Remark 7.45. Steenrod and Epstein (1962) write $\mathscr{A}_h$ for our $A(h+1)$. Adams (Math. Proc. Camb. Phil. Soc., 1966) writes A_r for our $A(r)$. Clearly $E(0) = A(0)$, and $E(n) \subset A(n)$ for $n \ge 1$. This can also be seen from the inclusion $I(n) \subset J(n)$.

((Write $P_s^t = Sq^{(0,\dots,0,2^t)}$ for the dual of $\xi_s^{2^t}$, so that $P_1^t = Sq^{2^t}$ and $P_{s+1}^0 = Q_s$? Review Adams–Margolis classification of Hopf ideals in $\mathscr{A}_*$ and Hopf subalgebras of $\mathscr{A}$, in terms of profile functions.))

7.6 Spectral realizations

Definition 7.46. Brown and Peterson (Topology, 1966) construct a spectrum BP such that $H^*(BP) \cong \mathscr{A}//E$ as an $\mathscr{A}$-module. Johnson and Wilson (Topology, 1973) construct spectra $BP\langle n\rangle$ such that $H^*(BP\langle n\rangle) \cong \mathscr{A}//E(n)$, for each $n \ge 0$. As a convention, one may define $BP\langle -1\rangle = H$.

The connective cover $k(n)$ of the n-th Morava K-theory spectrum $K(n)$ has cohomology $H^*(k(n)) \cong \mathscr{A}//E(Q_n)$, for each $n \ge 1$. By convention, $k(0) = H\mathbb{Z}_{(2)}$ and $K(0) = H\mathbb{Q}$.

Remark 7.47. Baker and Jeanneret (HHA, 2002), using methods of Lazarev (K-Theory, 2001), show that there is a diagram

$$BP \to \cdots \to BP\langle n\rangle \to \cdots \to BP\langle 0\rangle \to H$$

of S-algebras, or equivalently, of A_∞ ring spectra, inducing the surjections

$$\mathscr{A} \to \mathscr{A}//E(0) \to \cdots \to \mathscr{A}//E(n) \to \cdots \to \mathscr{A}$$

in cohomology. Naumann and Lawson (J. Topology, 2011) prove (for $p = 2$ only) that $BP\langle 2\rangle$ can be realized as a commutative S-algebra, or equivalently as an E_∞ ring spectrum, like the realizations $BP\langle 0\rangle_2^\wedge \simeq H\mathbb{Z}_2$ and $BP\langle 1\rangle_2^\wedge \simeq ku_2^\wedge$. It is an open problem whether BP can be realized as a commutative S-algebra.

Baas and Madsen (Math. Scand., 1972) realize $k(n)$. Angeltveit (Compos. Math., 2011) proves that $K(n)$ has a unique S-algebra structure. For $n = 1$ (and $p = 2$) one can take $k(1) = ku/2$ and $K(1) = KU/2$. None of the $k(n)$ for $n \ge 1$ admit commutative S-algebra structures, since the map $k(n) \to H$ induces a homomorphism $H_*(k(n)) \to \mathscr{A}_*$ that cannot commute with the Dyer–Lashof operations in the target.

Proposition 7.48. *The Adams spectral sequence for BP collapses at the E_2-term*

$$E_2^{*,*} \cong \operatorname{Ext}_E^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(v_k \mid k \ge 0)$$

to the abutment

$$\pi_*(BP_2^\wedge) \cong \mathbb{Z}_2[v_k \mid k \ge 1]\,,$$

where v_k in degree $2^{k+1}-2$ is detected in $E_\infty^{1,2^{k+1}-1}$ by the dual of $Q_k \in E$.

Similarly, the Adams spectral sequence for $BP\langle n\rangle$ collapses at

$$E_2^{*,*} \cong \operatorname{Ext}_{E(n)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(v_0, \dots, v_n)$$

to the abutment

$$\pi_*(BP\langle n\rangle) = \mathbb{Z}_2[v_1, \dots, v_n]\,,$$

and the Adams spectral sequence for $k(n)$ collapses at

$$E_2^{*,*} \cong \operatorname{Ext}_{E(Q_n)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(v_n)$$

to the abutment

$$\pi_*(k(n)) = \mathbb{F}_2[v_n]\,.$$

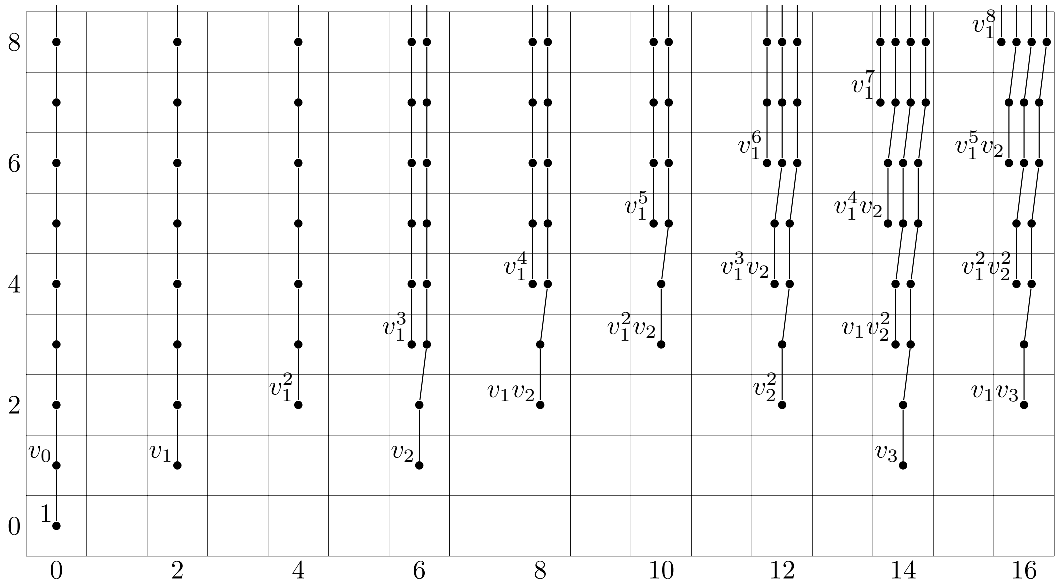


Figure 7: Adams spectral sequence for BP

Proof. The E_2-term can be computed using change-of-rings:

$$\mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(BP), \mathbb{F}_2) \cong \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathscr{A}//E, \mathbb{F}_2) \cong \mathrm{Ext}_E^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \cong P(v_k \mid k \geq 0)$$

where v_k is dual to the indecomposable $Q_k \in E$. In particular, $v_0 = h_0$ is dual to $Q_0 = Sq^1$. Since the E_2-term is concentrated in even total degrees, there is no room for differentials. There is also no room for other multiplicative extensions than the h_0-towers, since $\mathbb{Z}_2[v_k \mid k \geq 1]$ is free as a graded commutative algebra. ((This presumes that $\pi_*(BP)$ is commutative.)) □

Remark 7.49. Let MU be the complex bordism spectrum. Milnor (Ann. Math., 1960) and Novikov ((ref?)) shows that $H^*(MU)$ is a direct sum of suspensions of copies of $H^*(BP) = \mathscr{A}//E$. Brown and Peterson (Topology, 1966) showed that $MU_{(p)}$ splits as a wedge sum of suspensions of BP. One finds that $\pi_*(MU) \cong \mathbb{Z}[x_k \mid k \geq 1]$ with $|x_k| = 2k$. Quillen (Bull. Amer. Math. Soc., 1969) relates $\pi_*(MU)$ to formal group laws, in such a way that $\pi_*(BP)$ corresponds to p-typical formal group laws. The introduction of spectra like $BP\langle n\rangle$, $E(n)$, $k(n)$ and $K(n)$ is then motivated by the classification of formal group laws according to height, which in turn leads to the chromatic perspective on stable homotopy theory, which seeks to organize the homotopy groups of S and related spectra in periodic families of varying wave-lengths.

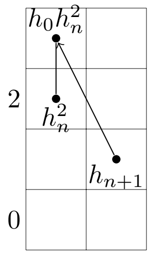
Remark 7.50. Starting with the Hopkins–Miller obstruction theory for A_∞ ring structures, continued by Goerss–Hopkins–Miller and Lurie for E_∞ ring structures, Hopkins and Mahowald (preprint, 1994) produce a connective E_∞ ring spectrum tmf with $H^*(tmf) \cong \mathscr{A}//A(2)$. We have already discussed the realizations $H^*(ko) \cong \mathscr{A}//A(1)$ and $H^*(H\mathbb{Z}) \cong \mathscr{A}//A(0)$. (The Davis–Mahowald proof of the non-realizability of $\mathscr{A}//A(2)$ (Amer. J. Math., 1982) contains an error.)

There is no spectrum with cohomology $H^*(X) \cong \mathscr{A}//A(n)$ for $n \geq 3$, since the unit map $S \to X$ would induce a map of Adams spectral sequences

$$E_2^{*,*}(S) = \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{A(n)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2) = E_2^{*,*}(X)$$

mapping $h_n \mapsto h_n$ and $h_{n+1} \mapsto 0$. This contradicts the Adams differential $d_2(h_{n+1}) = h_0 h_n^2$, since

$h_0 h_n^2 \neq 0$ on the right hand side for $n \geq 3$. ((Elaborate?))



(($B_* = A_*/(\xi_1^4, \xi_2^2, \xi_3^2, \xi_4, ...)$ has dual $B = A(1) \otimes E(Q_2)$ and Ext_B is $Ext_{A(1)} \otimes P(v_2)$.))

8 Ext **over** $A(1)$ **and** $A(2)$

8.1 The Iwai–Shimada generators

((Edit.)) Our next aim is to compute the homotopy $\pi_*(tmf)^\wedge_2$ of the spectrum of topological modular forms, which is a connective commutative S-algebra of finite type, with cohomology $H^*(tmf) \cong \mathscr{A}//A(2)$. We shall use the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(tmf), \mathbb{F}_2) \Longrightarrow \pi_*(tmf)^\wedge_2 \,.$$

Using change-of-rings, the E_2-term

$$\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(tmf), \mathbb{F}_2) = \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathscr{A}//A(2), \mathbb{F}_2) \cong \mathrm{Ext}_{A(2)}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$$

can be rewritten as Ext over the finite Hopf subalgebra

$$A(2) = \langle Sq^1, Sq^2, Sq^4 \rangle \subset \mathscr{A} \,,$$

which is dual to the finite Hopf quotient algebra

$$A(2)_* = P(\xi_1, \xi_2, \xi_3)/(\xi_1^8, \xi_2^4, \xi_3^2)$$

of $\mathscr{A}_*$. It has dimension $8 \cdot 4 \cdot 2 = 64$ as $\mathbb{F}_2$-vector space.

The first computation of Ext over $A(2)$ was done by Iwai and Shimada (Nagoya Math. J., 1967). The answer is complicated, but interesting. The graded commutative algebra $\mathrm{Ext}_{A(2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ has 13 generators

gen.	$(t-s, s)$	alt.
h_0	$(0,1)$	h_0
h_1	$(1,1)$	h_1
h_2	$(3,1)$	h_2
ω_0	$(8,4)$	$w_1 = v_1^4$
ω_1	$(20,4)$	g
α_0	$(48,8)$	$w_2 = v_2^8$
α_1	$(8,3)$	c_0
α_2	$(12,3)$	α
α_3	$(15,3)$	β
α_4	$(14,4)$	d_0
α_5	$(17,4)$	e_0
α_6	$(25,5)$	γ
α_7	$(32,7)$	δ

that are subject to a list of 54 relations, which we do not list here. In particular, it is a free $P(\omega_0, \alpha_0)$-module. The part in topological degrees $0 \leq t-s \leq 70$ is displayed in Figure 8, which was created by Christian Nassau (2001).

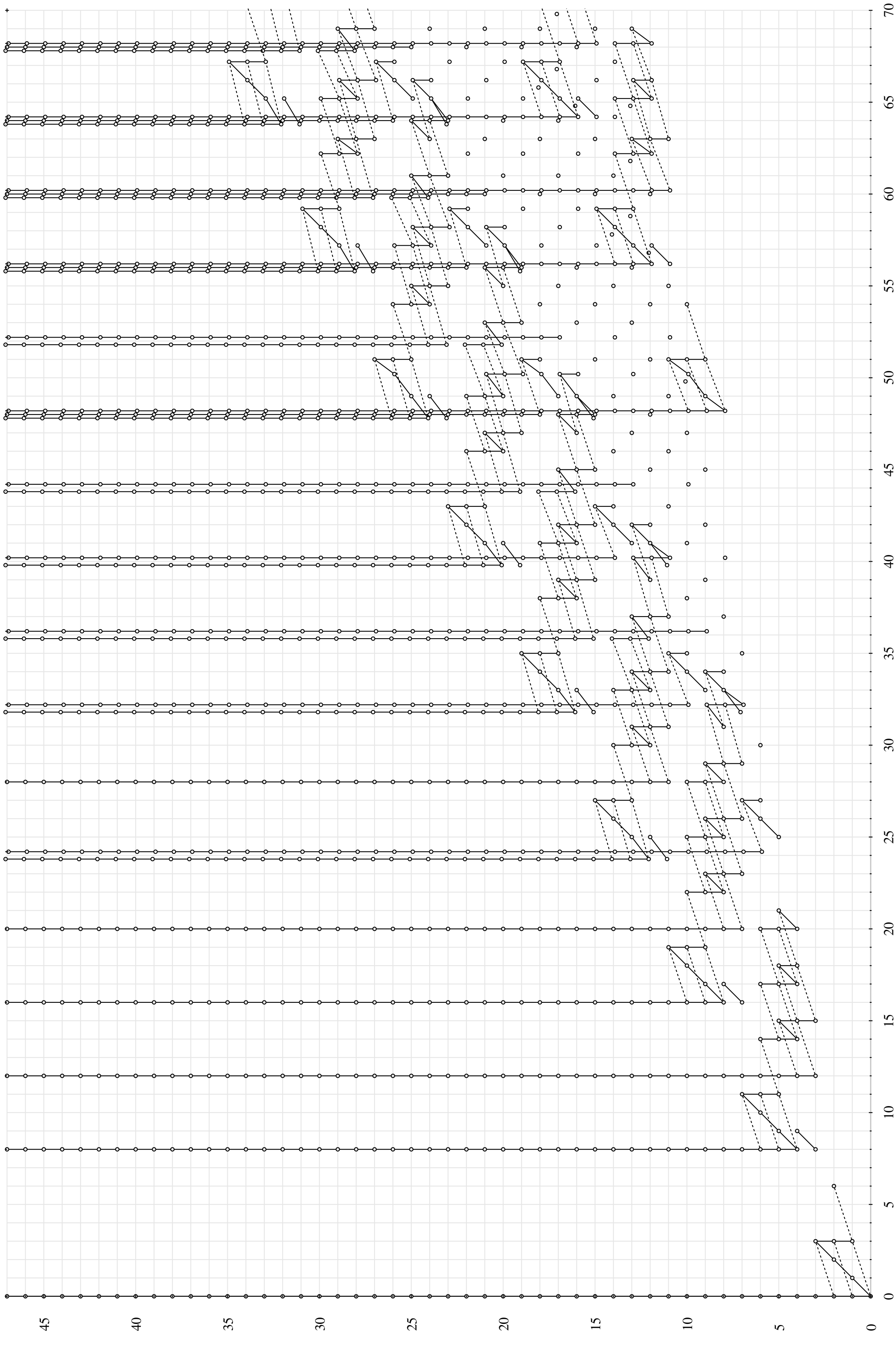


Figure 8: Ext over $A(2)$ by Christian Nassau (2001)

There are (commutative S-algebra) maps $S \to tmf \to BP\langle 2\rangle$ that induce surjections $\mathscr{A}//E(2) \to \mathscr{A}//A(2) \to \mathbb{F}_2$ in cohomology, and the restriction homomorphisms

$$\mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{*,*}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{*,*}_{E(2)}(\mathbb{F}_2, \mathbb{F}_2)$$

at the level of Adams E_2-terms. The classes h_0, h_1, h_2, c_0, d_0, e_0 and g in the Adams spectral sequence for S, detecting 2, η, ν, ϵ, κ, (no homotopy element) and $\bar{\kappa}$, map to the Iwai–Shimada generators h_0, h_1, h_2, α_1, α_4, α_5 and ω_1, respectively. The Iwai–Shimada generators h_0, ω_0 and α_0 map to v_0, v_1^4 and v_2^8 in the Adams spectral sequence for $BP\langle 2\rangle$, respectively. We may follow notes of André Hernandez (Talbot workshop, 2007), writing w_1 and w_2 for ω_0 and α_1, and writing α, β, γ and δ for the remaining algebra generators α_2, α_3, α_6 and α_7. With this notation, the E_2-term for tmf is free as a $P(w_1, w_2)$-module.

8.2 The Davis–Mahowald resolution

To make this calculation, we shall instead follow section 5 of Davis and Mahowald (CMS Conf. Proc., 1982) and use a Koszul-type resolution of $\mathbb{F}_2$ by $A(2)$-modules of the form $A(2)//A(1) \otimes N$, with the diagonal action. By the shearing lemma below, these are isomorphic to induced modules of the form $A(2) \otimes_{A(1)} N$, and using the change-of-rings isomorphism

$$\mathrm{Ext}^{s,t}_{A(2)}(A(2) \otimes_{A(1)} N, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{A(1)}(N, \mathbb{F}_2)$$

we are reduced to the problem of computing Ext over $A(1)$, which is quite straightforward.

Lemma 8.1 ((Reference?))**.** *Let A be a Hopf algebra with conjugation, N a left A-module and $B \subset A$ a ((Hopf?)) subalgebra. There is an isomorphism of left A-modules*

$$\theta \colon A \otimes_B N \xrightarrow{\cong} A//B \otimes N$$

where the left hand side has the A-module structure induced up from the restricted B-module structure on N, and the right hand side has the diagonal A-module structure.

Proof. This is analogous to the homeomorphism $G \times_H X \cong G/H \times X$ for a G-space X and a subgroup H. The shear map taking $[g, x]$ to $([g], gx)$ has inverse taking $([g], y)$ to $[g, g^{-1}y]$, for $g \in G$, $x, y \in X$. Similarly, the composite homomorphism

$$A \otimes N \xrightarrow{\psi\otimes 1} A \otimes A \otimes N \xrightarrow{\pi\otimes\lambda} A//B \otimes N$$

coequalizes the two homomorphisms

$$A \otimes B \otimes N \underset{1\otimes\lambda}{\overset{\rho\otimes 1}{\rightrightarrows}} A \otimes N$$

to induce θ, while the composite homomorphism

$$A \otimes N \xrightarrow{\psi\otimes 1} A \otimes A \otimes N \xrightarrow{1\otimes\chi\otimes 1} A \otimes A \otimes N \xrightarrow{1\otimes\lambda} A \otimes N \xrightarrow{\pi} A \otimes_B N$$

vanishes on $A \cdot I(B) \otimes N$ to induce θ^{-1}. These maps are mutual inverses; see Adams (1974, p. 338) and Anderson, Brown and Peterson (1969, Prop. 3.1). ((Thanks to Bruner for these references.)) $\square$

Corollary 8.2. *Let R and Y be spectra that are bounded below, with $H_*(R)$ and $H_*(Y)$ of finite type. Suppose furthermore that $H^*(R) \cong \mathscr{A}//B$, for some subalgebra $B \subset \mathscr{A}$ such that $\mathscr{A}$ is free as a right B-module. For instance, B might be a Hopf subalgebra. Then the E_2-term for the Adams spectral sequence converging to $\pi_*(R \wedge Y) = R_*(Y)$ is*

$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{\mathscr{A}}(H^*(R \wedge Y), \mathbb{F}_2) \cong \mathrm{Ext}^{*,*}_{B}(H^*(Y), \mathbb{F}_2)\,.$$

In particular, it only depends on the restricted B-module structure on $H^*(Y)$.

Proof. $H^*(R \wedge Y) \cong H^*(R) \otimes H^*(Y) \cong \mathscr{A}//B \otimes H^*(Y) \cong \mathscr{A} \otimes_B H^*(Y)$, and $\mathrm{Ext}_{\mathscr{A}}^{*,*}(sA \otimes_B H^*(Y), \mathbb{F}_2) \cong \mathrm{Ext}_B^{*,*}(H^*(Y), \mathbb{F}_2)$. □

Example 8.3. This shows that the E_2-terms of the Adams spectral sequences computing $H_*(Y;\mathbb{Z})$, $ku_*(Y)$ and $ko_*(Y)$ only depend on the $A(0)$-, $E(1)$- and $A(1)$-module structures on $H^*(Y)$. The later differentials in the spectral sequences may depend on more than these module structures. See Bayen and Bruner (1996, p. 2205).

Definition 8.4. Let $\bar{\xi}_k \in \mathscr{A}_*$ denote the conjugate Milnor generator $\bar{\xi}_k = \chi(\xi_k)$, for each $k \geq 0$. Thus $\bar{\xi}_0 = 1$, $\bar{\xi}_1 = \xi_1$ (for $p = 2$) and $\bar{\xi}_2 = \xi_2 + \xi_1^3$. The coproduct $\psi \colon \mathscr{A}_* \to \mathscr{A}_* \otimes \mathscr{A}_*$ satisfies

$$\psi(\bar{\xi}_k) = \sum_{i+j=k} \bar{\xi}_i \otimes \bar{\xi}_j^{2^i} .$$

We are interested in the case $n = 2$ of the following general result. Recall how the dual of a left $A(n)$-module can be viewed as a left $A(n)_*$-comodule or as a right $A(n)$-module.

Lemma 8.5. *The left $A(n)$-module $A(n)//A(n-1) = A(n) \otimes_{A(n-1)} \mathbb{F}_2$ of $A(n)$ is dual to the right $A(n)$-module subalgebra*

$$(A(n)//A(n-1))^* = A(n)_* \,\Box_{A(n-1)_*}\, \mathbb{F}_2 = E(\bar{\xi}_1^{2^n}, \bar{\xi}_2^{2^{n-1}}, \dots, \bar{\xi}_{n+1})$$

of $A(n)_* = P(\xi_1, \dots, \xi_{n+1})/(\xi_1^{2^{n+1}}, \xi_2^{2^n}, \dots, \xi_{n+1}^2)$. Hence $\pi \colon A(n) \to A(n)//A(n-1)$ is a surjection of left $A(n)$-module coalgebras.

Proof. By a dimension count, it suffices to prove that $\bar{\xi}_{k+1}^{2^{n-k}} \in A(n)_*$ lies in $A(n)_* \,\Box_{A(n-1)_*}\, \mathbb{F}_2$, for each $0 \leq k \leq n$. The coaction $\rho = (1 \otimes \pi)\psi \colon : A(n)_* \to A(n)_* \otimes A(n-1)_*$ takes this element to

$$\sum_{i+j=k+1} \bar{\xi}_i^{2^{n-k}} \otimes \bar{\xi}_j^{2^{i+n-k}} ,$$

and the terms with $j > 0$ vanish. □

Definition 8.6. Let $E_n = E(\bar{\xi}_1^{2^n}, \bar{\xi}_2^{2^{n-1}}, \dots, \bar{\xi}_{n+1}) = E(x_0, x_1, \dots, x_n)$ be the right $A(n)$-module graded exterior algebra generated by $x_k = \bar{\xi}_{k+1}^{2^{n-k}}$ for $0 \leq k \leq n$. Here $|x_k| = 2^{n-k}(2^{k+1} - 1) = 2^{n+1} - 2^{n-k}$, and $x_k \cdot Sq^{2^{n-k}} = x_{k-1}$ for $1 \leq k \leq n$, while $x_0 \cdot Sq^{2^n} = 1$. The inclusion $E_n \to A(n)_*$ is a homomorphism of right $A(n)$-module algebras.

Lemma 8.7. *The left $A(n)_*$-coaction on E_n is given by*

$$\lambda(x_k) = \bar{\xi}_{k+1}^{2^{n-k}} \otimes 1 + \sum_{i+j=k} \bar{\xi}_i^{2^{n-k}} \otimes x_j$$

for $0 \leq k \leq n$.

Proof. This is clear from the coproduct on $A(n)_*$, which is given by the same formula as the coproduct on $\mathscr{A}_*$. □

Example 8.8. For $n = 1$ we have $E_1 = E(\xi_1^2, \bar{\xi}_2) = E(x_0, x_1)$ concentrated in homological degrees $t \in \{0, 2, 3, 5\}$, with the right $A(1)$-module structure:

$$\begin{tikzcd}
1 & x_0 \arrow[l, bend right, "\cdot Sq^2"'] & x_1 \arrow[l, "\cdot Sq^1"] & x_0 x_1 \arrow[l, bend right, "\cdot Sq^2"']
\end{tikzcd}$$

The left $A(1)_*$-coaction is given by $\lambda(x_0) = 1 \otimes x_0 + \xi_1^2 \otimes 1$ and $\lambda(x_1) = 1 \otimes x_1 + \xi_1 \otimes x_0 + \bar{\xi}_2 \otimes 1$.

For $n = 2$ we have $E_2 = E(\xi_1^4, \bar\xi_2^2, \bar\xi_3) = E(x_0, x_1, x_2)$ concentrated in homological degrees $t \in \{0, 4, 6, 7, 10, 11, 13, 17\}$, with the following right $A(2)$-module structure:

$$\begin{tikzcd}
1 & x_0 \arrow[l, bend right, "\cdot Sq^4"'] & x_1 \arrow[l, bend right, "\cdot Sq^2"'] & x_2 \arrow[l, "\cdot Sq^1"] & x_0x_1 \arrow[ll, bend right, "\cdot Sq^4"'] & x_0x_2 \arrow[l, "\cdot Sq^1"'] \arrow[ll, bend left, "\cdot Sq^4"] & x_1x_2 \arrow[l, bend left, "\cdot Sq^2"] & x_0x_1x_2 \arrow[l, bend left, "\cdot Sq^4"]
\end{tikzcd}$$

The left $A(2)_*$-coaction is given by $\lambda(x_0) = 1 \otimes x_0 + \xi_1^4 \otimes 1$, $\lambda(x_1) = 1 \otimes x_1 + \xi_1^2 \otimes x_0 + \bar\xi_2^2 \otimes 1$ and $\lambda(x_2) = 1 \otimes x_2 + \xi_1 \otimes x_1 + \bar\xi_2 \otimes x_0 + \bar\xi_3 \otimes 1$.

((NOTE: The part in degrees $4 \le * \le 13$ occurs as $L[2]$ in $H_*(THH(tmf))$. Get sequences relating $\mathrm{Ext}_{A(2)}$ for $L[2]$ to those for $\mathbb{F}_2$ and $A(2)//A(1)$.))

Remark 8.9. $A(n-1)$ is not normal in $A(n)$, so $A(n)//A(n-1)$ is not a quotient Hopf algebra of $A(n)$, and E_n is not a Hopf subalgebra of $A(n)_*$. Nonetheless, E_n is a primitively generated Hopf algebra on its own. There is a standard way to resolve E_n-comodules using a twisted tensor product (Brown, Ann. of Math., 1959), which in this case specializes to a kind of dual Koszul resolution. This turns out to produce a useful right $A(n)$-module resolution. ((What is the general picture behind this??))

Definition 8.10. Let $R_n = P(y_0, y_1, \dots, y_n)$ be the right $A(n)$-module bigraded polynomial algebra generated by y_k of bigrading $(\sigma, t) = (1, |x_k|)$, for $0 \le k \le n$. It decomposes additively as

$$R_n = \bigoplus_{\sigma \ge 0} R_n^\sigma \,,$$

where R_n^σ is spanned by the monomials of degree σ in the y_k's. In particular, $R_n^0 = \mathbb{F}_2$ and $R_n^1 = \mathbb{F}_2\{y_0, \dots, y_n\}$. The right $A(n)$-module action on R_n^1 is given by $y_k \cdot Sq^{2^{n-k}} = y_{k-1}$ for $1 \le k \le n$, and extends to a right $A(n)$-action on R_n^σ for each $\sigma \ge 0$, since $A(n)$ is cocommutative.

Lemma 8.11. *The left $A(n)_*$-coaction on R_n is given by*

$$\lambda(y_k) = \sum_{i+j=k} \bar\xi_i^{2^{n-k}} \otimes y_j$$

for $0 \le k \le n$.

Proof. This is clear from the coaction on E_n, and the fact that $d(1) = 0$. □

Definition 8.12. Let $(E_n \otimes R_n, d)$ be the right $A(n)$-module differential bigraded algebra given by the tensor product of $E_n = E(x_0, \dots, x_n)$ (in degree $\sigma = 0$) and $R_n = P(y_0, \dots, y_n)$, with the diagonal right $A(n)$-module structure and with the differential given by $d(x_k) = y_k$ for all $0 \le k \le n$.

Remark 8.13. Our numbering of the x_k is reversed compared to that of Davis–Mahowald. Furthermore, they do not distinguish notationally between the x_k and the y_k.

Lemma 8.14. *The differential $d\colon E_n \otimes R_n^\sigma \to E_n \otimes R_n^{\sigma+1}$ is right $A(n)$-linear.*

Proof. For $e \in E_n$ and $r \in R_n$ we have $(d(e \cdot r))Sq^c = \sum_{a+b=c}(d(e))Sq^a \cdot (r)Sq^b$ since $d(r) = 0$, and $d((e \cdot r)Sq^c) = \sum_{a+b=c} d((e)Sq^a) \cdot (r)Sq^b$ since $d((r)Sq^b) = 0$, so it suffices to check that $d\colon E_n = E_n \otimes R_n^0 \to E_n \otimes R_n^1$ is $A(n)$-linear. When $n = 1$ we have that $d(x_0x_1) = x_0y_1 + x_1y_0$ is mapped by Sq^1 to $x_0y_0 + x_0y_0 = 0$ and by Sq^2 to y_1, while $d(x_1) = y_1$ is mapped by Sq^1 to y_0.

((Check for $n = 2$, or give general formula.)) □

Definition 8.15. For a fixed n, let $N_\sigma = (R_n^\sigma)^*$ be the dual left $A(n)$-module, so that $N = \bigoplus_\sigma N_\sigma$ is a left $A(n)$-module differential bigraded coalgebra. In particular, $N_0 = \mathbb{F}_2$.

Lemma 8.16. *$H_*(E_n \otimes R_n, d) \cong \mathbb{F}_2$, so*

$$0 \to \mathbb{F}_2 \xrightarrow{\eta} E_n \otimes R_n^0 \xrightarrow{d} E_n \otimes R_n^1 \xrightarrow{d} \dots \xrightarrow{d} E_n \otimes R_n^\sigma \xrightarrow{d} \dots$$

is an exact complex of right $A(n)$-modules. Dually,

$$\dots \to A(n)//A(n-1) \otimes N_\sigma \xrightarrow{\partial_\sigma} \dots \xrightarrow{\partial_2} A(n)//A(n-1) \otimes N_1 \xrightarrow{\partial_1} A(n)//A(n-1) \otimes N_0 \xrightarrow{\epsilon} \mathbb{F}_2 \to 0$$

is an exact complex of left $A(n)$-modules.

Proof. It is clear that $E(x_k) \otimes P(y_k)$ with $d(x_k) = y_k$ has homology $\mathbb{F}_2\{1\}$ concentrated in degree 0, for each $0 \le k \le n$. The lemma follows from the Künneth formula. □

For each σ, the short exact sequence of left $A(n)$-modules

$$0 \to \operatorname{im}(\partial_{\sigma+1}) \to A(n)//A(n-1) \otimes N_\sigma \to \operatorname{im}(\partial_\sigma) \to 0$$

(with $\partial_0 = \epsilon$) generates a long exact sequence

$$\begin{aligned} \to \operatorname{Ext}^{s-1,t}_{A(n)}(\operatorname{im}(\partial_{\sigma+1}), \mathbb{F}_2) \xrightarrow{\delta} \operatorname{Ext}^{s,t}_{A(n)}(\operatorname{im}(\partial_\sigma), \mathbb{F}_2) \to \\ \to \operatorname{Ext}^{s,t}_{A(n)}(A(n)//A(n-1) \otimes N_\sigma, \mathbb{F}_2) \to \operatorname{Ext}^{s,t}_{A(n)}(\operatorname{im}(\partial_{\sigma+1}), \mathbb{F}_2) \to \end{aligned}$$

in Ext. These can be linked together, for varying $\sigma \ge 0$, to an unrolled exact couple of (s,t)-bigraded abelian groups

$$\begin{tikzcd} \dots \arrow[r] & \operatorname{Ext}^{s-2,t}_{A(n)}(\operatorname{im}(\partial_2), \mathbb{F}_2) \arrow[r, "\delta"] \arrow[d] & \operatorname{Ext}^{s-1,t}_{A(n)}(\operatorname{im}(\partial_1), \mathbb{F}_2) \arrow[r, "\delta"] \arrow[d] & \operatorname{Ext}^{s,t}_{A(n)}(\mathbb{F}_2, \mathbb{F}_2) \arrow[d] \\ & \operatorname{Ext}^{s-2,t}_{A(n-1)}(N_2, \mathbb{F}_2) \arrow[ul, dashed] & \operatorname{Ext}^{s-1,t}_{A(n-1)}(N_1, \mathbb{F}_2) \arrow[ul, dashed] & \operatorname{Ext}^{s,t}_{A(n-1)}(N_0, \mathbb{F}_2) \arrow[ul, dashed] \end{tikzcd}$$

with

$$A^{\sigma,s,t} = \operatorname{Ext}^{s-\sigma,t}_{A(n)}(\operatorname{im}(\partial_\sigma), \mathbb{F}_2)$$

and

$$E^{\sigma,s,t} = \operatorname{Ext}^{s-\sigma,t}_{A(n)}(A(n)//A(n-1) \otimes N_\sigma, \mathbb{F}_2) \cong \operatorname{Ext}^{s-\sigma,t}_{A(n-1)}(N_\sigma, \mathbb{F}_2)\,.$$

Here we have used the shearing isomorphism $A(n)//A(n-1) \otimes N_\sigma \cong A(n) \otimes_{A(n-1)} N_\sigma$ and the change-of-rings isomorphism for $A(n-1) \subset A(n)$. Note that the E_1-term only depends on the restricted $A(n-1)$-module structure of the N_σ's.

Proposition 8.17 (Davis–Mahowald (1982, Cor. 5.3))**.** *There is an algebra spectral sequence converging to $\operatorname{Ext}^{*,*}_{A(n)}(\mathbb{F}_2, \mathbb{F}_2)$, with*

$$E_1^{\sigma,s,t} = \operatorname{Ext}^{s-\sigma,t}_{A(n-1)}(N_\sigma, \mathbb{F}_2)\,.$$

More generally, let M be a left $A(n)$-module. There is a spectral sequence converging to $\operatorname{Ext}^{*,*}_{A(n)}(M, \mathbb{F}_2)$, with

$$E_1^{\sigma,s,t} = \operatorname{Ext}^{s-\sigma,t}_{A(n-1)}(N_\sigma \otimes M, \mathbb{F}_2)\,.$$

The differential $d_1 \colon E_1^{\sigma,s,t} \to E_1^{\sigma+1,s+1,t}$ is induced on $\operatorname{Ext}^{*,*}_{A(n)}((-) \otimes M, \mathbb{F}_2)$ by the homomorphism $\partial_{\sigma+1} \colon A(n)//A(n-1) \otimes N_{\sigma+1} \longrightarrow A(n)//A(n-1) \otimes N_\sigma$.

Proof. The algebra structure can be seen from the right $A(n)$-module algebra resolution $\eta \colon \mathbb{F}_2 \to E_n \otimes R_n$, which we can also think of as a left $A(n)_*$-comodule algebra resolution. Applying $\operatorname{Ext}^{*,*}_{A(n)_*}(\mathbb{F}_2, -)$ for the category of left $A(n)_*$-comodules, we get an algebra spectral sequence

$$E_1^{\sigma,s,t} = \operatorname{Ext}^{s-\sigma,t}_{A(n)_*}(\mathbb{F}_2, E_n \otimes R_n^\sigma) \Longrightarrow \operatorname{Ext}^{s,t}_{A(n)_*}(\mathbb{F}_2, \mathbb{F}_2)\,.$$

((Elaborate??)) The contravariant duality equivalence gives isomorphisms $\mathrm{Ext}^{s-\sigma,t}_{A(n)_*}(\mathbb{F}_2, E_n \otimes R_n^\sigma) \cong \mathrm{Ext}^{s-\sigma,t}_{A(n)}(A(n)//A(n-1) \otimes N_\sigma, \mathbb{F}_2)$ and $\mathrm{Ext}^{s,t}_{A(n)_*}(\mathbb{F}_2,\mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{A(n)}(\mathbb{F}_2,\mathbb{F}_2)$, which identify the two spectral sequences.

The case with coefficients in a module M arises in the same way, from the short exact sequences

$$0 \to \mathrm{im}(\partial_{\sigma+1}) \otimes M \to A(n)//A(n-1) \otimes N_\sigma \otimes M \to \mathrm{im}(\partial_\sigma) \otimes M \to 0$$

of left $A(n)$-modules. □

Remark 8.18. ((Added April 25th 2012)) The Davis–Mahowald resolution for $n = 2$ may be closely related to the resolution coming from the Amitsur complex for $tmf \to ko$, meaning the cosimplicial commutative S-algebra

$$[k] \mapsto ko \wedge_{tmf} ko \wedge_{tmf} \cdots \wedge_{tmf} ko$$

with coface maps induced by the unit $tmf \to ko$ and codegeneracies induced by the multiplication $ko \wedge_{tmf} ko \to ko$. Its totalization is the completion of tmf along ko, which should be tmf again, since ko is connective with $\pi_0(ko) = \mathbb{Z}$. ((Explain $H_*(ko \wedge_{tmf} ko) \cong H_*(ko) \otimes_{H_*(tmf)} H_*(ko) = H_*(ko)[y_0, y_1, y_2]/(\sim)$ where $y_0^2 = \xi_1^8$, $y_1^2 = \bar{\xi}_2^4$ and $y_2^2 = \bar{\xi}_3^2$, with $\mathscr{A}_*$-coaction like in $E_2 \otimes R_2$. Probably the Amitsur complex gives a cobar type resolution, while $E_2 \otimes R_2$ is a minimal resolution.)) Similarly for $n = 1$, using $ko \to H\mathbb{Z}$.

8.3 Ext over $A(1)$, revisited

As a warm-up, we compute $\mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ using the Davis–Mahowald resolution.

Let $n = 1$. We have $R_1 = P(y_0, y_1)$ with $y_0 = d(\xi_1^2)$ and $y_1 = d(\bar{\xi}_2)$ in bidegrees $(\sigma, t) = (1, 2)$ and $(1, 3)$, respectively, with $y_1 \cdot Sq^1 = y_0$ and $(y_0^i y_1^j) Sq^1 = j \cdot y_0^{i+1} y_1^{j-1}$. Hence $R_1^\sigma = \mathbb{F}_2\{y_0^i y_1^j \mid i + j = \sigma\}$ is dual to $N_\sigma = \mathbb{F}_2\{a_{i,j} \mid i + j = \sigma\}$, where $a_{i,j}$ is dual to $y_0^i y_1^j$ of degree $2i + 3j$, and $Sq^1(a_{i,j}) = (j + 1) a_{i-1,j+1}$. Thus N_σ is a sum of free $A(0)$-modules on generators $a_{i,\sigma-i}$ for $0 < i \le \sigma$ with $i \equiv \sigma$ mod 2, plus a trivial $A(0)$-module on the generator $a_{0,\sigma}$ in the cases when σ is even.

$$
\begin{array}{llll}
N_0 : & a_{0,0} & & \\[2em]
N_1 : & a_{1,0} \xrightarrow{Sq^1} a_{0,1} & & \\[2em]
N_2 : & a_{2,0} \longrightarrow a_{1,1} & a_{0,2} & \\[2em]
N_3 : & a_{3,0} \longrightarrow a_{2,1} & a_{1,2} \longrightarrow a_{0,3} & \\[2em]
N_4 : & a_{4,0} \longrightarrow a_{3,1} & a_{2,2} \longrightarrow a_{1,3} & a_{0,4}
\end{array}
$$

Thus $\mathrm{Ext}^{*,*}_{A(0)}(N_\sigma, \mathbb{F}_2)$ is the sum of a copy of $\mathbb{F}_2$ on the generator $y_0^i y_1^{\sigma-i}$ in internal degree $t = 3\sigma - i$ dual to $a_{i,\sigma-i}$, for each $0 < i \le \sigma$ with $i \equiv \sigma$ mod 2, plus a copy of $\mathrm{Ext}^{*,*}_{A(0)}(\mathbb{F}_2, \mathbb{F}_2) = P(h_0)$ on the generator y_1^σ in internal degree $t = 3\sigma$ dual to $a_{0,\sigma}$, in the cases where σ is even.

The E_1-term is displayed in Figure 9 as an Adams chart in the $(t-s, s)$-plane. Vertical lines indicate h_0-multiplications, and the σ-filtration is indicated at the bottom of each h_0-tower.

The d_1-differentials $d_1 \colon E^{\sigma,s,t} \to E^{\sigma+1,s+1,t}$ are generated by $d_1(y_0) = 0$ and $d_1(y_1^2) = y_0^3$. This leaves the E_2-term shown in Figure 10.

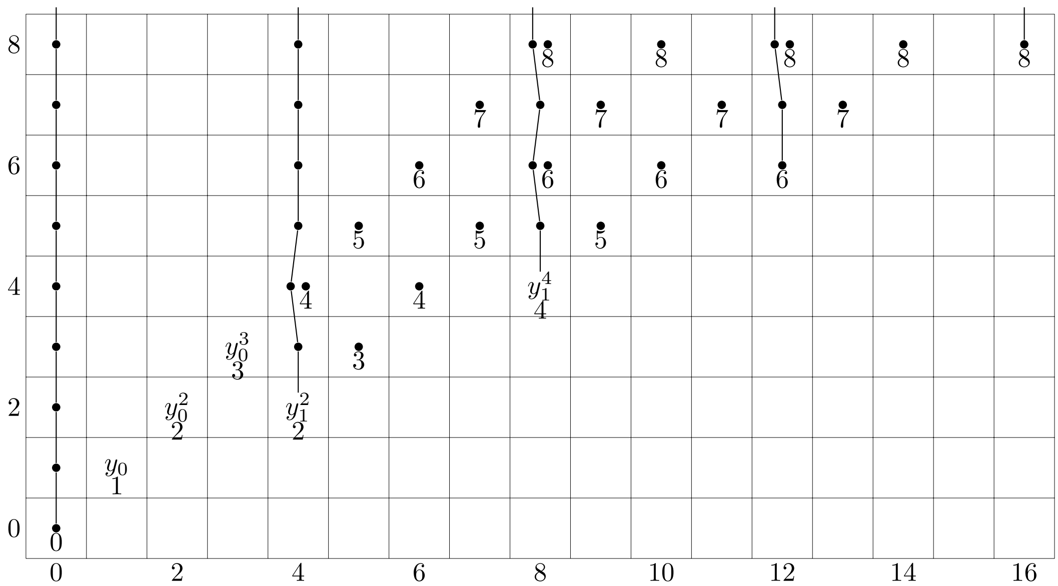


Figure 9: $E_1^{*,*,*} \Longrightarrow \mathrm{Ext}_{A(1)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$

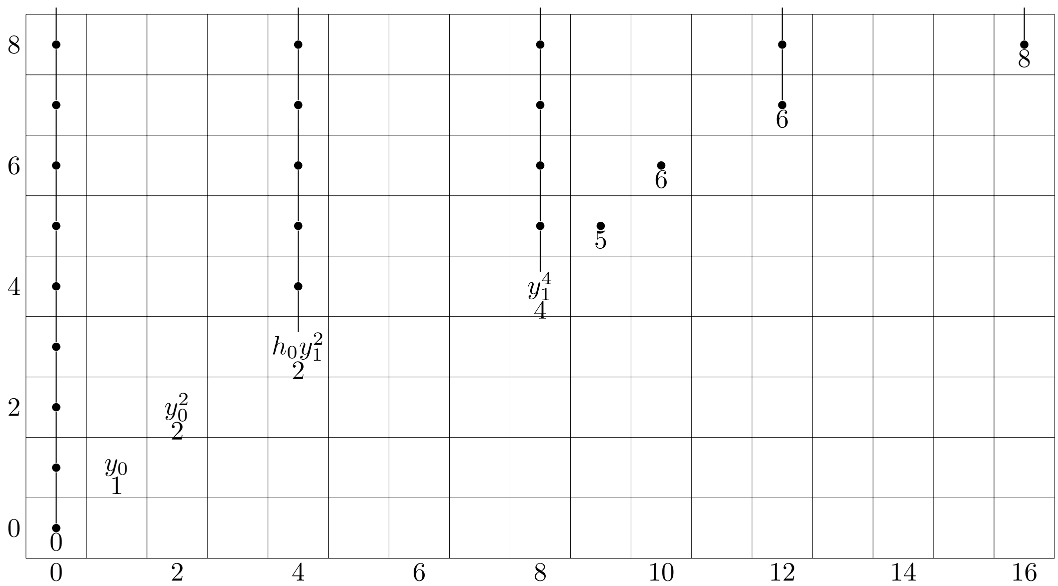


Figure 10: $E_2^{*,*,*} = E_\infty^{*,*,*}$

There is no room for further differentials, since $d_r\colon E_r^{\sigma,s,t} \to E_r^{\sigma+r,s+r,t}$ increases the σ-filtration by r. It follows that $\mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$ has the following algebra generators

gen.	$(t-s, s)$	rep.
h_0	$(0,1)$	h_0
h_1	$(1,1)$	y_0
v	$(3,4)$	$h_0 y_1^2$
w_1	$(8,4)$	y_1^4

that are subject to the relations $h_0h_1 = 0$, $h_1^3 = 0$, $h_1 v = 0$ and $v^2 = h_0^2 w_1$. In particular, it is free as a $P(w_1)$-module.

To find the differential, recall that the differential $d_1\colon E_1^{\sigma,s,t} \to E_1^{\sigma+1,s+1,t}$ is the composite homomorphism

$$\mathrm{Ext}^{s-\sigma,t}_{A(n)}(A(n)//A(n-1) \otimes N_\sigma, \mathbb{F}_2) \to \mathrm{Ext}^{s-\sigma,t}_{A(n)}(\mathrm{im}(\partial_{\sigma+1}), \mathbb{F}_2) \to \mathrm{Ext}^{s-\sigma,t}_{A(n)}(A(n)//A(n-1) \otimes N_{\sigma+1}, \mathbb{F}_2)$$

induced by the composite $A(n)$-module homomorphism

$$\partial_{\sigma+1}\colon A(n)//A(n-1) \otimes N_{\sigma+1} \twoheadrightarrow \mathrm{im}(\partial_{\sigma+1}) \rightarrowtail A(n)//A(n-1) \otimes N_\sigma\,.$$

In the case $\sigma = s$, $\mathrm{Ext}^{0,*}_{A(n)}(A(n)//A(n-1) \otimes N_\sigma, \mathbb{F}_2) \cong \mathrm{Hom}_{A(n)}(A(n)//A(n-1) \otimes N_\sigma, \mathbb{F}_2)$ is the subspace of $(A(n)//A(n-1) \otimes N_\sigma)^* \cong E_n \otimes R_n^\sigma$ where the right $A(n)$-module action is trivial (factors through the augmentation). This is the same as the subspace of left $A(n)_*$-comodule primitives. Hence the d_1-differential is given by the restriction of the composite

$$d\colon E_n \otimes R_n^\sigma \twoheadrightarrow \mathrm{im}(d) \rightarrowtail E_n \otimes R_n^{\sigma+1}$$

to the subspaces of $A(n)_*$-comodule primitives.

Example 8.19. For $n = 1$ and $\sigma = s = 2$, the class y_1^2 is represented by the $A(1)_*$-comodule primitive $y_1^2 + x_0 y_0^2$ in $E_1 \otimes R_1^2$. Hence $d_1(y_1^2)$ is represented by $d(y_1^2 + x_0y_0^2) = y_0^3$.

The commutative S-algebra maps $S \to ko \to ku$ induce surjections $\mathscr{A}//E(1) \to \mathscr{A}//A(1) \to \mathbb{F}_2$ in cohomology and restriction homomorphisms

$$\mathrm{Ext}^{*,*}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2)$$

of Adams E_2-terms. The classes h_0 and h_1 in the Adams spectral sequence for S, detecting 2 and η, map to the generators with the same names in $\mathrm{Ext}^{*,*}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$. The classes v and w_1 map to $v_0 v_1^2$ and v_1^4, respectively, in $\mathrm{Ext}^{*,*}_{E(1)}(\mathbb{F}_2, \mathbb{F}_2) = P(v_0, v_1)$.

8.4 Ext over $A(2)$

Let $n = 2$. We wish to calculate $\mathrm{Ext}^{*,*}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ using the Davis–Mahowald spectral sequence

$$E_1^{\sigma,s,t} = \mathrm{Ext}^{s-\sigma,t}_{A(1)}(N_\sigma \otimes M, \mathbb{F}_2) \Longrightarrow \mathrm{Ext}^{s,t}_{A(2)}(M, \mathbb{F}_2)$$

for $M = \mathbb{F}_2$, where $N_\sigma = (R_2^\sigma)^*$.

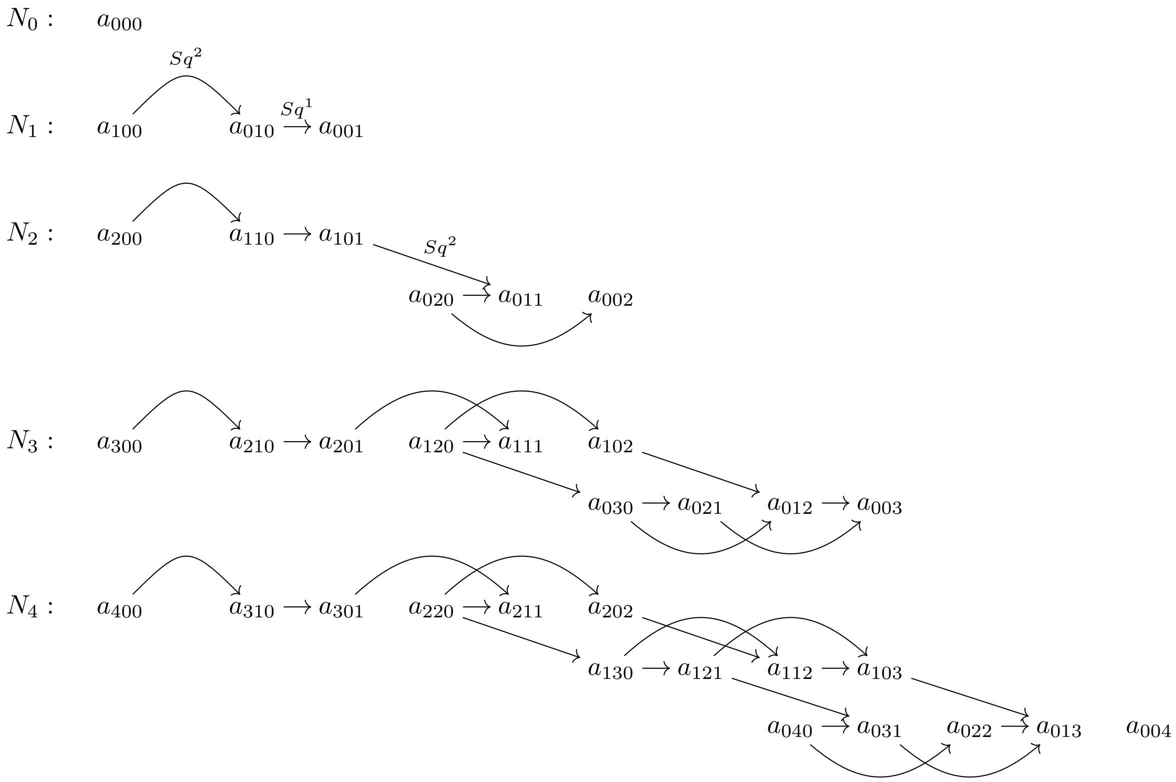
We have $R_2 = P(y_0, y_1, y_2)$ with $y_0 = d(\xi_1^4)$, $y_1 = d(\bar\xi_2^2)$ and $y_2 = d(\bar\xi_3)$ in bidegrees $(1,4)$, $(1,6)$ and $(1,7)$, respectively, with $Sq^1_*(y_2) = y_2 \cdot Sq^1 = y_1$ and $Sq^2_*(y_1) = y_1 \cdot Sq^2 = y_0$. Hence

$$\begin{aligned}(y_0^i y_1^j y_2^k) \cdot Sq^1 &= k \cdot y_0^i y_1^{j+1} y_2^{k-1} \\ (y_0^i y_1^j y_2^k) \cdot Sq^2 &= j \cdot y_0^{i+1} y_1^{j-1} y_2^k + \binom{k}{2} y_0^i y_1^{j+2} y_2^{k-2}\,.\end{aligned}$$

The $y_0^i y_1^j y_2^k$ with $i+j+k=\sigma$ give a basis for R_2^σ. Let $a_{i,j,k}$ of degree $4i+6j+7k$ be the dual basis element for N_σ. The left $A(1)$-module structure on N_σ is given by

$$\begin{aligned} Sq^1(a_{i,j,k}) &= (k+1)a_{i,j-1,k+1} \\ Sq^2(a_{i,j,k}) &= (j+1)a_{i-1,j+1,k} + \binom{k+2}{2} a_{i,j-2,k+2}\,. \end{aligned}$$

Here are the first few instances, where we abbreviate $a_{i,j,k}$ to a_{ijk}:



In particular, $N_0 = \mathbb{F}_2$ so that $\mathscr{A} \otimes_{A(1)} N_0 = H^*(ko)$, and $N_1 = \Sigma^4(A(1)/A(1)\{Sq^1, Sq^2Sq^3\})$ so that $\mathscr{A} \otimes_{A(1)} N_1 \cong H^*(bspin)$.

Notice that $a_{0,0,4}$ is left $A(1)$-module indecomposable. Dually, y_2^4 is left $A(1)_*$-comodule primitive. The same applies to $a_{0,0,k}$ and y_2^k for $k \equiv 0 \mod 4$, since R_2 is a left $A(1)_*$-comodule algebra (or by the formulas above).

Let $'R_2^\sigma \subset R_2^\sigma$ be the subspace generated by the $y_0^i y_1^j y_2^k$ with $0 \le k \le 3$ (and $i+j+k=\sigma$). Then

$$R_2 \cong \bigoplus_{\sigma \ge 0} {'R_2^\sigma} \otimes P(y_2^4)$$

as bigraded left $A(1)_*$-comodules, where y_2^4 has bidegree $(\sigma, t) = (4, 28)$. In filtration σ we get

$$R_2^\sigma = \bigoplus_{\substack{0 \le i \le \sigma \\ i \equiv \sigma \mod 4}} {'R_2^i}\{y_2^{\sigma - i}\} \cong \bigoplus_{\substack{0 \le i \le \sigma \\ i \equiv \sigma \mod 4}} \Sigma^{7(\sigma-i)}\,{'R_2^i}\,.$$

Here is the dual statement:

Lemma 8.20. *Let $N'_\sigma = N_\sigma / \mathbb{F}_2\{a_{i,j,k} \mid k \ge 4\}$ be the quotient space generated by $a_{i,j,k}$ with $0 \le k \le 3$ (and $i+j+k=\sigma$). Then*

$$N_\sigma \cong \bigoplus_{\substack{0 \le i \le \sigma \\ i \equiv \sigma \mod 4}} \Sigma^{7(\sigma-i)} N'_i$$

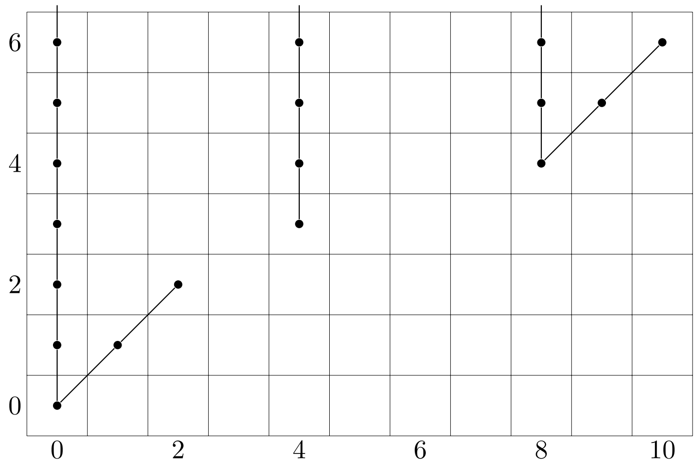


Figure 11: G_0, the Adams chart for ko

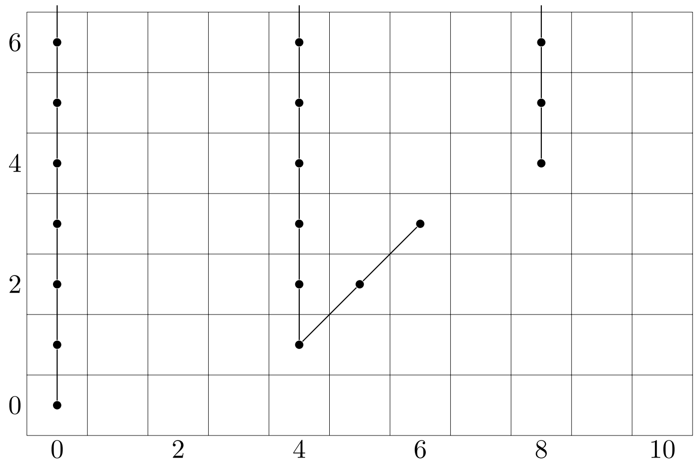


Figure 12: G_1, the Adams chart for $ksp = \Sigma^{-4} bspin$

as a left $A(1)$-module. Hence

$$\mathrm{Ext}^{*,*}_{A(1)}(N_\sigma \otimes M, \mathbb{F}_2) \cong \bigoplus_{\substack{0 \le i \le \sigma \\ i \equiv \sigma \mod 4}} \mathrm{Ext}^{*,*}_{A(1)}(\Sigma^{7(\sigma - i)} N_i' \otimes M, \mathbb{F}_2)\,.$$

Definition 8.21. For $i \geq 0$, let G_i be the following Adams chart, with lines indicating h_0- and h_1-multiplications. Each chart is free as a $P(v_1^4)$-module. Let $\Sigma^t G_i$ be the same chart as G_i, but shifted t units to the right.

Proposition 8.22. $\mathrm{Ext}^{*,*}_{A(1)}(N_\sigma', \mathbb{F}_2) = \Sigma^{4\sigma} G_\sigma$ *for each* $\sigma \geq 0$, *so*

$$\mathrm{Ext}^{*,*}_{A(1)}(N_\sigma, \mathbb{F}_2) = \bigoplus_{\substack{0 \le i \le \sigma \\ i \equiv \sigma \mod 4}} \Sigma^{7\sigma - 3i} G_i\,.$$

Proof. This is verified directly for $0 \le \sigma \le 2$. For $\sigma = 0$ we have $N_0' = N_0 = \mathbb{F}_2$ and $\mathscr{A} \otimes_{A(1)} N_0 \cong H^*(ko)$, so G_0 is the same as the Adams chart for ko. For $\sigma = 1$ we have $N_1' = N_1 = \Sigma^4 A(1)/A(1)\{Sq^1, Sq^2Sq^3\}$ so $\mathscr{A} \otimes_{A(1)} N_1 \cong H^*(bspin)$ and $\Sigma^4 G_1$ is the same as the Adams chart for $bspin$. Both of these are well known to be v_1^4-periodic. For $\sigma = 2$ we can write $N_2' = N_2$ as an extension

$$0 \to \Sigma^{12} A(1)//E(Q_1) \to N_2 \to \Sigma^8 A(1)//A(0) \to 0\,,$$

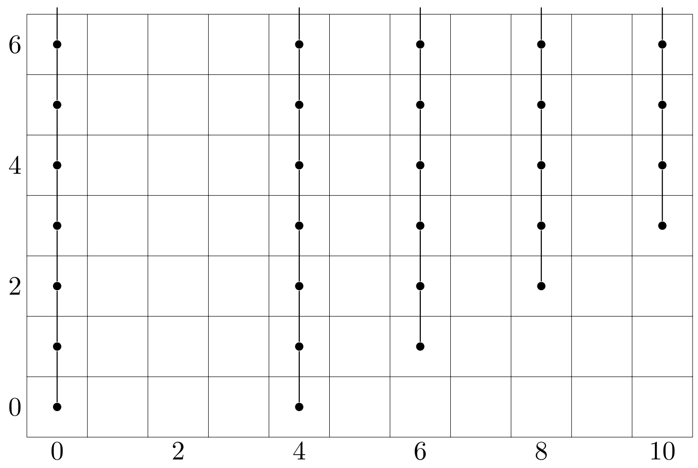


Figure 13: G_2

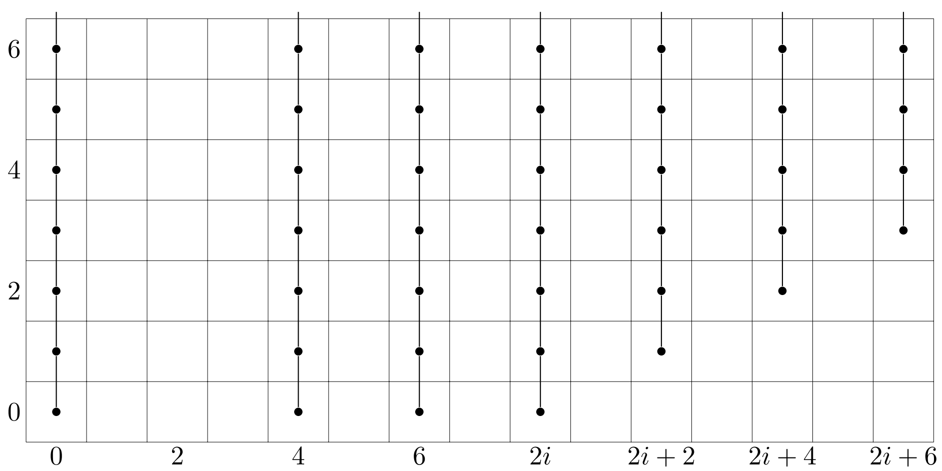


Figure 14: G_i for $i \geq 2$

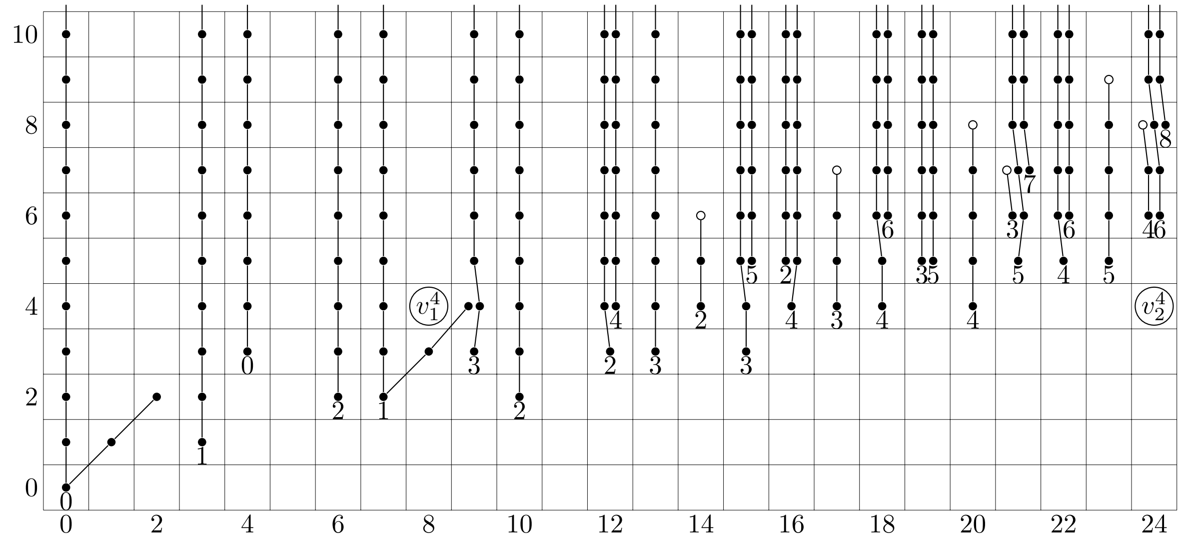


Figure 15: $P(v_1^4, v_2^4)$-basis for $E_1^{*,*,*} \Longrightarrow \mathrm{Ext}^{*,*}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$, $0 \le t - s \le 24$

so G_2 sits in a long exact sequence with the Adams charts for $H\mathbb{Z}$ and $\Sigma^4 ku/2$ (since $H^*(H\mathbb{Z}) = \mathscr{A}//A(0)$ and $H^*(ku/2) = \mathscr{A}//E(Q_1)$). The connecting homomorphism is trivial for bidegree reasons, so G_2 is additively the sum of these two charts. One only needs to check that the v_1^4-multiplication from bidegree $(0,0)$ is nonzero.

For $\sigma \ge 3$ there is an extension

$$0 \to \Sigma^{6\sigma} A(1)//E(Q_1) \longrightarrow N'_\sigma \longrightarrow \Sigma^4 N'_{\sigma-1} \to 0\,.$$

The submodule on the left is generated by $a_{0,j,k}$ for $j + k = \sigma$ and $0 \le k \le 3$. The projection to the quotient takes $a_{i,j,k}$ to $\Sigma^4 a_{i-1,j,k}$, for $i + j + k = \sigma$, $i \ge 1$ and $0 \le k \le 3$

The associated long exact sequence in Ext over $A(1)$ is

$$\cdots \to \Sigma^{4\sigma} G_{\sigma-1} \to \mathrm{Ext}^{*,*}_{A(1)}(N'_\sigma, \mathbb{F}_2) \to \Sigma^{6\sigma} P(v_1) \to \dots\,.$$

Here $\mathrm{Ext}^{*,*}_{A(1)}(\Sigma^{6\sigma} A(1)//E(Q_1), \mathbb{F}_2) \cong \Sigma^{6\sigma} \mathrm{Ext}^{*,*}_{E(Q_1)}(\mathbb{F}_2, \mathbb{F}_2) = \Sigma^{6\sigma} P(v_1)$. The sequence splits additively, for degree reasons, but there are nonzero h_0-extensions. ((Should discuss these.)) $\square$

((One should make the pairing $G_i \otimes G_j \to G_{i+j}$ explicit.))

Corollary 8.23. *There is an algebra spectral sequence*

$$E_1^{*,*,*} = P(v_2^4) \otimes \bigoplus_{i \ge 0} G_i\{h_2^i\} \Longrightarrow \mathrm{Ext}^{*,*}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$$

where $h_2^i v_2^{4k}$ has σ-filtration $i + 4k$ and bidegree $(t - s, s) = (3i + 24k, i + 4k)$, for $i, k \ge 0$.

The Davis–Mahowald E_1-term is displayed in degrees $0 \le t \le 48$ in Figures 15 and 16. It is free over $P(v_1^4, v_2^4)$, and only the generators are shown (as bullets), with the exception that v_1^4 times a generator is shown as a circle when it is also h_0 times a generator. This way the h_0-extensions are not hidden from the picture.

Theorem 8.24. *The classes h_0, h_1, v_1^4, h_2 and v_2^8 are infinite cycles. There are nonzero differentials*

$$d_1(\alpha_{2,0} h_2^2) = h_2^3$$
$$d_1(\alpha_{5,0} h_2^5) = \alpha_{3,0} h_2^6$$
$$d_1(v_2^4) = \alpha_{4,0} h_2^5$$

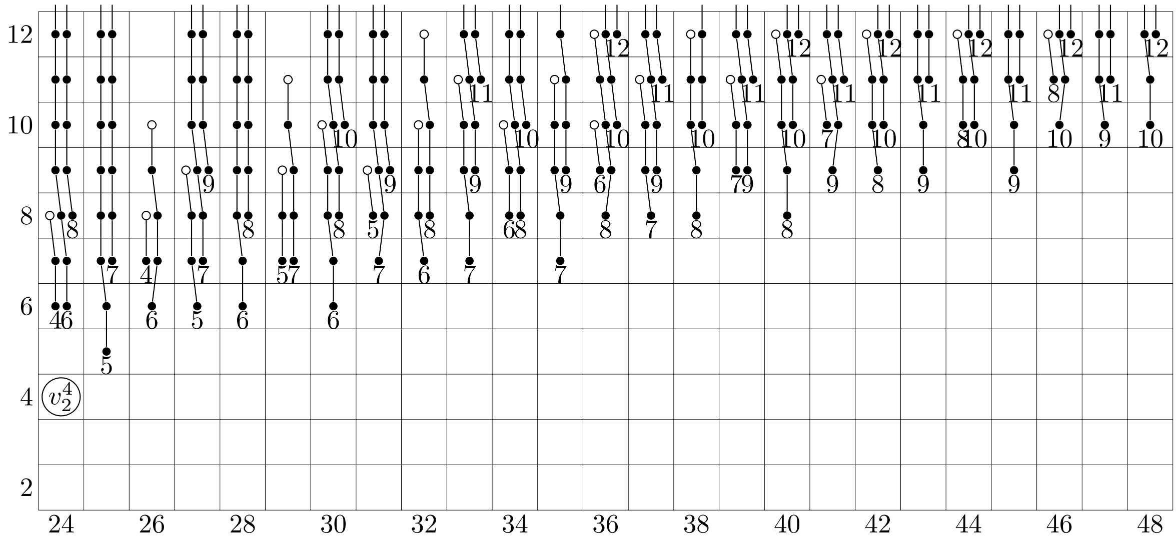


Figure 16: $P(v_1^4, v_2^4)$-basis for $E_1^{*,*,*} \Longrightarrow \mathrm{Ext}_{A(2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, $24 \le t - s \le 48$

where $\alpha_{k,s}h_2^i$ denotes a generator in bidegree $(2k+3i, s+i)$ of $G_i\{h_2^i\}$. The spectral sequence collapses at the E_2-term.

Proof. To determine the d_1-differential on classes in Adams filtration $s = \sigma$, we use the identification

$$E_1^{\sigma,\sigma,*} = \mathrm{Hom}_{A(1)}^*(N_\sigma, \mathbb{F}_2) \cong \mathrm{Hom}_{A(2)}^*(A(2)//A(1) \otimes N_\sigma, \mathbb{F}_2) \cong \mathbb{F}_2 \,\square_{A(2)_*}\, (E_2 \otimes R_2^\sigma)$$

of this part of the E_1-term with the left $A(2)_*$-comodule primitives on the right hand side. The differential $d_1\colon E_1^{\sigma,\sigma,*} \to E_1^{\sigma+1,\sigma+1,*}$ is then induced by the derivation

$$d\colon E_2 \otimes R_2^\sigma \to E_2 \otimes R_2^{\sigma+1}$$

by restriction to the left $A(2)_*$-comodule primitives. The formulas

$$\begin{aligned}
\lambda(x_0) &= 1 \otimes x_0 + \xi_1^4 \otimes 1 \\
\lambda(x_1) &= 1 \otimes x_1 + \xi_1^2 \otimes x_0 + \bar{\xi}_2^2 \otimes 1 \\
\lambda(x_2) &= 1 \otimes x_2 + \xi_1 \otimes x_1 + \bar{\xi}_2 \otimes x_0 + \bar{\xi}_3 \otimes 1 \\
\lambda(y_0) &= 1 \otimes y_0 \\
\lambda(y_1) &= 1 \otimes y_1 + \xi_1^2 \otimes y_0 \\
\lambda(y_2) &= 1 \otimes y_2 + \xi_1 \otimes y_1 + \bar{\xi}_2 \otimes y_0
\end{aligned}$$

are useful.

The generator $\alpha_{2,0}h_2^2$ in bidegree $(t-s, s) = (10, 2)$ is represented by the $A(1)_*$-comodule primitive y_1^2 in R_2^2, which corresponds to the $A(2)_*$-comodule primitive $y_1^2 + x_0y_0^2$ in $E_2 \otimes R_2^2$. The d_1-differential maps this to the $A(2)_*$-comodule primitive $d(y_1^2 + x_0y_0^2) = y_0^3$ in $E_2 \otimes R_2^3$, which represents h_2^3.

The generator $\alpha_{5,0}h_2^5$ in bidegree $(25, 5)$ is represented by the $A(1)_*$-comodule primitive $y_1^5 + y_0y_1^2y_2^2$ in R_2^5, which corresponds to the $A(2)_*$-comodule primitive $y_1^5 + y_0y_1^2y_2^2 + x_0y_0^3y_2^2 + ((ETC))$ in $E_2 \otimes R_2^5$. The d_1-differential maps this to ((ETC)), which represents $\alpha_{3,0}h_2^6$.

((Exercise: Compute left $A(2)_*$-coaction in $E_2 \otimes R_2^5$ in internal degree 30 to find the $A(2)_*$-comodule primitive.))

When combined with h_0-, h_1-, h_2- and v_1^4-linearity, these two differentials imply many others. The reader might draw them in Figures 15 and 16. The result is shown in Figures 17 and 18.

Next we bring v_2^4 into the picture. It is represented by the $A(1)_*$-comodule primitive y_2^4, which corresponds to the $A(2)_*$-comodule primitive $y_2^4 + x_0y_1^4$. The d_1-differential takes this to the class

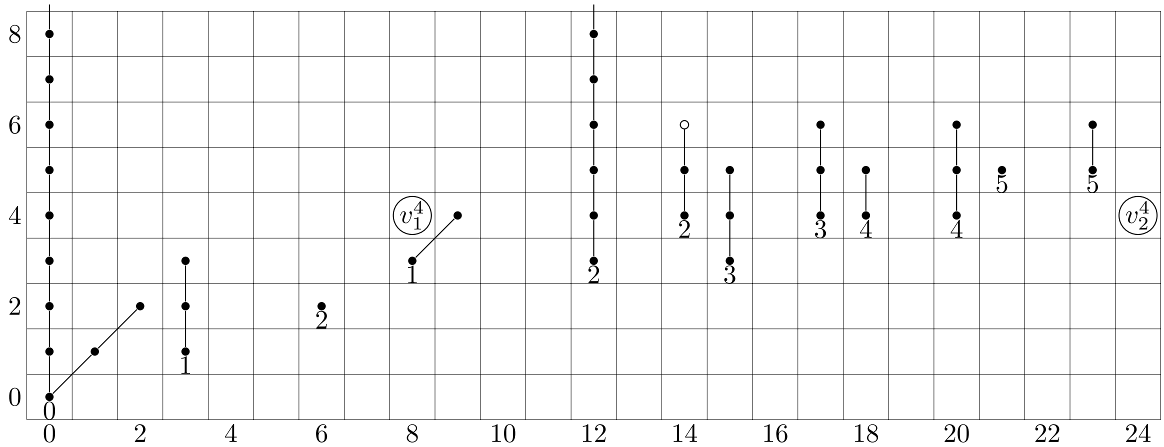


Figure 17: $E_1^{*,*,*}$ after first two d_1-differentials, $0 \leq t-s \leq 24$

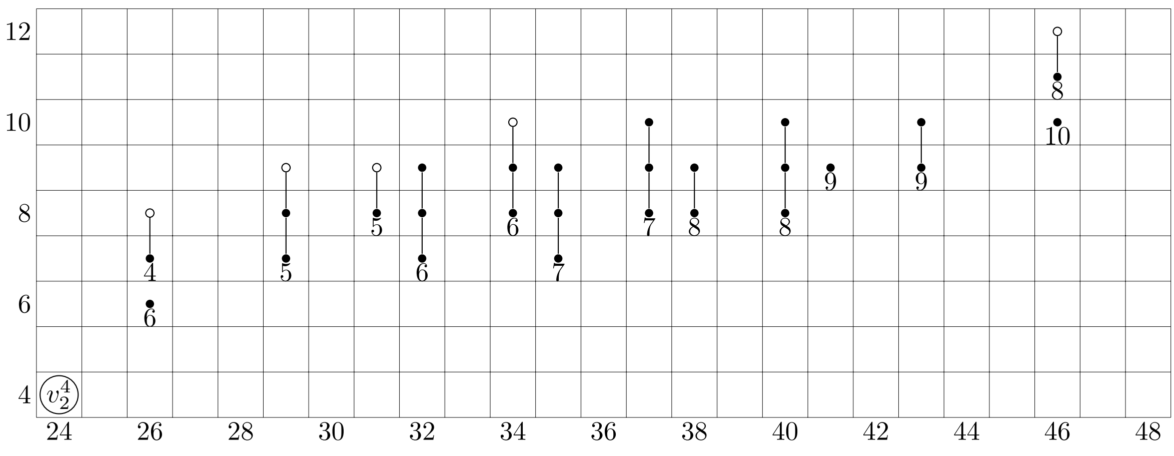


Figure 18: $E_1^{*,*,*}$ after first two d_1-differentials, $24 \leq t-s \leq 48$

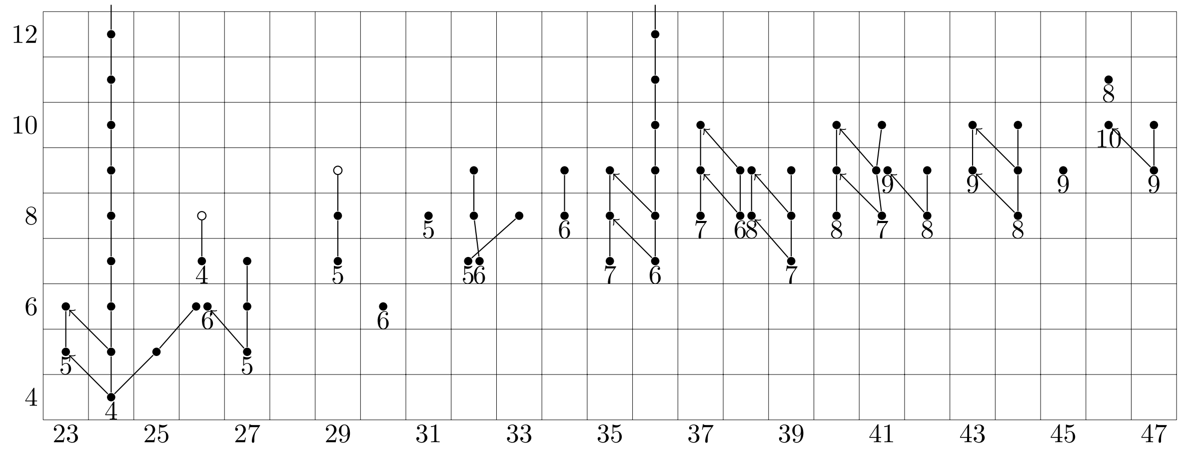


Figure 19: $E_1^{*,*,*}$ last d_1-differentials, $23 \le t-s \le 47$

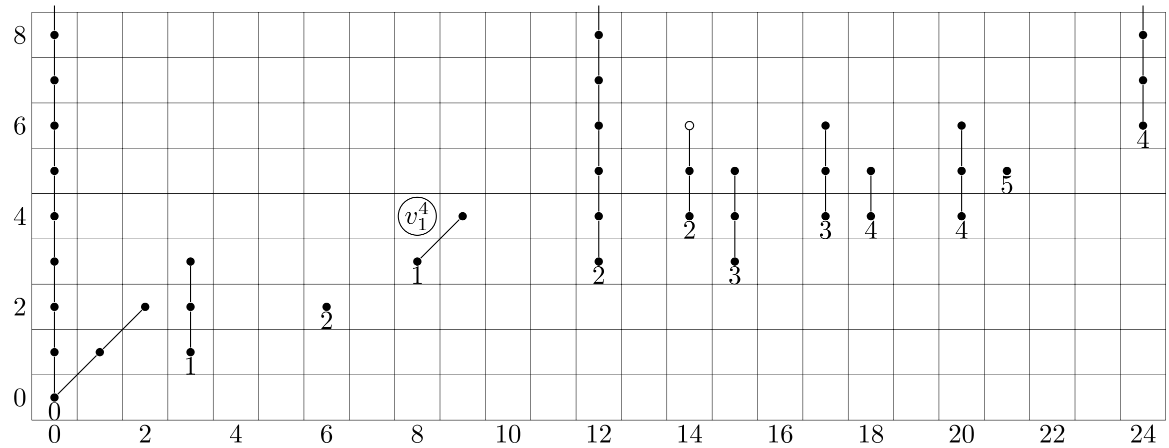


Figure 20: $E_\infty^{*,*,*} \Longrightarrow \mathrm{Ext}_{A(2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, $0 \le t-s \le 24$

represented by $d(y_2^4 + x_0 y_1^4) = y_0 y_1^4$, namely $\alpha_{4,0} h_2^5$. The further differentials implied by the multicative structure are illustrated in Figure 19, which is obtained by superimposing Figure 18 with a copy of Figure 17 shifted by v_2^4.

The remaining E_2-term is displayed in Figures 20 and 21. It is a free $P(v_1^4, v_2^8)$-module, and there is no room for further differentials, so $E_2 = E_\infty$.

$\square$

Remark 8.25. The wedge-shaped pattern that begins in bidegree $(t-s, s) = (35, 7)$ can be shown to continue. It is a free $P(v_1, h_{21})$-module, where $v_1 = h_{20}$ and h_{21} are detected by $Q_1 = Sq^{(0,1)}$ and $Sq^{(0,2)}$, dual to ξ_2 and ξ_2^2, respectively. A similar pattern in $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ was described by Mahowald and Tangora (Trans. Amer. Math. Soc., 1968).

Davis and Mahowald also determine the h_1- and h_2-multiplications in $\mathrm{Ext}_{A(2)}^{*,*}$ that are hidden by filtration shifts in the E_∞-term. These can also be determined by machine computation in this range, and lead to the charts in Figures 22 and 23. Sometimes v_1^4-multiples become h_0-divisible; this is indicated by the small circles. Remarkably, v_1^4-multiples never become more h_1- or h_2-divisible.

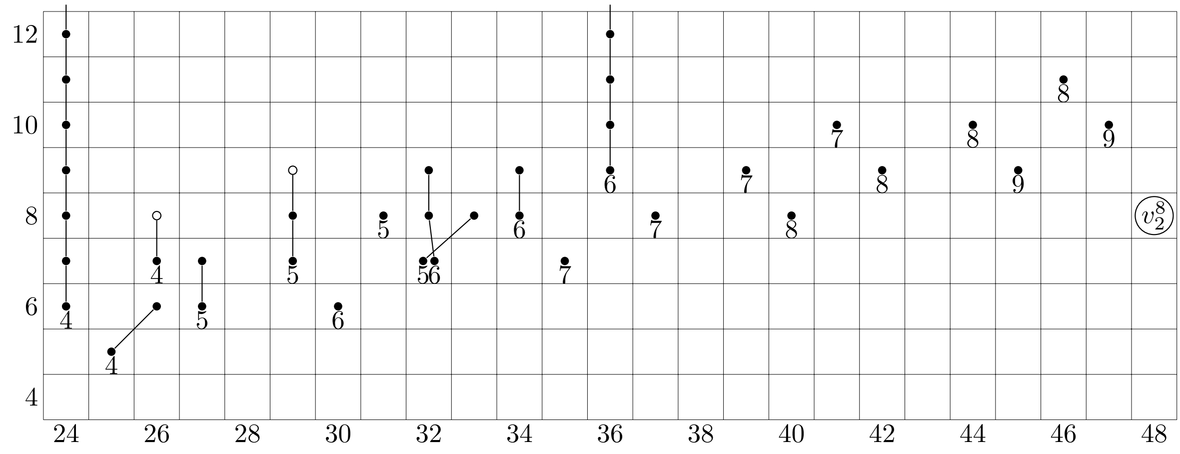


Figure 21: $E_\infty^{*,*,*} \Longrightarrow \mathrm{Ext}_{A(2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, $24 \le t-s \le 48$

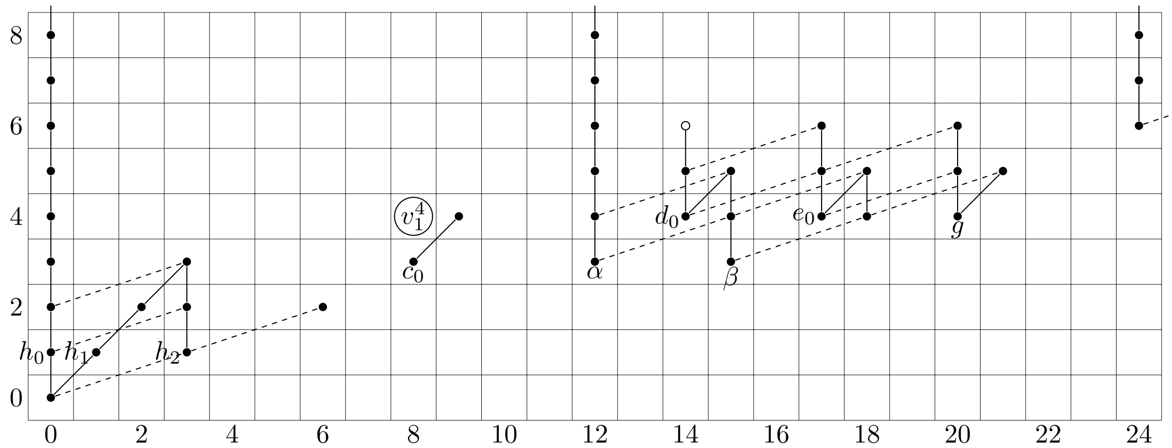


Figure 22: $P(v_1^4, v_2^8)$-basis for $\mathrm{Ext}_{A(2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, $0 \le t-s \le 24$

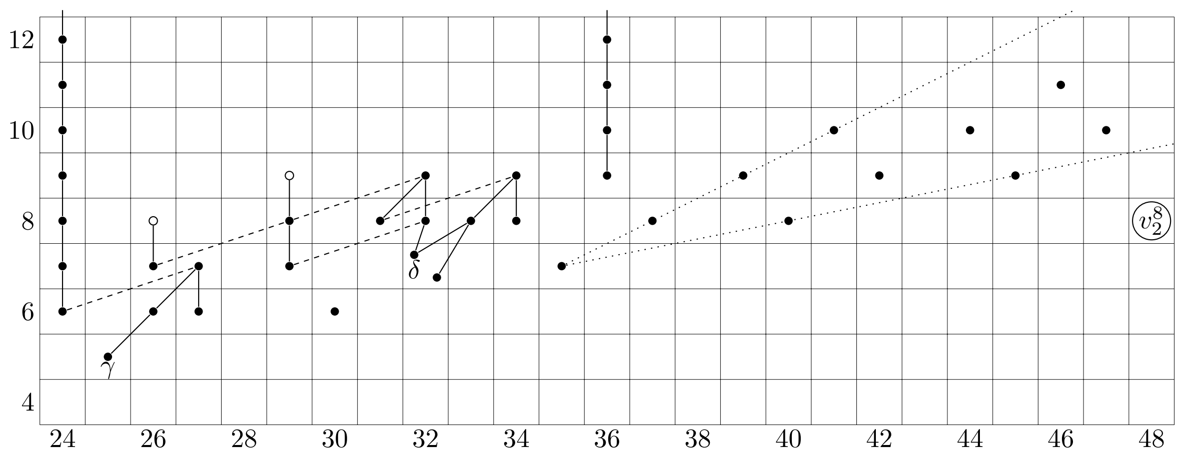


Figure 23: $P(v_1^4, v_2^8)$-basis for $\mathrm{Ext}_{A(2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, $24 \le t-s \le 48$

Definition 8.26. We name the following generators of $\mathrm{Ext}^{*,*}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$.

IS-gen.	$(t-s,s)$	alt.	DM-rep.	`ext`
h_0	$(0,1)$	h_0	h_0	1_0
h_1	$(1,1)$	h_1	h_1	1_1
h_2	$(3,1)$	h_2	y_0	1_2
ω_0	$(8,4)$	$w_1 = v_1^4$	w_1	4_1
ω_1	$(20,4)$	g	y_1^4	4_8
α_0	$(48,8)$	$w_2 = v_2^8$	y_2^8	8_{19}
α_1	$(8,3)$	c_0	(?)	3_2
α_2	$(12,3)$	α	(?)	3_3
α_3	$(15,3)$	β	(?)	3_4
α_4	$(14,4)$	d_0	(?)	4_4
α_5	$(17,4)$	e_0	(?)	4_6
α_6	$(25,5)$	γ	$h_1 v_2^4 + (?)$	5_{11}
α_7	$(32,7)$	δ	$c_0 v_2^4 + (?)$	7_{11}

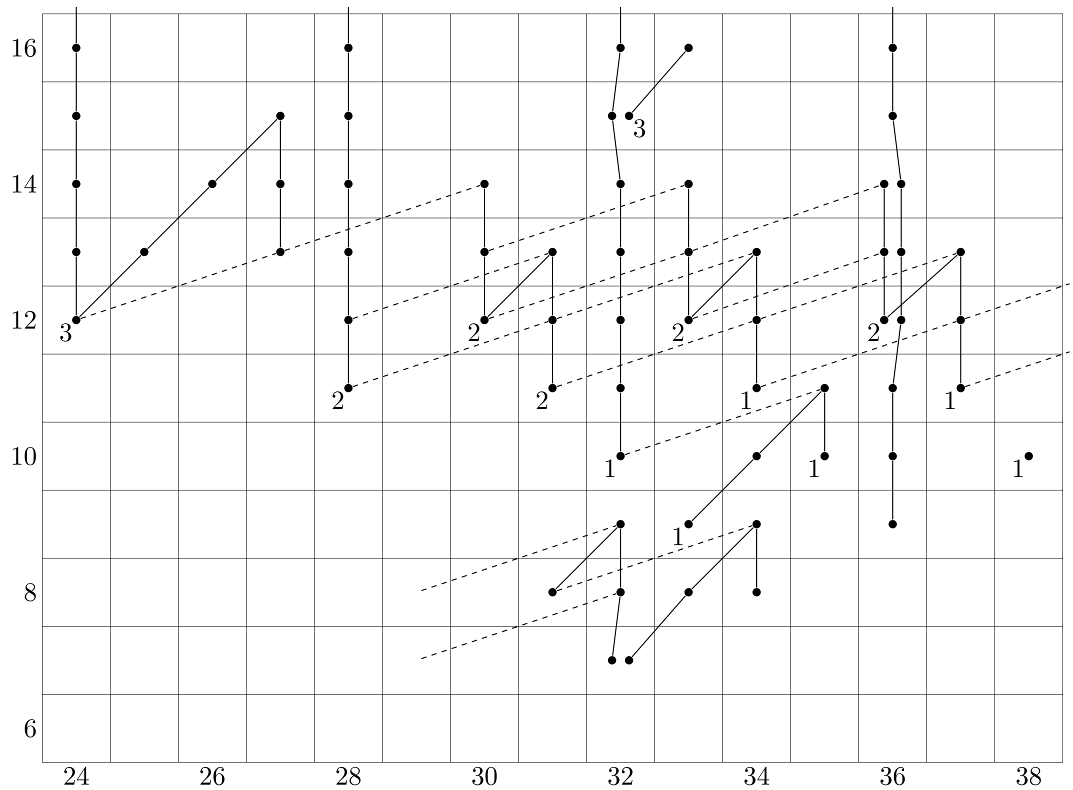
With the exception of $\alpha_7 = \delta$, each class is the unique nonzero class in its bidegree. The class $\alpha_7 = \delta$ is characterized by the properties $h_0\delta \neq 0$ and $h_1\delta \neq 0$. Bruner's `ext`-program uses the name s_g for the g'th generator in Adams filtration s, counting from $g = 0$.

Instead of displaying the module generators, Davis and Mahowald (1982) use the following convention to encode Adams charts that are free over $P(w_1) = P(v_1^4)$. ((They do not take h_2-multiples into account.))

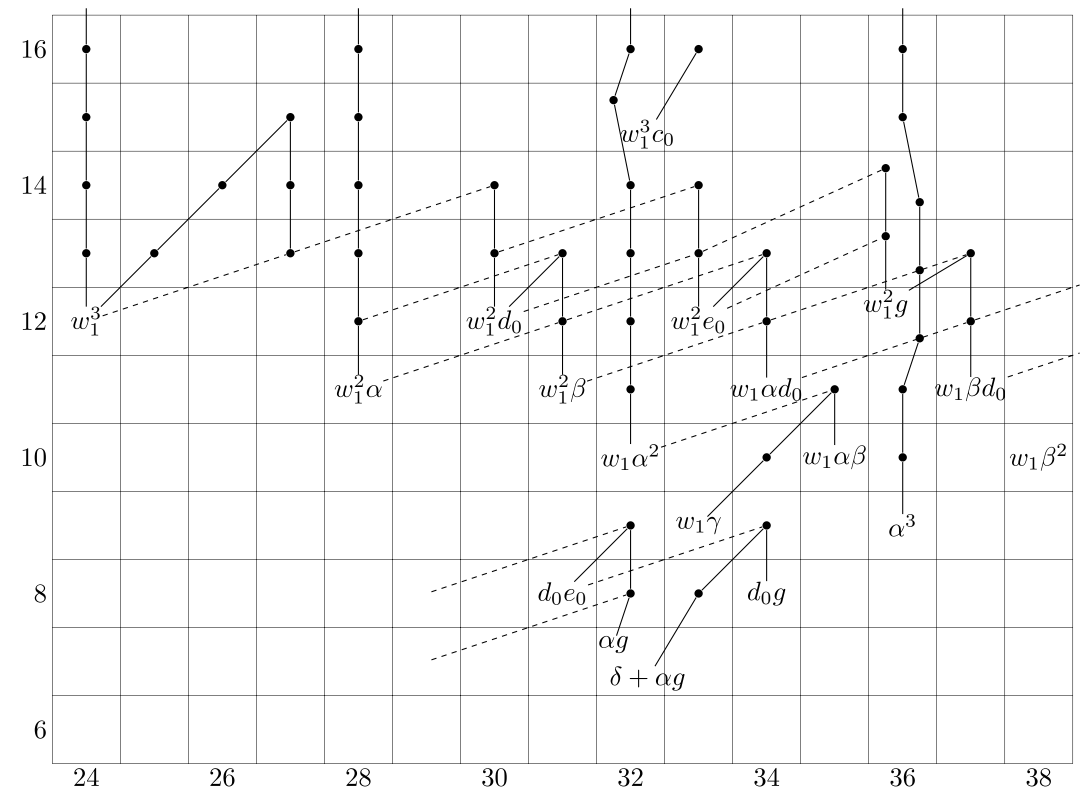
Definition 8.27. An indexed chart is a chart in which some elements x are labeled with integers $\ell(x)$. Each unlabeled element x is implicitly given the maximal label of a labeled element y such that $x = h_0^i h_1^j h_2^k y$, or 0 if no such y exists. Each indexed chart C generates an Adams chart $\langle C\rangle$, consisting of all elements $v_1^{4i}x$ such that $i + \ell(x) \geq 0$. In other words, each element x in C generates a free $P(v_1^4)$-module in $\langle C\rangle$ on a generator $v_1^{-4\ell(x)}x$.

((Use the modified chart, better suited for the tmf-differentials.))

Definition 8.28. Let E_0 be the following indexed chart:



Here is the same chart with named generators:



The dashed lines that exit the chart mean that h_2 times $w_1\beta d_0$ is w_1 times $h_0\alpha g = h_0\delta$, and similarly after multiplication by h_0.

Theorem 8.29. $\mathrm{Ext}_{A(2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ *is free over* $P(w_2) = P(v_2^8)$ *on* $\langle E_0 \rangle \oplus P(v_1, h_{21})\{g_{35,7}\}$*, where* $g_{35,7} = \beta g$.

This compact statement should be compared with the full Ext chart (in a finite range of degrees), as in Figure 8.

8.5 Coefficients in $A(0)$

The Adams E_2-term for the homotopy of $tmf/2 = tmf \wedge S/2$ is

$$E_2^{*,*} = \mathrm{Ext}_{\mathscr{A}}^{*,*}(H^*(tmf/2), \mathbb{F}_2) \cong \mathrm{Ext}_{A(2)}^{*,*}(A(0), \mathbb{F}_2)$$

since $H^*(tmf/2) \cong H^*(tmf) \otimes H^*(S/2) \cong \mathscr{A}//A(2) \otimes A(0) \cong \mathscr{A} \otimes_{A(2)} A(0)$, where $A(0)$ denotes the $\mathscr{A}$-module $H^*(S/2)$. It is, after all, free of rank 1 as an $A(0)$-module, and admits a unique $\mathscr{A}$-module structure. Note that $S/2$ is not a ring spectrum, and this is not an algebra spectral sequence, but it is a module spectral sequence over the Adams spectral sequence for tmf.

Computing $\mathrm{Ext}_{A(2)}^{*,*}(A(0), \mathbb{F}_2)$ will also be useful in proving Adams periodicity, saying that the part of $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ over a line of slope $1/5$ repeats periodically along lines (rays) of slope $1/2$.

We use the Davis–Mahowald spectral sequence

$$E_1^{\sigma,s,t} = \mathrm{Ext}_{A(1)}^{s-\sigma,t}(N_\sigma \otimes A(0), \mathbb{F}_2) \Longrightarrow \mathrm{Ext}_{A(2)}^{s,t}(A(0), \mathbb{F}_2)$$

for $M = A(0)$. It is not an algebra spectral sequence, since $A(0)$ is not an $A(2)$-comodule coalgebra, but it is a module spectral sequence over the Davis–Mahowald spectral sequence computing $\mathrm{Ext}_{A(2)}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.

Let $GA(0)_i$ be the chart so that $\mathrm{Ext}_{A(1)}^{*,*}(N_i' \otimes A(0), \mathbb{F}_2) = \Sigma^{4i} GA(0)_i$. Then

$$E_1^{*,*,*} = P(v_2^4) \otimes \bigoplus_{i \geq 0} GA(0)_i\{h_2^i\}\,.$$

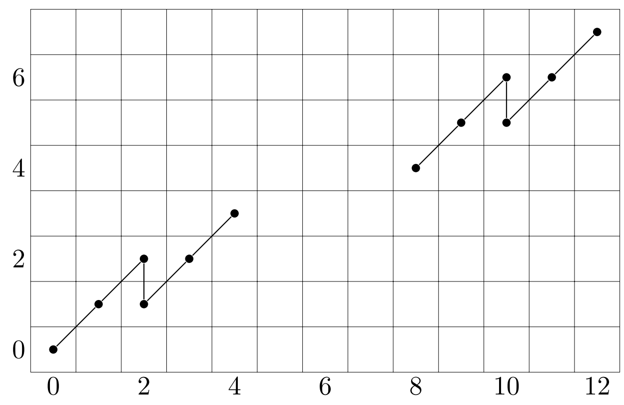


Figure 24: $GA(0)_0$, the Adams chart for $ko/2$

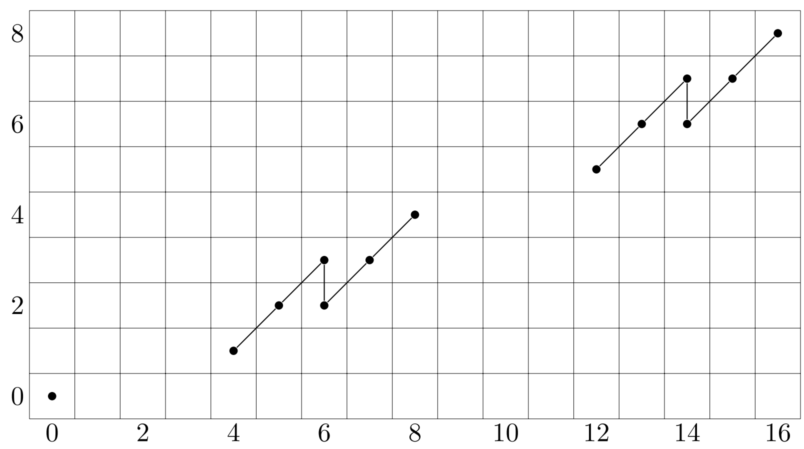


Figure 25: $GA(0)_1$, the Adams chart for $\Sigma^{-4}bspin/2$

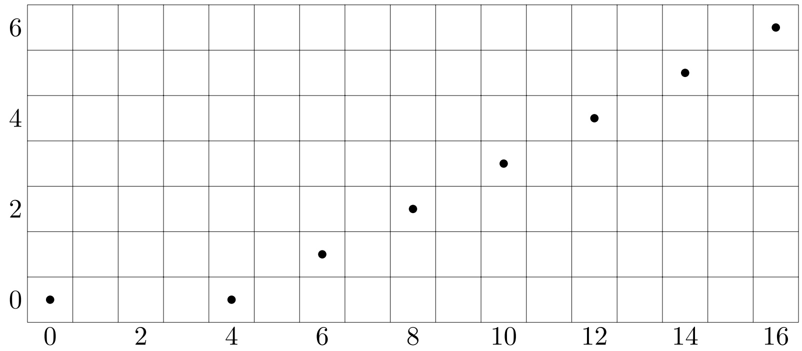


Figure 26: $GA(0)_2$

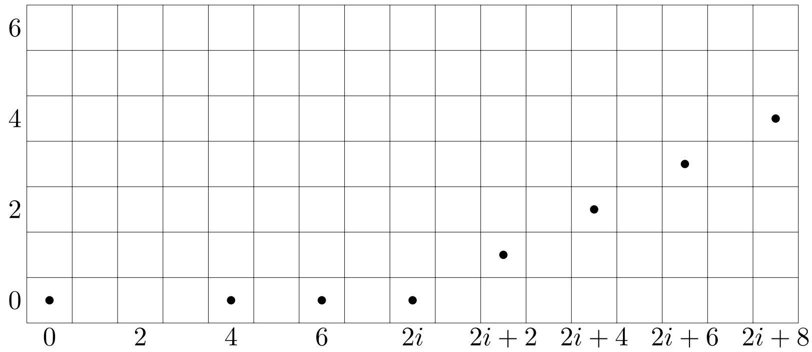


Figure 27: $GA(0)_i$ for $i \geq 2$

These charts can be readily computed. The first two are free as $P(w_1) = P(v_1^4)$-modules.

Thereafter there are $(i-1)$ v_1^4-torsion classes, before periodicity kicks in.

The Davis–Mahowald E_1-term for $A(0)$ as $A(2)$-module is displayed for $0 \leq t \leq 48$ in Figures 28 and 29. Most classes are only represented by their σ-filtration.

The augmentation $A(0) \to \mathbb{F}_2$ (corresponding to the map $tmf \to tmf/2$) induces a map of spectral sequences from the one computed in the previous subsection to this one. The differentials implied by $d_1(\alpha_{2,0}h_2^2) = h_2^3$ and $d_1(\alpha_{5,0}h_2^5) = \alpha_{3,0}h_2^6$ leave the classes displayed in Figures 30 and 31. Only the $P(v_1^4)$-module generators are shown. Most of them generate a free copy of $P(v_1^4)$, but some only generate a trivial module. The latter are labeled σ' in place of σ. The circle indicates a v_1^4-multiple that is h_1-divisible.

Superimposing Figure 31 with a copy of Figure 30 multiplied by v_2^4, and taking the differential $d_1(v_2^4) = \alpha_{4,0}h_2^5$ into account, we obtain Figure 32. The remaining E_2-term is shown in Figures 33 and 34. For σ-filtration reasons, this equals the E_∞-term.

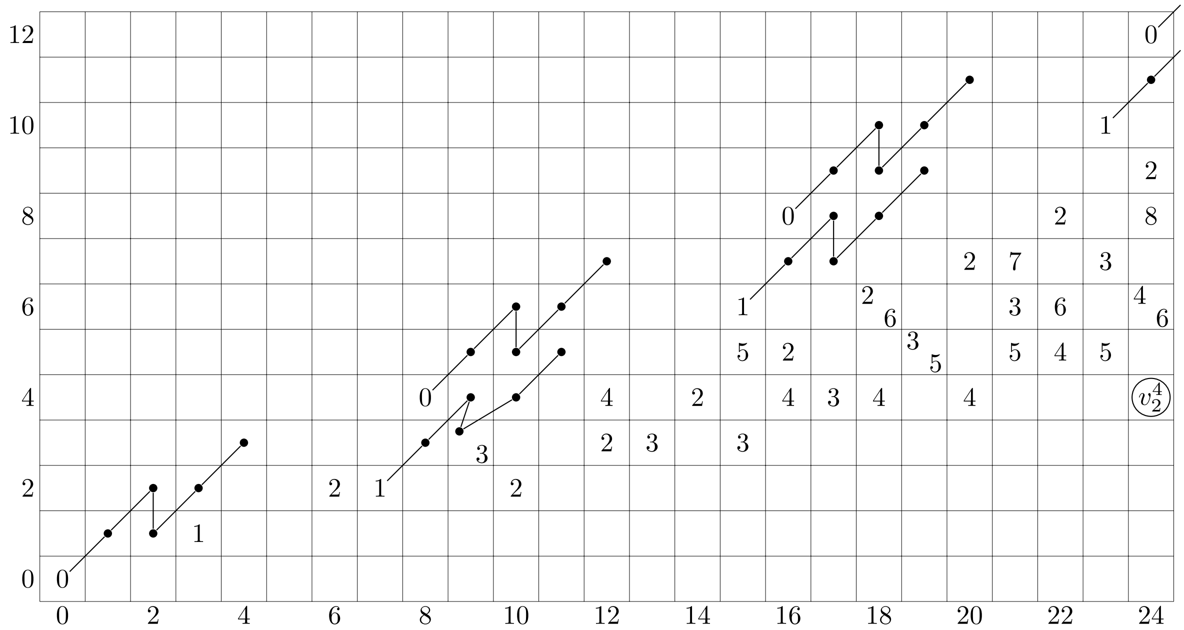


Figure 28: $P(v_2^4)$-basis for $E_1^{*,*,*} \Longrightarrow \mathrm{Ext}_{A(2)}^{*,*}(A(0), \mathbb{F}_2)$, $0 \leq t-s \leq 24$

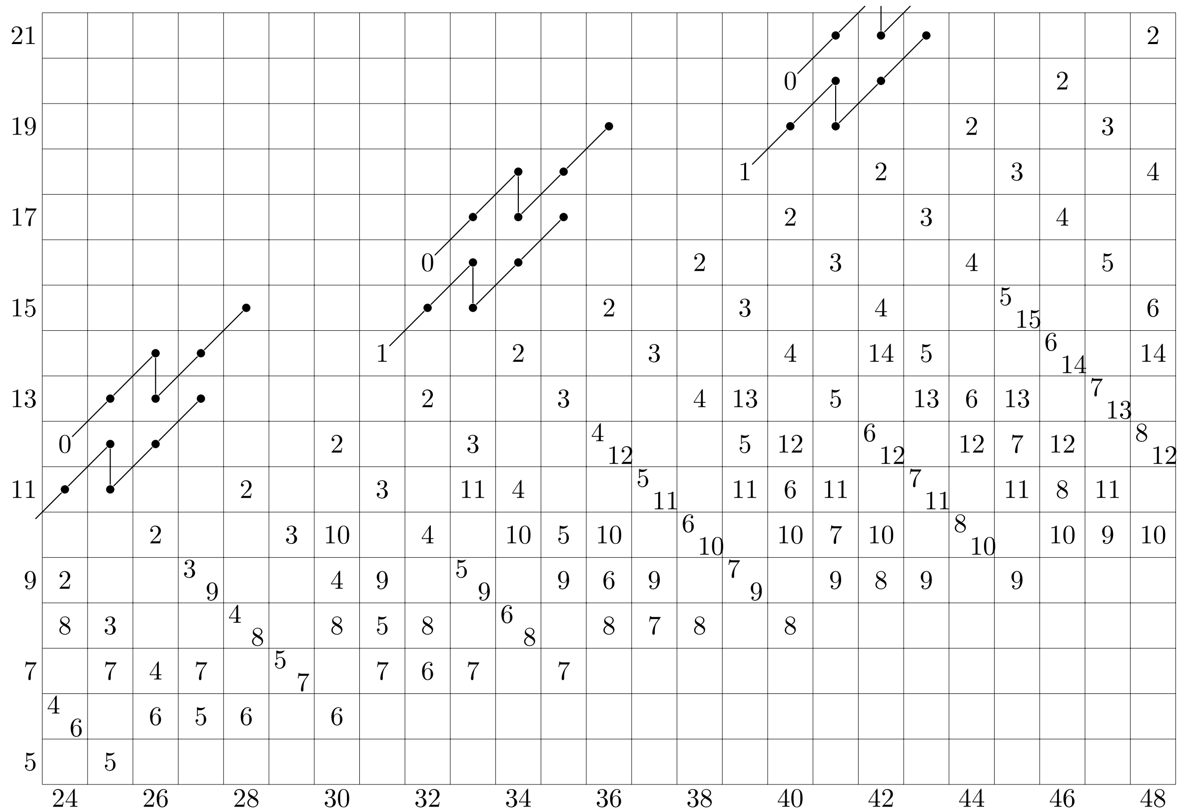


Figure 29: $P(v_2^4)$-basis for $E_1^{*,*,*} \Longrightarrow \mathrm{Ext}_{A(2)}^{*,*}(A(0), \mathbb{F}_2)$, $24 \leq t-s \leq 48$

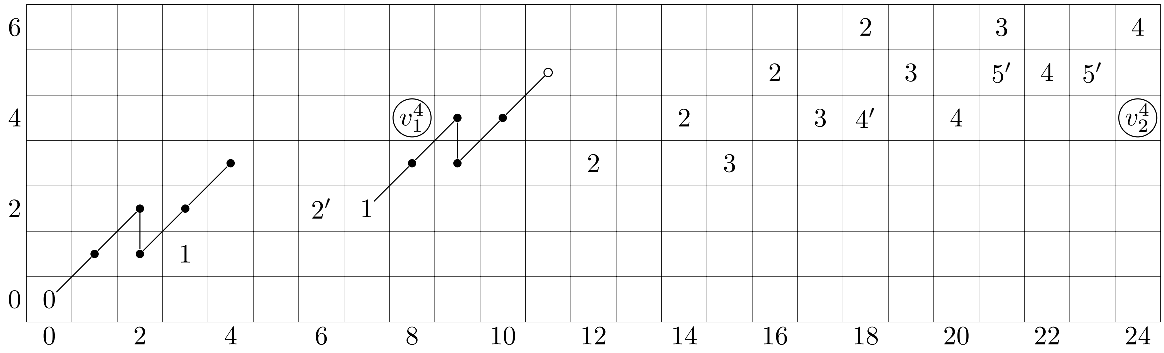


Figure 30: $P(v_1^4, v_2^4)$-generators for $E_1^{*,*,*}$ after first two differentials, $0 \leq t-s \leq 24$

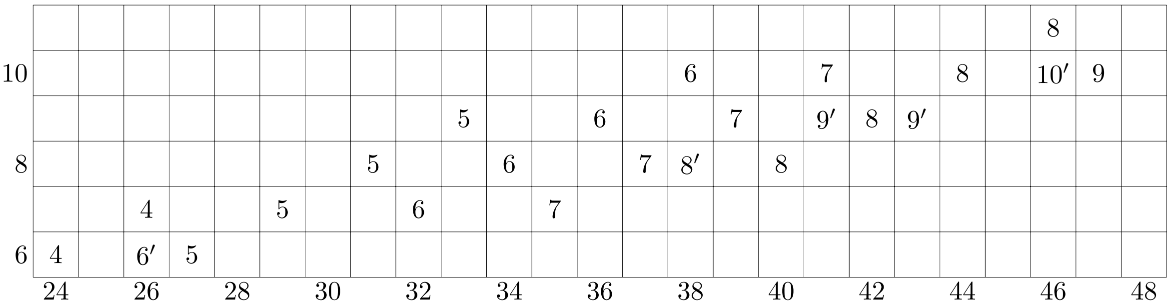


Figure 31: $P(v_1^4, v_2^4)$-generators for $E_1^{*,*,*}$ after first two differentials, $24 \leq t-s \leq 48$

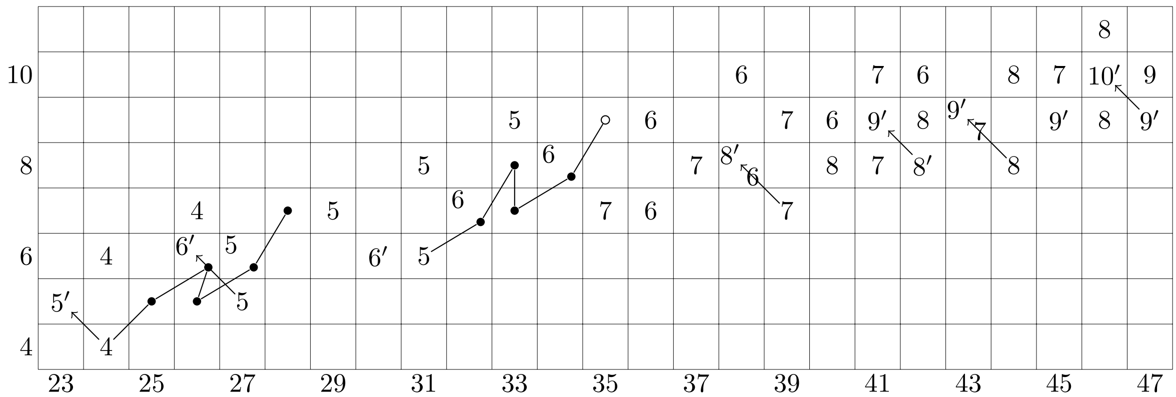


Figure 32: $E_1^{*,*,*}$ last differentials, $23 \leq t-s \leq 47$

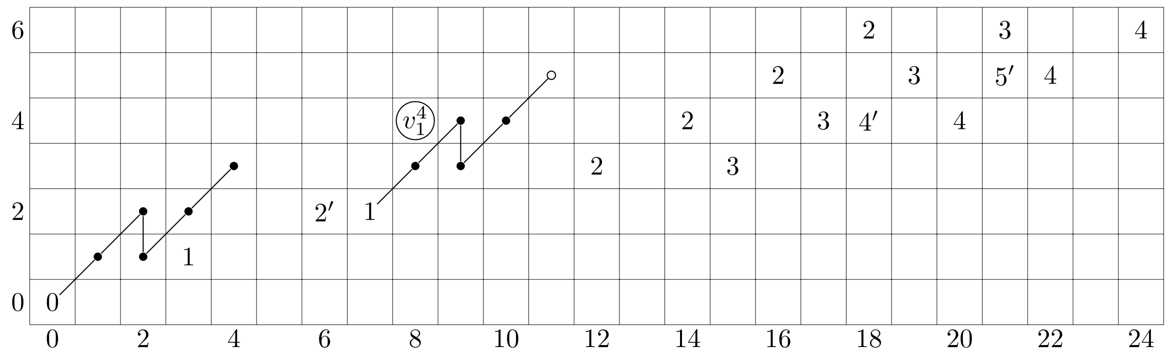


Figure 33: $P(v_1^4, v_2^8)$-generators for $E_\infty^{*,*,*} \Longrightarrow \mathrm{Ext}_{A(2)}^{*,*}(A(0), \mathbb{F}_2)$, $0 \le t-s \le 24$

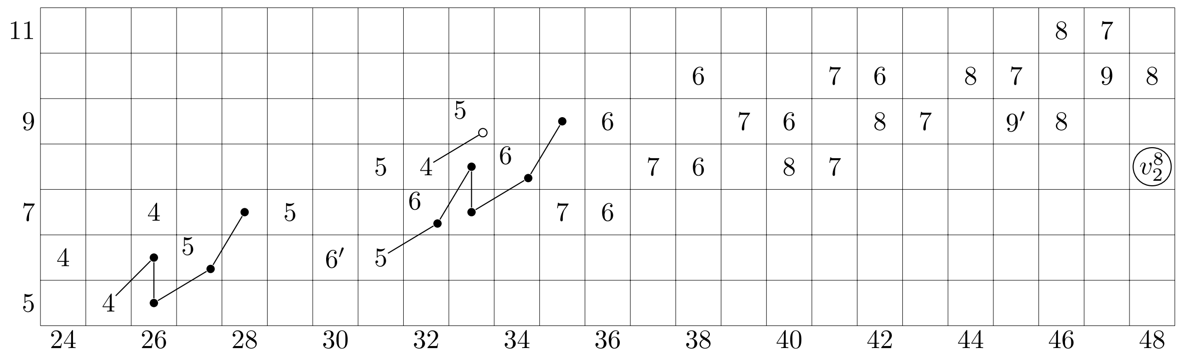


Figure 34: $P(v_1^4, v_2^8)$-generators for $E_\infty^{*,*,*} \Longrightarrow \mathrm{Ext}_{A(2)}^{*,*}(A(0), \mathbb{F}_2)$, $24 \le t-s \le 48$

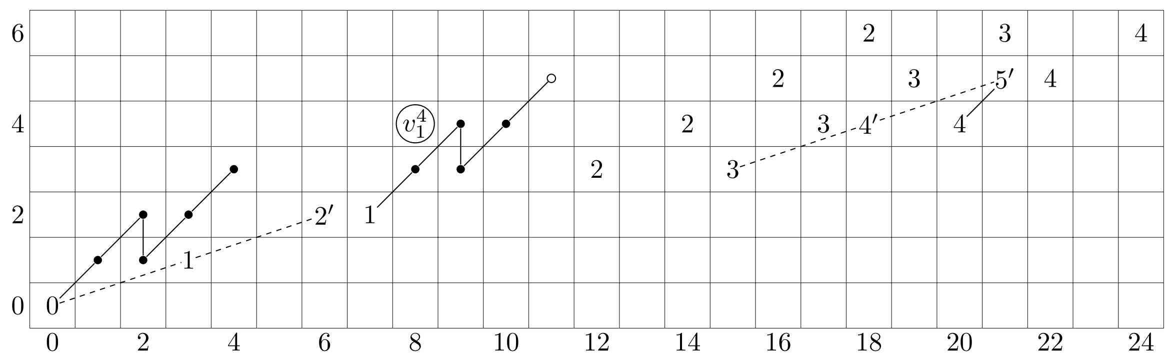


Figure 35: $P(v_1^4, v_2^8)$-generators for $\mathrm{Ext}_{A(2)}^{*,*}(A(0), \mathbb{F}_2)$, $0 \le t-s \le 24$

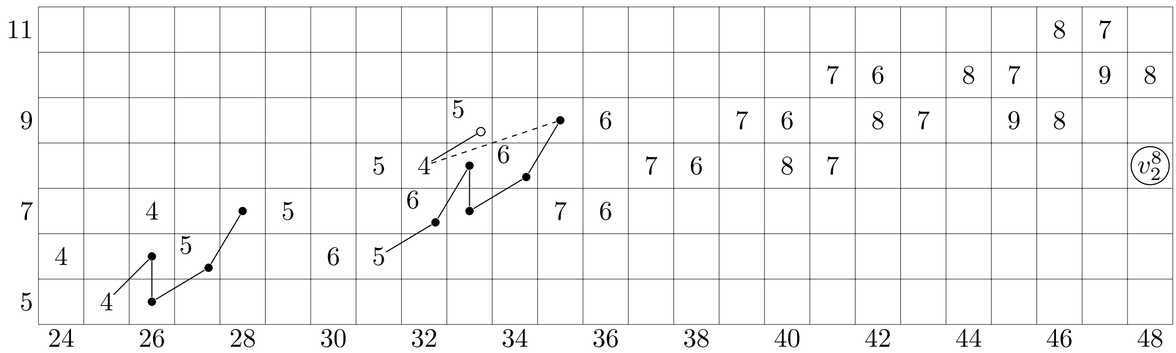
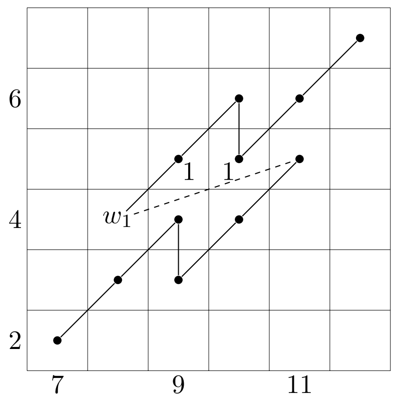
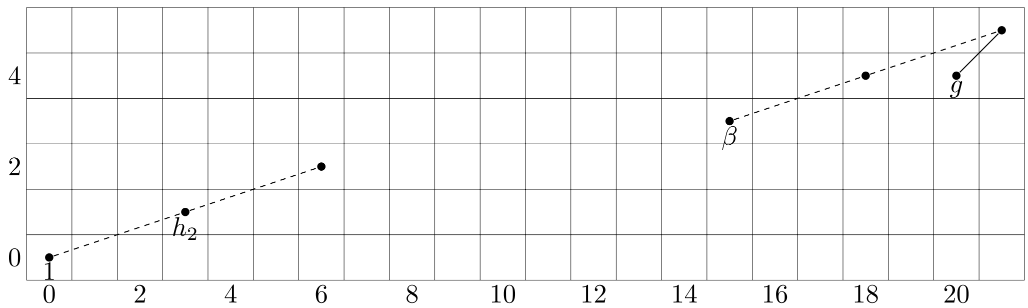


Figure 36: $P(v_1^4, v_2^8)$-generators for $\mathrm{Ext}^{*,*}_{A(2)}(A(0), \mathbb{F}_2)$, $24 \le t-s \le 48$

Definition 8.30. Let E_1 be the following indexed chart, where $w_1 = v_1^4$ is unlabeled:



and let F_1 be the (unindexed) chart:



Theorem 8.31. $\mathrm{Ext}^{*,*}_{A(2)}(A(0), \mathbb{F}_2)$ *is free over* $P(w_2) = P(v_2^8)$ *on the direct sum of* F_1 *and the free* $P(w_1) = P(v_1^4)$*-module*

$$\langle E_1 \rangle \{1, v_2^4\} \oplus P(v_1, h_{21})\{g_{12,3}, v_2^4 g_{12,3}\} \oplus P(h_{21})\{g_{30,6}\}\,.$$

where $v_1^4 g_{30,6} = v_1^3 h_{21}^4 g_{12,3}$.

Multiplication by h_1 *takes the class* 1 *in bidegree* $(t-s, s) = (0,0)$ *of* F_1 *into* $\langle E_1 \rangle$. *Multiplication by* v_1^4 *takes the classes* 1 *and* h_2 *in of* F_1 *into* E_1*, annihilates the classes* h_2^2*,* $h_2\beta$ *and* $h_2^2\beta = h_1 g$*, and takes the classes* β *and* g *into* $P(v_1, h_{21})\{g_{12,3}\}$.

The Ext*-homomorphism induced by the augmentation* $A(0) \to \mathbb{F}_2$ *takes* $\alpha = \alpha_2$ *to* $g_{12,3}$ *and* $\beta^2 = \alpha_3^2$ *to* $g_{30,6}$. *The class* $v_2^4 g_{12,3}$ *is not in the image of this homomorphism.*

8.6 Adams periodicity

We now discuss v_1-periodicity in $\mathrm{Ext}_{\mathscr{A}}^{*,*}$, following Adams (1966), with improved estimates due to Peter May, as presented in Ravenel (1986, section 3.4). Adams obtained periodicity above a line of slope 1/3 in the $(t-s, s)$-plane, which May improved to a line of optimal slope 1/5.

Proposition 8.32. *Let the functions v and w be defined by*

s	-2	-1	0	1	2	3	4	5	≥ 6
$v(s)$	-6	-4	1	8	6	18	18	21	$5s+3$
$w(s)$	-6	-4	1	8	10	18	23	25	$5s+3$

Then Yoneda multiplication by $v_1^4 \in \mathrm{Ext}_{A(2)}^{4,12}(\mathbb{F}_2, \mathbb{F}_2)$ induces an isomorphism

$$v_1^4 \colon \mathrm{Ext}_{A(2)}^{s,t}(A(0), \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{A(2)}^{s+4,t+12}(A(0), \mathbb{F}_2)$$

for $t - s < v(s)$, and a surjection for $v(s) \leq t - s < w(s)$.

Proof. This follows by inspection of the calculation of $\mathrm{Ext}_{A(2)}^{*,*}(A(0), \mathbb{F}_2)$. Surjectivity fails for $s \geq 6$ and $t - s = 5s + 3$ since $v_1^3 h_{21}^{s-5} g_{30,6} = v_1^2 h_{21}^{s-1} g_{12,3}$ is not divisible by v_1^4. The line $t - s = 5s + 3$ has slope 1/5. The multiples by powers of $w_2 = v_2^8$ lie on the line $t - s = 6s$ of slope 1/6, so they do not reduce the region of periodicity. □

Proposition 8.33. *Let the functions $\tilde{v}$ and $\tilde{w}$ be defined by*

s	-1	0	1	2	3	4	5	6	≥ 7
$\tilde{v}(s)$	-6	-4	1	6	10	18	21	25	$5s-2$
$\tilde{w}(s)$	-4	1	7	10	18	22	25	33	$5s+3$

Let M be an $A(2)$-module that is free as an $A(0)$-module, and concentrated in degrees $* \geq 0$. Then Yoneda multiplication by $v_1^4 \in \mathrm{Ext}_{A(2)}^{4,12}(\mathbb{F}_2, \mathbb{F}_2)$ induces an isomorphism

$$v_1^4 \colon \mathrm{Ext}_{A(2)}^{s,t}(M, \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{A(2)}^{s+4,t+12}(M, \mathbb{F}_2)$$

for $t - s < \tilde{v}(s)$, and a surjection for $\tilde{v}(s) \leq t - s < \tilde{w}(s)$.

Proof. Consider an extension $0 \to M' \to M \to M'' \to 0$ of $A(0)$-free $A(2)$-modules, with M' concentrated in degrees $* \geq 1$ and M'' free on generators in degree 0. We may inductively assume that the result holds for $\Sigma^{-1} M'$. Multiplication by v_1^4 induces a map of long exact sequences

$$\begin{tikzcd}
\mathrm{Ext}_{A(2)}^{s-1,t}(M') \arrow[r] \arrow[d, "v_1^4"] & \mathrm{Ext}_{A(2)}^{s,t}(M'') \arrow[r] \arrow[d, "v_1^4"] & \mathrm{Ext}_{A(2)}^{s,t}(M) \arrow[r] \arrow[d, "v_1^4"] & \mathrm{Ext}_{A(2)}^{s,t}(M') \arrow[r] \arrow[d, "v_1^4"] & \mathrm{Ext}_{A(2)}^{s+1,t}(M'') \arrow[d, "v_1^4"] \\
\mathrm{Ext}_{A(2)}^{s+3,t+12}(M') \arrow[r] & \mathrm{Ext}_{A(2)}^{s+4,t+12}(M'') \arrow[r] & \mathrm{Ext}_{A(2)}^{s+4,t+12}(M) \arrow[r] & \mathrm{Ext}_{A(2)}^{s+4,t+12}(M') \arrow[r] & \mathrm{Ext}_{A(2)}^{s+5,t+12}(M'')
\end{tikzcd}$$

where the second argument to each Ext-group is $\mathbb{F}_2$. We apply the five-lemma: The third (middle) map is surjective if the second and fourth maps are surjective and the fifth map is injective. This holds if $t - s < w(s)$ and $t - (s+1) < v(s+1)$, so we can let $\tilde{w}(s) = \min\{w(s), v(s+1) + 1\}$. The third map is injective if the second and fourth maps are injective and the first map is surjective. This holds if $t - s < v(s)$ and $(t-1) - (s-1) < \tilde{w}(s-1)$, so we can let $\tilde{v}(s) = \min\{v(s), \tilde{w}(s-1)\}$. □

In the following result we may interpret $A(n)$ for $n = \infty$ as $\mathscr{A}$. We are principally interested in the case $r = 2$.

Theorem 8.34 (Adams approximation). *Let $0 \leq r \leq n \leq \infty$ and let M be an $A(0)$-free $A(n)$-module that is concentrated in degrees $* \geq 0$. Restriction along $A(r) \subset A(n)$ induces an isomorphism*

$$\mathrm{Ext}_{A(n)}^{s,t}(M, \mathbb{F}_2) \xrightarrow{\cong} \mathrm{Ext}_{A(r)}^{s,t}(M, \mathbb{F}_2)$$

for $t-s < 2s + 2^{r+1} - \tilde{\epsilon}(s)$*, where*

$$\tilde{\epsilon} = \begin{cases} 5 & \text{for } s \equiv 0, 3 \mod 4, \\ 3 & \text{for } s \equiv 1 \mod 4, \\ 4 & \text{for } s \equiv 2 \mod 4. \end{cases}$$

Proof. We have an extension

$$0 \to K \to A(n) \otimes_{A(r)} M \to M \to 0$$

of $A(n)$-modules, with K concentrated in degrees $* \geq 2^{r+1}$, since $A(r) \to A(n)$ is an isomorphism in degrees $* < 2^{r+1}$. Adams (1966, Proposition 2.6) proves that the assumption that M is free over $A(0)$ implies that $A(n) \otimes_{A(r)} M$ and K are also $A(0)$-free. The argument is standard, as Bruner has kindly pointed out: The $A(n)$-module structure on M gives an isomorphism $A(n) \otimes_{A(r)} M \cong A(n)//A(r) \otimes M$. When M is $A(0)$-free, so is the tensor product, hence also the middle term in the extension. This implies that the kernel K is stably free, but this is the same as free for $A(0)$-modules.

We have an exact sequence

$$\mathrm{Ext}^{s-1,t}_{A(n)}(K, \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{A(n)}(M, \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{A(n)}(A(n) \otimes_{A(r)} M, \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{A(n)}(K, \mathbb{F}_2).$$

Under the change of rings isomorphism

$$\mathrm{Ext}^{s,t}_{A(n)}(A(n) \otimes_{A(r)} M, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{A(r)}(M, \mathbb{F}_2)$$

the middle homomorphism corresponds to the restriction homomorphism. We have a change-of-rings isomorphism

$$\mathrm{Ext}^{s,t}_{A(n)}(K, \mathbb{F}_2) \cong \mathrm{Ext}^{s,t}_{\mathscr{A}}(\mathscr{A} \otimes_{A(n)} K, \mathbb{F}_2),$$

with $\mathscr{A} \otimes_{A(n)} K$ concentrated in degrees $* \geq 2^{r+1}$ and $A(0)$-free. By Adams vanishing (Proposition 6.20) the displayed Ext-group is zero for $(t - 2^{r+1}) - s < 2s - \epsilon(s)$, where

$$\epsilon(s) = \begin{cases} 0 & \text{for } s \equiv 0 \mod 4, \\ 1 & \text{for } s \equiv 1 \mod 4, \\ 2 & \text{for } s \equiv 2, 3 \mod 4. \end{cases}$$

Hence the middle homomorphism is an isomorphism if $(t-2^{r+1}) - s < 2s - \epsilon(s)$ and $(t-2^{r+1}) - (s-1) < 2(s-1) - \epsilon(s-1)$. The second condition implies the first, since $\tilde{\epsilon}(s) = 3 + \epsilon(s-1) \geq \epsilon(s)$. □

It follows from the calculations for $A(2)$ that there are isomorphisms

$$\mathrm{Ext}^{s,t}_{A(n)}(M, \mathbb{F}_2) \xrightarrow{\cong} \mathrm{Ext}^{s+4,t+12}_{A(n)}(M, \mathbb{F}_2)$$

for $s \geq 0$, $t - s < \tilde{v}(s)$ and $t - s < 2s + 8 - \tilde{\epsilon}(s)$. The latter condition dominates for $s \geq 3$. When $n = 2$, this isomorphism is induced by the Yoneda product with $v_1^4 \in \mathrm{Ext}^{4,12}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$, but this class does not lift to $\mathrm{Ext}_{A(n)}$ for $n \geq 3$. However, there is a power of v_1^4 that does lift to $\mathrm{Ext}_{A(n)}$.

Theorem 8.35 (Adams)**.** *For each* $n \geq 2$ *there is a class* $\varpi_n \in \mathrm{Ext}^{2^n, 3 \cdot 2^n}_{A(n)}(\mathbb{F}_2, \mathbb{F}_2)$ *that restricts to* $w_1^{2^{n-2}} = v_1^{2^n} \in \mathrm{Ext}^{2^n, 3\cdot 2^n}_{A(1)}(\mathbb{F}_2, \mathbb{F}_2)$.

The proof is given in Adams (1966, Section 4) or Ravenel (1986, Lemma 3.4.10), and uses algebraic Steenrod operations in the cobar construction on $\mathscr{A}_*$. We must omit it, for now. The periodicity class ϖ_2 is the unique class in its bidegree, also known as $w_1 = v_1^4 = \omega_0$.

Proposition 8.36. *Let the functions* $\hat{v}$ *and* $\hat{w}$ *be defined by*

s	0	1	2	3	4	5	6	7	≥ 8
$\hat{v}(s)$	-4	0	6	9	16	21	24	31	$5s-3$
$\hat{w}(s)$	1	7	10	17	22	25	32	38	$5s+3$

Let $n \ge 2$ and let M be an $A(0)$-free $A(n)$-module that is concentrated in degrees $* \ge 0$. Yoneda multiplication by $\varpi_n \in \operatorname{Ext}_{A(n)}^{2^n, 3\cdot 2^n}(\mathbb{F}_2, \mathbb{F}_2)$ induces an isomorphism

$$\operatorname{Ext}_{A(n)}^{s,t}(M, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{A(n)}^{s+2^n, t+3\cdot 2^n}(M, \mathbb{F}_2)$$

for $t - s < \hat{v}(s)$, and a surjection for $\hat{v}(s) \le t - s < \hat{w}(s)$.

Proof. The claim for $s = 0$ follows by Proposition 8.33 and Adams approximation. For larger s we proceed by induction. Define K by the short exact sequence

$$0 \to K \to A(n) \otimes_{A(2)} M \to M \to 0$$

of $A(n)$-modules. Then K is $A(0)$-free and concentrated in degrees $* \ge 8$. By induction on t, we may assume that the proposition applies to $\Sigma^{-8}K$. Multiplication by ϖ_n induces a map of exact sequences

$$\begin{tikzcd}
\operatorname{Ext}_{A(2)}^{s-1,t}(M) \arrow[r] \arrow[d, "v_1^{2^n}"'] & \operatorname{Ext}_{A(n)}^{s-1,t}(K) \arrow[r] \arrow[d, "\varpi_n"'] & \operatorname{Ext}_{A(n)}^{s,t}(M) \arrow[r] \arrow[d, "\varpi_n"'] & \operatorname{Ext}_{A(2)}^{s,t}(M) \arrow[r] \arrow[d, "v_1^{2^n}"'] & \operatorname{Ext}_{A(n)}^{s,t}(K) \arrow[d, "\varpi_n"'] \\
\operatorname{Ext}_{A(2)}^{s'-1,t'}(M) \arrow[r] & \operatorname{Ext}_{A(n)}^{s'-1,t'}(K) \arrow[r] & \operatorname{Ext}_{A(n)}^{s',t'}(M) \arrow[r] & \operatorname{Ext}_{A(2)}^{s',t'}(M) \arrow[r] & \operatorname{Ext}_{A(n)}^{s',t'}(K)
\end{tikzcd}$$

where we have suppressed $\mathbb{F}_2$ in the second arguments, let $s' = s + 2^n$ and $t' = t + 3 \cdot 2^n$, and have used change-of-rings in the first and fourth columns.

By the five-lemma, the middle map is surjective if $(t-8) - (s-1) < \hat{w}(s-1)$, $t - s < \tilde{w}(s)$ and $(t-8) - s < \hat{v}(s)$, so we must have $\hat{w}(s) \le \min\{7 + \hat{w}(s-1), \tilde{w}(s), 8 + \hat{v}(s)\}$.

Furthermore, the middle map is injective if $(t-8) - (s-1) < \hat{v}(s-1)$, $t - s < \tilde{v}(s)$ and $t - (s-1) < \tilde{w}(s-1)$, so we must have $\hat{v}(s) \le \min\{7 + \hat{v}(s-1), \tilde{v}(s), -1 + \tilde{w}(s-1)\}$.

The above-defined functions $\hat{v}$ and $\hat{w}$ satisfy these conditions. We note that $\hat{w}(s) = \hat{v}(s+1) + 1$. □

((This agrees with Ravenel (1986, Lemma 3.4.14), except that his surjectivity for $t - s < h(s) - 1$ should probably be replaced by $t - s < h(s+2)$.))

Theorem 8.37. *Let M be an $A(0)$-free $\mathscr{A}$-module, concentrated in degrees $* \ge 0$. There is an isomorphism*

$$\Pi_n \colon \operatorname{Ext}_{\mathscr{A}}^{s,t}(M, \mathbb{F}_2) \xrightarrow{\cong} \operatorname{Ext}_{\mathscr{A}}^{s+2^n, t+3\cdot 2^n}(M, \mathbb{F}_2)$$

for $s \ge 0$ and $t - s < \min\{2s + 2^{n+1} - \tilde{\epsilon}(s), \hat{v}(s)\}$.

Remark 8.38. More precisely, the isomorphism is given in this range by the Massey product

$$\Pi_n(x) = \langle h_{n+1}, h_0^{2^n}, x \rangle .$$

This follows from a more precise description of the periodicity class ϖ_n, namely as the restriction along $A(n) \subset \mathscr{A}$ of a cochain with coboundary expressing the relation $h_0^{2^n} h_{n+1} = 0$. Following Tangora (1970), we write

$$P(x) = \langle h_3, h_0^4, x \rangle$$

for this operator in the case $n = 2$, when defined.

This leads to the following periodicity theorem, in the improved version due to May. See Ravenel (1986, Theorem 3.4.6).

Theorem 8.39 (Adams periodicity)**.** *Let v^* be defined by*

s	1	2	3	4	5	6	7	8	≥ 9
$v^*(s)$	-3	1	7	10	17	22	25	32	$5s - 7$

and let

$$\epsilon^*(s) = \begin{cases} 6 & \text{for } s \equiv 0, 1 \mod 4, \\ 4 & \text{for } s \equiv 2 \mod 4, \\ 5 & \text{for } s \equiv 3 \mod 4. \end{cases}$$

Let $n \geq 2$. There is an isomorphism

$$\Pi_n \colon \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\cong} \operatorname{Ext}_{\mathscr{A}}^{s+2^n, t+3\cdot 2^n}(\mathbb{F}_2, \mathbb{F}_2)$$

for $s \geq 1$ and $0 < t - s < \min\{2s + 2^{n+1} - \epsilon^*(s), v^*(s)\}$.

Remark 8.40. A direct computation shows that Π_2 is an isomorphism for $1 \leq s \leq 4$ and $0 < t - s < 2s + 2^3 - \epsilon^*(s)$, so we may improve the result a little by redefining $v^*(1) = 4$, $v^*(2) = 8$ and $v^*(3) = 9$. A further initial improvement might be possible by computing Π_3 for $1 \leq s \leq 8$ and $0 < t - s < 2s + 2^4 - \epsilon^*(s)$.

Proof. We use the short exact sequence

$$0 \to I(\mathscr{A}/\mathscr{A}Sq^1) \to \mathscr{A}/\mathscr{A}Sq^1 \to \mathbb{F}_2 \to 0$$

with $I(\mathscr{A}/\mathscr{A}Sq^1) = \Sigma^2 M$ free over $A(0)$ and concentrated in degrees $* \geq 2$. The connecting homomorphism

$$\operatorname{Ext}_{\mathscr{A}}^{s-1,t-2}(M, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$$

is then an isomorphism for all $t - s > 0$. We find that Π_n is an isomorphism for $s - 1 \geq 0$ and $(t-2) - (s-1) < \min\{2(s-1) + 2^{n+1} - \tilde{\epsilon}(s-1), \hat{v}(s-1)\}$. This translates to the conditions $s \geq 1$ and $t - s < \min\{2s + 2^{n+1} - 1 - \tilde{\epsilon}(s-1), 1 + \hat{v}(s-1)\}$, so we let $\epsilon^*(s) = 1 + \tilde{\epsilon}(s-1) = 4 + \epsilon(s-2)$ and $v^*(s) = 1 + \hat{v}(s-1)$, as above. $\square$

9 The homotopy groups of S and tmf

9.1 The image-of-J spectra

Let KU be the periodic complex K-theory spectrum, with homotopy groups $\pi_*(KU) = KU_* = \mathbb{Z}[u^{\pm 1}]$, given by inverting the complex Bott element u in $\pi_*(ku)$. Similarly, let KO be the periodic real K-theory spectrum, with homotopy groups

$$\pi_*(KO) = \mathbb{Z}[\eta, \alpha, \beta^{\pm 1}]/(2\eta, \eta^3, \eta\alpha, \alpha^2 = 4\eta)$$

given by inverting the real Bott element β in $\pi_*(ko)$, with image u^4 in $\pi_*(ku)$. These spectra represent complex and real topological K-theory, so that $KU^0(X)$ (resp. $KO^0(X)$) is the ring completion of the semiring of isomorphism classes of complex (resp. real) vector bundles over X, with respect to direct sum and tensor product, at least for finite CW complexes X. It is known that KU and KO admit (essentially unique) commutative S-algebra structures that realize these ring structures. The unit map $d \colon S \to KO$ is related to Adams' K-theory d-invariant.

For each integer k the Adams operation $\psi^k \colon KU^0(X) \to KU^0(X)$ is a natural ring homomorphism. By the splitting principle it is characterized by its value on complex line bundles $L \to X$, namely $\psi^k(L) = L^{\otimes k}$. Similarly for $\psi^k \colon KO^0(X) \to KO^0(X)$, which satisfies the same formula for real line bundles $L \to X$. As a consequence of this characterization, we have the relation $\psi^k \circ \psi^\ell = \psi^{k\ell}$. We also note that $\psi^{-1} = 1$ (the identity map) in the real case.

The Adams operations ψ^k do not commute with Bott periodicity, but map $u \in \tilde{KU}^0(S^2)$ to ku and $\beta \in \tilde{KO}^0(S^8)$ to $k^4\beta$. Hence it is necessary to localize, by inverting k, in order to extend ψ^k to stable operations $KU^*(X) \to KU^*(X)$ and $KO^*(X) \to KO^*(X)$. For $k \neq 0$ there are spectrum maps $\psi^k \colon KU[1/k] \to KU[1/k]$ and $\psi^k \colon KO[1/k] \to KO[1/k]$, such that $\psi^k \circ \psi^\ell = \psi^{k\ell}$ after inverting $k\ell \neq 0$.

Fix a prime p, and let k be an integer prime to p. After p-completion, $\psi^k \colon KU_p^\wedge \to KU_p^\wedge$ can be realized as a map of commutative S-algebras, with $\psi^k_* \colon \mathbb{Z}_p[u^{\pm 1}] \to \mathbb{Z}_p[u^{\pm 1}]$ taking u to ku. Similarly, $\psi^k \colon KO_p^\wedge \to KO_p^\wedge$ maps $\pi_*(KO_p^\wedge) \to \pi_*(KO_p^\wedge)$ by taking η to $k\eta$, α to $k^2\alpha$ and β to $k^4\beta$. Furthermore, these operations can be extended to p-adic integer values of k (still prime to p), so as to define an action of the p-adic units $\mathbb{Z}_p^\times$ on $KU_p^\wedge$, and similarly on $KO_p^\wedge$. These actions define multiplicative homomorphisms $\mathbb{Z}_p^\times \to (KU_p^\wedge)^0(KU_p^\wedge)$ and $\mathbb{Z}_p^\times/\pm 1 \to (KO_p^\wedge)^0(KO_p^\wedge)$, taking k to the homotopy class of ψ^k. These can be combined by the scalar multiplications of $(KU_p^\wedge)^* = \pi_{-*}(KU_p^\wedge)$ and $(KO_p^\wedge)^* = \pi_{-*}(KO_p^\wedge)$, to get the following ring isomorphisms:

Theorem 9.1. $(KU_p^\wedge)^*\langle\langle\mathbb{Z}_p^\times\rangle\rangle \cong (KU_p^\wedge)^*(KU_p^\wedge)$ *and* $(KO_p^\wedge)^*\langle\langle\mathbb{Z}_p^\times/\pm 1\rangle\rangle \cong (KO_p^\wedge)^*(KO_p^\wedge)$.

For odd p, any integer k that represents a generator of $(\mathbb{Z}/p^2)^\times$ is a topological generator of $\mathbb{Z}_p^\times$. Similarly, any integer k that represents a generator of $(\mathbb{Z}/8)^\times/\pm 1$ is a topological generator of $\mathbb{Z}_2^\times/\pm 1$. For $p = 2$ it is traditional in topology to pick $k = 3$, while the tradition in number theory may be to pick $k = 5$. Hereafter we assume that k is chosen as such a generator.

Definition 9.2. For odd p, let $J_p^\wedge = (KU_p^\wedge)^{h\psi^k}$ be the homotopy fixed points of the ψ^k-action on $KU_p^\wedge$. For $p = 2$, let $J_2^\wedge = (KO_2^\wedge)^{h\psi^k}$ be the homotopy fixed points of the ψ^k-action on $KO_2^\wedge$. These (p-complete) *image-of-J spectra* are commutative S-algebras, and there are commutative S-algebra maps $J_p^\wedge \to KU_p^\wedge$ and $J_2^\wedge \to KO_2^\wedge$.

((Can get commutative S-algebra actions by Goerss–Hopkins–Miller obstruction theory, which generalizes from $E_1 = KU_p^\wedge$ to the Lubin–Tate spectra E_n. The formation of homotopy fixed points for continuous actions by profinite groups (like $\mathbb{Z}_p^\times$) is technically complex, see Devinatz–Hopkins, Fausk, Behrens–Davis. In this case it suffices to work with the action by the free discrete monoid generated by ψ^k.))

The homotopy fixed points above can be rewritten as the homotopy equalizers of ψ^k and $1\colon KU_p^\wedge \to KU_p^\wedge$, or as the homotopy fiber of the difference map $\psi^k - 1\colon KU_p^\wedge \to KU_p^\wedge$, and similarly for $p = 2$. Applying $KU_p^\wedge$-cohomology to the cofiber sequence

$$J_p^\wedge \longrightarrow KU_p^\wedge \xrightarrow{\psi^k-1} KU_p^\wedge$$

we get the long exact sequence

$$\dots \to (KU_p^\wedge)^*\langle\langle\mathbb{Z}_p^\times\rangle\rangle \xrightarrow{\psi^k-1} (KU_p^\wedge)^*\langle\langle\mathbb{Z}_p^\times\rangle\rangle \longrightarrow (KU_p^\wedge)^*(J_p^\wedge) \to \dots$$

that induces an isomorphism

$$(KU_p^\wedge)^*\langle\langle\mathbb{Z}_p^\times\rangle\rangle/(k \sim 1) = (KU_p^\wedge)^* \cong (KU_p^\wedge)^*(J_p^\wedge)\,.$$

It follows that the unit map $e\colon S \to J_p^\wedge$ induces an isomorphism in $KU_p^\wedge$-cohomology, i.e., that it is a $KU_p^\wedge$-local equivalence. Similarly, for $p = 2$ we get an isomorphism

$$(KO_2^\wedge)^*\langle\langle\mathbb{Z}_2^\times/\pm 1\rangle\rangle/(k \sim 1) = (KO_2^\wedge)^* \cong (KO_2^\wedge)^*(J_2^\wedge)\,,$$

so that the unit map $e\colon S \to J_2^\wedge$ is a $KO_2^\wedge$-local equivalence.

((The map e is related to Adams' K-theory e-invariant. The role of these equivalences can be clarified in terms of Bousfield localizations. Theorems of Mahowald (for $p = 2$) and Haynes Miller (for p odd) prove that $e\colon S \to J_p^\wedge$ induces isomorphisms $\pi_*(S/p)[v_1^{-1}] \cong \pi_*(J/p)[v_1^{-1}]$, where $J/p = J_p^\wedge \wedge S/p$.))

Theorem 9.3. *For p odd,*

$$\pi_*(J_p^\wedge) \cong \begin{cases} \mathbb{Z}_p & \textit{if } * = 0 \textit{ or } * = -1, \\ \mathbb{Z}_p/(k^i - 1) & \textit{if } * = 2i - 1 \neq -1, \\ 0 & \textit{otherwise.} \end{cases}$$

For $p = 2$,

$$\pi_*(J_2^\wedge) \cong \begin{cases} \mathbb{Z}_2 & \textit{if } * = -1, \\ \mathbb{Z}_2 \oplus \mathbb{Z}/2 & \textit{if } * = 0, \\ \mathbb{Z}_2/(k^{4i} - 1) & \textit{if } * = 8i - 1 \neq -1, \\ \mathbb{Z}/2 & \textit{if } * = 8i \neq 0, \\ (\mathbb{Z}/2)^2 & \textit{if } * = 8i + 1, \\ \mathbb{Z}/2 & \textit{if } * = 8i + 2, \\ \mathbb{Z}/8 & \textit{if } * = 8i + 3, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. This is almost straightforward from the long exact sequences

$$\cdots \to \pi_*(J_p^\wedge) \longrightarrow \pi_*(KU_p^\wedge) \xrightarrow{\psi_*^k - 1} \pi_*(KU_p^\wedge) \longrightarrow \dots$$

and

$$\cdots \to \pi_*(J_2^\wedge) \longrightarrow \pi_*(KO_2^\wedge) \xrightarrow{\psi_*^k - 1} \pi_*(KO_2^\wedge) \longrightarrow \dots$$

where the action ψ_*^k of ψ^k on $\pi_*(KU_p^\wedge)$ and $\pi_*(KO_2^\wedge)$ has been discussed above. The only thing to check is that the extension giving $\pi_*(J_2^\wedge)$ for $* = 8i+1$ is split. ((Prove this!)) □

Remark 9.4. We have $\mathbb{Z}_p/(k^i - 1) = 0$ when $p - 1 \nmid i$, while $\mathbb{Z}_p/(k^i - 1) = \mathbb{Z}/p^{v+1}$ if $p - 1 \mid i$ and $v = v_p(i)$. Furthermore, $\mathbb{Z}_2/(k^{4i} - 1) = \mathbb{Z}/2^{v+4} = \mathbb{Z}_2/(16i)$ if $v = v_2(i)$.

Definition 9.5. For p odd, let the *connective image-of-J spectrum* $j_p^\wedge$ be the connective cover of $J_p^\wedge$. For $p = 2$, let $jo_2^\wedge$ be the connective cover of $J_2^\wedge$. These are commutative S-algebras, and there are commutative S-algebra maps $j_p^\wedge \to ku_p^\wedge$ and $jo_2^\wedge \to ko_2^\wedge$.

((Can also get E_∞ ring spectrum structure on $j_p^\wedge$ by discrete models, by taking k to be a prime power and using the algebraic K-theory of a finite field with k elements, following Quillen and May et al.))

There are cofiber sequences

$$j_p^\wedge \longrightarrow ku_p^\wedge \xrightarrow{\psi^k - 1} bu_p^\wedge$$

for p odd, and

$$jo_2^\wedge \longrightarrow ko_2^\wedge \xrightarrow{\psi^k - 1} bo_2^\wedge$$

for $p = 2$, where $\psi^k - 1$ denotes the unique lift of $\psi^k - 1\colon ku_p^\wedge \to ku_p^\wedge$ through the connected cover $bu_p^\wedge \to ku_p^\wedge$, and similarly for the connected cover $bo_2^\wedge \to ko_2^\wedge$.

For p odd the completed unit map $S_p^\wedge \to j_p^\wedge$ induces a split surjection on homotopy groups, as we shall discuss below. For $p = 2$, the lowest homotopy groups $\pi_0(jo_2^\wedge) = \mathbb{Z}_2 \oplus \mathbb{Z}/2$ and $\pi_1(jo_2^\wedge) \cong (\mathbb{Z}/2)^2$ are too large for this claim to hold, so we make an adjustment in these degrees to define the connective image-of-J spectrum at $p = 2$.

Definition 9.6. Let P^1X denote the first Postnikov section of X. We get a diagram of commutative S-algebras

$$\begin{tikzcd}
S_2^\wedge \arrow[dr, dashed, "e"] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& j_2^\wedge \arrow[r, dashed] \arrow[d, dashed] & jo_2^\wedge \arrow[r] \arrow[d] & ko_2^\wedge \arrow[d] \\
& P^1 S_2^\wedge \arrow[r] & P^1 jo_2^\wedge \arrow[r] & P^1 ko_2^\wedge
\end{tikzcd}$$

and define $j_2^\wedge$ to be the homotopy pullback in the left hand quadrangle.

The maps $S_2^\wedge \to j_2^\wedge \to ko_2^\wedge$ then induce equivalences of first Postnikov sections, which implies that there is a cofiber sequence

$$j_2^\wedge \longrightarrow ko_2^\wedge \xrightarrow{\psi^k - 1} bspin_2^\wedge$$

where $\psi^k - 1$ denotes the unique lift up to homotopy of $\psi^k - 1\colon ko_2^\wedge \to ko_2^\wedge$ over the 2-connected, hence 3-connected, cover $bspin_2^\wedge \to ko_2^\wedge$. This is usually taken as the definition of $j_2^\wedge$, but does not make the commutative S-algebra structure quite clear.

((Can also get E_∞ ring spectrum structure on $j_2^\wedge$ by a discrete model, as the algebraic K-theory of a suitable bipermutative category, following May et al.))

Proposition 9.7.

$$\pi_*(j_2^\wedge) \cong \begin{cases} \mathbb{Z}_2 & \textit{if } * = 0, \\ \mathbb{Z}/2 & \textit{if } * = 1, \\ \mathbb{Z}/2 & \textit{if } * = 8i+2 > 0, \\ \mathbb{Z}/8 & \textit{if } * = 8i+3 > 0, \\ \mathbb{Z}_2/(k^{4i}-1) & \textit{if } * = 8i-1 > 0, \\ \mathbb{Z}/2 & \textit{if } * = 8i > 0, \\ (\mathbb{Z}/2)^2 & \textit{if } * = 8i+1 > 1, \\ 0 & \textit{otherwise.} \end{cases}$$

The connecting homomorphism $\pi_{*+1}(bspin_2^\wedge) \to \pi_*(j_2^\wedge)$ is surjective for $* > 0$, except in degrees $* = 8i+r$ for $i \ge 0$ and $r = 1, 2$, where the cokernel maps isomorphically to $\pi_*(ko_2^\wedge) \cong \mathbb{Z}/2$. There are classes $\mu_{8i+r} \in \pi_{8i+r}(j_2^\wedge)$ of order 2 that map to the generators $\eta^r\beta^i$ of these groups.

((The classes $\mu_1 = \eta$, $\mu_2 = \eta^2$ and μ_{8i+2} for $i > 0$ are uniquely determined in $\pi_*(j_2^\wedge)$. How to characterize μ_{8i+1}?))

((Name the generators and order 2 classes?))

Corollary 9.8. *$e\colon S_2^\wedge \to j_2^\wedge$ is 6-connected.*

To prove that $\pi_*(e)$ is split surjective, we need a number of unstable (space level) constructions.

Definition 9.9. Let $F(n) \subset \Omega^n S^n$ be the monoid of base-point preserving homotopy equivalences $S^n \to S^n$, and let $O(n) \to F(n)$ be the monoid homomorphism taking an isometry $\mathbb{R}^n \to \mathbb{R}^n$ to the induced map $S^n \to S^n$ of one-point compactifications. These homomorphisms are compatible with the stabilizations $O(n) \to O(n+1)$ and $F(n) \to F(n+1)$, and induce a monoid homomorphism $j\colon O \to F = GL_1(S)$. The induced homomorphism

$$J = \pi_*(j)\colon \pi_*(O) \to \pi_*(F) \cong \pi_*(S)$$

(for $* > 0$) is called the *J-homomorphism*, after J.H.C. Whitehead.

Recall that

$$\pi_*(O) \cong \pi_{*+1}(BO) \cong \begin{cases} \mathbb{Z}/2 & \text{if } * \equiv 0,1 \mod 8, \\ \mathbb{Z} & \text{if } * \equiv 3,7 \mod 8, \\ 0 & \text{otherwise} \end{cases}$$

for $* > 0$, so that the image $\mathrm{im}(J) \subset \pi_*(S)$ is (trivial or) cyclic of order two for $* \equiv 0,1 \mod 8$ and (trivial or) finite cyclic for $* \equiv 3,7 \mod 8$.

We get a Puppe fiber sequence

$$O \xrightarrow{j} F \longrightarrow F/O \longrightarrow BO \xrightarrow{Bj} BF\,,$$

where F/O is defined as the homotopy fiber of $Bj\colon BO \to BF$. The right hand map represent a homomorphism

$$\widetilde{KO}^0(X) = [X, BO] \to [X, BF]$$

(for connected CW complexes X), that takes a vector bundle $E \to X$ to the stable spherical fibration class of its fiberwise one-point compactification. Its image is the group $J(X)$ studied by Adams in a series of papers.

((Discuss how $Bj\colon BO \to BF$ is a map of infinite loop spaces.))

There is a close relation between the subgroup $\mathrm{im}(J) \subset \pi_*(S)$ of the stable homotopy groups of spheres and the homotopy groups of the image-of-J spectrum j, given as quotient group of $\pi_*(S)$ via the unit map $e\colon S \to j$.

We sketch the presentation of May et al. Start with the lift $j\colon Spin \to SF = SL_1(S)$ of $j\colon O \to F$, and form the middle horizontal Puppe fiber sequence in the following diagram, implicitly completed at a prime p:

$$
\begin{tikzcd}
& J \arrow[r] \arrow[d, dashed, "\alpha^k"'] & BO \arrow[r, "\psi^k-1"] \arrow[d, dashed, "\gamma^k"'] & BSpin \arrow[d, "="'] & \\
Spin \arrow[r, "j"] & SF \arrow[r] \arrow[d, "e"'] & SF/Spin \arrow[r] \arrow[d, "\sigma^k"'] & BSpin \arrow[r, "Bj"] \arrow[d, "\rho^k"'] & BSF \\
& J_\otimes \arrow[r] & BO_\otimes \arrow[r, "\psi^k/1"] & BSpin_\otimes &
\end{tikzcd}
$$

(The solid arrows are infinite loop maps, when the spaces labeled $\otimes$ are given multiplicative infinite loop space structures.)

The next step is similar to Thom's construction of Stiefel–Whitney characteristic classes using Steenrod operations in mod 2 cohomology, replacing cohomology and Steenrod operations by real K-theory and Adams operations, respectively. The Atiyah–Bott–Shapiro *ko*-orientation of *Spin*-bundles specifies a Thom class $u\colon MSpin \to ko$ in the *ko*-cohomology of the Thom spectrum of the tautological vector bundle over $BSpin$. Applying the Adams operation $\psi^k\colon ko \to ko$, the composite class $\psi^k(u)$ corresponds under the *ko*-cohomology Thom isomorphism $ko^*(BSpin) \cong ko^*(MSpin)$ to a characteristic class $\rho^k\colon \Sigma^\infty BSpin_+ \to ko$ satisfying $u \cup \rho^k = \psi^k(u)$. The space level adjoint $BSpin \to BO_\otimes$ lifts to an infinite loop map $\rho^k\colon BSpin \to BSpin_\otimes$, known as the ((Bott?)) *cannibalistic class*. There is a corresponding operation $\sigma^k\colon SF/Spin \to BO_\otimes$ making the displayed square commute. The infinite loop map $\psi^k/1$ is the restriction of $\psi^k - 1\colon ko \to bspin$ to the 1-component $BO_\otimes = SL_1(ko)$, so its homotopy fiber is identified with the 1-component $J_\otimes = SL_1(j)$ of j, all after p-completion.

Turning to the upper half of the diagram, Adams proved that the composite $Bj \circ (\psi^k - 1)$ on some classes in $\widetilde{KO}^0(X) = [X, BO]$ is zero in $[X, BSF]^\wedge_p$, and conjectured that this is always so. The *Adams conjecture*, that $Bj \circ (\psi^k - 1)$ is null-homotopic after p-completion, was proved by Quillen and by Sullivan, and leads to the existence of the (p-complete) space level maps α^k and γ^k. Such map α^k is sometimes called a solution to the Adams conjecture. It is known ((Madsen, Tornehave?)) that these maps cannot be delooped for $p = 2$. ((Positive result for odd p by Friedlander.)) By Adams' calculations in the $J(X)$-papers, the square with corners BO, $BSpin$, $BO_\otimes$ and $BSpin_\otimes$ is homotopy cartesian, so that the composite map $e\alpha^k\colon J \to J_\otimes$ of homotopy fibers is a homotopy equivalence.

We write $\mu_{8i+r} \in \pi_{8i+r}(S) \cong \pi_{8i+r}(SF)$ for the image under α^k_* of the class with the same name in $\pi_{8i+r}(J)$, which is detected by $\eta^r\beta^i \in \pi_{8i+r}(BO_\otimes) \cong \pi_{8i+r}(ko)$.

Theorem 9.10 (Adams, Quillen, Sullivan)**.** *The homomorphism $e_*\colon \pi_*(S^\wedge_p) \to \pi_*(j^\wedge_p)$ is split surjective. A section for $* > 0$ is given (after implicit p-completion) by a solution $\alpha^k_*\colon \pi_*(J) \to \pi_*(SF) \cong \pi_*(S)$ to the Adams conjecture.*

The image $\mathrm{im}(\alpha^k_*) \cong \pi_*(J)$ of that section is the direct sum of two parts: The first part is the image $\mathrm{im}(J)$ of the J-homomorphism $J = \pi_*(j)\colon \pi_*(Spin) \to \pi_*(SF) \cong \pi_*(S)$. The second part is $\mathbb{Z}/2\{\mu_{8i+r} \mid i \ge 0, r = 1, 2\}$, which is detected by $d_*\colon \pi_*(S) \to \pi_*(ko)$.

Adams calls the μ-classes "honorary members" of the image of J.

Lemma 9.11. *(We implicitly work completed at $p = 2$.) $e_*\colon \pi_*(S) \to \pi_*(j)$ is an isomorphism in degrees $* \le 13$, except in degrees $* = 6$, 8 and 9:*
$\pi_6(e)\colon \pi_6(S) \to \pi_6(j)$ takes ν^2 to 0.
$\pi_7(e)\colon \pi_8(S) \to \pi_8(j)$ takes $\eta\sigma$ and ϵ to $\eta\sigma$.
$\pi_9(e)\colon \pi_9(S) \to \pi_9(j)$ takes $\eta^2\sigma$ and $\eta\epsilon$ to $\eta^2\sigma$, and $\mu = \mu_9$ to μ.

Proof. The claim about $\pi_6(e)$ is obvious, and implies that $\pi_9(e)$ takes $\nu^3 = \eta^2\sigma + \eta\epsilon$ to 0. ((Cite Toda for that relation?)) Hence both $\eta^2\sigma$ and $\eta\epsilon$ map to $\eta^2\sigma$, which implies the claim for $\pi_8(e)$. □

((It follows that $\eta\sigma$ must have Adams filtration ≥ 3 in $\pi_*(j)$.))

9.2 The image of J in the Adams spectral sequence

To describe the role of the image of J as a subgroup of the stable homotopy groups of spheres, viewed as the abutment of the Adams spectral sequence for S, we need to have an image of the latter in the

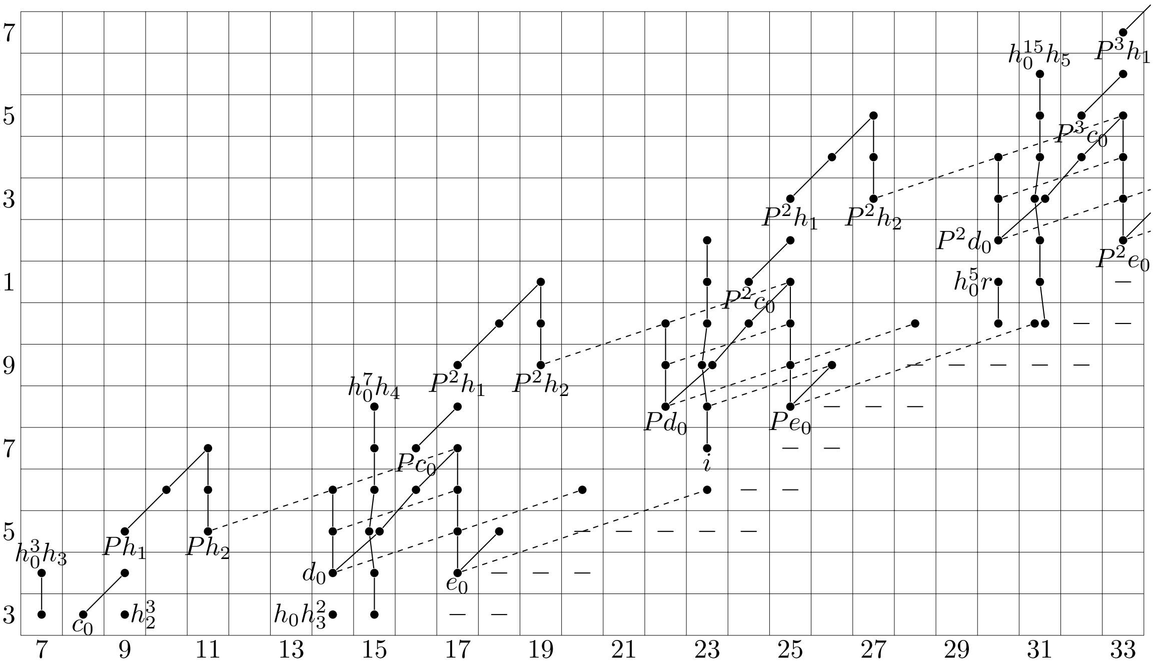


Figure 37: Π_3-periodic Adams chart for S

relevant region, which is a diagonal band parallel to the Adams vanishing line of slope 1/2. For this, we need to appeal to the Adams periodicity theorem, proved above in Theorem 8.39. The contribution of the image of J at the Adams E_∞-term is largely contained in the periodicity range for the operator $P = \Pi_2$ that increases $t-s$ by 8. However, some of our arguments involving $S/2$ fall outside of that range, so that it seems best to work in the periodicity range for the operator Π_3 that increases $t-s$ by 16, and which equals P^2 where the latter is defined.

We use the following form of the Adams periodicity theorem, proved above in Theorem 8.39 for $s \ge 7$. The claim for $3 \le s \le 6$ must be checked directly.

Theorem 9.12. *There is an isomorphism*

$$\Pi_3\colon \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \xrightarrow{\cong} \operatorname{Ext}_{\mathscr{A}}^{s+8,t+24}(\mathbb{F}_2,\mathbb{F}_2)$$

for $s \ge 3$ and

$$0 < t-s < 2s + \begin{cases} 10 & \textit{for } s \equiv 0,1 \mod 4, \\ 12 & \textit{for } s \equiv 2 \mod 4, \\ 11 & \textit{for } s \equiv 3 \mod 4. \end{cases}$$

Hence the pattern above the dashes in Figure 37 repeats every $t-s = 16$ degrees. We are most interested in the uppermost part, close to the line $t-s = 2s$.

As a consequence of the proven Adams conjecture, we get the following theorem. See Ravenel (1986, Theorem 3.4.16) and Davis–Mahowald (1989, Theorem 1.1).

Theorem 9.13. *The classes c_0, h_1c_0, Ph_1, h_1Ph_1, Ph_2, h_0Ph_2 and $h_0^2Ph_2 = h_1^2Ph_1$, as well as all of their images under powers of P, survive to E_∞ in the Adams spectral sequence (meaning that they are infinite cycles and not boundaries). They represent subgroups $\mathbb{Z}/2 \subset \pi_{8i}(S)$, $(\mathbb{Z}/2)^2 \subset \pi_{8i+1}(S)$, $\mathbb{Z}/2 \subset \pi_{8i+2}(S)$ and $\mathbb{Z}/8 \subset \pi_{8i+3}(S)$ that map isomorphically to $\pi_{8i}(j_2^\wedge)$, $\pi_{8i+1}(j_2^\wedge)$, $\pi_{8i+2}(j_2^\wedge)$ and $\pi_{8i+3}(j_2^\wedge)$, respectively.*

In topological degree $t-s = 8i-1$, for $i \ge 1$, there is a class surviving to E_∞ in each of the $(v+4)$ Adams filtrations s with $4i-v-3 \le s \le 4i$, where $v = v_2(i)$. These represent a subgroup $\mathbb{Z}/2^{v+4} = \mathbb{Z}_2/(16i) \subset \pi_{8i-1}(S)$ that maps isomorphically to $\pi_{8i-2}(j_2^\wedge)$.

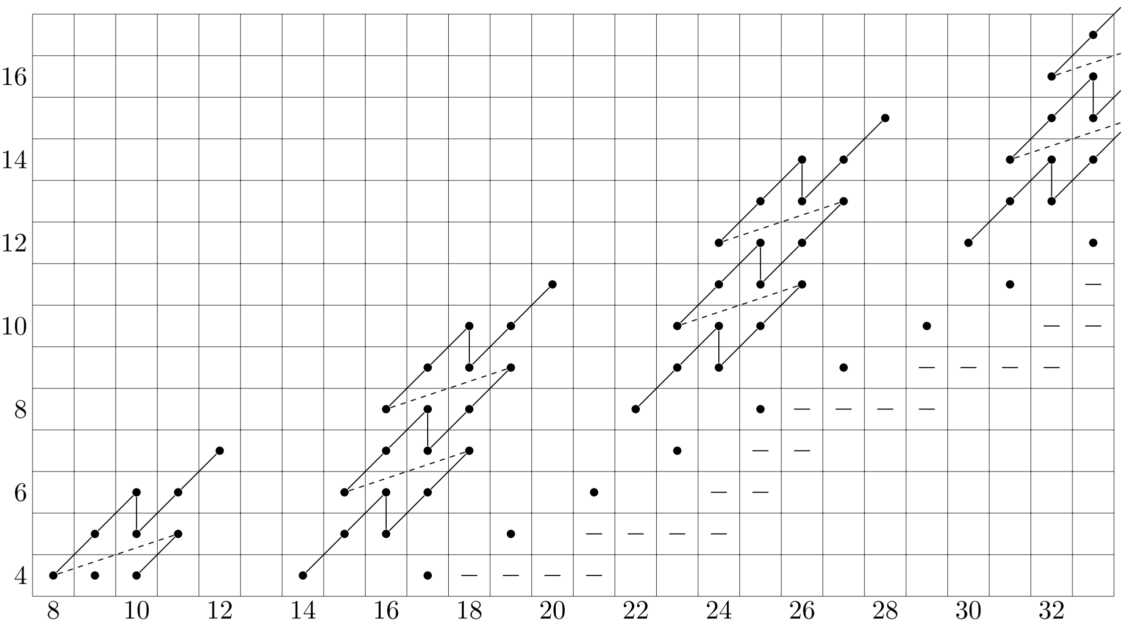


Figure 38: Π_3-periodic Adams chart for $S/2$

There is a hidden η-multiplication from the generator in degree $t-s = 8i-1$ and filtration $s = 4i-v-3$ to $P^{i-1}c_0$.

To prove (a part of) this, we shall compare S and j with $S/2$ and $j/2$. We proved the following version of the periodicity theorem for $S/2$ in Theorem 8.37, at least for $s \geq 5$. The case $s = 4$ can be checked directly.

Theorem 9.14. *There is an isomorphism*

$$\Pi_3 \colon \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(S/2), \mathbb{F}_2) \xrightarrow{\cong} \operatorname{Ext}_{\mathscr{A}}^{s+4,t+12}(H^*(S/2), \mathbb{F}_2)$$

for $s \geq 4$ and

$$t - s < 2s + \begin{cases} 10 & \text{for } s \equiv 0 \mod 4, \\ 11 & \text{for } s \equiv 1, 3 \mod 4, \\ 12 & \text{for } s \equiv 2 \mod 4. \end{cases}$$

Hence the pattern above the dashes in Figure 38 repeats every $t - s = 16$ degrees.

Associated to the cofiber sequence $S \to S/2 \to \Sigma S = S^1$, we have an extension

$$0 \to \Sigma\mathbb{F}_2 \to H^*(S/2) \to \mathbb{F}_2 \to 0$$

and a long exact sequence of Ext-groups:

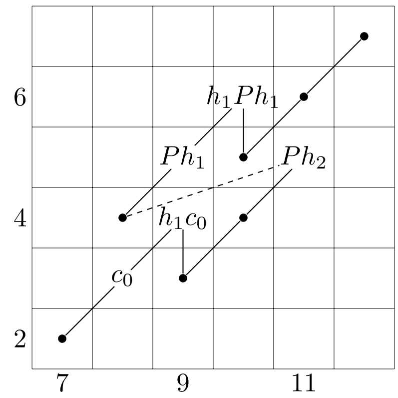
$$\cdots \to \operatorname{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s,t}(H^*(S/2), \mathbb{F}_2) \longrightarrow \operatorname{Ext}_{\mathscr{A}}^{s,t-1}(\mathbb{F}_2, \mathbb{F}_2) \xrightarrow{\delta} \operatorname{Ext}_{\mathscr{A}}^{s+1,t}(\mathbb{F}_2, \mathbb{F}_2) \to \dots$$

where the connecting homomorphism is given by the Yoneda product with $h_0 \in \operatorname{Ext}_{\mathscr{A}}^{1,1}(\mathbb{F}_2, \mathbb{F}_2)$.

Lemma 9.15. *The map $E_2^{*,*}(S) \to E_2^{*,*}(S/2)$ of Adams spectral sequences takes the h_0-indecomposable classes c_0, h_1c_0, Ph_1, h_1Ph_1 and Ph_2, as well as all of their images under powers of P, injectively to linearly independent classes in the target.*

The morphism $E_2^{*,*}(S/2) \to E_2^{*,*}(\Sigma S) \cong E_2^{*,*-1}(S)$ maps classes in the source surjectively to the h_0-annihilated classes $h_0^3h_3$, c_0, h_1c_0, Ph_1, h_1Ph_1 and $h_1^2Ph_1 = h_0^2Ph_2$, as well as all of their P-power images.

Proof. ((By inspection.))



□

Remark 9.16. This lemma accounts for 11 of the 12 generators in the two uppermost families of lightning flashes in the Adams chart for $S/2$. The remaining generator, in degree $(8i-1)$, is exceptional: The h_0-indecomposable class at the bottom of the tower leading up to $P^{i-1}(h_0^3h_3)$ can have very low Adams filtration, and does often not contribute to $E_2^{*,*}(S/2)$ within the Adams periodic range. Instead, the class $h_2P^{i-1}h_2$ is annihilated by h_0 and contributes a class x in $E_2^{4i-2,12i-3}(S/2)$ with h_1x equal to the image of c_0.

We can compare these charts with the Adams charts for j and $j/2$. See Mahowald–Milgram, Davis and Angeltveit–Rognes for the following calculation

Proposition 9.17. *The lift* $\theta\colon ko \to bspin$ *of* $\psi^3 - 1$ *induces the homomorphism*

$$\theta^*\colon \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^1, Sq^2Sq^3\} = H^*(bspin) \to H^*(ko) = \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2\} = \mathscr{A}//A(1)$$

that takes the generator $\Sigma^4 1$ *to the class of* Sq^4*. There are isomorphisms*

$$\Sigma K = \ker(\theta^*) \cong \Sigma^8\mathscr{A}/\mathscr{A}\{Sq^1, Sq^7, Sq^4Sq^6 + Sq^6Sq^4\}$$
$$C = \operatorname{cok}(\theta^*) \cong \mathscr{A}/\mathscr{A}\{Sq^1, Sq^2, Sq^4\} = \mathscr{A}//A(2)\,.$$

The extension

$$0 \to C \to H^*(j) \to K \to 0$$

is nontrivial, and

$$H^*(j) \cong \mathscr{A}\{1, x\}/\mathscr{A}\{Sq^1, Sq^2, Sq^4, Sq^8 + Sq^1x, Sq^7x, (Sq^4Sq^6 + Sq^6Sq^4)x\}$$

with generators 1 *and* x *in degrees* 0 *and* 7*, respectively.*

((The isomorphism $C \cong H^*(tmf)$ is incidental; there is no map $j \to tmf$ inducing the inclusion $C \to H^*(j)$ in cohomology. See also Bruner's note (2012). Check if $Sq^4Sq^6 + Sq^6Sq^4 = Sq^{(0,1,1)} + Sq^{(4,2)}$.))

Proposition 9.18 (Bruner)**.** *The map* $\theta/2\colon ko/2 \to bspin/2$ *induces the homomorphism*

$$(\theta/2)^*\colon \Sigma^4\mathscr{A}/\mathscr{A}\{Sq^2Sq^3\} = H^*(bspin/2) \to H^*(ko/2) = \mathscr{A}/\mathscr{A}\{Sq^2, Q_1\}$$

that takes the generator $\Sigma^4 1$ *to the class of ((ETC)).*

The extension

$$0 \to C \otimes H^*(S/2) \to H^*(j/2) \to K \otimes H^*(S/2) \to 0$$

induces a long exact sequence of Ext-groups:

$$\dots \to \operatorname{Ext}^{s,t}_{\mathscr{A}}(K \otimes H^*(S/2), \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s,t}_{\mathscr{A}}(H^*(j/2), \mathbb{F}_2) \longrightarrow \operatorname{Ext}^{s,t}_{\mathscr{A}}(C \otimes H^*(S/2), \mathbb{F}_2) \to \dots$$

where we can rewrite the right hand term as $\operatorname{Ext}^{s,t}_{A(2)}(H^*(S/2), \mathbb{F}_2)$, which we computed above.

The Adams spectral sequence for j was studied by Davis (1975). The sequence for $j/2$ is simpler, and is implicitly described on page 41 of Davis–Mahowald (1989). A more direct argument has been studied by Bruner:

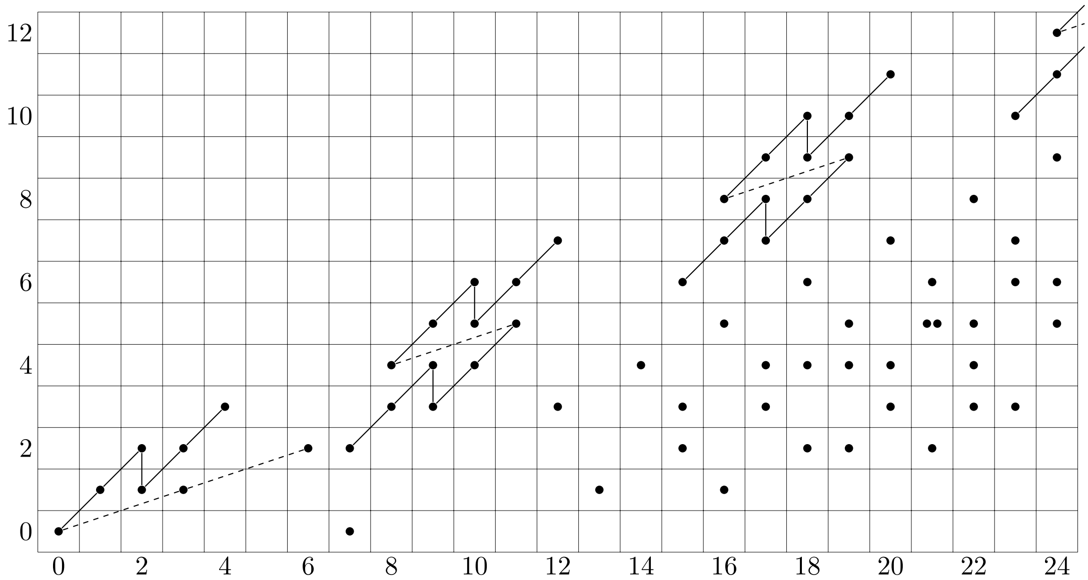


Figure 39: Adams chart for $j/2$

Proposition 9.19 (Bruner). *The exact sequence above splits, so that the Adams E_2-term for $j/2$ is*

$$E_2^{s,t}(j/2) \cong \mathrm{Ext}_{\mathscr{A}}^{s,t}(C \otimes H^*(S/2), \mathbb{F}_2) \oplus \mathrm{Ext}_{\mathscr{A}}^{s,t}(K \otimes H^*(S/2), \mathbb{F}_2)\,.$$

There is a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}_{\mathscr{A}}^{s-2,t-1}(K \otimes H^*(S/2), \mathbb{F}_2) &\longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t}(C \otimes H^*(S/2), \mathbb{F}_2) \\ &\longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(ko/2), \mathbb{F}_2) \oplus \mathrm{Ext}_{\mathscr{A}}^{s-1,t}(H^*(bspin/2), \mathbb{F}_2) \to 0 \end{aligned}$$

and the Adams d_2-differental is given by the left hand homomorphism, so that

$$E_3^{s,t}(j/2) \cong Ext_{\mathscr{A}}^{s,t}(H^*(ko/2), \mathbb{F}_2) \oplus \mathrm{Ext}_{\mathscr{A}}^{s-1,t}(H^*(bspin/2), \mathbb{F}_2)$$

is concentrated in bidegrees $(t-s, s)$ with $t - s \le 2s + 3$. There are no further differentials, so $E_3 = E_\infty$ for bidegree reasons.

This means that the Adams E_2-term for $j/2$ contains a copy of the charts for $ko/2$ and for $bspin/2$ (shifted up one filtration), consisting of two lightning flashes every eight degrees, plus two copies of $\mathrm{Ext}_{\mathscr{A}}$ for $K \otimes H^*(S/2)$, starting in bidegrees $(t-s, s) = (7, 0)$ and $(6, 2)$, respectively. The d_2-differentials make these two copies cancel, leaving only the lightning flashes at E_3 and beyond. See Figure 39. ((Add differentials to chart?))

Lemma 9.20. *The map $e/2\colon S/2 \to j/2$ induces a surjective homomorphism $H^*(j/2) \to H^*(S/2)$, and the induced homomorphism*

$$\mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(S/2), \mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s,t}(H^*(j/2), \mathbb{F}_2)$$

is an isomorphism for $s \ge 4$ and $t - s \le 2s + 3$.

Proof. We write c for the homotopy fiber of $e\colon S \to j$, so that there is a cofiber sequence

$$c \to S \xrightarrow{e} j \to \Sigma c$$

inducing the short exact sequences

$$0 \to H^*(\Sigma c) \to H^*(j) \to H^*(S) \to 0$$

and

$$0 \to H^*(\Sigma c/2) \to H^*(j/2) \to H^*(S/2) \to 0\,.$$

We get a long exact sequence

$$\cdots \to \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(S/2), \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(j/2), \mathbb{F}_2) \longrightarrow \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(\Sigma c/2), \mathbb{F}_2) \to \dots\,.$$

Here $H^*(\Sigma c/2) = H^*(\Sigma c) \otimes H^*(S/2)$ is $A(0)$-free and concentrated in degrees $* \ge 7$, so

$$\mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(\Sigma c/2), \mathbb{F}_2) = 0$$

for $(t-7)-s < 2s - \epsilon(s)$, by Adams vanishing in the form of Proposition 6.20. In particular,

$$\mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(S/2), \mathbb{F}_2) \to \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(j/2), \mathbb{F}_2)$$

is surjective for $t-s \le 2s+3$. For $s \ge 4$ the dimensions of the Ext-groups agree in this range, so these surjections are in fact isomorphisms. □

Remark 9.21. We call c the cokernel-of-J spectrum, to go with the image-of-J spectrum j. The composite $\pi_*(c) \to \pi_*(S) \to \mathrm{cok}(J)$ is almost an isomorphism, except for the μ-classes.

Proposition 9.22. *In the Adams spectral sequence*

$$E_2^{s,t}(S/2) = \mathrm{Ext}^{s,t}_{\mathscr{A}}(H^*(S/2), \mathbb{F}_2) \Longrightarrow \pi_{t-s}(S/2)$$

the classes in bidegree $(t-s, s)$ with $s \ge 4$ and $t-s \le 2s+3$ survive to E_∞. In degrees $t-s \ge 10$ they represent subgroups $\mathbb{Z}/2 \subset \pi_{8i-1}(S/2)$, $(\mathbb{Z}/2)^2 \subset \pi_{8i}(S/2)$, $\mathbb{Z}/2 \oplus \mathbb{Z}/4 \subset \pi_{8i+1}(S/2)$, $\mathbb{Z}/4 \oplus \mathbb{Z}/2 \subset \pi_{8i+2}(S/2)$, $(\mathbb{Z}/2)^2 \subset \pi_{8i+3}(S/2)$ and $\mathbb{Z}/2 \subset \pi_{8i+4}(S/2)$ that map isomorphically to $\pi_{8i-1}(j/2)$ through $\pi_{8i+4}(j/2)$, respectively.

Proof. The classes are infinite cycles for bidegree reasons. They cannot be boundaries, since we have a map of Adams spectral sequences

$$E_r^{*,*}(S/2) \longrightarrow E_r^{*,*}(j/2)$$

and their images in the Adams spectral sequence for $j/2$ are not boundaries. They represent subgroups in the abutment $\pi_*(S/2)$ that map isomorphically to the corresponding subgroups in the abutment $\pi_*(j/2)$, since the map of E_∞-terms is an isomorphism in the relevant filtrations. □

Proposition 9.23. *In the Adams spectral sequence for S, the five classes c_0, h_1c_0, Ph_1, h_1Ph_1 and Ph_2, as well as all of their images under powers of P, survive to E_∞. They represent classes in $\pi_*(S_2^\wedge)$ that map to generators of $\pi_*(j_2^\wedge)/2$ in degrees $8i \le * \le 8i+3$.*

((We are omitting the difficult degrees $* = 8i-1$ here.))

Proof. The classes are infinite cycles for bidegree reasons. They cannot be boundaries, since we have a map of Adams spectral sequences

$$E_r^{*,*}(S) \longrightarrow E_r^{*,*}(S/2)$$

that takes these classes to survivors in the right hand spectral sequence. The claim about abutments follows from the commutative square

$$\begin{tikzcd}
\pi_*(S_2^\wedge)/2 \arrow[r] \arrow[d, tail] & \pi_*(j_2^\wedge)/2 \arrow[d, tail] \\
\pi_*(S/2) \arrow[r] & \pi_*(j/2)\,.
\end{tikzcd}$$

□

Let us write $A[n] = \{x \in A \mid nx = 0\}$ for the exponent n subgroup of an abelian group A.

Proposition 9.24. *In the Adams spectral sequence for S, the six classes $h_0^3h_3$, c_0, h_1c_0, Ph_1, h_1Ph_1 and $h_1^2Ph_1 = h_0^2Ph_2$, as well as all of their images under powers of P, survive to E_∞. They represent classes (of order 2) in $\pi_*(S_2^\wedge)$ that map to generators of $\pi_*(j_2^\wedge)[2]$ in degrees $8i-1 \le * \le 8i+3$.*

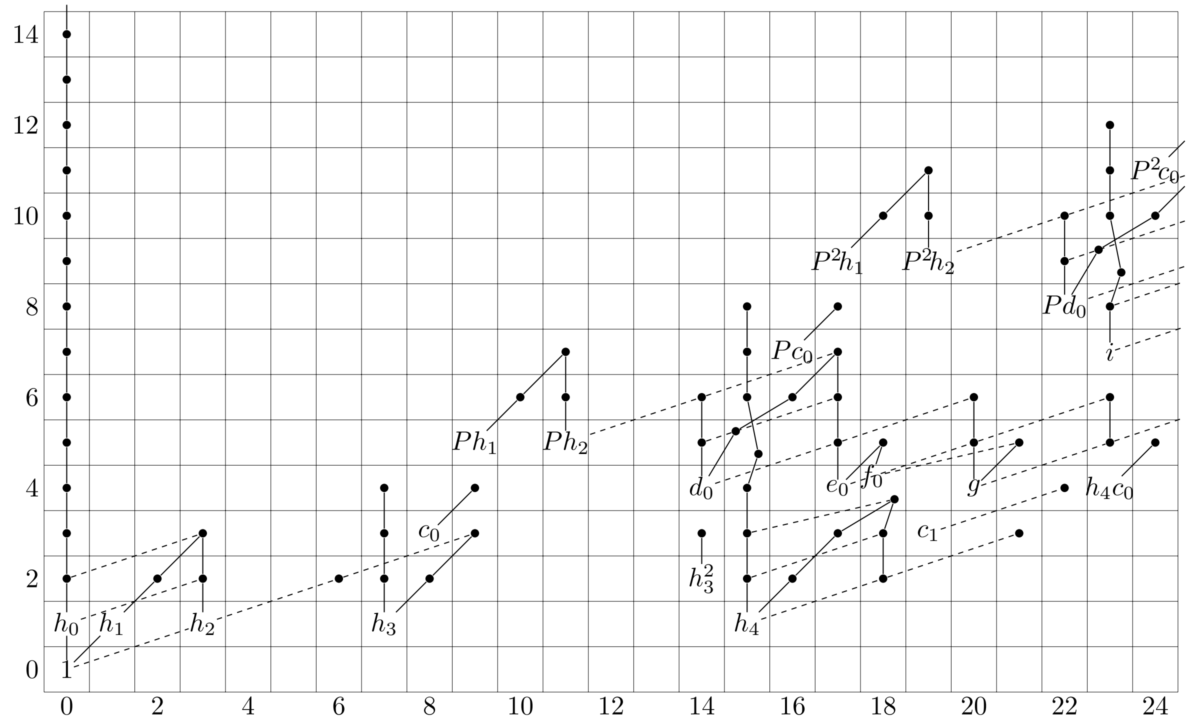


Figure 40: Adams E_2-term for S, $0 \le t-s \le 24$

Proof. The classes are too close to the vanishing line to support differentials. There are infinite survivors in bidegrees $(t-s,s)$ with $t-s \le 2s+3$ in the Adams spectral sequence for $S/2$, that map to these classes under the map of Adams spectral sequences

$$E_r^{*,*}(S/2) \longrightarrow E_r^{*,*}(\Sigma S) = E_r^{*,*-1}(S)\,.$$

Those infinite survivors represent a subgroup of $\pi_{*+1}(S/2)$ that maps isomorphically to the subgroup $\pi_*(j_2^\wedge)[2]$ of $\pi_*(j_2^\wedge)$, via the maps $S/2 \to j/2 \to \Sigma j_2^\wedge$. Hence the six classes represent a subgroup of $\pi_*(S_2^\wedge)[2] \subset \pi_*(S_2^\wedge)$ that maps onto $\pi_*(j_2^\wedge)[2]$, in view of the commutative square

$$\begin{tikzcd}
\pi_{*+1}(S/2) \arrow[r] \arrow[d, two heads] & \pi_{*+1}(j/2) \arrow[d, two heads] \\
\pi_*(S_2^\wedge)[2] \arrow[r] & \pi_*(j_2^\wedge)[2]\,.
\end{tikzcd}$$

If follows that the six classes remain linearly independent at E_∞, so none of them are hit by Adams differentials. $\square$

(($tmf/2$ and $tmf/(2, v_1^4)$?))

9.3 The Adams spectral sequence for S

Machine computation of $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$, e.g. using Bruner's `ext` program, gives the Adams E_2-term for the sphere given in Figures 40 and 41.

((Beware: f_0 and y ambiguous; $e_1' = e_1 + h_0^2 h_3 h_5$ and $Q' = Q + Pu$. Check f_1 and B_2.))

In this range we have the algebra generators given in Table 8 for the Yoneda product, grouped by Adams filtration s and topological degree $t-s$. The generators are named as in Mahowald–Tangora (1967), Mahowald–Tangora (1968) and Tangora (1970), extending the notation from May's thesis (1964). ((Explain `ext` name.))

We now discuss the Adams spectral sequence differentials implied by the image-of-J splitting.

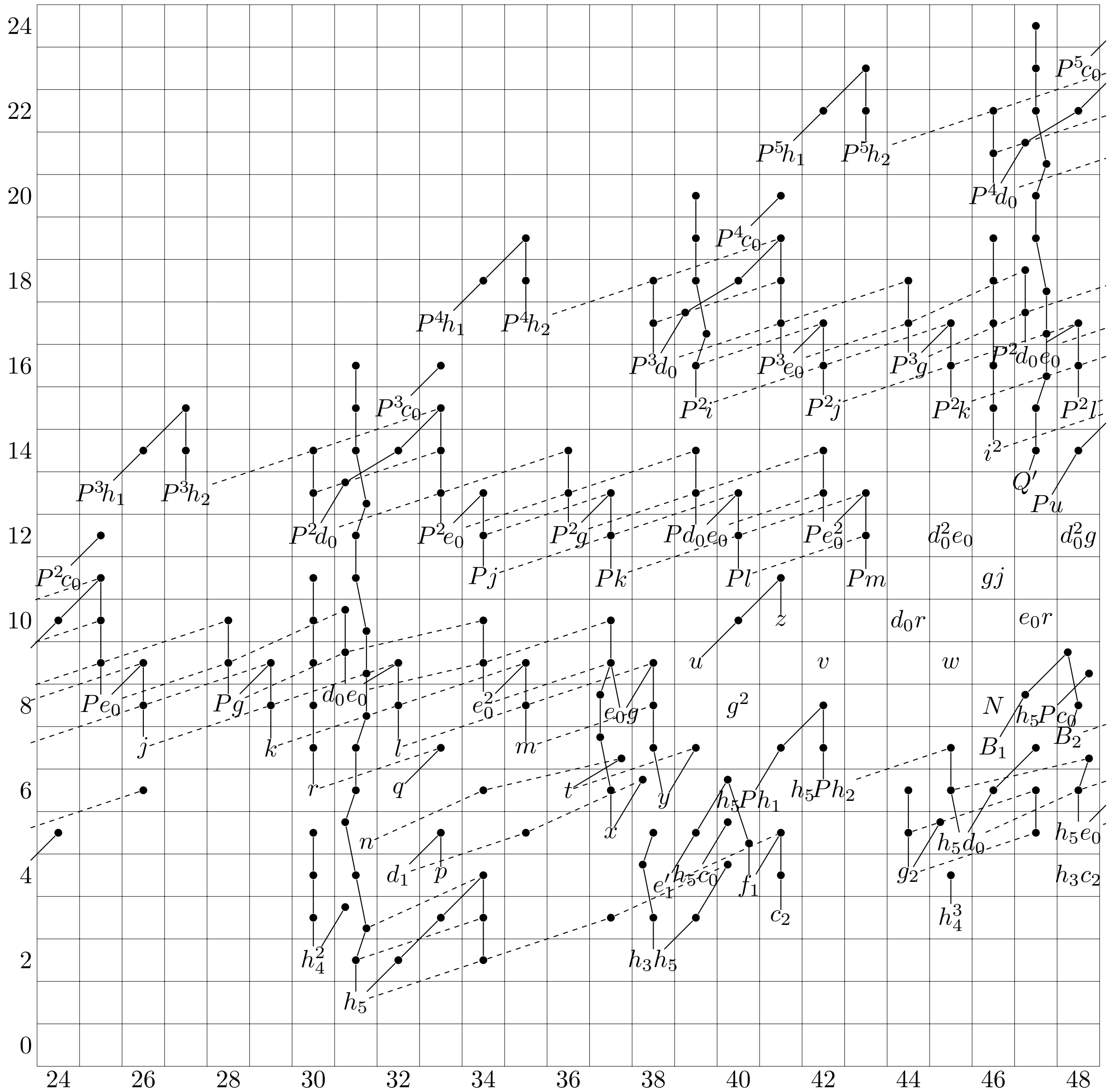


Figure 41: Adams E_2-term for S, $24 \leq t-s \leq 48$

Name	$t-s$	s	t	`ext`
h_0	0	1	1	1_0
h_1	1	1	2	1_1
h_2	3	1	4	1_2
h_3	7	1	8	1_3
h_4	15	1	16	1_4
h_5	31	1	32	1_5
c_0	8	3	11	3_3
c_1	19	3	22	3_9
c_2	41	3	44	3_{19}
d_0	14	4	18	4_3
e_0	17	4	21	4_5
f_0	18	4	22	$4_6(?)$
g	20	4	24	4_8
d_1	32	4	36	4_{13}
p	33	4	37	4_{14}
e_1	38	4	42	$4_{16}+4_{17}(?)$
f_1	40	4	44	$4_{19}+4_{20}(?)$
g_2	44	4	48	4_{22}
Ph_1	9	5	14	5_1
Ph_2	11	5	16	5_2
n	31	5	36	5_{13}
x	37	5	42	5_{17}
r	30	6	36	6_{10}
q	32	6	38	6_{12}
t	36	6	42	6_{14}
y	38	6	44	$6_{16}(?)$
Pc_0	16	7	23	7_3
i	23	7	30	7_5
j	26	7	33	7_6
k	29	7	36	7_7
l	32	7	39	7_{10}
m	35	7	42	7_{12}
B_1	46	7	53	7_{20}
B_2	48	7	55	$7_{22}(?)$

Name	$t-s$	s	t	`ext`
Pd_0	22	8	30	8_3
Pe_0	25	8	33	8_5
N	46	8	54	8_{20}
P^2h_1	17	9	26	9_1
P^2h_2	19	9	28	9_2
u	39	9	48	9_{18}
v	42	9	51	9_{19}
w	45	9	54	9_{20}
z	41	10	51	10_{14}
P^2c_0	24	11	35	11_3
Pj	34	11	45	11_7
P^2d_0	30	12	42	12_3
P^2e_0	33	12	45	12_5
P^3h_1	25	13	38	13_1
P^3h_2	27	13	40	13_2
Q	47	13	60	$13_{14}(?)$
Pu	47	13	60	$13_{15}(?)$
P^3c_0	32	15	47	15_3
P^2i	39	15	54	15_5
P^2j	42	15	57	15_6
P^3d_0	38	16	54	16_3
P^3e_0	41	16	57	16_5
P^4h_1	33	17	50	17_1
P^4h_2	35	17	52	17_2
P^4c_0	40	19	59	19_3
P^4d_0	46	20	66	20_3
P^4e_0	49	20	69	20_5
P^5h_1	41	21	62	21_1
P^5h_2	43	21	64	21_2
P^5c_0	48	23	71	21_3

Table 8: Algebra generators of $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$ for $t-s \leq 48$

Theorem 9.25. *There are nontrivial differentials $d_2(h_4) = h_0h_3^2$, $d_3(h_0h_4) = h_0d_0$ and $d_3(h_0^2h_4) = h_0^2d_0$.*

Proof. The image of J in $\pi_{15}(S_2^\wedge)$ is isomorphic to $\mathbb{Z}/32$, and we know that a generator is represented in Adams filtration $s = 4$, where the only nonzero class is $h_0^3h_4$. Hence the classes $h_0^ih_4$ for $3 \leq i \leq 7$ survive to E_∞, while the classes $h_0^ih_4$ for $0 \leq i \leq 2$ do not survive to E_∞. They cannot be boundaries for degree reasons, so they must support differentials.

The Adams differential $d_2(h_4) = h_0h_3^2$ is a consequence of the homotopy commutativity of S. The classes 2 and σ are represented by h_0 and h_3, respectively, so $2\sigma^2$ must be represented by the infinite cycle $h_0h_3^2$. By homotopy commutativity, $2\sigma^2 = 0$, which means that $h_0h_3^2$ must represent zero at E_∞, meaning that it is a boundary. The only possible class x to support a differential $d_r(x) = h_0h_3^2$ for $r \geq 2$ is $x = h_4$, giving the stated Adams differential.

It follows that $d_2(h_0h_4) = h_0^2h_3^2 = 0$, so h_0h_4 survives to E_3. If $d_3(h_0h_4) = 0$ then $d_3(h_0^2h_4) = 0$, and then $h_0^2h_4$ would have to be an infinite cycle, since there are no targets for later differentials on that class. This contradicts the order of the image of J in π_{15}, so we deduce that $d_3(h_0h_4)$ is nonzero. The only possible value is h_0d_0. Multiplication by h_0 then gives the value of $d_3(h_0^2h_4)$. □

Theorem 9.26. *There is a nontrivial d_2-differential $d_2(i) = h_0Pd_0$, which implies the nonzero differentials $d_2(h_0i) = h_0^2Pd_0$, $d_2(Pe_0) = h_1^2Pd_0$, $d_2(j) = h_0Pe_0$, $d_2(h_0j) = h_0^2Pe_0$, $d_2(h_0^2j) = h_0^3Pe_0$, $d_2(k) = h_0Pg$, $d_2(h_0k) = h_0^2Pg$, $d_2(l) = h_0d_0e_0$, $d_2(h_0l) = h_0^2d_0e_0$, $d_2(m) = h_0e_0^2$, $d_2(h_0m) = h_0^2e_0^2$, $d_2(y) = h_0^3x$, $d_2(h_0y) = h_0^4x$ and $d_2(h_0^2y) = h_0^5x$.*

Proof. We know that of the h_0-tower in topological degree $t - s = 23$ starting with i, only the top four classes survive to E_∞, since these generate a cyclic summand $\mathbb{Z}/16 \subset \pi_{23}(S)$ that maps isomorphically to $\pi_{23}(j_2^\wedge)$. Thus the classes i and h_0i cannot survive to E_∞. They cannot be boundaries, since any differential $d_r(x) = i$ would imply that $d_r(h_0^2x) = h_0^2i$ is a boundary, and similarly any differential $d_r(x) = h_0i$ would imply that $d_r(h_0x) = h_0^2i$ is a boundary. (For this part of the argument, it suffices to know that the top class, h_0^5i in Adams filtration $s = 12$ is not a boundary.) Hence i and h_0i must support nonzero differentials. The only possibilities for i are $d_2(i) = h_0Pd_0$ or ($d_2(i) = 0$ and) $d_3(i) = h_0^2Pd_0$. In the latter case, $d_2(h_0i) = 0$ and $d_3(h_0i) = 0$, which would make h_0i an infinite cycle. Since we know this does not happen, we must have $d_2(i) = h_0Pd_p$.

We claim that $d_2(t) = 0$. The alternative, $d_2(t) = h_0m$, would imply that $d_2(h_0t) = h_0^2m \neq 0$, which contradicts the relation $h_0t = 0$. It follows that $h_1y = h_2t$ is a d_2-cycle, so $h_1d_2(y) = 0$. This implies that $d_2(y) = 0$ or h_0^3x. Since $h_0^2y = h_2m$ supports the nonzero differential $d_2(h_0^2y) = h_0^5x$, we deduce that $d_2(y) = h_0^3x$. □

Theorem 9.27. *There are nontrivial differentials $d_2(h_5) = h_0h_4^2$, $d_3(h_0^3h_5) = h_0r$ and $d_4(h_0^8h_5) = h_0P^2d_0$, which imply the nonzero differentials $d_2(h_0h_5) = h_0^2h_4^2$, $d_2(h_0^2h_5) = h_0^4h_4^2$, $d_3(h_0^4h_5) = h_0^2r$, $d_3(h_0^5h_5) = h_0^3r$, $d_3(h_0^6h_5) = h_0^4r$, $d_3(h_0^7h_5) = h_0^5r$ and $d_4(h_0^9h_5) = h_0^2P^2d_0$.*

Proof. The image of J in $\pi_{31}(S_2^\wedge)$ is isomorphic to $\mathbb{Z}/64$, hence is represented by six classes in E_∞ in Adams filtrations $11 \leq s \leq 16$. In particular, a generator is represented by $h_0^{10}h_5$, so the classes $h_0^ih_5$ for $10 \leq i \leq 16$ survive to E_∞, while the ten classes for $0 \leq i \leq 9$ do not. They cannot be boundaries, as before, so they must support d_r-differentials for $r \geq 2$. The possible targets for these differentials are the 12 classes given by h_0-power multiples of $h_0h_4^2$, r and P^2d_0. The relations $h_0 \cdot h_0^3h_4^2 = 0$ and $h_0 \cdot h_0^5r = 0$ imply that at most one of the two classes $h_0^3h_4^2$ and r can be hit by these differentials, and likewise at most one of the two classes h_0^5r and P^2d_0 can be hit. Since there are at most ten targets for the ten classes that must support differentials, it follows that all the other possible targets are hit.

Starting in low filtrations, this tells us that $h_0h_4^2$ is a boundary, and $d_2(h_5) = h_0h_4^2$ is the only possibility. This implies $d_2(h_0h_5) = h_0^2h_4^2$, $d_2(h_0^2h_5) = h_0^3h_4^2$ and $d_2(h_0^ih_5) = 0$ for $i \geq 3$.

The seven remaining classes $h_0^ih_5$ with $3 \leq i \leq 9$ must support d_r-differentials, for $r \geq 3$, that hit all but one of the eight classes given by h_0-multiples of h_0r and P^2d_0. Since at most one of h_0^5r and P^2d_0 can be hit, the other possible targets, including h_0r, must be hit, which implies that $d_3(h_0^3h_5) = h_0r$. This tells us that $d_3(h_0^4h_5) = h_0^2r$, $d_3(h_0^5h_5) = h_0^3r$, $d_3(h_0^6h_5) = h_0^4r$, $d_3(h_0^7h_5) = h_0^5r$ and $d_3(h_0^ih_5) = 0$ for $i \geq 8$. We should argue that all but the last of these are in fact nonzero differentials. This can only fail if the target classes h_0^ir for $1 \leq r \leq 5$ were d_2-boundaries. The only candidates for such d_2-differentials would be $d_2(d_0e_0) = h_0^4r$ or $d_2(h_0d_0e_0) = h_0^5r$, but we have seen above that $d_0e_0 = d_2(l)$ and $h_0d_0e_0 = d_2(h_0l)$, so this would contradict the fact that $d_2 \circ d_2 = 0$ in any spectral sequence.

The two remaining classes $h_0^8h_5$ and $h_0^9h_5$ must support d_r-differentials for $r \geq 4$, and the only candidates for targets are $h_0P^2d_0$ and $h_0^2P^2d_0$. Hence $d_4(h_0^8h_5) = h_0P^2d_0$ and $d_4(h_0^9h_5) = h_0^2P^2d_0$. □

((A more complicated pattern occurs for $t-s = 63$, where other differentials intervene.))

Theorem 9.28. *There is a nontrivial d_2-differential $d_2(P^2i) = h_0P^3d_0$, which implies the nonzero differentials $d_2(h_0P^2i) = h_0^2P^3d_0$, $d_2(P^3e_0) = h_1^2P^3d_0$, $d_2(P^2j) = h_0P^3e_0$, $d_2(h_0P^2j) = h_0^2P^3e_0$, $d_2(h_0^2P^2j) = h_0^3P^3e_0$, $d_2(P^2k) = h_0P^3g$, $d_2(h_0P^2k) = h_0^2P^3g$, $d_2(P^2l) = h_0P^2d_0e_0$, $d_2(h_0P^2l) = h_0^2P^2d_0e_0$, $d_2(P^2m) = h_0P^2e_0^2$, $d_2(h_0P^2m) = h_0^2P^2e_0^2$, $d_2(R_1) = h_0^2x'$, $d_2(h_0R_1) = h_0^3x'$, $d_2(h_0^2R_1) = h_0^4x'$, $d_2(h_0^3R_1) = h_0^5x'$, $d_2(h_0^4R_1) = h_0^6x'$, $d_2(h_0^5R_1) = h_0^7x'$, $d_2(h_0^6R_1) = h_0^8x'$, $d_2(Q_1) = h_1^2x'$ and $d_2(h_1Q_1) = h_1^3x'$.*

Proof. Up to the statement about $d_2(R_1)$, this is very similar to the proof of the theorem about $d_2(i)$ and its consequences. ((The rest is easy, given Ext in this range.)) □

Theorem 9.29. *There are nontrivial differentials $d_2(Q') = h_0i^2$ and $d_3(h_0^5Q') = h_0P^4d_0$, which imply the nonzero differentials $d_2(h_0Q') = h_0^2i^2$, $d_2(h_0^2Q') = h_0^3i^2$, $d_2(h_0^3Q') = h_0^4i^2$, $d_2(h_0^4Q') = h_0^5i^2$, $d_3(h_0^5Q') = h_0P^4d_0$ and $d_3(h_0^6Q') = h_0^2P^4d_0$.*

Proof. The image of J in $\pi_{47}(S_2^\wedge)$ is isomorphic to $\mathbb{Z}_2/96 = \mathbb{Z}/32$, hence is represented by five classes in E_∞ in Adams filtrations $20 \leq s \leq 24$. In particular, a generator is represented by h_0^7Q', so the classes h_0^iQ' for $7 \leq i \leq 11$ survive to E_∞, while the seven classes for $0 \leq i \leq 6$ do not. They cannot be boundaries, as before, so they must support d_r-differentials for $r \geq 2$.

The possible targets for these differentials are the eight classes given by h_0-power multiples of h_0i^2 and P^4d_0. The relation $h_0 \cdot h_0^5i^2 = 0$ implies that at most one of the two classes $h_0^5i^2$ and P^4d_0 can be hit by these differentials. Since there are at most seven targets for the seven classes that must support differentials, it follows that all the other possible targets are hit.

In order of increasing Adams filtration, it follows that h_0i^2 must be hit by some $d_r(h_0^iQ')$ for $r \geq 2$, and $d_2(Q') = h_0i^2$ is the only possibility. This implies $d_2(h_0Q') = h_0^2i^2$, $d_2(h_0^2Q') = h_0^3i^2$, $d_2(h_0^3Q') = h_0^4i^2$ and $d_2(h_0^4Q') = h_0^5i^2$, while $d_2(h_0^5Q') = 0$. The remaining two classes h_0^5Q' and h_0^6Q' can now only hit $h_0P^4d_0$ and $h_0^2P^4d_0$, which means that $d_3(h_0^5Q') = h_0P^4d_0$ and $d_3(h_0^6Q') = h_0^2P^4d_0$. □

Theorem 9.30. *There is a nontrivial d_2-differential $d_2(P^4i) = h_0P^5d_0$, which implies the nonzero differentials $d_2(h_0P^4i) = h_0^2P^5d_0$, $d_2(P^5e_0) = h_1^2P^5d_0$, $d_2(P^4j) = h_0P^5e_0$, $d_2(h_0P^4j) = h_0^2P^5e_0$, $d_2(h_0^2P^4j) = h_0^3P^5e_0$, $d_2(P^4k) = h_0P^5g$, $d_2(h_0P^4k) = h_0^2P^5g$, $d_2(P^4l) = h_0P^4d_0e_0$, $d_2(h_0P^4l) = h_0^2P^4d_0e_0$, $d_2(P^4m) = h_0P^4e_0^2$, $d_2(h_0P^4m) = h_0^2P^4e_0^2$ ((ETC)).*

Proof. Through the statement about $d_2(h_0P^4m)$, this is very similar to the proof of the theorem about $d_2(i)$ and its consequences. ((Need Ext for $69 \leq t-s \leq 80+$ for full statement.)) □

Theorem 9.31. *There are nontrivial differentials $d_2(e_0) = h_1^2d_0$, $d_2(f_0) = h_0^2e_0$ and $d_2(h_0f_0) = h_0^3e_0$.*

Proof. There is a multiplicative relation $h_0^2y = f_0g$. Since $d_2(g) = 0$ and $d_2(h_0^2y) = h_0^5x \neq 0$, it follows from the Leibniz rule that $d_2(f_0) \neq 0$. The only possibility is $d_2(f_0) = h_0^2e_0$. Multiplying by h_0 gives $d_2(h_0f_0) = h_0^3e_0$, and dividing by h_1 gives $d_2(e_0) = h_1^2d_0$. □

Theorem 9.32. *There is a nontrivial differential $d_2(h_0Pj) = h_0^2P^2e_0$, which implies the nonzero differentials $d_2(P^2e_0) = h_1^2P^2d_0$, $d_2(Pj) = h_0P^2e_0$, $d_2(h_0^2Pj) = h_0^3P^2e_0$, $d_2(Pk) = h_0P^2g$, $d_2(h_0Pk) = h_0^3P^2g$, $d_2(h_0^2Pk) = h_0^3P^2g$, $d_2(Pl) = h_0Pd_0e_0$, $d_2(h_0Pl) = h_0^3Pd_0e_0$, $d_2(h_0^2Pl) = h_0^3Pd_0e_0$, $d_2(Pm) = h_0Pe_0^2$, $d_2(h_0Pm) = h_0^3Pe_0^2$ and $d_2(h_0^2Pm) = h_0^3Pe_0^2$.*

Proof. This follows as above from the multiplicative relation $h_0^6R_1 = g \cdot h_0Pj$, where $d_2(h_0^6R_1) \neq 0$ and $d_2(g) = 0$. □

Theorem 9.33. *There is a nontrivial differential $d_2(h_0P^3j) = h_0^2P^4e_0$, which implies the nonzero differentials $d_2(P^4e_0) = h_1^2P^4d_0$, $d_2(P^3j) = h_0P^4e_0$, $d_2(h_0^2P^3j) = h_0^3P^4e_0$ ((ETC)).*

Proof. ((Use differential on $g \cdot h_0P^3j$, or periodicity.)) □

Here are the nonobvious multiplicative consequences of these differentials, for $t-s \leq 49$.

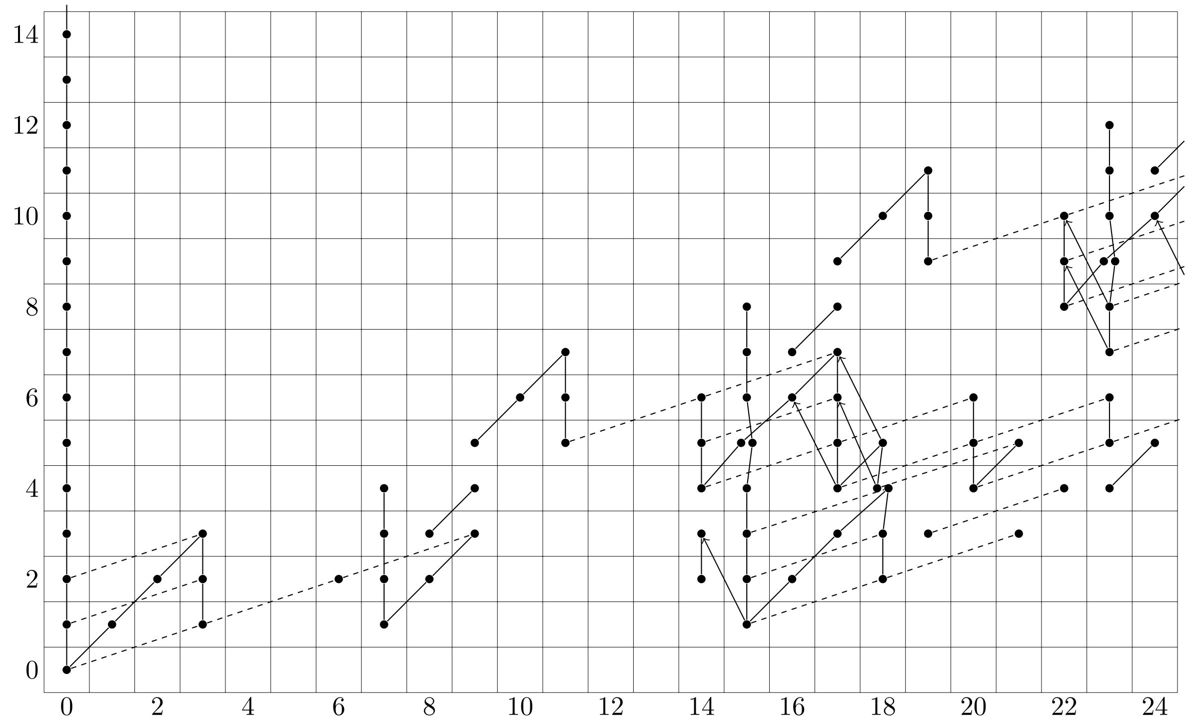


Figure 42: Adams d_2-differentials for S, $0 \le t - s \le 24$

Lemma 9.34. *$d_2(d_0e_0) = h_1^2Pg = 0$, $d_2(h_3h_5) = h_0h_3h_4^2 = 0$, $d_2(h_5c_0) = h_0h_4^2c_0 = 0$, $d_2(h_4^3) = h_0h_3^2h_4^2 = 0$, $d_2(h_5Pc_0) = h_0h_4^2Pc_0 = 0$, $d_2(P^2d_0e_0) = h_1^2P^3g = 0$ and $d_2(ij) = h_0Pd_0j + h_0Pe_0i = 0$.*

Lemma 9.35. *The differential d_2 is zero on the remaining algebra generators in degrees $t - s \le 49$, except for the three cases c_2, v and B_1.*

Proof. The differential d_2 is zero on h_1, n, d_1, q, t, e_1, z, Pu by h_0-linearity. It is zero on p by h_1-linearity. It vanishes on c_1 and r since the possible targets support nonzero d_2-differentials. It is zero on the remaining algebra generators in degrees $t - s \le 49$, with the exception of c_2, v and B_1, since the target groups are trivial. □

Theorem 9.36. *There are nontrivial differentials $d_2(c_2) = h_0f_1$ and $d_2(v) = h_0z$, while $d_2(B_1) = 0$.*

This implies the nonzero differentials $d_2(h_0c_2) = h_0^2f_1$, $d_2(h_3c_2) = h_0h_2g_2$, $d_2(h_5e_0) = h_1^2h_5d_0$ and $d_2(h_5f_0) = h_0^2h_5e_0$.

((Proof postponed.))

Remark 9.37. The differential $d_2(c_2) = h_0f_1$ was overlooked in Mahowald–Tangora (1967), but discovered by means of Steenrod operations in $\mathrm{Ext}_{\mathscr{A}}$ by Milgram (1972), and also corrected in Barratt–Mahowald–Tangora (1970).

We draw these d_2-differentials in Figures 42 and 44, with bullets replacing the named classes.

This leads to the E_3-term given in Figures 43 and 45.

Theorem 9.38. *The classes h_1h_4 and h_2h_4 survive to E_∞.*

((Can be proved using H_∞ structure, see Bruner (1986) Proposition VI.1.6.))

Theorem 9.39. *The class h_4c_0 survives to E_∞.*

Proof. Assume, for a contradiction, that $d_4(h_4c_0) = Pd_0$. Then $d_4(h_1h_4c_0) = h_1Pd_0$ is nonzero at E_4. But h_1h_4 and c_0 are permanent cycles, hence so is their product. □

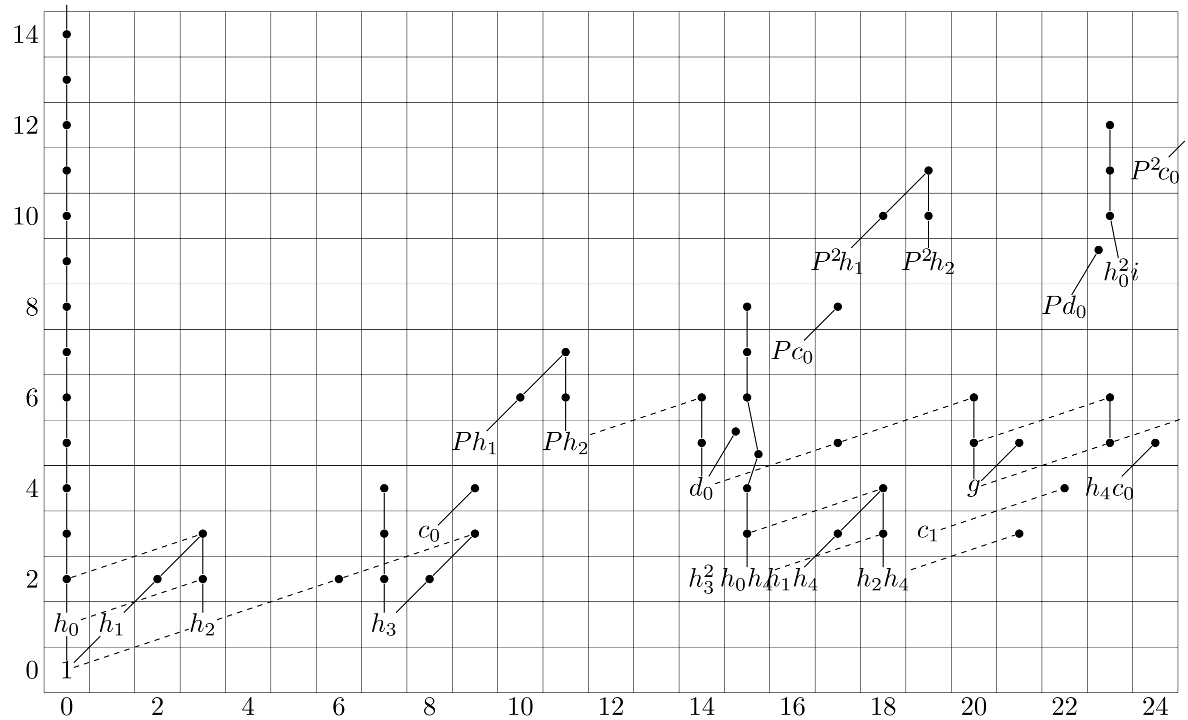


Figure 43: Adams E_3-term for S, $0 \le t - s \le 24$

We draw the d_3-differentials in dimensions $0 \le t \le 24$ in Figure 46, leaving the $E_3 = E_\infty$-term shown in Figure 47. The dotted lines represent hidden h_0- and h_1-extensions, to be explained in the following theorem.

Theorem 9.40. *The table lists $\pi_n(S_2^\wedge)$ for $0 \le n \le 24$, together with generators of the cyclic summands*

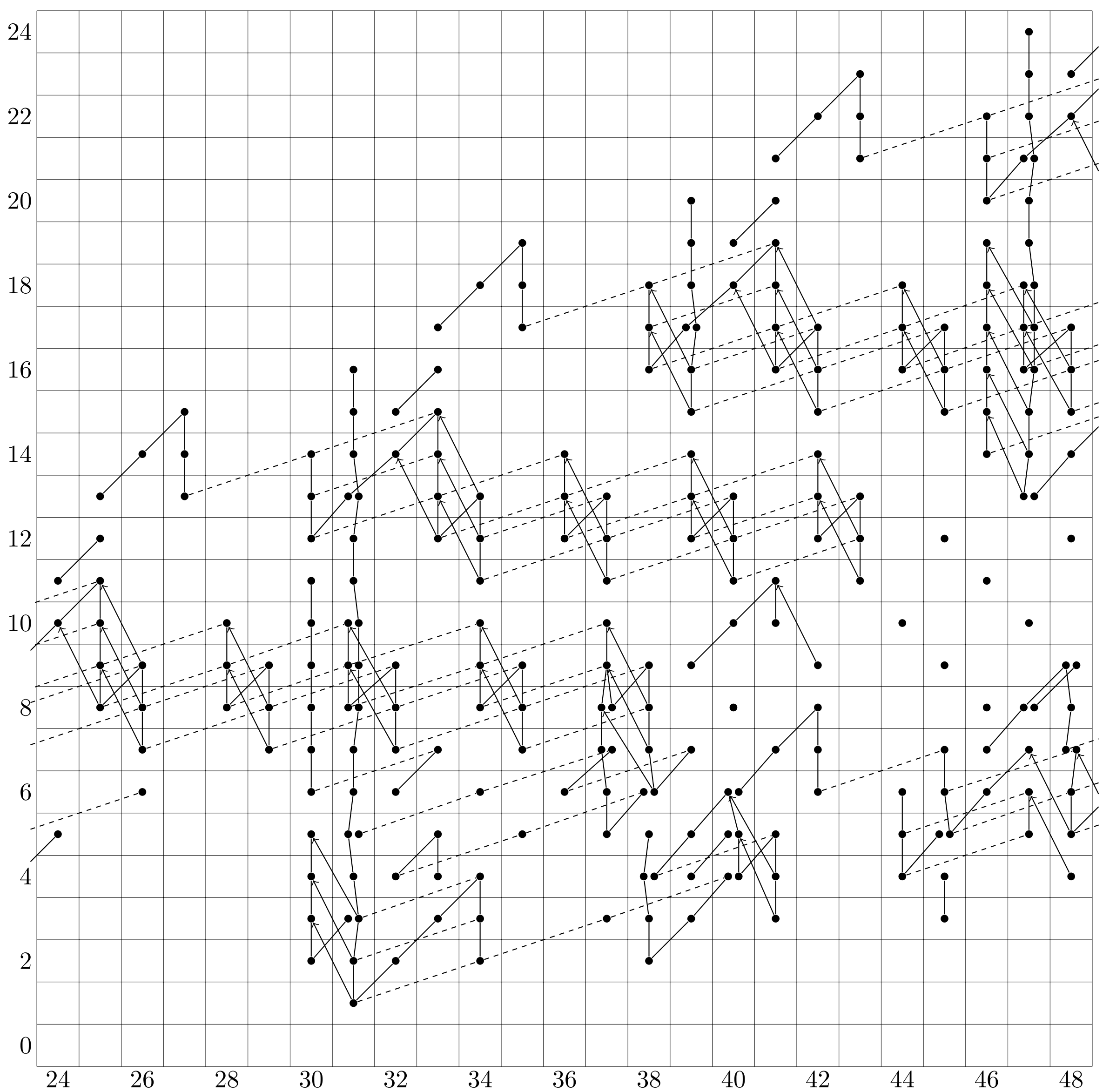


Figure 44: Adams d_2-differentials for S, $24 \leq t - s \leq 48$

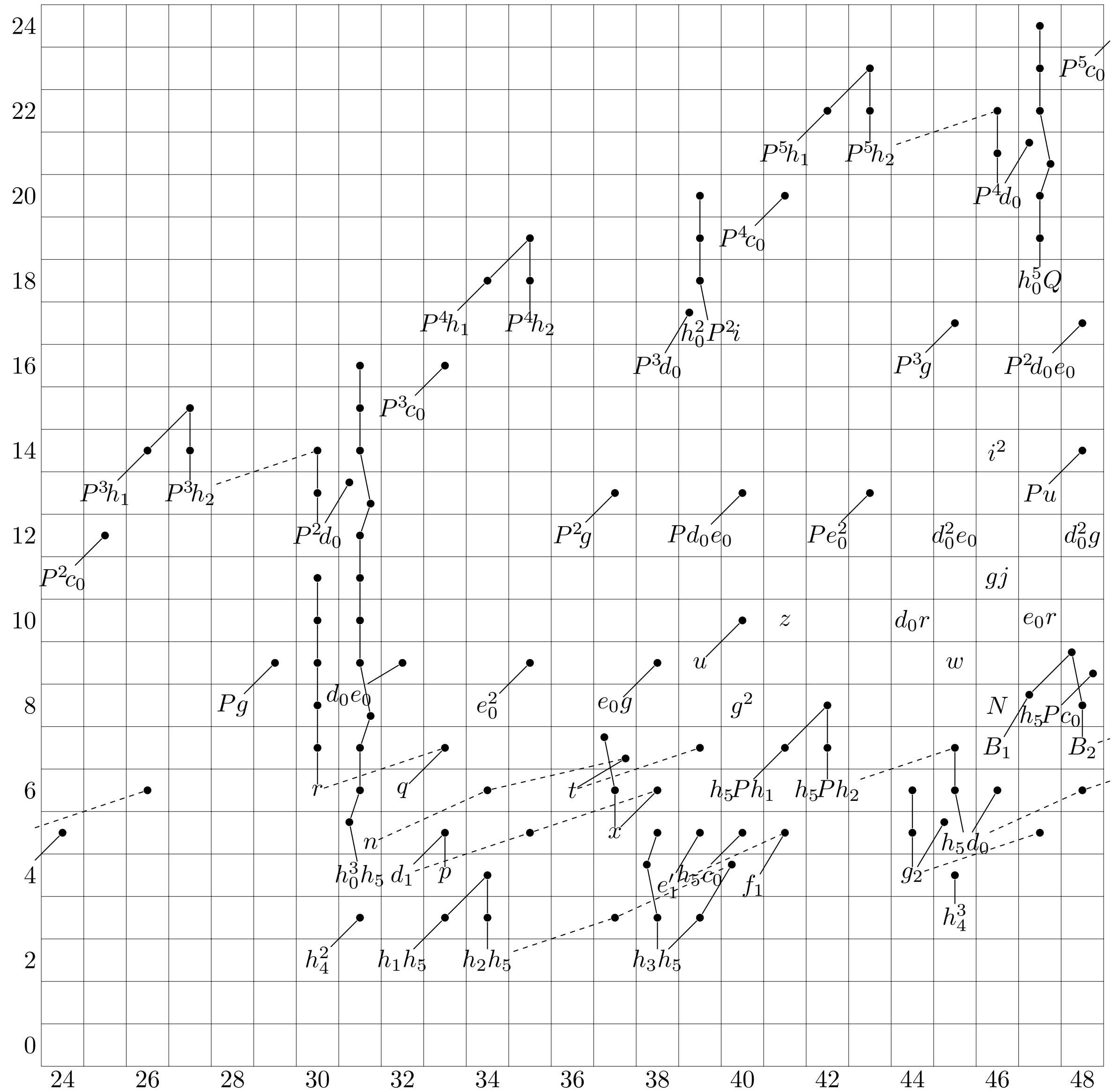


Figure 45: Adams E_3-term for S, $24 \leq t-s \leq 48$

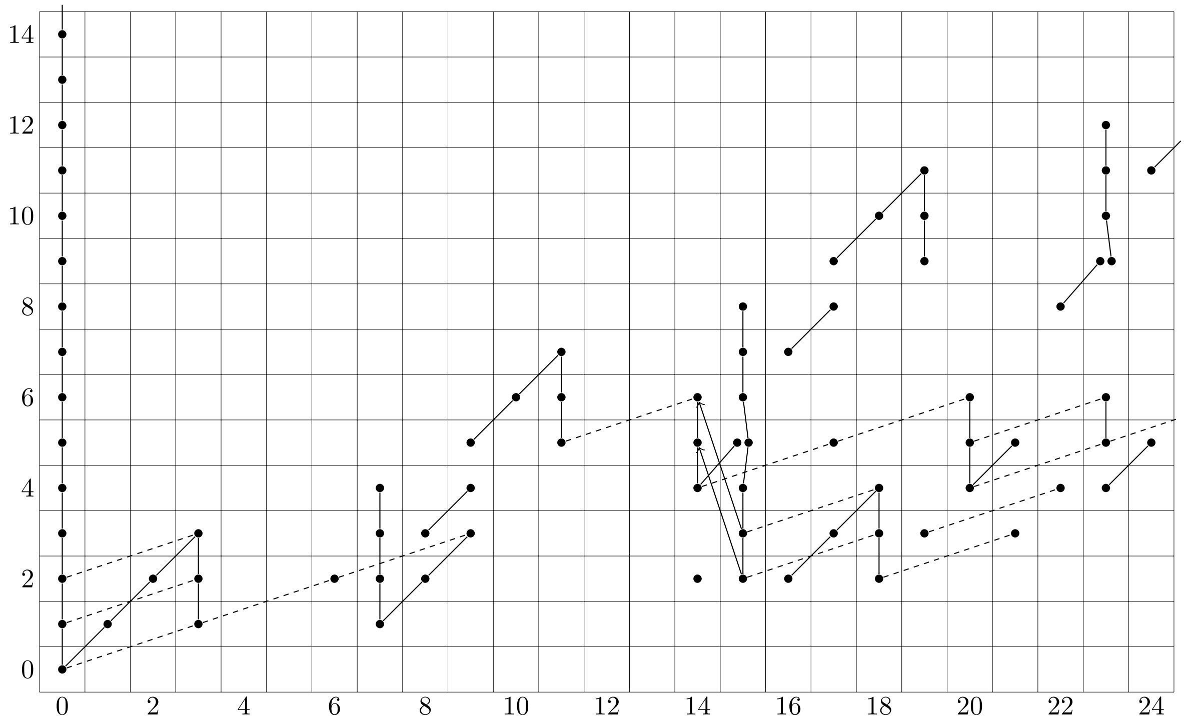


Figure 46: Adams d_3-differentials for S, $0 \le t - s \le 24$

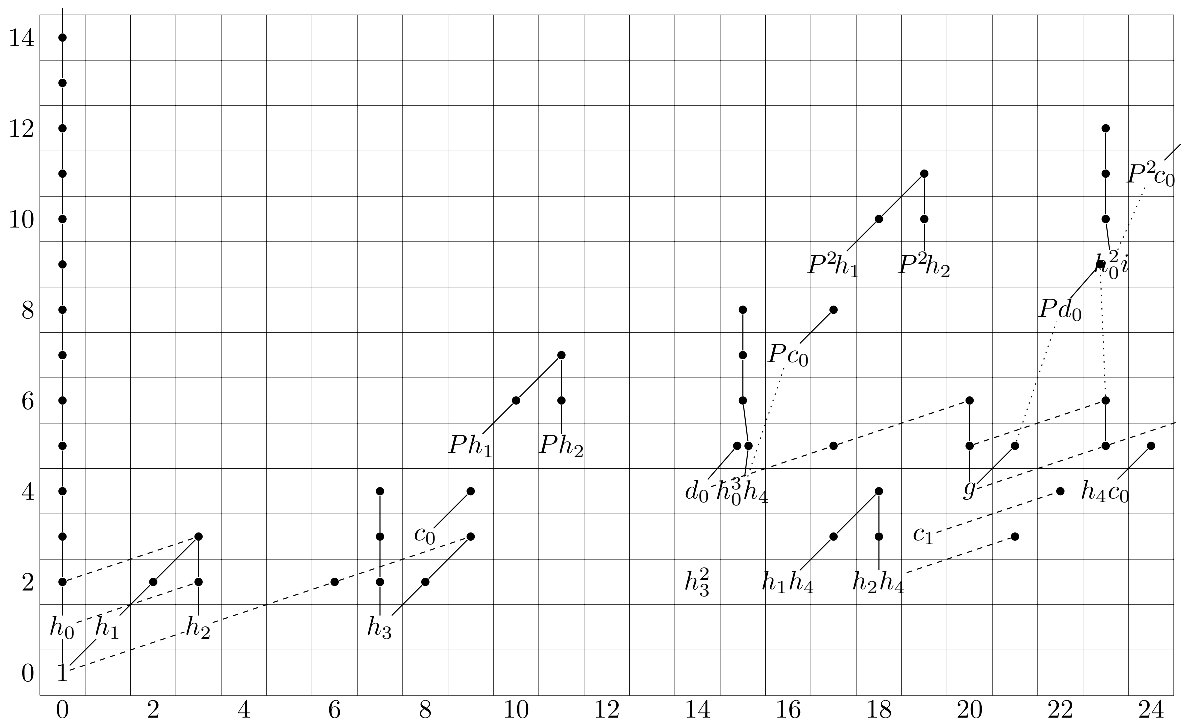


Figure 47: Adams E_∞-term for S, $0 \le t - s \le 24$

and Adams E_∞ classes representing these generators.

n	$\pi_n(S_2^\wedge)$	*gen.*	E_∞*-rep.*
0	$\mathbb{Z}_2$	1	1
1	$\mathbb{Z}/2$	η	h_1
2	$\mathbb{Z}/2$	η^2	h_1^2
3	$\mathbb{Z}/8$	ν	h_2
4	0		
5	0		
6	$\mathbb{Z}/2$	ν^2	h_2^2
7	$\mathbb{Z}/16$	σ	h_3
8	$(\mathbb{Z}/2)^2$	ϵ, $\eta\sigma$	c_0, h_1h_3
9	$(\mathbb{Z}/2)^3$	μ, $\eta\epsilon$, $\eta^2\sigma$	Ph_1, h_1c_0, $h_1^2h_3$
10	$\mathbb{Z}/2$	$\eta\mu$	h_1Ph_1
11	$\mathbb{Z}/8$	ζ	Ph_2
12	0		
13	0		
14	$(\mathbb{Z}/2)^2$	κ, σ^2	d_0, h_3^2
15	$\mathbb{Z}/2 \oplus \mathbb{Z}/32$	$\eta\kappa$, ρ	h_1d_0, $h_0^3h_4$
16	$(\mathbb{Z}/2)^2$	$\eta\rho$, η^*	Pc_0, h_1h_4
17	$(\mathbb{Z}/2)^4$	$\bar\mu$, $\eta^2\rho$, $\nu\kappa$, $\eta\eta^*$	P^2h_1, h_1Pc_0, h_2d_0, $h_1^2h_4$
18	$\mathbb{Z}/2 \oplus \mathbb{Z}/8$	$\eta\bar\mu$, ν^*	$h_1P^2h_1$, h_2h_4
19	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$	$\bar\zeta$, $\bar\sigma$	P^2h_2, c_1
20	$\mathbb{Z}/8$	$\bar\kappa$	g
21	$(\mathbb{Z}/2)^2$	$\eta\bar\kappa$, $\nu\nu^*$	h_1g, $h_2^2h_4$
22	$(\mathbb{Z}/2)^2$	$\eta^2\bar\kappa$, $\nu\bar\sigma$	Pd_0, h_2c_1
23	$\mathbb{Z}/16 \oplus \mathbb{Z}/8 \oplus \mathbb{Z}/2$	?, $\nu\bar\kappa$, ?	h_0^2i, h_2g, h_4c_0
24	$(\mathbb{Z}/2)^2$		P^2c_0, $h_1h_4c_0$

Proof. Using the splitting of $\pi_*(j_2^\wedge)$ off from $\pi_*(S_2^\wedge)$, the additive structure in degrees $0 \le n \le 20$ is straightforward. For instance, $2 \cdot \eta\eta^* = 0$ since $2\eta = 0$. The nontrivial fact is that there is a hidden η-multiplication from $\eta\bar\kappa$, represented by h_1g, to $\eta^2\bar\kappa$, represented by Pd_0. See Mahowald–Tangora (1967) Theorem 2.1.1. This implies that $2 \cdot \nu\nu^* = 0$, and that $2 \cdot 2\nu\bar\kappa \ne 0$. □

((Explain hidden η-multiplications by comparison with the Adams spectral sequence for $C\eta = S \cup_\eta e^2$?))

Theorem 9.41. *There are nontrivial differentials $d_3(r) = h_1Pg$, $d_3(d_0e_0) = h_0^5r$, $d_3(h_2h_5) = h_0p$, $d_3(e_1) = h_1t$ and $d_3(i^2) = h_1P^3g$.*

This implies the nonzero differential $d_3(d_0r) = h_1Pe_0^2$.

Remark 9.42. The differentials on r, e_1 and $i^2 = P^2r$ can be found from the H_∞ structure.

Corollary 9.43. *The class h_4^2 survives to E_∞, representing θ_4 in $\pi_{30}(S_2^\wedge)$.*

Theorem 9.44. *There are nontrivial differentials $d_4(\alpha) = P^2d_0$, where $\alpha = d_0e_0 + h_0^7h_5$, $d_4(e_0g) = P^2g$, $d_4(h_3h_5) = h_0x$, $d_4(Pd_0e_0) = P^3d_0$ and $d_4(P^2d_0e_0) = P^4d_0$.*

This implies the nonzero differentials $d_4(h_1d_0e_0) = h_1P^2d_0$, $d_4(h_1e_0g) = h_1P^2g$, $d_4(h_0h_3h_5) = h_0^2x$, $d_4(h_1Pd_0e_0) = h_1P^3d_0$, $d_4(d_0^2e_0) = P^3g$ and $d_4(h_1P^2d_0e_0) = h_1P^4d_0$.

Theorem 9.45. *The classes h_1h_5, $h_0h_2h_5$, t, f_1, h_5Ph_1, z, h_5Ph_2 and h_4^3 survive to E_∞.*

((To be confirmed: Are h_5Pc_0 and B_2 infinite cycles?))

We draw the d_3-differentials in dimensions $24 \le t \le 48$ in Figure 48, leaving the E_4-term shown in Figure 49. ((This assumes that $d_3 = 0$ on h_5Pc_0 and B_2.))

Next we draw the d_4-differentials in dimensions $24 \le t \le 48$ in Figure 50, leaving the E_5-term shown in Figure 51. If B_2 survives to E_∞ (as it does according to Kochman), then this is also the E_∞-term in this range of dimensions.

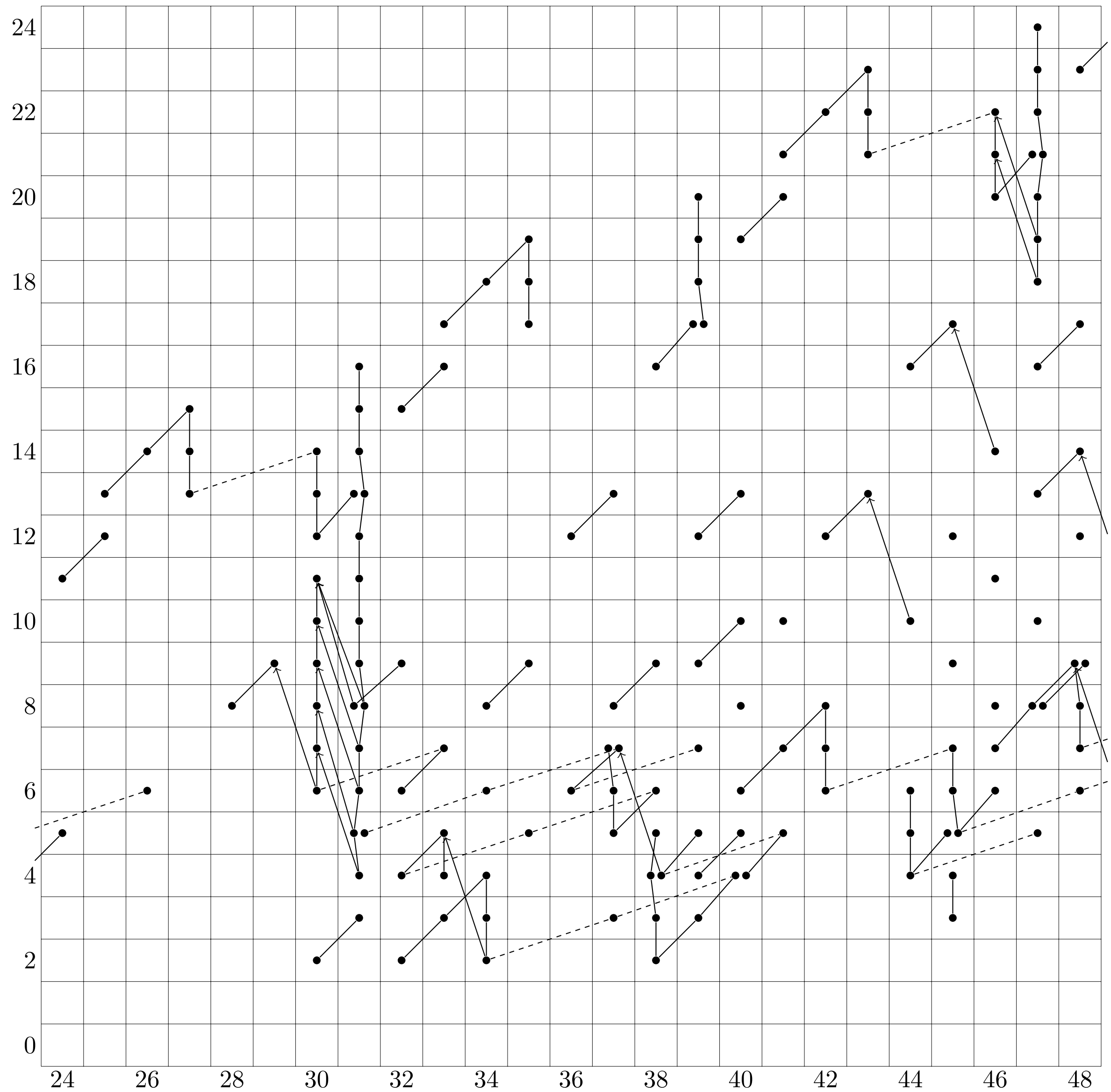


Figure 48: Adams d_3-differentials for S, $24 \leq t - s \leq 48$

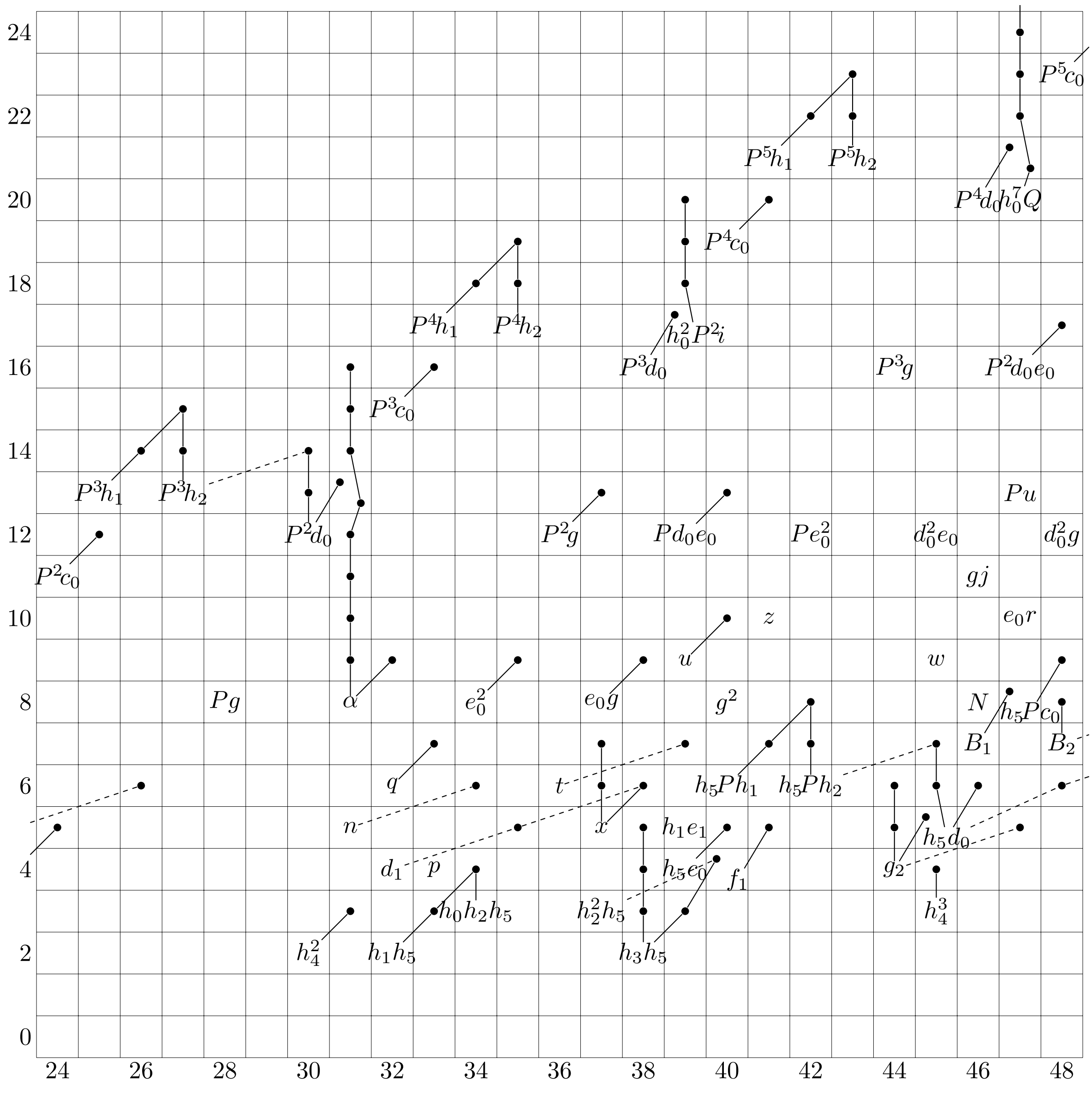


Figure 49: Adams E_4-term for S, $24 \leq t-s \leq 48$ ($\alpha = d_0e_0 + h_0^7h_5$)

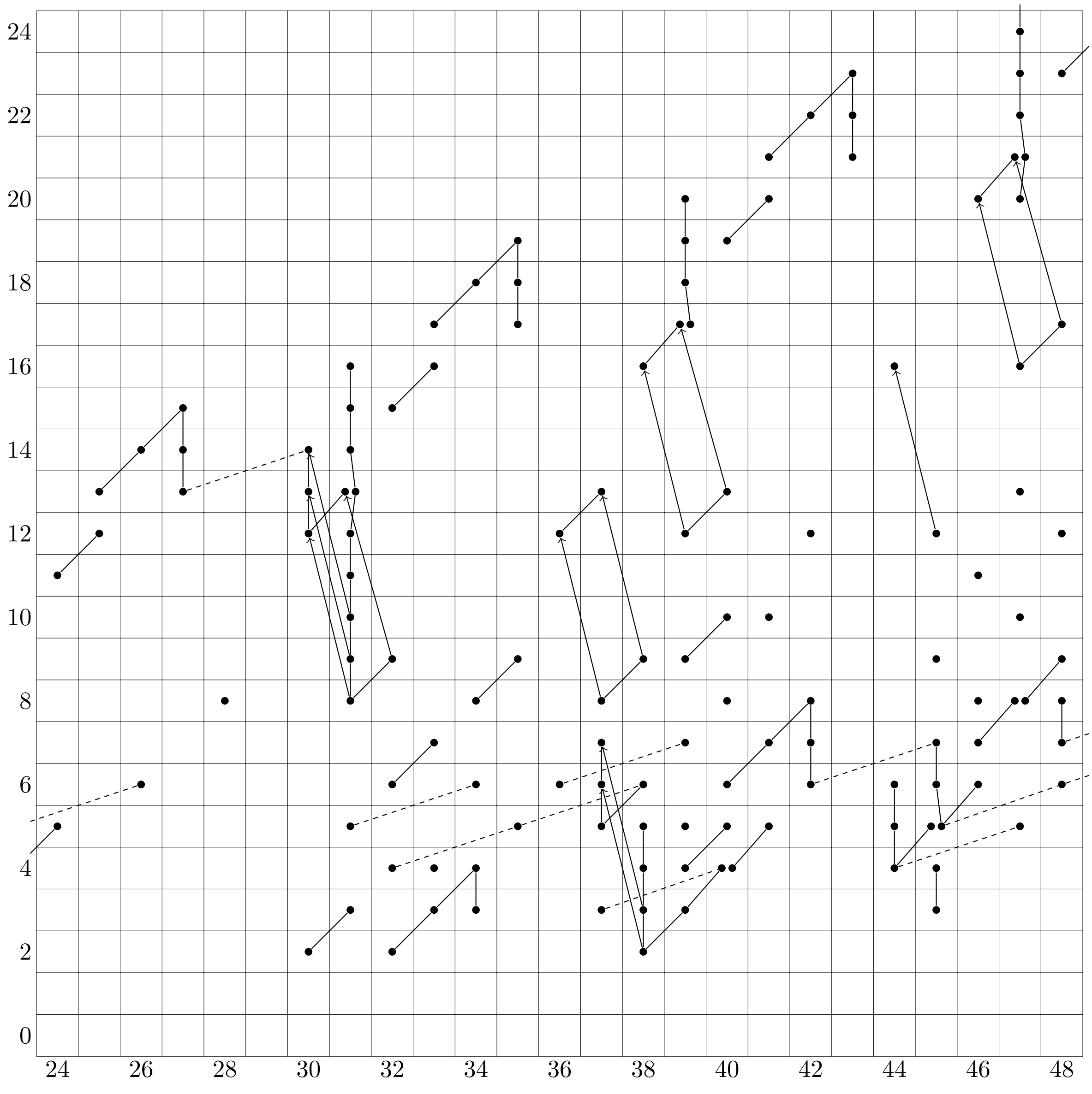


Figure 50: Adams d_4-differentials for S, $24 \leq t - s \leq 48$

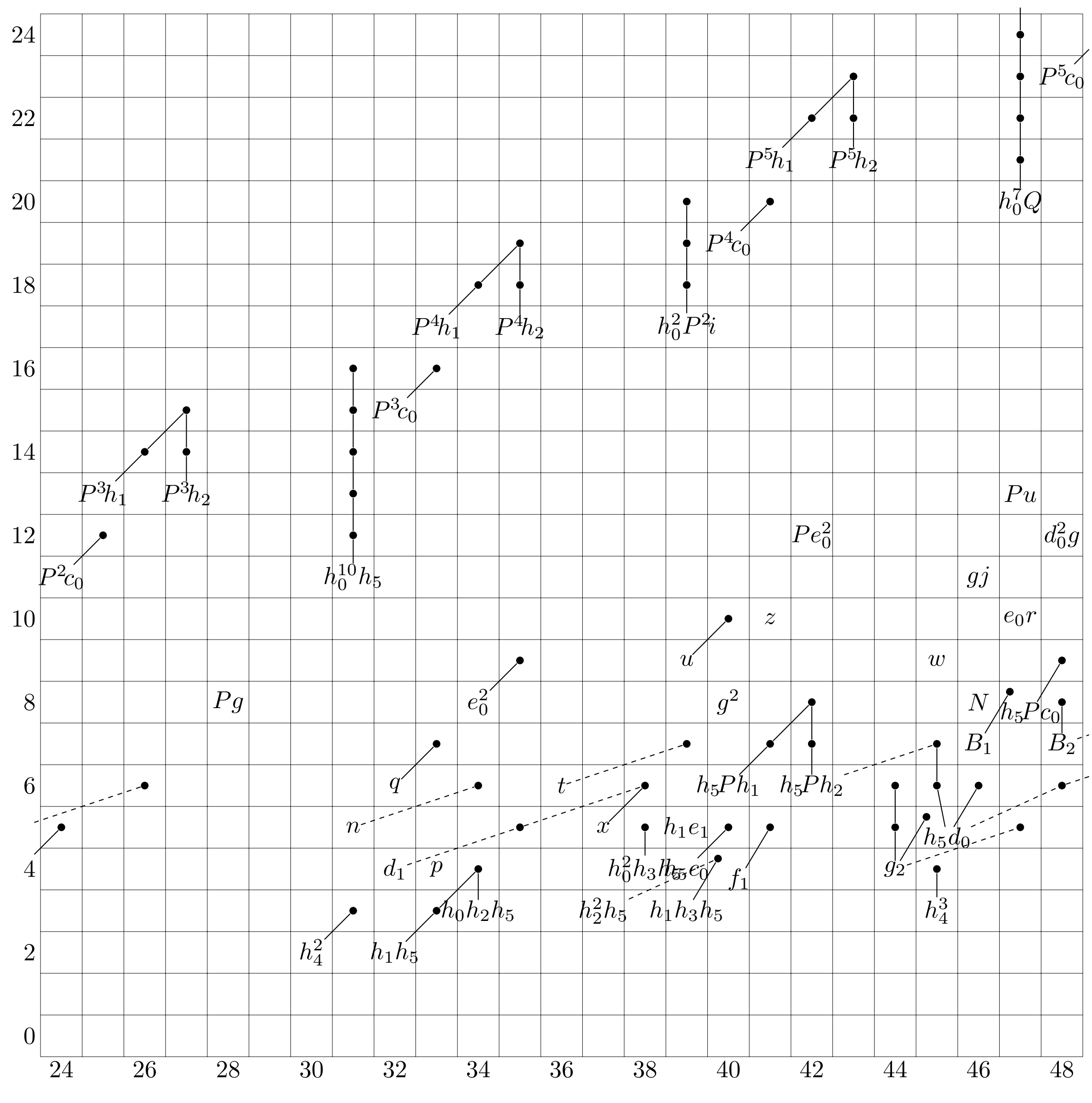


Figure 51: Adams $E_5 = E_\infty$-term for S, $24 \leq t - s \leq 48$

Theorem 9.46. *The table lists $\pi_n(S_2^\wedge)$ for $25 \le n \le 31$, together with generators of the cyclic summands and Adams E_∞ classes representing these generators.*

n	$\pi_n(S_2^\wedge)$	*gen.*	E_∞*-rep.*
25	$(\mathbb{Z}/2)^2$	$?,\ ?$	$P^3h_1,\ h_1P^2c_0$
26	$(\mathbb{Z}/2)^2$	$?,\ \nu^2\bar\kappa$	$h_1P^3h_1,\ h_2^2g$
27	$\mathbb{Z}/8$	$?$	P^3h_2
28	$\mathbb{Z}/2$	$?$	Pg
29	0		
30	$\mathbb{Z}/2$	θ_4	h_4^2
31	$\mathbb{Z}/64 \oplus (\mathbb{Z}/2)^2$	$?,\ ?,\ \eta\theta_4$	$h_0^{10}h_5,\ n,\ h_1h_4^2$

9.4 Power operations in $\pi_*(S)$

9.5 Steenrod operations in the Adams spectral sequence

The (graded) commutativity of the Yoneda product in the E_2-term $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2,\mathbb{F}_2)$ of the Adams spectral sequence for S can be seen as a consequence of the cocommutativity of the Hopf algebra $\mathscr{A}$. Moreover, this cocommutativity implies that there are Steenrod operations

$$Sq^i\colon \mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2,\mathbb{F}_2) \longrightarrow \mathrm{Ext}_{\mathscr{A}}^{s+t-i,2t}(\mathbb{F}_2,\mathbb{F}_2)$$

that double the internal degree (from t to $2t$) and increase the topological degree by i (from $t-s$ to $t-s+i = 2t-(s+t-i)$). This is the grading convention used by Bruner (1986), which is compatible with the grading for the power operations in homotopy that come from the H_∞ structure on S. (Other authors let Sq^i map Ext^s to Ext^{s+i}.)

It is known that $Sq^i(x) = 0$ for $i < t-s$, $Sq^{t-s}(x) = x^2$ and $Sq^i(x) = 0$ for $i > t$. We have $Sq^{2^i}(h_i) = h_{i+1}$ for $i \ge 0$ and the Cartan formula

$$Sq^k(xy) = \sum_{i+j=k} Sq^i(x)Sq^j(k)$$

holds.

Suppose that $x \in E_2^{s,t}$ survives to E_r for $r \ge 2$. By work of Kahn (1970), Milgram (1972), Mäkinen (1973) and Bruner (1986), we have formulas for the generically first differential on $Sq^i(x)$, in terms of $d_r(x)$, the Steenrod operations and the Adams spectral sequence representatives of the generators of $\mathrm{im}(J) \subset \pi_*(S)$.

Let $B_1 \dotplus B_2$ mean B_1, $B_1 + B_2$ or B_2 if B_1 has lower, equal or greater Adams filtration than B_2, respectively. Here is the first result in this general direction.

Theorem 9.47. *Let $x \in E_r^{s,t}$ is in topological degree $n = t-s$, and consider $x^2 = Sq^n(x) \in E_2^{2s,2t}$. Then*

$$d_{r+1}(x^2) = Sq^n(d_r(x)) \dotplus h_0 x d_r(x)$$

if n is even, and

$$d_{2r-1}(x^2) = Sq^n(d_r(x))$$

if n is odd.

These expressions imply that x^2 survives to E_{r+1} in the even case, and to E_{2r-1} in the odd case. The expressions may, of course, be zero in particular cases, in which case x^2 may survive to even later terms.

((See Bruner (1986) Theorem VI.1.1 for the general result.))

9.6 The Adams spectral sequence for tmf

The computation of $\mathrm{Ext}_{A(2)}^{*,*}(\mathbb{F}_2,\mathbb{F}_2)$, by Iwai–Shimada, Davis–Mahowald, Bruner or Nassau, gives the Adams E_2-term for tmf given in Figures 52, 53, 54 and 55.

((MT-wedge missing in Figure 55.))

((Recall algebra generators h_0, h_1, h_2, c_0, α, β, w_1, d_0 e_0, g, γ, δ and w_2. Maybe recall some common relations not visible in the charts.))

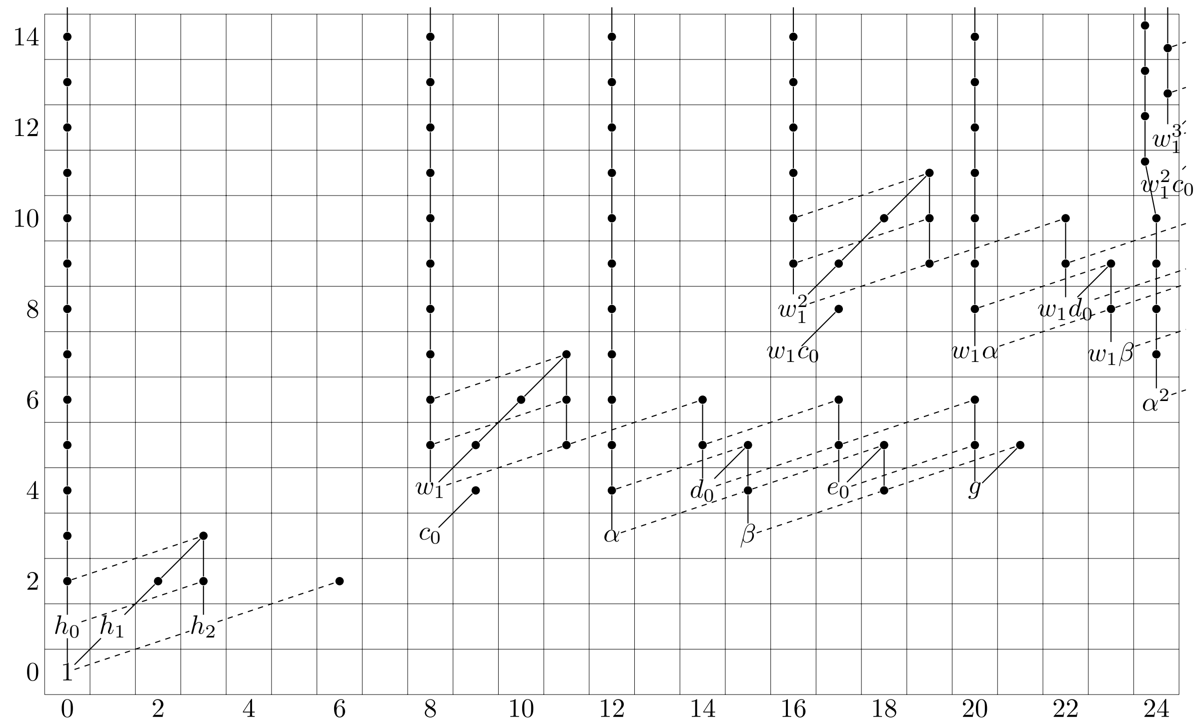


Figure 52: Adams E_2-term for tmf, $0 \le t - s \le 24$

Proposition 9.48. *The Steenrod operations Sq^i on $\mathrm{Ext}^{s,t}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$, for $t - s \le i \le t$, are given by*

$$\begin{aligned}
Sq^*(h_0) &= (h_0^2, h_1)\\
Sq^*(h_1) &= (h_1^2, h_2)\\
Sq^*(h_2) &= (h_2^2, 0)\\
Sq^*(c_0) &= (0, h_0 e_0, h_2\beta, 0)\\
Sq^*(\alpha) &= (\alpha^2, \gamma, 0, 0)\\
Sq^*(\beta) &= (\beta^2, 0, 0, 0)\\
Sq^*(w_1) &= (w_1^2, 0, 0, 0, ?)\\
Sq^*(d_0) &= (w_1 g, 0, \beta^2, 0, 0)\\
Sq^*(e_0) &= (d_0 g, \beta g, 0, 0, 0)\\
Sq^*(g) &= (g^2, 0, 0, 0, 0)\\
Sq^*(\gamma) &= (\gamma^2, ?, 0, 0, 0, 0)\\
Sq^*(\delta) &= (0, ?, ?, 0, 0, 0, 0, 0)\\
Sq^*(w_2) &= (w_2^2, 0, 0, 0, 0, 0, 0, 0, 0)\,.
\end{aligned}$$

Proof. Adams gives the Steenrod operations on the h_i, where we note that $h_3 = 0$ in Ext over $A(2)$. Bruner (Theorem VI.1.9) gives the Steenrod operations on c_0, d_0 and e_0, quoting Mukohda (1969) and Milgram (1972), where we note that $c_0^2 = 0$, f_0 maps to $h_2\beta$, $c_1 = 0$, $d_0^2 = w_1 g$, r maps to β^2, $d_1 = 0$, $e_0^2 = d_0 g$ and m maps to βg, all in Ext over $A(2)$. Applying Sq^{14} to $h_1\alpha = 0$ gives $h_2\alpha^2 = h_1^2 Sq^{13}(\alpha)$, which implies $Sq^{13}(\alpha) = \gamma$. ((ETC: Is $Sq^{12}(w_1) = g$? What is $Sq^{26}(\gamma)$?)) We also note that $\gamma^2 = \beta^2 g + h_1^2 w_2$ is nonzero, while $\delta^2 = 0$. □

Theorem 9.49. *There are nontrivial differentials $d_2(\alpha) = h_2 w_1$, $d_2(h_0\alpha) = w_1 h_0 h_2$, $d_2(h_0^2\alpha) = w_1 h_0^2 h_2$, $d_2(\beta) = h_0 d_0$, $d_2(h_0\beta) = h_0^2 d_0$, $d_2(h_2\beta) = h_0^2 e_0$, $d_2(\alpha d_0) = w_1 h_0 e_0$, $d_2(\beta d_0) = w_1 h_0 g$, $d_2(h_0\beta d_0) =$*

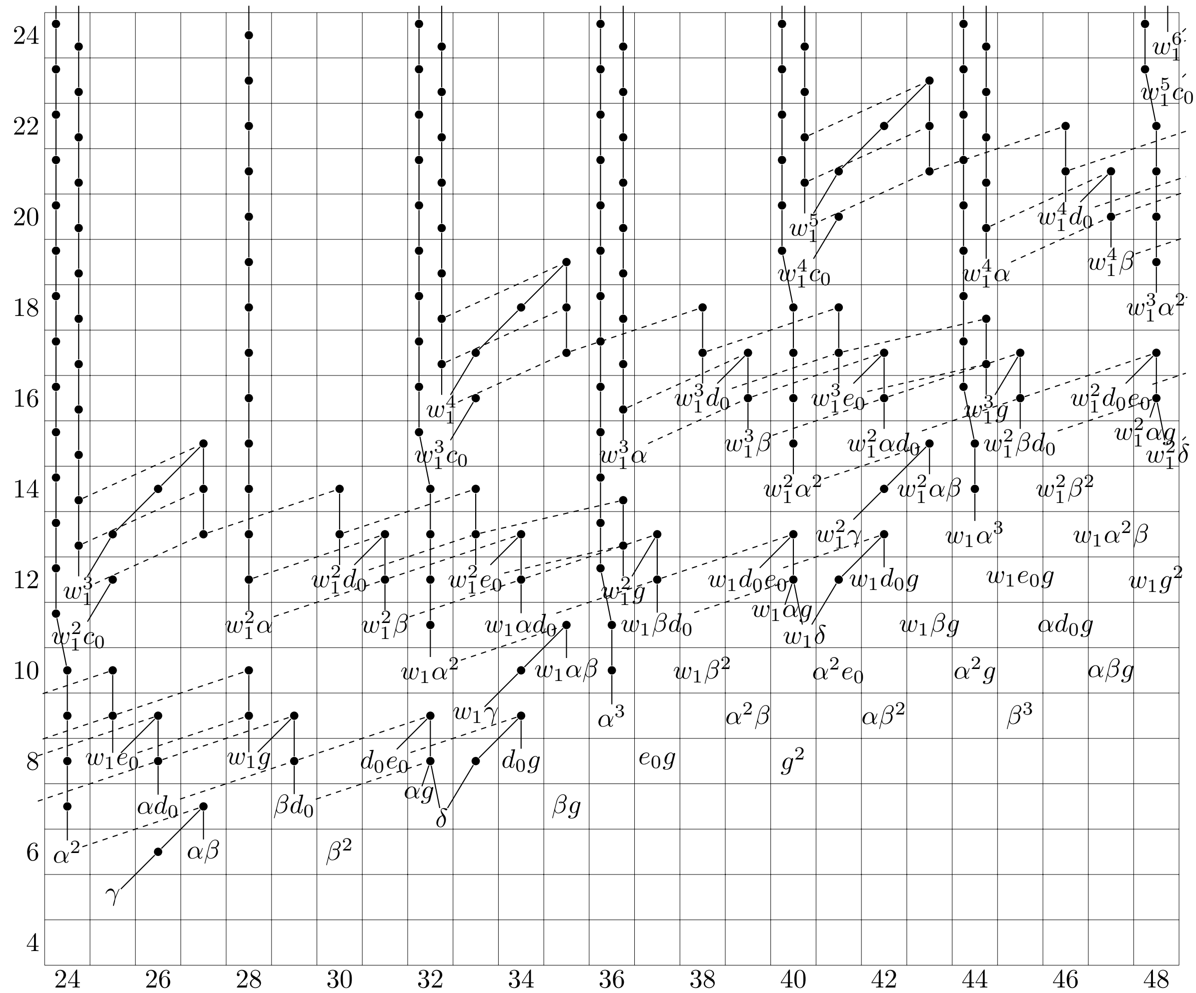


Figure 53: Adams E_2-term for tmf, $24 \leq t-s \leq 48$ (v_2^8-multiplies omitted)

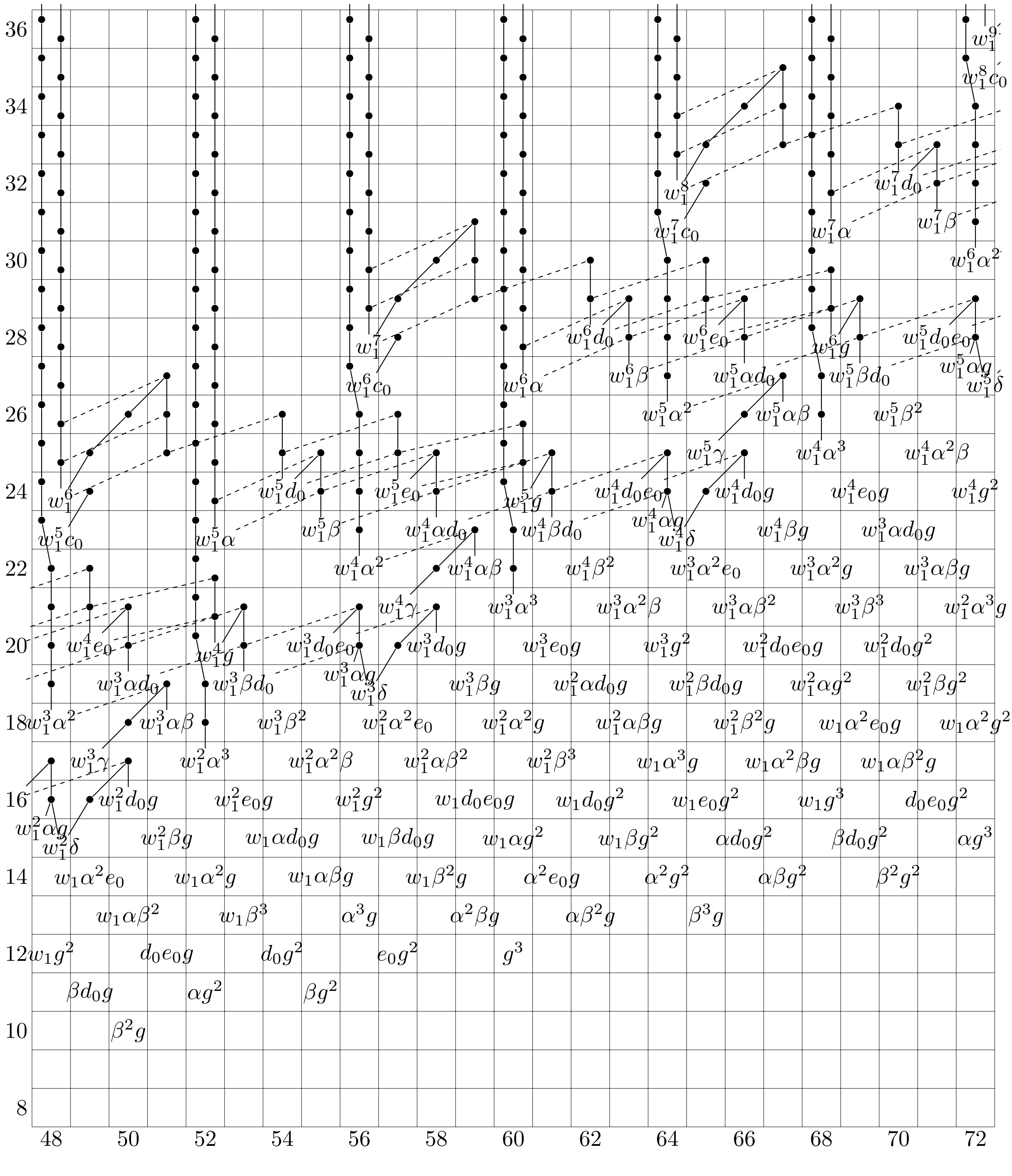


Figure 54: Adams E_2-term for tmf, $48 \le t-s \le 72$ (v_2^8-multiplies omitted)

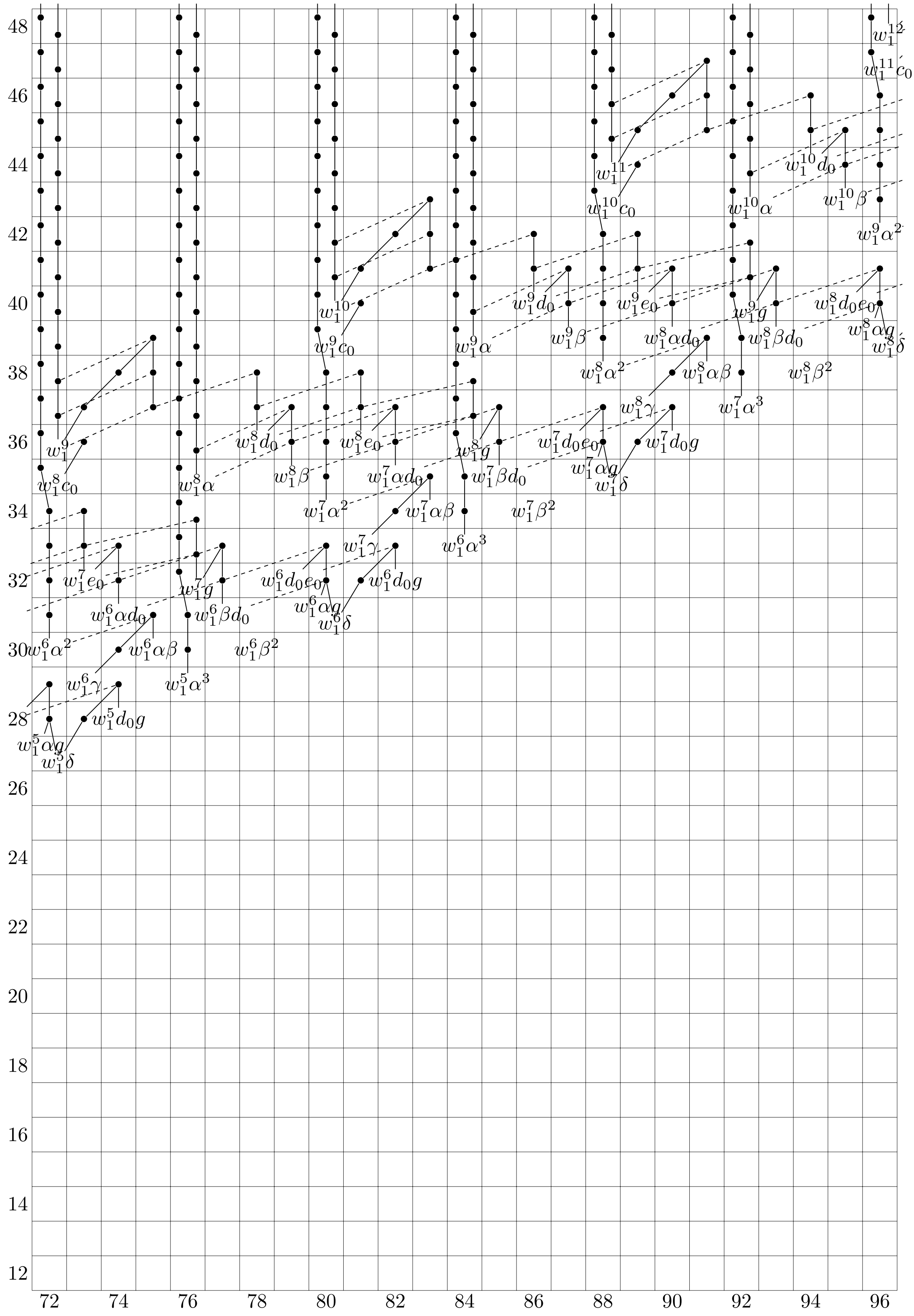


Figure 55: Adams E_2-term for tmf, $72 \leq t-s \leq 96$ (v_2^8-multiplies omitted)

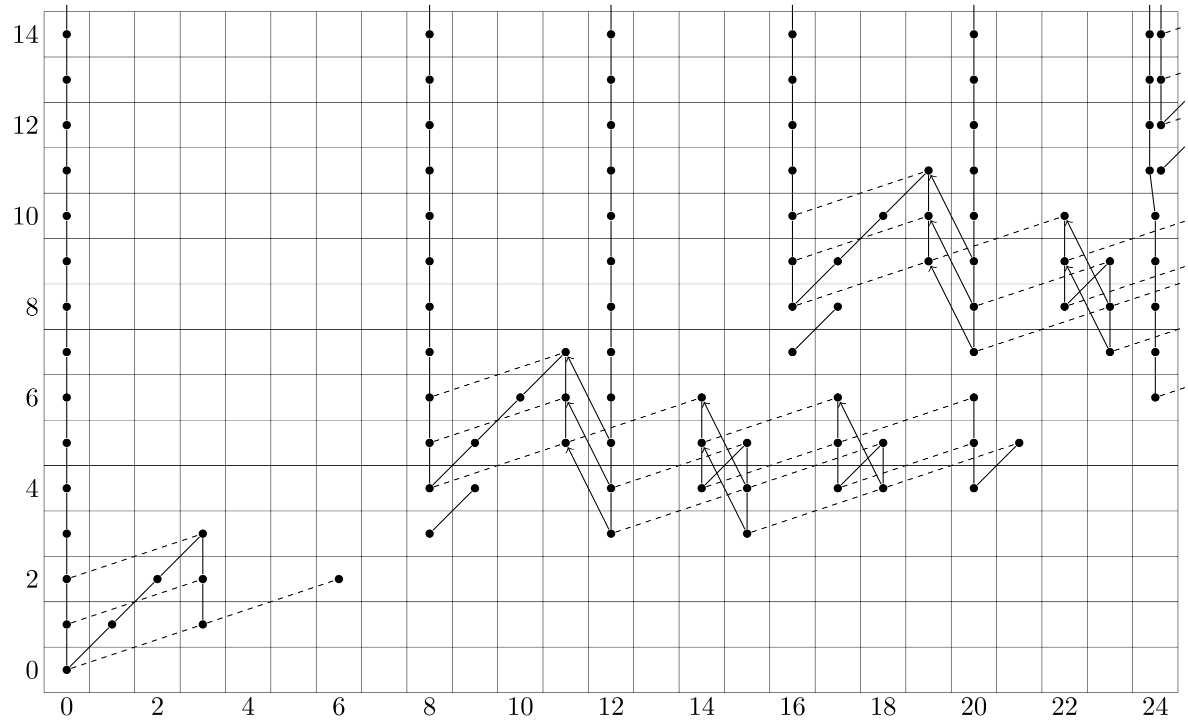


Figure 56: Adams d_2-differentials for tmf, $0 \le t-s \le 24$

$w_1h_0^2g$, $d_2(\beta g) = h_0d_0g$ and $d_2(\alpha^3) = w_1h_0\alpha\beta$, together with all of their w_1-power multiples.

Proof. These all follow from the differential $d_2(h_2\beta) = h_0^2e_0$, which either follows by naturality with respect to the map $S \to tmf$ (taking f_0 to $h_2\beta$, using the known differential $d_2(f_0) = h_0^2e_0$ for the sphere) or directly from the H_∞ structure on tmf (using the formula $d_3(Sq^{10}(c_0)) = h_0Sq^9(c_0)$ for the infinite cycle c_0, where $Sq^9(c_0) = h_0d_0$ and $Sq^{10}(c_0) = h_2\beta$, see Bruner (1986) §VI.1). □

(($d_2 = 0$ on h_0, h_1, h_2, c_0, w_1, d_0, e_0, g, γ and δ.))

Theorem 9.50. *There are nontrivial differentials $d_2(w_2) = \alpha\beta g$, $d_2(w_2\alpha) = \alpha^2\beta g + w_2w_1h_2$, $d_2(w_2h_0\alpha) = w_2w_1h_0h_2$, $d_2(w_2h_0^2\alpha) = w_2w_1h_0^2h_2$, $d_2(w_2d_0) = \alpha\beta d_0g = \alpha^2e_0g$, $d_2(w_2\beta) = \alpha\beta^2g + w_2h_0d_0$, $d_2(w_2h_0\beta) = w_2h_0^2d_0$, $d_2(w_2e_0) = \alpha\beta e_0g = \alpha^2g^2$, $d_2(w_2h_2\beta) = w_2h_0^2e_0$, $d_2(w_2g) = \alpha\beta g^2$, $d_2(w_2\gamma) = \alpha\beta\gamma g = \alpha g^3$, $d_2(w_2\alpha^2) = \alpha^3\beta g = d_0e_0g^2$, $d_2(w_2\alpha\beta) = \alpha^2\beta^2 g = d_0g^3$, $d_2(w_2\beta^2) = \alpha\beta^3 g = e_0g^3$, $d_2(w_2\alpha d_0) = \alpha^2\beta d_0g + w_2w_1h_0e_0$, $d_2(w_2\beta d_0) = \alpha^3g^2 + w_2w_1h_0g$, $d_2(w_2\alpha g) = \alpha^2\beta g^2 + w_2w_1h_2g$, $d_2(w_2\beta g) = \alpha\beta^2g^2 + w_2h_0d_0g$, ((ETC)), together with all their w_1-power multiples.*

Proof. We use the relation

$$\gamma^2 = \beta^2 g + w_2h_1^2$$

in $\mathrm{Ext}_{A(2)}$. By h_0-linearity, γ survives (at least) to E_6. We shall prove in Theorem 9.56 below that $d_4(\beta^2g) = w_1\alpha^2e_0 \ne 0$. This implies that $d_4(w_2h_1^2) = w_1\alpha^2e_0 \ne 0$. Suppose, for a contradiction, that $d_2(w_2) = 0$. Then w_2 survives at least to E_5, since $d_3(w_2)$ and $d_4(w_2)$ live in trivial groups, and this implies that $w_2h_1^2$ survives to E_5, contradicting the fact that $d_4(w_2h_1^2) \ne 0$. Hence $d_2(w_2)$ is nonzero, and the only possible value is $\alpha\beta g$.

The other differentials follow from $d_2(w_2) = \alpha\beta g$ by the Leibniz rule. □

The d_2-differentials are displayed in Figures 56, 57, 58 and **??**. The resulting E_3-terms appear in Figures 59, 60, 61 and **??**.

Theorem 9.51. *There are nontrivial differentials $d_3(\alpha^2) = w_1h_1d_0$, $d_3(\beta^2) = w_1h_1g$, $d_3(e_0) = w_1c_0$ and $d_3(h_1e_0) = w_1h_1c_0$, together with all their w_1-power multiples.*

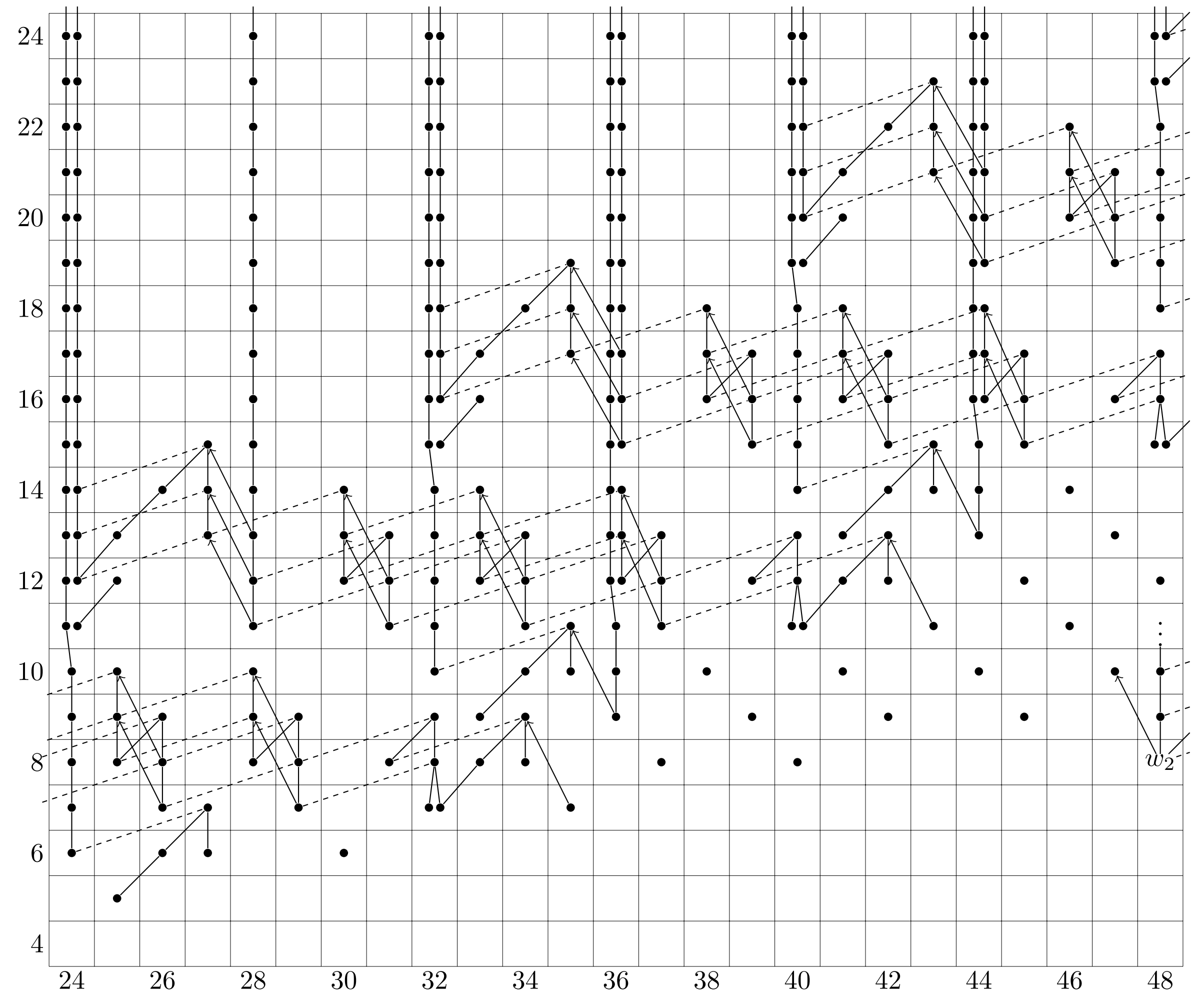


Figure 57: Adams d_2-differentials for tmf, $24 \le t - s \le 48$

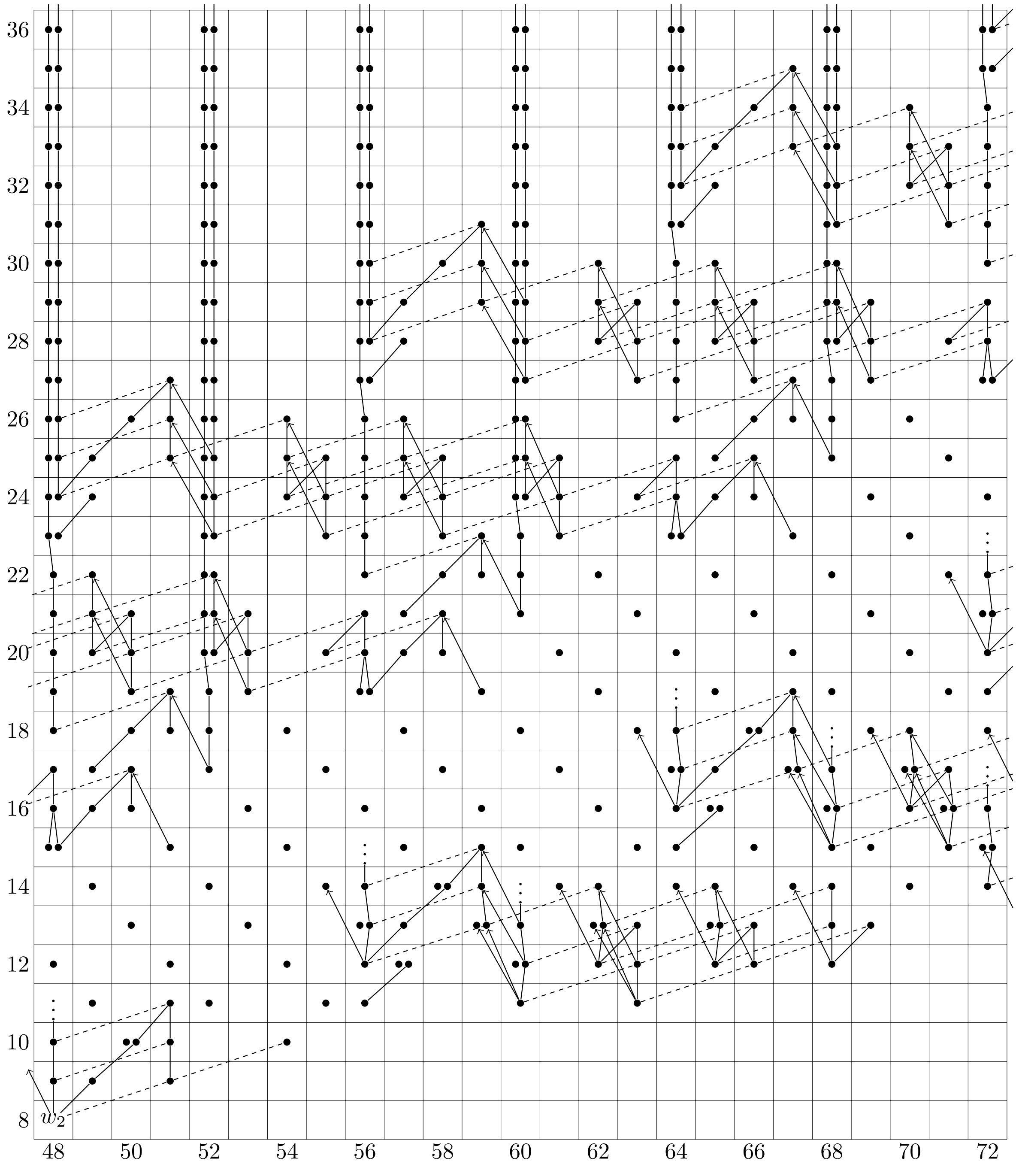


Figure 58: Adams d_2-differentials for tmf, $48 \leq t - s \leq 72$

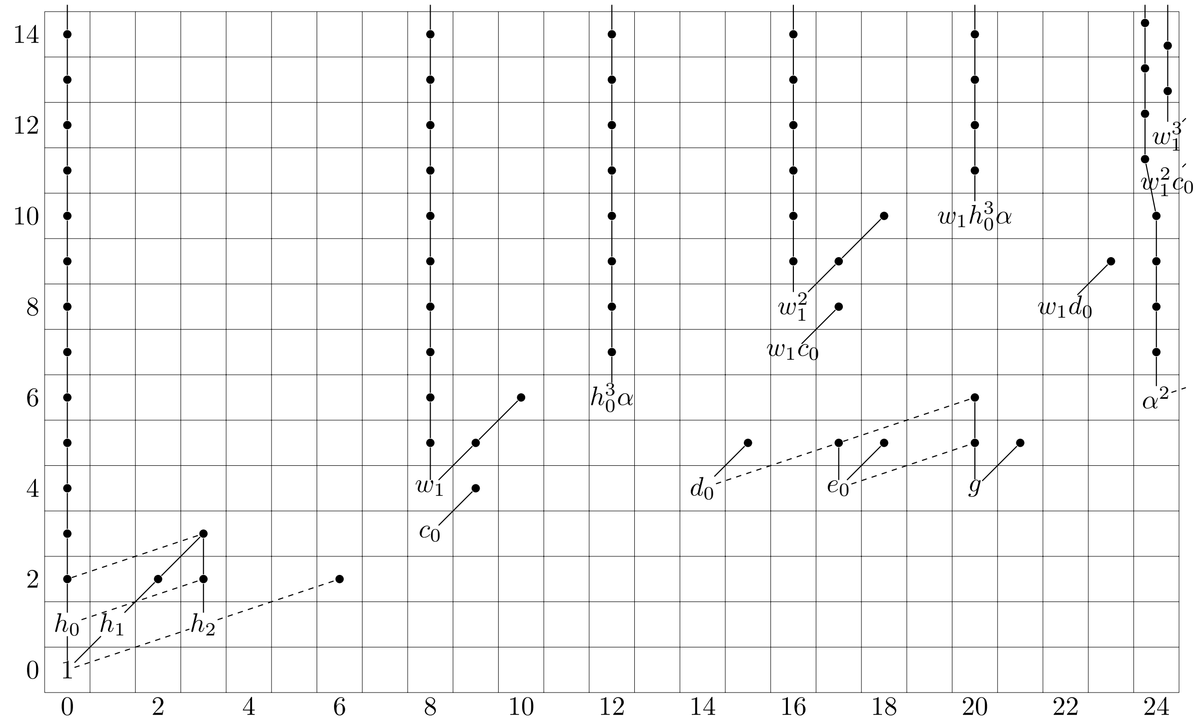


Figure 59: Adams E_3-term for tmf, $0 \le t - s \le 24$

Proof. The first two follow from the H_∞ structure on tmf, using Bruner's formulas $d_3(Sq^{12}(\alpha)) = Sq^{12}(w_1h_2)+h_0\cdot\alpha\cdot w_1h_2 = w_1h_1d_0$ (here $Sq^{12}(w_1h_2) = Sq^9(w_1)h_2^2 = 0$) and $d_3(Sq^{15}(\beta)) = Sq^{15}(h_0d_0) = h_1d_0^2 = w_1h_1g$.

The other two differentials follow from considerations of the image of J. The class $\eta^2\rho$ in $\mathrm{im}(J) \subset \pi_{17}(S)$ is detected by h_1Pc_0 in the Adams spectral sequence for S, which maps to $w_1h_1c_0$ in the Adams spectral sequence for tmf. The class ρ in $\mathrm{im}(J) \subset \pi_{15}(S)$ maps to a class in $\pi_{15}(tmf) = \mathbb{Z}/2$ that is either 0 or the image of $\eta\kappa$. Hence $\eta^2\rho$ maps either to 0 or the image of $\eta^3\kappa = 4\nu\kappa$. But $\nu\kappa$ is detected by h_2d_0 in Adams filtation 5, and there are no infinite cycles in Adams filtrations 6 or 7 for tmf, so $4\nu\kappa$ cannot be detected by $w_1h_1c_0$ in Adams filtration 8. Hence $\eta^2\rho$ maps to 0 in tmf, and $w_1h_1c_0$ must be a boundary. The only possibility is $d_3(h_1e_0) = w_1h_1c_0$, which also implies $d_3(e_0) = w_1c_0$.

Alternatively, we can use the relation $\eta\rho = \sigma\mu$ in $\pi_*(S)$, and the fact that σ maps to 0 in tmf, do deduce that $\eta\rho$ maps to 0 in tmf. This class is detected by Pc_0 in S, which maps to w_1c_0 in the Adams spectral sequence for tmf, so that infinite cycle cannot survive to E_∞, and must be a boundary. The only possibility is $d_3(e_0) = w_1c_0$. □

This accounts for all the possible d_3-differentials starting above the Mahowald–Tangora wedge. The possible d_3-differentials going out of that wedge are the w_1-power multiples of the following two cases.

Theorem 9.52. *$d_3(\alpha^2e_0) = 0$ and $d_3(\alpha\beta^2) = w_1h_1\delta$.*

Proof. We shall prove below that $d_4(\beta^2g) = w_1\alpha^2e_0$, so that $w_1\alpha^2e_0$ is an infinite cycle. We may divide by w_1 to deduce that α^2e_0 is an infinite cycle.

We shall prove below that $d_4(\alpha\beta^2g) = w_1^2\beta^3$, which is nonzero at E_4, by inspection of $\mathrm{Ext}_{A(2)}$. Suppose that $d_3(\alpha\beta^2) = 0$. We cannot have $d_4(\alpha\beta^2) = w_1^2\gamma$, since $h_1\alpha\beta^2 = 0$, but $w_1^2h_1\gamma \neq 0$ at E_4. The other possibility is $d_4(\alpha\beta^2) = 0$, which would imply $d_4(\alpha\beta^2g) = 0$, contradicting the formula above. Hence $d_3(\alpha\beta^2) \neq 0$, and $w_1h_1\delta$ is the only possible value. □

Theorem 9.53. *$d_3(w_2h_1) = w_1g^2$ and $d_3(w_2^2) = \beta g^4$. ((ETC: Consequences.))*

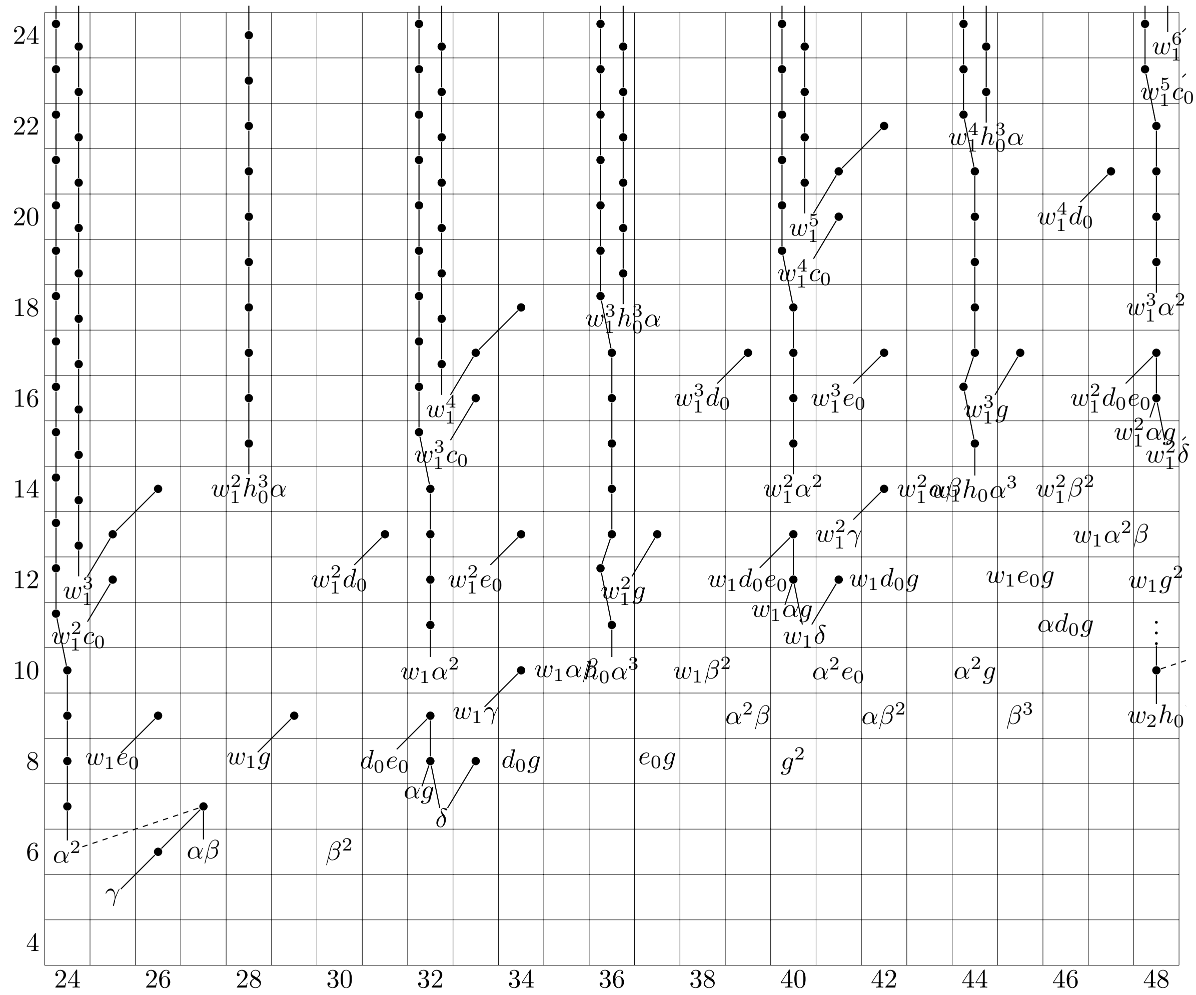


Figure 60: Adams E_3-term for tmf, $24 \leq t-s \leq 48$ (h_0-tower on w_2h_0 truncated)

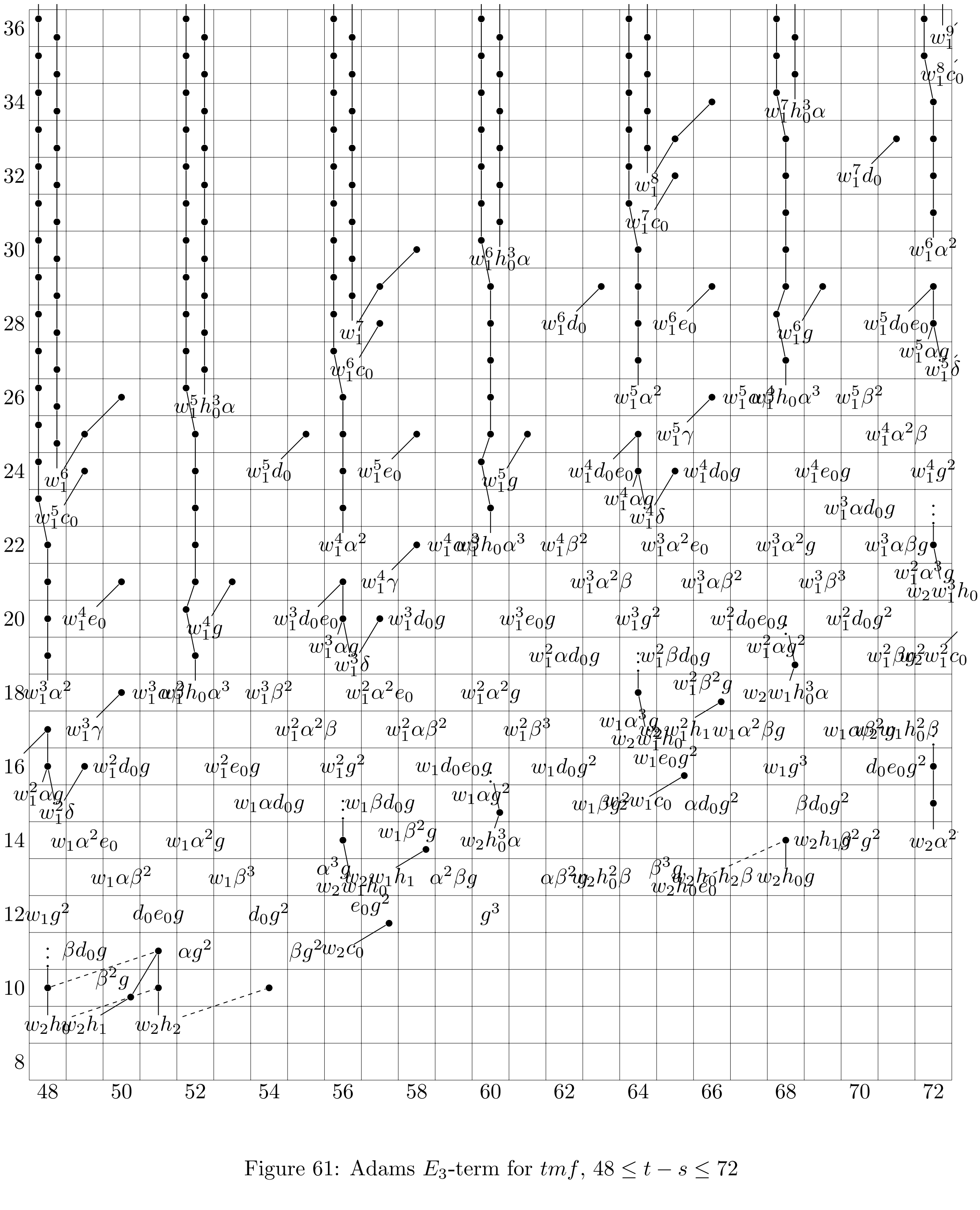


Figure 61: Adams E_3-term for tmf, $48 \leq t - s \leq 72$

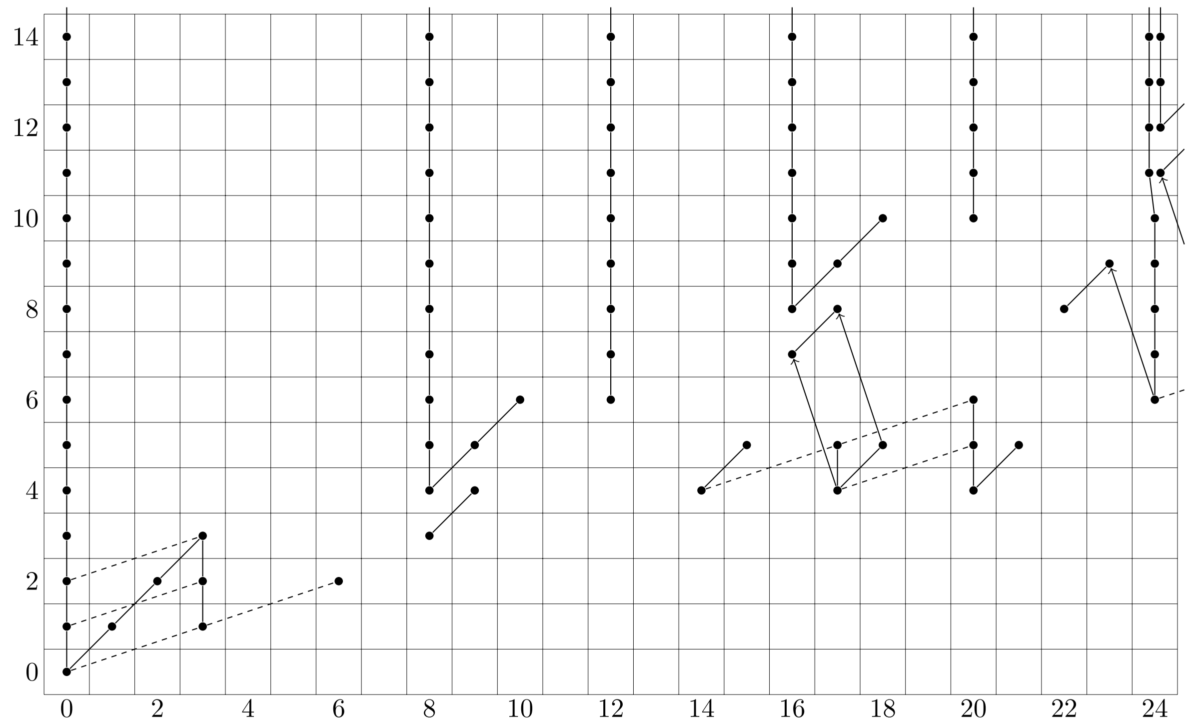


Figure 62: Adams d_3-differentials for tmf, $0 \le t-s \le 24$

Proof. The first differential follows from the relation $\gamma^2 = \beta^2 g + w_2 h_1^2$. We saw above that $d_4(w_2 h_1^2) = d_4(\beta^2 g) = w_1 \alpha^2 e_0 \neq 0$. Suppose for a contradiction that $d_3(w_2 h_1) = 0$. Then $d_4(w_2 h_1) = 0$ by h_0-linearity, which would imply that $d_4(w_2 h_1^2) = 0$. This shows that $d_3(w_2 h_1) \neq 0$, and by h_0-linearity again the only possible value is $w_1 g^2$.

The second differential follows from Bruner's formula $d_3(Sq^{48}(w_2)) = Sq^{48}(\alpha\beta g) + h_0 \cdot w_2 \cdot \alpha\beta g = Sq^{13}(\alpha)\beta^2 g^2 = \beta g^4$, where we use that $Sq^{13}(\alpha) = \gamma$ and $\beta\gamma = g^2$. $\square$

((Transport d_3-differentials back to S.))

The d_3-differentials are displayed in Figures 62, 63, 64 and **??**. The resulting E_4-terms appear in Figures 65, 66, **??** and **??**.

Remark 9.54. The differential $d_4(e_0 g) = P^2 g$ in the Adams spectral sequence for S is one of the key results of Mahowald–Tangora (1967). One could use naturality with respect to the map $S \to tmf$ to deduce the corresponding differential $d_4(e_0 g) = w_1^2 g$ in the Adams spectral sequence for tmf, but in fact it is far easier to deduce the tmf-differential directly. Using naturality in the opposite direction then gives a simplified proof of the Mahowald–Tangora differential.

Theorem 9.55. *There are nontrivial d_4-differentials $d_4(d_0 e_0) = w_1^2 d_0$ and $d_4(e_0 g) = w_1^2 g$, together with all their w_1-power multiples. ((Also g-multiples. When are these nonzero?))*

Proof. We know that $\kappa \in \pi_{14}(S)$ and $\eta^2\bar{\kappa} \in \pi_{22}(S)$ are detected by d_0 and Pd_0, respectively, in the Adams spectral sequence for S. The images in $\pi_{14}(tmf)$ and $\pi_{22}(tmf)$ are then detected by d_0 and $w_1 d_0$, respectively, in the Adams spectral sequence for tmf. Hence the product $\kappa \cdot \eta^2\bar{\kappa}$ is detected by $w_1 d_0^2 = w_1^2 g$ in the Adams spectral sequence for tmf. But $\eta^2\kappa = 0$ in $\pi_{16}(tmf) \cong \mathbb{Z}_2$, so this product is 0 and $w_1^2 g$ must be a boundary. The only possibility is $d_4(e_0 g) = w_1^2 g$.

Multiplying with w_1, we get $d_4(w_1 e_0 g) = w_1^3 g$. We can rewrite this as $d_4(d_0^2 e_0) = w_1^2 d_0^2$. We can divide by d_0 to deduce $d_4(d_0 e_0) = w_1^2 d_0$. $\square$

((Display the differential behavior in the indexed chart E_0?))

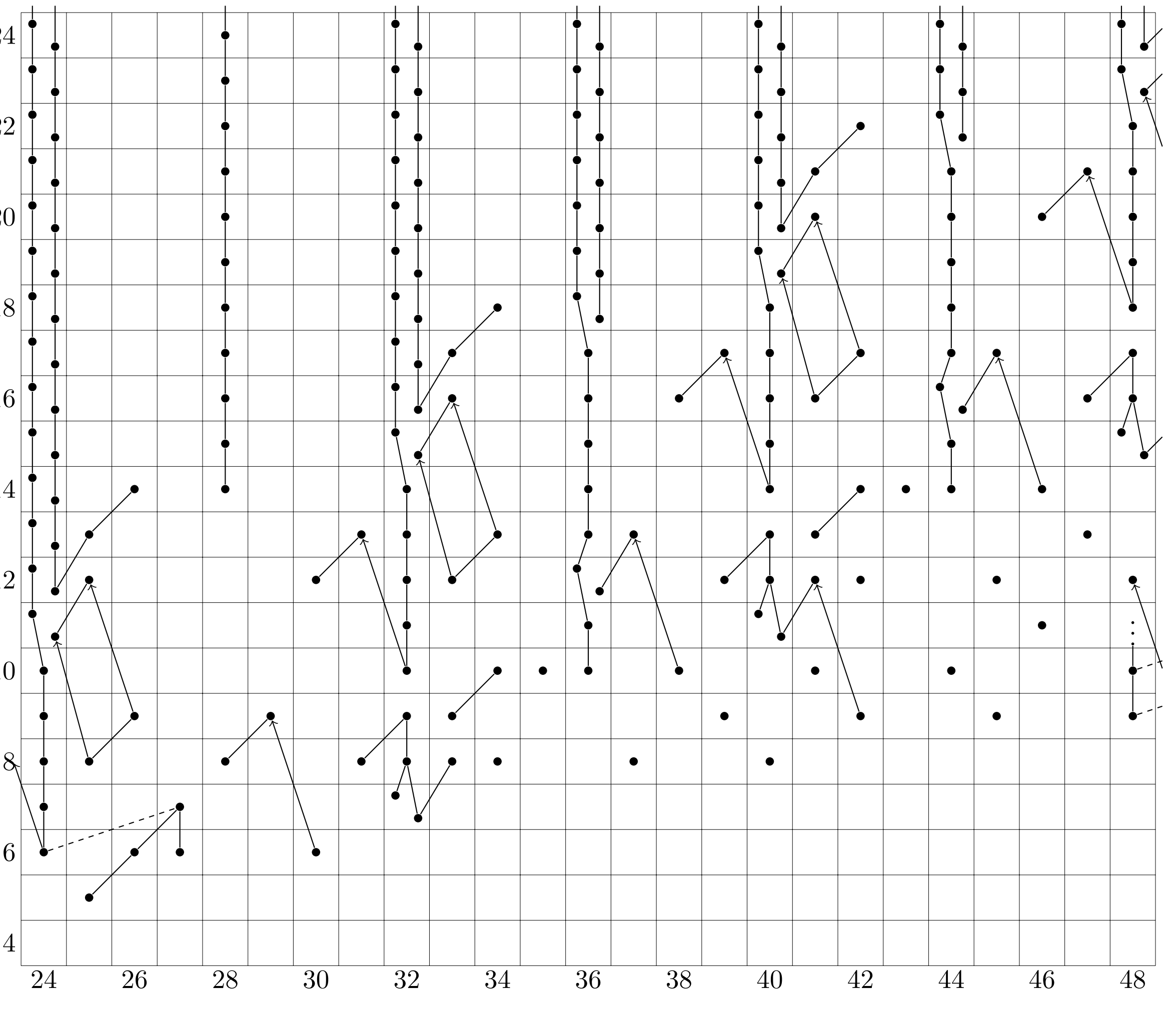


Figure 63: Adams d_3-differentials for tmf, $24 \leq t-s \leq 48$ (h_0-tower on w_2h_0 truncated)

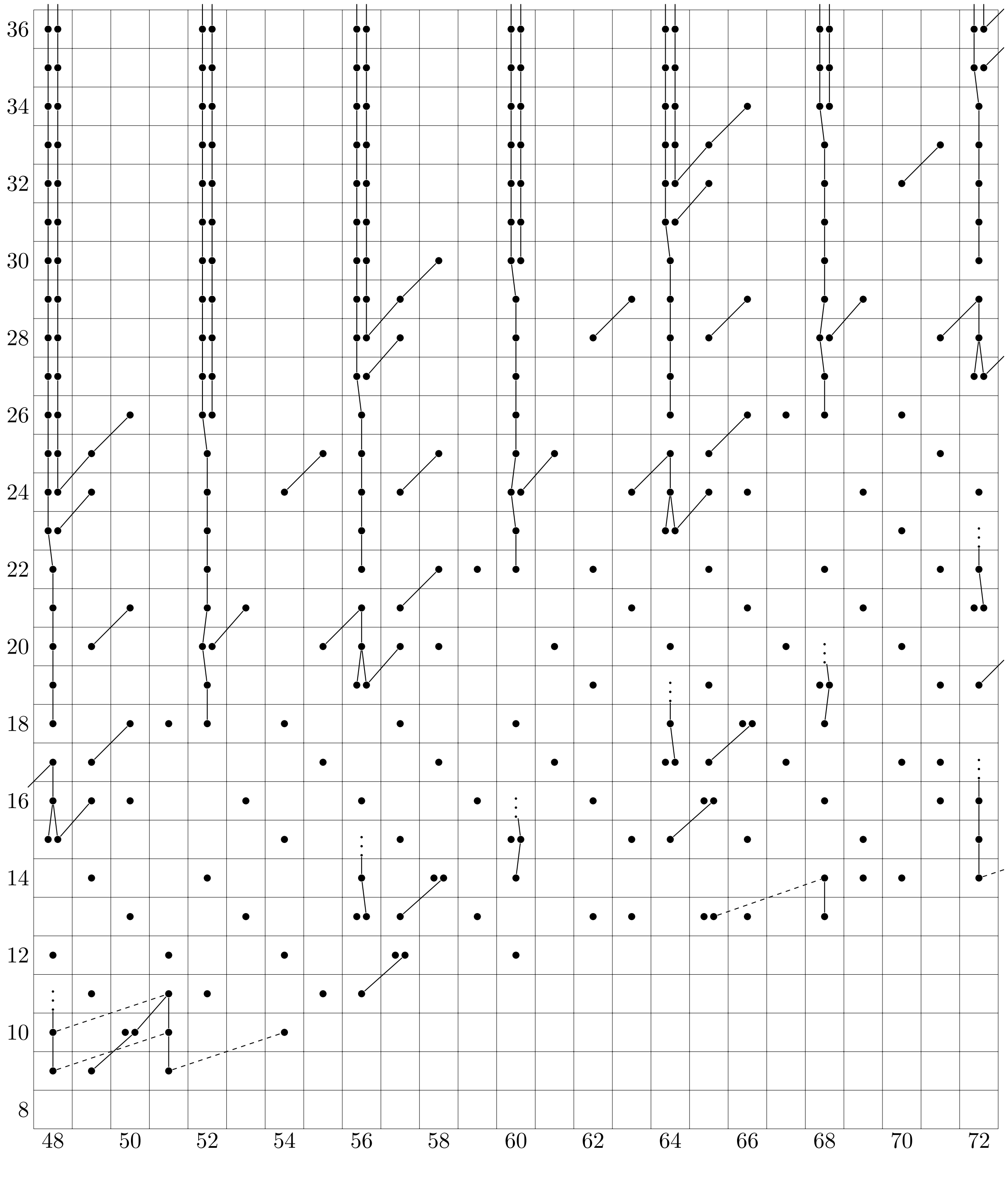


Figure 64: Adams d_3-differentials for tmf, $48 \leq t-s \leq 72$ ((incomplete))

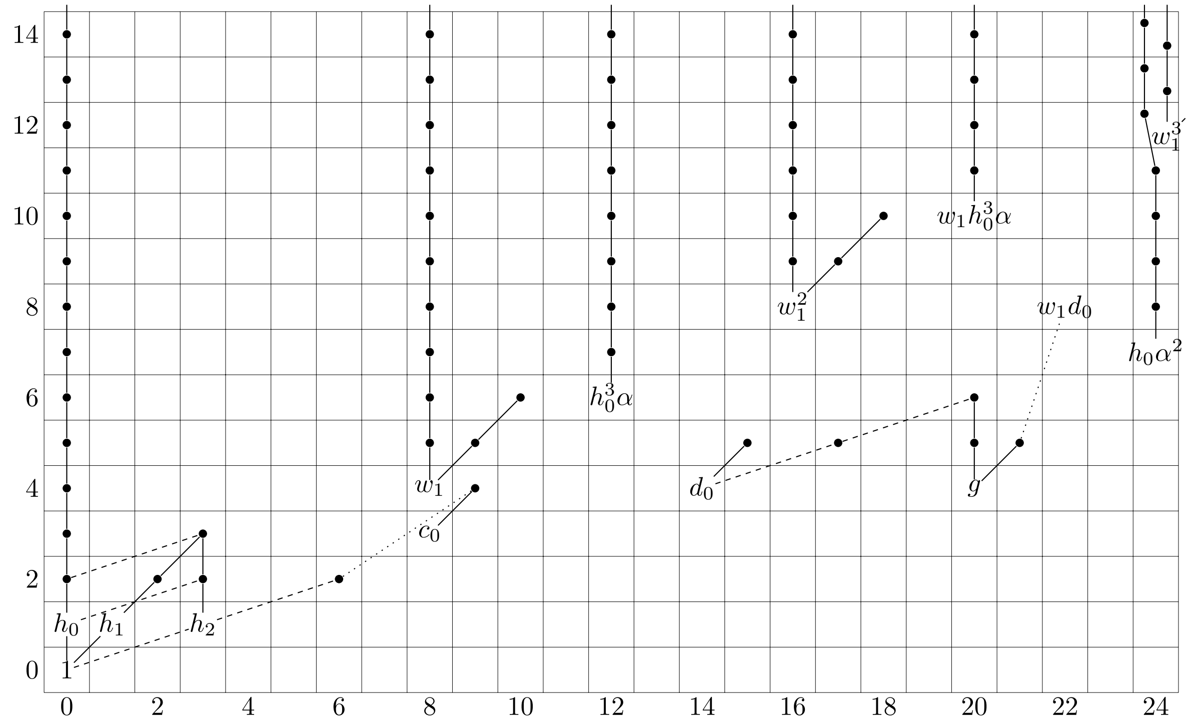


Figure 65: Adams $E_4 = E_\infty$-term for tmf, $0 \le t-s \le 24$

We can propagate these differentials to cover the Mahowald–Tangora wedge, as in their 1968 paper. See Figure 67.

Theorem 9.56. *There are d_4-differentials $d_4(e_0g) = w_1^2 g$, $d_4(d_0e_0) = w_1^2 d_0$, $d_4(\alpha^3 g) = w_1^2\alpha^2\beta$, $d_4(\alpha\beta^2 g) = w_1^2\beta^3$, $d_4(\beta^2 g) = w_1\alpha^2 e_0$, $d_4(\alpha^2 g) = w_1\alpha\beta$, $d_4(\beta d_0 g) = w_1^2\alpha g$ and $d_4(\beta g^2) = w_1\alpha d_0 g$, together with all their w_1- and g-power multiples. Not all of these multiples are nonzero, since the target classes may be d_2- or d_3-boundaries.*

Proof. The differentials originating in Adams filtration $s \equiv 0 \mod 4$, on d_0e_0 and e_0g, are already known.

The class $\alpha^2\beta \in E_4^{9,48}$ is an infinite cycle, so we get differentials $d_4(\alpha^3g^2) = w_1^2\alpha^2\beta g$ and $d_4(w_1\alpha\beta^2 g) = w_1^2\alpha^2\beta d_0 = w_1^3\beta^3$ in filtrations $s \equiv 1 \mod 4$, since $\alpha^2\beta \cdot e_0 g = \alpha^3 g^2$ and $\alpha^2\beta \cdot d_0e_0 = w_1\alpha\beta^2 g$. We can divide these by g and w_1, respectively.

The class $\alpha^2 e_0 \in E_4^{10,51}$ is an infinite cycle, so we get differentials $d_4(w_1\beta^2g^2) = w_1^2\alpha^2 e_0 g$ and $d_4(w_1\alpha^2 g^2) = w_1^4\alpha^2 d_0 e_0 = w_1^3\alpha\beta g$ in filtrations $s \equiv 2 \mod 4$, since $\alpha^2 e_0 \cdot e_0 g = w_1\beta^2 g^2$ and $\alpha^2 e_0 \cdot d_0 e_0) = w_1\alpha^2 g^2$. We can divide both of these by $w_1 g$.

The class $\alpha g \in E_4^{7,39}$ is an infinite cycle, so we get differentials $d_4(\beta d_0 g^2) = w_1^2\alpha g^2$ and $d_4(w_1\beta g^2) = w_1^2\alpha d_0 g$ in filtrations $s \equiv 3 \mod 4$, since $\alpha g \cdot e_0 g = \beta d_0 g^2$ and $\alpha g \cdot d_0 e_0 = w_1\beta g^2$. We can divide these by g and w_1, respectively. □

Theorem 9.57. *$d_4(w_2h_0) = w_1\alpha^2\beta$ and $d_4(w_2h_1^2) = w_1\alpha^2 e_0$.*

Proof. ((TODO: How to prove the first differential?))

The second differential has been discussed before; it follows from the relation $\gamma^2 = \beta^2 g + w_2h_1^2$, the fact that γ is an infinite cycle, and the Mahowald–Tangora differential $d_4(\beta^2 g) = w_1\alpha^2 e_0$. □

The d_4-differentials are displayed in Figures 68, **??** and **??**. The resulting E_5-terms appear in Figures 69, **??** and **??**.

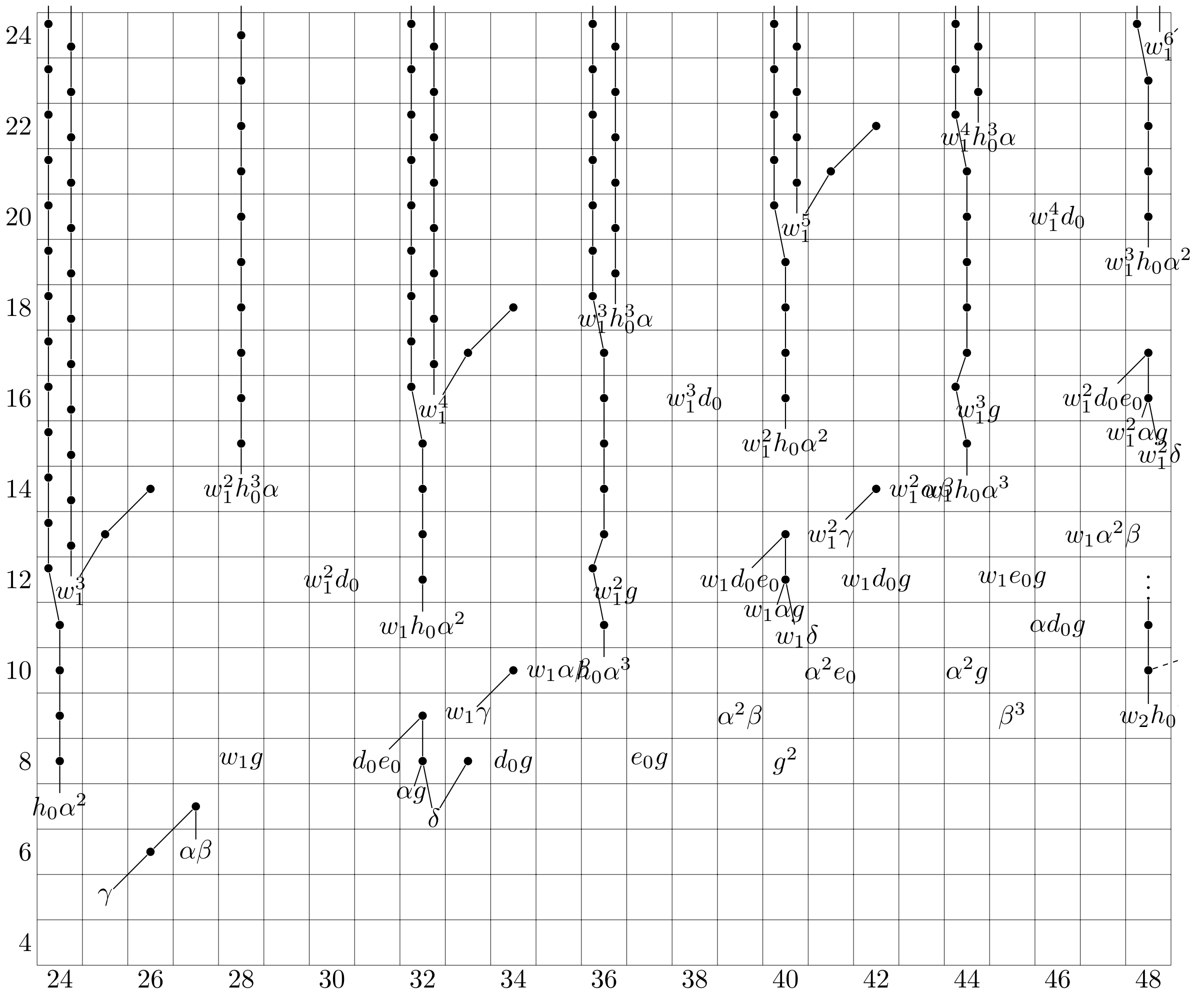


Figure 66: Adams E_4-term for tmf, $24 \le t-s \le 48$ (h_0-tower on w_2h_0 truncated)

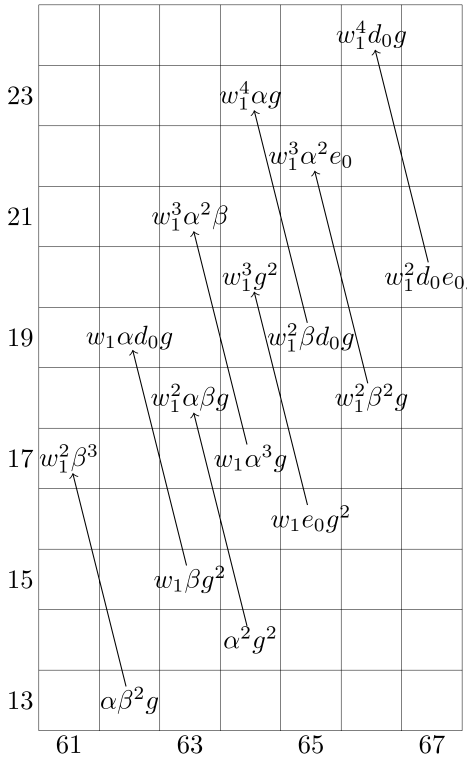


Figure 67: Ideal d_4-differentials in Mahowald–Tangora wedge

9.7 The Adams spectral sequences for $tmf/2$ and tmf/η

((Determine Adams differentials. Get hidden multiplications by 2 or η.))

10 Low filtrations

10.1 Quotient algebras

((Quotients of $\mathscr{A}$ dual to $P(\xi_1, \ldots, \xi_n) \subset \mathscr{A}_*$).))

((Ext-calculations.))

10.2 The bar and cobar complexes

(((Co-)bar resolution. (Co-)bar complex.))

((Discuss the free resolution that arises from the canonical Adams resolution.))

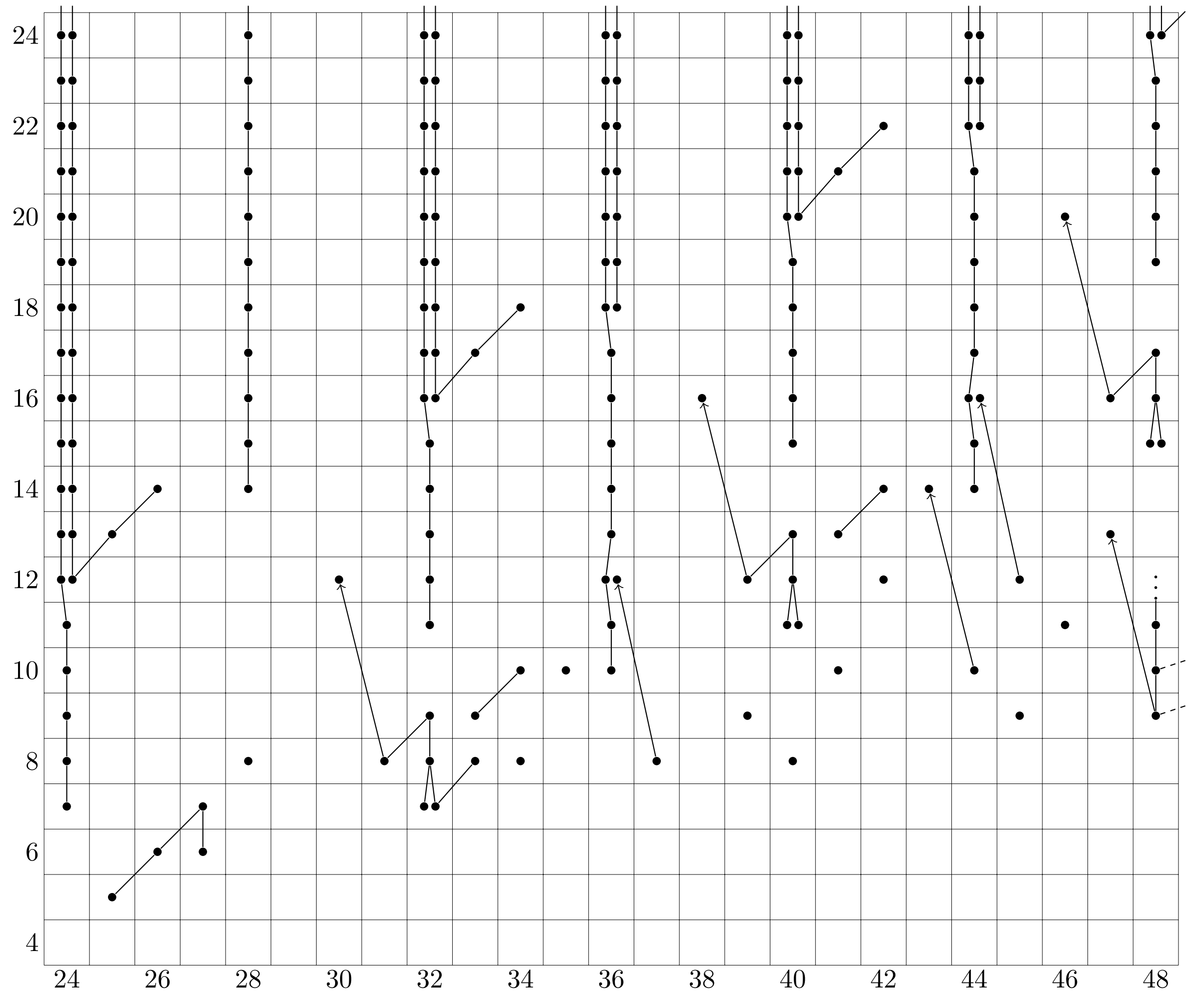


Figure 68: Adams d_4-differentials for tmf, $24 \le t - s \le 48$ (h_0-tower on $w_2 h_0$ truncated)

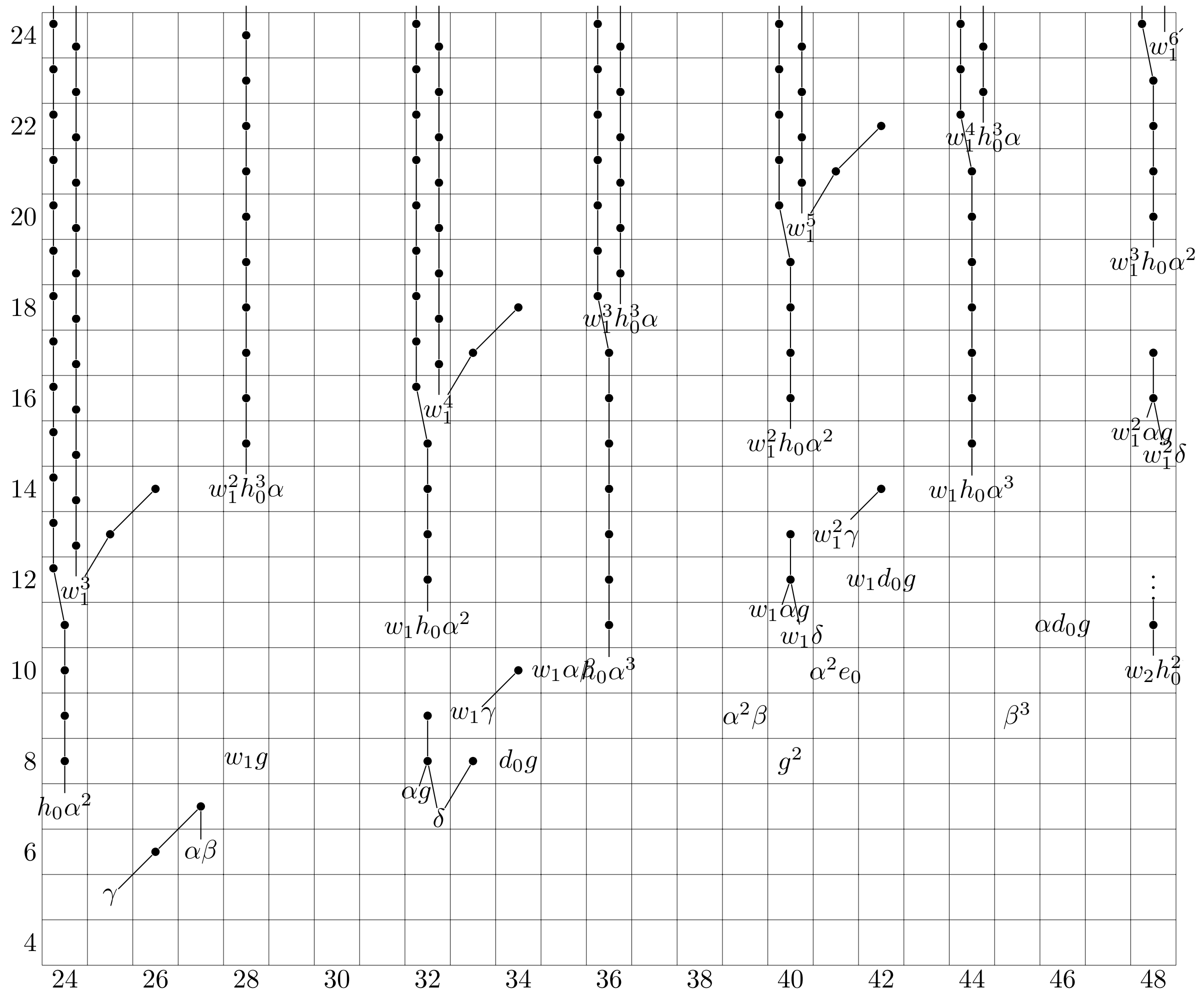


Figure 69: Adams E_5-term for tmf, $24 \leq t-s \leq 48$ (h_0-tower on $w_2 h_0^2$ truncated)